Comparing Symmetries of Nonlinear PDEs and their Linearizations

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Introduction and Background

OBJECTIVE: Our research objective for this project was to calculate the symmetry groups for the Reduced Gibbons-Tsarev equation and the Born-Infeld equation, and then compare them to the symmetry groups for their linearizations.

THE EQUATIONS: Both the Born-Infeld and Reduced Gibbons-Tsarev equations are nonlinear, second-order partial differential equations in two independent variables. The Born-Infeld equation originally appeared in the context of nonlinear electrodynamics and is still found today in string theory in the description of the action of open strings. The Reduced-Gibbons Tsarev equation arose in the context of parametrizations of the Benney moment equations with finitely many (two) dependent variables. It finds use in the study of dispersionless systems.

Born-Infeld: $(u_y^2 - 1)u_{xx} - 2u_x u_y u_{xy} + (u_x^2 + 1)u_{yy} = 0$

Reduced Gibbons-Tsarev: $u_{xx} - u_y u_{xy} + u_x u_{yy} = 0$

Linearized Born-Infeld: $u_{xy} + 2(\frac{x u_x - y u_y}{x^2 - y^2}) = 0$

Linearized Reduced Gibbons-Tsarev: $u_{xy} + \frac{u_x - u_y}{x - y} = 0$

Methods

Approach

1. Define the total derivative operator.
2. Define the prolongated infinitesimal operator.
3. Calculate Lie's Invariance Condition.
4. Separate into a system of determining equations.
5. Solve for the symmetry-generating infinitesimals.
6. * Calculate the vector fields generated by the symmetries.
7. * Lie Bracket the resulting vector fields together.
8. * Characterize the resulting symmetry group.

Diagram

Classical Symmetries

Reduced Gibbons-Tsarev

$$X(x,y,u) = -2c_5 y + (-c_2 + 2c_4)x + c_7$$
$$Y(x,y,u) = -\frac{c_1 x}{2} + c_4 y + c_5 u + c_6$$
$$U(x,y,u) = c_1 y + c_2 u + c_3$$

Linearized Reduced Gibbons-Tsarev

$$X(x,y,u) = c_1 x^2 + c_3 x + c_4$$
$$Y(x,y,u) = c_1 y^2 + c_3 y + c_4$$
$$\begin{aligned} U(x,y,u) = {} & \frac{(-x+y)(\frac{d}{dx}f(x)}{2} + \frac{(-x+y)(\frac{d}{dy}g(y)}{2} \\ & + f(x) - g(y) + (c_1 x + c_1 y + c_2)u \end{aligned} \tag{1}$$

Born-Infeld

$$X(x,y,u) = c_1 x + c_2 u + c_3 y + c_4$$
$$Y(x,y,u) = c_1 y + c_3 x + c_5 u + c_6$$
$$U(x,y,u) = c_1 u - c_2 x + c_5 y + c_7$$

Linearized Born-Infeld

$$X(x,y,u) = \frac{1}{2}c_1 x^2 + c_4 x - \frac{1}{2}c_3$$
$$Y(x,y,u) = \frac{1}{2}c_1 y^2 + c_4 y - \frac{1}{2}c_3$$
$$\begin{aligned} U(x,y,u) = {} & \frac{1}{2(x+y)}((x-y)(\frac{d}{dx}f(x)) \\ & + (-x+y)(\frac{d}{dy}g(y)) - 2f(x) - 2g(y) \\ & + 2u(c_1 y + c_2)x + 2c_2 uy + 2c_3 u) \end{aligned} \tag{2}$$

Nonclassical Symmetries

Reduced Gibbons-Tsarev

Linearized Reduced Gibbons-Tsarev

Born-Infeld

Linearized Born-Infeld

Conclusion

References

[1] Arrigo, Daniel J. *Symmetry Analysis for Differential Equations: An Introduction.* Wiley (2015)