Comparing Classical and Nonclassical Symmetries of Nonlinear Partial Differential Equations

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Table of Contents

Project Goal

Our research objective for this project was to calculate the classical and nonclassical symmetry groups for the Born-Infeld equation and the reduced Gibbons-Tsarev equation and compare them.

What is a Symmetry?

Definition

A symmetry is a transformation that leaves an object invariant.

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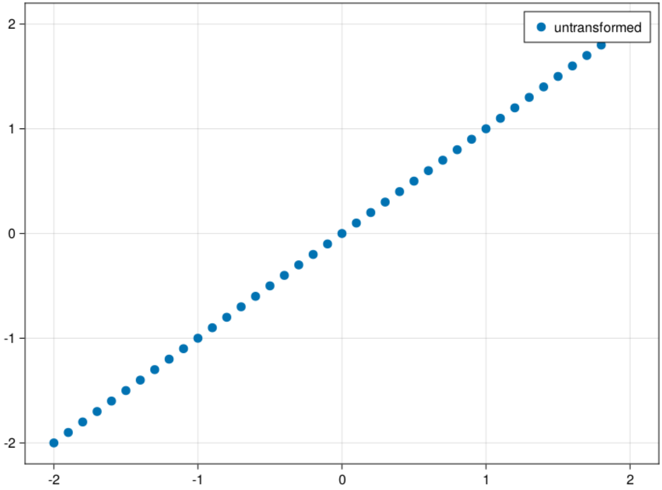
Definition

A symmetry is a change that doesn't change anything.

What is a Symmetry?

Let's see this in action using the simple linear equation $x - y = 0$.



What is a Symmetry?

Example (A Non-Example)

- For our first transformation, let's define new variables $\bar{x} = x + 1$ and $\bar{y} = y$.
- Now let's rewrite our equation using these new variables.

$$\begin{aligned} \bar{x} - \bar{y} &= 0 && \text{by definition} \\ x + 1 - y &= 0 && \text{by substitution} \\ y &= x + 1 && \text{by rewriting in slope-intercept form} \end{aligned}$$

- This transformation is not a symmetry: $x - y + 1 \neq x - y$

What is a Symmetry?

A Transformation that is a Symmetry

Example (2)

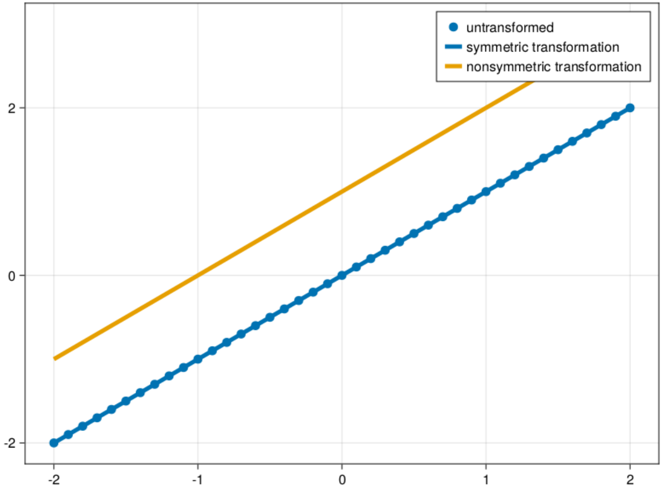
- Let's define some new variables again
  $\bar{x} = x + 1$ and $\bar{y} = y + 1$.
- Now let's rewrite our equation using these new variables.

$$\begin{aligned} \bar{x} - \bar{y} &= 0 && \text{by definition} \\ (x+1) - (y+1) &= 0 && \text{by substitution} \\ (x-y) + (1-1) &= 0 && \text{by algebra} \\ x - y &= 0 && \text{by algebra} \\ y &= x && \text{by rewriting in slope-intercept form} \end{aligned}$$

- This transformation is a symmetry:
  $x - y = x - y$

What is a Symmetry?

The graphs of our three equations.



What is a Symmetry?

Who cares?

- Symmetries help us to understand and to solve equations that we wouldn't normally be able to solve.

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- Symmetries help us to understand and to solve equations that we wouldn't normally be able to solve.
- Symmetries also encode physically meaningful aspects of equations, like conservation laws.

What is a Differential Equation?

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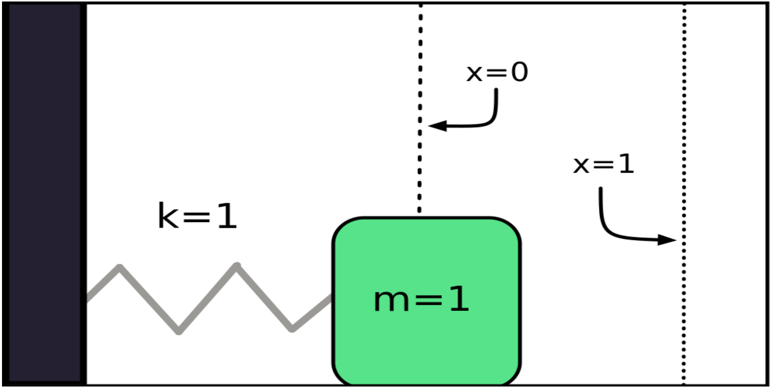
- Differential equations show up everywhere we model something using information about how that thing changes. This includes everything from population dynamics to planetary orbits.
- Differential equations are different from algebraic equations, and they can't be solved in the same ways.

What is a Differential Equation?

As an example, let's look a frictionless spring-mass system.



What is a Differential Equation?

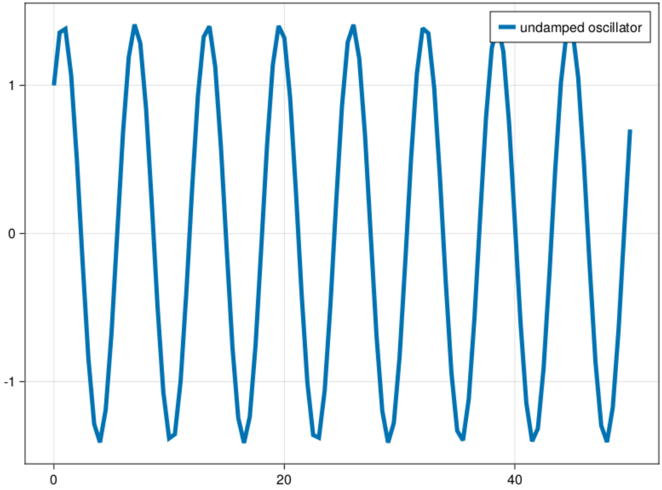
This system can be modeled by the following differential equation.

$$m\ddot{x} = -\kappa x$$

- Here m is the mass, $\ddot{x}$ is the acceleration, κ is the spring constant, and x is the position.
- If we set both m and κ to 1, we get $\ddot{x} = -x$.
- This is a simple differential equation with the algebraic solution of $x(t) = c_1 \cos(t) + c_2 \sin(t)$.

What is a Differential Equation?

This is the graph of our spring system where $c_1 = c_2 = 1$.



The History of the Born-Infeld and the reduced Gibbons-Tsarev Equations

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The goal of our project was to find symmetry transformations for these two PDEs, both classical and nonclassical, and compare them.
- It turns out that there's a standard method for calculating both kinds of symmetry transformations for differential equations.

$$\begin{aligned} \text{Input Equation} &\rightarrow \text{Calculate Lie's invariance condition} \\ &\rightarrow \text{Solve linear system of PDEs} \\ &\rightarrow \text{Symmetries!} \end{aligned}$$

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

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- For a differential equation, this means that the function and any of its derivatives that are present in the equation must not change under the transformation.
- Lie's invariance condition gives us a way to generate transformations that meet all these requirements.
- If you can solve Lie's invariance condition for your equation, you can find its symmetries.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

In general, Lie's invariance condition is the falsely unassuming

$$\Gamma^{(n)}\Delta|_{\Delta=0} = 0$$

But there's a lot of complexity hiding under this notation.

Here Γ is the infinitesimal operator, Δ is your differential equation, and n is its order.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

For our equations, the infinitesimal operator Γ looks like this:

$$T\frac{\partial}{\partial t}+X\frac{\partial}{\partial x}+U\frac{\partial}{\partial u}+U_{[t]}\frac{\partial}{\partial u_t}+U_{[x]}\frac{\partial}{\partial u_x}+U_{[tt]}\frac{\partial}{\partial u_{tt}}+U_{[tx]}\frac{\partial}{\partial u_{tx}}+U_{[xx]}\frac{\partial}{\partial u_{xx}}$$

$$\begin{aligned}
U_{[t]} &= D_t(U) - u_t D_t(T) - u_x D_t(X)\\
U_{[x]} &= D_x(U) - u_t D_x(T) - u_x D_x(X)\\
U_{[tt]} &= D_t(U_{[t]}) - u_{tt} D_t(T) - u_{tx} D_t(X)\\
U_{[tx]} &= D_t(U_{[x]}) - u_{tx} D_t(T) - u_{xx} D_t(X)\\
U_{[xx]} &= D_t(U_{[x]}) - u_{tx} D_x(T) - u_{xx} D_x(X)
\end{aligned}$$

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x}\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x}
\end{aligned}$$

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The result of putting our PDEs through Lie's invariance condition was a system of 24 linear PDEs for the Born-Infeld equation and 21 for the reduced Gibbons-Tsarev equation.

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- Though difficult, these systems were easier to solve than the original single equations because of their linearity.
- Solving them resulted in the following symmetries.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

Classical Symmetries of the Born-Infeld Equation

- $X(x, y, u) = c_1 x + c_2 u + c_3 y + c_4$
- $Y(x, y, u) = c_1 y + c_3 x + c_5 u + c_6$
- $U(x, y, u) = c_1 u - c_2 x + c_5 y + c_7$

Classical Symmetries of the reduced Gibbons-Tsarev Equation

- $X(x, y, u) = -2c_5 y + (-c_2 + 2c_4)x + c_7$
- $Y(x, y, u) = -\frac{c_1 x}{2} + c_4 y + c_5 u + c_6$
- $U(x, y, u) = c_1 y + c_2 u + c_3$

The Nonclassical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

Future Work: Does Integrability Imply Equivalence of Classical and Nonclassical Symmetries?

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