Comparing Classical and Nonclassical Symmetries of Nonlinear Partial Differential Equations

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Project Goal

Our research objective for this project was to calculate the classical and nonclassical symmetry groups for the Born-Infeld equation and the reduced Gibbons-Tsarev equation and compare them.

What is a Symmetry?

Definition

A symmetry is a transformation that leaves an object invariant.

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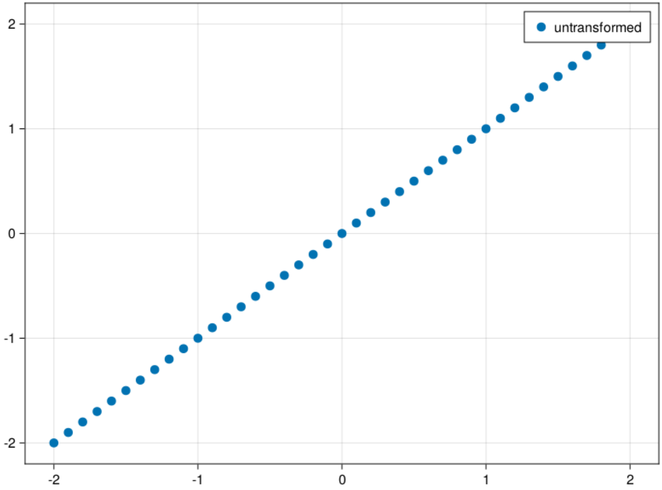
A symmetry is a transformation that leaves an object invariant.

Definition

A symmetry is a change that doesn't change anything.

What is a Symmetry?

Let's see this in action using the simple linear equation $x - y = 0$.



What is a Symmetry?

A Non-Example

Example (1)

- For our first transformation, let's define new variables $\bar{x} = x + 1$ and $\bar{y} = y$.
- Now let's rewrite our equation using these new variables.

$$\begin{aligned} \bar{x} - \bar{y} &= 0 && \text{by definition} \\ x + 1 - y &= 0 && \text{by substitution} \\ y &= x + 1 && \text{by rewriting in slope-intercept form} \end{aligned}$$

- This transformation is not a symmetry: $x - y + 1 \neq x - y$

Example (2)

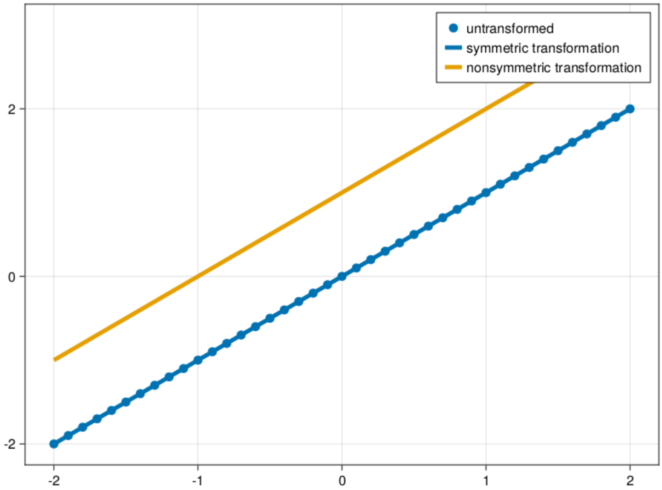
- Let's define some new variables again
  $\bar{x} = x + 1$ and $\bar{y} = y + 1$.
- Now let's rewrite our equation using these new variables.

$$
\begin{aligned}
\bar{x} - \bar{y} &= 0 && \text{by definition} \\
(x+1) - (y+1) &= 0 && \text{by substitution} \\
(x-y) + (1-1) &= 0 && \text{by algebra} \\
x - y &= 0 && \text{by algebra} \\
y &= x && \text{by rewriting in slope-intercept form}
\end{aligned}
$$

- This transformation is a symmetry:
  $x - y = x - y$

What is a Symmetry?

The graphs of our three equations.



What is a Symmetry?

Who cares?

- Symmetries help us to understand and to solve equations that we wouldn't normally be able to solve.

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- Symmetries help us to understand and to solve equations that we wouldn't normally be able to solve.
- Symmetries also encode physically meaningful aspects of equations, like conservation laws.

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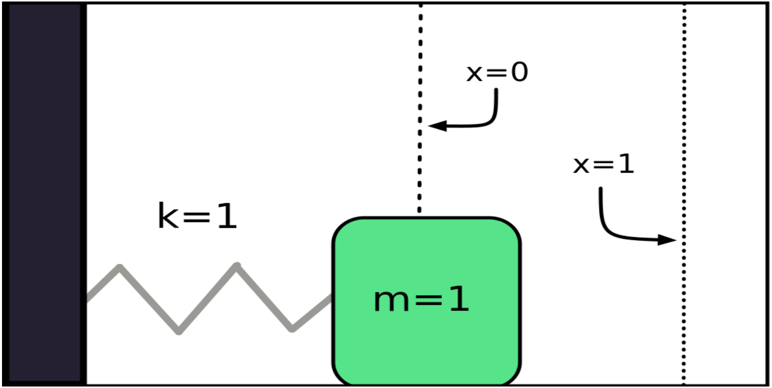
- Differential equations show up everywhere we model something using information about how that thing changes. This includes everything from population dynamics to planetary orbits.
- Differential equations are different from algebraic equations, and they can't be solved in the same ways.

What is a Differential Equation?

As an example, let's look a frictionless spring-mass system.



What is a Differential Equation?

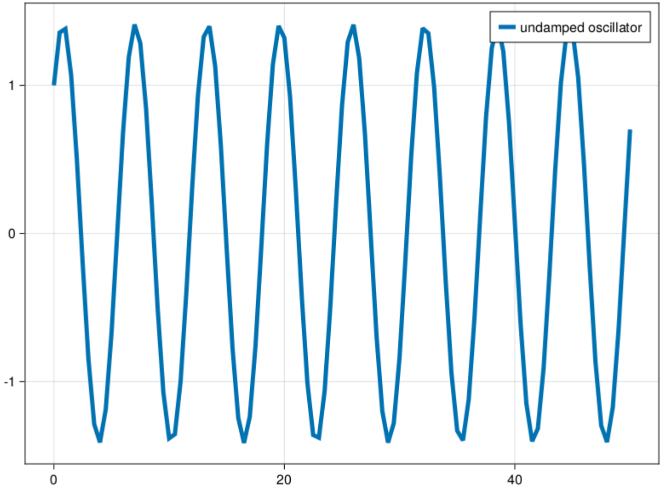
This system can be modeled by the following differential equation.

$$m\ddot{x} = -\kappa x$$

- Here m is the mass, $\ddot{x}$ is the acceleration, κ is the spring constant, and x is the position.
- If we set both m and κ to 1, we get $\ddot{x} = -x$.
- This is a simple differential equation with the algebraic solution of $x(t) = c_1 \cos(t) + c_2 \sin(t)$.

What is a Differential Equation?

This is the graph of our spring system where $c_1 = c_2 = 1$.



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- The Born-Infeld equation comes from a relativistic formulation of quantum electrodynamics (circa 1930).
- In classical electrodynamics the energy of a particle grows to infinity as its radius becomes very small.
- This is a problem if you want to study electrodynamics on the quantum scale.
- Max Born and Leopold Infeld's "fix" was to introduce a relativistic factor, similar to how $E = mc^2$ becomes $E = mc^2(1 - \sqrt{1 - v^2/c^2})$ in special relativity.

The Born-Infeld Equation

- From this new theory, we can derive an equation for electromagnetic waves in space. We get

$$\left(1-\frac{1}{b^2}\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial x^2}+2\,\frac{1}{b^2}\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y}\,\frac{\partial^2 u}{\partial x\partial y}-\left(1-\frac{1}{b^2}\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial y^2}=0,$$

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  where b is a constant.
- Setting $b=1$ for simplicity, we have our Born-Infeld equation

$$\left(1-\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial x^2}+2\,\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}-\left(1-\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial y^2}=0.$$

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- This is called a partial differential equation, because it makes use of these "partial derivatives" where $\partial u/\partial x$ reads as "the partial derivative of u with respect to x".

The Born-Infeld Equation

$$\left(1 - \left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial x^2} + 2\,\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} - \left(1 - \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial y^2} = 0$$

- Note that this is now a differential equation of a function of two variables, where $u = u(x, y)$.
- This is called a partial differential equation, because it makes use of these "partial derivatives" where $\partial u/\partial x$ reads as "the partial derivative of u with respect to x".
- Not only is this a PDE, but it's also nonlinear. These equations are usually impossible to solve, and may not possess a general solution.

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- Thus, it comes from studying some special behavior of light, gravitational waves, and plasma oscillations.
- Aside from its physical meaning, it has experienced a lot of research interest in a purely mathematical context.
- By studying the behavior and geometry of equations like Gibbons-Tsarev, we can learn more about how PDE work in general.

The Gibbons-Tsarev Equation

- Our equation is a reduced form of the Gibbons-Tsarev equation, dropping a $+1$, term as it arises naturally that way from its so-called "quotient construction". We studied

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} = 0.$$

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

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$$
\begin{aligned}
\text{Input Equation} &\rightarrow \text{Calculate Lie's invariance condition}\\
&\rightarrow \text{Solve linear system of PDEs}\\
&\rightarrow \text{Symmetries!}
\end{aligned}
$$

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- If you can solve Lie's invariance condition for your equation, you can find its symmetries.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

In general, Lie's invariance condition is the falsely unassuming

$$\Gamma^{(n)}(\Delta)\Big|_{\Delta=0} = 0$$

But there's a lot of complexity hiding under this notation.

Here Γ is the infinitesimal operator, Δ is your differential equation, and n is its order.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

For our equations, the infinitesimal operator Γ we need is

$$\Gamma^{(2)} = Y\frac{\partial}{\partial y} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u} + U_{[y]}\frac{\partial}{\partial u_y} + U_{[x]}\frac{\partial}{\partial u_x} + U_{[yy]}\frac{\partial}{\partial u_{yy}} + U_{[xy]}\frac{\partial}{\partial u_{xy}} + U_{[xx]}\frac{\partial}{\partial u_{xx}} = 0,$$

where

$$\begin{aligned}
U_{[y]} &= D_y(U) - u_y D_y(Y) - u_x D_y(X),\\
U_{[x]} &= D_x(U) - u_y D_x(Y) - u_x D_x(X),\\
U_{[yy]} &= D_y(U_{[y]}) - u_{yy} D_y(Y) - u_{xy} D_y(X),\\
U_{[xy]} &= D_y(U_{[x]}) - u_{xy} D_y(Y) - u_{xx} D_y(X),\\
U_{[xx]} &= D_y(U_{[x]}) - u_{xy} D_x(Y) - u_{xx} D_x(X),\\
D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{yy}\frac{\partial}{\partial u_y} + u_{xy}\frac{\partial}{\partial u_x},\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_y} + u_{xx}\frac{\partial}{\partial u_x}.
\end{aligned}$$

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The result of putting our PDEs through Lie's invariance condition was a system of 24 linear PDEs for the Born-Infeld equation and 21 for the reduced Gibbons-Tsarev equation.

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- Though difficult, these systems were easier to solve than the original single equations because of their linearity.
- Solving them resulted in the following symmetries.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

Classical Symmetries of the Born-Infeld Equation

- $X(x, y, u) = c_1 x + c_2 u + c_3 y + c_4$
- $Y(x, y, u) = c_1 y + c_3 x + c_5 u + c_6$
- $U(x, y, u) = c_1 u - c_2 x + c_5 y + c_7$

Classical Symmetries of the reduced Gibbons-Tsarev Equation

- $X(x, y, u) = -2c_5 y + (-c_2 + 2c_4)x + c_7$
- $Y(x, y, u) = -\dfrac{c_1 x}{2} + c_4 y + c_5 u + c_6$
- $U(x, y, u) = c_1 y + c_2 u + c_3$

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- We need an equation called the invariant surface condition, which is a linear PDE, given by

$$\Delta_I = X(x, y, u)\frac{\partial u}{\partial x} + Y(x, y, u)\frac{\partial u}{\partial y} - U(x, y, u) = 0.$$

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- We then seek the symmetries of the PDE system of our equation and the invariant surface condition. Our invariance condition then looks like

$$\Gamma^{(n)}(\Delta)\Big|_{\Delta=0,\ \Delta_I=0} = 0.$$

The Nonclassical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The result of running our PDE through the new invariance condition was a system of 28 nonlinear PDE for the Born-Infeld equation and 12 for the reduced Gibbons-Tsarev equation.

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- Unlike the classical condition, this was a nonlinear system, and the equations were much harder to solve.
- But we did it!

The Nonclassical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

Nonclassical Symmetries of the Born-Infeld Equation

- $X(x,y,u) = c_2 x - c_5 u + c_7 y + c_3$
- $Y(x,y,u) = c_1 u + c_2 y + c_7 x + c_4$
- $U(x,y,u) = c_1 y + c_2 u + c_5 x + c_6$

Nonclassical Symmetries of the reduced Gibbons-Tsarev Equation

- $X(x,y,u) = -2c_5 y + (2c_4 - c_2)x + c_7$
- $Y(x,y,u) = c_4 y + c_5 u - \dfrac{c_1 x}{2} + c_6$
- $U(x,y,u) = c_1 y + c_2 u + c_3$

Something Unexpected

Do you notice anything?

Classical Symmetries

- $X = c_1x + c_2u + c_3y + c_4$
- $Y = c_1y + c_3x + c_5u + c_6$
- $U = c_1u - c_2x + c_5y + c_7$

Nonclassical Symmetries

- $X = c_2x - c_5u + c_7y + c_3$
- $Y = c_1u + c_2y + c_7x + c_4$
- $U = c_1y + c_2u + c_5x + c_6$

Future Work: The Interplay of Darboux Integrability and Symmetry Equivalence

- For each of our equations that we studied, the nonclassical and classical symmetries matched.

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- For each of our equations that we studied, the nonclassical and classical symmetries matched.
- Somehow, the invariant surface condition and the equations we studied share enough of the same geometry that we get no new symmetries.

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- Naturally, the next step is to find more examples of equations with this unique property, and try to understand why this is occurring.
- The Born-Infeld and Gibbons-Tsarev equations are both "Darboux integrable".
- It turns out, that every DI equation that we've checked also has this property.

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- Just maybe, we could construct a proof that every DI equation has matching classical and nonclassical symmetries.
- Even further, what if we could prove the converse?
- Then we'd have a test for whether or not our equation is DI.
- This would be a big result, and great progress toward a test for whether or not a differential equation is solvable in general!

Thank you!