Comparing Classical and Nonclassical Symmetries of Nonlinear Partial Differential Equations

William Helman and Daniel Sinderson

Southern Oregon University

2023

Table of Contents

Introduction

- Project Goal
- What is a Symmetry?
- What is a Differential Equation?
- Background on our Equations

Results

- Classical Symmetries
- Nonclassical Symmetries

Conclusion

- Discussion
- Open Questions
- Next Steps

Project Goal

Our research objective for this project was to calculate the classical and nonclassical symmetry groups for the reduced Gibbons-Tsarev equation and the Born-Infeld equation and compare them.

What is a Symmetry?

Definition

A symmetry is a transformation that leaves an object invariant.

What is a Symmetry?

Definition

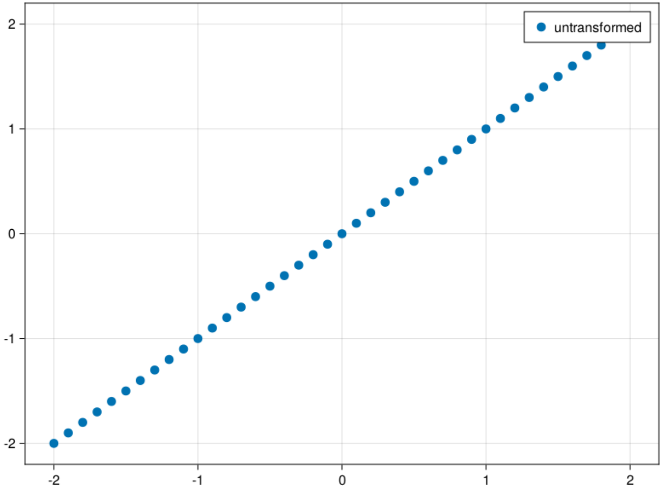
A symmetry is a transformation that leaves an object invariant.

Definition

A symmetry is a change that doesn't change anything.

What is a Symmetry?

Let's see this in action using the simple linear equation $x - y = 0$.



What is a Symmetry?

Example (A Non-Example)

- For our first transformation, let's define new variables $\bar{x} = x + 1$ and $\bar{y} = y$.
- Now we rewrite our equation using these new variables.

$$
\begin{aligned}
\bar{x} - \bar{y} &= 0 && \text{by definition} \\
x + 1 - y &= 0 && \text{by substitution} \\
y &= x + 1 && \text{by rewriting in slope-intercept form}
\end{aligned}
$$

- This transformation is not a symmetry: $x - y + 1 \neq x - y$

What is a Symmetry?

A Transformation that is a Symmetry

Example (2)

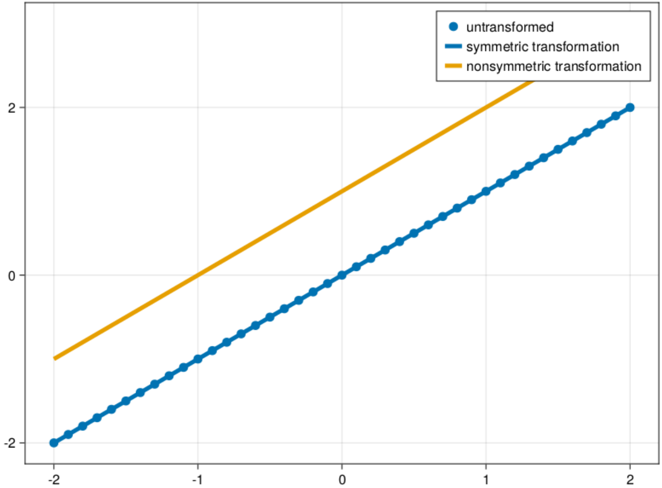
- Let's define some new variables again $\bar{x} = x + 1$ and $\bar{y} = y + 1$.
- Now we rewrite our equation using these new variables.

$$
\begin{aligned}
\bar{x} - \bar{y} &= 0 && \text{by definition} \\
(x + 1) - (y + 1) &= 0 && \text{by substitution} \\
(x - y) + (1 - 1) &= 0 && \text{by algebra} \\
x - y &= 0 && \text{by algebra} \\
y &= x && \text{by rewriting in slope-intercept form}
\end{aligned}
$$

- This transformation is a symmetry: $x - y = x - y$

What is a Symmetry?

The graphs of our three equations.



What is a Symmetry?

Who cares?

- Symmetries help us understand and solve equations that we wouldn't normally be able to.

What is a Symmetry?

Who cares?

- Symmetries help us understand and solve equations that we wouldn't normally be able to.
- Symmetries encode physically meaningful aspects of equations, like conservation laws in physics.

What is a Differential Equation?

Definition

- A differential equation is an equation that contains both an unknown function and information about how that function relates to its rates of change.

What is a Differential Equation?

Definition

- A differential equation is an equation that contains both an unknown function and information about how that function relates to its rates of change.
- Differential equations show up everywhere we model something using information about how that thing changes. This includes everything from population dynamics to planetary orbits.

What is a Differential Equation?

Definition

- A differential equation is an equation that contains both an unknown function and information about how that function relates to its rates of change.
- Differential equations show up everywhere we model something using information about how that thing changes. This includes everything from population dynamics to planetary orbits.
- Differential equations are different from algebraic equations, and can't be solved in the same ways.

What is a Differential Equation?

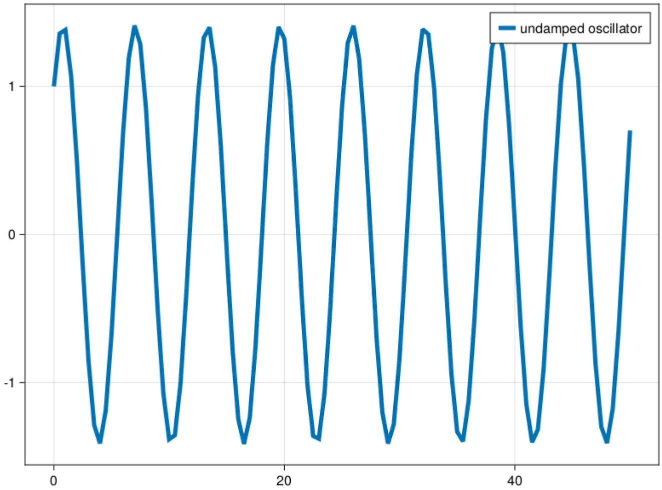
As an example, we'll use the equation for an undamped spring.

$$m\ddot{x} = -\kappa x$$

- Here m is the mass, $\ddot{x}$ is the acceleration, κ is the spring constant, and x is the position.
- If we set both m and κ to 1, we get $\ddot{x} = -x$.
- This is a simple differential equation with the algebraic solution of $x(t) = c_1 \cos(t) + c_2 \sin(t)$.

What is a Differential Equation?

The graph of our spring system $x(t) = \cos(t) + \sin(t)$, where $c_1 = c_2 = 1$.



The History of the Born-Infeld and the reduced Gibbons-Tsarev Equations

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The goal of our project was to find symmetry transformations for these two PDEs, both classical and nonclassical, and compare them.
- It turns out that there's a standard method for calculating both kinds of symmetry transformations for differential equations.

$$\begin{aligned} \text{Input Equation} &\rightarrow \text{Calculate Lie's invariance condition} \\ &\rightarrow \text{Solve linear system of PDEs} \\ &\rightarrow \text{Symmetries!} \end{aligned}$$

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To guarantee that a transformation is a symmetry, we need to ensure that all aspects of the equation remain the same under the transformation.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To guarantee that a transformation is a symmetry, we need to ensure that all aspects of the equation remain the same under the transformation.
- For a differential equation, this means that the function and any derivatives of that function that are present in the equation must not change under the transformation.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To guarantee that a transformation is a symmetry, we need to ensure that all aspects of the equation remain the same under the transformation.
- For a differential equation, this means that the function and any derivatives of that function that are present in the equation must not change under the transformation.
- Lie's invariance condition gives us a way to generate transformations that meet all these requirements.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To guarantee that a transformation is a symmetry, we need to ensure that all aspects of the equation remain the same under the transformation.
- For a differential equation, this means that the function and any derivatives of that function that are present in the equation must not change under the transformation.
- Lie's invariance condition gives us a way to generate transformations that meet all these requirements.
- If you can solve Lie's invariance condition for your equation, you can find its symmetries.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

The general form of Lie's invariance condition is the falsely unassuming

$$\Gamma^{(n)}\Delta|_{\Delta=0} = 0$$

Here Γ is the infinitesimal operator, n is the order of the equation, and Δ is the equation itself set to equal 0 (this is standard form).

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

For our equations, the infinitesimal operator Γ looks like the following:

$$T\frac{\partial}{\partial t}+X\frac{\partial}{\partial x}+U\frac{\partial}{\partial u}+U_{[t]}\frac{\partial}{\partial u_t}+U_{[x]}\frac{\partial}{\partial u_x}+U_{[tt]}\frac{\partial}{\partial u_{tt}}+U_{[tx]}\frac{\partial}{\partial u_{tx}}+U_{[xx]}\frac{\partial}{\partial u_{xx}}$$

$$\begin{aligned}
U_{[t]} &= D_t(U) - u_t D_t(T) - u_x D_t(X) \\
U_{[x]} &= D_x(U) - u_t D_x(T) - u_x D_x(X) \\
U_{[tt]} &= D_t(U_{[t]}) - u_{tt} D_t(T) - u_{tx} D_t(X) \\
U_{[tx]} &= D_t(U_{[x]}) - u_{tx} D_t(T) - u_{xx} D_t(X) \\
U_{[xx]} &= D_t(U_{[x]}) - u_{tx} D_x(T) - u_{xx} D_x(X)
\end{aligned}$$

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} \\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x}
\end{aligned}$$

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The result of putting our PDEs through Lie's invariance condition was a system of ... linear PDEs for the Born-Infeld equation and ... for the reduced Gibbons-Tsarev equation.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The result of putting our PDEs through Lie's invariance condition was a system of ... linear PDEs for the Born-Infeld equation and ... for the reduced Gibbons-Tsarev equation.
- Though difficult, these systems were easier to solve than the original single equations because of their linearity. This resulted in the following symmetries.

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

Classical Symmetries of the Born-Infeld Equation

- .
- .
- .

Classical Symmetries of the reduced Gibbons-Tsarev Equation

- .
- .
- .

The Nonclassical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

Future Work: Does Integrability Imply Equivalence of Classical and Nonclassical Symmetries?

Future Work: Does Equivalence of Classical and Nonclassical Symmetries Imply Integrability?