Comparing Symmetries of Nonlinear PDEs and their Linearizations

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Introduction and Background

OBJECTIVE: Our research objective for this project was to calculate the symmetry groups for the Reduced Gibbons-Tsarev equation and the Born-Infeld equation, and then determine if they were the same as the symmetry groups for their linearizations.

THE EQUATIONS: Both the Born-Infeld and Reduced Gibbons-Tsarev equations are nonlinear, second-order partial differential equations in two independent variables. The Born-Infeld equation originally popped up in the context of nonlinear electrodynamics and is still found today in string theory in the description of the action of open strings. The Reduced-Gibbons Tsarev equation arose in the context of parametrizations of the Benney moment equations with finitely many (two) dependent variables. It finds use in the study of dispersionless systems.

Born-Infeld: $(u_y^2 - 1)u_{xx} - 2u_x u_y u_{xy} + (u_x^2 + 1)u_{yy} = 0$

Reduced Gibbons-Tsarev: $u_{xx} - u_y u_{xy} + u_x u_{yy} = 0$

Linearized Born-Infeld: $u_{xy} + 2(\frac{xu_x - yu_y}{x^2 - y^2}) = 0$

Linearized Reduced Gibbons-Tsarev: $u_{xy} + \frac{u_x - u_y}{x - y} = 0$

Methods

Mathematical Tools

- The total derivative operator
- The prolongated infinitesimal operator
- Lie's Invariance Condition
- Symmetry Groups
- Lie Bracket

Computational Tools

- Maple (computer algebra software)
- DEtools (Maple package)
- DifferentialGeometry (Maple package)
- LieAlgebras (Maple package)

Approach

1. Define the total derivative operator in Maple.
2. Define the prolongated infinitesimal operator for second-order differential equations in two independent variables.
3. Calculate Lie's Invariance Condition for the equation.
4. Collect like terms and separate into a system of determining equations for your symmetry generating infinitesimals.
5. Solve the system of determining equations.
6. Use the resulting infinitesimals to find the number of symmetries that the equation has by counting the number of arbitrary constants.
7. * Calculate vector fields by one-hot encoding the constants.
8. * Lie Bracket the resulting vector fields together.
9. * Characterize the resulting symmetry group.

Results

Reduced Gibbons-Tsarev

$$X(x,y,u) = -2c_5 y + (-c_2 + 2c_4)x + c_7$$
$$Y(x,y,u) = -\frac{c_1 x}{2} + c_4 y + c_5 u + c_6$$
$$U(x,y,u) = c_1 y + c_2 u + c_3$$

Linearized Reduced Gibbons-Tsarev

$$X(x,y,u) = c_1 x^2 + c_3 x + c_4$$
$$Y(x,y,u) = c_1 y^2 + c_3 y + c_4$$
$$U(x,y,u) = \frac{(-x+y)(\frac{d}{dx}f(x))}{2} + \frac{(-x+y)(\frac{d}{dy}g(y))}{2} + f(x) - g(y) + (c_1 x + c_1 y + c_2)u$$

Born-Infeld

$$X(x,y,u) = c_1 x + c_2 u + c_3 y + c_4$$
$$Y(x,y,u) = c_1 y + c_3 x + c_5 u + c_6$$
$$U(x,y,u) = c_1 u - c_2 x + c_5 y + c_7$$

Linearized Born-Infeld

$$X(x,y,u) = \frac{1}{2}c_1 x^2 + c_4 x - \frac{1}{2}c_3$$
$$Y(x,y,u) = \frac{1}{2}c_1 y^2 + c_4 y - \frac{1}{2}c_3$$
$$U(x,y,u) = \frac{1}{2(x+y)}((x-y)(\frac{d}{dx}f(x)) + (-x+y)(\frac{d}{dy}g(y)) - 2f(x) - 2g(y) + 2u(c_1 y + c_2)x + 2c_2 uy + 2c_3 u)$$

Conclusion

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References

References