Comparing Classical and Nonclassical Symmetries of Nonlinear Partial Differential Equations

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Project Goal

Our research objective for this project was to calculate the classical and nonclassical symmetry groups for the reduced Gibbons-Tsarev equation and the Born-Infeld equation and compare them.

What is a Symmetry?

Definition

A symmetry is a transformation that leaves an object invariant.

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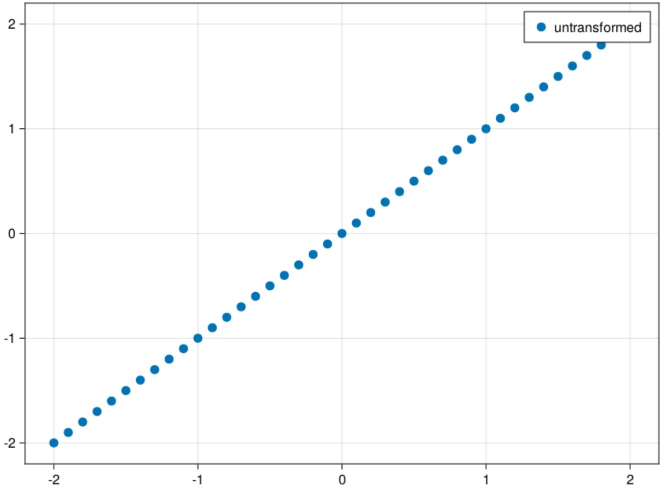
Definition
A symmetry is a change that doesn't change anything.

What is a Symmetry?

Let's see this in action using the simple linear equation $x - y = 0$.



What is a Symmetry?

Example (A Non-Example)

- For our first transformation, let's define new variables $\bar{x} = x + 1$ and $\bar{y} = y$.
- Now we rewrite our equation using these new variables.

$$\begin{aligned} \bar{x} - \bar{y} &= 0 && \text{by definition} \\ x + 1 - y &= 0 && \text{by substitution} \\ y &= x + 1 && \text{by rewriting in slope-intercept form} \end{aligned}$$

- This transformation is not a symmetry: $x - y + 1 \neq x - y$

What is a Symmetry?

A Transformation that is a Symmetry

Example (2)

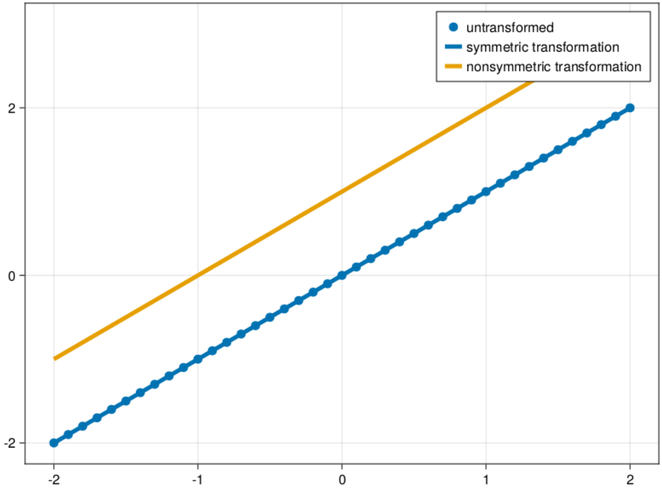
- Let's define some new variables again
  $\bar{x} = x + 1$ and $\bar{y} = y + 1$.
- Now we rewrite our equation using these new variables.

$$
\begin{aligned}
\bar{x} - \bar{y} &= 0 && \text{by definition} \\
(x+1) - (y+1) &= 0 && \text{by substitution} \\
(x-y) + (1-1) &= 0 && \text{by algebra} \\
x - y &= 0 && \text{by algebra} \\
y &= x && \text{by rewriting in slope-intercept form}
\end{aligned}
$$

- This transformation is a symmetry:
  $x - y = x - y$

What is a Symmetry?

The graphs of our three equations.



What is a Symmetry?

Who cares?

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- Differential equations show up everywhere we model something using information about how that thing changes over time. This includes everything from population dynamics to the motion of the planets.
- Differential equations are different from algebraic equations, and can't be solved in the same ways.

What is a Differential Equation?

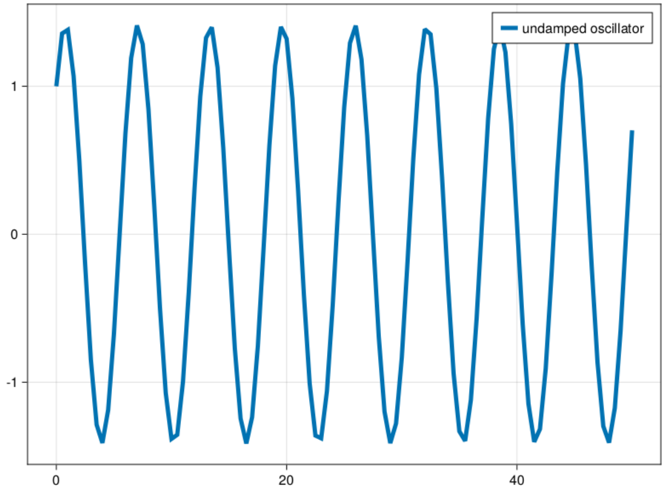
As an example, we'll use the equation for an undamped spring.

$$m\ddot{x} = -\kappa x$$

- Here m is the mass, $\ddot{x}$ is the acceleration, κ is the spring constant, and x is the position.
- If we set both m and κ to 1, we get $\ddot{x} = -x$.
- This is a simple differential equation with the algebraic solution of $x(t) = c_1 \cos(t) + c_2 \sin(t)$.

What is a Differential Equation?

The graph of our spring system $x(t) = \cos(t) + \sin(t)$, where $c_1 = c_2 = 1$.



The History of the Born-Infeld and the reduced Gibbons-Tsarev Equations

The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To find the classical symmetries for a differential equation, you have to solve for Lie's Invariance Condition.
- Lie's Invariance Condition changes based on the type and order of differential equation.
- For second order partial differential equations like we had, Lie's Invariance condition is as follows

$$\Gamma^2(\Delta)$$

The Nonclassical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

Future Work: Does Integrability Imply Equivalence of Classical and Nonclassical Symmetries?

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