# Comparing Classical and Nonclassical Symmetries of Nonlinear Partial Differential Equations

William Helman and Daniel Sinderson

Southern Oregon University

2023

# Table of Contents

# Project Goal

Our research objective for this project was to calculate the classical and nonclassical symmetry groups for the reduced Gibbons-Tsarev equation and the Born-Infeld equation and compare them.

# What is a Symmetry?

### Definition

A symmetry is a transformation that leaves an object invariant.

# What is a Symmetry?

### Definition

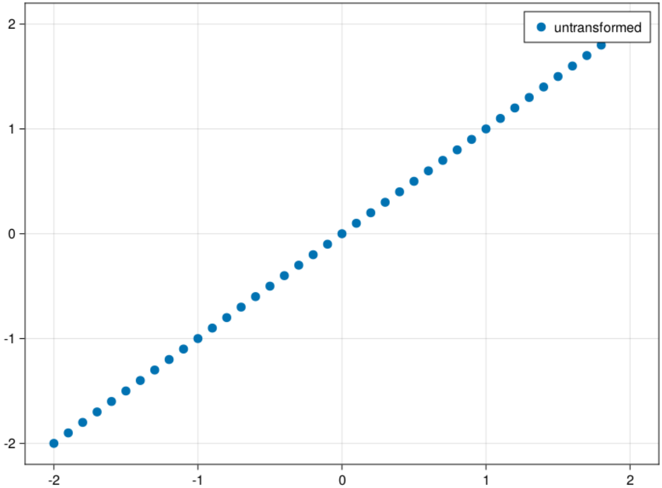
A symmetry is a transformation that leaves an object invariant.

### Definition

A symmetry is a change that doesn't change anything.

# What is a Symmetry?

Let's see this in action using the simple linear equation $x - y = 0$.



# What is a Symmetry?

## Example (A Non-Example)

- For our first transformation, let's define new variables $\bar{x} = x + 1$ and $\bar{y} = y$.
- Now we rewrite our equation using these new variables.

$$
\begin{aligned}
\bar{x} - \bar{y} &= 0 && \text{by definition} \\
x + 1 - y &= 0 && \text{by substitution} \\
y &= x + 1 && \text{by rewriting in slope-intercept form}
\end{aligned}
$$

- This transformation is not a symmetry: $x - y + 1 \neq x - y$

# What is a Symmetry?

## A Transformation that is a Symmetry

### Example (2)

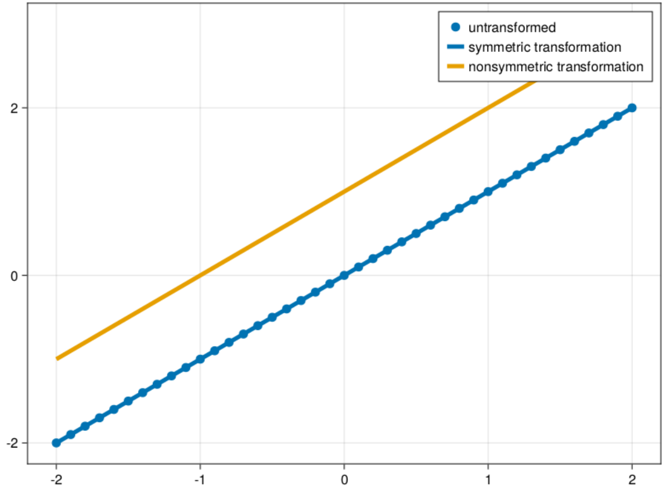
- Let's define some new variables again
  $\bar{x} = x + 1$ and $\bar{y} = y + 1$.
- Now we rewrite our equation using these new variables.

$$
\begin{aligned}
\bar{x} - \bar{y} &= 0 && \text{by definition} \\
(x+1) - (y+1) &= 0 && \text{by substitution} \\
(x-y) + (1-1) &= 0 && \text{by algebra} \\
x - y &= 0 && \text{by algebra} \\
y &= x && \text{by rewriting in slope-intercept form}
\end{aligned}
$$

- This transformation is a symmetry:
  $x - y = x - y$

# What is a Symmetry?

The graphs of our three equations.



# What is a Symmetry?

## Who cares?

- Symmetries help us understand and solve equations that we wouldn't normally be able to.

# What is a Symmetry?

Who cares?

- Symmetries help us understand and solve equations that we wouldn't normally be able to.
- Symmetries encode physically meaningful aspects of equations, like conservation laws in physics.

# What is a Differential Equation?

## Definition

- A differential equation is an equation that contains both an unknown function and information about how that function relates to its rates of change.

# What is a Differential Equation?

## Definition

- A differential equation is an equation that contains both an unknown function and information about how that function relates to its rates of change.
- Differential equations show up everywhere we model something using information about how that thing changes. This includes everything from population dynamics to planetary orbits.

# What is a Differential Equation?

## Definition

- A differential equation is an equation that contains both an unknown function and information about how that function relates to its rates of change.
- Differential equations show up everywhere we model something using information about how that thing changes. This includes everything from population dynamics to planetary orbits.
- Differential equations are different from algebraic equations, and can't be solved in the same ways.

# What is a Differential Equation?

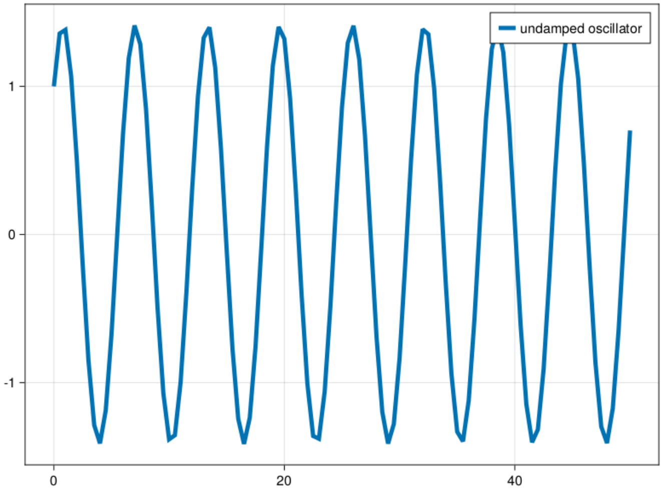
As an example, we'll use the equation for an undamped spring.

$$m\ddot{x} = -\kappa x$$

- Here $m$ is the mass, $\ddot{x}$ is the acceleration, $\kappa$ is the spring constant, and $x$ is the position.
- If we set both $m$ and $\kappa$ to 1, we get $\ddot{x} = -x$.
- This is a simple differential equation with the algebraic solution of $x(t) = c_1 \cos(t) + c_2 \sin(t)$.

# What is a Differential Equation?

The graph of our spring system $x(t) = \cos(t) + \sin(t)$, where $c_1 = c_2 = 1$.



# The History of the Born-Infeld and the reduced Gibbons-Tsarev Equations

## The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The goal of our project was to find symmetry transformations for these two PDEs, both classical and nonclassical, and compare them.
- It turns out that there's a standard method for calculating both kinds of symmetry transformations for differential equations.

$$\begin{aligned} \text{Input Equation} &\rightarrow \text{Calculate Lie's invariance condition} \\ &\rightarrow \text{Solve linear system of PDEs} \\ &\rightarrow \text{Symmetries!} \end{aligned}$$

# The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To guarantee that a transformation is a symmetry, we need to ensure that all aspects of the equation remain the same under the transformation.

# The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To guarantee that a transformation is a symmetry, we need to ensure that all aspects of the equation remain the same under the transformation.
- For a differential equation, this means that the function and any derivatives of that function that are present in the equation must not change under the transformation.

## The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To guarantee that a transformation is a symmetry, we need to ensure that all aspects of the equation remain the same under the transformation.
- For a differential equation, this means that the function and any derivatives of that function that are present in the equation must not change under the transformation.
- Lie's invariance condition gives us a way to generate transformations that meet all these requirements.

## The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- To guarantee that a transformation is a symmetry, we need to ensure that all aspects of the equation remain the same under the transformation.
- For a differential equation, this means that the function and any derivatives of that function that are present in the equation must not change under the transformation.
- Lie's invariance condition gives us a way to generate transformations that meet all these requirements.
- If you can solve Lie's invariance condition for your equation, you can find its symmetries.

# The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

The general form of Lie's invariance condition is the falsely unassuming

$$\Gamma^{(n)}\Delta|_{\Delta=0} = 0$$

Here $\Gamma$ is the infinitesimal operator, $n$ is the order of the equation, and $\Delta$ is the equation itself set to equal 0 (this is standard form).

## The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

For our equations, which are both second order PDEs in two independent variables, Lie's invariance condition looks like the following:

$$\dots$$

# The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The result of putting our PDEs through Lie's invariance condition was a system of ... linear PDEs for the Born-Infeld equation and ... for the reduced Gibbons-Tsarev equation.

# The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

- The result of putting our PDEs through Lie's invariance condition was a system of . . . linear PDEs for the Born-Infeld equation and . . . for the reduced Gibbons-Tsarev equation.
- Though difficult, these systems were easier to solve than the original single equations because of their linearity. This resulted in the following symmetries.

# The Classical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

Classical Symmetries of the Born-Infeld Equation

- .
- .
- .

Classical Symmetries of the reduced Gibbons-Tsarev Equation

- .
- .
- .

# The Nonclassical Symmetries of the Born-Infeld and the reduced Gibbons-Tsarev Equations

# Future Work: Does Integrability Imply Equivalence of Classical and Nonclassical Symmetries?

# Future Work: Does Equivalence of Classical and Nonclassical Symmetries Imply Integrability?