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TRANSFORMING ARC ROUTING INTO NODE ROUTING PROBLEMS

WEN-LEA PEARN*
Department of Mathematics, California State University, Fresno, CA 93740, U.S.A.

ARJANG ASSAD† and BRUCE L. GOLDEN‡
College of Business and Management, University of Maryland, College Park, MD 20742, U.S.A.

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Scope and Purpose—A routing problem for a fleet of vehicles may be classified as a node routing or an arc routing problem, depending on where the demand occurs on the underlying road network. In node routing problems, which comprise most traditional routing applications, the demands are localized at customer sites, represented by nodes of the road network. In arc routing, the service activity involves the traversal or coverage of an arc. In this paper, we describe how a generic arc routing problem can be formulated as a standard vehicle (node) routing problem. This allows us to transform arc routing problems into node routing ones. This is of interest since node routing problems have received so much more attention than arc routing problems in the research literature.

Abstract—In this paper, we describe how the Capacitated Arc Routing Problem can be formulated as a standard vehicle routing problem. This allows us to transform arc routing into node routing problems and, therefore, establishes the equivalence of these two classes of problems.

INTRODUCTION

A routing problem for a fleet of vehicles may be classified as a node routing or an arc routing problem, depending on where the demand occurs on the underlying road network. In node routing problems, which comprise most traditional routing applications, demands are localized at customer sites, represented by nodes of the road network. In arc routing, the service activity involves the traversal or coverage of an arc. Examples of arc routing problems include street sweeping or spraying, meter reading, and mail collection activities (see [1–3]). The routing literature contains a number of algorithms specifically designed for arc routing problems (see [4–6]).

A generic arc routing problem that subsumes a number of well-known routing problems is the Capacitated Arc Routing Problem (CARP). This problem, introduced by Golden and Wong [7], may be briefly defined as follows: Given a network where costs and demands are associated with each arc, find a set of minimum cost vehicle cycles, based at a distinguished node called the depot, and traversing all arcs of positive demand, so that the total demand serviced by each cycle does not exceed the vehicle capacity W. As pointed out by Golden and Wong [7], a number of well-known node routing problems can be viewed as special cases of the CARP simply by splitting any original node into two new nodes joined by an arc, and assigning the demand of the original node to the arc.

In this paper, we describe how the CARP can be formulated as a standard vehicle (node) routing problem. This allows us to transform arc routing problems into node routing ones and thus, in view of the preceding remark, establishes the equivalence of these two classes of problems.

*Wen-Lea Pearn is an Assistant Professor of Mathematics at California State University of Fresno. He received his Ph.D. from the University of Maryland where he studied applied mathematics.

†Arjang Assad is Associate Professor of Management Science at the University of Maryland. He holds a Ph.D. in Management Science from the Sloan School of Management at M.I.T. His previous degrees in operations research, chemical engineering, and mathematics are also all from M.I.T. His research interests include distribution planning and transportation networks. He is currently an associate editor for *Operations Research*. His publications include a special issue of *Computers & Operations Research* devoted to vehicle routing and scheduling.

‡Bruce L. Golden is Westinghouse Professor and Chairman of the Department of Management Science and Statistics at the University of Maryland. Dr Golden received his Ph.D. in Operations Research from M.I.T. His research interests include network optimization, mathematical programming, and applied statistics, and he has published numerous articles in these fields. He is currently an Editor or Associate Editor of *Networks, Management Science, Computers & Operations Research, Decision Sciences, Operations Research* and the *American Journal of Mathematical and Management Sciences*.

THE TRANSFORMATION

Consider an undirected network, $G = (V, A)$, with a set of nodes, V, and a set of arcs, A. Each arc (i,j) in A has a cost, c_{ij}, and a demand, q_{ij}, specified. We represent the depot by node 1 and let dist(i,j) stand for the shortest path cost between nodes i and j in G with respect to arc costs c_{ij}. The Capacitated Arc Routing Problem (CARP) on G involves finding a set of vehicle cycles, based at node 1, covering all arcs of positive demand at minimum cost, subject to the constraint that the demands serviced on each cycle not exceed the vehicle capacity W. The Capacitated Chinese Postman Problem (CCPP) refers to the special case of the CARP where all arcs in A have positive demand; i.e. $q_{ij} > 0$ for all (i, j). For convenience, we describe the transformation for the CCPP first, and then remark on how it carries over to the CARP.

To transform a CCPP (arc) routing problem on G to a standard vehicle (node) routing problem (VRP), we replace each arc in A with three new nodes, called side and middle nodes. Specifically, arc (i, j) is replaced with three nodes, s_{ij}, m_{ij} and s_{ji}. The first and third nodes listed are referred to as side nodes below; m_{ij} is the middle node corresponding to arc (i, j). The vehicle routing problem is defined on the set of nodes $N = \{1\} \cup \{s_{ij}, m_{ij}, s_{ji} \mid (i, j) \in A\}$. Note that other than node 1 (the depot), the original nodes in V are discarded. To define the inter-node distances in the VRP, one can think of the new nodes as being placed along the original arc (i, j) at equal "spacings", as shown in Fig. 1. The distance function $d(\,.\,,\,.)$ on $N \times N$ is defined as follows:

$$d(s_{ij}, s_{kl}) = \begin{cases} \frac{1}{4}(c_{ij} + c_{kl}) + \mathrm{dist}(i, k) & \text{if } (i,j) \neq (k,l), \\ 0 & \text{if } (i,j) = (k,l) \end{cases} \tag{1}$$

$$d(1, s_{ij}) = \tfrac{1}{4}c_{ij} + \mathrm{dist}(i, 1) \tag{2}$$

$$d(m_{ij}, v) = \begin{cases} \frac{1}{4}c_{ij} & \text{if } v = s_{ij} \text{ or } s_{ji}, \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

Note that dist(i, k) is the shortest path length from i to k in the original network G. Finally, the node demands in the VRP are defined by

$$q(s_{ij}) = q(s_{ji}) = q(m_{ij}) = \tfrac{1}{3}q_{ij} \quad \text{for all } (i, j), \tag{4}$$

where $q(n)$ is the demand on a node $n \in N$. The depot node has no demand associated with it.

Figure 2 illustrates the preceding definitions for a simple arc routing problem. Note that the 5 arcs in the original network give rise to $5 \times 3 + 1 = 16$ nodes in the transformed problem. We can interpret the definition in (1) in the following simple way: To compute the distance between two side nodes, each is first connected to the original node "closest" to it. The resulting two nodes in V are then joined by the shortest path with respect to the original network G. Table 1 provides the distances between pairs of side nodes for the example depicted in Fig. 2. For instance,

$$d(s_{12}, s_{31}) = \tfrac{1}{4}(8 + 16) + \mathrm{dist}(1, 3) = 6 + 12 = 18,$$

since the shortest path in G between nodes 1 and 3 goes through node 2. Similarly,

$$d(s_{34}, s_{43}) = \tfrac{1}{4}(20 + 20) + 16 = 26.$$

The definition in (3) ensures that a middle node m_{ij} is accessible only from nodes s_{ij} and s_{ji} in any solution with finite cost.

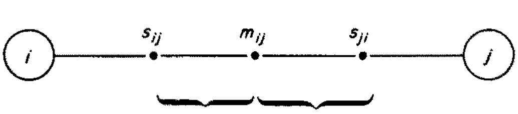

Fig. 1. Introducing new nodes for each original arc.

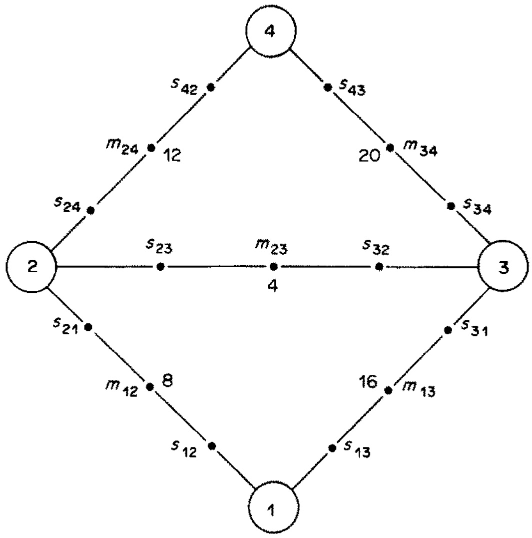

Fig. 2. A small arc routing example with new nodes introduced (original arc costs are shown next to each arc).

Table 1. Inter-node distances for s_{ij} nodes of the network in Fig. 2

	1	s_{12}	s_{13}	s_{21}	s_{23}	s_{24}	s_{31}	s_{32}	s_{34}	s_{42}	s_{43}
1	—	2	4	10	9	11	16	13	17	23	25
s_{12}		—	6	12	11	13	18	15	19	25	27
s_{13}			—	14	13	15	20	17	21	27	29
s_{21}				—	3	5	10	7	11	17	19
s_{23}					—	4	9	6	10	16	18
s_{24}						—	11	8	12	18	20
s_{31}							—	5	9	23	25
s_{32}								—	6	20	22
s_{34}									—	24	26
s_{42}										—	8
s_{43}											—

EQUIVALENCE OF THE ORIGINAL AND TRANSFORMED PROBLEMS

The goal of this section is to demonstrate the equivalence of the VRP problem and the original arc routing problem on G. To this end, first note that any feasible solution to the CCPP can be transformed into a feasible set of tours for the VRP of no greater cost.

Indeed, given a cycle occurring in the CCPP solution, one can replace each arc (i, j) that is serviced in the cycle by the three nodes s_{ij}, m_{ij}, s_{ji} associated with it. These node triplets are visited in the same order as the arc is traversed in the original solution. Arcs that are traversed but not serviced in the CCPP solution, called deadhead arcs, do not figure in this transformation.

To see that the resulting VRP tours do not cost more than the original CCPP solution, note that the cost c_{ij} of any serviced arc (i, j) is fully incurred in the VRP tour: one-half of this cost is paid by going from s_{ij} to s_{ji} via m_{ij}, while the other half is accounted for in travel costs into s_{ij} and out of s_{ji}.

Next, consider deadhead arcs in the CCPP solution. Suppose that (i, j) and (k, l) are two consecutive serviced arcs with $j \neq k$ so that a vehicle has to deadhead from j to k. In the transformed VRP tour, node s_{ji} is connected to s_{kl} at a cost of $\frac{1}{4}(c_{ij} + c_{kl}) + \text{dist}(j, k)$, as shown in Fig. 3. The $\text{dist}(j, k)$ term cannot exceed the cost of deadheading between j and k in G, while the terms with the $\frac{1}{4}$ factor are absorbed into traversal costs for the serviced arcs (i, j) and (k, l), as outlined above.

To illustrate this transformation, consider the CCPP cycle $\underline{1,2,4},3,1$ in Fig. 2, where the underlining denotes serviced arcs (1,2) and (2,4) in this case. The cost of this cycle is $8 + 12 + 20 + 16 = 56$. The transformed VRP tour is 1, s_{12}, m_{12}, s_{21}, s_{24}, m_{24}, s_{42}, 1 and its cost is $8 + 12 + 20 = 40$. Thus, the transformed solution actually has a lower cost. Of course, this difference occurs since the original CCPP cycle does not minimize the cost of deadheading back to the depot.

The above shows that any feasible CCPP solution can be transformed into a VRP solution of no greater cost. Conversely, the optimal VRP tours correspond to CCPP cycles of equal cost. This

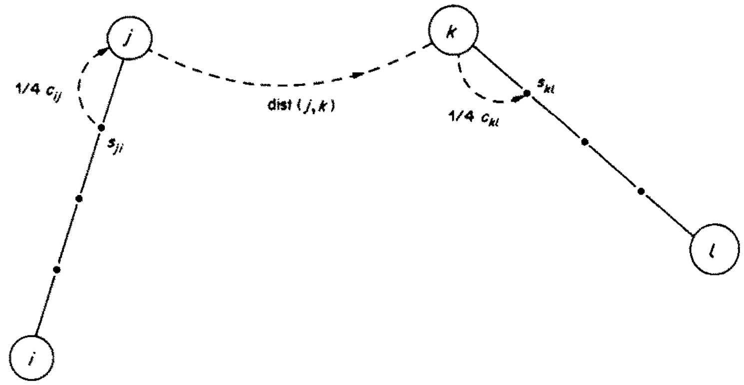

Fig. 3. Illustration of the cost of going directly from s_{ij} to s_{kl} in the transformed network.

follows since any middle node m_{ij} must always be visited in the sequence s_{ij}, m_{ij}, s_{ji} or s_{ji}, m_{ij}, s_{ij} to avoid encountering infinite distances or visiting either of the side nodes twice. Thus, the triplets (s_{ij}, m_{ij}, s_{ji}) in each tour of the VRP solution are in 1:1 correspondence with the serviced arcs of a CCPP cycle of equal cost. This establishes the equivalence of the original arc routing problem and its node routing counterpart.

COMMENTS AND CONCLUSIONS

Based on the demonstration in the preceding section, one can easily see that the following remarks apply to the transformation.

(a) The original arc cost c_{ij} can be broken down in ways other than (1)–(3) without affecting the validity of the transformation. For instance, the $\frac{1}{4}$ factors in (1) and (2) can be replaced by α and the one in (3) by β, provided $2(\alpha + \beta) = 1$ and $\alpha, \beta > 0$.

(b) Since the three nodes s_{ij}, m_{ij}, s_{ji} are always visited together in the optimal VRP solution, the allocation of the arc demand q_{ij} to these nodes can be made arbitrarily in (4). One only needs to ensure that $q(s_{ij}) + q(s_{ji}) + q(m_{ij}) = q_{ij}$.

(c) The transformation applies equally well to the CARP where some arc demands vanish. In this case, the node triplets are introduced *only* for those arcs that have positive demand ($q_{ij} > 0$). The other features of the transformation carry through. Following the transformation process described in this paper, an arc routing problem, such as the CARP, can be transformed into a VRP with 3 NA + 1 nodes, where NA equals the number of arcs with positive demand in the original problem. Thus, in the case of the CCPP transformation, the resulting number of nodes is roughly three times the number of arcs in the original problem. This increase reflects the greater inherent difficulty of solving arc routing problems as compared to their node routing counterparts.

The transformation introduced in this paper may be viewed as a means for linking the two classes of node and arc routing problems, so that theoretical results on the former (more thoroughly investigated) class can be extended to the latter class. No claims are made, however, as to the computational utility of this transformation. Future computational work has to determine whether the strategy of transforming an arc routing problem, and solving it with an exact node routing algorithm, is computationally effective.

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