Single Machine Scheduling

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Some details about algorithm

Lower bound - the solution found by hungarian algorithm. Upper bound - the best feasible solution found at the moment. Branching rule: from all elements causing infeasibility we choose one with the minimum tolerance and branch on it. We interrupt recursion if:

- found solution is feasible
- ► lower bound ≥ upper bound
- we forcibly included infeasible pair

Input

```
1 job:
p_1 = 3
available: [[1, 3], [5, \infty]]
cost: f(t) = t - 2
2 job:
p_2 = 2
available: [[2, \infty]]
cost: f(t) = t - 1
3 job:
p_3 = 2
available: [[2, \infty]]
cost: f(t) = t - 2
```

Relaxation

time	1	2	3	4	5	6	7
1 job	0	0	0	∞	0	∞	∞
	∞	0	0	∞	0	0	∞
	∞	∞	1	∞	3	4	5
2 job	∞	0	0	0	0	0	∞
	∞	∞	2	3	4	5	6
3 job	∞	0	0	0	0	0	∞
	∞	∞	1	2	3	4	5

Upper bound $= \infty$

Forcibly included: []

Forcibly excluded: []



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (6, 5) and (7, 4).

$$(6, 5) - 0$$

$$(7, 4) - 0$$

 $\mathsf{Upper}\;\mathsf{bound} = \infty$

Forcibly included: []

Forcibly excluded: [(6, 5)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (6, 6) and (7, 4).

$$(6, 6) - 0$$

$$(7, 4) - 0$$

Upper bound $= \infty$

Forcibly included: []

Forcibly excluded: [(6, 5), (6, 6)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (2, 5) and (3, 3).

$$(2, 5) - 0$$

$$(3, 3) - 0$$

 $\mathsf{Upper}\;\mathsf{bound} = \infty$

Forcibly included: []

Forcibly excluded: [(6, 5), (6, 6), (2, 5)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (2, 6) and (3, 3).

$$(2, 6) - 1$$

$$(3, 3) - 0$$

Upper bound $= \infty$

Forcibly included: []

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 5) and (5, 3).

$$(4, 5) - 1$$

$$(5, 3) - 0$$

Upper bound $= \infty$

Forcibly included: []

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 5) and (5, 4).

$$(4, 5) - 1$$

$$(5, 4) - 1$$

Upper bound $= \infty$

Forcibly included: []

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5)]



The optimal value equals 10.

Lower bound = 10

The solution is infeasible: (2, 6) and (3, 5).

$$(2, 6) - 0$$

$$(3, 5) - 0$$

Upper bound $= \infty$

Forcibly included: []

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6)]



The optimal value equals 10.

Lower bound = 10

The solution is infeasible: (6, 4) and (7, 3).

$$(6, 4) - 1$$

$$(7, 3) - 1$$

Upper bound $= \infty$ Forcibly included: []

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2,



The optimal value equals 11.

Lower bound = 11

The solution is infeasible: (4, 6) and (5, 4).

$$(4, 6) - 2$$

$$(5, 4) - 0$$

```
Upper bound = \infty
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2,
6), (6, 4), (5, 4)]
   999
                           999
                      999
                               999
                      999
   999
       999
   qqq
                      999
                           999
                               qqq
       999
```

The optimal value equals 11.

Lower bound = 11

The solution is feasible!

The optimal value equals 11.

Lower bound = 11

Lower bound ≥ Upper bound

Upper bound = 11

Forcibly included: [(6, 4)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6)]



The optimal value equals 10.

Lower bound = 10

The solution is infeasible: (6, 4) and (7, 3).

The optimal value equals 13.

Lower bound = 13

Lower bound \geq Upper bound

```
Upper bound = 11
Forcibly included: [(6, 4), (7, 3)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6)]
Infeasible pair included forcibly!
```

Upper bound = 11

Forcibly included: [(2, 6)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5)]

0	0	0	999	0	999	999
999	999	999	999	999	0	999
999	999	999	999	3	999	5
999	0	0	0	999	999	999
999	999	999	3	4	999	6
999	0	0	0	999	999	999
999	999	1	2	3	999	5

The optimal value equals 10.

Lower bound = 10

The solution is infeasible: (2, 6) and (3, 5); (6, 4) and (7, 3).

$$(3, 5) - 0$$

$$(6, 4) - 0$$

$$(7, 3) - 1$$

Upper bound = 11

Forcibly included: [(2, 6)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (3, 5)]



The optimal value equals 10.

Lower bound = 10

The solution is infeasible: (6, 4) and (7, 3).

$$(6, 4) - 0$$

$$(7, 3) - 1$$

The optimal value equals 10.

999

999

Lower bound = 10

The solution is feasible!

999

999

Upper bound = 10

Forcibly included: [(2, 6), (6, 4)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (3, 5)]



The optimal value equals 10.

Lower bound = 10

Lower bound ≥ Upper bound

```
Upper bound = 10
```

Forcibly included: [(3, 5)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5)]



The optimal value equals 10.

Lower bound = 10

Lower bound ≥ Upper bound

```
Upper bound = 10
```

Forcibly included: [(4, 5)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 5) and (5, 4)

```
Upper bound = 10
```

Forcibly included: [(4, 5)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (5, 4)]



The optimal value equals 11.

Lower bound = 11

Lower bound \geq Upper bound

```
Upper bound = 10
Forcibly included: [(4, 5), (5, 4)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3)]
Infeasible pair included forcibly!
```

Upper bound = 10

Forcibly included: [(5, 3)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 5) and (5, 3).

```
Upper bound = 10
```

Forcibly included: [(5, 3)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (4, 5)]



The optimal value equals 10.

Lower bound = 10

Lower bound \geq Upper bound

```
Upper bound = 10
Forcibly included: [(5, 3), (4, 5)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3)]
Infeasible pair included forcibly!
```

```
Upper bound = 10
```

Forcibly included: [(3, 3)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5)]



The optimal value equals 9.

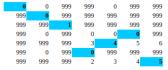
Lower bound = 9

The solution is infeasible: (2, 6) and (3, 3)

```
Upper bound = 10
```

Forcibly included: [(3, 3)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (2, 6)]



The optimal value equals 10.

Lower bound = 10

 $Lower\ bound \geq Upper\ bound$

Upper bound = 10Forcibly included: [(3, 3), (2, 6)]Forcibly excluded: [(6, 5), (6, 6), (2, 5)]Infeasible pair included forcibly!

Upper bound = 10

Forcibly included: [(2, 5)]

Forcibly excluded: [(6, 5), (6, 6)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (2, 5) and (3, 3); (4, 6) and (5, 4)

- (3, 3): 0
- (4, 6): 2
- (6, 4): 0

Upper bound = 10

Forcibly included: [(2, 5)]

Forcibly excluded: [(6, 5), (6, 6), (3, 3)]

0	0	0	999	999	999	999
999	999	999	999	0	999	999
999	999	999	999	999	4	5
999	0	0	0	999	0	999
999	999	2	3	999	5	6
999	0	0	0	999	999	999
999	999	1	2	999	4	5

The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 6) and (5, 3)

Tolerances:

(4, 6): 2

(5, 3): 0

```
Upper bound = 10
```

Forcibly included: [(2, 5)]

Forcibly excluded: [(6, 5), (6, 6), (3, 3), (5, 3)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 6) and (5, 4)

Tolerances:

(4, 6): 2

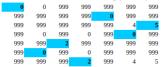
(5, 4): 2

No sense to branch because Lower bound $+ \min\{\text{tolerances}\} \ge \text{Upper bound}$.

Upper bound = 10

Forcibly included: [(2, 5), (5, 3)]

Forcibly excluded: [(6, 5), (6, 6), (3, 3)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 6) and (5, 3)

```
Upper bound = 10
```

Forcibly included: [(2, 5), (5, 3)]

Forcibly excluded: [(6, 5), (6, 6), (3, 3), (4, 6)]

0	0	999	999	999	999	999
999	999	999	999	0	999	999
999	999	999	999	999	4	5
999	0	999	0	999	999	999
999	999	2	999	999	999	999
999	0	999	0	999	999	999
999	999	999	2	999	4	5

The optimal value equals 11.

Lower bound = 11

Lower bound \geq Upper bound

```
Upper bound = 10
Forcibly included: [(2, 5), (5, 3), (4, 6)]
Forcibly excluded: [(6, 5), (6, 6), (3, 3)]
Infeasible pair included forcibly!
```

Upper bound = 10 Forcibly included: [(2, 5), (3, 3)] Forcibly excluded: [(6, 5), (6, 6)] Infeasible pair included forcibly!

Upper bound = 10

Forcibly included: [(6, 6)]

Forcibly excluded: [(6, 5)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (6, 6) and (7, 4)

Upper bound = 10

Forcibly included: [(6, 6)]

Forcibly excluded: [(6, 5), (7, 4)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 5) and (5, 4)

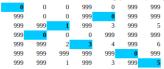
$$(4, 5) - 0$$

$$(5, 4) - 1$$

Upper bound = 10

Forcibly included: [(6, 6)]

Forcibly excluded: [(6, 5), (7, 4), (4, 5)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (2, 5) and (3, 3)

$$(2, 5) - 1$$

$$(3, 3) - 0$$

Upper bound = 10

Forcibly included: [(6, 6)]

Forcibly excluded: [(6, 5), (7, 4), (4, 5), (3, 3)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (6, 6) and (7, 3)

Tolerances:

(7, 3) - 1 So there is no sense to branch.

Upper bound = 10

Forcibly included: [(6, 6), (3, 3)]

Forcibly excluded: [(6, 5), (7, 4), (4, 5)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (2, 5) and (3, 3)

Tolerances:

(2, 5) - 1 So there is no sense to branch.

Upper bound = 10

Forcibly included: [(6, 6), (4, 5)]

Forcibly excluded: [(6, 5), (7, 4)]

0	0	0	999	999	999	999
999	0	0	999	999	999	999
999	999	1	999	999	999	5
999	999	999	999	0	999	999
999	999	2	3	999	999	6
999	999	999	999	999	0	999
999	999	1	999	999	999	5

The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 5) and (5, 4)

Tolerances:

(5, 4) - ∞ So there is no sense to branch.

Upper bound = 10 Forcibly included: [(6, 6), (7, 4)] Forcibly excluded: [(6, 5)] Infeasible pair included forcibly!

Upper bound = 10 Forcibly included: [(6, 5)] Forcibly excluded: []



The optimal value equals 9.

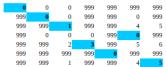
Lower bound = 9

The solution is infeasible: (6, 5) and (7, 4)

Upper bound = 10

Forcibly included: [(6, 5)]

Forcibly excluded: [(7, 4)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 6) and (5, 4)

$$(4, 6) - 0$$

$$(5, 4) - 2$$

Upper bound = 10

Forcibly included: [(6, 5)]

Forcibly excluded: [(7, 4), (4, 6)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (2, 6) and (3, 3)

$$(2, 6) - 2$$

$$(3, 3) - 0$$

Upper bound = 10

Forcibly included: [(6, 5)]

Forcibly excluded: [(7, 4), (4, 6), (3, 3)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (6, 5) and (7, 3)

Tolerances:

$$(7, 3) - 2$$

So there is no sense to branch.

Upper bound = 10

Forcibly included: [(6, 5), (3, 3)]

Forcibly excluded: [(7, 4), (4, 6)]



The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (2, 6) and (3, 3)

Tolerances:

$$(2, 6) - 2$$

So there is no sense to branch.

Upper bound = 10

Forcibly included: [(6, 5), (4, 6)]

Forcibly excluded: [(7, 4)]

999	999	999	999	0	0	0
999	999	999	999	0	0	999
5	999	999	999	1	999	999
999	0	999	999	999	999	999
6	999	999	3	2	999	999
999	999	0	999	999	999	999
5	999	999	999	1	999	999

The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 6) and (5, 4)

Tolerances:

$$(5, 4)$$
 - ∞

So there is no sense to branch.

```
Upper bound = 10
Forcibly included: [(6, 5), (7, 4)]
Forcibly excluded: []
Infeasible pair included forcibly!
```

Recursion tree

