Single Machine Scheduling

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Some details about algorithm

Lower bound - the solution found by hungarian algorithm.
Upper bound - the best feasible solution found at the moment.
Branching rule: from all elements causing infeasibility we choose one with the minimum tolerance and branch on it.
We interrupt recursion if:

- found solution is feasible
- lower bound $\geq$ upper bound
- we forcibly included infeasible pair

Input

1 job:
$p_1 = 3$
available: $[[1, 3], [5, \infty]]$
cost: $f(t) = t - 2$

2 job:
$p_2 = 2$
available: $[[2, \infty]]$
cost: $f(t) = t - 1$

3 job:
$p_3 = 2$
available: $[[2, \infty]]$
cost: $f(t) = t - 2$

Relaxation

time	1	2	3	4	5	6	7
1 job	0	0	0	∞	0	∞	∞
	∞	0	0	∞	0	0	∞
	∞	∞	1	∞	3	4	5
2 job	∞	0	0	0	0	0	∞
	∞	∞	2	3	4	5	6
3 job	∞	0	0	0	0	0	∞
	∞	∞	1	2	3	4	5

Solution: step 1

Upper bound = ∞
Forcibly included: []
Forcibly excluded: []

0	0	0	999	0	999	999
999	**0**	0	999	0	0	999
999	999	**1**	999	3	4	5
999	0	0	0	0	**0**	999
999	999	2	3	4	5	**6**
999	0	0	0	**0**	0	999
999	999	1	**2**	3	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (6, 5) and (7, 4).
Tolerances:
(6, 5) - 0
(7, 4) - 0

Solution: step 2

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5)]

0	0	0	999	0	999	999
999	**0**	0	999	0	0	999
999	999	**1**	999	3	4	5
999	0	0	0	**0**	0	999
999	999	2	3	4	5	**6**
999	0	0	0	999	**0**	999
999	999	1	**2**	3	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (6, 6) and (7, 4).
Tolerances:
(6, 6) - 0
(7, 4) - 0

Solution: step 3

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6)]

0	0	0	999	0	999	999
999	0	0	999	**0**	0	999
999	999	**1**	999	3	4	5
999	0	0	0	0	**0**	999
999	999	2	3	4	5	**6**
999	**0**	0	0	999	999	999
999	999	1	**2**	3	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (2, 5) and (3, 3).
Tolerances:
(2, 5) - 0
(3, 3) - 0

Solution: step 4

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6), (2, 5)]

0	0	0	999	0	999	999
999	0	0	999	999	**0**	999
999	999	**1**	999	3	4	5
999	0	0	0	**0**	0	999
999	999	2	3	4	5	**6**
999	**0**	0	0	999	999	999
999	999	1	**2**	3	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (2, 6) and (3, 3).
Tolerances:
(2, 6) - 1
(3, 3) - 0

Solution: step 5

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3)]

0	0	0	999	0	999	999
999	0	0	999	999	**0**	999
999	999	999	999	3	4	**5**
999	0	0	0	**0**	0	999
999	999	**2**	3	4	5	6
999	**0**	0	0	999	999	999
999	999	1	**2**	3	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (4, 5) and (5, 3).
Tolerances:
(4, 5) - 1
(5, 3) - 0

Solution: step 6

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3)]

0	0	0	999	0	999	999
999	0	0	999	999	**0**	999
999	999	999	999	3	4	**5**
999	0	0	0	**0**	0	999
999	999	999	**3**	4	5	6
999	**0**	0	0	999	999	999
999	999	**1**	2	3	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (4, 5) and (5, 4).
Tolerances:
(4, 5) - 1
(5, 4) - 1

Solution: step 7

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5)]

0	0	0	999	0	999	999
999	0	0	999	999	**0**	999
999	999	999	999	**3**	4	5
999	0	0	**0**	999	0	999
999	999	999	3	4	5	**6**
999	**0**	0	0	999	999	999
999	999	**1**	2	3	4	5

The optimal value equals 10.

Lower bound = 10
The solution is infeasible: (2, 6) and (3, 5).
Tolerances:
(2, 6) - 0
(3, 5) - 0

Solution: step 8

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6)]

0	0	0	999	0	999	999
999	**0**	0	999	999	999	999
999	999	999	999	**3**	4	5
999	0	0	0	999	**0**	999
999	999	999	3	4	5	**6**
999	0	0	**0**	999	999	999
999	999	**1**	2	3	4	5

The optimal value equals 10.

Lower bound = 10
The solution is infeasible: (6, 4) and (7, 3).
Tolerances:
(6, 4) - 1
(7, 3) - 1

Solution: step 9

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6), (6, 4)]

0	0	0	999	0	999	999
999	**0**	0	999	999	999	999
999	999	999	999	**3**	4	5
999	0	0	0	999	**0**	999
999	999	999	**3**	4	5	6
999	0	**0**	999	999	999	999
999	999	1	2	3	4	**5**

The optimal value equals 11.

Lower bound = 11
The solution is infeasible: (4, 6) and (5, 4).
Tolerances:
(4, 6) - 2
(5, 4) - 0

Solution: step 10

Upper bound = ∞
Forcibly included: []
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6), (6, 4), (5, 4)]

0	0	0	999	0	999	999
999	**0**	0	999	999	999	999
999	999	999	999	**3**	4	5
999	0	0	0	999	**0**	999
999	999	999	999	4	5	**6**
999	0	**0**	999	999	999	999
999	999	1	**2**	3	4	5

The optimal value equals 11.

Lower bound = 11
The solution is feasible!

Solution: step 11

Upper bound = 11
Forcibly included: [(5, 4)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6), (6, 4)]

0	0	0	999	0	999	999
999	**0**	0	999	999	999	999
999	999	999	999	**3**	4	5
999	0	0	999	999	**0**	999
999	999	999	**3**	999	999	999
999	0	**0**	999	999	999	999
999	999	1	999	3	4	**5**

The optimal value equals 11.

Lower bound = 11
Lower bound $\geq$ Upper bound

Solution: step 12

Upper bound = 11

Forcibly included: [(6, 4)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6)]

0	0	0	999	0	999	999
999	**0**	0	999	999	999	999
999	999	999	999	**3**	4	5
999	0	0	999	999	**0**	999
999	999	999	999	4	5	**6**
999	999	999	**0**	999	999	999
999	999	**1**	999	3	4	5

The optimal value equals 10.

Lower bound = 10

The solution is infeasible: (6, 4) and (7, 3).

Solution: step 13

Upper bound = 11
Forcibly included: [(6, 4)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6), (7, 3)]

0	0	0	999	0	999	999
999	**0**	0	999	999	999	999
999	999	999	999	**3**	4	5
999	0	**0**	999	999	0	999
999	999	999	999	4	**5**	6
999	999	999	**0**	999	999	999
999	999	999	999	3	4	**5**

The optimal value equals 13.

Lower bound = 13
Lower bound $\geq$ Upper bound

Solution: step 14

Upper bound = 11
Forcibly included: [(6, 4), (7, 3)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (2, 6)]
Infeasible pair included forcibly!

Solution: step 15

Upper bound = 11

Forcibly included: [(2, 6)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5)]

0	0	0	999	0	999	999
999	999	999	999	999	**0**	999
999	999	999	999	**3**	999	5
999	**0**	0	0	999	999	999
999	999	999	3	4	999	**6**
999	0	0	**0**	999	999	999
999	999	**1**	2	3	999	5

The optimal value equals 10.

Lower bound = 10

The solution is infeasible: (2, 6) and (3, 5); (6, 4) and (7, 3).

Tolerances:

(3, 5) - 0

(6, 4) - 0

(7, 3) - 1

Solution: step 16

Upper bound = 11
Forcibly included: [(2, 6)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (3, 5)]

0	0	0	999	0	999	999
999	999	999	999	999	**0**	999
999	999	999	999	999	999	**5**
999	**0**	0	0	999	999	999
999	999	999	3	**4**	999	6
999	0	0	**0**	999	999	999
999	999	**1**	2	3	999	5

The optimal value equals 10.

Lower bound = 10
The solution is infeasible: (6, 4) and (7, 3).
Tolerances:
(6, 4) - 0
(7, 3) - 1

Solution: step 17

Upper bound = 11
Forcibly included: [(2, 6)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (3, 5), (6, 4)]

0	0	0	999	0	999	999
999	999	999	999	999	**0**	999
999	999	999	999	999	999	**5**
999	0	0	**0**	999	999	999
999	999	999	3	**4**	999	6
999	**0**	0	999	999	999	999
999	999	**1**	2	3	999	5

The optimal value equals 10.

Lower bound = 10
The solution is feasible!

Solution: step 18

Upper bound = 10

Forcibly included: [(2, 6), (6, 4)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5), (3, 5)]

0	0	0	999	0	999	999
999	999	999	999	999	**0**	999
999	999	999	999	999	999	**5**
999	**0**	0	999	999	999	999
999	999	999	999	**4**	999	6
999	999	999	**0**	999	999	999
999	999	**1**	999	3	999	5

The optimal value equals 10.

Lower bound = 10

Lower bound $\geq$ Upper bound

Solution: step 19

Upper bound = 10

Forcibly included: [(3, 5)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (4, 5)]

0	0	0	999	999	999	999
999	0	0	999	999	**0**	999
999	999	999	999	**3**	999	999
999	0	0	**0**	999	0	999
999	999	999	3	999	5	**6**
999	**0**	0	0	999	999	999
999	999	**1**	2	999	4	5

The optimal value equals 10.

Lower bound = 10

Lower bound $\geq$ Upper bound

Solution: step 20

Upper bound = 10

Forcibly included: [(4, 5)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3)]

0	0	0	999	999	999	999
999	0	0	999	999	**0**	999
999	999	999	999	999	4	**5**
999	999	999	999	**0**	999	999
999	999	999	**3**	999	5	6
999	**0**	0	0	999	999	999
999	999	**1**	2	999	4	5

The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 5) and (5, 4)

Solution: step 21

Upper bound = 10

Forcibly included: [(4, 5)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3), (5, 4)]

0	0	0	999	999	999	999
999	**0**	0	999	999	0	999
999	999	999	999	999	**4**	5
999	999	999	999	**0**	999	999
999	999	999	999	999	5	**6**
999	0	0	**0**	999	999	999
999	999	**1**	2	999	4	5

The optimal value equals 11.

Lower bound = 11

Lower bound $\geq$ Upper bound

Solution: step 22

Upper bound = 10
Forcibly included: [(4, 5), (5, 4)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (5, 3)]
Infeasible pair included forcibly!

Solution: step 23

Upper bound = 10

Forcibly included: [(5, 3)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3)]

0	0	999	999	0	999	999
999	0	999	999	999	**0**	999
999	999	999	999	3	4	**5**
999	0	999	0	**0**	0	999
999	999	**2**	999	999	999	999
999	**0**	999	0	999	999	999
999	999	999	**2**	3	4	5

The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 5) and (5, 3).

Solution: step 24

Upper bound = 10

Forcibly included: [(5, 3)]

Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3), (4, 5)]

0	0	999	999	0	999	999
999	0	999	999	999	**0**	999
999	999	999	999	**3**	4	5
999	0	999	**0**	999	0	999
999	999	**2**	999	999	999	999
999	**0**	999	0	999	999	999
999	999	999	2	3	4	**5**

The optimal value equals 10.

Lower bound = 10

Lower bound $\geq$ Upper bound

Solution: step 25

Upper bound = 10
Forcibly included: [(5, 3), (4, 5)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (3, 3)]
Infeasible pair included forcibly!

Solution: step 26

Upper bound = 10
Forcibly included: [(3, 3)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5)]

0	0	999	999	0	999	999
999	0	999	999	999	**0**	999
999	999	**1**	999	999	999	999
999	0	999	0	**0**	0	999
999	999	999	3	4	5	**6**
999	**0**	999	0	999	999	999
999	999	999	**2**	3	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (2, 6) and (3, 3)

Solution: step 27

Upper bound = 10
Forcibly included: [(3, 3)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5), (2, 6)]

0	0	999	999	0	999	999
999	**0**	999	999	999	999	999
999	999	**1**	999	999	999	999
999	0	999	0	0	**0**	999
999	999	999	3	**4**	5	6
999	0	999	**0**	999	999	999
999	999	999	2	3	4	**5**

The optimal value equals 10.

Lower bound = 10
Lower bound $\geq$ Upper bound

Solution: step 28

Upper bound = 10
Forcibly included: [(3, 3), (2, 6)]
Forcibly excluded: [(6, 5), (6, 6), (2, 5)]
Infeasible pair included forcibly!

Solution: step 29

Upper bound = 10
Forcibly included: [(2, 5)]
Forcibly excluded: [(6, 5), (6, 6)]

0	0	0	999	999	999	999
999	999	999	999	**0**	999	999
999	999	**1**	999	999	4	5
999	0	0	0	999	**0**	999
999	999	2	**3**	999	5	6
999	**0**	0	0	999	999	999
999	999	1	2	999	4	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (2, 5) and (3, 3); (4, 6) and (5, 4)
Tolerances:
(3, 3): 0
(4, 6): 2
(6, 4): 0

Solution: step 30

Upper bound = 10
Forcibly included: [(2, 5)]
Forcibly excluded: [(6, 5), (6, 6), (3, 3)]

0	0	0	999	999	999	999
999	999	999	999	**0**	999	999
999	999	999	999	999	4	**5**
999	0	0	0	999	**0**	999
999	999	**2**	3	999	5	6
999	**0**	0	0	999	999	999
999	999	1	**2**	999	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (4, 6) and (5, 3)
Tolerances:
(4, 6): 2
(5, 3): 0

Solution: step 31

Upper bound = 10
Forcibly included: [(2, 5)]
Forcibly excluded: [(6, 5), (6, 6), (3, 3), (5, 3)]

0	0	0	999	999	999	999
999	999	999	999	**0**	999	999
999	999	999	999	999	4	**5**
999	0	0	0	999	**0**	999
999	999	999	**3**	999	5	6
999	**0**	0	0	999	999	999
999	999	**1**	2	999	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (4, 6) and (5, 4)
Tolerances:
(4, 6): 2
(5, 4): 2
No sense to branch because Lower bound + min{tolerances} $\geq$ Upper bound.

Solution: step 32

Upper bound = 10

Forcibly included: [(2, 5), (5, 3)]

Forcibly excluded: [(6, 5), (6, 6), (3, 3)]

0	0	999	999	999	999	999
999	999	999	999	**0**	999	999
999	999	999	999	999	4	**5**
999	0	999	0	999	**0**	999
999	999	**2**	999	999	999	999
999	**0**	999	0	999	999	999
999	999	999	**2**	999	4	5

The optimal value equals 9.

Lower bound = 9

The solution is infeasible: (4, 6) and (5, 3)

Solution: step 33

Upper bound = 10

Forcibly included: [(2, 5), (5, 3)]

Forcibly excluded: [(6, 5), (6, 6), (3, 3), (4, 6)]

0	0	999	999	999	999	999
999	999	999	999	**0**	999	999
999	999	999	999	999	**4**	5
999	**0**	999	0	999	999	999
999	999	**2**	999	999	999	999
999	0	999	**0**	999	999	999
999	999	999	2	999	4	**5**

The optimal value equals 11.

Lower bound = 11

Lower bound $\geq$ Upper bound

Solution: step 34

Upper bound = 10
Forcibly included: [(2, 5), (5, 3), (4, 6)]
Forcibly excluded: [(6, 5), (6, 6), (3, 3)]
Infeasible pair included forcibly!

Solution: step 35

Upper bound = 10
Forcibly included: [(2, 5), (3, 3)]
Forcibly excluded: [(6, 5), (6, 6)]
Infeasible pair included forcibly!

Solution: step 36

Upper bound = 10
Forcibly included: [(6, 6)]
Forcibly excluded: [(6, 5)]

0	0	0	999	0	999	999
999	**0**	0	999	0	999	999
999	999	**1**	999	3	999	5
999	0	0	0	**0**	999	999
999	999	2	3	4	999	**6**
999	999	999	999	999	**0**	999
999	999	1	**2**	3	999	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (6, 6) and (7, 4)

Solution: step 37

Upper bound = 10
Forcibly included: [(6, 6)]
Forcibly excluded: [(6, 5), (7, 4)]

0	0	0	999	0	999	999
999	**0**	0	999	0	999	999
999	999	**1**	999	3	999	5
999	0	0	0	**0**	999	999
999	999	2	**3**	4	999	6
999	999	999	999	999	**0**	999
999	999	1	999	3	999	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (4, 5) and (5, 4)
Tolerances:
(4, 5) - 0
(5, 4) - 1

Solution: step 38

Upper bound = 10
Forcibly included: [(6, 6)]
Forcibly excluded: [(6, 5), (7, 4), (4, 5)]

0	0	0	999	0	999	999
999	0	0	999	**0**	999	999
999	999	**1**	999	3	999	5
999	**0**	0	0	999	999	999
999	999	2	**3**	4	999	6
999	999	999	999	999	**0**	999
999	999	1	999	3	999	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (2, 5) and (3, 3)
Tolerances:
(2, 5) - 1
(3, 3) - 0

Solution: step 39

Upper bound = 10
Forcibly included: [(6, 6)]
Forcibly excluded: [(6, 5), (7, 4), (4, 5), (3, 3)]

0	0	0	999	0	999	999
999	0	0	999	**0**	999	999
999	999	999	999	3	999	**5**
999	**0**	0	0	999	999	999
999	999	2	**3**	4	999	6
999	999	999	999	999	**0**	999
999	999	**1**	999	3	999	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (6, 6) and (7, 3)
Tolerances:
(7, 3) - 1 So there is no sense to branch.

Solution: step 40

Upper bound = 10
Forcibly included: [(6, 6), (3, 3)]
Forcibly excluded: [(6, 5), (7, 4), (4, 5)]

0	0	999	999	0	999	999
999	0	999	999	**0**	999	999
999	999	**1**	999	999	999	999
999	**0**	999	0	999	999	999
999	999	999	**3**	4	999	6
999	999	999	999	999	**0**	999
999	999	999	999	3	999	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (2, 5) and (3, 3)
Tolerances:
(2, 5) - 1 So there is no sense to branch.

Solution: step 41

Upper bound = 10
Forcibly included: [(6, 6), (4, 5)]
Forcibly excluded: [(6, 5), (7, 4)]

0	0	0	999	999	999	999
999	**0**	0	999	999	999	999
999	999	**1**	999	999	999	5
999	999	999	999	**0**	999	999
999	999	2	**3**	999	999	6
999	999	999	999	999	**0**	999
999	999	1	999	999	999	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (4, 5) and (5, 4)
Tolerances:
(5, 4) - ∞ So there is no sense to branch.

Solution: step 42

Upper bound = 10
Forcibly included: [(6, 6), (7, 4)]
Forcibly excluded: [(6, 5)]
Infeasible pair included forcibly!

Solution: step 43

Upper bound = 10
Forcibly included: [(6, 5)]
Forcibly excluded: []

0	0	0	999	999	999	999
999	**0**	0	999	999	0	999
999	999	**1**	999	999	4	5
999	0	0	0	999	**0**	999
999	999	2	3	999	5	**6**
999	999	999	999	**0**	999	999
999	999	1	**2**	999	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (6, 5) and (7, 4)

Solution: step 44

Upper bound = 10
Forcibly included: [(6, 5)]
Forcibly excluded: [(7, 4)]

0	0	0	999	999	999	999
999	**0**	0	999	999	0	999
999	999	**1**	999	999	4	5
999	0	0	0	999	**0**	999
999	999	2	**3**	999	5	6
999	999	999	999	**0**	999	999
999	999	1	999	999	4	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (4, 6) and (5, 4)
Tolerances:
(4, 6) - 0
(5, 4) - 2

Solution: step 45

Upper bound = 10
Forcibly included: [(6, 5)]
Forcibly excluded: [(7, 4), (4, 6)]

0	0	0	999	999	999	999
999	0	0	999	999	**0**	999
999	999	**1**	999	999	4	5
999	**0**	0	0	999	999	999
999	999	2	**3**	999	5	6
999	999	999	999	**0**	999	999
999	999	1	999	999	4	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (2, 6) and (3, 3)
Tolerances:
(2, 6) - 2
(3, 3) - 0

Solution: step 46

Upper bound = 10
Forcibly included: [(6, 5)]
Forcibly excluded: [(7, 4), (4, 6), (3, 3)]

0	0	0	999	999	999	999
999	0	0	999	999	**0**	999
999	999	999	999	999	4	**5**
999	**0**	0	0	999	999	999
999	999	2	**3**	999	5	6
999	999	999	999	**0**	999	999
999	999	**1**	999	999	4	5

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (6, 5) and (7, 3)
Tolerances:
(7, 3) - 2
So there is no sense to branch.

Solution: step 47

Upper bound = 10
Forcibly included: [(6, 5), (3, 3)]
Forcibly excluded: [(7, 4), (4, 6)]

0	0	999	999	999	999	999
999	0	999	999	999	**0**	999
999	999	**1**	999	999	4	5
999	**0**	999	0	999	999	999
999	999	999	**3**	999	5	6
999	999	999	999	**0**	999	999
999	999	1	999	999	4	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (2, 6) and (3, 3)
Tolerances:
(2, 6) - 2
So there is no sense to branch.

Solution: step 48

Upper bound = 10
Forcibly included: [(6, 5), (4, 6)]
Forcibly excluded: [(7, 4)]

0	0	0	999	999	999	999
999	**0**	0	999	999	999	999
999	999	**1**	999	999	999	5
999	999	999	999	999	**0**	999
999	999	2	**3**	999	999	6
999	999	999	999	**0**	999	999
999	999	1	999	999	999	**5**

The optimal value equals 9.

Lower bound = 9
The solution is infeasible: (4, 6) and (5, 4)
Tolerances:
(5, 4) - ∞
So there is no sense to branch.

Solution: step 49

Upper bound = 10
Forcibly included: [(6, 5), (7, 4)]
Forcibly excluded: []
Infeasible pair included forcibly!

Recursion tree