519 Problem Set 6: Central Limit Theorem and Laws of Large Numbers

Problem 1. (August 2014) Let $X_1, X_2, \ldots$ be an infinite sequence of iid uniform random variables on $[0,1]$. Find the limit

$$\lim_{n\to\infty} P\left(\frac{(\prod_{i=1}^n X_i)^{1/n}}{\frac{1}{n}\sum_{i=1}^n X_i} > \frac{3}{4}\right)$$

Problem 2. (Jan 2013) Ten independently picked $U[0,100]$ numbers are each rounded to the nearest integer. Use the central limit theorem to approximate the probability that the sum of the ten rounded numbers equals the rounded value of the sum of the ten original numbers

Problem 3. (Aug 2012) m balls are distributed into n urns in a completely random and mutually independent way. Let $W_{m,n}$ denote the number of bins that remain empty.

a) Find the mean and variance of $W_{m,n}$.

b) Suppose there is a fixed $\lambda \in (0,1)$ such that $m/n \to \lambda$ as $n \to \infty$. Show that there is a constant $c(\lambda)$ such that

$$\frac{W_{m,n}}{n} \xrightarrow{p} c(\lambda)$$

and identify $c(\lambda)$.

Problem 4. A man has had much too much to drink, but is still strong enough to walk and to see where he is trying to go. He starts at the origin in $\mathbb{R}^2$. He takes iid steps, each step equally likely to be "up," "down," or "right," always of length 1: in other words, the three steps $(0,1)$, $(0,1)$, and $(1,0)$ are of probability $1/3$ each. At the first time that his horizontal position equals 100, what is the approximate numerical probability that his vertical position is greater than or equal to 10?

Problem 5. A fair coin is tossed n times. Suppose X heads are obtained. Given $X = x$, let Y be generated according to the Poisson distribution with mean x. Find the unconditional variance of Y , and then find the limit of the probability $P(|Y - \frac{n}{2}| > n^{3/4})$, as $n \to \infty$.

Problem 6. Anirban plays a game repeatedly. On each play he wins an amount uniformly distributed in $(0,1)$ (in dollars), and then he tips the lady in charge of the game the square of the amount he has won. Then he plays again, tips again, and so on. What is the approximate probability that if he plays and tips six hundred times, his total winnings minus his total tips will exceed 105 dollars?

Problem 7. Half of the many stones on a beach are perfect cubes of edge length one, and half are perfect cubes of edge length two. Out of the vast number of stones, Tom picks one hundred at random and Mary also picks one hundred at random. Estimate the probability that the total volume of the stones Tom chose exceeds 450, and estimate the probability that the total volume of the stones Tom chose exceeds the total volume of the stones that Mary chose by at least 100.

Problem 8. A fair die is repeatedly rolled until the sum of the rolls exceeds 500 for the first time. Find approximately the probability that at most 140 tosses will be necessary. You may leave your answer in the form of an integral.