519 Problem Set 3: Densities and Probabilities

Problem 1. (Jan 2015) Suppose that X and Y are jointly distributed on the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. For $0 \le x \le 1$, the joint probability density function $f_{X,Y}(x,y)$ is proportional to x. Let W denote the product of X and Y, i.e., $W := XY$. Find the probability density function of W.

Problem 2. Jan 2012 The random variables X, Z, and W are independent. X is geometric with parameter $\frac{1}{3}$, and Z and W are exponential with parameters 2 and 3, respectively. Find $P(X < Z < W)$.

Problem 3. (Aug 2012) Suppose X and Y have joint density:

$$f_{X,Y}(x,y) = \begin{cases} 1/x^3 \text{ if } x > 1 \text{ and } 0 \le y \le x \\ 0 \text{ otherwise} \end{cases}$$

Find the density of $X - Y$.

Problem 4. (Jan 2011) Let X_i, $1 \le i \le 10$, be independent exponential random variables. Let Y_1 and Y_2 be independent random variables which are both discrete uniform on $\{0,1,2\}$ and which are also independent of the exponentials. Find the probability that of these twelve random variables, the Y_j's are the second and third order statistics, that is the second and third smallest of the twelve numbers.

Problem 5. Suppose that a solution now contains a single living bacterium. This organism has the property that, after 24 hours, it will give rise to a random number N_1 of descendants with a Geometric(p) number of failures distribution: $P(N_1 = k) = q^k p$, $k = 0,1,2,\dots$ with $p \in (0,1)$ and $q = 1-p$ and $E[N_1] = \frac{q}{p}$. If it gives rise to no descendants after 24 hours, it dies (So N_1 is the population size after 24 hours). Furthermore, each bacterium present in 24 hours will give rise after another 24 hours to a random number of descendants with the same Geometric(p) number of failures distribution, with different bacteria having independent numbers of descendants. Let N_2 be the population size 48 hours from now. Find $E(N1|N2 = 0)$. (Hint: The maximum value of this quantity as p varies between 0 and 1 is 1/3.)

Problem 6. Joe's dog got mad at him and broke his walking cane, first uniformly into two pieces, and then the longer piece again uniformly into two pieces. Find the probability that Joe can make a triangle out of the three pieces of his cane.

Problem 7. A needle of length $L < 1$ is dropped randomly onto a sheet of paper with parallel lines drawn at unit distance apart. Find explicitly the probability that the needle will cut one of the parallel lines.

Problem 8. Suppose that U is a standard Cauchy random variable, that is, its density function is $f_U(x) = \frac{1}{\pi}\frac{1}{1+x^2}$

i What is the law of $1/U$?

ii Show that for any $\varepsilon \in [0,1]$, it holds that $e[|U|^{1-\varepsilon}] \ge 1$.