519 Problem Set 4: Uniform and Exponential

Problem 1. (based on August 2014) Suppose X and Y are independent exponential random variables with parameters λ_1 and λ_2, respectively. Find the CDF and mean of $\frac{X}{X+Y}$.

Problem 2. (August 2014) Suppose for some given $m \geq 2$, we choose m iid $U[0,1]$ variables $X_1, X_2, \ldots, X_m$. Let $X_{(1)}$ denote their minimum and $X_{(m)}$ their maximum. Now continue sampling $X_{m+1}, X_{m+2}, \ldots$ from the $U[0,1]$ density. Let N be the first index k such that X_{m+k} falls outside of the interval $[X_{(1)}, X_{(m)}]$.

a) Find a formula for $P(N > n)$ for a general n.

b) Hence, explicitly find $E(N)$.

Problem 3. (Jan 2013) Three points are chosen independently and uniformly inside the unit square in the plane. Find the expected area of the smallest closed rectangle that has sides parallel to the
coordinate axes and that contains the three points.

Problem 4. (Aug 2013) Let X_i, for $1 \leq i \leq 5$, be independent exponential (1) random variables. Let N be the number of these random variables which are a nearest neighbor of one of the others (meaning their distance from one of the others is the least of the four distances). Find the expectation of N.

Problem 5. (Jan 2011) Ten points are chosen independently according to the uniform distribution on the interval $[0, 1]$.

a. Find the probability that at least 8 of the other 9 points are within .01 of the minimum point.

b. Find the probability that there is an interval of length .01 which contains at least 9 of the 10 points.

Problem 6. Let X, Y, and Z be iid uniform $(0, 1)$ random variables. Let min be the smallest of X, Y, Z, Max be the largest, and Med be the median. Find $P(\text{Med} - \min < Y - X)$.

Problem 7. Three lightbulbs have independent exponential parameter one lifetimes. Find the probability that none of the three failure times are within one of each other.

Problem 8. (a) Find the density of the median of three independent uniform $(0, 1)$ variables.

(b) Is there a density g such that if X_1, X_2, X_3 are independent and have density g then the median of the X_i has a uniform (0, 1) distribution?

Problem 9. Kelly began working as an Uber driver t weeks ago, and due to her job she frequently comes into contact with people. She tested negative for the coronavirus and for antibodies when she started her job, but hasn't been tested since. Assume the time it takes for Kelly to contract the 'rona, starting from when she began her job, is an exponential random variable with (unknown) parameter λ, and that once she gets it, she sheds the virus for two weeks. You talked to her today with no masks or social distancing, and you want to know the probability that she was shedding the virus.

- Based on this model, what is the largest possible probability that she is shedding the virus today?
- What are some reasons why you can and cannot be confident that your model is reasonable?

Problem 10. (Aug 2019) In Boilermaker dining hall, the weight X of the sirloin steak follows exponential distribution with mean 12 oz. If the steak is less than 15 oz, it is already included in your meal plan, which means that you do not pay extra. If it is more than 15 oz., you need to pay an extra $X - 15$ dollars. Let Y be the extra amount you need to pay.

(a) Find and plot the cdf of Y.

(b) Find $E[Y]$.

(c) Five friends walk into the dining hall and order steak. Assuming the sizes of their steaks are independent, what is the probability that at least two of them have to pay extra?

Problem 11. (Aug 2019)
$X_1, X_2, \ldots$ are i.i.d. random variables drawn according to Unif$(0, \theta)$. Let $Y_n = \max\{X_1, \ldots, X_n\}$ and $Z_n = \min\{X_1, \ldots, X_n\}$.

(a) Find $E[Y_n]$.

(b) Let $W_n = n(\theta - Y_n)$. Find $g(\omega) = \lim_{n\to\infty} P(W_n \leq w)$.

(c) Find $P(X_{n+1} < Y_n)$.

(d) Find $(X_{n+1} > Y_n | X_{n+1} > Z_n)$.