519 Problem Set 1: Mostly Combinatorics

Problem 1. (Jan 2015) Suppose that $U_0, U_1, \dots, U_n$ are independent continuous random variables, each uniformly distributed on the interval $[0, 2]$. Let T_n denote the number of sets $\{i, j\}$, with $1 \leq i < j \leq n$, such that $\max(Ui, Uj) < U_0$. Find the probability mass function of T_n.

Problem 2. (Jan 2014) Roll a fair, six-sided die 10 times. What is the probability that each side appears at least once?

Problem 3. (August 2014)

a) 3 balls are distributed one by one and at random in 3 boxes. What is the probability that exactly one box remains empty?

b) n balls are distributed one by one and at random in n boxes. Find the probability that exactly one box remains empty.

c) n balls are distributed one by one and at random in n boxes. Find the probability that exactly two boxes remain empty.

Problem 4. Consider a collection of $2n$ people: n women and their n husbands. Randomly put the people into n groups consisting of 2 people each. (A group may consist of any two people, regardless of sex.) What is the probability that exactly j of the n groups consist of a married couple? Give an exact formula.

Problem 5. (Aug 2013) An urn contains 5 balls numbered 1, 2, 3, 4, 5. The balls are identical, aside from the numbering. Draw two balls from the urn at random, record their numbers, and return the balls to the urn. Repeat three times. Thus, you obtain four independent Simple Random Samples of size 2 from the urn. Find the numerical probability that every one of the 5 balls appears at least once in your four Simple Random Samples of size 2.

Problem 6. A particle performing a simple, symmetric three-dimensional random walk starts at time $n = 0$ at the origin $(0, 0, 0)$. At each subsequent time $n = 1, 2, 3, \dots,$ the particle moves exactly 1 unit in one direction: either right, left, forward, backward, up, or down. Each of these six possible moves occurs with an equal probability of $1/6$. The particle makes independent movement decisions at different times. Calculate the probability that, at time $n = 6$, the particle is at the origin.

Problem 7. Toss a coin with probability p of heads repeatedly. Let X, Y, Z, and W be the tosses on which the first, second, third and fourth heads respectively appeared. For example if the tosses are TTTHTHTTHH then $X = 4$, $Y = 6$, $Z = 9$, and $W = 10$. Find $P(Y - X > X)$. Find $P(\text{either } X = 6 \text{ or } Y = 6 \text{ or } Z = 6 | W = 15)$.

Problem 8. A die is rolled until two different numbers appear. Let T be the total number of times the die is rolled. Find $E[T]$ and also find the probability that exactly one three is rolled up to and including roll number T.

Problem 9. The birthdays of 5 people are known to fall in exactly 3 calendar months. What is the probability that exactly two of the 5 were born in January ?

Problem 10. Let N be a positive integer. Choose an integer at random from $\{1, \ldots, N\}$. Let E be the event that your chosen random number is divisible by 3, and divisible by at least one of 4 and 6, but not divisible by 5. Find, explicitly, $\lim_{N\to\infty} P(E)$.

Problem 11. Steve and Kyle, with 5 other men, and 5 other women, are lined up at random for taking a picture. Let X be the number of men between Steve and Kyle in the lineup. Compute, exactly, $P(X = 2)$.

Problem 12. Suppose the number of children in a particular family is 1, 2, 3 or 4, with probabilities .1, .3, .4, .2 respectively. Nells, a child in this family, has no sisters. What is the probability he is the only child?

Problem 13. Ten white and ten black balls are divided into two boxes. What is the division that maximizes the probability that if one of the boxes is picked at random and then one ball is drawn from the box at random, the chosen ball is white? Justify your answer.

Problem 14. Ten men and ten women are randomly seated around a round table. Let N be the number of men whose immediate neighbors are both women. Find $E[N]$, $P(N = 10)$, $P(N = 9)$.

Problem 15. A city has n families which have at least 3 children in the family. Give a good estimate of the minimal value of n so that there is a probability of at least $1/2$ that, for some pair of families, the firstborns will have a common birthday, the secondborns have a common birthday, and the thirdborns will have a common birthday. As usual in birthday problems, you should ignore leap days and assume that birthdays are independent for different people and uniformly distributed on the 365 possible days.

Problem 16. There are 30 chairs around a (very large) circular table. People arrive one by one. As each person arrives, he or she takes one of the empty seats at random (with all empty seats having equal probability, of course). After 7 people have arrived and seated themselves, what is the probability that no two people are adjacent?