519 Problem Set 2: Mean and Variance

Problem 1. (August 2014) n players each roll a fair six-sided die. For each pair of players who roll the same number, the group is awarded a point. Find the mean and variance of the total number of points the group receives.

Problem 2. (Jan 2015) Fix positive integers m and n. A person sitting at a circular table is called isolated if nobody is sitting to the persons immediate left nor to the persons immediate right. If m people sit at a table with n seats, with all seatings equally likely, find the variance of the number of people who are isolated.

Problem 3. (Jan 2015) Suppose that $U_0, U_1, \ldots, U_n$ are independent continuous random variables, each uniformly distributed on the interval $[0, 2]$. Let T_n denote the number of sets $\{i, j\}$, with $1 \leq i < j \leq n$, such that $\max(Ui, Uj) < U_0$.

a. Find the probability mass function of T_n.

b. Find the expected value of T_n.

c. Find the variance of T_n.

Problem 4. (Jan 2013) Let $X_1, X_2, \ldots, X_n$ denote a sequence of independent Bernoulli(p) random variables. Let Y denote the number of ”runs” (i.e. strings of consecutive 0s or 1s). Find the expectation and variance of Y.

Problem 5. Jan 2012 Ten points are chosen at random on a circle of radius 1. A point is lonely if no other point is within a circle distance of $2\pi/100$ from it. Find the expectation and variance of the number of lonely points.

Problem 6. (Aug 2012) m balls are distributed into n urns in a completely random and mutually independent way. Let $W_{m,n}$ denote the number of bins that remain empty.

a) Find the mean and variance of $W_{m,n}$.

b) Suppose there is a fixed $\lambda \in (0, 1)$ such that $m/n \to \lambda$ as $n \to \infty$. Show that there is a constant $c(\lambda)$ such that

$$\frac{W_{m,n}}{n} \xrightarrow{p} c(\lambda)$$

and identify $c(\lambda)$.

(Do part (a) only for this problem set).

Problem 7. (Aug 2012) Assume that X is a positive random variable such that

$$\lim_{x\to\infty} e^{\sqrt{t}} P(X > t) = L > 0$$

for some finite L. Compute $\sup\{p > 0 : E[e^{X^p}] < \infty\}$.

Problem 8. (Jan 2011) A pond contains M golden fish and K silver fish. The fish are removed one at a time at random until all the fish remaining in the pond are the same color. Find the expectation and the variance of the number of fish remaining in the pond.

Problem 9. (Jan 2010) Let U_i be uniform on $(i-1, i+1)$ for $i = 1, 2, \ldots, n$, and let the U_i be independent. Let N be the number of integers i for which $U_{i-1} > U_i$. Find the mean of N. Also find the mean and variance of $\sum_{i=2}^{n}(U_{i-1} - Ui)^+$. ($a^+$ equals a if a is positive and equals 0 if a is not positive).

Problem 10. Twelve dots are arranged in four rows, with three dots in each row. Randomly choose four of the twelve dots. Let N be the number of rows with no chosen dot. Find the mean and variance of N.

Problem 11. Coupons are drawn, independently, with replacement, one at a time, from a set of 10 coupons. Find, explicitly, the expected number of draws a) until the first drawn coupon is drawn again; b) until a duplicate occurs.

Problem 12. Burgess is going to Moose Pass, Alaska. He is driving his Dodge. He puts his car on cruise control at 70 miles per hour. Gas stations are located every 30 miles, starting from his home. His car runs out of gas at a time distributed as an Exponential with mean 4 hours. When that happens, he gets out, takes his bike out of his trunk, and bikes to the NEXT gas station, say M, at 10 miles per hour. Let the time elapsed between when Burgess starts his trip and when he arrives at the gas station M be T. Find $E[T]$. (For simplicity, assume Moose Pass, Alaska is infinitely far away from Burgess's home.)

Problem 13. A fair coin is tossed n times. Suppose X heads are obtained. Given $X = x$, let Y be generated according to the Poisson distribution with mean x. Find the unconditional variance of Y , and then find the limit of the probability $P(|Y - \frac{n}{2}| > n^{3/4})$, as $n \to \infty$. (just do the first part for this problem set).

Problem 14. A balanced, six-sided die is rolled ten times. Add the total of all the numbers which were rolled immediately preceding those times a six is rolled. Call this total the score. (So if the first five rolls are two and the last five are six the score is 26.) Find the mean and variance of the score.

Problem 15. Toss a quarter n times. Each time the quarter comes up heads toss a nickel. Let X_n equal the number of heads on the quarter, and Y_n be the number of heads on the nickel. Find $E[X_n + Y_n]$ and $E[X_n Y_n]$.

Problem 16. Let X and Y be random variables. Let $m = \min(X, Y)$ and $M = \max(X, Y)$.

(a) Prove that $E[X]E[Y] \geq E[m]E[M]$

(b) Prove that if $\mathrm{Cov}(X, Y) > 0$, then $\mathrm{Cov}(m, M) > 0$.

Problem 17. (Aug 2019) Let $ABCD$ be a square with are 1. Let $\alpha, \beta, \gamma, \delta$ be random points on $\overline{AB}$, $\overline{AB}$, $\overline{CD}$, $\overline{DA}$, respectively. Let S be the area of quadrangle $\alpha\beta\gamma\delta$. Find $E[S]$ and $\mathrm{Var}(S)$.