519 Problem Set 5: Joint Densities

Problem 1. (Jan 2014) Let X and Y be independent exponential random variables with mean 1. Let $U = e^X + 2e^Y$, and $V = 2e^{X^2} + e^{Y^2}$. Let $g(u, v)$ be a joint density function for U and V which is as continuous as possible. Find $g(3e, 3e)$.

Problem 2. (Jan 2013) Let X be uniformly distributed on the interval $(0, 1)$. Consider a sequence $\{Y_k\}_{k\geq 1}$ of independent random variables that each have density $\frac{1}{(1+y)^2}$ for $y > 0$. Also assume that the Y_k are independent of X. Let N be the index of the first Y_k which is strictly larger than X, and for succinctness, define $Z := Y_N$. Find the joint probability density function of X and Z.

Problem 3. (Aug 2010) Let X and Y be independent unit exponential random variables, and let $W = X/Y$. Find the density of W.

Problem 4. Suppose X_1 and X_2 are random variables with joint density

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 3x_1 \text{ if } 0 < x_2 < x_1 < 1 \\ 0 \text{ otherwise} \end{cases}$$

Let $Y_1 = \frac{1}{X_1}$ and $Y_2 = \frac{1}{X_2}$. Find the joint density $f_{Y_1,Y_2}(y_1, y_2)$ of Y_1 and Y_2.

Problem 5. Let X and Y be independent exponential random variables each with parameter one. Find the density of $|X - Y|$ and the joint density of X and $|X - Y|$.

Problem 6. Find the greatest possible value of $E(XY)$, if X is Exponential ($\lambda = 1$) and Y is discrete uniform on $\{1, 2\}$. Justify your answer.

Problem 7. Let T be an exponential random variable with parameter λ. Let $X|T$ be uniformly distributed on $[0, T]$. Find the joint density of T and X, and find the mean and variance of X.

Problem 8. Suppose X is $N(0, 1)$ (i.e. standard normal) and $Y|X = x$ is $N(x + 1, 1)$.

1. Find the marginal distribution of Y.
2. Find the correlation between X and Y.
3. Find $E[X|Y = y]$.