# 519 Problem Set 1: Proofs

**Problem 1.** (Jan 2013) Assume that $X_1$, $X_2$ are two iid random variables such that $E[X_1^2] < \infty$, and there exists $c > 0$ such that $\frac{X_1+X_2}{c}$ has the same probability distribution as $X_1$.

a. Compute c.

b. Prove that $X_1$ is normally distributed.

**Problem 2.** Jan 2012 Prove that if $X$ has a normal distribution and $\theta$ is a number, then the variance of $\max(X, \theta)$ is smaller than the variance of $X$.

**Problem 3.** There are $N$ molecules of benzine mixed in with a quart of water in a jar. Half the mixture is removed and replaced with half a quart of pure water. The liquid in the jar is mixed and again half of it is removed and replaced with water, and so on. Show that there is a positive number $c$ not depending on $N$ such that for every $N$ there is a probability of at least $c$ that at some time during this process exactly one molecule of benzine remains in the jar.

**Problem 4.** a) Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ be an infinite sequence of events, where each one is contained in the next. Prove the following:

$$\lim_{n\to\infty} P(E_n) = P\left(\bigcup_{i=1}^{\infty} E_i\right)$$

b) Let $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$ be an infinite sequence of events, where each one contains the next. Use the previous result to prove:

$$\lim_{n\to\infty} P(E_n) = P\left(\bigcap_{i=1}^{\infty} E_i\right)$$

**Problem 5.** Prove Chebychev's Inequality: Let $X$ be a random variable with mean $\mu$ and variance $\sigma^2$. Then for all $a > 0$:

a)

$$P(|X| \geq a) \leq \frac{\mu}{a}$$

(This version is also known as Markov's Inequality.)

b)

$$P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

**Problem 6.** Let $E_1, E_2, \ldots$ be events. Define:

$$A = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_i$$

and

$$B = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} E_i$$

a) If a fair coin is flipped infinitely many times, and $E_i$ is the event that the $i$th flip comes up heads, what is $P(A)$? What is $P(B)$?

b) Going back to the general setup, show that $P(B) \leq P(A)$.

c) Show that if $\sum_{i=1}^{\infty} P(E_i) < \infty$, then $P(A) = 0$.

Note: Part (c) is the famous Borel-Cantelli lemma. There are two ways to prove it. One uses the result of Problem 4, and the other uses expectation.

**Problem 7.** Customers entering a store spend nothing half the time and the other half the time they spend a continuous uniform on the interval (0, 1) number of dollars. Use Chebyshevs inequality to bound the probability that the average number of dollars spent by the first one hundred customers to enter the store is between .23 and .27.

**Problem 8.** Let $X$ and $Y$ be jointly distributed random variables. Prove that

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$$

**Hint.** *You may use the fact that* $E[X] = E[E[X|Y]]$.

**Problem 9.** A gamma random variable with parameters $(\lambda, s)$ has density

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)}$$

where $\Gamma(s)$ is the Gamma Function, which is equal to $(s-1)!$ for positive integer values of $s$. Note that an exponential random variable with parameter $\lambda$ is a Gamma random variable with parameters $(\lambda, 1)$.

a) Let $X \sim \text{Gamma}(\lambda, s)$ and $Y \sim \text{Gamma}(\lambda, t)$. Prove $X \sim \text{Gamma}(\lambda, s)$

b) Let $X_1, X_2, X_3, \ldots$ be iid $\text{Exp}(\lambda)$ random variables. Fix $T \in (0, \infty)$ and let $N$ be the largest $k$ such that $\sum_{i=1}^{k} X_i < T$. Prove that $N$ has a Poisson distribution. What is its mean?

**Problem 10.** a) Let $X$ have the discrete uniform distribution on $\{1, 2\}$. Let $Y$ have the discrete uniform distribution on $\{1, 2, 3\}$, and let $X$ and $Y$ be independent. What is $E[\frac{X}{X+Y}]$? How does it compare to $\frac{E[X]}{E[X+Y]}$?

b) Let $\{X_i\}_{i \in \mathbb{N}}$ be iid, and let $k \leq N$. Find

$$E\left[\frac{\sum_{i=1}^{k} X_i}{\sum_{i=1}^{N} X_i}\right]$$

Justify your answer.

**Problem 11.** Let $X$ and $Y$ be positive random variables, and let $Z$ be a nonzero random variable.

a) Prove that $\text{Cov}\left(X, \frac{1}{X}\right) \leq 0$, with equality if and only if $X$ is constant with probability 1.

b) Does the above necessarily hold for $Z$?

c) Prove that if $\text{Cov}(Y, X) > 0$, then $\text{Cov}\left(Y, \frac{1}{X}\right) < 0$, or provide a counterexample.

## 0.1 Bonus Questions

**Problem 12.** Use problems 5 and 6 to prove the Strong Law of Large Numbers for Bernoulli random variables. That is, if $(X_n)_n$ is an iid sequence where each $X_i$ is 1 with probability $p$ and 0 with probability $1-p$, show that:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = p$$

.

*Hint: For each* $\varepsilon$*, consider the event* $E_{n,\varepsilon} = \{|\frac{1}{n}\sum_{i=1}^n X_i - p| > \varepsilon\}$*. Show that it is equal to the event* $\{(\sum_{i=1}^n (x_i - p))^4 > n^4\varepsilon^4\}$*. Apply Problem 5, compute the relevant expectation, and then apply Problem 6, part (c).*

**Problem 13.** Suppose $X$ is a random variable, and $f$ is an increasing function. Assuming $E[X] < \infty$ and $E[f(X)] < \infty$, prove that $\mathrm{Cov}(X, f(X)) \geq 0$.

**Problem 14.** Egorov's Theorem says that if $(X_n)_{n\in\mathbb{Z}}$ is a sequence of random variables, and if $\lim_{n\to\infty} X_n = 0$ with probability 1, then $\lim_{n\to\infty} P(|X_n| > \varepsilon) = 0$ for all $\varepsilon$. In other words, *almost sure convergence* implies *convergence in probability.* In this problem, we will show that the converse of Egorov's Theorem is not true.

Say $X$ is a *St. Petersburg random variable* if $X = 2^k$ with probability $2^{-k}$ for all $k \geq 1$. Consider an infinite iid sequence $(X_n)_{n\in\mathbb{Z}}$ of St. Petersburg random variables, and then define $Y_n = \mathbb{1}_{\{X_n \geq n\}}$.

a) Prove that $Y_n$ converges to 0 in probability.

b) Prove that $Y_n$ does not converge to 0 almost surely.

*Hint: For part (b), it will help to recall that* $(1-x) \leq e^{-x}$ *for all* $x \geq 0$.