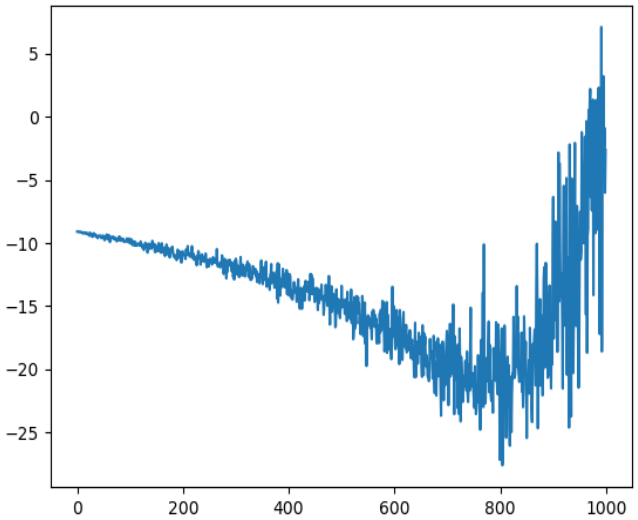
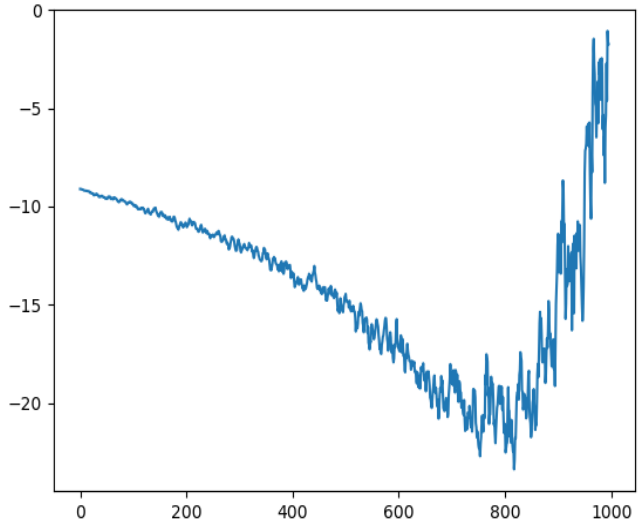
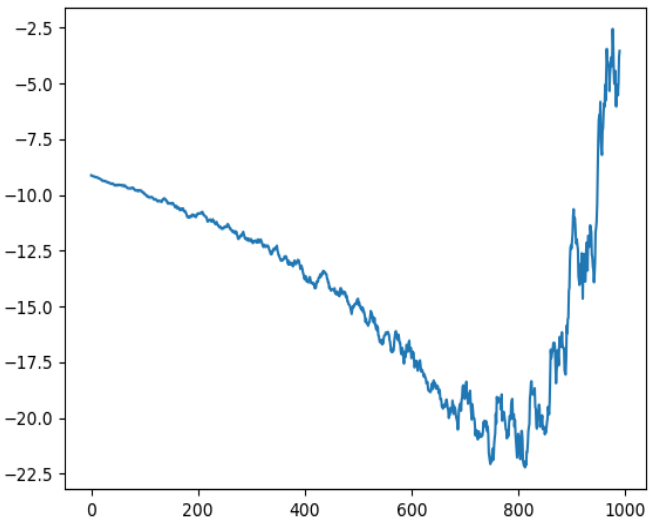
Social Welfare and Stable Matching

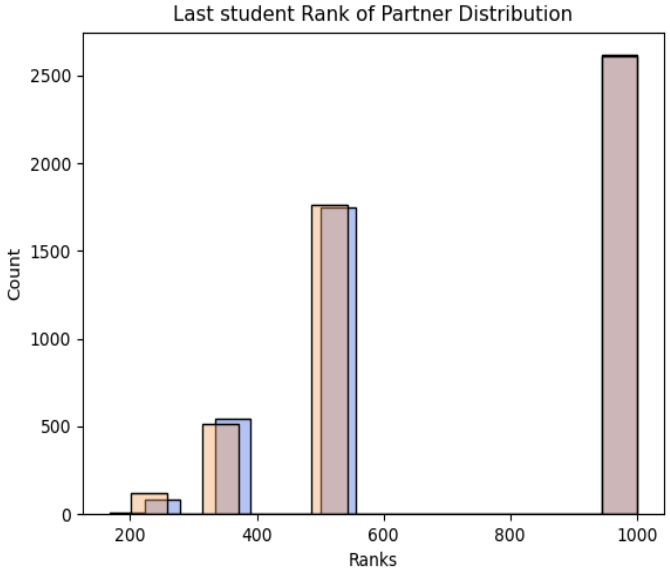
Industrial Engineering and Operations Research

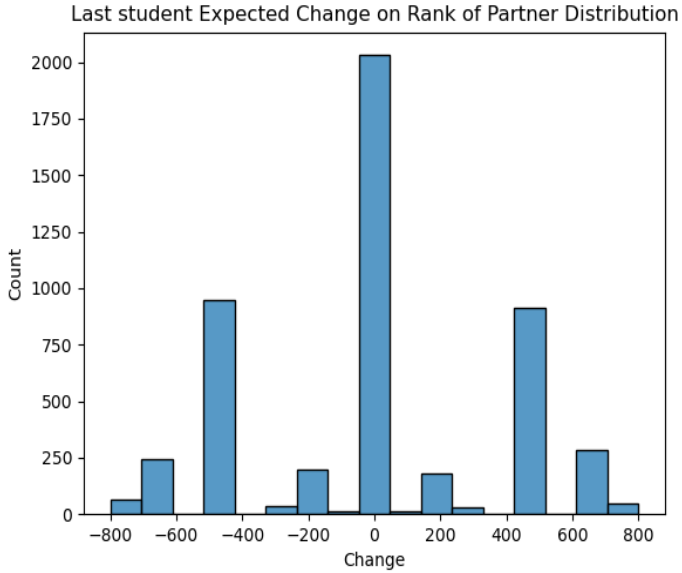
October 9, 2023

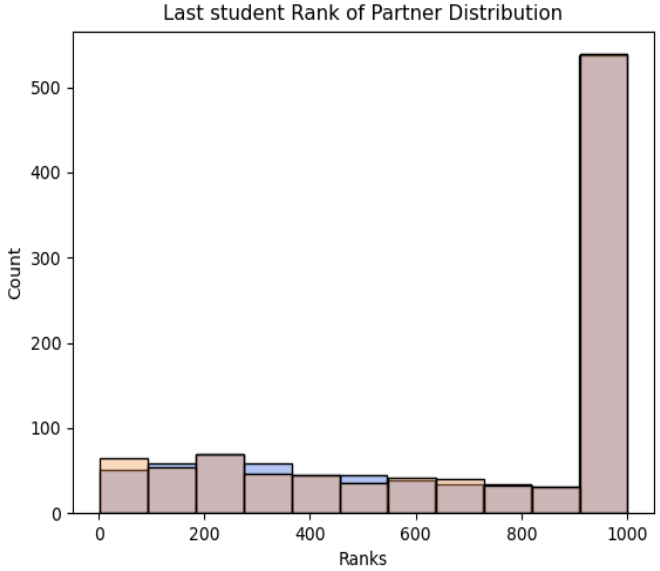

5
0
−5
−10
−15
−20
−25
0
200
400
600
800
1000

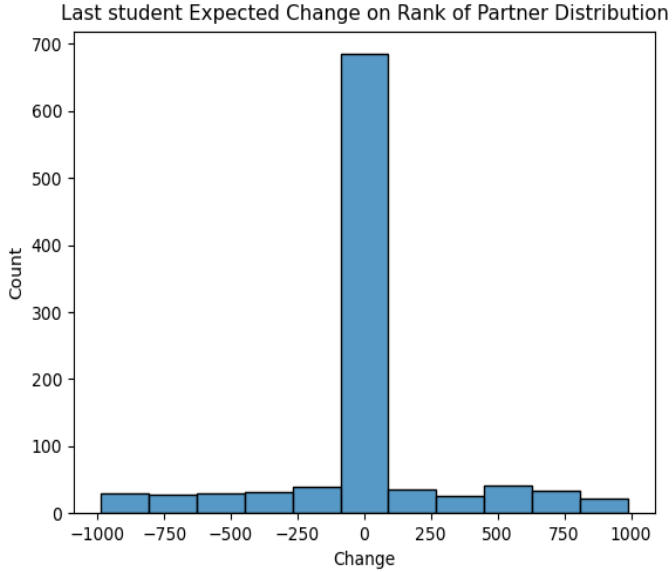

0
−5
−10
−15
−20
0
200
400
600
800
1000

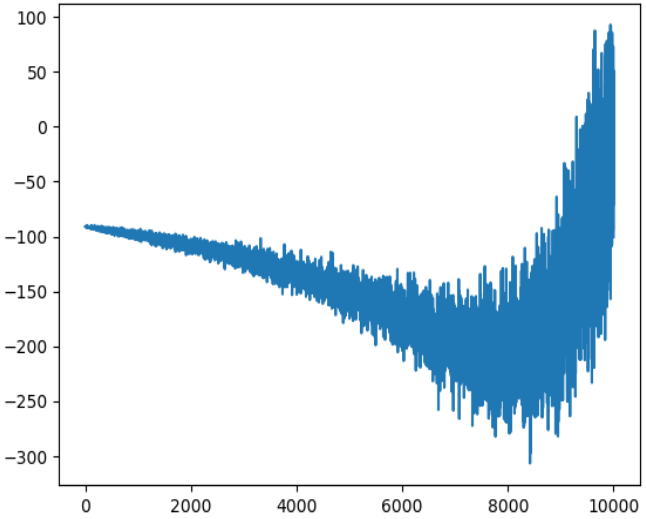

−2.5
−5.0
−7.5
−10.0
−12.5
−15.0
−17.5
−20.0
−22.5
0
200
400
600
800
1000

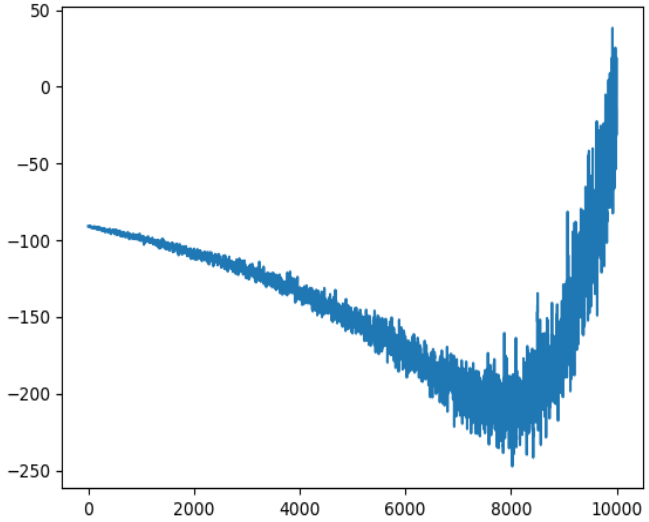

Last student Rank of Partner Distribution
Count
Ranks
0
500
1000
1500
2000
2500
200
400
600
800
1000

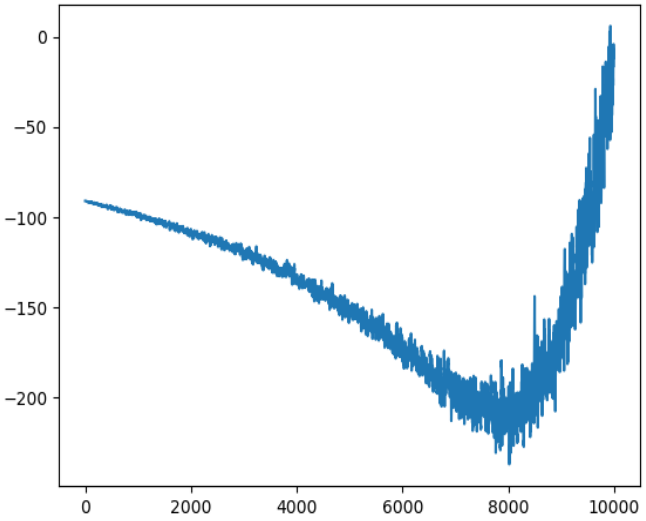

Last student Expected Change on Rank of Partner Distribution
Count
Change
0
250
500
750
1000
1250
1500
1750
2000
−800
−600
−400
−200
0
200
400
600
800

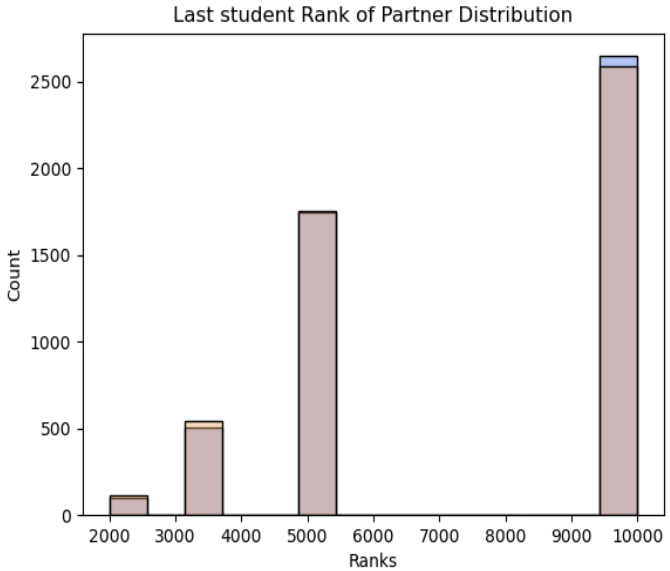

Last student Rank of Partner Distribution
Count
Ranks
0
100
200
300
400
500
0
200
400
600
800
1000

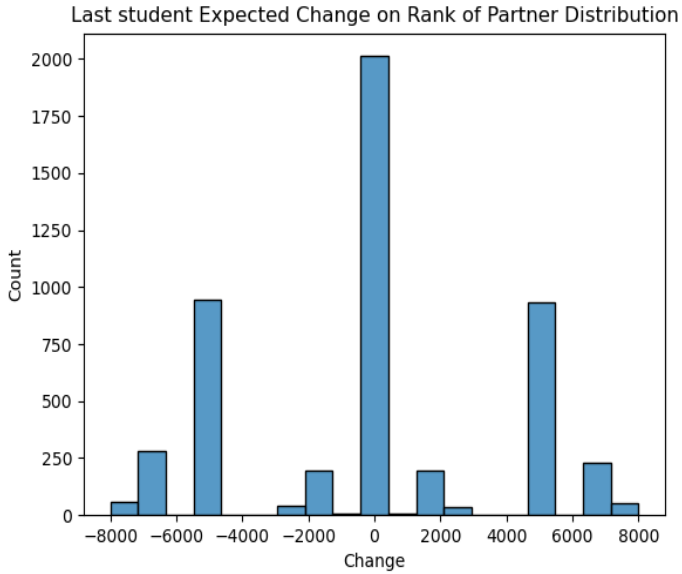

Last student Expected Change on Rank of Partner Distribution
Count
Change
0
100
200
300
400
500
600
700
−1000
−750
−500
−250
0
250
500
750
1000

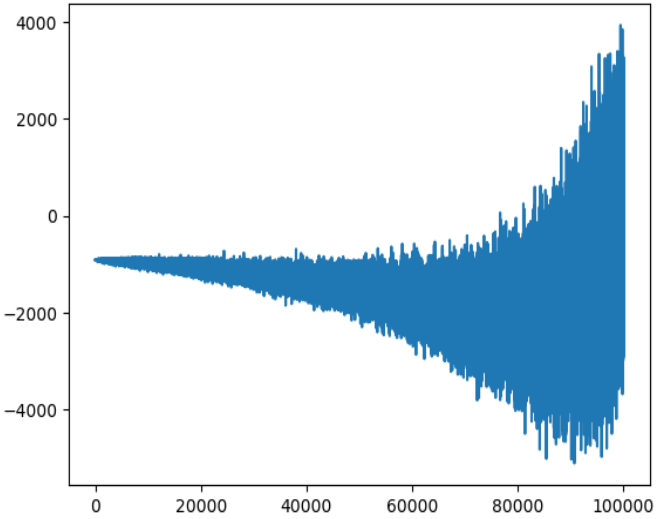

100
50
0
−50
−100
−150
−200
−250
−300
0
2000
4000
6000
8000
10000

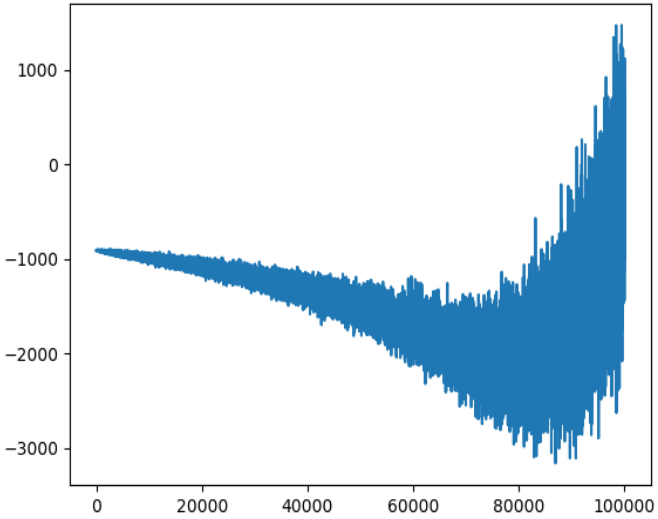

50
0
−50
−100
−150
−200
−250
0
2000
4000
6000
8000
10000

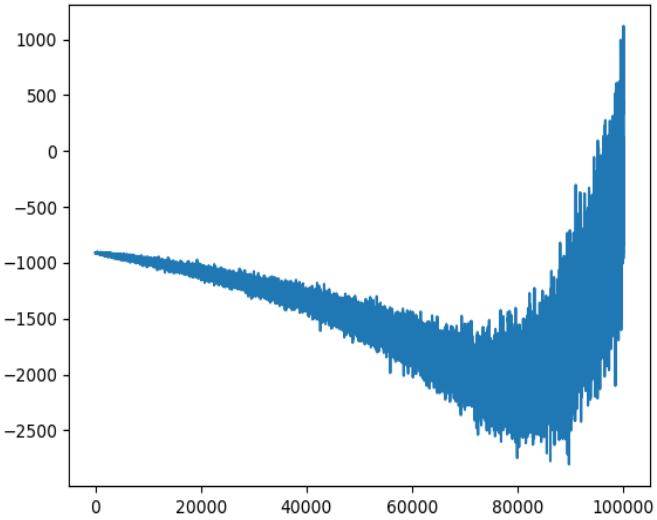

0
−50
−100
−150
−200
0
2000
4000
6000
8000
10000

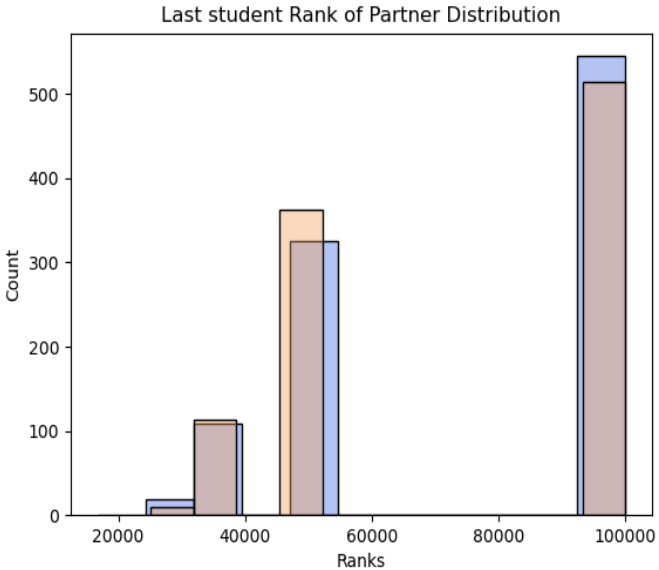

Last student Rank of Partner Distribution
Count
Ranks
0
500
1000
1500
2000
2500
2000
3000
4000
5000
6000
7000
8000
9000
10000

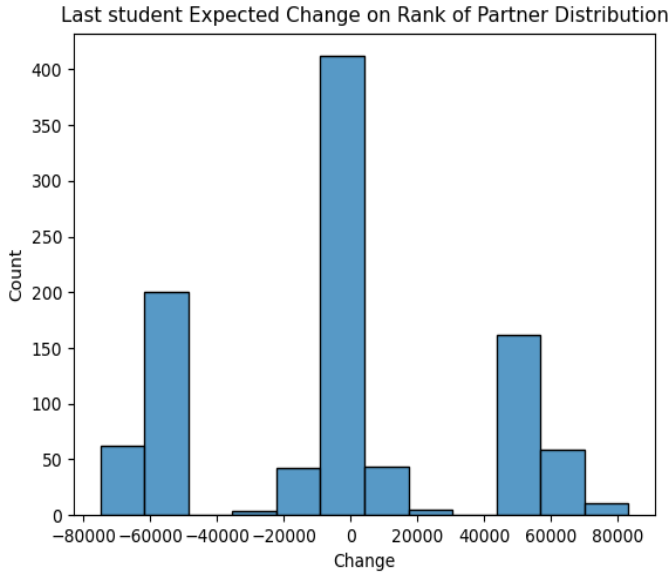

Last student Expected Change on Rank of Partner Distribution
Count
Change
0
250
500
750
1000
1250
1500
1750
2000
−8000
−6000
−4000
−2000
0
2000
4000
6000
8000

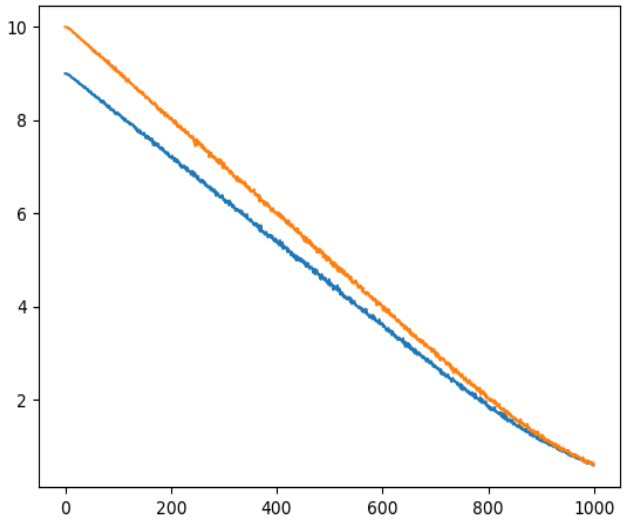

4000
2000
0
−2000
−4000
0
20000
40000
60000
80000
100000

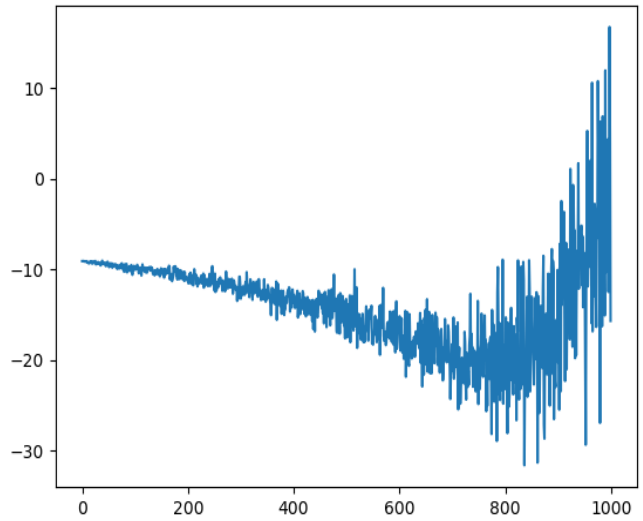

1000
0
−1000
−2000
−3000
0
20000
40000
60000
80000
100000

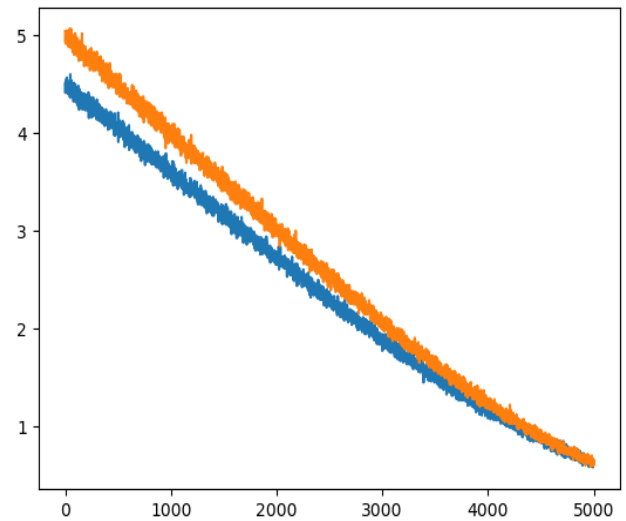

1000
500
0
−500
−1000
−1500
−2000
−2500
0
20000
40000
60000
80000
100000

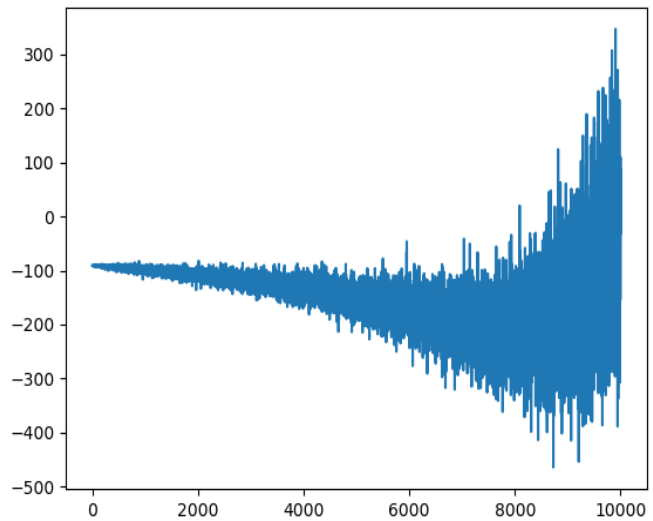

Last student Rank of Partner Distribution
Count
Ranks
0
100
200
300
400
500
20000
40000
60000
80000
100000

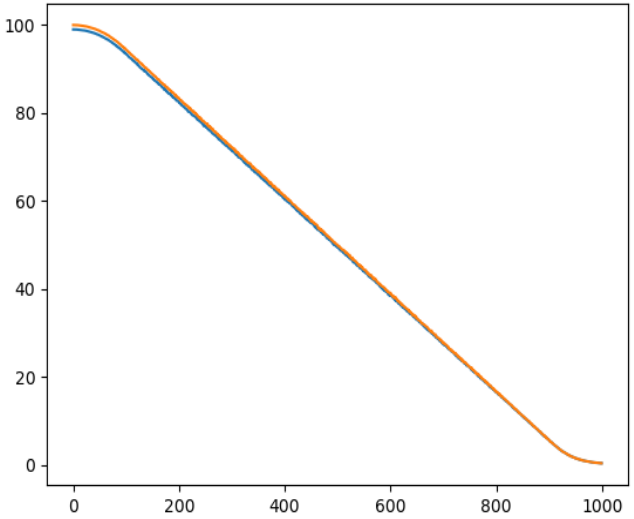

Last student Expected Change on Rank of Partner Distribution
Count
Change
0
50
100
150
200
250
300
350
400
−80000
−60000
−40000
−20000
0
20000
40000
60000
80000

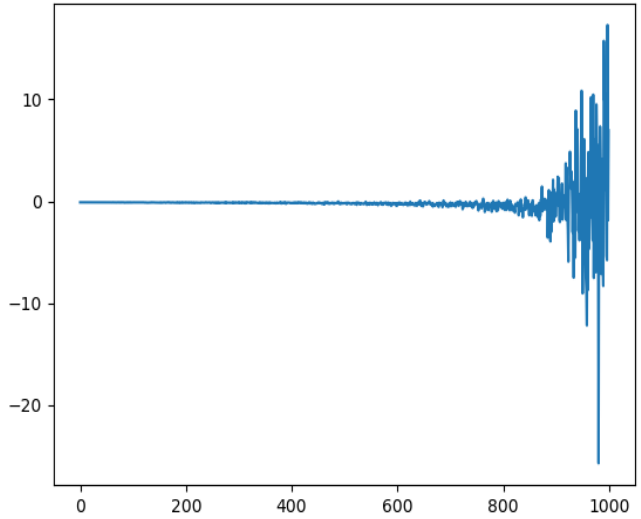

10
8
6
4
2
0
200
400
600
800
1000

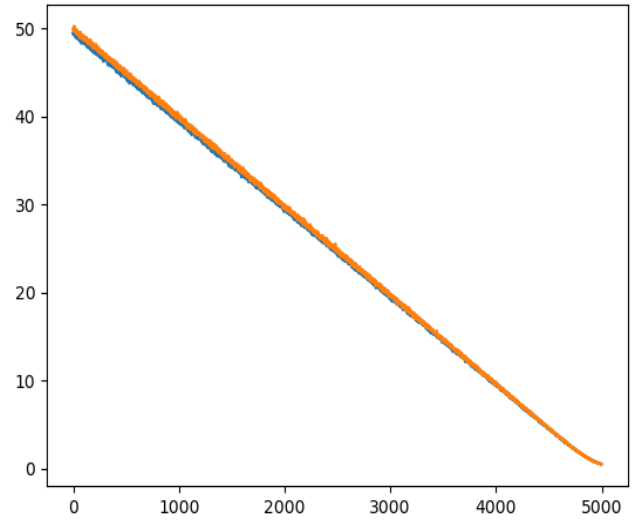

10
0
−10
−20
−30
0
200
400
600
800
1000

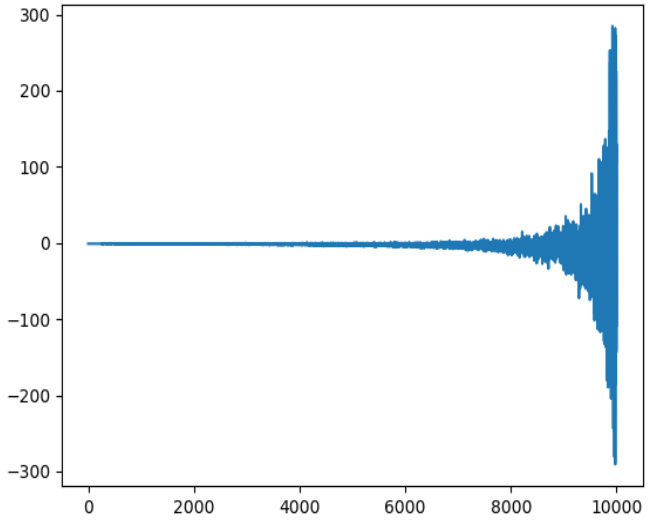

5
4
3
2
1
0
1000
2000
3000
4000
5000

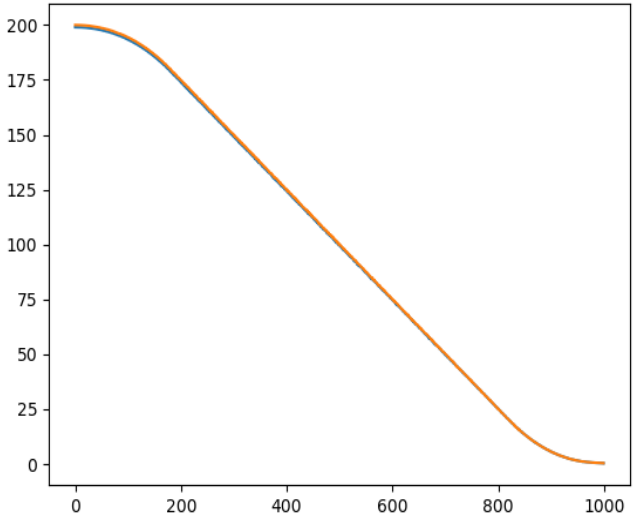

300
200
100
0
−100
−200
−300
−400
−500
0
2000
4000
6000
8000
10000

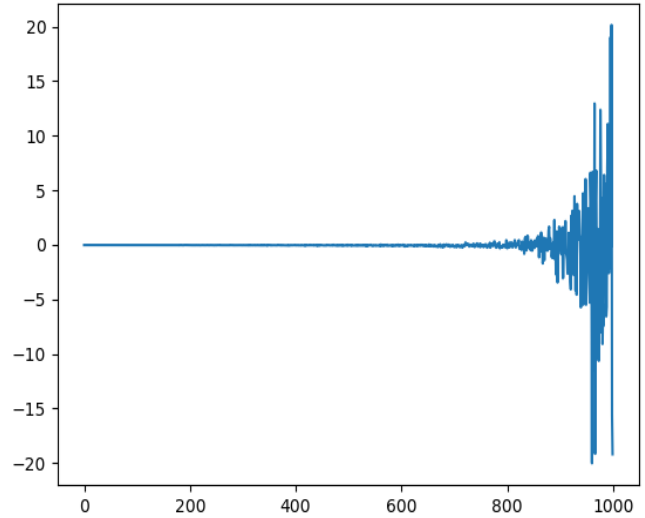

100
80
60
40
20
0
0
200
400
600
800
1000

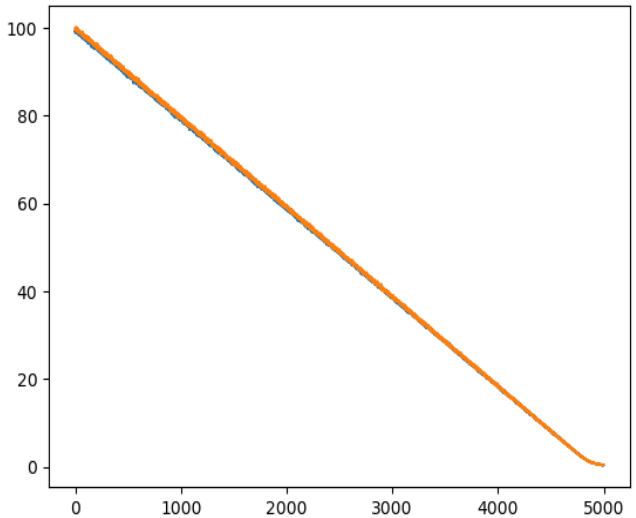

10
0
−10
−20
0
200
400
600
800
1000

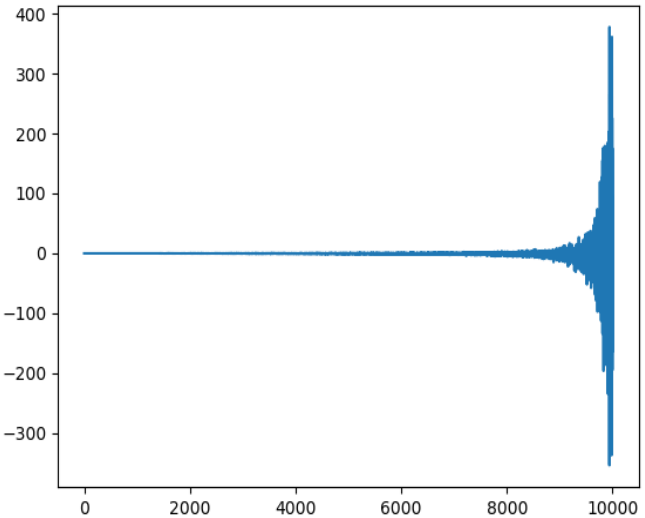

50
40
30
20
10
0
0
1000
2000
3000
4000
5000

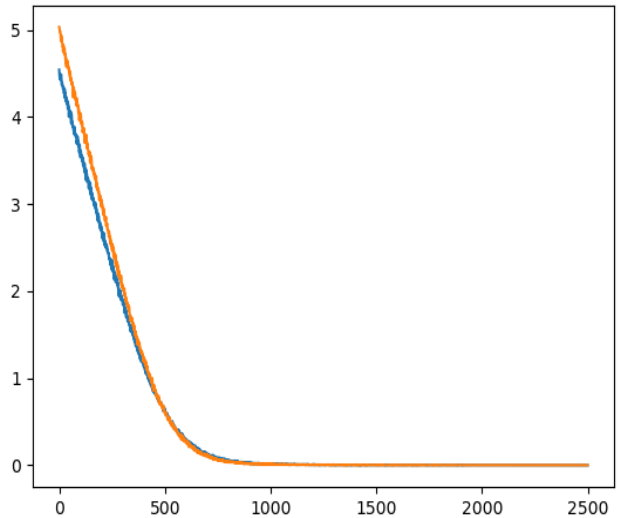

300
200
100
0
−100
−200
−300
0
2000
4000
6000
8000
10000

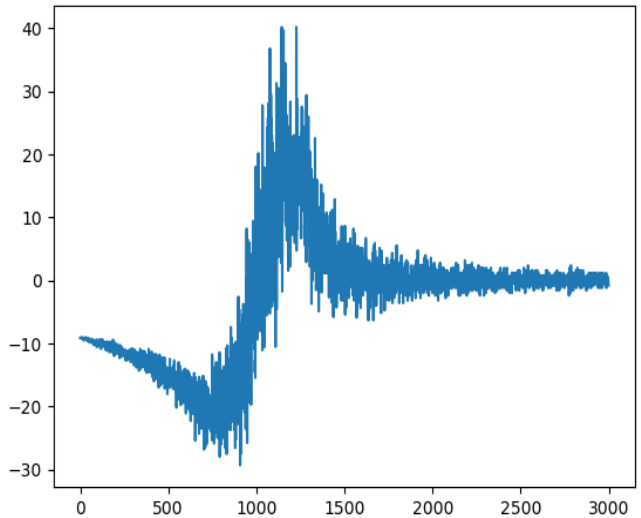

200
175
150
125
100
75
50
25
0
0
200
400
600
800
1000

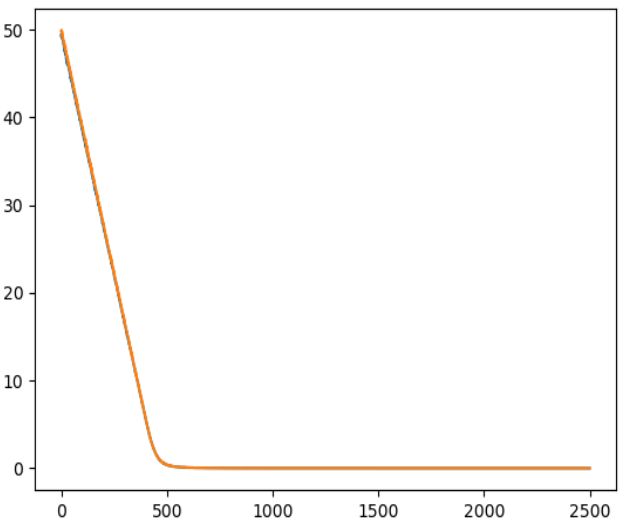

20
15
10
5
0
−5
−10
−15
−20
0
200
400
600
800
1000

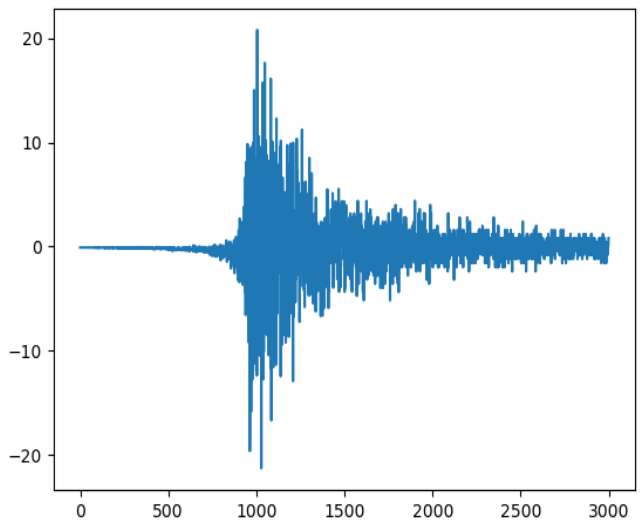

100
80
60
40
20
0
0
1000
2000
3000
4000
5000


400
300
200
100
0
−100
−200
−300
0
2000
4000
6000
8000
10000


5
4
3
2
1
0
0
500
1000
1500
2000
2500


40
30
20
10
0
−10
−20
−30
0
500
1000
1500
2000
2500
3000

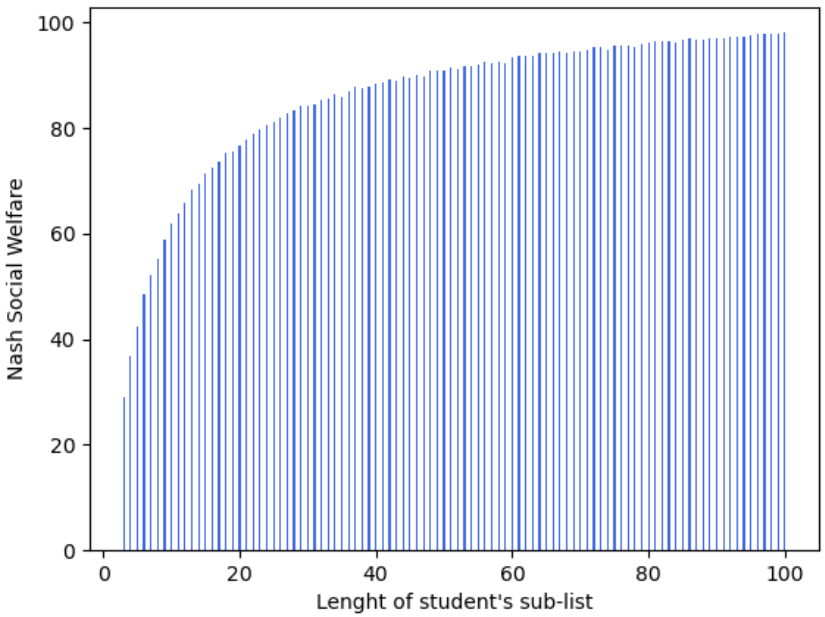

50
40
30
20
10
0
0
500
1000
1500
2000
2500


20
10
0
−10
−20
0
500
1000
1500
2000
2500
3000

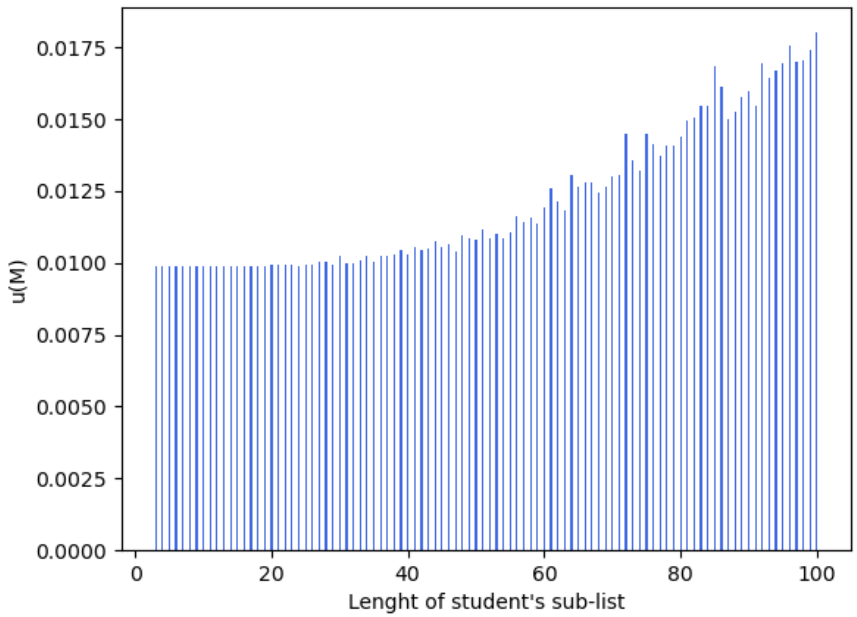
Outline

1. Utility Functions and Average Ranks
   - TTC Algorithm
   - GS Algorithm
2. Pareto Optimality and The GS Algorithm
   - 100 Simulations
   - 10 Simulations
   - 1 Simulation
   - 90 student 100 schools

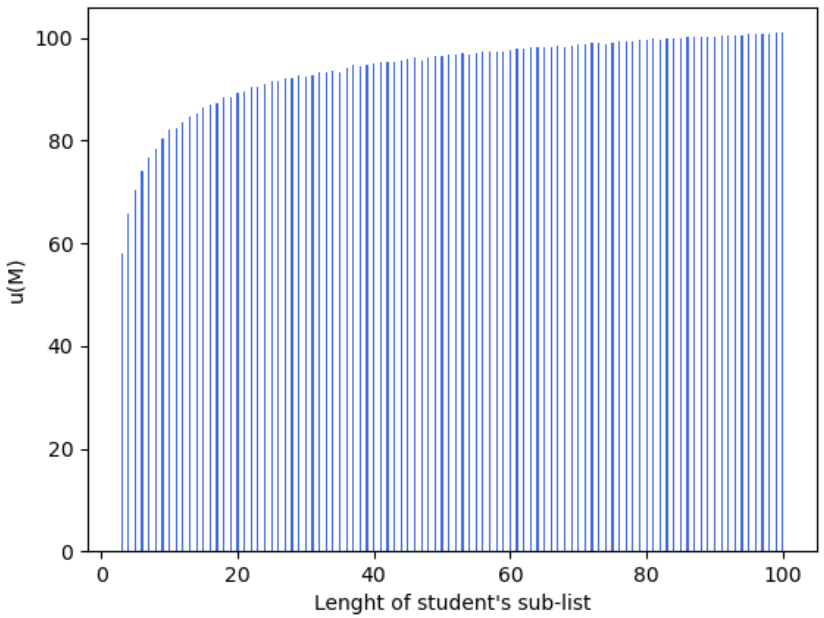
Sensible Utility Functions



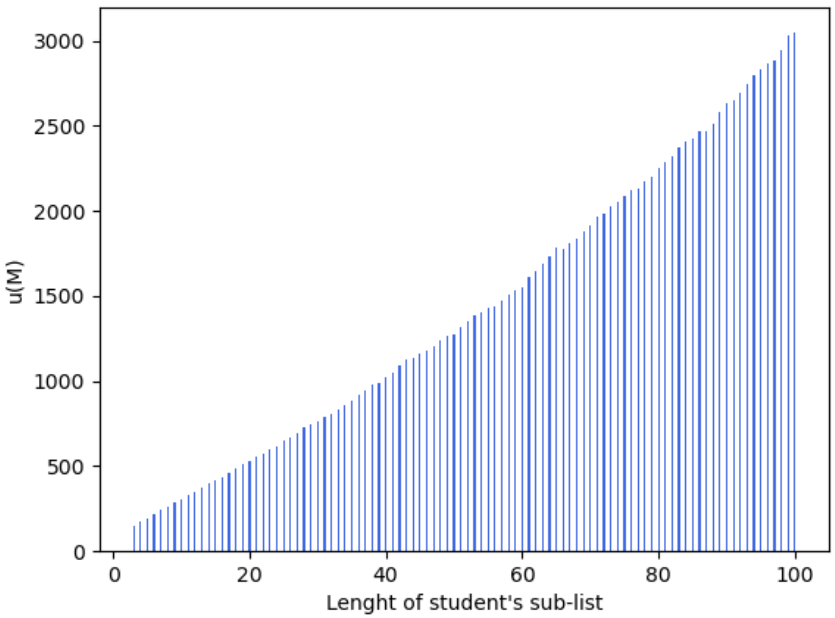
(a) $u(M) = (\prod_{i \in S} u(i, \mu(i)))^{\frac{1}{n}}$
$u(i, \mu(i))) = n + 2 - rank_i$



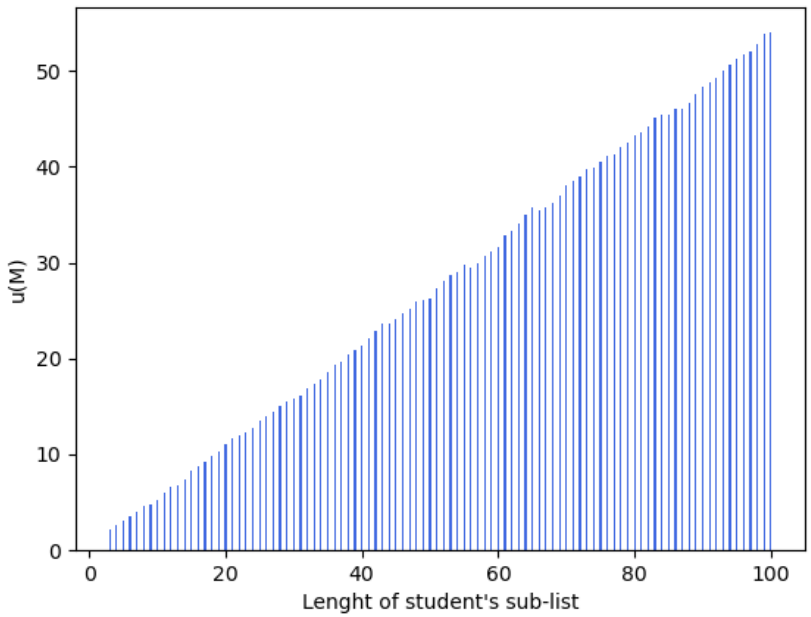
(b) $u(M) = min(u(i, \mu(i))))$
$u(i, \mu(i))) = 1/rank_i$



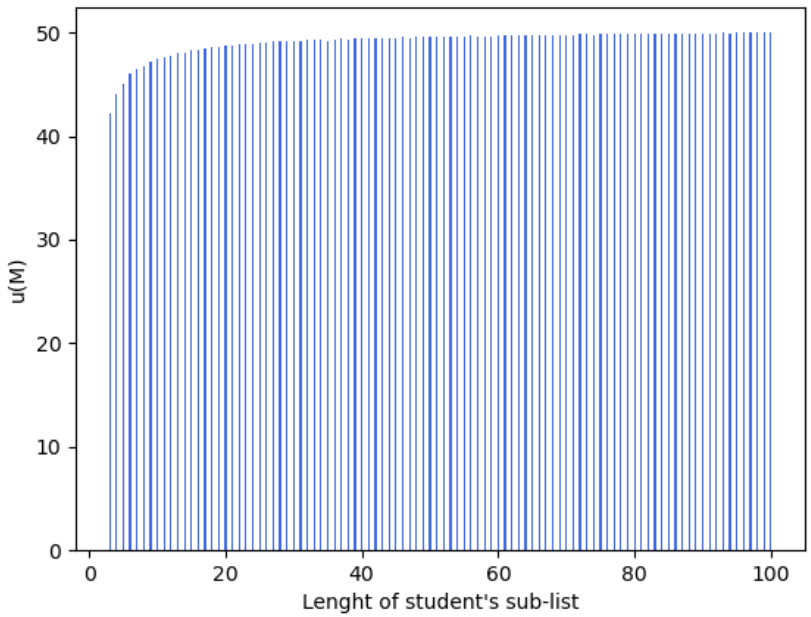
(a) $u(M) = \sum_{i \in ranks} x_i^{(n+1-rank_i)/n+1} - x_{n+1}$



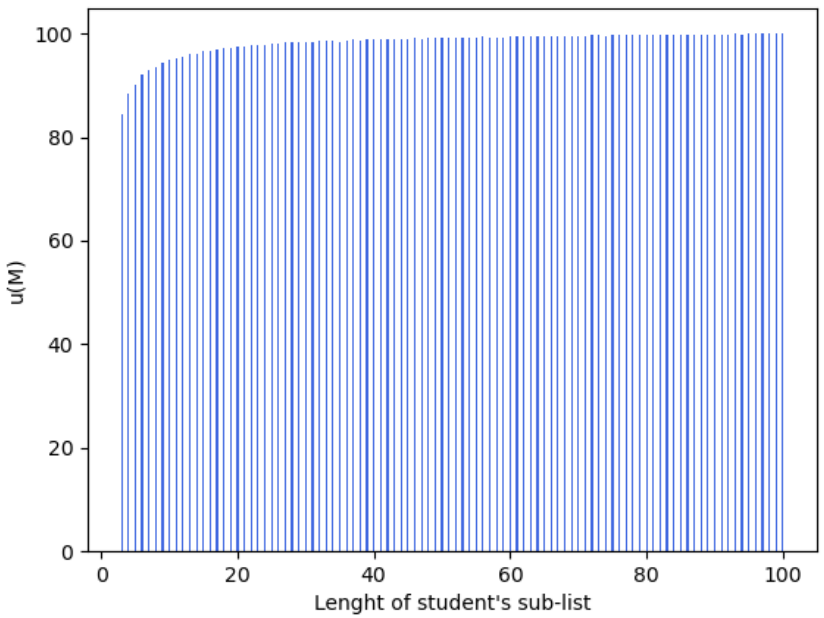
(b) $u(M) = \sum_{i \in ranks} x_i^2 - x_{n+1}$



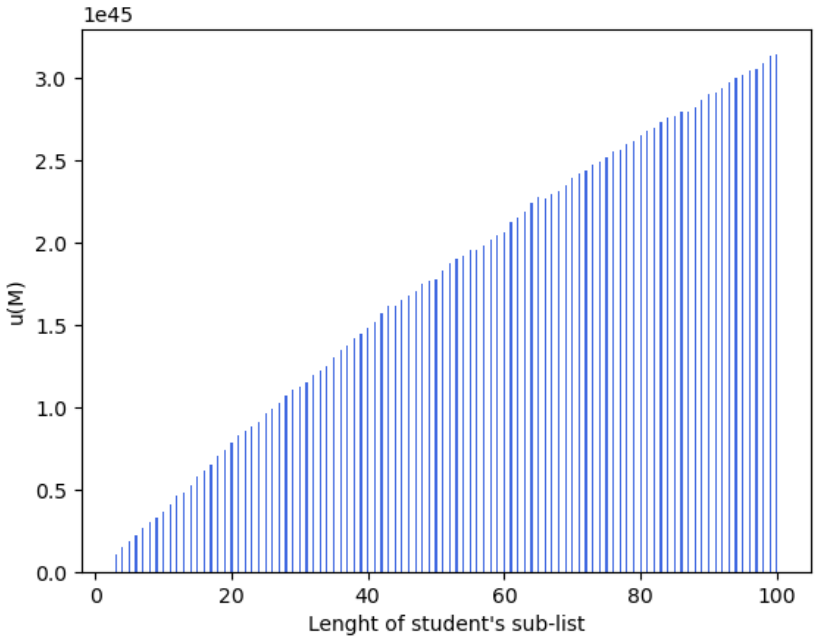
(a) $\prod_{i \in ranks} \frac{x_i^{(n+1-rank_i)/n+1}}{x_{i+1}^{(n+1-rank_{i+1})/n+1}}$



(b) $\prod_{i \in S} \frac{1}{4} - \frac{1}{2(n+1-rank_i)^2}$



(a) $\sum_{i \in students} 1 - e^{-(n+1-rank_i-(n+1)/2))}$



(b) $\sum_{i \in students_{>.5}} 1 - e^{-(n+1-rank_i-(n+1)/2)} - \sum_{i \in students_{<.5}} 2 * e^{(n+1-rank_i-(n+1)/2)}$

Average Rank of Match

For a Match μ between students $\mathcal{S}$ and schools $\mathcal{H}$, the average rank (corresponding to original full list) for students is:

$$R(\mu)=\frac{1}{n}\Big(|\bar{\mathcal{S}}(\mu)|(|H|+1)+\sum_{s\in\mathcal{S}\setminus\bar{\mathcal{S}}(\mu)} Rank_s(\mu(s))\Big)$$

where $\bar{s}(\mu)$ are the unmatched students. Average Rank for schools are computed with the equivalent function.

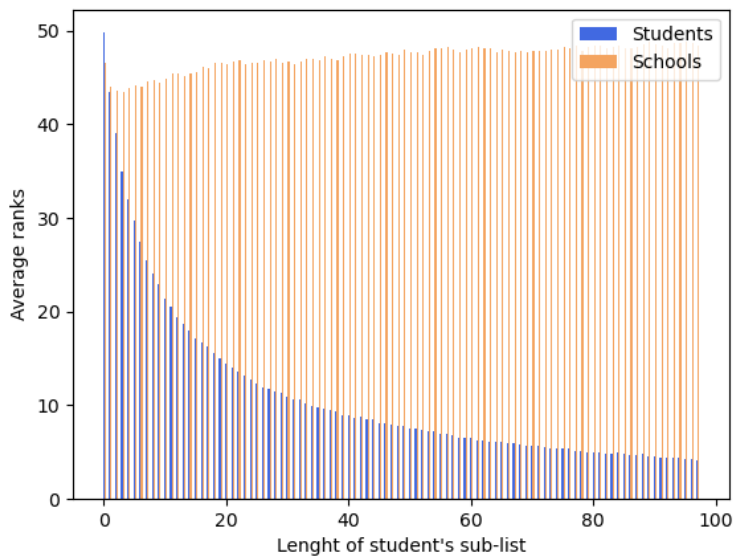
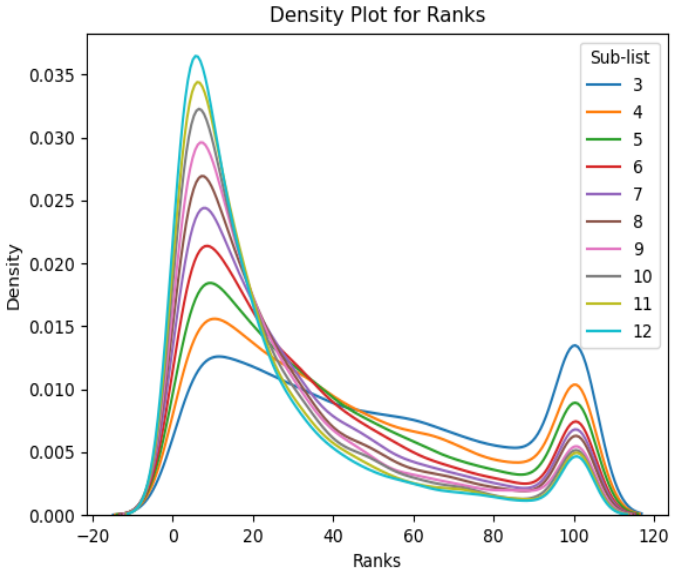
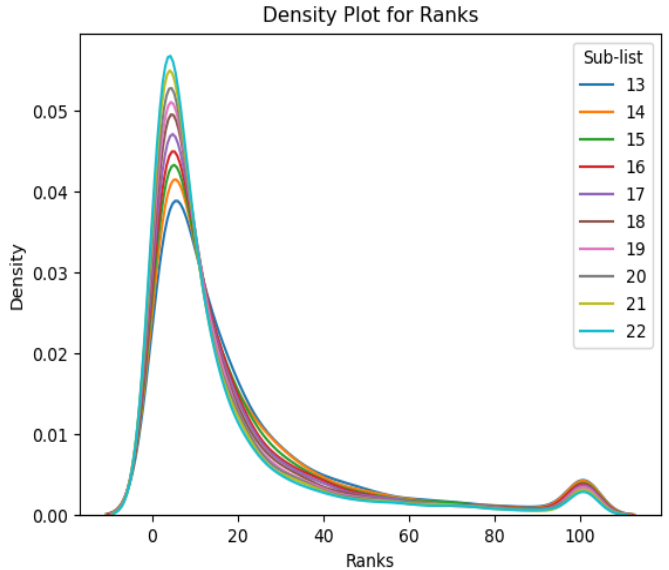
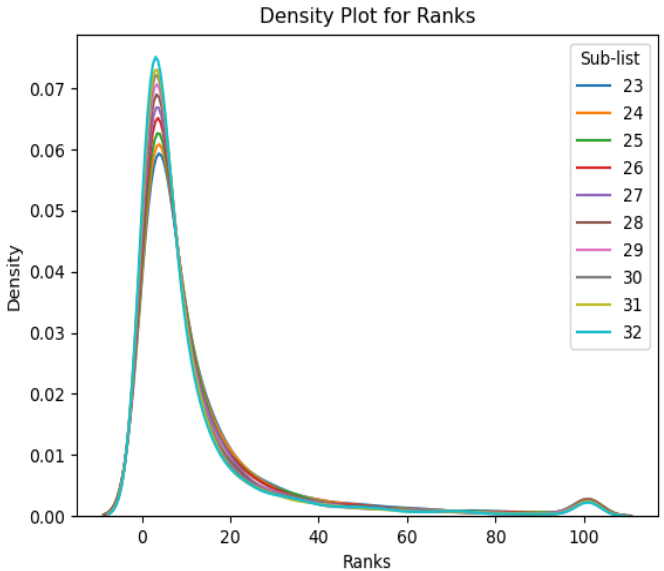
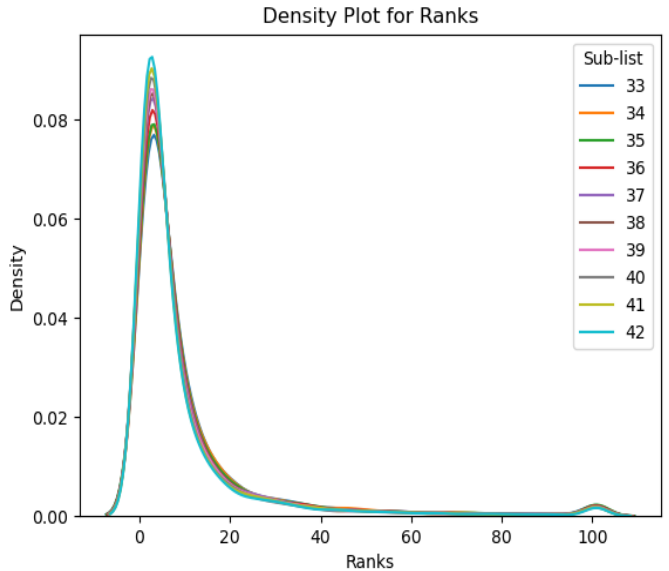


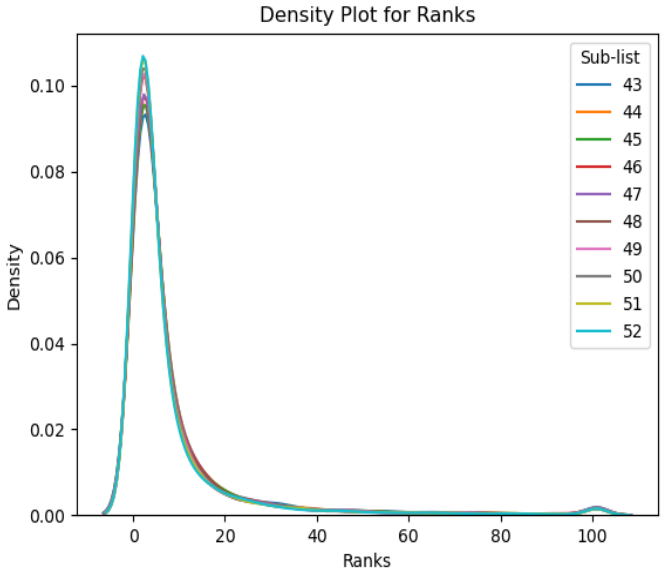
Figure: Average rank of match for the market

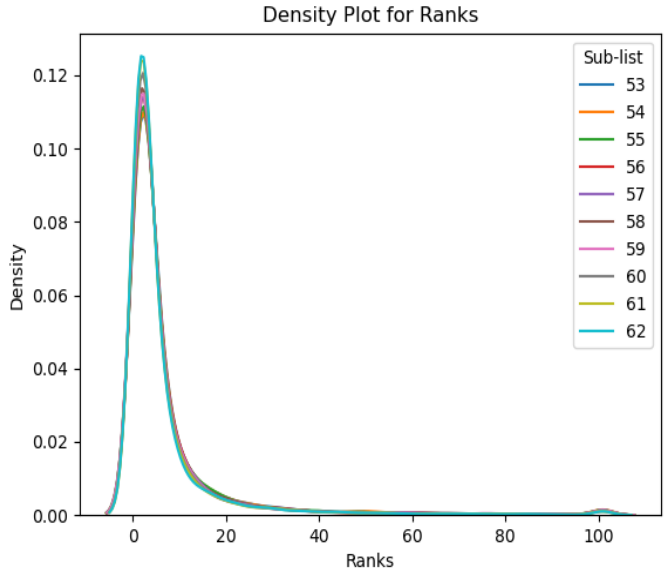

Density Plot for Ranks
Sub-list
3
4
5
6
7
8
9
10
11
12
Density
Ranks
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
−20
0
20
40
60
80
100
120

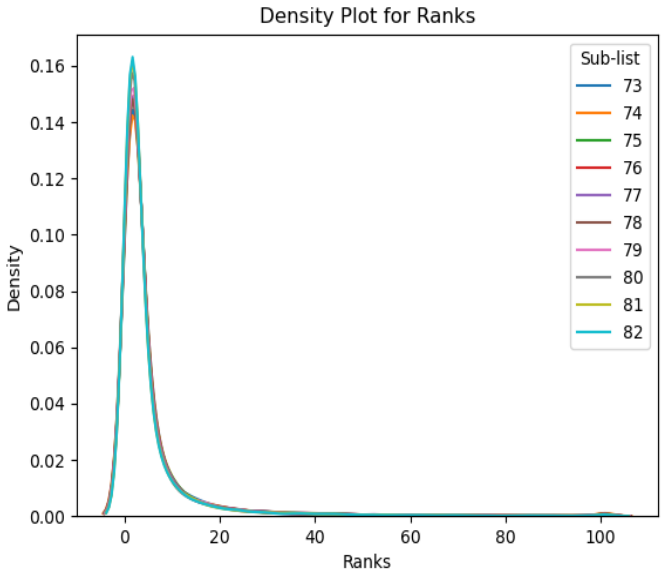

Density Plot for Ranks
Sub-list
13
14
15
16
17
18
19
20
21
22
Density
Ranks
0.05
0.04
0.03
0.02
0.01
0.00
0
20
40
60
80
100

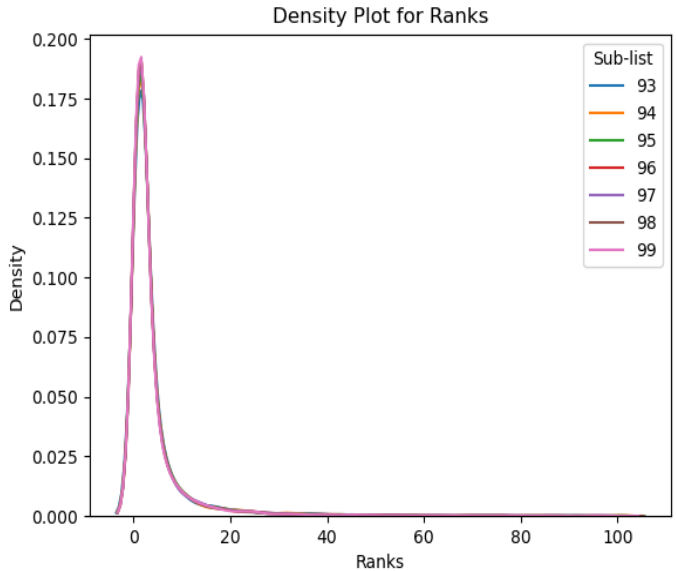

Density Plot for Ranks
Sub-list
23
24
25
26
27
28
29
30
31
32
Density
Ranks
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0
20
40
60
80
100


Density Plot for Ranks
Sub-list
33
34
35
36
37
38
39
40
41
42
Density
Ranks
0.08
0.06
0.04
0.02
0.00
0
20
40
60
80
100


Density Plot for Ranks
Sub-list
43
44
45
46
47
48
49
50
51
52
Density
Ranks
0.00
0.02
0.04
0.06
0.08
0.10
0
20
40
60
80
100


Density Plot for Ranks
Sub-list
53
54
55
56
57
58
59
60
61
62
Density
Ranks
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0
20
40
60
80
100

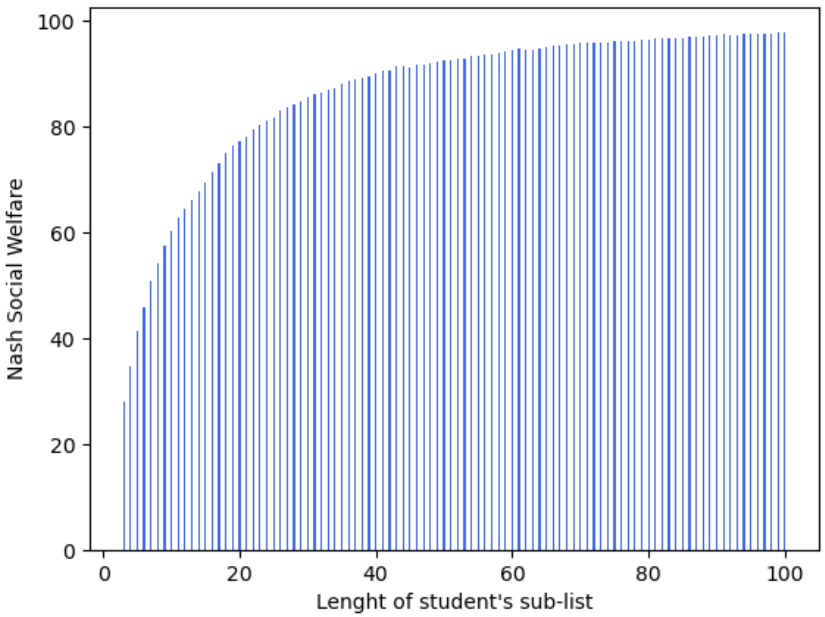

Density Plot for Ranks
Sub-list
73
74
75
76
77
78
79
80
81
82
Density
Ranks
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0
20
40
60
80
100


Density Plot for Ranks
Sub-list
93
94
95
96
97
98
99
Density
Ranks
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
0
20
40
60
80
100

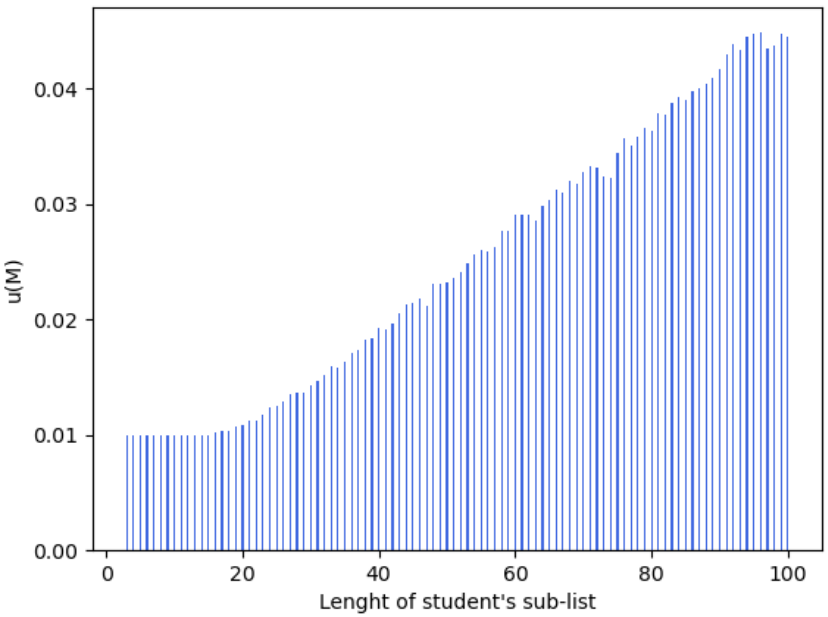
Outline

1. Utility Functions and Average Ranks
   - TTC Algorithm
   - GS Algorithm
2. Pareto Optimality and The GS Algorithm
   - 100 Simulations
   - 10 Simulations
   - 1 Simulation
   - 90 student 100 schools

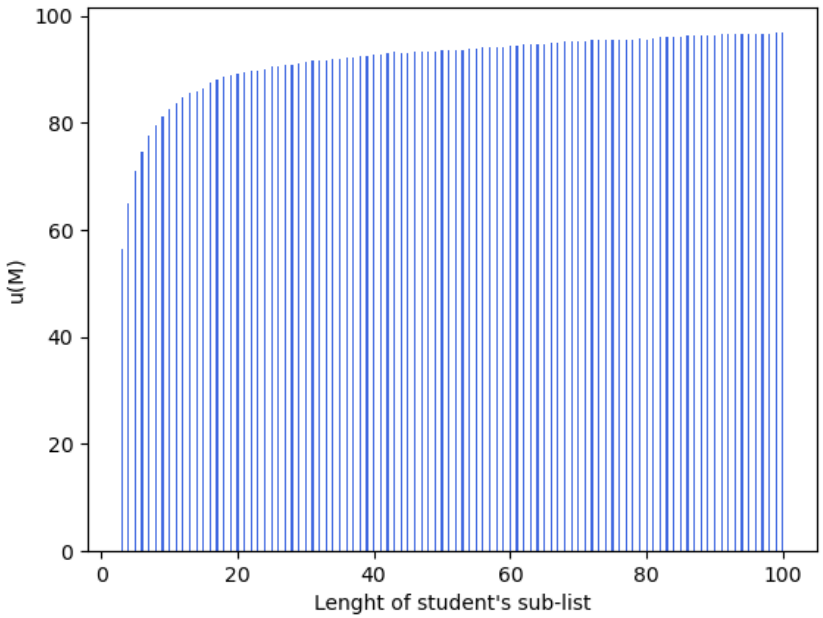
Sensible Utility Functions



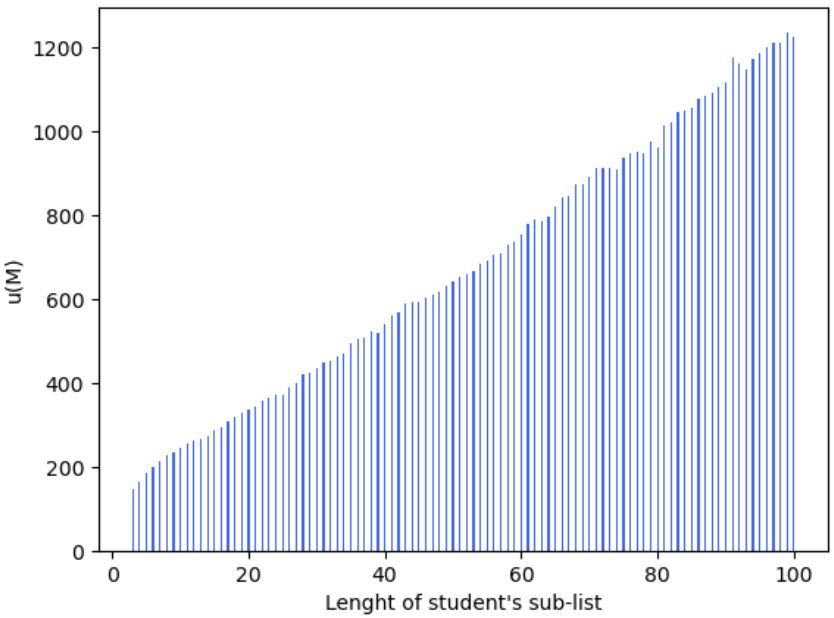
(a) $u(M) = (\prod_{i \in S} u(i, \mu(i)))^{\frac{1}{n}}$
$u(i, \mu(i))) = n + 2 - rank_i$



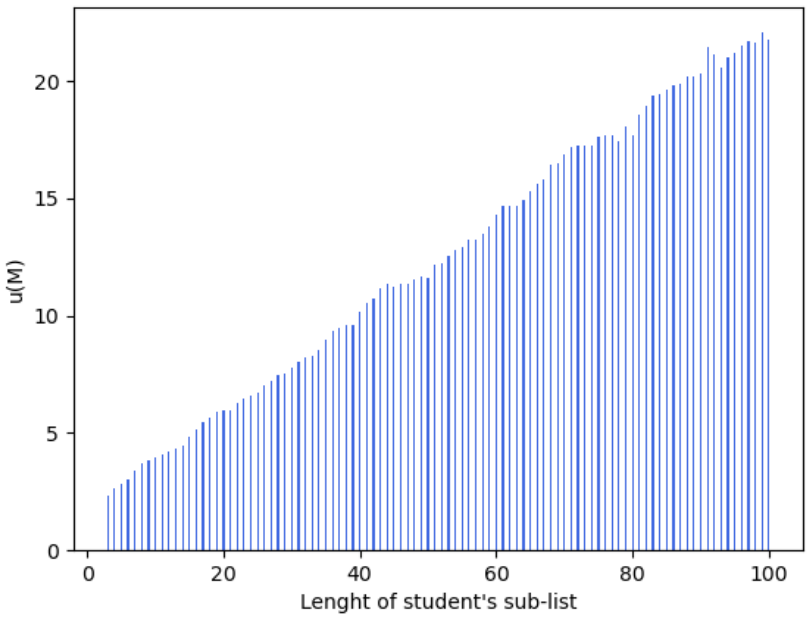
(b) $u(M) = min(u(i, \mu(i))))$
$u(i, \mu(i))) = 1/rank_i$



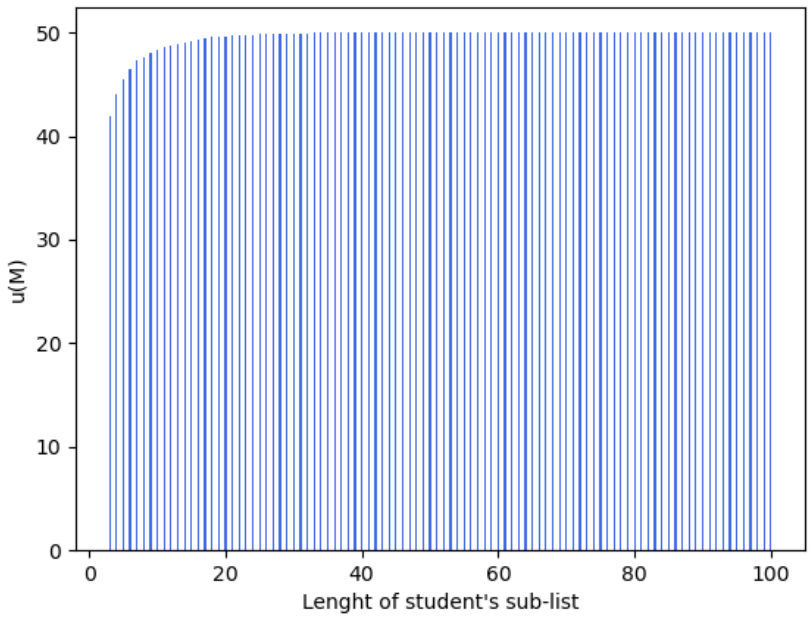
(a) $u(M) = \sum_{i \in ranks} x_i^{(n+1-rank_i)/n+1} - x_{n+1}$



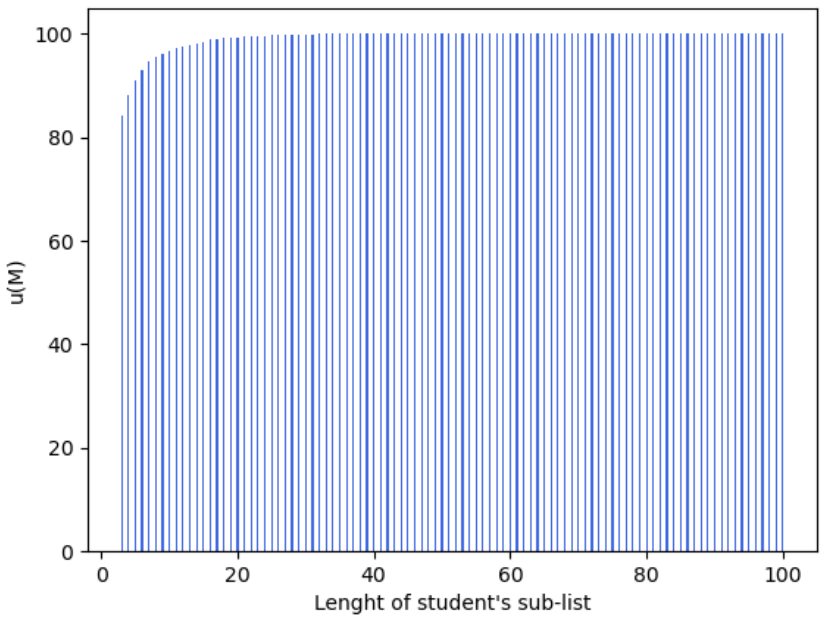
(b) $u(M) = \sum_{i \in ranks} x_i^2 - x_{n+1}$



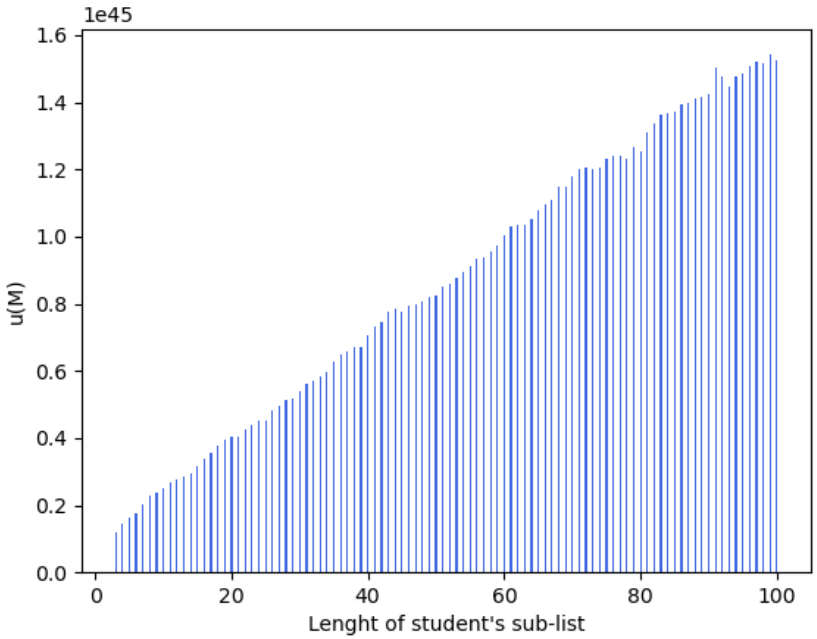
(a) $\prod_{i \in ranks} \frac{x_i^{(n+1-rank_i)/n+1}}{x_{i+1}^{(n+1-rank_{i+1})/n+1}}$



(b) $\prod_{i \in S} \frac{1}{4} - \frac{1}{2(n+1-rank_i)^2}$



(a) $\sum_{i \in students} 1 - e^{-(n+1-rank_i-(n+1)/2))}$



(b) $\sum_{i \in students_{>.5}} 1 - e^{-(n+1-rank_i-(n+1)/2)} - \sum_{i \in students_{<.5}} 2 * e^{(n+1-rank_i-(n+1)/2)}$

Average Rank of Match

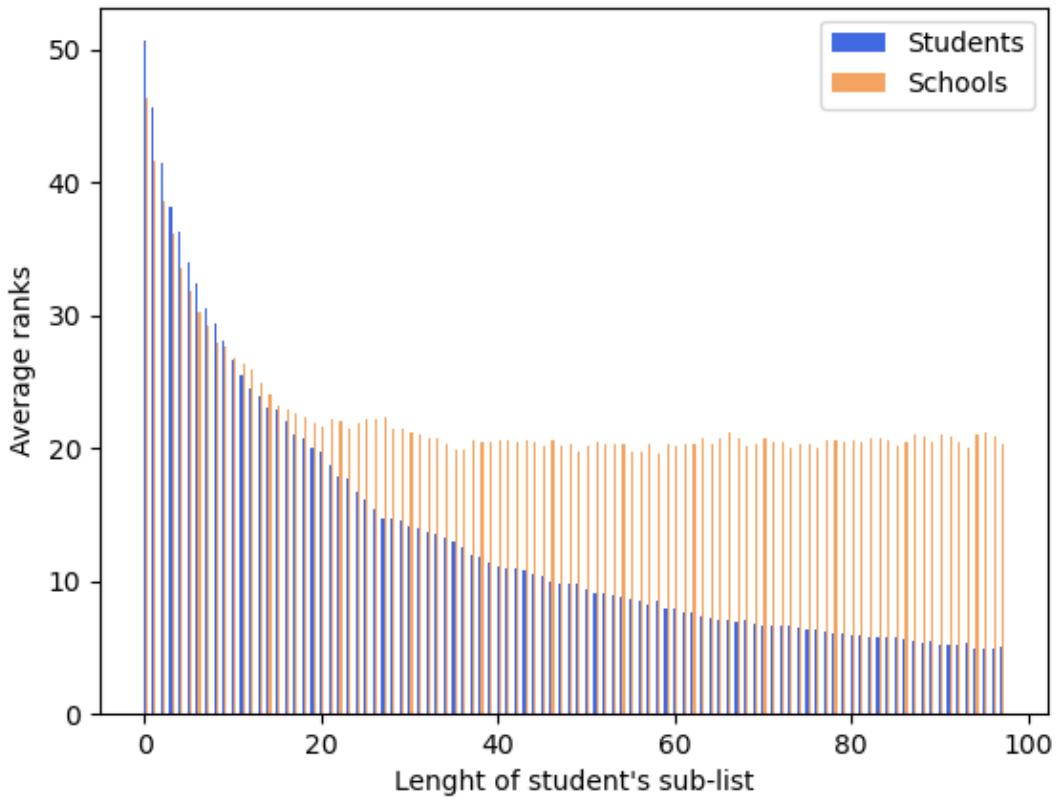
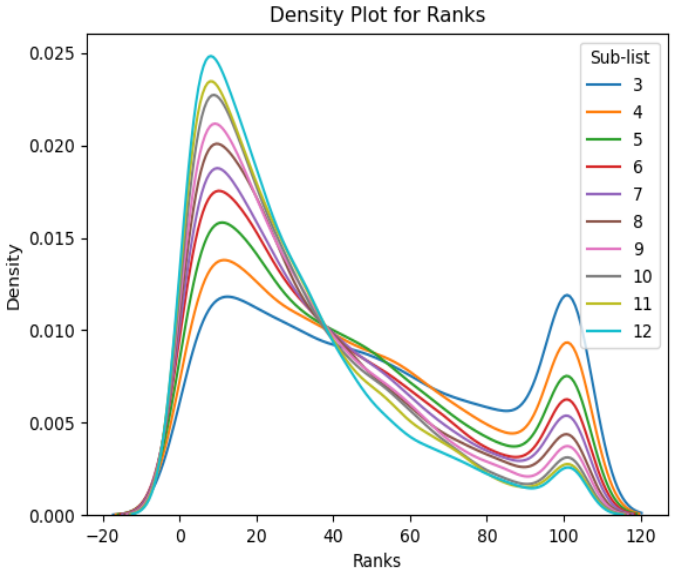
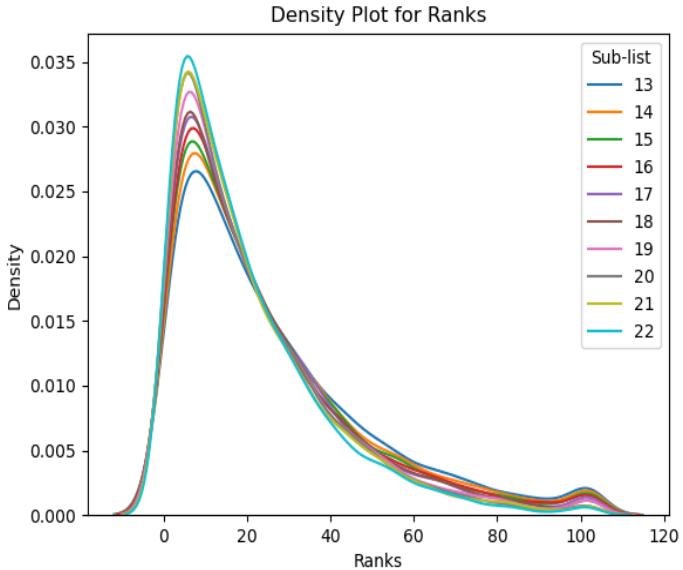
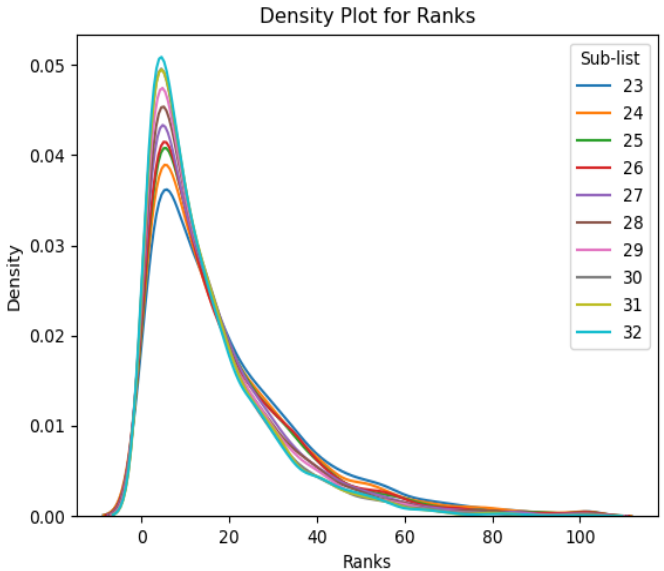
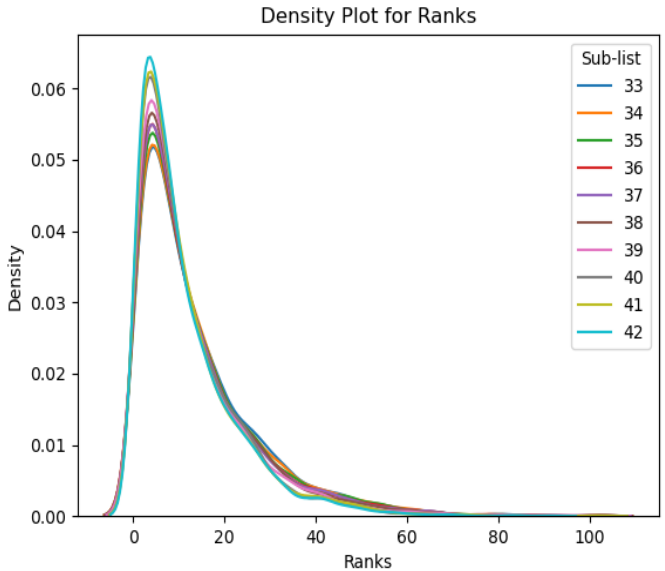


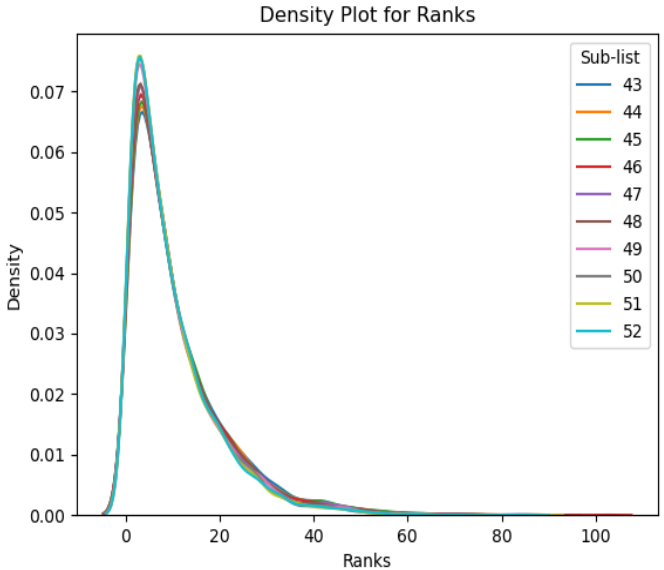
Figure: Average rank of match for the market

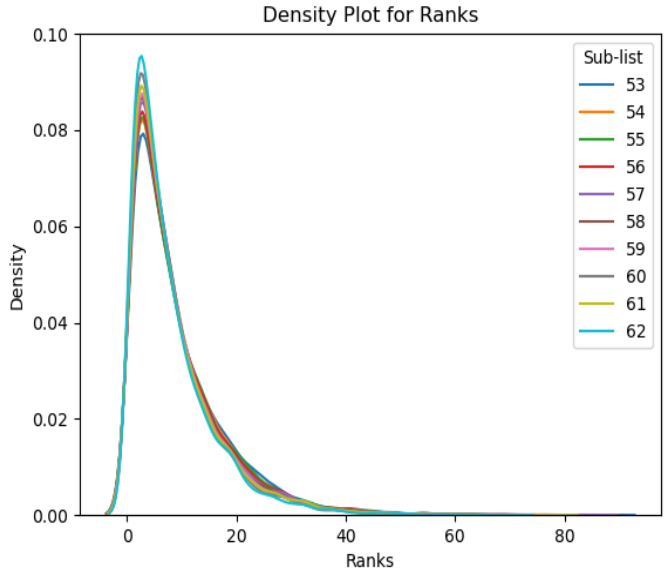

Density Plot for Ranks
Density
Ranks
Sub-list
3
4
5
6
7
8
9
10
11
12

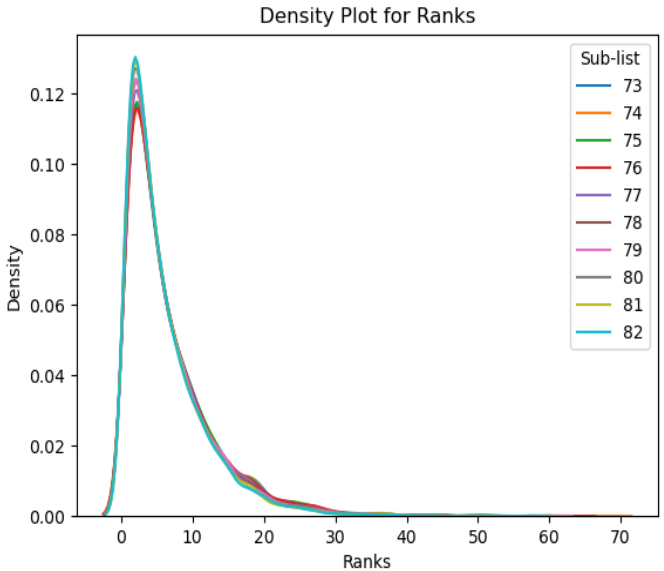

Density Plot for Ranks
Density
Ranks
Sub-list
13
14
15
16
17
18
19
20
21
22

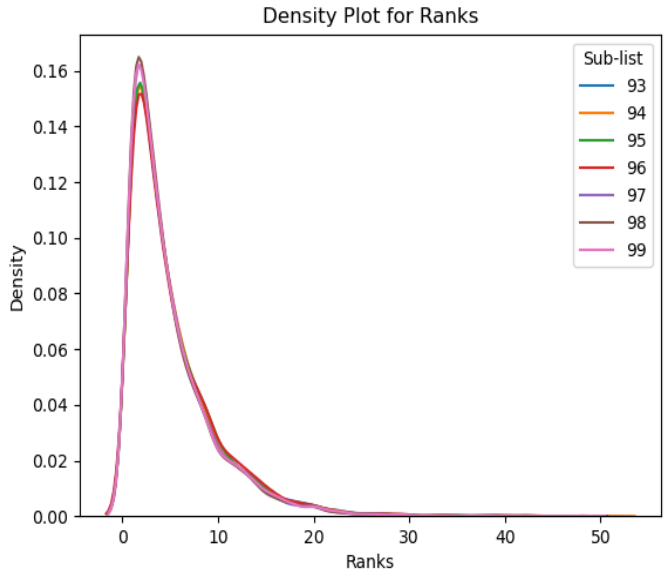

Density Plot for Ranks
Density
Ranks
Sub-list
23
24
25
26
27
28
29
30
31
32


Density Plot for Ranks
Density
Ranks
Sub-list
33
34
35
36
37
38
39
40
41
42


Density Plot for Ranks
Density
Ranks
Sub-list
43
44
45
46
47
48
49
50
51
52


Density Plot for Ranks
Density
Ranks
Sub-list
53
54
55
56
57
58
59
60
61
62

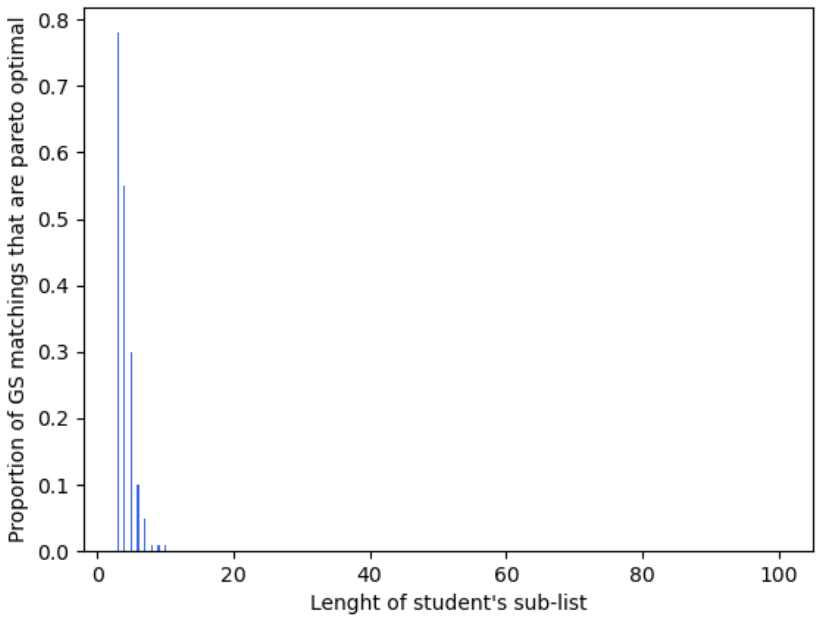

Density Plot for Ranks
Density
Ranks
Sub-list
73
74
75
76
77
78
79
80
81
82


Density Plot for Ranks
Density
Ranks
Sub-list
93
94
95
96
97
98
99

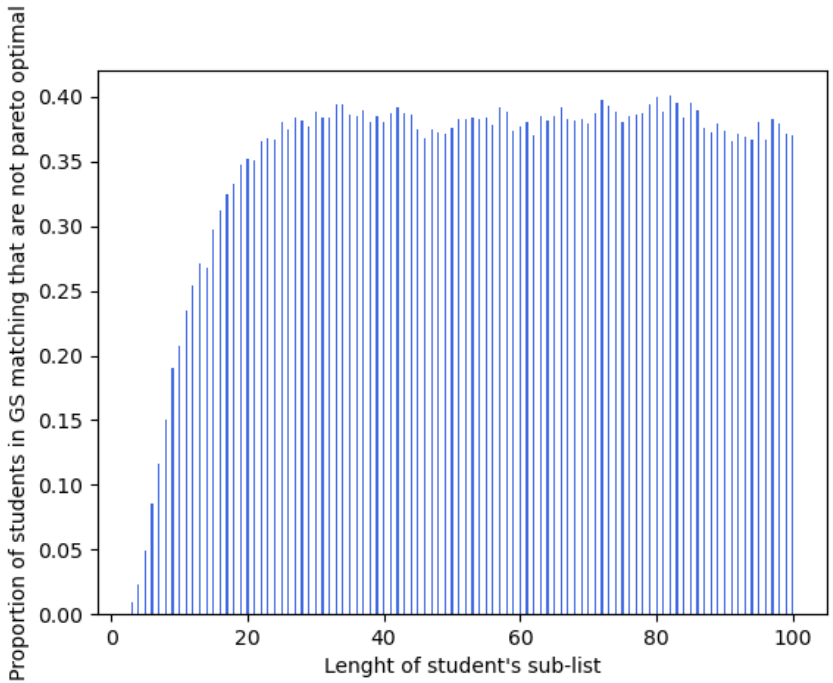
Outline

1. Utility Functions and Average Ranks
   - TTC Algorithm
   - GS Algorithm
2. Pareto Optimality and The GS Algorithm
   - 100 Simulations
   - 10 Simulations
   - 1 Simulation
   - 90 student 100 schools

The GS output and Pareto Optimality

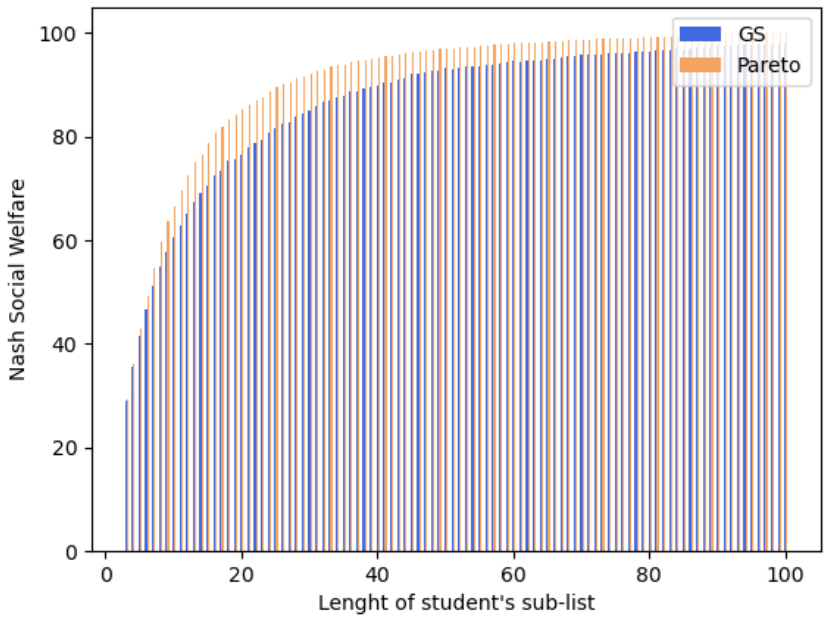


(a) Proportion of experiments with a Pareto Optimal matching



(b) Proportion of students that weren't Pareto Optimal

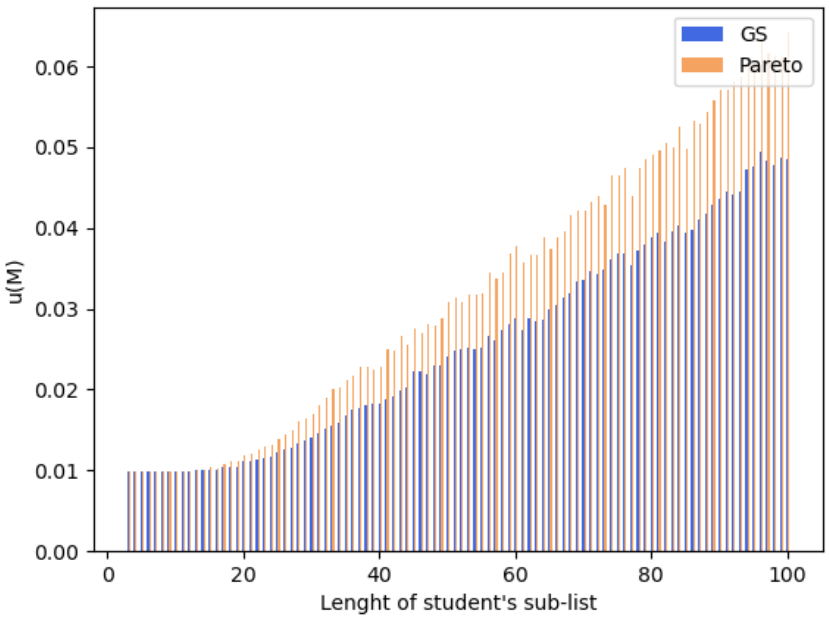
Welfare in GS Output vs Pareto Optimality



(a) Nash Welfare

$u(M) = (\prod_{i \in S} u(i, \mu(i)))^{\frac{1}{n}}$

$u(i, \mu(i))) = n + 2 - rank_i$



(b) Leontief Utilities

$u(M) = min(u(i, \mu(i))))$

$u(i, \mu(i))) = 1/rank_i$

Average Rank of Match

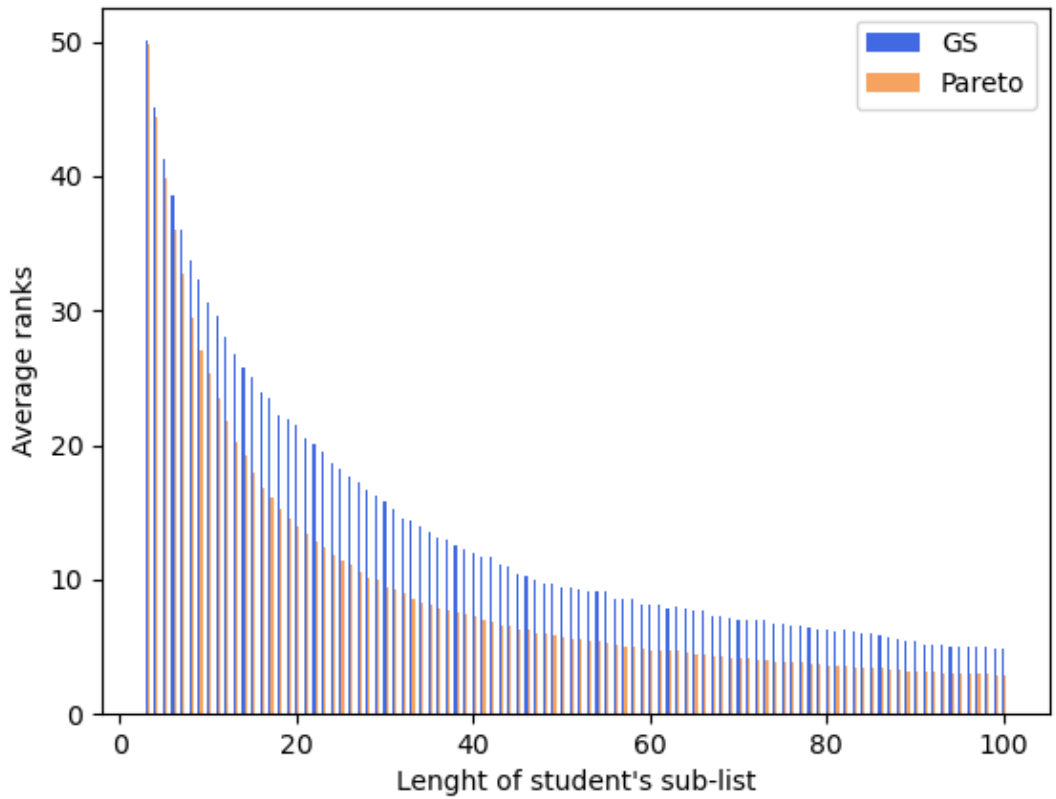
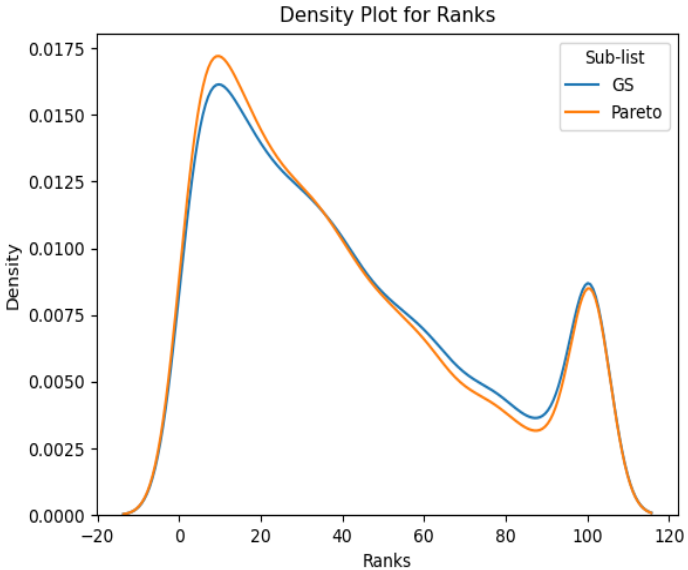
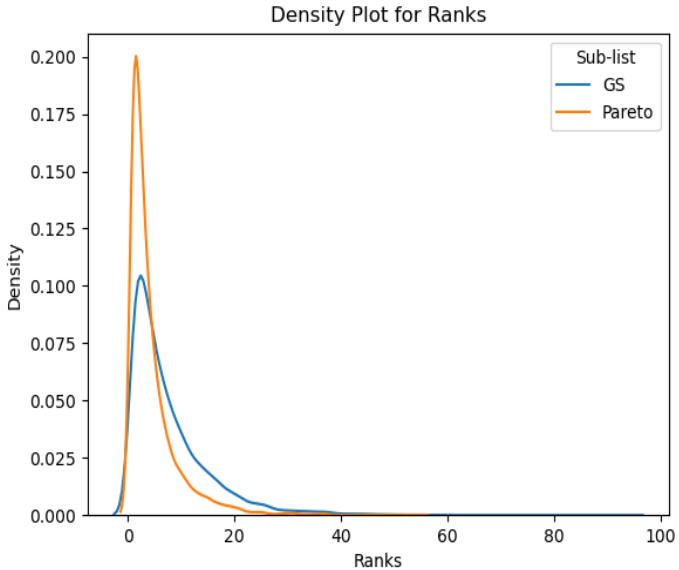
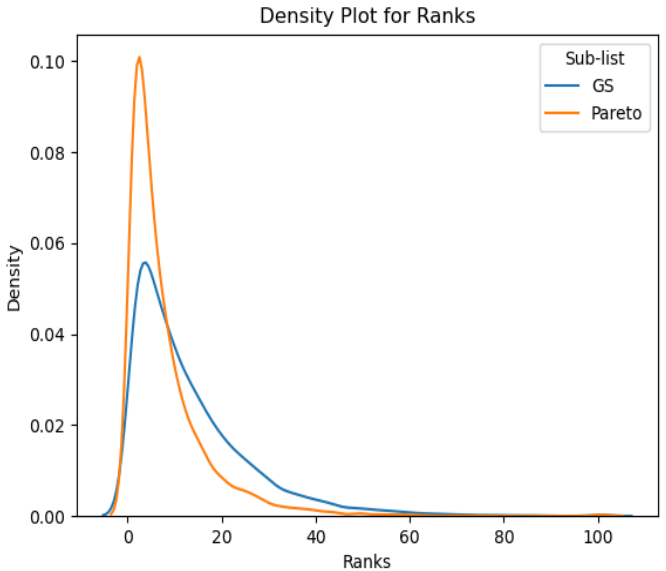
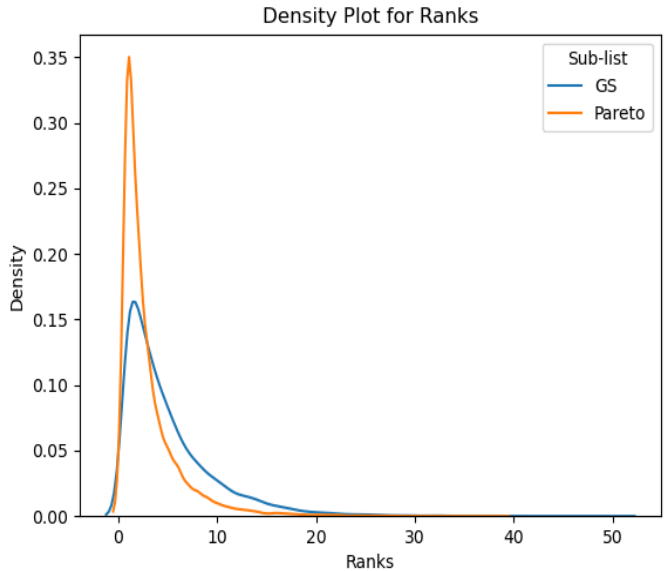


Figure: Average rank of match


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.0175
0.0150
0.0125
0.0100
0.0075
0.0050
0.0025
0.0000
−20
0
20
40
60
80
100
120


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.200
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
0
20
40
60
80
100

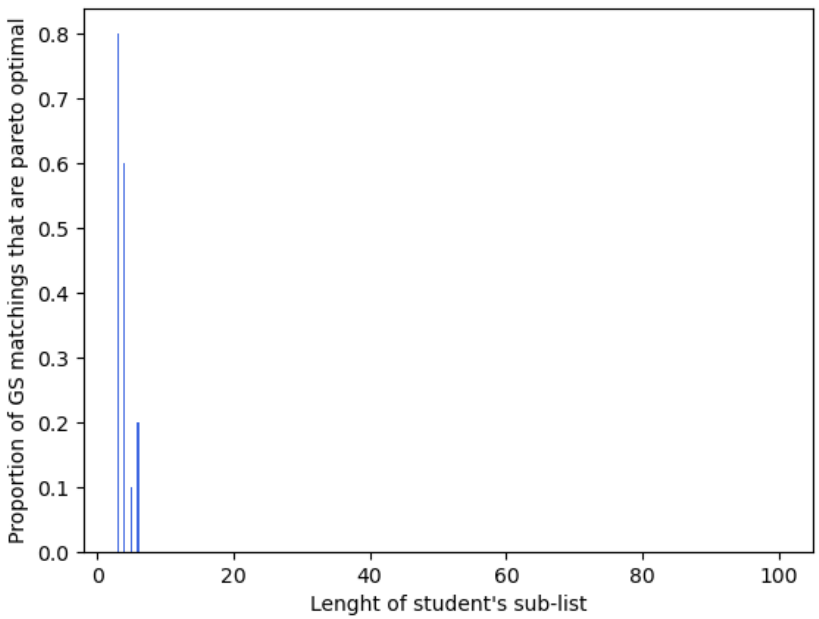

Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.10
0.08
0.06
0.04
0.02
0.00
0
20
40
60
80
100


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
10
20
30
40
50

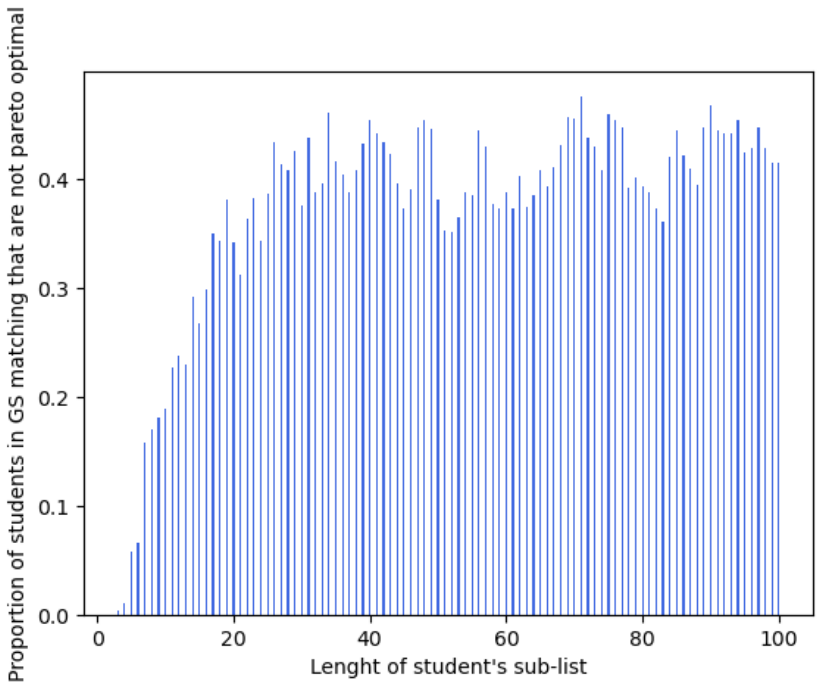
Outline

1. Utility Functions and Average Ranks
   - TTC Algorithm
   - GS Algorithm
2. Pareto Optimality and The GS Algorithm
   - 100 Simulations
   - 10 Simulations
   - 1 Simulation
   - 90 student 100 schools

The GS output and Pareto Optimality

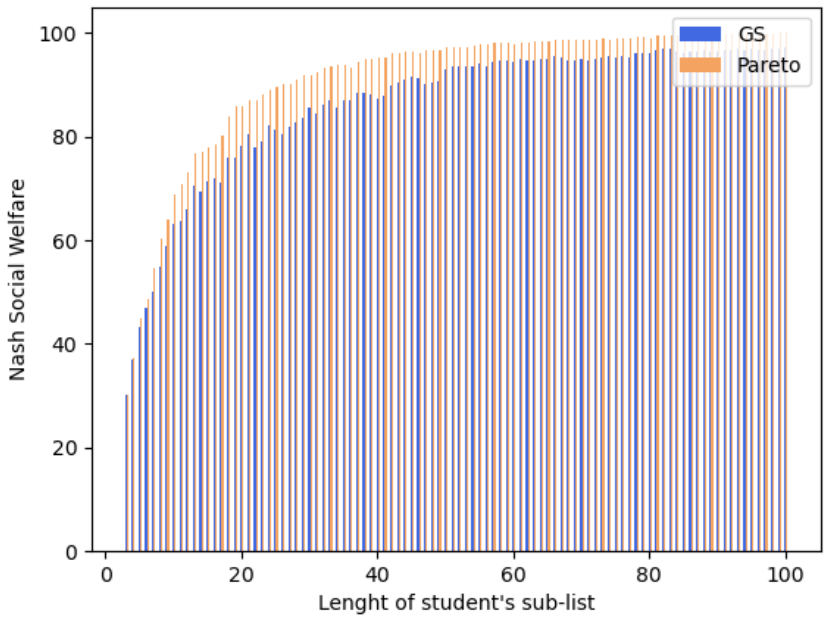


(a) Proportion of experiments with a Pareto Optimal matching



(b) Proportion of students that weren't Pareto Optimal

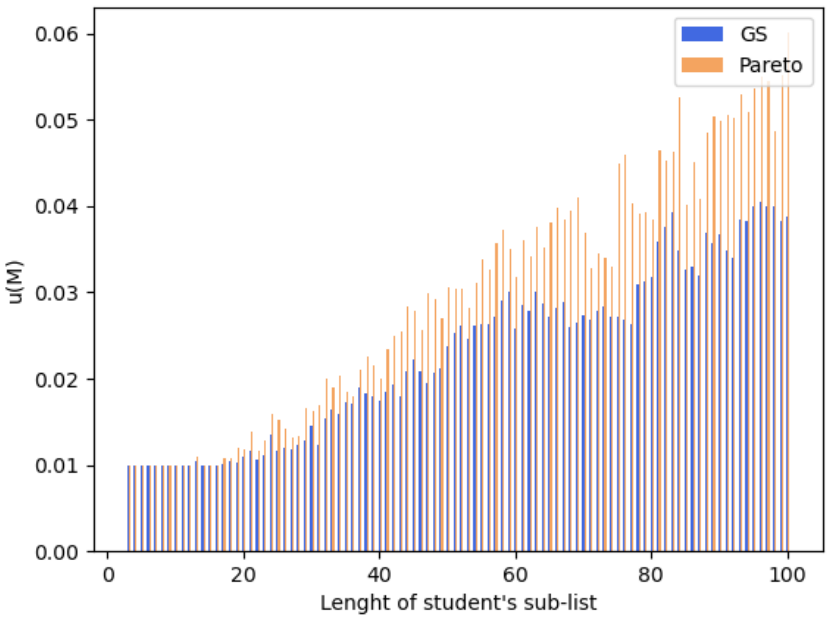
Welfare in GS Output vs Pareto Optimality



(a) Nash Welfare

$u(M) = (\prod_{i \in S} u(i, \mu(i)))^{\frac{1}{n}}$

$u(i, \mu(i))) = n + 2 - rank_i$



(b) Leontief Utilities

$u(M) = min(u(i, \mu(i))))$

$u(i, \mu(i))) = 1/rank_i$

Average Rank of Match

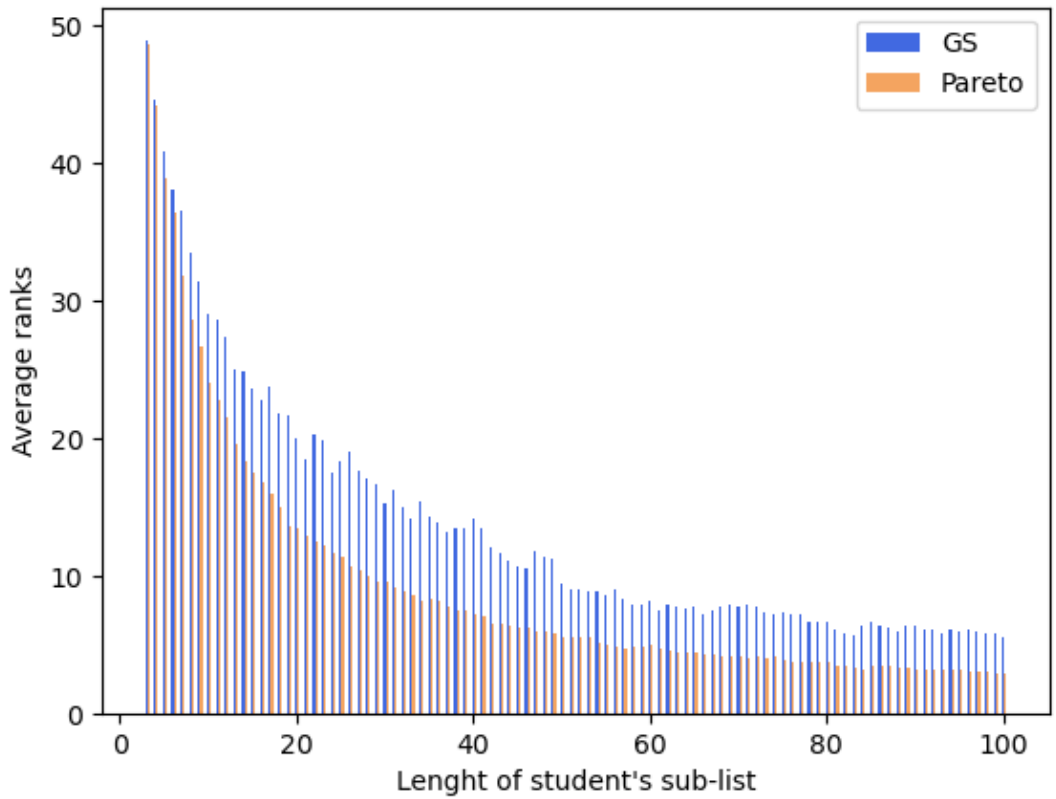
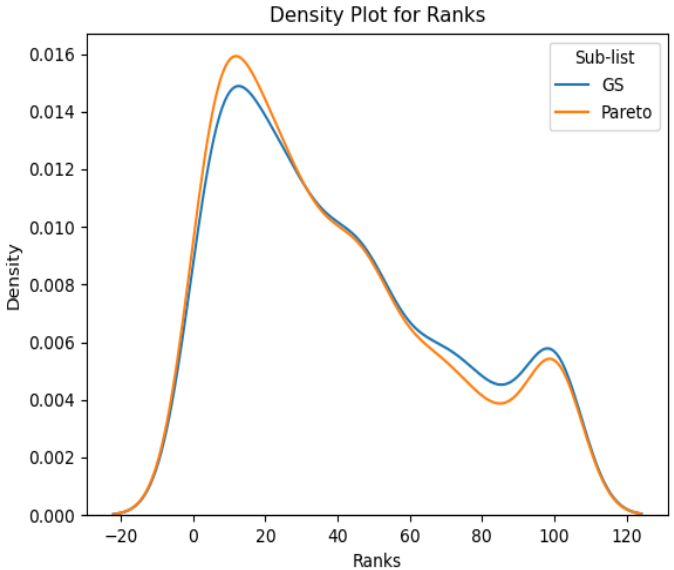
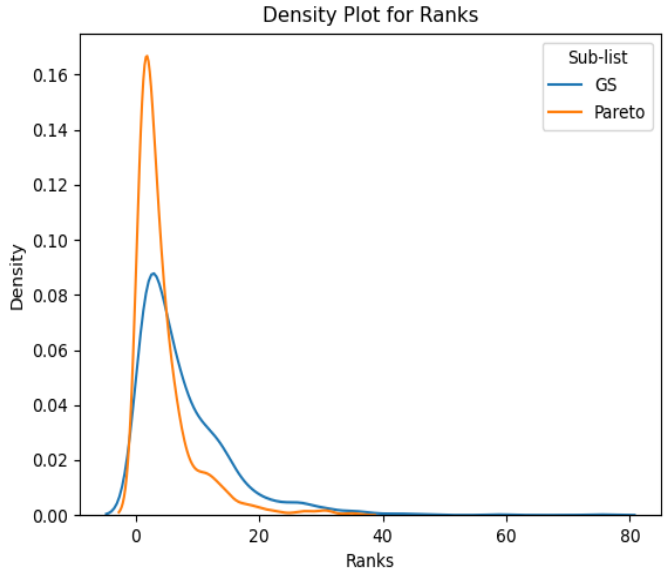
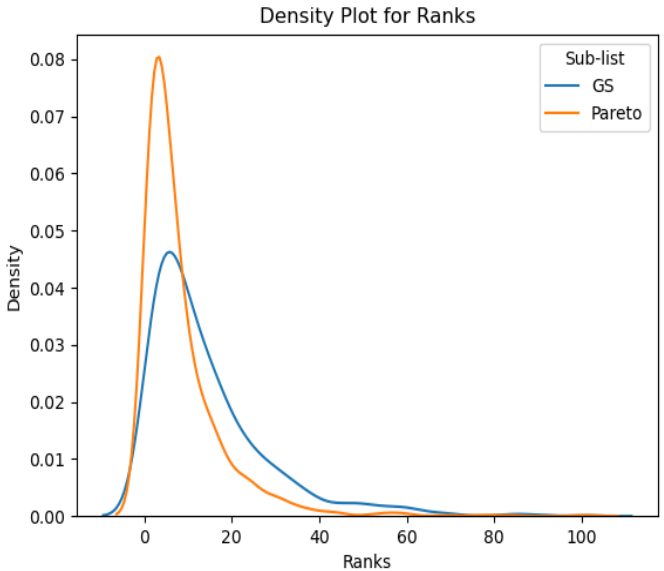
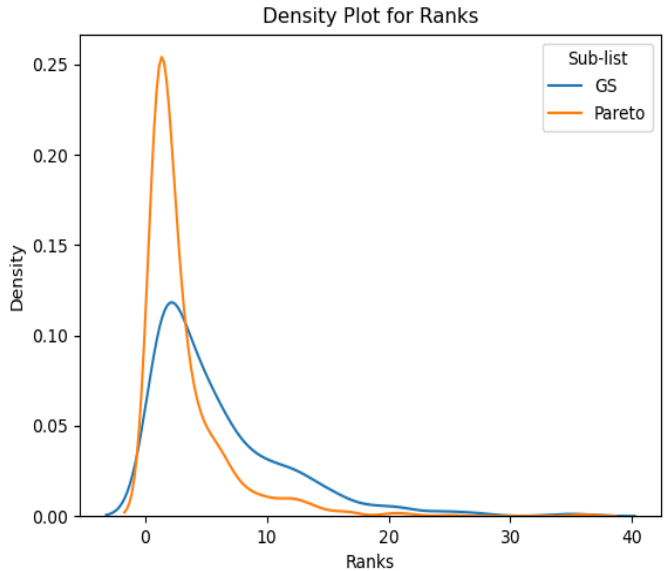


Figure: Average rank of match


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.016
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
−20
0
20
40
60
80
100
120


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
20
40
60
80

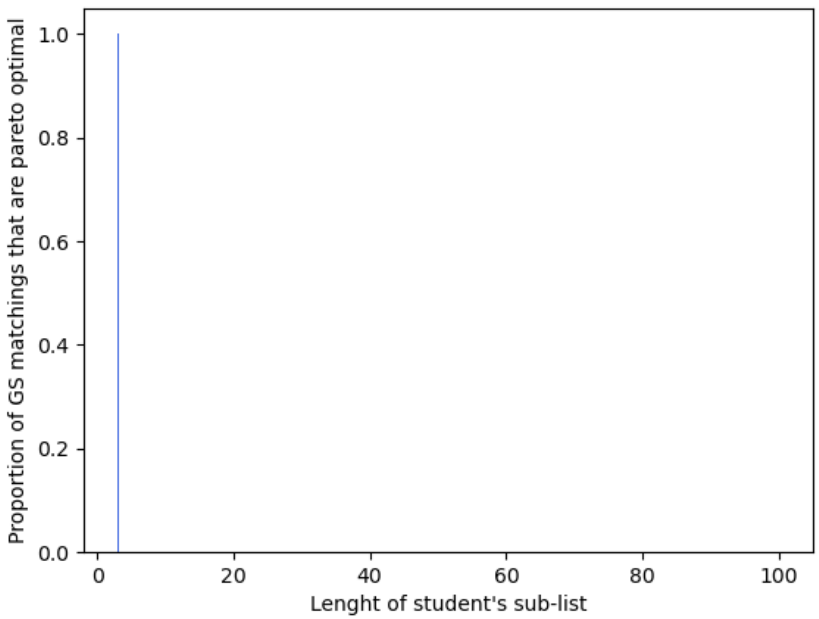

Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0
20
40
60
80
100


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.25
0.20
0.15
0.10
0.05
0.00
0
10
20
30
40

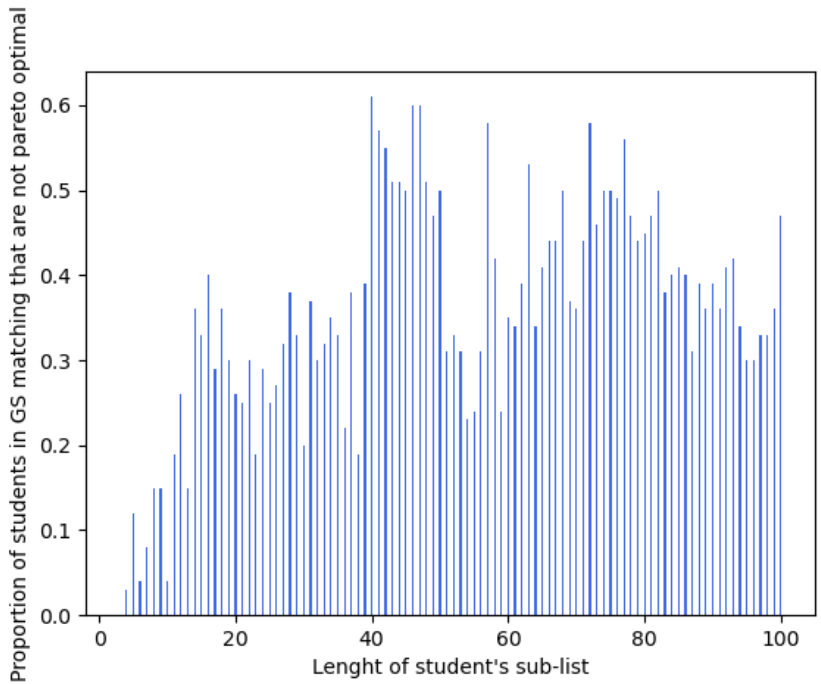
Outline

1. Utility Functions and Average Ranks
   - TTC Algorithm
   - GS Algorithm
2. Pareto Optimality and The GS Algorithm
   - 100 Simulations
   - 10 Simulations
   - 1 Simulation
   - 90 student 100 schools

The GS output and Pareto Optimality

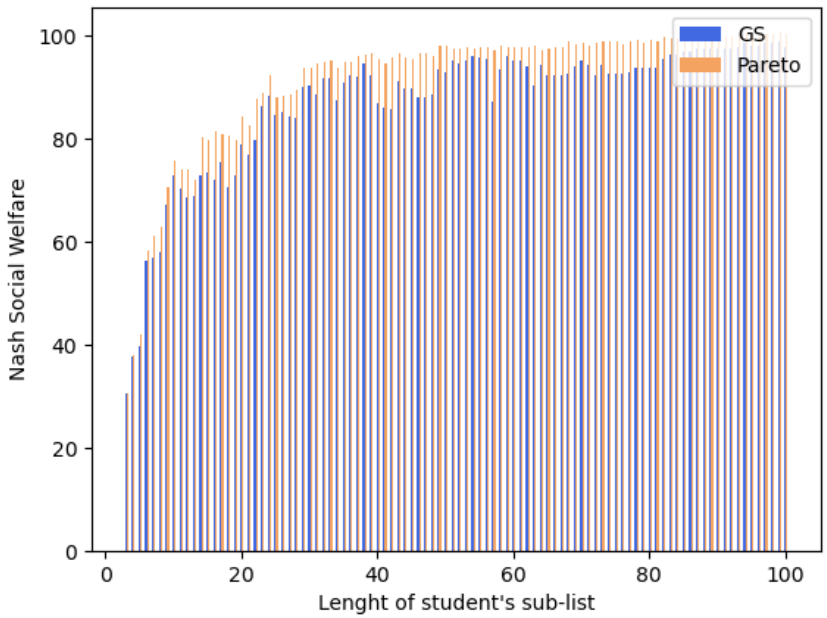


(a) Proportion of experiments with a Pareto Optimal matching



(b) Proportion of students that weren't Pareto Optimal

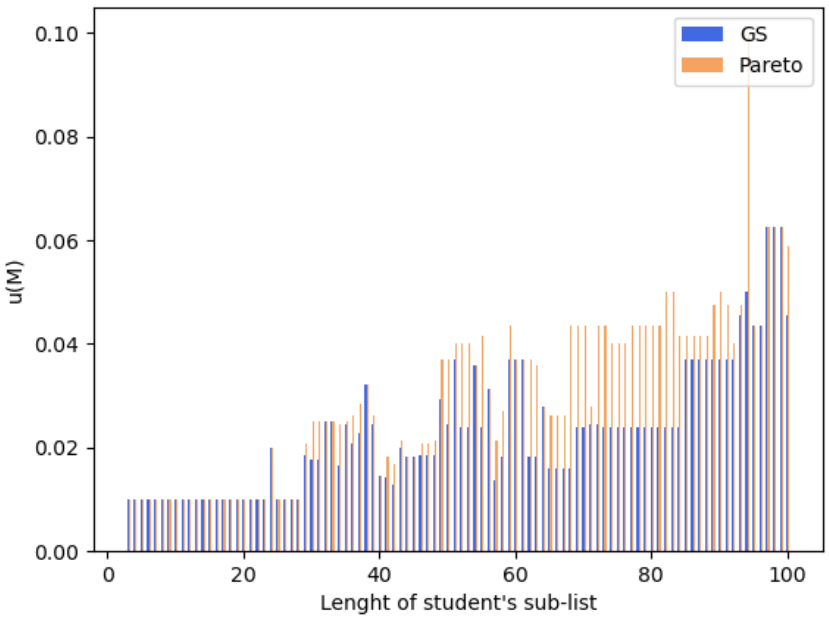
Welfare in GS Output vs Pareto Optimality



(a) Nash Welfare

$u(M) = (\prod_{i \in S} u(i, \mu(i)))^{\frac{1}{n}}$

$u(i, \mu(i))) = n + 2 - rank_i$



(b) Leontief Utilities

$u(M) = min(u(i, \mu(i))))$

$u(i, \mu(i))) = 1/rank_i$

Average Rank of Match

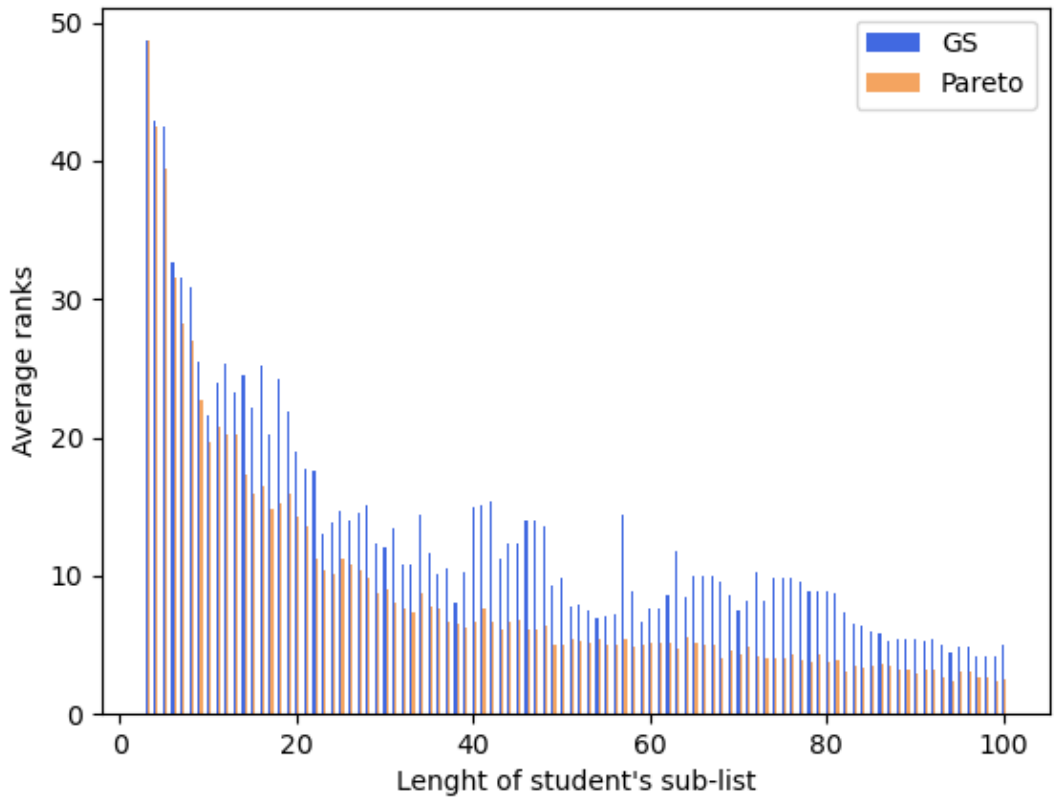
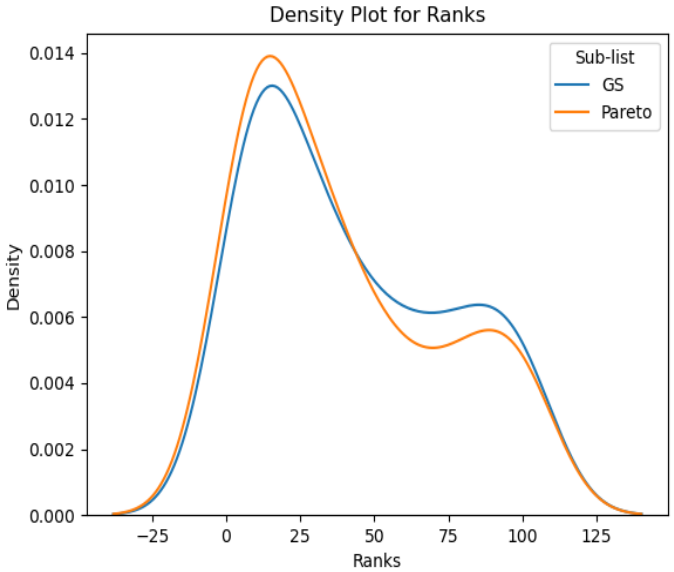
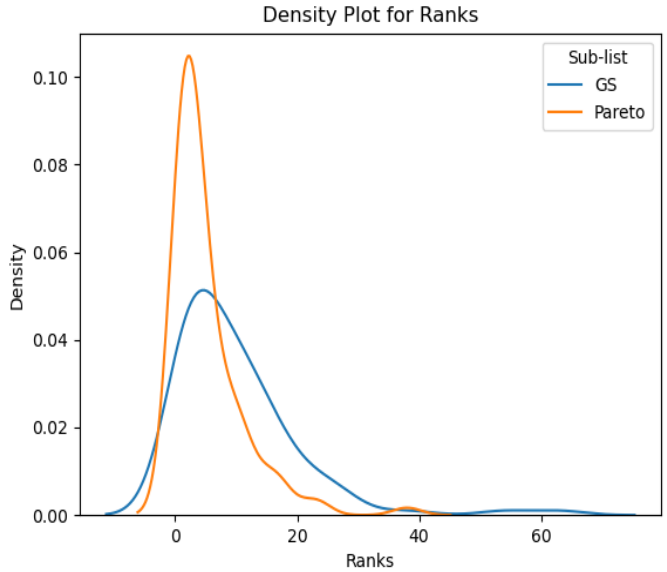
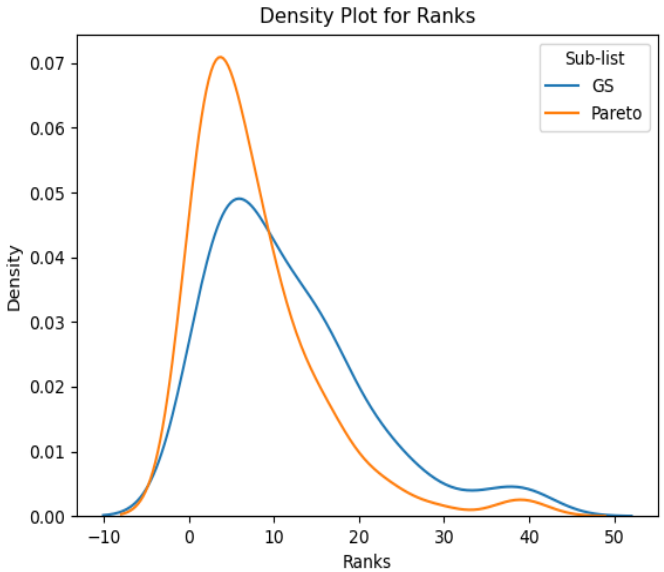
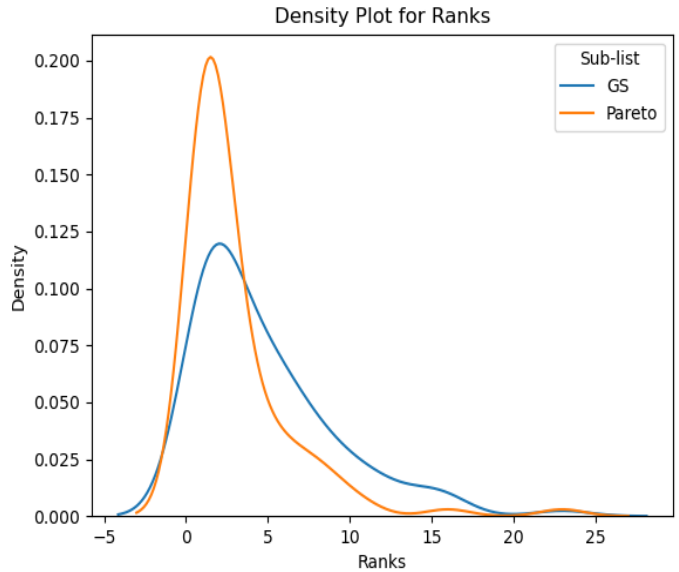


Figure: Average rank of match


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
−25
0
25
50
75
100
125


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.10
0.08
0.06
0.04
0.02
0.00
0
20
40
60

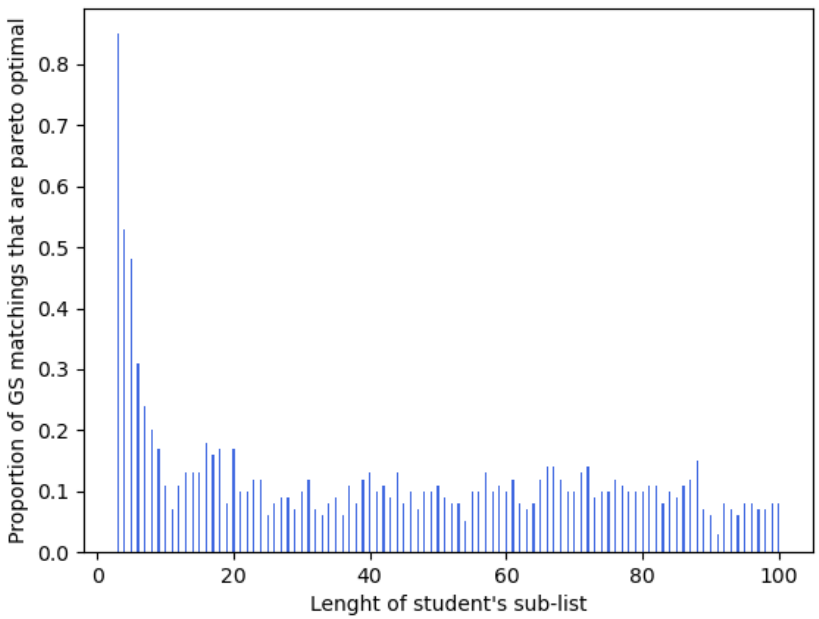

Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
−10
0
10
20
30
40
50


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.200
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
−5
0
5
10
15
20
25

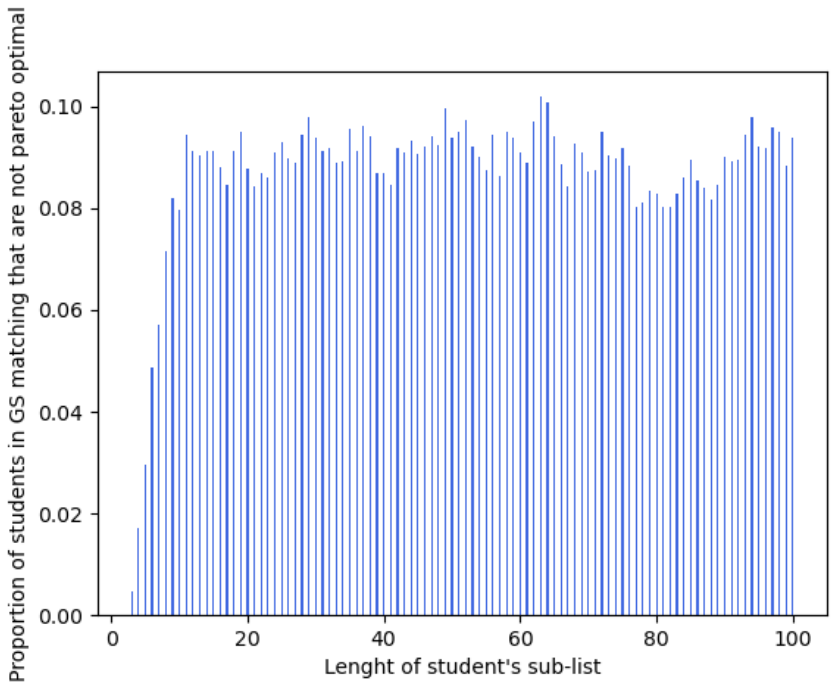
Outline

1. Utility Functions and Average Ranks
   - TTC Algorithm
   - GS Algorithm
2. Pareto Optimality and The GS Algorithm
   - 100 Simulations
   - 10 Simulations
   - 1 Simulation
   - 90 student 100 schools

The GS output and Pareto Optimality

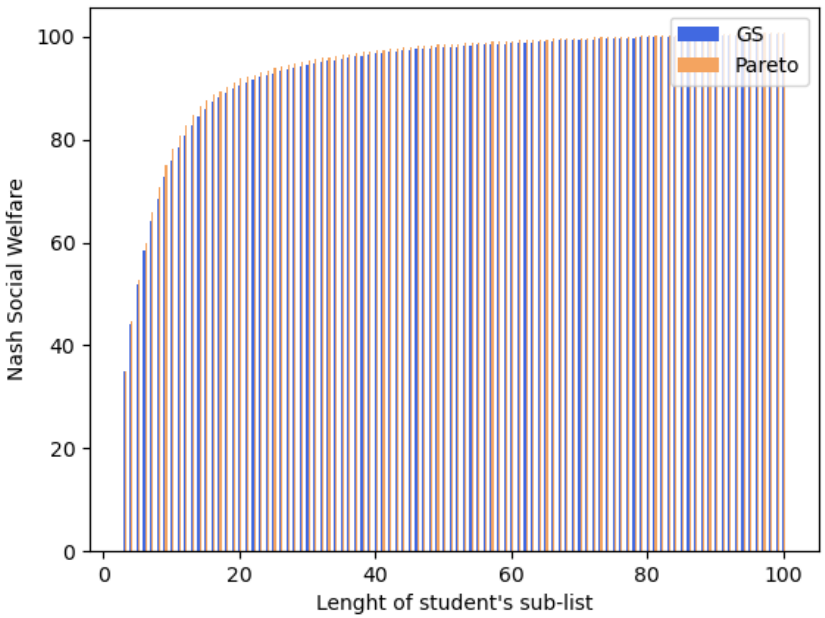


(a) Proportion of experiments with a Pareto Optimal matching



(b) Proportion of students that weren't Pareto Optimal

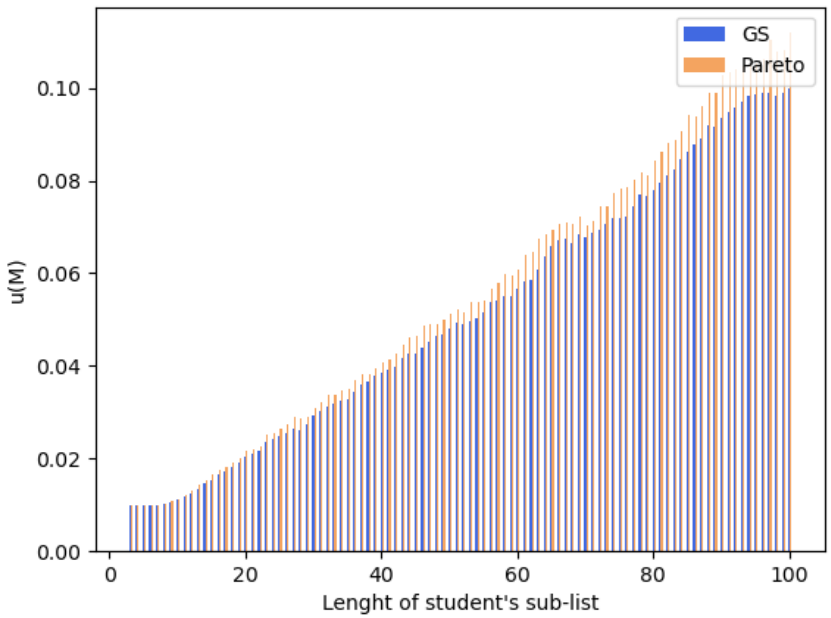
Welfare in GS Output vs Pareto Optimality



(a) Nash Welfare

$u(M) = (\prod_{i \in S} u(i, \mu(i)))^{\frac{1}{n}}$

$u(i, \mu(i))) = n + 2 - rank_i$



(b) Leontief Utilities

$u(M) = min(u(i, \mu(i))))$

$u(i, \mu(i))) = 1/rank_i$

Average Rank of Match

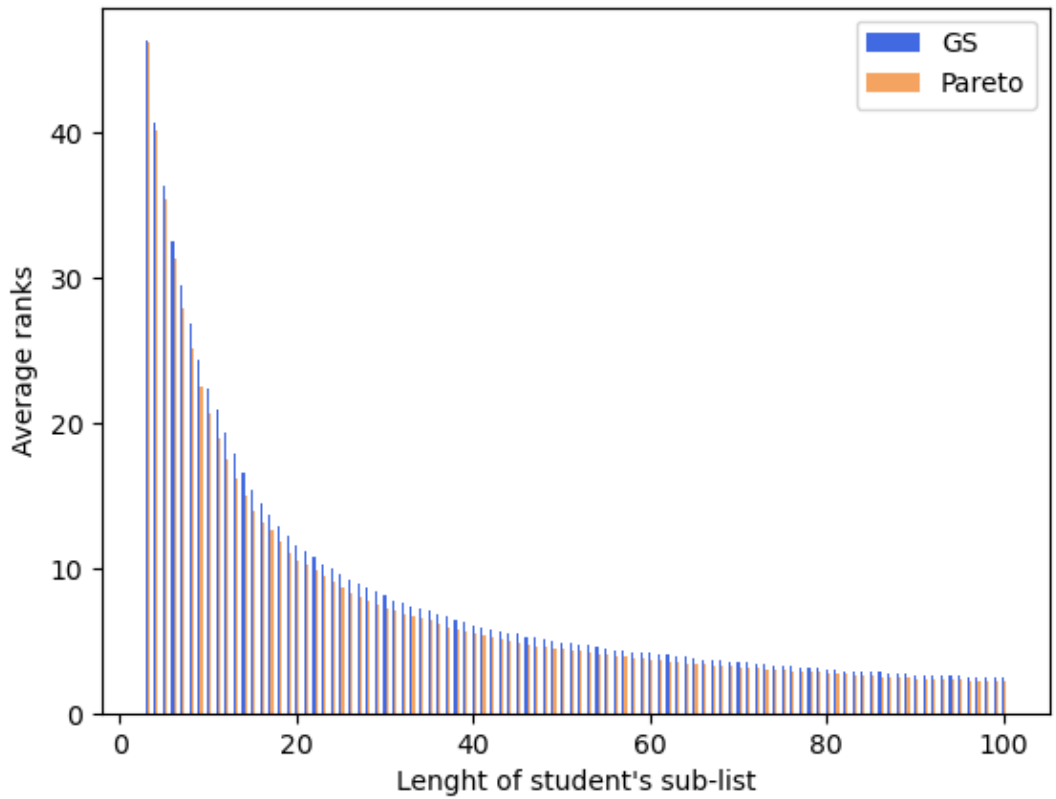
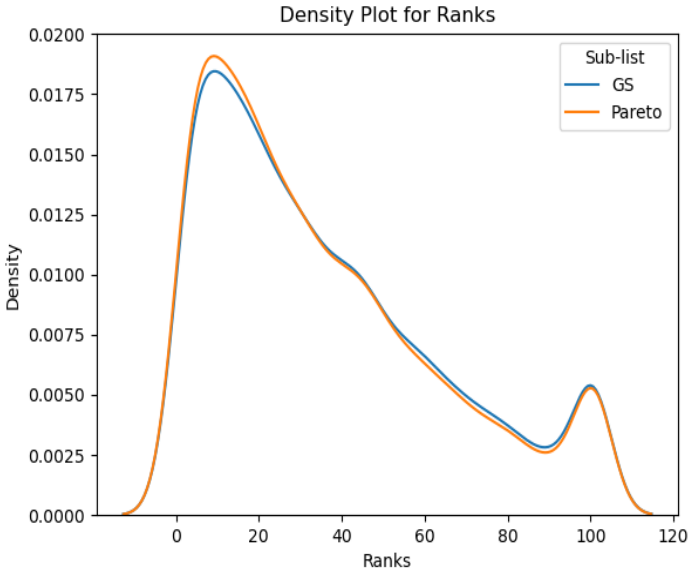
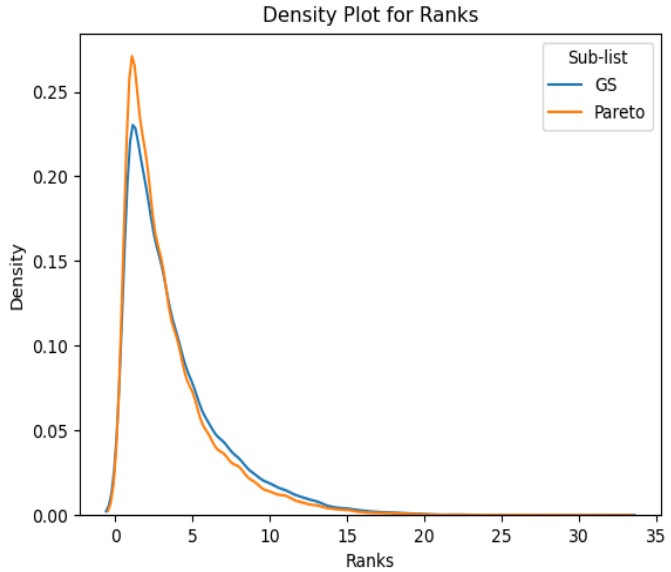
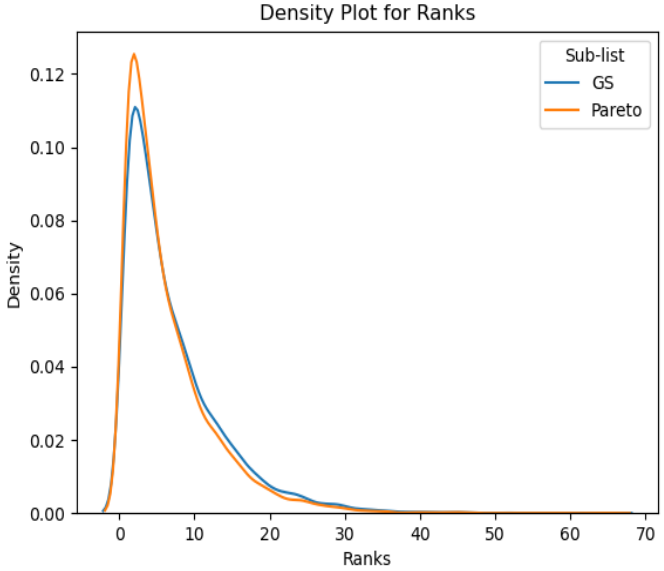
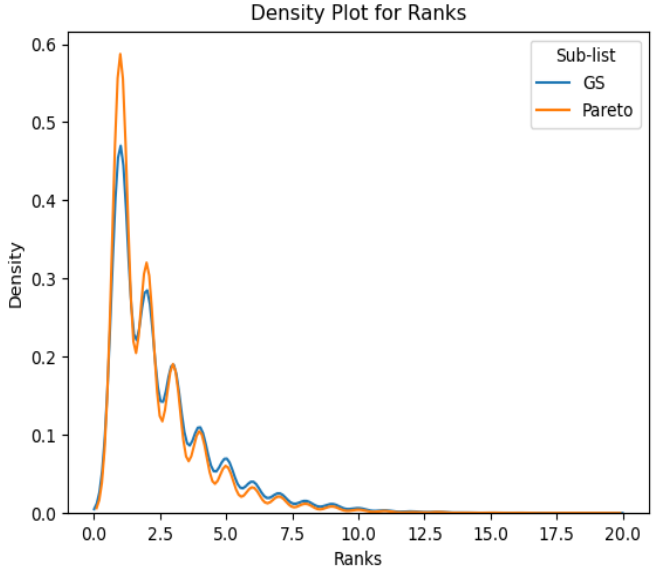


Figure: Average rank of match


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.0200
0.0175
0.0150
0.0125
0.0100
0.0075
0.0050
0.0025
0.0000
0
20
40
60
80
100
120


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.25
0.20
0.15
0.10
0.05
0.00
0
5
10
15
20
25
30
35


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
10
20
30
40
50
60
70


Density Plot for Ranks
Sub-list
GS
Pareto
Density
Ranks
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0