# Elliptic-curve Diffie–Hellman

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# Section 1

# Introduction

1. In this presentation I am going to brief about ECDH.
2. First we will be discussing ECC.
3. Then we will discuss ECDH algorithm.
4. Finally I will be discussing the ECDH example.

# Section 2

# What are Elliptic Curves ?

$$E = \{(x, y) \mid y^2 = x^3 + ax + b\}$$

$a, b \in K$

point at infinity: $\mathcal{O}$

$4a^3 + 27b^2 \neq 0$

Examples of fields

$K$ : $\mathbb{R}$

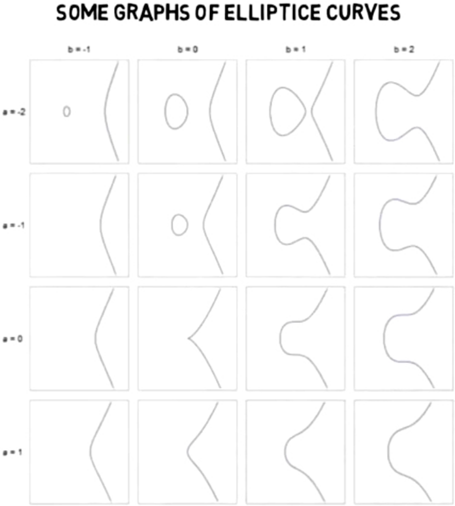
$\mathbb{Q}$

$\mathbb{C}$

$\mathbb{Z}/p\mathbb{Z}$

# Elliptic Curves Graph

$\mathbb{R}$



# Why Elliptic Curves ?

Shorter encryption keys use fewer memory and CPU resources.

smaller keys

| Symmetric Encryption (Key Size in bits) | RSA and Diffie-Hellman (modulus size in bits) | ECC Key Size in bits |
|---|---|---|
| 56 | 512 | 112 |
| 80 | 1024 | 160 |
| 112 | 2048 | 224 |
| 128 | 3072 | 256 |
| 192 | 7680 | 384 |
| 256 | 15360 | 512 |

Notice the Ratio

$$\frac{1024}{160} \approx \frac{6.4}{1}$$

$$\frac{3072}{256} = \frac{12}{1}$$

$$\frac{15360}{512} = \frac{30}{1}$$

COMPARABLE SECURITY

$\mathbb{Z}/p\mathbb{Z}$

ELLIPTIC CURVES

# Section 3

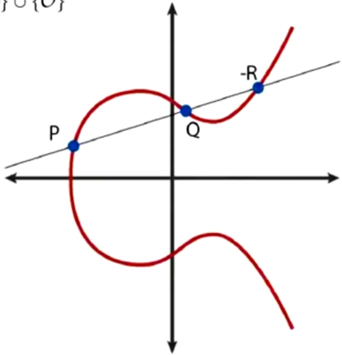
# Addition

## Group Operations

**+ ADDITION**

Given two points in the set $E = \{(x, y) \mid y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$ point at infinity

P+Q=?



# Addition P+Q

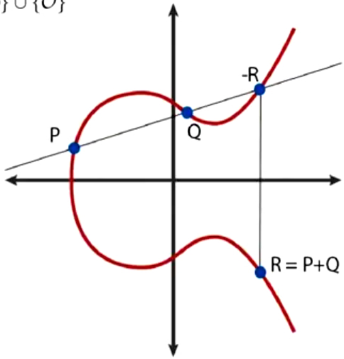
## Group Operations

**+ ADDITION**

point at infinity

Given two points in the set $E = \{(x, y) \mid y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$

P+Q = ?



# Addition P+Q Formula

## Group Operations

+ **ADDITION**

point at infinity

Given two points in the set $E = \{(x,y) \mid y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$

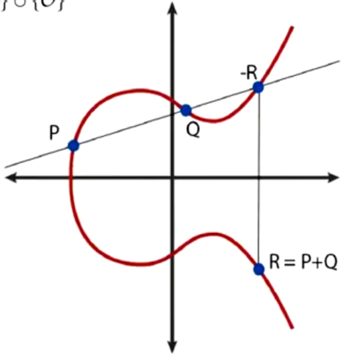
P+Q=?

Algebraically

$$s = \frac{y_P - y_Q}{x_P - x_Q}$$

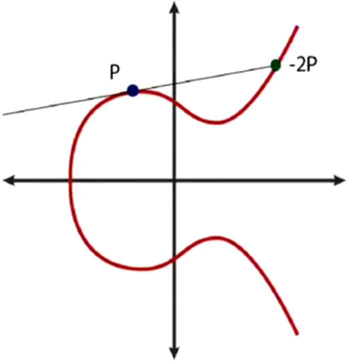
$$x_R = s^2 - (x_P + x_Q)$$

$$y_R = s(x_P - x_R) - y_P$$



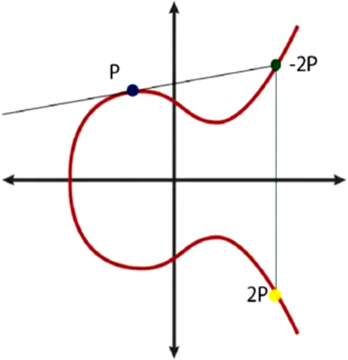
# Point Doubling 1

Point Doubling $P + P = R = 2P$



# Point Doubling 2

Point Doubling $P + P = R = 2P$



# Point Doubling Formula

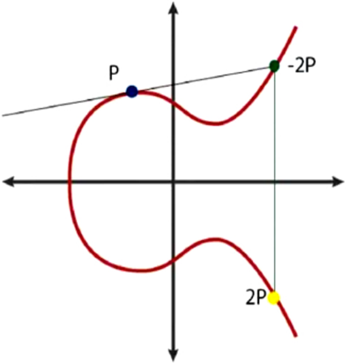
Point Doubling $P + P = R = 2P$

Algebraically

$$s = \frac{3x_P^2 + a}{2y_P}$$

$$x_R = s^2 - 2x_P$$

$$y_R = s(x_P - x_R) - y_P$$

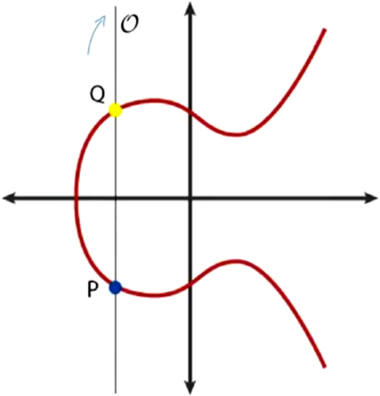


# Adding Vertical Points

## Adding Vertical Points

$P + Q = \mathcal{O}$ if $x_P = x_Q$

$P + P = \mathcal{O}$ if $x_P = 0$



# Scalar Multiplication

## Scalar Multiplication

$P \in E$

$k \in \mathbb{Z}$

$Q = kP$

**REPEATED ADDITION**

$$Q = P + P + \ldots + P \Big\} \text{ K times}$$

# ECDLP

## Elliptic Curve Discrete Log Problem

Scalar Multiplication → One Way Function

$E(\mathbb{Z}/p\mathbb{Z})$

**GIVEN**

$Q, P \in E(\mathbb{Z}/p\mathbb{Z})$ Q is a multiple of P

**FIND**

$k$ such that $Q = kP$

# Section 4

# Basepoint

The Base Point (Generator)

$G \in E(\mathbb{Z}/p\mathbb{Z})$ GENERATES A CYCLIC GROUP

$ord(G) = n$ size of subgroup smallest positive integer s.t. $kG = \mathcal{O}$

number of points on the curve

Cofactor: $h = \dfrac{|E(\mathbb{Z}/p\mathbb{Z})|}{n}$

IDEALLY: $h = 1$

# Parameters

Domain Parameters

$\{p, a, b, G, n, h\}$

$p$ : field (modulo p)

$a, b$ : curve parameters

$G$ : Generator Point

$n$ : ord(G)

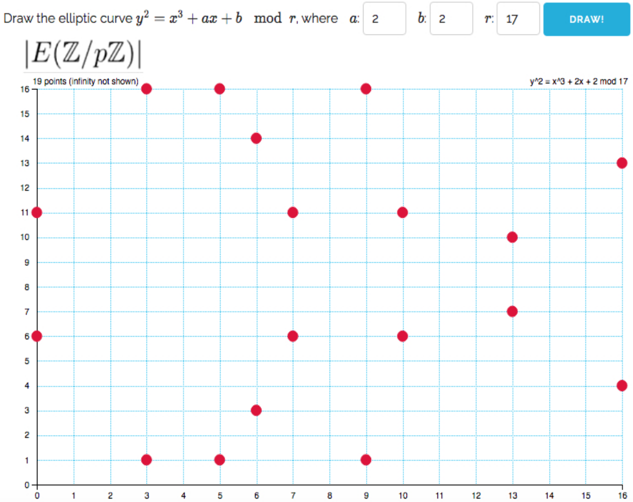
$h$ : cofa

# Example 1

## An Example

$$E: \ y^2 \equiv x^3 + 2x + 2 \pmod{17}$$

# Example 2


Draw the elliptic curve $y^2 = x^3 + ax + b \mod r$, where $a$: 2 $b$: 2 $r$: 17
DRAW!
$|E(\mathbb{Z}/p\mathbb{Z})|$
19 points (infinity not shown)
y^2 = x^3 + 2x + 2 mod 17
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

# Example 3

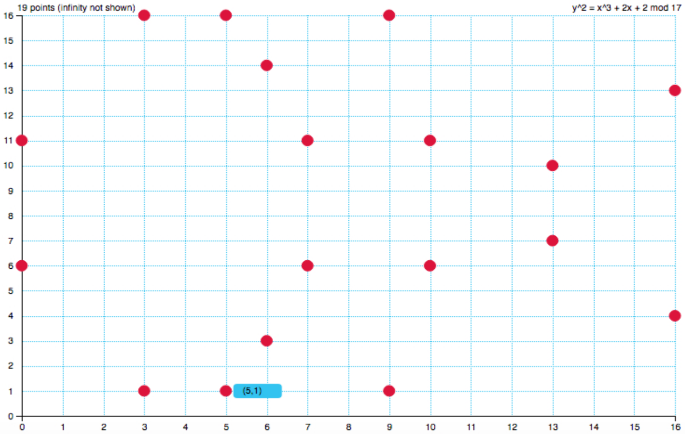
## An Example

$E: \ y^2 \equiv x^3 + 2x + 2 \pmod{17}$

$G = (5, 1)$

# Example 4

Draw the elliptic curve $y^2 = x^3 + ax + b \mod r$, where $a$: 2 $b$: 2 $r$: 17 DRAW!



# Computing 2G

The Cyclic Group

COMPUTE $2G = G + G$

$$s = \frac{3x_G^2 + a}{2y_G} \qquad s \equiv \frac{3(5^2)+2}{2(1)} \equiv 77 \cdot 2^{-1} \equiv 9 \cdot 9 \equiv 13 \pmod{17}$$

$$x_{2G} = s^2 - 2x_G \qquad x_{2G} \equiv 13^2 - 2(5) \equiv 16 - 10 \equiv 6 \pmod{17}$$

$$y_{2G} = s(x_G - x_{2G}) - y_G \qquad y_{2G} \equiv 13(5-6) - 1 \equiv -13 - 1 \equiv -14 \equiv 3 \pmod{17}$$

$$2G = (6, 3)$$

# Computing Subgroup of G

## An Example

$$E: \; y^2 \equiv x^3 + 2x + 2 \pmod{17}$$

$G = (5, 1)$
$2G = (6, 3)$
$3G = (10, 6)$
$4G = (3, 1)$
$5G = (9, 16)$
$6G = (16, 13)$
$7G = (0, 6)$
$8G = (13, 7)$
$9G = (7, 6)$
$10G = (7, 11)$
$11G = (13, 10)$
$12G = (0, 11)$
$13G = (16, 4)$
$14G = (9, 1)$
$15G = (3, 16)$
$16G = (10, 11)$
$17G = (6, 14)$
$18G = (5, 16)$
$19G = \mathcal{O}$

$n = 19$

$h = 1$

$$h = \frac{|E(\mathbb{Z}/p\mathbb{Z})|}{n}$$

h=19/19

# ECDH Demonstration

Bob

Bob picks

$\beta = 9$

Computes

$B = 9G = (7,6)$

Receives

$A = (10,6)$

Computes

Eve

$y^2 \equiv x^3 + 2x + 2 \pmod{17}$

$G = (5,1)$

$n = 19$

$A = (10,6)$

$B = (7,6)$

Alice

Alice pices

$\alpha = 3$

Computes

$A = 3G = (10,6)$

Receives

$B = (7,6)$

Computes

$\beta A = 9A = 9(3G) = 27G = 8G = (13,7)$

$\alpha B = 3B = 3(9G) = 27G = 8G = (13,7)$

# ECDH Algorithm

Elliptic Curce Diffie Hellmann

**Bob**

Bob picks private key $\beta$

$1 \leq \beta \leq n-1$

Computes

$B = \beta G$

Receives

$A = (x_A, y_A)$

Computes

$P = \beta\alpha G$

**Eve**

$y^2 = x^3 + ax + b$

$p$

$a$

$b$

$G$

$n$

$h$

$A$

$B$

$P = ?$

**Alice**

Alice picks private key $\alpha$

$1 \leq \alpha \leq n-1$

Computes

$A = \alpha G$

Receives

$B = (x_B, y_B)$

Computes

$P = \alpha\beta G$

# Section 5

# Conclusion

- We understood the ECC.
- We understood the working of ECDH.
- I was able to implement ECDH Algorithm.

*Thank You!*