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Singularities, interfaces and cracks in dissimilar anisotropic media

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For a non-pathological bimaterial in which an interface crack displays no oscillatory behaviour, it is observed that, apart possibly from the stress intensity factors, the structure of the near-tip field in each of the two blocks is independent of the elastic moduli of the other block. Collinear interface cracks are analysed under this non-oscillatory condition, and a simple rule is formulated that allows one to construct the complete solutions from mode III solutions in an isotropic, homogeneous medium. The general interfacial crack-tip field is found to consist of a two-dimensional oscillatory singularity and a one-dimensional square root singularity. A complex and a real stress intensity factors are proposed to scale the two singularities respectively. Owing to anisotropy, a peculiar fact is that the complex stress intensity factor scaling the oscillatory fields, however defined, does not recover the classical stress intensity factors as the bimaterial degenerates to be non-pathological. Collinear crack problems are also formulated in this context, and a strikingly simple mathematical structure is identified. Interactive solutions for singularity–interface and singularity–interface–crack are obtained. The general results are specialized to decoupled antiplane and in-plane deformations. For this important case, it is found that if a material pair is non-pathological for one set of relative orientations of the interface and the two solids, it is non-pathological for any set of orientations. For bonded orthotropic materials, an intuitive choice of the principal measures of elastic anisotropy and dissimilarity is rationalized. A complex-variable representation is presented for a class of degenerate orthotropic materials. Throughout the paper, the equivalence of the Lekhnitskii and Stroh formalisms is emphasized. The article concludes with a formal statement of interfacial fracture mechanics for anisotropic solids.

1. INTRODUCTION

The interactive solutions for singularity–interface and singularity–crack serve as building blocks for many micromechanics models and computational methods, as indicated by the references cited in Suo (1989*a*). The impetus to undertake this work came from the recent experimental and theoretical investigations of several research groups, to one of which the writer belongs, on fracture behaviours of woods, composites, bicrystals and oriented polymers.

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Another fact that motivates the present work comes from fundamental concerns of fracture mechanics. About forty years ago, G. R. Irwin, in his pioneering work, identified the three independent singular fields at a crack tip in a homogeneous, isotropic body, which have since been referred to as the three modes of singularities. For a given material, each mode is universal for all cracked specimens under arbitrary loadings, except for a normalizing constant, or stress intensity factor, which depends on the specific specimen geometry and external loading. Based on this mathematical artifact, Irwin was able to define a material property, toughness, a loosely termed jargon among metallurgists then, as the critical value (or combinations) of the stress intensity factor that a material can sustain.

During the last four decades, Irwin's agenda has been pursued for both brittle and ductile solids. Here we will only keep track of the development for brittle solids. Williams (1959) discovered the so-called oscillatory near-tip behaviour for an interface crack between two isotropic materials. In this situation, for a given bimaterial the near-tip field is universal up to a complex and a real normalizing factors. The in-plane deformations are coupled and oscillatory, and can be normalized by the complex stress intensity factor. The mode III field has a square root singularity and can be defined separately with the real stress intensity factor. Significant progress has been made in interfacial fracture mechanics, which is assessed by Rice (1988), Hutchinson (1990), Suo (1989) and Shih *et al.* (1990).

In another direction, Stroh (1958) and Sih *et al.* (1965), among others, investigated the crack-tip fields in an anisotropic homogeneous body. The full singular fields were tabulated by Hoenig (1982) for the most general anisotropy. The near-tip fields can be normalized by three real stress intensity factors, and moreover, for this situation, the three modes can be separately defined.

The next logical target, a crack along an interface between dissimilar anisotropic media, has been tackled by Bassani & Qu (1989), Clements (1971), Gotoh (1967), Qu & Bassani (1989), Tewary *et al.* (1989), Ting (1986), Wang (1984) and Willis (1971). Several basic crack problems have been solved, and calculation of the oscillatory index has been emphasized. However, it is not clear, from the work published to date, what the structure of the near-tip fields is, and, in particular, whether the near-tip fields in general can be normalized by one complex and one real factors. This is not self-evident. Willis (1971), for example, defined a 'stress concentration vector', which involved three complex, or six real, quantities. A similar situation is found in Wang (1984) and Tewary *et al.* (1989). Such conceptual ambiguity in defining stress intensity factors, and thus toughness, stands in the way of further development of Irwin's fracture mechanics scheme. One aim of this paper is to identify the precise structure of the near-tip field.

A breakthrough was made recently by considering bimaterials with no pathological behaviour at interface crack tips (Qu & Bassani 1989; Bassani & Qu 1989). These authors proved a necessary and sufficient non-oscillatory condition. Under this condition they could define the three modes separately in the conventional fashion, and found the Irwin-type energy release rate expression is simply the average of the corresponding results for the two homogeneous materials. Indeed, in §5 of this paper, we observe that for such an interface crack,

the near-tip fields (stresses and displacements) in the two solids do not interact with each other in the sense that, apart from the stress intensity factors, the field in each solid is independent of the moduli of the other solid. Thus the structure of the field in each solid is identical to those for a crack in the corresponding homogeneous material. Collinear interface cracks are also considered, and a simple rule is formulated to construct the complete solutions from the mode III potentials for an isotropic, homogeneous material.

The general near-tip field is derived in §6, which consists of a two-dimensional coupled oscillatory singularity and a one-dimensional square root singularity. Each of the two types of singularities may contain mixed in-plane and antiplane deformations. A definition of stress intensity factors is proposed. It is found that the complex stress intensity factor for the two-dimensional oscillatory field, however defined, does not recover the classical stress intensity factor as the bimaterial degenerates to be non-pathological. This peculiar feature has not been encountered in the previous experience with isotropic bimaterials. A revealing example is given in §9 for orthotropic bimaterials. A constructive formula is devised to obtain stress intensity factors and complete field solutions for collinear crack problems.

Several known results of dislocation mechanics, such as the solution of a dislocation in a homogeneous material and dislocation–interface interaction are included in this paper for ease of reference. These results, in conjunction with the basic crack solutions, are used to solve singularity–interface-crack interaction problems in §7. The latter problems were also studied recently by Tewary *et al.* (1989).

For an important situation in which in-plane and antiplane deformations decouple, it is found in §10 that a material pair is non-pathological in any relative orientation if it is so in one relative orientation. Non-pathology for this special case, as a consequence, is a property that an anisotropic material pair may or may not have, regardless of the relative orientation.

A formal statement of interfacial fracture mechanics for anisotropic solids is stated at the end of the article on the basis the structure of the near-tip singularities.

Throughout the paper the equivalence is emphasized among several apparently different complex-variable representations of two-dimensional elasticity, such as those by Lekhnitskii (1963), Green & Zerna (1954) and Eshelby *et al.* (1953). In other words, for instance, the formulation due to Eshelby *et al.*, which is better known as the Stroh (1958) formalism, is none other than an alternative derivation of the earlier Russian work contained in Lekhnitskii's book. Workers in this field seem to be aware of this fact, but no explicit discussions are found in the established literature. The equivalence allows one to take advantage of all these theories, so that one can enjoy many remarkable algebraic results in the Stroh formalism, and also benefit from the explicitness of Lekhnitskii's derivation, as well as many solution techniques developed in his book. In particular, analytic continuation techniques will be exploited throughout the paper, which simplify many earlier works using integral transforms.

One thing that makes the subject chaotic is notation. In this paper I have

tried to unify the notation by using smallest possible set of 3×3 matrices, i.e. A , L , B and H . The first three are matrices only involving elastic constants of one material, whereas the last one, H , is a bimaterial matrix. All of them are complex-valued, and B and H are positive-definite Hermitian matrices.

2. LEKHNITSKII-ESHELBY-STROH (LES) REPRESENTATION

Hooke's law connecting the stresses σ_{ij} and strains ϵ_{ij} for a generally anisotropic material can be written in one of the following forms

$$\left. \begin{aligned} \epsilon_{ij} &= \sum_{k,l=1}^3 S_{ijkl} \sigma_{kl}, & \sigma_{ij} &= \sum_{k,l=1}^3 C_{ijkl} \epsilon_{kl}, \\ \epsilon_i &= \sum_{j=1}^6 s_{ij} \sigma_j, & \sigma_i &= \sum_{j=1}^6 c_{ij} \epsilon_j. \end{aligned} \right\} \quad (2.1)$$

The standard correspondence is adopted, i.e.

$$\{\epsilon_i\} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{31}, 2\epsilon_{12}]^T \quad \text{and} \quad \{\sigma_i\} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}]^T.$$

The superscript T denotes the transpose. The fourth-order tensors S and C are referred to as the compliance and stiffness tensors respectively. The 6×6 matrices s and c ($s = c^{-1}$) are conventional compliance and stiffness matrices. The tensor C_{ijkl} can be replaced by the matrix c_{ij} correspondingly. The relation between s_{ij} and S_{ijkl} is analogous except that numerical factors are needed, e.g. $s_{11} = S_{1111}$, $s_{14} = 2S_{1123}$, $s_{44} = 4S_{2323}$. To avoid confusion, in this paper no summation is assumed implicitly for repeated indices.

It has been shown by Lekhnitskii (1963) and Eshelby *et al.* (1953), that for a two dimensional problem, i.e. with geometry and external loading invariant in the direction normal to xy -plane, the elastic field can be represented in terms of three functions $f_1(z_1)$, $f_2(z_2)$, $f_3(z_3)$, each of which is holomorphic in its argument $z_j = x + \mu_j y$. Here μ_j are three distinct complex numbers with positive imaginary part, which can be solved as roots of a sixth-order polynomial to be listed shortly. With these holomorphic functions, or complex potentials, the representation for displacements u_i , stresses σ_{ij} , and resultant forces on an arc T_i (the medium is kept on the left-hand side as an observer travels in the positive direction of the arc) is

$$\left. \begin{aligned} u_i &= 2 \operatorname{Re} \left[\sum_{j=1}^3 A_{ij} f_j(z_j) \right], & T_i &= -2 \operatorname{Re} \left[\sum_{j=1}^3 L_{ij} f_j(z_j) \right], \\ \sigma_{2i} &= 2 \operatorname{Re} \left[\sum_{j=1}^3 L_{ij} f'_j(z_j) \right], & \sigma_{1i} &= -2 \operatorname{Re} \left[\sum_{j=1}^3 L_{ij} \mu_j f'_j(z_j) \right]. \end{aligned} \right\} \quad (2.2)$$

Here, ()' is designated as the derivative with respect to the associated arguments, and A and L are two 3×3 matrices depending on elastic constants, which will be defined shortly. Curiously, the derivations by Lekhnitskii and Eshelby *et al.*, and some other authors, notably Green & Zerna (1954), gave entirely different schemes to compute the numbers μ_α and matrices A and L .

On the basis of two Airy-type stress functions, Lekhnitskii found that μ_α satisfies the sixth-order characteristic equation

$$l_2(\mu) l_4(\mu) - [l_3(\mu)]^2 = 0, \quad (2.3)$$

where

$$\left. \begin{aligned} l_2(\mu) &= s_{55}\mu^2 - 2s_{45}\mu + s_{44}, \\ l_4(\mu) &= s_{11}\mu^4 - 2s_{16}\mu^3 + (2s_{12} + s_{66})\mu^2 - 2s_{26}\mu + s_{22}, \\ l_3(\mu) &= s_{15}\mu^3 - (s_{14} + s_{56})\mu^2 + (s_{25} + s_{46})\mu - s_{24}. \end{aligned} \right\} \quad (2.4)$$

By requiring the compliance matrix to be positive definite, he was able to prove that (2.3) has no real root. If one assumes the roots are distinct, the six roots form three complex conjugate pairs, from which three μ_α with positive imaginary part can be selected. The elements of the matrices A and L are given by

$$L = \begin{bmatrix} -\mu_1 & -\mu_2 & -\mu_3 \eta_3 \\ 1 & 1 & \eta_3 \\ -\eta_1 & -\eta_2 & -1 \end{bmatrix}, \quad (2.5)$$

and

$$\left. \begin{aligned} A_{1\alpha} &= s_{11}\mu_\alpha^2 + s_{12} - s_{16}\mu_\alpha + \eta_\alpha(s_{15}\mu_\alpha - s_{14}), \\ A_{2\alpha} &= s_{21}\mu_\alpha + s_{22}/\mu_\alpha - s_{26} + \eta_\alpha(s_{25} - s_{24}/\mu_\alpha), \\ A_{3\alpha} &= s_{41}\mu_\alpha + s_{42}/\mu_\alpha - s_{46} + \eta_\alpha(s_{45} - s_{44}/\mu_\alpha), \end{aligned} \right\} \quad (2.6a)$$

for $\alpha = 1, 2$, and

$$\left. \begin{aligned} A_{13} &= \eta_3(s_{11}\mu_3^2 + s_{12} - s_{16}\mu_3) + s_{15}\mu_3 - s_{14}, \\ A_{23} &= \eta_3(s_{21}\mu_3 + s_{22}/\mu_3 - s_{26}) + s_{25} - s_{24}/\mu_3, \\ A_{33} &= \eta_3(s_{41}\mu_3 + s_{42}/\mu_3 - s_{46}) + s_{45} - s_{44}/\mu_3, \end{aligned} \right\} \quad (2.6b)$$

where

$$\eta_\alpha = -l_3(\mu_\alpha)/l_2(\mu_\alpha) \quad (\alpha = 1, 2) \quad \eta_3 = -l_3(\mu_3)/l_4(\mu_3). \quad (2.7)$$

Equations (2.5)–(2.7) are valid for plane stress deformation. Plane strain deformation can be treated by a change of compliances

$$s'_{ij} = s_{ij} - s_{i3}s_{j3}/s_{33}. \quad (2.8)$$

Evidently unaware of the Russian work, Eshelby *et al.* (1953) presented their more elegant formalism based on the Navier–Cauchy equations. Their representation has the same structure as (2.2). However, each of the characteristic roots μ_α , as well as each column of A is solved from the eigenvalue problem

$$\sum_{k=1}^3 [\mathcal{C}_{i1k1} + \mu_\alpha(\mathcal{C}_{i1k2} + \mathcal{C}_{i2k1}) + \mu_\alpha^2 \mathcal{C}_{i2k2}] A_{k\alpha} = 0. \quad (2.9a)$$

Each column of A may be normalized arbitrarily. Thus μ_α are the roots with positive imaginary parts of the sixth-order polynomial

$$|\mathcal{C}_{i1k1} + \mu_\alpha(\mathcal{C}_{i1k2} + \mathcal{C}_{i2k1}) + \mu_\alpha^2 \mathcal{C}_{i2k2}| = 0. \quad (2.9b)$$

The matrix L is given by

$$L_{i\alpha} = \sum_{k=1}^3 [C_{i2k1} + \mu_{\alpha} C_{i2k2}] A_{k\alpha}. \quad (2.10)$$

Plane strain deformation is assumed in (2.9a) and (2.10). For plane stress problems the following substitution has to be made

$$c'_{ij} = c_{ij} - c_{i3}c_{j3}/c_{33} \quad (2.11)$$

Now the question of equivalence of the two formulations arises naturally: are μ_{α} , A and L defined in the two entirely different ways actually identical? The answer is yes. It is clear in Eshelby *et al.* (1953) that the representation (2.2) is uniquely determined by the elastic constants of a material (up to the three normalization factors for the matrix A), however one derives it. Therefore the Lekhnitskii derivation gives, explicitly, a specially normalized A . In the remainder of the paper, the basic formula (2.2) will be referred to as the LES representation. Fundamental results known in different formalisms will be cited freely as needed.

Assuming that the roots of the characteristic equation (2.3) or (2.9b) (they are equivalent) form three distinct complex conjugate pairs, Stroh (1958) showed that A and L are non-singular, and moreover, the matrix B is a positive definite hermitian matrix, where

$$B = iAL^{-1}. \quad (2.12)$$

Here $i = \sqrt{-1}$. The matrix B will appear in various solutions. For convenience, a positive-definite hermitian matrix H involving bimaterial elastic constants is defined as

$$H = B_1 + \bar{B}_2. \quad (2.13)$$

Here, and throughout the paper, an overbar denotes the complex conjugation, and the subscripts 1 and 2 attached to matrices and vectors are reserved exclusively to indicate the two materials.

Under an in-plane coordinate rotation

$$[R_{ij}] \equiv \begin{bmatrix} \partial x_i^* \\ \partial x_j \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2.14)$$

where (*) indicates the new coordinate system, Ting (1982) showed that the characteristic numbers transform as

$$\mu_j^* = (\mu_j \cos \phi - \sin \phi) / (\mu_j \sin \phi + \cos \phi), \quad (2.15)$$

and furthermore, each column of A and L transforms like a vector, namely

$$A^* = RA, \quad L^* = RL. \quad (2.16)$$

It is obvious from (2.16) that under such an in-plane rotation, B and H defined in this paper transform like second-order tensors, i.e.

$$B^* = RBR^T, \quad H^* = RHR^T. \quad (2.17)$$

Some consequences of this transformation will be discussed in §10.

The LES representation may break down if the roots degenerate. A well-known

case is an isotropic material, which was treated extensively by Muskhelishvili (1953*a*) with his famous complex-variable representation. An analogous representation for a class of degenerate orthotropic materials is presented in §10. Note, however, the matrices B and H are independent of the normalizing factors for A , and have smooth limits even if A and L become singular. This point will be illustrated in §9.

Having listed the LES representation and stated the consistency of the two derivations, we now add some words about a particular method, analytic continuation, which will be used in the paper. Stated below is a trivial observation that makes analytic continuation arguments possible.

A function $h(z)$ is an analytic function of $z = x + \mu y$ for $y > 0$ (or $y < 0$) for any μ if it is analytic for $y > 0$ (or $y < 0$) for one μ , where μ is any complex number with positive imaginary part.

Consequently, when talking about a function analytic in the upper (or lower) half plane, one needs not refer to its argument, as long as the argument has the form $z = x + \mu y$ ($\text{Im } \mu > 0$).

Without loss of any information, we can and will present our solutions by the function vector $\mathbf{f}(z)$ defined as

$$\mathbf{f}(z) = [f_1(z), f_2(z), f_3(z)]^T, \quad (2.18)$$

where the argument has the generic form $z = x + \mu y$ ($\text{Im } \mu > 0$). Once the solution of $\mathbf{f}(z)$ is obtained for a given boundary value problem, a replacement of z_1, z_2 or z_3 should be made for each component function to calculate field quantities from (2.2). Of particular importance is the following set of vectors defined along the x -axis

$$\left. \begin{aligned} \mathbf{u}(x) &= \{u_j(x, 0)\} = A\mathbf{f}(x) + \bar{A}\bar{\mathbf{f}}(x), \\ \mathbf{T}(x) &= \{T_j(x, 0)\} = -L\mathbf{f}(x) - \bar{L}\bar{\mathbf{f}}(x), \\ \mathbf{t}(x) &= \{\sigma_{2j}(x, 0)\} = L\mathbf{f}'(x) + \bar{L}\bar{\mathbf{f}}'(x). \end{aligned} \right\} \quad (2.19)$$

They will be used extensively in the following sections.

3. LINE FORCE AND DISLOCATION IN A HOMOGENEOUS MEDIUM

Isolated singularities in an infinite homogeneous medium are building blocks for many subsequent interaction problems. Consider a dislocation line in the direction perpendicular to xy -plane, with Burgers vector \mathbf{b} , and consider a line force uniformly distributed along that direction, with force per unit length \mathbf{p} . Both singularities are at the point (x_0, y_0) . The solution is of the form (Eshelby *et al.* 1953)

$$f_j(z) = q_j \ln(z - s_j), \quad s_j = x_0 + \mu_j y_0, \quad (3.1)$$

where the complex coefficient vector $\mathbf{q} = \{q_j\}$ is to be determined in terms of \mathbf{b} and \mathbf{p} . The branch points for the \ln -functions are at s_j , while, for definiteness, the branch cuts are chosen in the negative x -direction, and the phase angle is measured from the positive x -direction.

With the aid of (2.2), by definition one has

$$\mathbf{b} = \mathbf{u}^+ - \mathbf{u}^- = 2\pi i(A\mathbf{q} - \bar{A}\bar{\mathbf{q}}), \quad \mathbf{p} = \mathbf{T}^- - \mathbf{T}^+ = 2\pi i(L\mathbf{q} - \bar{L}\bar{\mathbf{q}}). \quad (3.2)$$

Solving for \mathbf{q} from the above algebraic equations, one finds

$$\mathbf{q} = (2\pi)^{-1} L^{-1}(B + \bar{B})^{-1} \mathbf{b} - (2\pi)^{-1} A^{-1}(B^{-1} + \bar{B}^{-1})^{-1} \mathbf{p}. \quad (3.3)$$

Hence a complete description of the solution is achieved.

In the above, the subscripts of z are dropped with the understanding mentioned at the end of the last section. When calculating stresses and displacements from (2.2), one needs to reinterpret z by z_j accordingly. The merit of this scheme will be demonstrated in the next section.

Several well-known results are derived below to familiarize ourselves with the notation. Let a dislocation sit at the origin ($\mathbf{b} \neq 0, \mathbf{p} = 0$). The complex potentials are given by

$$L\mathbf{f}(z) = (2\pi)^{-1} (B + \bar{B})^{-1} \mathbf{b} \ln z. \quad (3.4)$$

The traction along the x -axis, calculated from (2.19), is given by

$$\mathbf{t}(x) = \frac{1}{2\pi} (B + \bar{B})^{-1} \mathbf{b} \left(\frac{1}{x + 0i} + \frac{1}{x - 0i} \right). \quad (3.5)$$

In deriving (3.5) one may imagine that the dislocation is infinitesimally off the x -axis (upper or below). The notation $x + 0i$ represents a point approaching the x -axis from the upper half-plane, with a similar convention for $x - 0i$. This distinction was suggested by Willis (1971), which is quite crucial when the dislocation solution is used as the kernel to formulate integral equations for cracks, especially for interface cracks. The strain energy of a dislocation is thereby (Stroh 1958)

$$U \equiv \frac{1}{2} \mathbf{b}^T \int_r^R \mathbf{t}(x) dx = \frac{1}{2\pi} \mathbf{b}^T (B + \bar{B})^{-1} \mathbf{b} \ln \frac{R}{r}. \quad (3.6)$$

4. A SINGULARITY IN A BIMATERIAL

Now the interaction problem in figure 1a is taken up. Suppose we know, somehow, the solution for an isolated singularity in an infinite homogeneous medium, designated as $\mathbf{f}_0(z)$, not necessarily of the form (3.1). The aim is to construct the solution for the same singularity embedded in bimaterials. Without loss of generality, the singularity is taken to be in material 2, and thus the material constants involved in $\mathbf{f}_0(z)$ are for material 2. This problem was posed and solved by Tucker (1969), and studied recently by Tewary *et al.* (1989). Adapted below is a derivation consistent with the present notation. The results will be used in §7 to examine singularity-interface-crack interactions.

Write the solution for the two blocks formally as

$$\mathbf{f}(z) = \begin{cases} \mathbf{f}^1(z), & z \in 1, \\ \mathbf{f}^2(z) + \mathbf{f}_0(z), & z \in 2, \end{cases} \quad (4.1)$$

where z is a complex-variable of the form $z = x + \mu y (\text{Im } \mu > 0)$. Here and

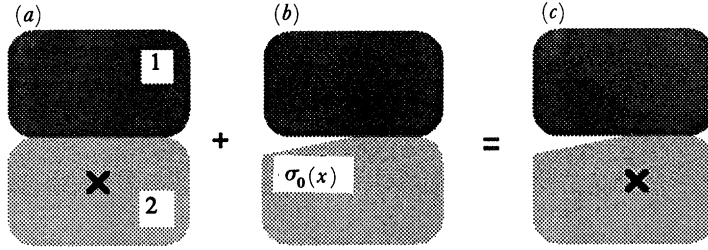


FIGURE 1. (a) A singularity embedded in material 2. (b) An interface crack with traction prescribed on the faces. (c) Interaction between a singularity and a traction-free crack.

throughout the paper, $z \in 1$ means $y > 0$, and $z \in 2$ means $y < 0$. The task is to solve for $f^1(z)$ and $f^2(z)$, analytic in upper and lower half planes, respectively, in terms of $f_0(z)$. From (2.19), the continuity of forces across the interface requires that

$$L_1 f^1(x) + \bar{L}_1 \bar{f}^1(x) = L_2 [f^2(x) + f_0(x)] + \bar{L}_2 [\bar{f}^2(x) + \bar{f}_0(x)]. \quad (4.2)$$

Rearranging the above, one obtains

$$L_1 f^1(x) - \bar{L}_2 \bar{f}^2(x) - L_2 f_0(x) = L_2 f^2(x) - \bar{L}_1 \bar{f}^1(x) - \bar{L}_2 \bar{f}_0(x). \quad (4.3)$$

Equation (4.3) holds along the whole x -axis. Moreover, the functions at the left-hand side are analytic in the upper half plane, whereas those on the right-hand side are analytic in the lower half plane. By standard analytic continuation arguments one reaches

$$L_1 f^1(z) - \bar{L}_2 \bar{f}^2(z) - L_2 f_0(z) = 0, \quad z \in 1. \quad (4.4)$$

Continuity of the displacements across the interface, with the same arguments, gives

$$A_1 f^1(z) - \bar{A}_2 \bar{f}^2(z) - A_2 f_0(z) = 0, \quad z \in 1. \quad (4.5)$$

Solving from (4.3) and (4.4) for $f^1(z)$ and $f^2(z)$, one finds

$$\left. \begin{aligned} f^1(z) &= L_1^{-1} H^{-1} (\bar{B}_2 + B_2) L_2 f_0(z), & z \in 1, \\ f^2(z) &= L_2^{-1} \bar{H}^{-1} (\bar{B}_2 - \bar{B}_1) \bar{L}_2 \bar{f}_0(z), & z \in 2, \end{aligned} \right\} \quad (4.6)$$

Substitution into (4.1) gives the complete solution. When calculating the field quantities via (2.2), one has to replace z by $z_j = x + \mu_j y$ respectively for each component of $f(z)$ in (4.1). Notice that this relation to construct a bimaterial solution from a one-material solution is universal in that no specific information about the singularity is needed.

A singularity in a half-space interacting with a traction-free surface, $y = 0$, can be constructed similarly on the basis of the infinite plane solution, $f_0(z)$. The result is

$$f(z) = f_0(z) - L^{-1} \bar{L} \bar{f}_0(z). \quad (4.7)$$

Another interesting case is a singularity in a half plane interacting with a rigidly held surface on $y = 0$. The solution is

$$f(z) = f_0(z) - A^{-1} \bar{A} \bar{f}_0(z). \quad (4.8)$$

The general solution developed above will be used in §7 to study the interaction between singularities and interfacial cracks. Dislocation solution in a half space was used by Suo (1990*b*) as kernels in an integral equation formulation of delamination in composites. Here we include several more elementary examples to further familiarize ourselves with notation.

Let a dislocation sit on a bimaterial interface at the origin, with Burgers vector \mathbf{b} . The corresponding solution for a homogeneous body, $\mathbf{f}_0(z)$, is given by (3.4), with the elastic constants for medium 2. The solution for bimaterial system is thereby

$$L_1 \mathbf{f}_1(z) = \bar{L}_2 \bar{\mathbf{f}}_2(z) = (2\pi)^{-1} H^{-1} \mathbf{b} \ln z. \quad (4.9)$$

A direct calculation using (2.19) gives the traction along the interface

$$\mathbf{t}(x) = \frac{1}{2\pi} \left(\frac{H^{-1}}{x + 0i} + \frac{\bar{H}^{-1}}{x - 0i} \right) \mathbf{b}. \quad (4.10)$$

The strain energy for a dislocation on the interface is then

$$U \equiv \frac{1}{2} \mathbf{b}^T \int_r^R \mathbf{t}(x) dx = \frac{1}{2\pi} \mathbf{b}^T H^{-1} \mathbf{b} \ln \frac{R}{r}. \quad (4.11)$$

Equation (4.10) was derived by Willis (1971) with Fourier transform, and used to formulate an interfacial crack problem. Different expressions for the strain energy of a dislocation sitting on the interface analogous to (4.11) can be found in the literature. The present expression seems to agree with a somewhat messy one presented in Bacon *et al.* (1979, p. 233, see also references cited therein).

5. INTERFACE CRACK: NON-OSCILLATORY FIELDS

Ting (1986) showed that an interface crack tip is free of oscillation if H is real (his notation is different from that used here). This turns out to be a necessary and sufficient condition (Qu & Bassani 1989). The structure of the crack-tip fields and the solution of several crack problems under this condition are strikingly simple.

We begin with an asymptotic problem. Consider a semi-infinite, traction-free crack lying along the interface between two anisotropic half spaces, with material 1 above, and material 2 below. The two half-spaces are bonded by the half-plane $y = 0$ and $x > 0$, and the crack is on the other half plane. Two-dimensional problems are considered in which quantities are invariant along the direction normal to the xy -plane, so that one can phrase everything by a two-dimensional picture. No specific length and load are present in this problem. Singular fields are sought to satisfy continuity of traction and displacement vectors $\mathbf{t}(x)$ and $\mathbf{u}(x)$ (defined in (2.19) for each material) across the bonded portion of the interface, as well as the traction-free condition on the cracked portion. This is a homogeneous boundary-value problem, or an eigenvalue problem.

Let the vector potentials defined in (2.18) in the two blocks be $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ respectively. Obviously the traction $\mathbf{t}(x)$ defined in (2.19) is continuous across the whole x -axis, both the bonded and cracked portions, so that

$$L_1 \mathbf{f}'_1(x) + \bar{L}_1 \bar{\mathbf{f}}'_1(x) = L_2 \mathbf{f}'_2(x) + \bar{L}_2 \bar{\mathbf{f}}'_2(x). \quad (5.1)$$

To facilitate the analytic continuation, (5.1) is rearranged as

$$L_1 \mathbf{f}'_1(x) - \bar{L}_2 \bar{\mathbf{f}}'_2(x) = L_2 \mathbf{f}'_2(x) - \bar{L}_1 \bar{\mathbf{f}}'_1(x). \quad (5.2)$$

By the standard analytic continuity argument it follows that

$$L_1 \mathbf{f}'_1(z) = \bar{L}_2 \bar{\mathbf{f}}'_2(z), \quad z \in 1. \quad (5.3)$$

Define the displacement jump across the interface as

$$\mathbf{d}(x) = \mathbf{u}(x, 0^+) - \mathbf{u}(x, 0^-). \quad (5.4)$$

With the aid of (5.3), a direct calculation gives

$$\mathbf{t}(x) = L_1 \mathbf{f}'_1(x) + L_2 \mathbf{f}'_2(x), \quad (5.5)$$

and

$$\mathbf{i}\mathbf{d}'(x) = H L_1 \mathbf{f}'_1(x) - \bar{H} L_2 \mathbf{f}'_2(x). \quad (5.6)$$

A very simple solution can be obtained if the hermitian matrix H is real, that is

$$H = \bar{H}. \quad (5.7)$$

The whole of the remainder of this section will assume (5.7), and the case that H is complex will be treated in the next section.

Continuity of the displacement across the bonded interface ($\mathbf{d} = 0$), as inferred from (5.6), implies that $L_1 \mathbf{f}'_1(z)$ and $L_2 \mathbf{f}'_2(z)$ can be analytically extended to the whole plane except on the crack line and satisfy

$$\mathbf{h}(z) = L_1 \mathbf{f}'_1(z) = L_2 \mathbf{f}'_2(z), \quad z \notin C, \quad (5.8)$$

where $\mathbf{h}(z)$ is introduced for convenience. Here and later, C is denoted as the crack line (or union of crack lines if several cracks are considered). The traction-free condition on the crack, using (5.5) and (5.8), leads to a homogeneous Hilbert problem

$$\mathbf{h}^+(x) + \mathbf{h}^-(x) = 0, \quad z \in C. \quad (5.9)$$

An obviously admissible singular solution to (5.9) is

$$\mathbf{h}(z) = \frac{1}{2}(2\pi z)^{-\frac{1}{2}} \mathbf{k}, \quad (5.10)$$

where the branch cut for \sqrt{z} is along the crack line. The undetermined constant vector \mathbf{k} appears to consist of three complex constants. However, upon requiring traction to be real, one concludes that \mathbf{k} is a real vector. The normalization adopted in (5.10) is consistent with the conventional definition of stress intensity factors, with the identification

$$\mathbf{k} = [K_{II}, K_I, K_{III}]^T \quad (5.11)$$

as will be clear in (5.13) below.

The complete asymptotic solution is then given by

$$L_1 \mathbf{f}'_1(z) = L_2 \mathbf{f}'_2(z) = \mathbf{h}(z) = \frac{1}{2}(2\pi z)^{-\frac{1}{2}} \mathbf{k}. \quad (5.12)$$

Assuming L_1 and L_2 are non-singular, one can readily obtain the elastic potentials for the two half spaces. The stresses and displacements can be calculated using the basic representation (2.2), with z properly reinterpreted of course. Examining

(5.12) one immediately discovers that the crack-tip fields (stresses, displacements) in each block do not depend on the elastic constants of the other block. In other words, the near-tip fields in each block are identical to those of a crack in the corresponding homogeneous anisotropic medium. The latter fields have been completely tabulated in Sih *et al.* (1965) and Hoenig (1982).

The traction in the interface a distance r ahead of the crack tip, and the displacement jump a distance r behind of the crack tip, calculated from (5.5) and (5.6) respectively, are given by

$$\mathbf{t}(r) = (2\pi r)^{-\frac{1}{2}}\mathbf{k}, \quad \mathbf{d}(r) = (2r/\pi)^{\frac{1}{2}}H\mathbf{k}. \quad (5.13)$$

Each one of the above may be taken as a defining equation for the stress intensity factors.

Irwin (1957) realized that the energy released for a unit area of interface to decohere can be written as

$$G \equiv \frac{1}{2A} \int_0^A \mathbf{t}^T(A-r) \mathbf{d}(r) dr, \quad (5.14)$$

where A is an arbitrary length scale. Incorporating (5.13) one reaches an Irwin-type expression connecting G and \mathbf{k} :

$$G = \frac{1}{4}\mathbf{k}^T H \mathbf{k}. \quad (5.15)$$

In obtaining (5.15) a special value of the beta function has been used

$$\int_0^1 \left(\frac{t}{1-t} \right)^q dt = \frac{q\pi}{\sin q\pi}, \quad (|\operatorname{Re} q| < 1) \quad (5.16)$$

with $q = \frac{1}{2}$. This identity will be invoked again in the next section.

Comparison of the strain energy of a dislocation on the interface (4.12) and the energy release rate at a crack tip (5.15) leads to a simple relation. Both are quadratic forms, and the matrices involved are inverse to each other. This relation, for the special case of a crack in a homogeneous medium, was presented by Barnett & Asaro (1972). The problem in the present context was formulated earlier by Bassani & Qu (1988) in a different approach.

Having obtained a complete description of the asymptotic fields, we turn our attention to a class of crack problems. Consider a set of collinear cracks along the interface between two dissimilar anisotropic half planes, with self-equilibrated traction $\mathbf{t}_0(x)$ prescribed on the crack faces. Suppose there are n finite cracks in the intervals (a_j, b_j) , $j = 1, 2, \dots, n$ and two semi-infinite cracks in the intervals $(-\infty, b_0)$ and $(a_0, +\infty)$. The Hilbert problem (5.9) is replaced by a non-homogeneous one

$$\mathbf{h}^+(x) + \mathbf{h}^-(x) = \mathbf{t}_0(x), \quad x \in C, \quad (5.17)$$

where C is the union of all cracks. Equation (5.17) does not have a unique solution. Several auxiliary conditions needed are: $\mathbf{h}(z)$ approaches zero faster than $1/z$ as $|z|$ goes to infinity; $\mathbf{h}(z)$ has a square-root singularity as in (5.10) at the crack tips; and moreover, the net Burgers vector for each of the n finite cracks is zero. From (5.6), this latter statement leads to

$$\int_{a_j}^{b_j} [\mathbf{h}^+(x) - \mathbf{h}^-(x)] dx = 0, \quad j = 1, 2, \dots, n. \quad (5.18)$$

Notice that the governing equations and the auxiliary conditions for the vector function $\mathbf{h}(z)$ are independent of any elastic constants, and exactly the same as those for the mode III potentials in an isotropic one-material. A straightforward method thus emerges to construct the complete solutions for the above interface crack problems, without much work, from the well-known mode III solutions for isotropic one-materials: each component of the vector function $\mathbf{h}(z)$ is the same (except for a factor of $-\frac{1}{2}$ due to different conventions) as the mode III potential for collinear cracks in a homogeneous isotropic material, and the latter can be found in Rice (1968); knowing $\mathbf{h}(z)$, one can obtain the complete solutions for the two blocks, $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$, from (5.8). In particular, the stress intensity factors, normalized as (5.10) or (5.13), are identical to the classical results for the same crack configurations in an isotropic one-material. Listed below are the general solutions so constructed for the collinear interfacial cracks:

$$\mathbf{h}(z) = \frac{\chi_0(z)}{2\pi i} \int_C \frac{\mathbf{t}_0(x) dx}{\chi_0^+(x)(x-z)} + \mathbf{P}(z) \chi_0(z), \quad (5.19)$$

where

$$\chi_0(z) = \prod_{j=0}^n (z-a_j)^{-\frac{1}{2}}(z-b_j)^{-\frac{1}{2}}. \quad (5.20)$$

Here the branch cuts are chosen along the crack lines so that the product for each finite crack behaves as $1/z$ for large z . And $\mathbf{P}(z)$ is a vector involving three polynomials, which should be chosen to satisfy the auxiliary conditions. The interfacial crack solutions for the two important configurations depicted in figure 2 are given below.

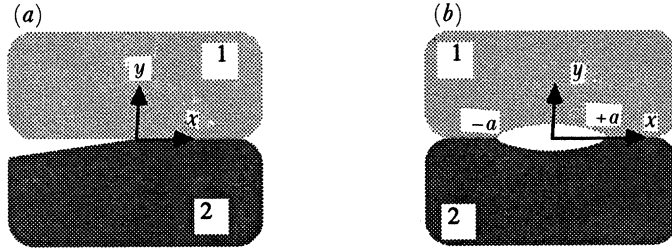


FIGURE 2. Two crack configurations.

(a) *Semi-infinite crack*

$$\chi_0(z) = z^{-\frac{1}{2}}, \quad \mathbf{P}(z) = 0. \quad (5.21)$$

The solutions for the two blocks are then given by

$$L_1 \mathbf{f}'_1(z) = L_2 \mathbf{f}'_2(z) = \frac{z^{-\frac{1}{2}}}{2\pi} \int_{-\infty}^0 \frac{(-x)^{\frac{1}{2}} \mathbf{t}_0(x)}{x-z} dx. \quad (5.22)$$

Comparing (5.21) with the asymptotic field (5.10), one finds the stress intensity factors

$$\mathbf{k} = -(2/\pi)^{\frac{1}{2}} \int_{-\infty}^0 (-x)^{-\frac{1}{2}} \mathbf{t}_0(x) dx. \quad (5.23)$$

(b) *Finite crack in the interval $(-a, a)$*

$$\chi_0(z) = (z^2 - a^2)^{-\frac{1}{2}}, \quad \mathbf{P}(z) = 0 \quad (5.24)$$

The full solution is given by

$$L_1 \mathbf{f}'_1(z) = L_2 \mathbf{f}'_2(z) = \frac{(z^2 - a^2)^{-\frac{1}{2}}}{2\pi} \int_{-a}^a \frac{(a^2 - x^2)^{\frac{1}{2}}}{x - z} \mathbf{t}_0(x) dx. \quad (5.25)$$

The stress intensity factors are

$$\mathbf{k} = -(\pi a)^{-\frac{1}{2}} \int_{-a}^a \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} \mathbf{t}_0(x) dx. \quad (5.26)$$

The solution to the second problem listed above was obtained earlier by Bassani & Qu (1989) with an integral transform.

6. INTERFACE CRACK: OSCILLATORY FIELDS

In this section we examine the general case when H is not real. The asymptotic fields associated with a traction-free, semi-infinite interface crack will be considered first. It is shown that the mathematical structure of the crack-tip behaviour can be decomposed into two types of singular fields. One shows a mixture of singularities $r^{-\frac{1}{2} \pm i\epsilon}$ in stresses, and the other has a $r^{-\frac{1}{2}}$ singularity in stresses. The two singularities are scaled by a complex and a real stress intensity factors, K and K_3 respectively. In standard notation, ϵ , referred to as the oscillatory index, is a non-dimensional real number measuring an aspect of elastic dissimilarity of the two materials. The collinear crack problems are also worked out for complex H , which provide a collection of stress intensity factors.

The same collinear crack problems have been solved by several authors (Gotoh 1967; Clements 1971; Willis 1971; Tewary 1989), and can be dated back even earlier if one accepts some of the contact problems in the book by Galin (1961) as limiting cases of interface crack problems (rigid-elastic interfaces). Yet these earlier authors seemed to have missed the inherent simplicity of the solutions, and thus they have not been able to interpret their results in the spirit of fracture mechanics. In particular, the 'stress concentration factor' used in these earlier works is a complex column vector with six real quantities. We will formulate a derivation that allows one to grasp the simple mathematical structure and read off the stress intensity factors trivially. The method used here, however, is actually not new. Mathematically, it is a variant of those contained in the above references, and was treated thoroughly in general terms by Muskhelishvili (1953*b*).

The results in §5 up to the introduction of $H = \bar{H}$ in (5.7) are still valid. Continuity of displacement across the bonded interface requires, from equation (5.6), the existence of a function $\mathbf{h}(z)$ analytic in the whole plane except on the crack lines, such that

$$\mathbf{h}(z) = L_1 \mathbf{f}'_1(z) = H^{-1} \bar{H} L_2 \mathbf{f}'_2(z), \quad z \notin C. \quad (6.1)$$

Hence one can focus on $\mathbf{h}(z)$, and once $\mathbf{h}(z)$ is obtained the full-field solution is given

by (6.1). In terms of $\mathbf{h}(z)$, the traction (5.5) and displacement jump (5.6) can be expressed as

$$\mathbf{t}(x) = \mathbf{h}^+(x) + \bar{H}^{-1} H \mathbf{h}^-(x), \quad (6.2)$$

and

$$\mathbf{id}'(x) = H[\mathbf{h}^+(x) - \mathbf{h}^-(x)]. \quad (6.3)$$

Consider the asymptotic problem first. With (6.2), the traction-free condition gives

$$\mathbf{h}^+(x) + \bar{H}^{-1} H \mathbf{h}^-(x) = 0, \quad z \in C. \quad (6.4)$$

This is a homogeneous Hilbert problem. Let a solution to (6.4) be of the form

$$\mathbf{h}(z) = \mathbf{w} z^{-\frac{1}{2} + i\epsilon}, \quad (6.5)$$

where \mathbf{w} is a constant vector and ϵ a constant number, both to be determined. The branch cut for the multivalued function in (6.5) is chosen to be along the crack line $x < 0$, and the phase angle of z is measured from the positive x -axis. Substituting (6.5) into (6.4), one obtains an algebraic eigenvalue problem

$$\bar{H} \mathbf{w} = e^{2\pi\epsilon} H \mathbf{w}. \quad (6.6)$$

It turns out that the specific structure of eigenpairs of this problem is the key to understanding the results below. We insert the following digression to explain this structure.

The following properties hold for a positive definite hermitian matrix H : the eigenvalue $e^{2\pi\epsilon}$ is positive, and thus ϵ is real; if (ϵ, \mathbf{w}) is an eigenpair so is $(-\epsilon, \bar{\mathbf{w}})$, and consequently, because there are only three eigenvalues, $\epsilon = 0$ is an eigenvalue and the associated eigenvector can be chosen to be real. Therefore, a real number ϵ ($\neq 0$ iff $H \neq \bar{H}$), a complex vector \mathbf{w} , and a real vector \mathbf{w}_3 can be found to form three distinct eigenpairs

$$(\epsilon, \mathbf{w}), \quad (-\epsilon, \bar{\mathbf{w}}), \quad (0, \mathbf{w}_3), \quad (6.7)$$

satisfying

$$\bar{H} \mathbf{w} = e^{2\pi\epsilon} H \mathbf{w}, \quad \bar{H} \mathbf{w}_3 = H \mathbf{w}_3. \quad (6.8)$$

These eigenvectors are orthogonal in the sense

$$\mathbf{w}^T H \mathbf{w} = \mathbf{w}^T H \mathbf{w}_3 = \bar{\mathbf{w}}^T H \mathbf{w}_3 = 0. \quad (6.9)$$

Orthogonality conditions involving \bar{H} can be obtained by taking the complex conjugation of the above.

The two vectors \mathbf{w} and \mathbf{w}_3 are fully determined by the eigenvalue problem up to a complex and a real normalizing constants respectively. In general they should be normalized to be dimensionless to give conventional dimensions for stress intensity factors (see (6.16) below). We choose to leave the normalization of these vectors otherwise unspecified at this point, because the results presented below are unaffected by the specific normalization.

Every complex-valued vector \mathbf{g} can be represented as a linear combination of the three eigenvectors, that is

$$\mathbf{g} = g_1 \mathbf{w} + g_2 \bar{\mathbf{w}} + g_3 \mathbf{w}_3, \quad (6.10)$$

where the complex numbers g_i can be thought of as the components of vector \mathbf{g} , and evaluated by taking inner products, i.e.

$$g_1 = \frac{\bar{\mathbf{w}}^T H \mathbf{g}}{\bar{\mathbf{w}}^T H \mathbf{w}}, \quad g_2 = \frac{\mathbf{w}^T H \mathbf{g}}{\mathbf{w}^T H \bar{\mathbf{w}}}, \quad g_3 = \frac{\mathbf{w}_3^T H \mathbf{g}}{\mathbf{w}_3^T H \mathbf{w}_3}. \quad (6.11)$$

When \mathbf{g} is real-valued (for example, \mathbf{g} is the traction in the interface or the prescribed traction along the crack faces), one may confirm that $g_2 = \bar{g}_1$, and g_3 is a real number, which is obvious from (6.10).

Now we return to the main problem. The admissible singular solution to (6.4) is then a linear combination of the three homogeneous solutions of form (6.5):

$$\mathbf{h}(z) = z^{-\frac{1}{2}}[a\mathbf{w}z^{i\epsilon} + b\bar{\mathbf{w}}z^{-i\epsilon} + c\mathbf{w}_3], \quad (6.12)$$

where a, b and c are three undetermined complex numbers. Substituting (6.12) into (6.2) and requiring traction to be real along the interface, one concludes

$$a = e^{2\pi\epsilon} \bar{b}, c = \text{real}, \quad (6.13)$$

thus, only one complex constant and one real constant are independent, chosen to be K and K_3 respectively, such that

$$\mathbf{h}(z) = \frac{e^{\pi\epsilon} K z^{i\epsilon} \mathbf{w} + e^{-\pi\epsilon} \bar{K} z^{-i\epsilon} \bar{\mathbf{w}}}{2(2\pi z)^{\frac{1}{2}} \cosh \pi\epsilon} + \frac{K_3 \mathbf{w}_3}{2(2\pi z)^{\frac{1}{2}}}. \quad (6.14)$$

Other constants in (6.14) are embedded in a manner similar to the isotropic bimaterial crack tip fields (see, for example, Rice 1988). The potentials for the two half spaces are thereby

$$\left. \begin{aligned} L_1 \mathbf{f}'_1(z) &= \frac{e^{\pi\epsilon} K z^{i\epsilon} \mathbf{w} + e^{-\pi\epsilon} \bar{K} z^{-i\epsilon} \bar{\mathbf{w}}}{2(2\pi z)^{\frac{1}{2}} \cosh \pi\epsilon} + \frac{K_3 \mathbf{w}_3}{2(2\pi z)^{\frac{1}{2}}}, \\ L_2 \mathbf{f}'_2(z) &= \frac{e^{-\pi\epsilon} K z^{i\epsilon} \mathbf{w} + e^{+\pi\epsilon} \bar{K} z^{-i\epsilon} \bar{\mathbf{w}}}{2(2\pi z)^{\frac{1}{2}} \cosh \pi\epsilon} + \frac{K_3 \mathbf{w}_3}{2(2\pi z)^{\frac{1}{2}}}. \end{aligned} \right\} \quad (6.15)$$

It is interesting to note that the structure of the singular fields is the same for the two half spaces, except for a change of π to $-\pi$ everywhere.

Substitution of (6.14) into (6.2) gives the traction in the bonded interface a distance r ahead of the crack tip

$$\mathbf{t}(r) = (2\pi r)^{-\frac{1}{2}}[K r^{i\epsilon} \mathbf{w} + \bar{K} r^{-i\epsilon} \bar{\mathbf{w}} + K_3 \mathbf{w}_3]. \quad (6.16)$$

It reads that the interface traction at each fixed point r can be decomposed into two components: one is along \mathbf{w}_3 and the other is in the plane spanned by $\text{Re}[\mathbf{w}]$ and $\text{Im}[\mathbf{w}]$. The components, in the sense of (6.11), are

$$t_1(r) \equiv \frac{\bar{\mathbf{w}}^T H \mathbf{t}(r)}{\bar{\mathbf{w}}^T H \mathbf{w}} = \frac{K r^{i\epsilon}}{(2\pi r)^{\frac{1}{2}}}, \quad t_3(r) \equiv \frac{\mathbf{w}_3^T H \mathbf{t}(r)}{\mathbf{w}_3^T H \mathbf{w}_3} = \frac{K_3}{(2\pi r)^{\frac{1}{2}}}, \quad (6.17)$$

where t_3 is the \mathbf{w}_3 component and t_1 is the (complex) planar component. These equations may be taken as defining equations for the complex K and real K_3 . As r approaches the tip, the \mathbf{w}_3 component has a square root singularity and the planar component is oscillatory, with K_3 and K measuring their intensities,

respectively. The results are clearly the analogue of the corresponding ones for isotropic bimaterial. For an anisotropic bimaterial, $t_1(r) \equiv \sigma_{yy} + i\sigma_{xy}$ is the in-plane traction, and $t_3(r) \equiv \sigma_{yz}$ is the anti-plane traction.

The displacement jump a distance r behind the tip is

$$d(r) = (H + \bar{H}) \left(\frac{r}{2\pi} \right)^{\frac{1}{2}} \left[\frac{K r^{1\epsilon} \mathbf{w}}{(1 + 2i\epsilon) \cosh \pi\epsilon} + \frac{\bar{K} r^{-1\epsilon} \bar{\mathbf{w}}}{(1 - 2i\epsilon) \cosh \pi\epsilon} + K_3 \mathbf{w}_3 \right]. \quad (6.18)$$

Due to anisotropy the matrix $(H + \bar{H})$ may rotate the base vectors in (6.18), implying that non-oscillatory direction of the displacement jump may not coincide with that of the interface traction, and similarly for the oscillatory planes.

The energy release rate defined in (5.14) is

$$G = \bar{\mathbf{w}}^T (H + \bar{H}) \mathbf{w} |K|^2 / (4 \cosh^2 \pi\epsilon) + \frac{1}{8} \mathbf{w}_3^T (H + \bar{H}) \mathbf{w}_3 K_3^2 \quad (6.19)$$

In deriving this the integral identity (5.16) has been used with $q = \frac{1}{2}, \frac{1}{2} \pm i\epsilon$. An analogous expression was presented in Willis (1971) connecting G and the stress concentration factors defined in his paper.

The structure of the near-tip fields around an interface crack has been identified, with only one real and one complex normalizing factors K_3 and K . In principle, for a given boundary value problem, these factors should be determined by the external geometry and load, and can be used the similar way as the conventional stress intensity factors in Irwin's fracture mechanics.

Remember that a normalization to the two eigenvectors has not been assigned yet. The different choices of normalization affect the definition of stress intensity factors by a real factor to K_3 and a complex factor to K . As clearly indicated by an example in §9, it is impossible to find a specific normalization such that the stress intensity factors so defined reduce to the classical stress intensity factors for a crack tip in a homogeneous, anisotropic medium as the two materials become the same. A tentative normalization is proposed as follows, which recovers the stress intensity factor definition for isotropic bimaterials

$$\mathbf{w} = [-\frac{1}{2}i, *, *]^T, \quad \mathbf{w}_3 = [*, *, 1]^T, \quad (6.20)$$

where $(*)$ signifies numbers determined by the eigenvalue problem (6.6). Note this normalization may not always be valid. For example, when \mathbf{w}_3 is an in-plane vector containing no anti-plane components, (6.20) is invalid.

Now consider the collinear crack problems specified in §5 for the complex H . From (6.2), the prescribed traction $\mathbf{t}_0(x)$ on the crack lines C results in the Hilbert problem

$$\mathbf{h}^+(x) + \bar{H}^{-1} H \mathbf{h}^-(x) = \mathbf{t}_0(x), \quad x \in C. \quad (6.21)$$

Writing the above equation in its components, or equivalently, taking the inner product of (6.21) with $\bar{\mathbf{w}}^T H$, $\mathbf{w}^T H$ and $\mathbf{w}_3^T H$, one obtains

$$\left. \begin{aligned} h_1^+(x) + e^{-2\pi\epsilon} h_1^-(x) &= t_{01}(x), \\ h_2^+(x) + e^{+2\pi\epsilon} h_2^-(x) &= \bar{t}_{01}(x), \\ h_3^+(x) + h_3^-(x) &= t_{03}(x), \end{aligned} \right\} \quad x \in C. \quad (6.22)$$

The components are defined in the sense of (6.10) and (6.11). Note that these equations are decoupled. Furthermore, because they contain no explicit material dependence besides ϵ , one may conjecture that they should be identical to those for isotropic bimetals. Indeed they are (cf. England 1965; Erdogan 1965; Rice & Sih 1965). Constructed from the known solutions for isotropic bimetals, the complete solution is

$$\mathbf{h}(z) = h_1(z) \mathbf{w} + h_2(z) \bar{\mathbf{w}} + h_3(z) \mathbf{w}_3, \quad (6.23)$$

where

$$\left. \begin{aligned} h_1(z) &= \frac{\chi(z)}{2\pi i} \int_C \frac{t_{01}(x) dx}{\chi^+(x)(x-z)}, \\ h_2(z) &= \frac{\bar{\chi}(z)}{2\pi i} \int_C \frac{\bar{t}_{01}(x) dx}{\bar{\chi}^+(x)(x-z)}, \\ h_3(z) &= \frac{\chi_0(z)}{2\pi i} \int_C \frac{t_{03}(x) dx}{\chi_0^+(x)(x-z)}. \end{aligned} \right\} \quad (6.24)$$

In the above $\chi_0(z)$ is the same as that of (5.20) and $\chi(z)$ is the standard function used in isotropic bimaterial interface crack problems, and is defined as

$$\chi(z) = \prod_{j=0}^n (z-a_j)^{-\frac{1}{2}-i\epsilon} (z-b_j)^{-\frac{1}{2}+i\epsilon}. \quad (6.25)$$

Knowing $\mathbf{h}(z)$, one can obtain the full field solution via (6.1).

The stress intensity factors can be easily extracted by comparison with the asymptotic solution. As a matter of fact, by construction the answer should have the same structure as their isotropic bimaterial counterparts. For example, the stress intensity factors for a semi-infinite crack are

$$\left. \begin{aligned} K_3 &= -(2/\pi)^{\frac{1}{2}} \int_{-\infty}^0 (-x)^{-\frac{1}{2}} t_{03}(x) dx, \\ K &= -(2/\pi)^{\frac{1}{2}} \cosh \pi\epsilon \int_{-\infty}^0 (-x)^{-\frac{1}{2}-i\epsilon} t_{01}(x) dx, \end{aligned} \right\} \quad (6.26)$$

and for an internal crack the stress intensity factors are

$$\left. \begin{aligned} K_3 &= -(\pi a)^{-\frac{1}{2}} \int_{-a}^a \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} t_{03}(x) dx, \\ K &= -(2/\pi)^{\frac{1}{2}} \cosh \pi\epsilon (2a)^{-\frac{1}{2}-i\epsilon} \int_{-a}^a \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}+i\epsilon} t_{01}(x) dx, \end{aligned} \right\} \quad (6.27)$$

where the complex t_{01} and real t_{03} are the components of the applied traction \mathbf{t}_0 in the sense of (6.11). Without actually re-solving the problems, one can easily write down any other solutions for anisotropic bimetals, providing one knows the solutions for an isotropic bimaterial. We emphasize that such a constructive rule only applies to the collinear crack problems specified in this article.

Notice that in the above the auxiliary conditions discussed in §5, as well as an arbitrary polynomial in each equation of (6.24), have been omitted. One may

confirm that the auxiliary conditions are identical to those for isotropic bimaterials.

7. SINGULARITY-INTERFACE-CRACK INTERACTION

Now the interaction problem illustrated in figure 1c can be readily solved by the superposition scheme illustrated in figure 1. The only relatively non-trivial part is the integral involved in the process. We will not pursue the interaction problem for the general singularity, as has been done for isotropic materials in Suo (1990a). Instead, we will concentrate on a special problem to illustrate the process. The technique, though, is generally applicable.

Consider, for example, a dislocation or a line force, embedded in material 2, interacting with the traction-free semi-infinite interface crack. The non-oscillatory condition $H = \bar{H}$ is assumed. For the problem in figure 1a, from (4.7) the traction in the interface is

$$\mathbf{t}(x) = C\mathbf{f}'_0(x) + \bar{C}\bar{\mathbf{f}}'_0(x), \quad (7.1)$$

where $\mathbf{f}_0(z)$ is of the form (3.1), and C is an abbreviation of

$$C = H^{-1}(B_2 + \bar{B}_2)L_2. \quad (7.2)$$

The negative of this traction is prescribed on the crack faces in figure 1b. The solution for this problem has been examined in §5. The key is to evaluate the integral (5.19). In the present context, it is

$$\mathbf{h}(z) = -\frac{\chi_0(z)}{2\pi i} \int_{-\infty}^0 \frac{C\mathbf{f}'_0(x) + \bar{C}\bar{\mathbf{f}}'_0(x)}{\chi_0^+(x)(x-z)} dx, \quad (7.3)$$

where $\chi_0(z) = 1/\sqrt{z}$, with the branch cut along the crack. The integral can be evaluated by a contour integral (for details see Suo 1989a). The final result is

$$\mathbf{h}(z) = -\frac{1}{2}\{C\mathbf{f}'_0(z) + \bar{C}\bar{\mathbf{f}}'_0(z) - z^{-\frac{1}{2}}CD\mathbf{f}'_0(z) - z^{-\frac{1}{2}}\bar{C}\bar{D}\bar{\mathbf{f}}'_0(z)\} \Bigg\} \quad (7.4)$$

$$D = \text{diag}[\sqrt{s_1}, \sqrt{s_2}, \sqrt{s_3}],$$

where s_j are defined in (3.1) and $\text{diag}[\]$ denotes a diagonal matrix. The complete solution can be obtained using (5.8). Comparison of (7.4) and (5.10) gives the stress intensity factors induced by a dislocation or line force

$$\mathbf{k} = -2(2\pi)^{\frac{1}{2}} \text{Re} \{H^{-1}(B_2 + \bar{B}_2)L_2[d_1 s_1^{-\frac{1}{2}}, d_2 s_2^{-\frac{1}{2}}, d_3 s_3^{-\frac{1}{2}}]^T\}, \quad (7.5)$$

where d_j and s_j are defined in (3.1) with elastic constants for material 2.

The dislocation interacting with a crack was treated in a homogeneous medium by Atkinson (1966). The Green function for an internal interface crack was obtained recently by Tewary *et al.* (1989). The basic solution developed here has been used as kernels in an integral equation formulation of cracks kinking out of interfaces in bicrystals (Wang & Shih 1990).

8. ANTIPLANE-FIELD

Consider materials with xy -plane as a mirror plane, in which the in-plane and antiplane deformations are decoupled. They will be treated separately in this and the next section.

In equation (2.3) $l_3(\mu)$ is identically zero for a material with such a symmetry. The characteristic equation for antiplane deformation thus becomes

$$l_2(\mu) \equiv s_{55}\mu^2 - 2s_{45}\mu + s_{44} = 0. \quad (8.1)$$

The expression $s_{44}s_{55} - (s_{45})^2$, a principal minor of the compliance matrix, is positive. Hence there are two complex conjugate roots to (8.1). According to the convention the root with positive imaginary part is chosen, i.e.

$$\mu_3 = (s_{45} + i(s_{44}s_{55} - s_{45}^2)^{\frac{1}{2}})/s_{55}. \quad (8.2)$$

Only one holomorphic function $f(z_3)$ is needed to represent antiplane deformations, with $z_3 = x + \mu_3 y$. The LES representation (2.2) reduces to

$$\left. \begin{aligned} u_3 &= 2 \operatorname{Re} [Af(z_3)], & T_3 &= -2 \operatorname{Re} [Lf(z_3)], \\ \sigma_{23} &= 2 \operatorname{Re} [Lf'(z_3)], & \sigma_{13} &= -2 \operatorname{Re} [L\mu_3 f'(z_3)]. \end{aligned} \right\} \quad (8.3)$$

Now all 3×3 matrices defined in §2 reduce to scalars. Keeping the same notation, one has

$$L = -1, \quad A = iB, \quad B = (s_{44}s_{55} - s_{45}^2)^{\frac{1}{2}}. \quad (8.4)$$

Clearly, B can be interpreted as the inverse of an equivalent shear modulus, which reduces to the standard shear modulus as the material degenerates to be transversely cubic, or tetragonal. Subscripts 1 and 2 will be attached to B to signify the two materials. Consistent with Ting's (1982) general results, B is an invariant under an in-plane rotation. The bimaterial matrix, now a scalar, $H = B_1 + B_2$, is identically real! Consequently, for the decoupled deformations, the mode III near-tip field of an interfacial crack is non-oscillatory.

The solution for a screw dislocation with Burgers vector b , and a line shear force with density p at point (x_0, y_0) in an infinite homogeneous medium is

$$\left. \begin{aligned} f(z) &= q \ln(z - s_3), & s_3 &= x_0 + \mu_3 y_0, \\ q &= -(4\pi B)^{-1}b + i(4\pi)^{-1}p. \end{aligned} \right\} \quad (8.5)$$

The stresses and displacements can be calculated using (8.3) and (8.5) with $z = z_3$.

The potential for a singularity in two bonded blocks, embedded in material 2, say, can be constructed by the potential, $f_0(z)$, for the same singularity in an infinite homogeneous medium as

$$f(z) = \left\{ \begin{aligned} &2B_2/(B_1 + B_2)f_0(z), & z \in 1, \\ &f_0(z) + (B_2 - B_1)/(B_1 + B_2)\bar{f}_0(z), & z \in 2. \end{aligned} \right\} \quad (8.6)$$

The energy of a screw dislocation with Burgers vector b lying on the interface is

$$U = [2\pi(B_1 + B_2)]^{-1}b^2 \ln(R/r). \quad (8.7)$$

Near-tip fields for an interfacial crack in the two blocks respectively, are

$$f'_1(z) = f'_2(z) = -\frac{1}{2}(2\pi z)^{-\frac{1}{2}}K_{III}. \quad (8.8)$$

The traction at a distance r ahead of the crack tip is

$$\sigma_{23}(r) = (2\pi r)^{-\frac{1}{2}}K_{III}, \quad (8.9)$$

and the displacement jump at a distance r behind the crack tip is

$$d_3(r) = (2r/\pi)^{\frac{1}{2}}(B_1 + B_2)K_{III}. \quad (8.10)$$

The energy release rate is related to the stress intensity factor by

$$G = \frac{1}{4}(B_1 + B_2)K_{III}^2 \quad (8.11)$$

Consider a set of collinear cracks lying along the interface, with self-equilibrated traction $\tau_0(x)$ prescribed on the crack faces. The potentials, with proper interpretation of the arguments for the two materials, are exactly the same (except for a factor of $-\frac{1}{2}$) as those for cracks in isotropic homogeneous materials, namely

$$f'_1(z) = f'_2(z) = -\frac{\chi(z)}{2\pi i} \int_C \frac{\tau_0(x) dx}{\chi^+(x)(x-z)} + P(z)\chi(z), \quad (8.12)$$

where $\chi(z)$ is given by (5.19), and the polynomial $P(z)$ should be determined to satisfy some auxiliary conditions (see §5 for detail).

The solution for a singularity interacting with traction-free cracks can be solved by the superposition scheme illustrated in §7. Taking the singularity of form (8.5) as an example, in addition to the contribution due to the singularity in well-bonded bimaterial system (equation (8.6) and figure 1a), one obtains the image contribution due to the presence of a semi-infinite crack (figure 1b)

$$f'_1(z) = f'_2(z) = \frac{2B_2}{B_1 + B_2} \left\{ \left(\left(\frac{s_3}{z} \right)^{\frac{1}{2}} - 1 \right) \frac{q}{z - s_3} + \left(\left(\frac{\bar{s}_3}{z} \right)^{\frac{1}{2}} - 1 \right) \frac{\bar{q}}{z - \bar{s}_3} \right\}, \quad (8.13)$$

where q and s_3 should be calculated from (8.5) and (8.6), with the elastic constants for material 2. The stress intensity factor is given by

$$K_{III} = 4(2\pi)^{\frac{1}{2}}B_2/(B_1 + B_2) \operatorname{Re} [qs_3^{-\frac{1}{2}}]. \quad (8.14)$$

For a screw dislocation, using (8.5) for q (setting $p = 0$), one obtains a nice result

$$K_{III} = -2b/(B_1 + B_2) \operatorname{Re} [(2\pi s_3)^{-\frac{1}{2}}]. \quad (8.15)$$

The isotropic one-material version of the interactive problem was documented in Thomson (1986).

9. IN-PLANE FIELDS: ORTHOTROPIC MATERIALS

For a homogeneous material with xy -plane as a mirror plane, the characteristic equation for in-plane deformation, specialized from (2.3), is

$$l_4(\mu) \equiv s_{11}\mu^4 - 2s_{16}\mu^3 + (2s_{12} + s_{66})\mu^2 - 2s_{26}\mu + s_{22} = 0. \quad (9.1)$$

It has been shown by Lekhnitskii (1963) that the roots of equation (9.1) can never be real, and thus they occur in two conjugate pairs. Assuming they are distinct, one can choose two different roots, μ_1 and μ_2 , with positive imaginary parts, to each of which a complex variable $z_j = x + \mu_j y$ is associated. The field quantities can be expressed by two holomorphic functions $f_1(z_1)$ and $f_2(z_2)$, as obtained by discarding $f_3(z_3)$ in (2.2). The matrices A , L , B and H are 2×2 now the elements for A and L can be specialized from (2.5) and (2.6) with $\eta_1 = \eta_2 = 0$, while

$$B \equiv iAL^{-1} = \begin{bmatrix} s_{11} \operatorname{Im}(\mu + \mu_2) & -i(\mu_1 \mu_2 s_{11} - s_{12}) \\ i(\bar{\mu}_1 \bar{\mu}_2 s_{11} - s_{12}) & -s_{22} \operatorname{Im}(\mu_1^{-1} + \mu_2^{-1}) \end{bmatrix}. \quad (9.2)$$

In deriving (9.2) the standard relations between roots and coefficients have been used. These algebraic results are basically all one needs to specialize various solutions in the previous sections.

To gain more insight, we consider below orthotropic materials. The principal axes of each material are taken to be in x and y axes, since other orientations may be treated by in-plane rotations and the associated tensor rules in §2. Given an orthotropic solid, since $s_{16} = s_{26} = 0$, only four elastic constants, s_{11} , s_{22} , s_{12} and s_{66} , enter the plane problem formulation. Following the notation introduced earlier (Suo 1990b), we define two non-dimensional parameters as

$$\lambda = s_{11}/s_{22}, \quad \rho = \frac{1}{2}(2s_{12} + s_{66})(s_{11}s_{22})^{-\frac{1}{2}}. \quad (9.3)$$

The two parameters measure the anisotropy in the sense that $\lambda = 1$ when the material has transversely cubic symmetry and $\lambda = \rho = 1$ when the material is transversely isotropic. The positive definiteness of the strain energy density requires that

$$\lambda > 0 \quad \text{and} \quad -1 < \rho < \infty.$$

The characteristic equation (9.1) is then

$$\lambda\mu^4 + 2\rho\lambda^{\frac{1}{2}}\mu^2 + 1 = 0. \quad (9.4)$$

The roots with positive imaginary parts are

$$\begin{aligned} \mu_1 &= i\lambda^{-\frac{1}{4}}(n+m), & \mu_2 &= i\lambda^{-\frac{1}{4}}(n-m), & \text{for } 1 < \rho < \infty, \\ \mu_1 &= \lambda^{-\frac{1}{4}}(in+m), & \mu_2 &= \lambda^{-\frac{1}{4}}(in-m), & \text{for } -1 < \rho < 1, \\ \mu_1 &= \mu_2 = i\lambda^{-\frac{1}{4}}, & & & \text{for } \rho = 1, \\ n &= [\tfrac{1}{2}(1+\rho)]^{\frac{1}{2}}, & m &= [\tfrac{1}{2}(1-\rho)]^{\frac{1}{2}} \end{aligned} \quad (9.5)$$

From the above we know the LES representation (2.2) does not hold for the degenerate case $\rho = 1$. The significance of this special case will be discussed in the next section.

The matrix B for an orthotropic material, reduced from (9.2), is

$$B = \begin{bmatrix} 2n\lambda^{\frac{1}{4}}(s_{11}s_{22})^{\frac{1}{2}} & i((s_{11}s_{22})^{\frac{1}{2}} + s_{12}) \\ -i((s_{11}s_{22})^{\frac{1}{2}} + s_{12}) & 2n\lambda^{-\frac{1}{4}}(s_{11}s_{22})^{\frac{1}{2}} \end{bmatrix}. \quad (9.6)$$

It is interesting to note that B is still well-behaved even if $\rho = 1$ (A and L are

singular for this case). The matrix H in (2.13) for two orthotropic materials with aligned principal axes is

$$H = \begin{bmatrix} H_{11} & -i\beta(H_{11}H_{22})^{\frac{1}{2}} \\ i\beta(H_{11}H_{22})^{\frac{1}{2}} & H_{22} \end{bmatrix}, \quad (9.7)$$

where

$$\left. \begin{aligned} H_{11} &= [2n\lambda^{\frac{1}{2}}(s_{11}s_{22})^{\frac{1}{2}}]_1 + [2n\lambda^{\frac{1}{2}}(s_{11}s_{22})^{\frac{1}{2}}]_2, \\ H_{22} &= [2n\lambda^{-\frac{1}{2}}(s_{11}s_{22})^{\frac{1}{2}}]_1 + [2n\lambda^{-\frac{1}{2}}(s_{11}s_{22})^{\frac{1}{2}}]_2, \\ (H_{11}H_{22})^{\frac{1}{2}}\beta &= [(s_{11}s_{22})^{\frac{1}{2}} + s_{12}]_2 - [(s_{11}s_{22})^{\frac{1}{2}} + s_{12}]_1. \end{aligned} \right\} \quad (9.8)$$

Here β is a generalization of one of the Dundurs (1969) parameters. The non-oscillatory fields can be obtained by the corresponding results in §5 if H is real, or $\beta = 0$. We will focus on the case $\beta \neq 0$ below.

The oscillatory index ϵ , solved from the eigenvalue problem in §6, is

$$\epsilon = (2\pi)^{-1} \ln((1 - \beta)/(1 + \beta)). \quad (9.9)$$

Values of oscillatory indices for several systems are given in table 1.

TABLE 1. OSCILLATORY INDEX ϵ

	Al ₂ O ₃	boron	carbon	E-glass	SiC
Al (FCC)	-0.047	-0.049	-0.062	0.017	-0.049
Cr (BCC)	-0.025	-0.031	-0.064	0.061	-0.028
Cu (FCC)	-0.029	-0.032	-0.049	0.040	-0.030
Pb (FCC)	-0.034	-0.035	-0.039	-0.006	-0.035
Zr (HCP)	-0.042	-0.046	-0.062	0.030	-0.045

The eigenvector normalized as (6.20) is

$$\mathbf{w} = [-\frac{1}{2}i, \frac{1}{2}(H_{11}/H_{22})^{\frac{1}{2}}]^T, \quad (9.10)$$

and the in-plane traction component defined in (6.11) is

$$t_1 = (H_{22}/H_{11})^{\frac{1}{2}}\sigma_{22} + i\sigma_{12}. \quad (9.11)$$

With the complex stress intensity factor K , the traction in the interface is given by

$$(H_{22}/H_{11})^{\frac{1}{2}}\sigma_{22} + i\sigma_{12} = (2\pi r)^{-\frac{1}{2}}Kr^{i\epsilon}. \quad (9.12)$$

The displacement jump across the crack is

$$(H_{11}/H_{22})^{\frac{1}{2}}d_2 + id_1 = \frac{2H_{11}Kr^{\frac{1}{2}+i\epsilon}(2\pi)^{-\frac{1}{2}}}{(1 + 2i\epsilon)\cosh \pi\epsilon}. \quad (9.13)$$

The energy release rate is thereby

$$G = H_{11}|K|^2/(4\cosh^2 \pi\epsilon). \quad (9.14)$$

The stress intensity factors for this case, however defined, may not reduce to the classical definition as the bimaterial degenerates to have $\epsilon = 0$, because

$H_{22}/H_{11} \neq 1$ if $\lambda \neq 1$. For the case $\epsilon = 0$ one may rescale $\text{Re}(K)$ by $(H_{22}/H_{11})^{\frac{1}{2}}$ to recover the classical stress intensity factor.

The stress intensity factor for an internal crack subjected to traction on the faces is

$$K = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cosh \pi \epsilon (2a)^{-\frac{1}{2}-i\epsilon} \int_{-a}^a \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}+i\epsilon} \left[\left(\frac{H_{22}}{H_{11}}\right)^{\frac{1}{2}} \sigma_{22} + i\sigma_{12}\right] dx. \quad (9.15)$$

For a traction-free internal crack under remote stresses σ_{22} and σ_{12} , the stress intensity factor at the right-hand side is

$$K = (1 + 2i\epsilon) [(H_{22}/H_{11})^{\frac{1}{2}} \sigma_{22} + i\sigma_{12}] (2a)^{-i\epsilon} (\pi a)^{\frac{1}{2}}. \quad (9.16)$$

In addition to β (or ϵ), another generalized Dundurs parameters α , or Σ , can be defined as

$$\Sigma = [(s_{11} s_{22})^{\frac{1}{2}}]_2 / [(s_{11} s_{22})^{\frac{1}{2}}]_1 = (1 + \alpha) / (1 - \alpha). \quad (9.17)$$

Table 2 gives values of stiffness ratios for several bimaterial systems.

TABLE 2. STIFFNESS RATIO Σ

	Al ₂ O ₃	boron	carbon	E-glass	SiC
Al (FCC)	5.5	6.2	9.1	1.0	6.8
Cr (BCC)	1.2	1.3	2.0	0.2	1.5
Cu (FCC)	5.0	5.6	8.3	0.9	6.2
Pb (FCC)	29.5	33.0	49.0	5.6	36.5
Zr (HCP)	3.2	3.5	5.3	0.6	4.0

Obviously, Σ (and α) measures the relative stiffness of the two materials. It can be shown that, for a composite of two aligned orthotropic materials, α and β (or ϵ and Σ) are the only bimaterial parameters needed for traction prescribed problems, in addition to two anisotropy measures, λ and ρ , for each material.

10. ON SOLUTIONS FOR $\epsilon = 0$ AND $\rho = 1$

The non-oscillatory condition for an interface crack between two anisotropic media is that the matrix H is real. A question is whether this condition is a property of a given material pair independent of the relative orientations. As the matrices B and H are second-order tensors under a rotation in the xy -plane, two important results can be easily inferred. First, for two materials with a fixed relative orientation, ϵ is invariant under the rotation of the interface in the xy -plane (Ting 1986). Secondly, for a material pair with the xy -plane as a mirror plane, if H is real ($\epsilon = 0$) for one relative orientation of the two materials, it is real for any relative orientation. A special case of the latter statement was contained in Qu & Bassani (1989), where bicrystals (mis-oriented but otherwise identical crystals) are considered. If one focuses on bimaterial composites with the xy -plane as a mirror plane, as a consequence of the two statements, one can talk about the non-pathology without referring to the relative orientations of the constituents with respect to the interface.

For two orthotropic materials so bonded that the xy -plane is a mirror plane of the composite, the non-oscillatory condition is $\beta = 0$, or equivalently, from (9.8)

$$[(s_{11}s_{22})^{\frac{1}{2}} + s_{12}]_2 = [(s_{11}s_{22})^{\frac{1}{2}} + s_{12}]_1, \quad (10.1)$$

where the compliances are referred to the principal material axes, although the two orthotropic materials may not be aligned with each other. If the two materials satisfy $\rho_1, \rho_2 > 1$, (most fibre-reinforced composite materials satisfy these conditions), it can be shown β is less than $\frac{1}{2}$. Sample calculations indicate that β is usually much smaller than this value.

The reward of the non-oscillatory condition is significant. The near-tip fields are decoupled in two senses: the three singularity modes can be separately defined; and the fields in the two materials do not interact with each other in the sense mentioned earlier. The inherent simplicity of various solutions, as indicated §5, makes it straightforward, at least in principle, to implement interfaces into many micromechanics models.

It appears that ρ defined in (9.3) for orthotropic solids is typically somewhere in the range from 0 to 5. A more important fact is that the known solutions only weakly depend on ρ (see, for example, Suo 1990b). Hence I believe that it will turn out that the primary measure of orthotropy for woods and composites is λ , the ratio of stiffnesses in two principal directions. In terms of compliances, the condition $\rho = 1$ is

$$s_{66} = 2[(s_{11}s_{22})^{\frac{1}{2}} - s_{12}]. \quad (10.2)$$

The benefit of this simplification is that for many problems of practical significance, the solutions for the case $\rho = 1$ but $\lambda \neq 1$ can be extracted, without much effort, from the existing solutions for an isotropic material by a rescaling technique, as illustrated in Suo (1989b).

With those two degenerate values taken, i.e. $\epsilon = 0$ and $\rho = 1$, only three constants (two one-material parameters λ and one bimaterial parameter Σ) are needed to characterize an aligned orthotropic bimaterial system.

Notice, however, that the case $\rho = 1$ makes the roots of the characteristic equation (9.4) degenerate. Although it can be treated as a limiting case of the general LES representation when the field quantities have been obtained, it may be more convenient to start from a degenerate formulation. The complex potential formulation is analogous to Muskhelishvili's for isotropic materials. The Airy stress function, $U(x, y)$, can be expressed in terms of two holomorphic functions $\phi(z)$ and $\psi(z)$ as

$$\left. \begin{aligned} U(x, y) &= \operatorname{Re} [\bar{z}\phi(z) + \int \psi(z) dz], \\ z &= x + i\lambda^{-\frac{1}{2}}y. \end{aligned} \right\} \quad (10.3)$$

The various field quantities can be derived from

$$\left. \begin{aligned} \sigma_y + \lambda^{\frac{1}{2}}\sigma_x &= 4 \operatorname{Re} [\phi'(z)], \\ \sigma_y - \lambda^{\frac{1}{2}}\sigma_x + 2i\lambda^{\frac{1}{2}}\tau_{xy} &= 2[\bar{z}\phi''(z) + \psi'(z)], \\ 2\tilde{\mu}(u_x + i\lambda^{\frac{1}{2}}u_y) &= \bar{\kappa}\phi(z) - z\bar{\phi}'(\bar{z}) - \bar{\psi}(\bar{z}), \\ i(\lambda^{\frac{1}{2}}f_x + if_y) &= \phi(z) + z\bar{\phi}'(\bar{z}) + \bar{\psi}(\bar{z}), \end{aligned} \right\} \quad (10.4)$$

where

$$\tilde{\mu} = \frac{1}{s_{66}} = \frac{1}{2((s_{11}s_{22})^{\frac{1}{2}} - s_{12})}, \quad \frac{1+\tilde{\kappa}}{4} = \frac{(s_{11}s_{22})^{\frac{1}{2}}}{(s_{11}s_{22})^{\frac{1}{2}} - s_{12}} = \frac{2(s_{11}s_{22})^{\frac{1}{2}}}{s_{66}}; \quad (10.5)$$

11. INTERFACIAL FRACTURE MECHANICS FOR ANISOTROPIC SOLIDS

The purpose of interfacial fracture mechanics is to define a measurable and usable material property, toughness, to quantify fracture resistance of interfaces. A comprehensive mechanics scheme, at least for brittle interfacial failure in isotropic solids, has emerged recently, including extraction of toughness curves from experimental raw data and application of the measured toughness curves in structural and material design. The basic notions of interfacial fracture mechanics are summarized in Rice (1988). Here I try to extend the theory to include anisotropic solids on the basis of the crack-tip singular fields developed in §§5 and 6. There are a lot of discussions in the last few years on the role of the oscillatory index. I favour an approach in which one treats $\epsilon = 0$ and $\epsilon \neq 0$ separately.

If the bimaterial under investigation is such that H is real, the crack-tip field is non-oscillatory ($\epsilon = 0$). An important example of this class is tilt grain boundaries, not necessarily symmetrically tilted but with both the tilt axis and interfacial crack front coincident with a principal crystal axis (Qu & Bassani 1989). When $\epsilon = 0$, the three real stress intensity factors, K_I, K_{II} and K_{III} , can be defined in the conventional way, i.e. by normalizing interface traction ahead of the crack tip like (5.13). The singular stress field is linear in these stress intensity factors. The fracture criterion may be stated as (J. W. Hutchinson, personal communication)

$$G = G_c(\psi, \phi). \quad (11.1)$$

Here G is the energy release rate, which may be calculated from (5.15) for a given cracked body if its K -calibration is available. The interfacial toughness G_c in general depends on the loading phases, defined as solid angles, ψ and ϕ , in K_I, K_{II}, K_{III} -space (figure 3a). One needs a toughness surface to fully characterize fracture resistance of an interface.

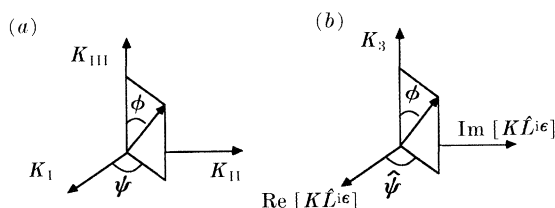


FIGURE 3. Definition of the loading phases. (a) When $\epsilon = 0$, the conventional K -space is used. (b) when $\epsilon \neq 0$, a fixed length \hat{L} is appealed in the same spirit of interfacial fracture mechanics for isotropic solids.

For a bimaterial with complex H , the crack-tip field consists of two singularities. The two-dimensional singularity is oscillatory and the one-dimensional singularity is non-oscillatory, with one complex K and one real K_3 , defined in the fashion

like (6.16). The near-tip stress and displacement fields are linear in $\text{Re}(Kr^{ie})$, $\text{Im}(Kr^{ie})$ and K_3 . The fracture criterion can be formulated as

$$G = G_c(\hat{\psi}, \phi). \quad (11.2)$$

In the above the energy release rate G may be calculated from (6.19). As illustrated in figure 3*b*, the loading phases, $\hat{\psi}, \phi$, are defined as the solid angles in $\text{Re}(K\hat{L}^{ie})$, $\text{Im}(K\hat{L}^{ie})$, K_3 -space, and \hat{L} is a fixed length (e.g. 0.1 mm). The underlying subtlety of such a formulation for isotropic solids can be found in Rice *et al.* (1989).

For this mechanics scheme to be useful in experimental investigation, a key question posed to the solid mechanics community is to design fracture specimens for anisotropic bimetals and to provide appropriate K -calibrations. An initial attempt has been made in Suo (1989*c*), where an analytic K -calibration is derived for a family of fracture specimens consisting of symmetric tilt double-layers.

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