HW2

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1 Problem 1

Let set A={n: $(3^{n+2}+2^{4n+2})\%13 \neq 0\}$. Since A is a subset of positive integers, according to WOP, there must be a least element in A. Denote that as b+1.

Then $3^{b+1+2}+2^{4(b+1)+2}$ ∈A.

$3^{b+3}+2^{4b+4+2} = 3*3^{b+2}+16*2^{4b+2} = 3*(3^{b+2}+2^{4b+2})+13*2^{4b+2}$

Since $13*2^{4b+2}$ is divisible by 13, then $3*(3^{b+2}+2^{4b+2})$ must not be divisible by 13 $\implies$ $(3^{b+2}+2^{4b+2})$ is not divisible by 13

Since $b < b+1$, contradict! $\implies$ there is no such set A $\implies$ $3^{n+2}+2^{4n+2}$ are all divisible by 13

Q.E.D.

2 Problem 2

a. A is well-ordered, with a lower bound of 0 (k = 1)

$$\forall k \geq 1, 2^k \geq 2^1 \implies \frac{1}{2^k} \leq \frac{1}{2^1} \implies \frac{1}{2} - \frac{1}{2^k} \geq 0 = \frac{1}{2} - \frac{1}{2^1}$$

$\implies$ 0 is the lower bound, and the lowest element in A is when k=1

Then we consider the subsets of A. Let M be any subset of A, and $N = \{k : \frac{1}{2} - \frac{1}{2^k} \in M\}$. We first prove that $\forall k, \frac{1}{2} - \frac{1}{2^k} < \frac{1}{2} - \frac{1}{2^{k+1}}$

Let $m = \frac{1}{2} - \frac{1}{2^k}$, $n = \frac{1}{2} - \frac{1}{2^{k+1}}$.

$2^{k+1} > 2^k \implies \frac{1}{2^{k+1}} < \frac{1}{2^k} \implies \frac{1}{2} - \frac{1}{2^k} < \frac{1}{2} - \frac{1}{2^{k+1}} \implies m < n$

Since N is a subset of positive integers, it is well-ordered $\implies$ There is a least element p in N. According to the proof above, $\frac{1}{2} - \frac{1}{2^p} < \frac{1}{2} - \frac{1}{2^q}, p < q \implies$ M has a least element as well

Q.E.D.

b. B is not well-ordered. Prove by contradict.

Lets assume B is well-ordered, then there is a smallest b = $\frac{1}{2} + \frac{1}{2^m}$ in B as a lower bound, where m is a positive integer.

Let c = $\frac{1}{2} + \frac{1}{2^{m+1}}$. $2^{m+1} > 2^m \implies \frac{1}{2^{m+1}} < \frac{1}{2^m} \implies \frac{1}{2} + \frac{1}{2^m} > \frac{1}{2} + \frac{1}{2^{m+1}} \implies b > c$

Contradict! $\implies$ there is no lower bounds

Q.E.D.

c. We first prove that $\forall$ x s.t. $0 < x < \frac{1}{2}$, there is an element in A that is larger then x.

Lets assume there is no such element $\implies x - a \geq 0$ $\forall$a that is in A

Then for the set of $x - a$, there must be a least element according to WOP.

Assume the least element is x-m, where $m = \frac{1}{2} - \frac{1}{2^k}$. Let $n = \frac{1}{2} - \frac{1}{2^{k+1}}$.

$2^{k+1} > 2^k \implies \frac{1}{2^{k+1}} < \frac{1}{2^k} \implies \frac{1}{2} - \frac{1}{2^k} < \frac{1}{2} - \frac{1}{2^{k+1}} \implies m < n \implies x - m > x - n$

Contradict! So there must be an element a in A s.t. $a > x$

Then, let the element that is greater than x be $\frac{1}{2} - \frac{1}{2^p}$. As we proved above, $\frac{1}{2} - \frac{1}{2^{p+1}} > \frac{1}{2} - \frac{1}{2^p}$, and similarly $\frac{1}{2} - \frac{1}{2^{p+2}} > \frac{1}{2} - \frac{1}{2^{p+1}}$.

Then for an arbitrary n, we have $\frac{1}{2} - \frac{1}{2^{p+n-1}} > \frac{1}{2} - \frac{1}{2^{p+n-2}} > \frac{1}{2} - \frac{1}{2^{p+n-3}} > ...... > \frac{1}{2} - \frac{1}{2^p} > x$

Q.E.D.

d. According to problem a, we proved that A is well-ordered. If there is an infinite decreasing sequence in A, let S be A's subset s.t. S = $\{a_1, a_2, a_3, ...\}$ and $a_1 > a_2 > a_3 > a_4$.... In other words, let S be the set of infinite decreasing sequence.

$\implies$ there is no least element in S $\implies$ Contradict with the definition of well-ordered set! $\implies$ There is no such sequence.

Q.E.D.

3 Problem 3

It is noteworthy that $A \oplus B \leftrightarrow A \cdot \overline{B} + \overline{A} \cdot B$

Proof:

A	B	$A \oplus B$	$A \cdot \overline{B} + \overline{A} \cdot B$
1	1	0	0
1	0	1	1
0	1	1	1
0	0	0	0

a. $(A \oplus B) \oplus C \leftrightarrow (A \cdot \overline{B} + \overline{A} \cdot B) \cdot \overline{C} + \overline{(A \cdot \overline{B} + \overline{A} \cdot B)} \cdot C \leftrightarrow A \cdot \overline{B} \cdot \overline{C} + \overline{A} \cdot B \cdot \overline{C} + \overline{A} \cdot \overline{B} \cdot C + A \cdot B \cdot C$

$A \oplus (B \oplus C) \leftrightarrow \overline{A} \cdot (C \cdot \overline{B} + \overline{C} \cdot B) + A \cdot \overline{(C \cdot \overline{B} + \overline{C} \cdot B)} \leftrightarrow A \cdot \overline{B} \cdot \overline{C} + \overline{A} \cdot B \cdot \overline{C} + \overline{A} \cdot \overline{B} \cdot C + A \cdot B \cdot C$

$\implies (A \oplus B) \oplus C \leftrightarrow A \oplus (B \oplus C)$

Q.E.D.

b. $A \oplus B \leftrightarrow A \cdot \overline{B} + \overline{A} \cdot B \leftrightarrow \overline{A} \cdot B + A \cdot \overline{B} \leftrightarrow B \oplus A$

Q.E.D.

c. $A \oplus A \leftrightarrow A \cdot \overline{A} + \overline{A} \cdot A \leftrightarrow 0 + 0 \leftrightarrow 0 \not\leftrightarrow A$

$\implies$ the operation is not idempotent.

Q.E.D

d. $P \oplus Q \leftrightarrow P \cdot \overline{Q} + \overline{P} \cdot Q \implies$ all we need to prove is $\overline{P} \cdot Q \leftrightarrow \overline{(Q \rightarrow P)}$

P	Q	$\overline{P} \cdot Q$	$\overline{(Q \rightarrow P)}$
1	1	0	0
1	0	0	0
0	1	1	1
0	0	0	0

Q.E.D

4 Problem 4

a. $P \cap Q \leftrightarrow P \cdot Q \leftrightarrow \overline{P} \# Q$

b. $P \cup Q \leftrightarrow P + Q \leftrightarrow \overline{P \# \overline{Q}}$

c. $P \rightarrow Q \leftrightarrow \overline{\overline{P} \# \overline{Q}}$

P	Q	$\overline{\overline{P} \# \overline{Q}}$
1	1	1
1	0	0
0	1	1
0	0	1