HW9

chaofan tao

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1 Problem 1

Set of Finite Subsets of Countable Set is Countable

Use B to denote the set of all finite subsets of the countable set A. We want to prove that B inj $\mathbb{N}$.

A is countable $\implies$ $\mathbb{N}$ inj A and $\mathbb{N}$ surj A $\implies$ $\forall a \in A$, there is a unique integer corresponding to it. Write the integer corresponding to a as i_a

$\forall b \in B$, consider the following function f: B $\rightarrow \mathbb{N}$.

$$f(b) = \prod_{b' \in b} P_{i_{b'}}$$

where $P_{i_{b'}}$ represents the $i_{b'}$th prime number. Additionally, if b is empty, we define $f(\emptyset) = 1$. Since the expression for integer as product of prime numbers is unique, each b has a unique f(b) value $\in \mathbb{N}$. $\implies$ $\forall x, y \in B$, if f(x) = f(y), then x=y $\implies$ B inj $\mathbb{N}$.

Q.E.D.

2 Problem 2

a. consider the following function f: A → B.

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{2^{n-1}} & x = \frac{1}{2} - \frac{1}{2^n}, n > 1 \\ \frac{1}{2} + \frac{1}{2^{n-1}} & x = \frac{1}{2} + \frac{1}{2^n}, n > 1 \\ x & otherwise \end{cases}$$

$\forall x, y \in A$, since all three functions are monotonous and the only overlap is when x=$\frac{1}{2}$, when f(x) = $\frac{1}{2}$ $\implies$ when f(x) = f(y), x=y $\implies$ the function is **injective**
$\forall b \in B$, if b $\notin [\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} + \frac{1}{2^n}]$, then f(b) = b. If b = $\frac{1}{2} + \frac{1}{2^{n-1}}$, then b = f($\frac{1}{2} + \frac{1}{2^n}$). If b = $\frac{1}{2} - \frac{1}{2^{n-1}}$, then f($\frac{1}{2} - \frac{1}{2^n}$)=b. If b = $\frac{1}{2}$, then f($\frac{1}{2}$)= b $\implies$ f is **surjective**.
Q.E.D.

b. consider the following function f: [a,b] → [c,d]

$$f(x) = \frac{d-c}{b-a}x + \frac{bc-ad}{b-a}$$

The function is linear and continuous and monotonous over [a, b] $\implies$ it is injective.
Since the function is monotonously increasing and continuous, and when f(x) = d, x = b, when f(x)=c, x =a $\implies$ $\forall m \in [c, d]$, there must be an x $\in [a, b]$ s.t. f(x)=m $\implies$ the function is surjective.
Q.E.D.

c. As shown in the problem, $\mathbb{R}$ bij (0,1), and as proven in (a) (0,1) bij [0,1] $\implies$ $\mathbb{R}$ bij [0,1]. As proven in (b), [0,1] bij [0, 2π].
C is the unit circle $\implies$ C = {(r, θ): r=1, $\theta \in [0, 2\pi)$} $\implies$ if $\mathbb{R}$ bij [0, 2π),

then $\mathbb{R}$ bij C.

Since R bij [0, 1], [0,1] bij [0, 2π] $\implies$ R bij [0, 2π]. Since infinite interval [0, 2π), it bij [0, 2π) $\cup\, 2\pi = [0, 2\pi]$ $\implies$ R bij [0, 2π]

Q.E.D.