HW4

chaofan tao

February 2019

Contents

1 Problem 1

a. it is irreflexive, transitive, anti-symmetric, and asymmetric

b. it is reflexive, transitive, anti-symmetric

c. it is reflexive, transitive, symmetric

d. it is reflexive, transitive, symmetric

Proof:

Reflexive: $\forall a \in \mathbb{R}, \exists b \in \mathbb{R} s.t. a > b$ Since $b \leq a \implies aRa \implies$ They are reflexive.

Transitive: $\forall a, b, c \in \mathbb{R} s.t. aRb, bRc, \exists d \in \mathbb{R} s.t. a > d, b > d, c > d \implies a > d, d \leq c \implies aRc \implies$ The relation is transitive

Symmetric: $\forall a, b \in \mathbb{R} s.t. aRb, \exists c \in \mathbb{R} s.t. a > c, b \geq c \implies a \geq c - 1, b > c - 1 \implies bRa \implies$ The relation is symmetric

e. R is empty, and thus it is transitive, symmetric, anti-symmetric, asymmetric, and irreflexive

Proof:

irreflexive: $\forall x \in \mathbb{R}$, xRx does not hold since R is an empty set $\implies$ it is not reflexive

transitive, symmetric, anti-symmetric, asymmetric Since the antecedent conditions are false, the statements would be true.

f. It is reflexive, transitive, symmetric

Proof:

Reflexive: $\forall x \in \mathbb{R}, x - x = 0 \in \mathbb{Q} \implies xRx$

Transitive: $\forall a, b, c \in \mathbb{R} s.t. aRb, bRc \implies \exists q_1, q_2 \in \mathbb{Q} s.t. a - b = q_1, b - c = q_2 \implies a - c = a - b + b - c = q_1 + q_2 \in \mathbb{Q} \implies aRc \implies$ the relation is transitive

Symmetric: $\forall a, b \in \mathbb{R} s.t. aRb \implies \exists q \in \mathbb{Q} s.t. a - b = q \implies b - a = -q \in \mathbb{Q} \implies bRa$

2 Problem 3

a. f is injective

$\forall x, y \in \mathbb{R}^+$, assume f(x) = f(y) $\implies 2^x = 2^y \implies \log_2 2^x = \log_2 2^y \implies x = y$

$\implies$ There is only one x that could lead to f(x)

f is injective

f is not surjective, since $2.5 \in \mathbb{R}^+$, but cannot be reached with the function.

b. f is not injective since 5 = 7-2 = 8 - 3, it could be reached from (7,2) and (8,3) $\in$ A

f is surjective

$\forall b \in B, b+1 \in \mathbb{Z}, 1 \in \mathbb{Z}^+ \implies (b+1, 1) \in A \implies \forall b \in B, \exists a \in A$

$s.t. f(a) = b \implies$ it is surjective

c. f is not surjective, since $-\sqrt{2}$ is never reachable from the f(A).

f is injective.

$\forall x, y \in \mathbb{R}, x > y$, assume f(x) = f(y). There are three conditions: x,y $\leq$ 1, or $x > 1, y \leq 1$, or $x, y > 1$

If x,y $\leq$ 1, f(x) = f(y) $\implies 2 - x = 2 - y \implies x = y$

If $x > 1, y \leq 1$, f(x) = f(y) $\implies \frac{1}{x} = 2 - y$ However, $x > 1 \implies \frac{1}{x} < 1$, and $y \leq 1 \implies 2 - y > 1 \implies$ This condition does not exist.

If $x, y > 1$, f(x) = f(y) $\implies \frac{1}{x} = \frac{1}{y} \implies x = y$

There is only one x that could lead to f(x)

Q.E.D.

3 Problem 4

a. Given f∘g is bijective over a finite set X, we first prove that f is bijective. Assume f∘g is defined over A $\rightarrow$ C, g is defined over A $\rightarrow$B, f is defined as B$\rightarrow$C, given f∘g is bijective, we know that every element on C has an in-order of 1. Since $\mid A \mid = \mid B \mid = \mid C \mid \implies$ function f is surjective and injective $\implies$ f is bijective.

Then we prove g is bijective. Since $\mid A \mid = \mid C \mid \implies \forall c \in C$, there is only one corresponding unique a $\in$ A, and $\forall a \in A$, there is one corresponding c. $\implies$ out-order of g=1.

f is bijective $\implies \forall b \in B$, there is one corresponding c and $\forall c \in C$, there is only one corresponding unique b $\in$ B

$\implies \forall b \in B$ there is only one corresponding a $\in$ A $\implies$ g is injective. Given the $out - order_g = 1$, g is injective $\implies$ g is surjective.

Q.E.D.

b. A counter example is X = $\mathbb{R}^+$, g=x+1, f=x-1 if $x > 1$, and f=0 if x=1. In this case, 1 is not in the range of g, thus g is not surjective.

4 Problem 5

Lets first define two students A and B in the DAG form a chain iff $A < B$, and thus the problem is just to prove that there exists a chain whose size is larger than 13.

According to Dilworth's Lemma, $\forall t > 0$ the students must have either a chain of size grater than t, or an antichain of size at least $\frac{170}{t}$. Let t=13.

If there is a chain of size 14, the problem is proved. Suppose that there is not a chain of size 14 $\implies$ there is an antichain of size at least 14. According to the definition of antichain, $\forall A, B \in antichain,\ A > B \implies$ the antichain forms a decreasing sequence. $\implies$ It also satisfies the requirements of the problem.

Q.E.D.

5 Problem 6

a. We first prove R is reflexive. $\forall$ (a,b) $\in \mathbb{Z}^+ \times \mathbb{Z}^+$, since ab = ba $\implies$ (a,b)R(a,b) $\implies$ reflexive

Then we prove R is transitive.$\forall$ (a,b), (c,d), (e,f) $\in \mathbb{Z}^+ \times \mathbb{Z}^+$, (a,b)R(c,d), (c,d)R(e,f) $\implies$ ad=bc, cf=ed $\implies$ $a = \frac{bc}{d}, f = \frac{ed}{c}$ $\implies$ $af = \frac{becd}{cd} = be$ $\implies$ (a, b)R(e,f)

Then we prove R is symmetric. $\forall$ (a,b), (c,d)$\in \mathbb{Z}^+ \times \mathbb{Z}^+$s,t, (a,b)R(c,d), $\implies$ $ad = bc \implies cb = da \implies$ (c,d)R(a,b)

Q.E.D

b. The function f is:

$\forall(a, b) \in$ the set of equivalence classes induced by R, if gcd(a,b) = 1, then f= $\frac{a}{b}$

Prove it is bijective: We first prove it is injective.

$\forall(a, b)(c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ that f(a,b) and f(c,d) are not None, if f(a,b) = f(c,d) $\implies$ $\frac{a}{b} = \frac{c}{d}$. Because gcd(a,b) = gcd(c,d) = 1 $\implies$ a=c, b=d $\implies$ (a,b) = (c,d)

Then prove that the function is surjective. $\forall r \in \mathbb{R}^+$, r could be represented as $\frac{m}{n}, m, n \in \mathbb{Z}^+$ and gcd(m,n)=1. Then according to the definition of the set of equivalence classes of R, m,n must be in one of the equivalence classes $\implies$ f(m,n)=r

Q.E.D.