HW7

chaofan tao

March 2019

Contents

1 Problem 1

Prove that a graph G = (V, E) is 2^k-colorable if and only if E can be partitioned into k sets such that for every $1 \leq i \leq k, G_i = (V, E_i)$ is a bipartite graph

We first prove that if E can be partitioned into k sets, then G is 2^k colorable. We color the nodes into black and white in each E_i. $\forall v \in V$, we use a binary number of length k to represent its color. For $1 \leq i \leq k$, if v is colored black in E_i, we write 0 at the i^{th} bit of its binary code. If v is colored white in E_i, we write 1 at the i^{th} bit of its binary code. After the coloring, each v has a binary code. Meanwhile, vertices with same binary code are labelled as the same color in every bipartite graph formed by the partitions of edges, and thus is not next to each other in all cases. Since for a binary number of length k, there are at most 2^k distinct values, the graph has at most 2^k distinct colors $\implies$ is 2^k colorable.

We then prove that if G is 2^k colorable, then E could be partitioned into k sets. Since the graph is 2^k colorable, we used a binary index for each node of length k, according to its color. Note that nodes of the same binary index, namely the same color, would never be included in any partition. The partitions, as a result, could be formed bit by bit according to the binary code. For each bit in the binary code, the nodes with same bit value at location i are placed in one side of the bipartite graph formed by E_i. In this case, every edge would be included exactly once since if there is an edge between vertices, their binary code(color) must be different, and if the edge is already used, we could just ignore it next time. Since there are k digits, there will be k different subsets of E.

Q.E.D.

2 Problem 2

a. Prove that the maximum number of edges in a matching contained in E is $|A| - def(G)$

If def(G)=0, then $\forall S \subseteq A, |S| \leq |N(S)|$, which satisfies Hall's condition. By Hall's theorem, There is a match that covers A. Since vertices in B are not connected to each other, the maximum number of edges in a matching in E is $|A| - 0 = |A|$

If def(G) > 0, we add def(G) vertices $\in$ B and connect each added vertices to every vertex $\in$ A. $\implies$ $\forall S \subseteq A$, write the new Neighborhood as N(S)'= N(S) + def(G). Since def(G) = max(def(S)), def(S) = max(0, $|S| - |N(S)|$) $\implies$ $|S| \leq |N(S)'|$ $\implies$ by Hall's theorem, there is a match that covers A. Since vertices in B are not connected to each other, the maximum number of edges in a matching in the new graph is $|A|$.

Note there are def(G) new vertices that we added. Each would contribute at most 1 to the total number of edges according to the definition of matching, and at least 1 to the total number of edges since each is connected to all vertices in A, and for the S s.t. def(S) is maximum, all add-on vertices contribute one edge to the total count. $\implies$ The add-on vertices produce def(G) more edges $\implies$ the maximum number of edges in a matching in E is $|A| - def(G)$

Q.E.D.

b. We say there is a solution in this grid if there is a way to choose a card from each column so that we have a card of at least 10 different ranks.

Thus, finding a solution is equivalent to selecting a card from each of the 13 columns so that cards have 10 different ranks. Now consider the bipartite graph G = ($C \cup R$, E) where the vertices in C correspond to columns, the vertices in

R correspond to the ranks of cards, and an edge is between column c $\in$ C and rank r $\in$ R if and only if there is a card of rank r in column c. The problem then is just to prove there is a match in the bipartite graph s.t. the number of edges $\geq$ 10

$\forall S \subseteq C, |S| = k$. Since there are 3 columns, there are in total 3k cards. Since there are 4 cards each rank, there must be at least $\frac{3k}{4}$ different ranks. $\implies$ the smallest N(S) is $\frac{3k}{4}$ $\implies$ $def(S) = max(0, k - \frac{3k}{4}) = \frac{k}{4}$ $\implies$ the largest $def(G) = max(\frac{k}{4}) = \frac{13}{4}$ $\implies$ since there could only be integer number of cards, the largest possible def(G) is 3 because when k=13 smallest N(S)=10.

According to the theorem we prove above, there is a match s.t. the number of edge is 13-3=10. If def(G) is smaller than 3, the there is always a match with edge number greater than 10.

Q.E.D.