HW8

chaofan tao

March 2019

Contents

1 Problem 1

a. Prove that the following equality holds: $y_1 = 2 + \sum_{k=3}^{n-1}(k-2) \cdot y_k$

Prove by induction. When n=3, the tree could only be a line graph, and thus there are two leaves $\implies$ the statement holds.

We assume it holds when $n = m$, where $m > 3$. When n=m+1, we divide the tree into two parts by breaking an edge of a leaf node, since a tree always has a leaf node: one part of m nodes, and the other part only contains 1 leaf vertex, v, of the tree. According to the assumption, the sub-tree with m nodes has $y_{m1} = 2 + \sum_{k=3}^{m-1}(k-2) \cdot y_k$. We then try to put the edge back between the two sub trees. We write degree of the node that v is connecting to as p. Note that the max possible p = m-1, while min(p)=1

After we put the two sub trees together, we observe that if p=1, $y_1 = y_{m1}$, since v replaces one of the leaves. If $p > 1$, $y_1 = y_{m1} + 1$, since v does not replace any of the leaves. We also observe that in the original tree with m+1 nodes, y_{p+1} increases by 1, and y_p decrease by 1. $\implies \forall p \geq 2, y_1 = y_{m1} + 1 = 2 + \sum_{k=3}^{m-1}(k-2) + 1 = 2 + \sum_{k=3}^{m-1}(k-2) + (p-1) \cdot 1 - (p-2) \cdot 1$ If there is a node of degree m in the original tree, then p must be m-1, then $y_m = 1, p-1 = m-2$; if there is no node of degree m, then $y_m = 0$ $\implies y_1 = 2 + \sum_{k=3}^{m-1}(k-2) + (p-1) \cdot 1 - (p-2) \cdot 1 = 2 + \sum_{k=3}^{m}(k-2) \cdot y_k$. When p=1, $y_1 = y_{m1} = 2 + \sum_{k=3}^{m-1}(k-2)$, and since there will not be any node of degree m, $2 + \sum_{k=3}^{m-1}(k-2) = 2 + \sum_{k=3}^{m-1}(k-2) + y_m \cdot m - 2 \implies y_1 = 2 + \sum_{k=3}^{m}(k-2) \cdot y_k$
By induction, the statement holds.

b. if T is a rooted full binary tree, then T has (n + 1)/2 leaves

Prove: T is a rooted full binary tree $\implies$ it only has degree three nodes and degree one nodes, except for the root, which has degree 2. $\implies$ according to

part a, $y_1 = 2 + \sum_{k=3}^{n-1} (k-2) \cdot y_k = 2 + y_3 = 2 + (n - y_1 - 1) = n + 1 - y_1 \implies$
$2y_1 = n + 1 \implies y_1 = \frac{n+1}{2}$

Q.E.D.

2 Problem 2

We prove the problem by induction. Let's assume T has more than 1 node.If k=2, then G has a minimum degree of at least 1, which means f could map all connected u,v to the two nodes in T $\implies$ the statement holds for base.

Assume the statement holds when k=m. When k=m+1, since every tree with more than one nodes has at least two leaves, we write the two leaves as l_1, l_2. We divide the tree into two parts by breaking the edge between l_1 and the tree. The results would be a sub tree with m nodes, whose copy is contained in G, and an isolated vertex l_1. We denote the parent node of l_1 as p_1, and the function that maps the sub tree to G as F. Since the sub tree has a copy in G, then F(p_1) is in G. Since the minimum degree of G is at least m, then F(p_1) has at least m nodes connecting to it. In the sub tree of m nodes, p_1 could at most have m-1 vertices connecting to it $\implies$ At most m-1 adjacency nodes of F(p_1) are mapped already $\implies$ there must be at least m-(m-1) = 1 node that is not mapped.

We could just map l_1 to one of the adjacent nodes of F(p_1) in G, and then every nodes in the tree could be mapped to G. $\implies$ When k=m+1, there is a copy in G.

By induction, the statement holds.

3 Problem 3

In any run of DFS on G, the vertex with the largest post-value belongs to a source component of G.

We prove by contradiction.

First note that every vertex is a source component if it could not be reached from others. If the vertex with the largest post-value, which we write as v, does not belong to a source component of G, then it means there must be some V that could reach v. Since v could be reached by V, that makes v one of the children of V, $\implies$ V has a higher post-value than v $\implies$ Contradict!

Q.E.D.