HW6

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1 Problem 1

Prove the following by induction: the number of ways to order n people is the product of the first n positive integers.

base: When n=1, there is only one way to order the person, which means Order = 1 = 1! = n!. The statement holds.

inductive hypothesis: Now assume when n=k, the statement holds. Denote the set of orders to be O(k).

inductive step:When n=k+1, for every order of k people, there are k+1 slots where we could insert the new person. $\implies \forall o \in O(k)$, there are k+1 ways to add a new person into o. $\implies |O(k+1)| = (k+1) * |O(k)| = (k+1) * k! = (k+1)! \implies$ the statement holds when n=k+1.

According to induction, Q.E.D.

2 Problem 2

Prove that the minimum value in a binary min-heap is stored at the root.

Assume the minimum value in a binary min-heap is not stored at the root. Denote the min value as A[m]. If there are multiple minimum values, we choose the one that is closest to the root.

Since m is not the root of the binary tree, it must have a parent by the definition of a binary tree. Write its parent as n. By the definition of min-heap, $A[n] \leq A[m]$

If $A[n] < A[m]$, then A[m] could not be the minimum value in the tree $\implies$ Contradict! Then the minimum value is stored at the root.

If $A[n] = A[m]$, then there are multiple minimum values. However, when choosing m, we chose the closest node to the root $\implies$ it is not possible for A[n] to equal A[m]

$\implies$ The minimum value is stored at the root Q.E.D.

3 Problem 3

G has an Eulerian circuit if and only if the degree of every vertex in G is even.

We prove it by two steps. We first prove that if G has an Eulerian circuit, then every vertex in G has an even degree.

Lets denote the Eulerian circuit to be E(k)=$\{v_1, v_2, v_3, ..., v_k\}$, where $v_1 = v_k$, and each v represents a node (not necessarily unique) in the walk. $\forall v_i \in E(k)$, we use an unused edge $< v_{i-1} - v_i >$ to visit the node, and unused edge $< v_i - v_{i+1} >$ to leave the node $\implies deg(v_i) = 2 \implies$ the degree of each vertex in G would always be a summation of 2 $\implies$ the degree is even

We then prove that the degree of every vertex in the graph is even, then G has an Eulerian circuit.

base. When n=1, m=0, there is no walk in the graph.When n=2, m could only be 1 or 0, which does not satisfy the assumption. When n=3, m=3, the degree of every vertices is 2, we assume the three vertices to be A,B,C. An Eulerian circuit would be a walk from A to B, B to C, C to A

inductive assumption. the statement holds when m≤k. We call the Eulerian circuit E(k)=$\{v_1, v_2, v_3, v_4, ..., v_k, v_1\}$, where each v represents a node in the walk(not necessarily unique).

inductive step. When m=k+1, we start from an arbitrary vertex V, and keep choosing random nodes to walk to until we reach back to V. Since the degree of each vertex is even, we could always go back to V because for each vertex on the walk, we use an unused edge to reach it, and there must be another unused edge to leave it. Same for V, there must be an edge that leads us back to it. We call the trail W.

If $|W| = k + 1$, the proof is done. If $|W| < k + 1$, then G is divided into several smaller connected graphs by the walk W, and each sub graph contains a vertex that is in W, the vertices in each graph have an even degree, and the number of edges of each sub graph is smaller than k+1. Denote the vertex in the sub graph that is also in W as v_i. According to the inductive hypothesis, each of the sub graph has an Eulerian circuit that could start at v_i. Write the Eulerian circuit as E_i. Now that we just need to replace each v_i in W with the corresponding E_i, and then we get the Eulerian circuit for m=k+1.

By induction, the statement holds.

Q.E.D.