HW10

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Contents

1 Problem 1

Let R denote the total area of the regions colored red, and let B denote the total area of regions colored blue. Give a closed form expression (i.e., no summations) for B/(R + B), and prove your claim

It is obvious that each strip of blue, if it is in the $2i$th cube, equals

$$(2i)^2-(2i-1)^2 \implies B=\sum_{i=1}^{\frac{n}{2}}((2i)^2-(2i-1)^2)=\sum_{i=1}^{\frac{n}{2}}((2i+2i-1)(2i-2i+1))=\sum_{i=1}^{\frac{n}{2}}(4i-1)$$

$$=4\cdot\sum_{i=1}^{\frac{n}{2}}i-\frac{n}{2}=4\cdot\frac{(2+n)n}{8}-\frac{n}{2}=\frac{(1+n)n}{2}$$

It is also obvious that R+B=n^2, since R+B is the total area.

$$\implies \frac{B}{R+B}=\frac{\frac{(1+n)n}{2}}{n^2}=\frac{1+n}{2n}$$

2 Problem 2

a. We first prove that for $n>1$, the n^{th} term in the sequence is strictly smaller than 2^{-n}. We prove by induction.

base: When n=2, $a_2=0.1<2^{-2}=0.25$. When n=3, $a_3=0.02<0.125 \implies$ the statement holds for base cases.

Inductive: we then assume that the statement holds for n and n-1. For n+1 th term, according to the assumption, $a_{n+1}=0.1\cdot a_n+0.01\cdot a_{n-1}<0.1\cdot 2^{-n}+0.01\cdot 2^{-n+1}=0.1\cdot 2^{-(n+1)}\cdot 2+0.01\cdot 2^{-(n+1)}\cdot 4=0.2\cdot 2^{-(n+1)}+0.04\cdot 2^{-(n+1)}=0.24\cdot 2^{-(n+1)}<2^{-(n+1)} \implies$ the statement holds according to induction,

Since we know the sum of the geometric sequence $\sum_2^\infty 2^{-n}=\frac{1}{2}$ from basic algebra, we could safely say that $\sum_{n=2}^\infty a_n<0.5 \implies S-a_1<0.5 \implies$

$S - 1 < 0.5 \implies S \leq 2$

Q.E.D.

b. $S = 1 + \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{5}{10^4} + \frac{8}{10^5} ...$

$10S = 10 + 1 + \frac{2}{10} + \frac{3}{10^2} + \frac{5}{10^3} + \frac{8}{10^4} ...$

$9S = 10 + \frac{1}{10} + \frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \frac{5}{10^5} ...$

$90S = 100 + 1 + \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{5}{10^4} + \frac{8}{10^5} ... = 100 + S$

$89S = 100$

$S = \frac{100}{89}$

3 Problem 3

The result that is unique among all results means either all other results are T, and the person got H, or other results are all H and the person get T. The two events are disjoint $\implies$ the total probability is the sum of the probability of 1 H in all n flips and 1 T in all n flips.

$$P = \binom{n}{1} p^{n-1}(1-p) + \binom{n}{1}(1-p)^{n-1}p$$

The probability formula is from PDF of the binomial distribution.

4 Problem 4

The streak could start an any coin from first flip to $\frac{n}{2}+1$ flip. If it happens at the first flip, it does not have to be different from the previous flip (there is no previous flip, anyway). If the streak happens in the middle, it has to be different from the previous. $\Longrightarrow$

$$P = \frac{2*2^{\frac{n}{2}}+(\frac{n}{2})*2*2^{\frac{n}{2}-1}-2}{2^n} = \frac{(2+\frac{n}{2})*2^{\frac{n}{2}}-2}{2^n}$$

where there are in total 2^n situations of coin flips, and $2*2^{\frac{n}{2}}$ situations if the streak starts at the first flip, $(\frac{n}{2})*2*2^{\frac{n}{2}-1}-2$ situations if streak starts in each of the later positions. They are disjoint so we could just add them up. Note that the factor of 2 is because the streak could be Head or Tail, in each situations. The minus 2 is because when the flip starts at $\frac{n}{2}+1$, we have to minus 1 possibility for both H and T that all previous flips are the same.

5 Problem 5

a. There are 4 situations, faulty odor, non-faulty odor, faulty non-odor, non-faulty non-odor. They are all disjoint, so we could just add the odor ones up. Also, since P(A) = P(A, B) + P(A, $\bar{B}$),

$$P = P(odor|faulty)*P(faulty)+P(odor|non-faulty)*P(non-faulty) = 0.005*0.95+0.995*0.1 = 0.10425$$

b. The conditional probability is calculated following the formula. Note that P(odor) is calculated first, using the fact that P(A) = P(A, B) + P(A, $\bar{B}$)

$$P(faulty|odor) = \frac{P(faulty \cap odor)}{P(odor)} = \frac{P(odor|faulty) * P(faulty)}{P(odor)} = \frac{0.005 * 0.95}{0.10425} = 0.0455635492$$

c. Similarly, the conditional probability is calculated using the formula. Note that P(non-odor)=1 - P(odor)

$$P(nonfaulty|nonodor) = \frac{P(nonfaulty \cap nonodor)}{P(nonodor)} = \frac{P(nonodor|nonfaulty) * P(nonfaulty)}{0.8957}$$

$$= \frac{P(nonodor|nonfaulty) * P(nonfaulty)}{0.8957} = \frac{0.9 * 0.995}{0.8957} = 0.999776711$$

d. He could be incorrect if the engine is correct but there is odor, or faulty but no odor. They are independent from each other and disjoint, so we could add them up.

$$P = P(faulty \cap nonodor) + P(nonfaulty \cap odor)$$

$$= P(nonodor|faulty) * P(faulty) + P(odor|nonfaulty) * P(nonfaulty)$$

$$= 0.05 * 0.005 + 0.1 * 0.995 = 0.09975$$

6 Problem 6

a. We prove it by finding a bijection function between the two sides of the equation. For every spanning tree, we could write it as a sequence of edges. That is, $s = \{\{v_1, v_2\}, \{v_2, v_3\}...\}$. If there is a function that could map the spanning tree s in G bijectively with the spanning trees in G-e + G\e, the proof is done. Write edge e as {u,v}, the set of spanning trees in G as A, the set of spanning trees in G-e + G\e as B.

Define the function $f : A \to B$ as follows:

$$\forall s \in A, f(s) = \begin{cases} s & \{u,v\} \notin s \\ s \setminus \{\{u,v\},\{x,v\}\} \cup \{\{u,x\}\} & \{u,v\} \in s \end{cases}$$

if spanning tree s does not contain {u, v}, it should be in G-e. We map it to itself. If it contains v, we map the two edges {x, v}, {v, u} in the spanning tree in G to {x, u} in the spanning tree in G\e. All other edges remain the same.

We then prove the function if bijective.

The function is injective since s either contains {u, v} or not contains it. The function is surjective since $\forall s \in B$, if s $\in G \setminus e$, it could be mapped to a spinning tree in A that has {u, v}; if s $\in G - e$, it could be mapped to a spinning tree in A that does not have {u, v}. $\implies$ Function is bijective

Q.E.D.

b. There are in total $\tau(G)$ spanning trees. According to what we showed in part a, all spanning trees containing edge e could be mapped bijectively to $G \setminus e \implies$ there are $\tau(G \setminus e_1)$ spanning trees containing e_1, $\tau(G \setminus e_2)$ spanning trees containing e_2. To calculate the number of spanning trees containing both e_1, e_2, just remove e_2 from $G \setminus e_1 \implies$ there are $\tau(G \setminus e_1 \setminus e_2)$ spanning trees containing both. The answer is thus

$$P = \frac{\tau(G \setminus e_2) + \tau(G \setminus e_1) - 2 \cdot \tau(G \setminus e_1 \setminus e_2)}{\tau(G)}$$