ON SPACES WITH σ-CLOSED DISCRETE DENSE SETS

RODRIGO R. DIAS, DANIEL T. SOUKUP

Abstract. The main purpose of this paper is to study *e-separable spaces*, originally introduced by Kurepa as K_0' spaces; a space X is e-separable iff X has a dense set which is the union of countably many closed discrete sets. We primarily focus on the behaviour of e-separable spaces under products and the cardinal invariants that are naturally related to e-separable spaces. Our main results show that the statement "the product of at most $\mathfrak{c}$ many e-separable spaces is e-separable" depends on the existence of certain large cardinals and hence independent of ZFC.

1. Introduction

The goal of this paper is to study a natural generalization of separability: let us call a space X *e-separable* iff X has a dense set which is the union of countably many closed discrete sets. This definition is due to Kurepa [12], who introduced this notion as property K_0' in his study of Suslin's problem. Later, e-separable spaces appear in multiple papers related to linearly ordered spaces [19, 20, 22]. In Kurepa's old notation, d-separable spaces (i.e. spaces with σ-discrete dense sets) were called K_0. d-separable spaces were investigated in great detail (see [1, 3, 10, 15, 21]) and they show very interesting behaviour in many aspects, in particular, regarding products. Recall that the Hewitt-Marcewski-Pondiczery theorem [7] states that the product of at most $\mathfrak{c}$ many separable spaces is separable again. A. Arhangel'skiĭ proved in [1] that any product of d-separable spaces is d-separable; in [10], the authors show that for every space X there is a cardinal κ so that X^κ is d-separable. Motivated by these results, one of our main objective is to determine the behaviour of e-separable spaces under products.

Our paper splits into three main parts. First, we make initial observations on e-separable spaces in Section 2. Then, in Section 3, we investigate if the existence of many large closed discrete sets suffices for a space to be e-separable. While this is not the case in general, we prove that once an infinite power X^κ has a closed discrete set of size $d(X^\kappa)$ (the density of X^κ) then X^κ is e-separable.

Date: November 14, 2016.

1991 *Mathematics Subject Classification.* 54D65, 54A25, 54A35.

Key words and phrases. sigma closed discrete, dense, weakly compact.

Second, in Section 4, we compare two natural cardinal functions: $d(X)$, the size of the smallest dense set in X, with $d_e(X)$, the size of the smallest σ-closed discrete dense set. In Theorem 4.2, we show that there is a 0-dimensional space X which satisfies $d(X) < d_e(X)$. We show that a similar example can be constructed for d-separable spaces, at least under $\aleph_1 < 2^{\aleph_0}$.

Our main results are finally presented in Section 5: we describe those cardinals κ such that the product of κ many e-separable spaces is e-separable again. First, note that $2^{\mathfrak{c}^+}$ is not e-separable (as a compact, non-separable space) and hence the question of preserving e-separability comes down to products of at most $\mathfrak{c}$ terms. We prove two results which show that the statement "the product of at most $\mathfrak{c}$ many e-separable spaces is e-separable" is independent of ZFC:

Corollary 5.8. *If the existence of a weakly compact cardinal is consistent with ZFC then so is the statement that there are less than $\mathfrak{c}$-many discrete spaces with non e-separable product.*

Corollary 5.10. *If there are no weakly inaccesible cardinals $\leq \mathfrak{c}$ then the product of at most $\mathfrak{c}$ many e-separable spaces is e-separable.*

Throughout this paper, all spaces are assumed to be T_1. Given a product of discrete spaces $X = \prod\{X_\alpha : \alpha < \lambda\}$, $x \in X$ and $a \in [\lambda]^{<\omega}$ we write

$$[x \upharpoonright a] = \{y \in X : x \upharpoonright a = y \upharpoonright a\}.$$

That is, $[x \upharpoonright a]$ is a basic open neighbourhood of x in X. We let $D(\kappa)$ denote the discrete space on a cardinal κ.

In general, we use standard notation and terminology consistent with Engelking [7].

2. Preliminaries

The main concept we study in this paper is the following:

Definition 2.1. *A topological space X is* e-separable *if there is a sequence $(D_n)_{n\in\omega}$ of closed discrete subspaces of X such that $\bigcup_{n\in\omega} D_n$ is dense in X.*

Let us start with simple observations:

Observation 2.2. *Every separable space is e-separable and every e-separable space is d-separable.*

Recall the following two well known cardinal functions: the density of X, denoted by $d(X)$, is the smallest possible size of a dense set in X. The extent of a space X, denoted by $e(X)$, is the supremum of all cardinalities $|E|$ where E is a closed discrete subset of X.

Observation 2.3. *$d(X) \le e(X)$ for every e-separable space X, moreover, if $cf(d(X)) > \omega$ then there is a closed discrete set of size $d(X)$ in X. In particular, a countably compact space is e-separable iff it is separable.*

Example 2.4. *$2^{\mathfrak{c}^+}$ is a compact, d-separable but non e-separable space.*

Proof. By Arhangleskii [1], d-separability is preserved by products. Also, $2^{\mathfrak{c}^+}$ is not e-separable as $d(2^{\mathfrak{c}^+}) > e(2^{\mathfrak{c}^+}) = \omega$ hence Observation 2.3 can be applied. □

What can we say about metric spaces?

Observation 2.5. *Every space with a σ-discrete π-base is e-separable. Hence, every metrizable space is e-separable.*

The following claim shows that actually a large class of generalized metric spaces are e-separable:

Proposition 2.6. *Every developable space is e-separable.*

Recall that a space X is *developable* iff there is a developement of X i.e. a sequence $(\mathscr{G}_n)_{n\in\omega}$ of open covers of X such that for every $x \in X$ and open V containing x there is an $n \in \mathbb{N}$ so that $st(x, \mathscr{G}_n) = \bigcup\{U \in \mathscr{G}_n : x \in U\} \subseteq V$.

Proof. Let $(\mathscr{G}_n)_{n\in\omega}$ be a development for a topological space X. For each $x \in X$ and $n \in \omega$, let $V_n^x = st(x, \mathscr{G}_n)$. By Proposition 1.3 of [2], there is a closed discrete $D_n \subseteq X$ such that $X = \bigcup_{x\in D_n} V_n^x$ for each $n \in \omega$. We claim that $\bigcup_{n\in\omega} D_n$ is dense in X.

Suppose, to the contrary, that there is $p \in X \setminus \overline{\bigcup_{n\in\omega} D_n}$, and let $m \in \omega$ be such that $V_m^p \cap \bigcup_{n\in\omega} D_n = \emptyset$. By the choice of D_m, there is $x \in D_m$ such that $p \in V_m^x$; but then $\{p, x\} \subseteq U$ for some $U \in \mathscr{G}_m$, which implies that $x \in V_m^p$, thus contradicting the fact that $V_m^p \cap D_m = \emptyset$. □

It is worth comparing the above result with Proposition 2.3 of [3], which states that every quasi-developable space is d-separable. Note also that the Michael line is a quasi-developable space (*cf.* [4]) that is not e-separable; this also shows, in particular, that e-separability is strictly stronger than d-separability.

The proof of Proposition 2.6 suggests that there might be a connection between D-spaces and e-separable space; recall that a space X is a D-space iff for every open neighbourhood assignemnt $N : X \to \tau$ there is a closed discrete $D \subseteq X$ so that $N[D]$ covers X. However, we note that the Alexandrov double circle is hereditarily D but not e-separable.

For later reference, we would like to state two results on the existence of closed discrete sets in products.

Theorem 2.7 ([8]). *ω^{2^κ} contains a closed discrete set of size κ for every κ less than the 1^{st} measurable cardinal.*

The above result was first proved by Lós [13] but the reference [8] is much more accessible.

Theorem 2.8 ([18]). *ω^κ contains a closed discrete set of size κ for every κ less than the 1^{st} weakly inaccesible cardinal.*

3. Density and extent for e-separable spaces

Recall that $d(X) \le e(X)$ for every e-separable space X. Note that this inequality does not imply that there are closed discrete sets of size $d(X)$:

Example 3.1. *There is a σ-closed discrete (hence e-separable) space X which contains no closed discrete sets of size $d(X)$.*

Proof. Let $X = \aleph_\omega + 1$; have all points in $\aleph_\omega$ isolated and let $\{\{\aleph_\omega\} \cup A : A \in [\aleph_\omega]^{<\aleph_\omega}\}$ form a neighborhood base at $\aleph_\omega$. □

Also, even significantly stronger assumptions than $d(X) \le e(X)$ fail to imply e-separability in general.

Example 3.2. *There is a 0-dimensional space X such that $|X| = \omega_1$, every somewhere dense subset of X contains a closed discrete subset of size ω_1 while X is not e-separable.*

Proof. Let $X = \omega_1^{<\omega}$ and declare $U \subseteq X$ to be open iff $x \in U$ implies that $\{\alpha < \omega_1 : x^\frown(\alpha) \in U\}$ contains a club. Now, X is a Hausdorff, 0-dimensional and dense-in-itself space.

Observation 3.3. *A set $E \subseteq X$ is closed discrete iff $\{\alpha < \omega_1 : x^\frown(\alpha) \in E\}$ is non stationary for every $x \in X$.*

This observation immediately implies that the σ-closed discrete sets are closed discrete and hence X cannot be e-separable.

Suppose that $Y \subseteq X$ is dense in a nonempty open set V; $I_x = \{\alpha \in \omega_1 : x^\frown\alpha \in Y\}$ must be stationary for any $x \in V$ and so we can select an uncountable but non stationary $I \subseteq I_x$. Hence $E(I_x)$ is an uncountable closed discrete subset of Y. □

Now, let us turn to powers a fixed space X. Could it be that $d(X^\kappa) \le e(X^\kappa)$ implies that X^κ is e-separable whenever κ is an infinite cardinal? The answer is no in general:

Example 3.4. *If κ is measurable then $d(\omega^\kappa) = e(\omega^\kappa)$ however ω^κ is not e-separable.*

Proof. It is clear that $d(\omega^\kappa) = \kappa$; also, $2^\lambda < \kappa$ whenever $\lambda < \kappa$ and so Theorem 2.7 implies that $e(\omega^\kappa) = \kappa$ as well.

If we show that ω^κ has no closed discrete sets of size κ then ω^κ cannot be e-separable. Suppose that $A = \{x_\alpha : \alpha < \kappa\} \subseteq \omega^\kappa$ and that $\mathcal{U}$ is a κ-complete non-principle ultrafilter on κ. Define $y \in \omega^\kappa$ by $y(\xi) = n \in \omega$ iff $\{\alpha < \kappa : x_\alpha(\xi) = n\} \in \mathcal{U}$. It is easy to see that $\{\alpha < \kappa : x_\alpha \in V\} \in \mathcal{U}$ for every open neighbourhood V of y; hence, y is an accumulation point of A. $\square$

However, if we suppose a bit more than $d(X^\kappa) \leq e(X^\kappa)$ then we get

Theorem 3.5. *Let X be any space and κ an infinite cardinal. If X^κ contains a closed discrete set of size $d(X^\kappa)$ then X^κ is e-separable.*

We will prove a somewhat technical lemma now which immediately implies Theorem 3.5 and will be of use later as well:

Lemma 3.6. *Let X be any space and κ an infinite cardinal. Suppose that $D \subseteq X^\kappa$ is dense in X^κ and X^κ contains a closed discrete set of size $|D|$. Then there is a dense set E in X^κ such that*

(1) $|D| = |E|$, $d(D) = d(E)$, and
(2) E is σ-closed discrete.

Proof. Pick a countable increasing sequence $\{I_n\}_{n\in\omega}$ of subsets of κ such that $\kappa = |I_n| = |\kappa \setminus I_n|$ for each $n \in \omega$ and $\kappa = \bigcup_{n\in\omega} I_n$. Fix closed discrete sets E_n of size $|D|$ in $X^{\kappa\setminus I_n}$ and bijections $\varphi_n : D \to E_n$ for each $n \in \omega$.

We define maps $\psi_n : D \to X^\kappa$ by

$$\psi_n(d)(\xi) = \begin{cases} d(\xi), & \text{for } \xi \in I_n, and \\ \varphi(d)(\xi), & \text{for } \xi \in \kappa \setminus I_n. \end{cases}$$

Let $E = \bigcup_{n\in\omega} \mathrm{ran}(\psi_n)$.Clearly $|D| = |E|$ holds.

It is easy to see that E is dense in X^κ; if $[\varepsilon]$ is a basic open set in X^κ then there is an $n \in \omega$ such that $\mathrm{dom}(\varepsilon) \subseteq I_n$ hence $\mathrm{ran}(\psi_n) \cap [\varepsilon] \neq \emptyset$. Next we show (2) by proving that $\mathrm{ran}(\psi_n)$ is closed discrete as well for each $n \in \omega$. Pick any $x \in X^\kappa$. There is a basic open set $[\varepsilon]$ in $X^{\kappa\setminus I_n}$ such that $x \restriction_{\kappa\setminus I_n} \in [\varepsilon]$ and $|[\varepsilon] \cap E_n| \leq 1$. Hence $x \in [\varepsilon]$ and $|[\varepsilon] \cap \mathrm{ran}(\psi_n)| \leq 1$.

Finally we prove $d(D) = d(E)$. Note that if D_0 is dense in D then $\bigcup_{n\in\omega} \psi_n'' D_0$ is dense in E hence $d(E) \leq d(D)$. Suppose that $A \in [E]^{<d(D)}$, we want to prove that A is not dense in E. Let

$$D_A = \{d \in D : \exists n \in \omega : \psi_n(d) \in A\};$$

then D_A cannot be dense in D as $|D_A| \leq |A| < d(D)$.

Fix a basic open set $U = [\varepsilon]$ such that $[\varepsilon] \cap D_A = \emptyset$. There is an $n^* \in \omega$ such that $\mathrm{dom}(\varepsilon) \subseteq I_{n^*}$.

Claim 3.7. *If $m \geq n^*$ then $[\varepsilon] \cap \{\psi_m(d) : d \in D_A\} = \emptyset$.*

Proof. Suppose that $m \geq n$ and $d \in D_A$. Then $d \restriction_{I_m} = \psi_m(d) \restriction_{I_m}$, $d \notin [\varepsilon]$ and $\mathrm{dom}(\varepsilon) \subseteq I_m$ thus $\psi_m(d) \notin [\varepsilon]$. □

Hence

$$U \cap \bigcup_{n \in \omega} \psi_n(D_A) \subseteq \bigcup_{n < n^*} \psi_n(D_A),$$

that is, $U \cap \bigcup_{n \in \omega} \psi_n(D_A)$ is closed discrete as each $\psi_n(D_A)$ is closed discrete. However, $A \subseteq \bigcup_{n \in \omega} \psi_n(D_A)$ which shows that $A \cap U$ cannot be dense in U. □

It is natural to ask if we can say something similar about products. Let us present a result in this direction:

Lemma 3.8. *Suppose that κ is an infinite cardinal and there is a decreasing sequence $\{I_n : n \in \omega\}$ of subsets of κ with empty intersection such that $\prod\{X_\alpha : \alpha \in I_n\}$ contains a closed discrete subset of size $\delta_n = d(\prod\{X_\alpha : \alpha \in \kappa \setminus I_n\})$ for every $n \in \omega$. Then $X = \prod\{X_\alpha : \alpha < \kappa\}$ is e-separable.*

Proof. Let $X(J)$ denote $\prod\{X_\alpha : \alpha \in J\}$ for $J \subset \kappa$. Pick $D_n = \{d^n_\xi : \xi < \delta_n\} \subseteq X(\kappa \setminus I_n)$ and let $F_n = \{f^n_\xi : \xi < \delta_n\} \subseteq X(I_n)$ closed discrete.

Now, we define $E_n \subseteq X$ as follows: we define e^n_ξ for $\xi < \delta_n$ by

$$e^n_\xi(\alpha) = \begin{cases} d^n_\xi(\alpha), & \text{for } \alpha \in \kappa \setminus I_n, \text{ and} \\ f^n_\xi((\alpha), & \text{for } \alpha \in I_n. \end{cases}$$

We claim that E_n is closed discrete. This follows from

Observation 3.9. *Suppose that $E \subset \prod\{X_\alpha : \alpha < \kappa\}$ and there is $I \subset \kappa$ such that π_I is 1-1 on E and the image $\pi_I'' E$ is closed discrete in $\prod\{X_\alpha : \alpha \in I\}$. Then E is closed discrete.*

Now, it is clear that $\cup\{E_n : n \in \omega\}$ is a dense and σ-closed discrete subset of X. □

4. The sizes of σ-discrete dense sets

Next we investigate the size of the smallest σ-discrete dense sets in e-separable spaces.

Definition 4.1. *For an e-separable space X we define*

$$d_e(X) = \min\{|E| : \ E \textit{ is a dense } \sigma\textit{-closed discrete subset of } X\}.$$

Clearly $d(X) \leq d_e(X) \leq e(X)$ for any e-separable space X and next we show that $d(X) = d_e(X)$ fails to hold in general:

Theorem 4.2. *There is a 0-dimensional e-saparable space X such that*

$$\mathfrak{c} = d(X) < d_e(X) = e(X) = w(X) = 2^{\mathfrak{c}}.$$

Proof. First note the following:

Claim 4.3. *Suppose that a space X can be written as $D\dot{\cup}E$ so that*

(1) D is dense in X,
(2) E is dense and σ-closed discrete in X,
(3) $d(D) < d(E)$, and
(4) every $A \in [D]^{\leq e(D)}$ is nowhere dense in X (or equivalently, in D).

Then X is e-separable and $d(X) < d_e(X)$.

Proof. X is e-separable by (2) and $d(X) \leq d(D)$ by (1). We prove that if $F \in [X]^{\leq d(X)}$ and F is σ-closed discrete then F is not dense in X; this proves the claim. Take $F \subseteq X$ as above and note that by (3) there is a nonempty open set $U \subseteq X$ such that $U \cap E \cap F = \emptyset$. As $|F \cap D| \leq e(D)$, $F \cap D$ must be nowhere dense in X. Thus there is a nonempty open $V \subseteq U$ such that $V \cap D \cap F = \emptyset$. Thus $V \cap F = \emptyset$ showing that F is not dense. □

Now, it suffices to construct a 0-dimensional space $X = D \cup E$ satisfying (1)-(4). Let us construct $X = D \cup E \subseteq \omega^{2^{\mathfrak{c}}}$ such that

(i) D is dense in $\omega^{2^{\mathfrak{c}}}$,
(ii) E is dense and σ-closed discrete in $\omega^{2^{\mathfrak{c}}}$,
(iii) $|D| = \mathfrak{c}$ and $d(E) = 2^{\mathfrak{c}}$, and
(iv) $e(D) = \omega$.

It is trivial to see that (i)-(iii) implies (1)-(3), respectively, while (iv) implies (4) using the fact that $d(\omega^{2^{\mathfrak{c}}}) = \mathfrak{c}$.

First we construct D. Costruct dense subsets $D_n \subseteq n^{2^{\mathfrak{c}}}$ of size $\mathfrak{c}$ which are countably compact, for each $n \in \omega$; this can be done by choosing a dense subset $D_n^0 \subseteq n^{2^{\mathfrak{c}}}$ of size $\mathfrak{c}$ and adding accumulation points recursively (ω_1-many times) for all countable subsets. Define $D = \bigcup_{n\in\omega} D_n$. Then D is dense in $\omega^{2^{\mathfrak{c}}}$ as $\bigcup_{n\in\omega} n^{2^{\mathfrak{c}}}$ is dense in $\omega^{2^{\mathfrak{c}}}$ and $e(D) = \omega$ as $e(D_n) = \omega$ for all $n \in \omega$; thus D satisfies (i), (iv) and the first part of (iii).

Now, we construct E satisfying (ii) and (iii) which finishes the proof. Let $S = \sigma(\omega^{2^{\mathfrak{c}}})$; then $d(S) = 2^{\mathfrak{c}}$ and S is dense in $\omega^{2^{\mathfrak{c}}}$. Recall that $\omega^{2^{\mathfrak{c}}}$ contain a closed discrete set of size $2^{\mathfrak{c}}$ by Theorem 2.8. Now, by applying Lemma 3.6, we find a σ-closed discrete E which is dense in $\omega^{2^{\mathfrak{c}}}$ and satisfies $d(E) = d(S) = 2^{\mathfrak{c}}$.

□

Naturally, one can consider the same problem for d-separable space. Let us present an example along the same lines under the assumption $2^\omega \geq \omega_2$:

Proposition 4.4. *Suppose that $2^\omega \geq \omega_2$. Then there is a d-separable space X with density ω_1 such that X contains no dense σ-discrete sets of size ω_1.*

Proof. J. Moore [14] proved that there is a coloring $c : [\omega_1]^2 \to \omega$ such that for every $n \in \omega$, uncountable pairwise disjoint $A \subset [\omega_1]^n$, uncountable $B \subset \omega_1$ and $h : n \to \omega$ there is $a \in A$ and $\beta \in B \setminus \max(a)$ such that $c(a(i), \beta) = h(i)$ where $a = \{a(i) : i < n\}$.

Suppose that $D = \{d_n : n \in \omega\}$ is any countable space.

Claim 4.5. *There is a dense and hereditarily Lindelöf subspace $Y \subseteq D^{\omega_1}$.*

Proof. Define $y_\beta \in D^{\omega_1}$ as follows:

$$y_\beta(\alpha) = \begin{cases} d_{c(\alpha,\beta)} & \text{if } \alpha < \beta, \\ d_0 & \text{if } \beta \leq \alpha. \end{cases} \tag{1}$$

Let $Y = \{y_\beta : \beta < \omega_1\}$. First, we claim that there is an $\alpha < \omega_1$ so that $Y \restriction \omega_1 \setminus \alpha$ is dense in $D^{\omega_1 \setminus \alpha}$. Suppose otherwise: then we can find basic open sets $[\varepsilon_\alpha]$ in $D^{\omega_1 \setminus \alpha}$ so that $Y \cap [\varepsilon_\alpha] = \emptyset$. By standard Δ-system arguments, we find $I \in [\omega_1]^{\omega_1}$, $n \in \omega$ and $h : n \to \omega$ so that $\mathrm{dom}(\varepsilon_\alpha) = \{a_\alpha(i) : i < n\}$ are pairwise disjoint for $\alpha \in I$ and $d_{h(i)} \in \varepsilon_\alpha(a_\alpha(i))$. Now, there is an $\alpha \in I$ and $\beta \in \omega_1 \setminus \max(\mathrm{dom}(\varepsilon_\alpha))$ so that $c(a_\alpha(i), \beta) = h(i)$. This means that $d_{c(a_\alpha(i),\beta)} \in \varepsilon_\alpha(a_\alpha(i))$ and so $y_\beta \in [\varepsilon_\alpha]$. This contradicts our assumption.

The proof that $Y \restriction \omega_1 \setminus \alpha$ is hereditarily Lindelöf is again rather standard (and similar to the above argument) hence we omit the details. $\square$

Now, by $2^{\aleph_0} \geq \aleph_2$, we can pick a countable dense $D \subseteq 2^{\omega_2}$. Thus D^{ω_1} is dense in $(2^{\omega_2})^{\omega_1} = 2^{\omega_2}$ as well. Hence the claim above gives a subspace Y which is dense in 2^{ω_2} but every discrete subset of Y is countable. Using Lemma 3.6, we find a σ-closed discrete E which is dense in 2^{ω_2} and satisfies $d(E) = 2^{\omega_2}$.

Let $X = Y \cup E$. An argument strictly analogue to what is done in Theorem 4.2 finishes the proof. $\square$

Question 4.6. *Is there a ZFC example of a d-separable space X with the property that every σ-discrete dense subset of X has cardinality bigger than $d(X)$?*

5. Preservation under products

As mentioned in the introduction, the behaviour of separable and d-separable spaces under products and powers are very well described. Hence our goal in this section is answering the following natural question: for which cardinals κ is it true that the product of κ many e-separable spaces is e-separable? As noted earlier, any such κ is at most the continuum.

Let us start with powers of a single e-separable space. The next proposition is due to Ofelia T. Alas (personal communication).

Proposition 5.1. *Let X be an e-separable space and $\kappa \le \mathfrak{c}$. Then the space X^κ is e-separable.*

Proof. Let $(D_n)_{n\in\omega}$ be a sequence of closed discrete subsets of X with $\bigcup_{n\in\omega} D_n$ dense in X. Fix a subspace $Y \subseteq \mathbb{R}$ with $|Y| = \kappa$, and let $\mathcal{B}$ be a countable base for Y. Now consider $T = \bigcup_{n\in\omega}(S_n \times {}^n\omega)$, where $S_n = \{(B_0, \ldots, B_{n-1}) \in {}^n\mathcal{B} : \forall i, j < n\ (i \ne j \Rightarrow B_i \cap B_j \ne \emptyset)\}$ for every $n \in \omega$.

Fix an arbitrary $p \in X$. For each $t = ((B_0, \ldots, B_{n-1}), (k_0, \ldots, k_{n-1})) \in T$, we define E_t to be the set of those $x \in X^Y$ so that there is an $(a_i)_{i<n} \in \prod_{i<n} D_{k_i}$ with

$$x(\alpha) = \begin{cases} a_i, & \text{for } \alpha \in B_i \text{ and } i < n, \text{ and} \\ p, & \text{for } \alpha \in Y \setminus \bigcup_{i=0}^{n-1} B_i. \end{cases}$$

It is routine to verify that each E_t is a closed discrete subspace of X^Y and that $\bigcup_{t\in T} E_t$ is dense in X^Y. Since T is countable and $|Y| = \kappa$, it follows that X^κ is e-separable. □

Now, we turn to arbitrary products of e-separable spaces. We will see that the heart of the matter is if we can find large closed discrete sets in the product of small discrete spaces.

In [17], Mrowka introduced a class of cardinals denoted by $\mathcal{M}^*$: we write $\lambda \in \mathcal{M}^*$ iff there is a product of λ many discrete spaces $X = \prod\{X_\alpha : \alpha < \lambda\}$ with each of size $< \lambda$ so that X has a closed discrete set of size λ. Equivalently, the product

$$\prod_{\nu\in\lambda\cap\mathrm{Card}} D(\nu)^\lambda$$

contains a closed discrete set of size λ.

If a cardinal λ is in $\mathcal{M}^*$ then some degree of compactess fails for λ. Let us make this statement precise: recall that $\mathcal{L}_{\lambda,\omega}$ is the infinitary language which allows conjunctions and disjunctions of $< \lambda$ formulas and universal or existential quantification over finitely many variables. The language $\mathcal{L}_{\lambda,\omega}$ is *weakly compact* by definition if every set of at most λ sentences Σ from $\mathcal{L}_{\lambda,\omega}$ has a model provided that every $S \in [\Sigma]^{<\lambda}$ has a model (see [9], p. 382).

Theorem 5.2 ([17],[6])**.** *$\lambda \notin \mathcal{M}^*$ if and only if $\mathcal{L}_{\lambda,\omega}$ is weakly compact.*

We remark that, as expected, $\lambda \notin \mathcal{M}^*$ has some large cardinal strength. This is summarized in the next two lemmas:

Lemma 5.3 ([9], Exercise 32.1 and 32.2)**.** *If $\mathcal{L}_{\lambda,\omega}$ is weakly compact then λ is weakly inaccessible.*

Lemma 5.4 ([9], Lemma 32.1)**.** *λ is a weakly compact cardinal iff it is strongly inaccessible and $\mathcal{L}_{\lambda,\omega}$ is weakly compact.*

Now, it is easy to derive our first main result about non preservation:

Lemma 5.5. *If $\lambda \leq \mathfrak{c}$ and $\lambda \notin \mathcal{M}^*$ then there are λ many discrete spaces with non e-separable product.*

Proof. $\lambda \notin \mathcal{M}^*$ implies that $\mathcal{L}_{\lambda,\omega}$ is weakly compact and hence λ is a regular limit cardinal. Now take discrete spaces X_α of size $< \lambda$ such that $\sup\{|X_\alpha| : \alpha < \kappa\} = \lambda$. $X = \prod\{X_\alpha : \alpha < \lambda\}$ contains no closed discrete subsets of size λ as $\lambda \notin \mathcal{M}^*$. We claim that $d(X) = \lambda$ which follows from the following more general observation:

Observation 5.6. *Suppose that $\kappa \leq \mathfrak{c}$ and X_α is discrete for $\alpha < \kappa$. Then $d(\prod\{X_\alpha : \alpha < \kappa\}) = \sup\{|X_\alpha| : \alpha < \kappa\}$.*

To prove this observation, simply apply the usual trick appearing in the proof of Proposition 5.1.

Now, we claim that X cannot be e-separable. Indeed, if X is e-separable then Observation 2.3 implies that X has a closed discrete subset of size $d(X) = \lambda = cf(\lambda) > \omega$ however this is not the case.

□

Recall that it is possible to realize the assumption of Lemma 5.5:

Theorem 5.7 ([6],[5])**.** *If λ is a weakly compact cardinal and $\mathbb{C}_{\lambda^+}$ is the poset for adding λ^+-many Cohen-reals then $V^{\mathbb{C}_{\lambda^+}} \models$ "$\mathcal{L}_{\lambda,\omega}$ is weakly compact hence $\mathfrak{c} \setminus \mathcal{M}^* \neq \emptyset$".*

Hence, we immediately get the following:

Corollary 5.8. *If the existence of a weakly compact cardinal is consistent with ZFC then so is the statement that there are less than $\mathfrak{c}$-many discrete spaces with with non e-separable product.*

Now, we will show that it is also consistent that the product of at most $\mathfrak{c}$ many e-separable spaces is e-separable. It suffices to show

Theorem 5.9. *Suppose that $\lambda \leq \mathfrak{c}$ is minimal so that there is a family of λ e-separable spaces with non e-separable product. Then $\lambda \notin \mathcal{M}^*$.*

Recall that $\lambda \notin \mathcal{M}^*$ implies that λ is weakly inaccessible (see Lemma 5.3). In turn, we have the following consistency result:

Corollary 5.10. *If there are no weakly inaccesible cardinals below $\mathfrak{c}$ then the product of at most $\mathfrak{c}$ many e-separable spaces is e-separable.*

Let us turn now to prove Theorem 5.9. First, we start by reducing the problem to products of discrete spaces again:

Lemma 5.11. *Suppose that $\kappa \leq \mathfrak{c}$. Then the following are equivalent:*

(*a*) *every product of at most κ many e-separable spaces is e-separable;*
(*b*) *every product of at most κ many discrete spaces is e-separable.*

Proof. The implication $(a) \Rightarrow (b)$ holds trivially. We prove $(b) \Rightarrow (a)$.

Let $X = \prod_{\alpha \in Y} X_\alpha$, where $Y \subseteq \mathbb{R}$ has cardinality at most κ and each X_α is e-separable. For each $\alpha \in Y$, fix a point $p_\alpha \in X_\alpha$ and a sequence $(E^\alpha_k)_{k\in\omega}$ of closed discrete subsets of X_α with $\overline{\bigcup_{k\in\omega} E^\alpha_k} = X_\alpha$.

Fix a countable base $\mathcal{B}$ for Y and, for each $n \in \omega$, consider

$$S_n = \{(B_i)_{i<n} \in {}^n\mathcal{B} : \forall i, j < n\ (i \neq j \Rightarrow B_i \cap B_j = \emptyset)\};$$

now, for each $t = ((B_i)_{i<n}, (k_0, \ldots, k_{n-1})) \in S_n \times {}^n\omega$, define Y_t to be the set of those $x \in X$ so that

$$x(\alpha) = \begin{cases} x'_\alpha & \text{for some } x'_\alpha \in E^\alpha_{k_i} \text{ for } \alpha \in B_i \text{ and } i < n, \text{ and} \\ p_\alpha, & \text{for } \alpha \in Y \setminus \bigcup_{i=0}^{n-1} B_i. \end{cases}$$

Note that each Y_t is homeomorphic to the product $\prod_{i<n}\prod_{\alpha\in B_i} E^\alpha_{k_i}$. Hence Y_t is is e-separable by (b). Let $(D^t_k)_{k\in\omega}$ be a sequence of closed discrete subsets of Y_t with $\overline{\bigcup_{k\in\omega} D^t_k} = Y_t$. Since each Y_t is closed in X, we have that each D^t_k is a closed discrete subset of X. Finally, as $\bigcup_{n\in\omega}\bigcup_{r\in S_n\times{}^n\omega} Y_t$ is dense in X, it follows that

$$\overline{\bigcup_{n\in\omega}\bigcup_{t\in S_n\times{}^n\omega}\bigcup_{k\in\omega} D^t_k} = X,$$

thus showing that X is e-separable. □

Note that we immediately get the following easy:

Corollary 5.12. *The product of finitely many e-separable spaces is e-separable.*

Second, we show that as long as we take the product of large discrete sets relative to the number of terms, we end up with an e-separable product:

Lemma 5.13. *Let κ be an infinite cardinal. Then the product of at most κ many discrete spaces of cardinality at least κ is e-separable.*

Proof. Let $X = \prod_{\alpha\in\lambda} X_\alpha$, where $\lambda \leq \kappa$ and each X_α is a discrete space with cardinality at least κ. We can assume that λ is infinite and that $X_\alpha = |X_\alpha|$ for all $\alpha \in \lambda$.

Define

$$P^i_j = \{(F, p) \in [\lambda]^i \times Fn(\lambda, \kappa) : |p| = j \text{ and } F \cap \mathrm{dom}(p) = \emptyset\}$$

for each $i, j \in \omega$ where $Fn(\lambda, \kappa)$ denotes the set of finite partial functions from λ to κ. Fix an injective function $\varphi : \bigcup_{i,j\in\omega} P^i_j \to \kappa$ such that $\varphi(F, p) > \max(\mathrm{ran}(p))$ for every $(F, p) \in \bigcup_{i,j\in\omega} P^i_j$.

Now, for every $i, j \in \omega$, let E^i_j be the set of all $x \in X$ such that there is $(F, p) \in P^i_j$ such that

(1) $x(\xi) \geq \kappa$ for all $\xi \in F$,
(2) $x \in [p]$, and
(3) $x(\xi) = \varphi(F, p)$ for all $\xi \in \lambda \setminus (F \cup \mathrm{dom}(p))$.

It is straightforward to verify that $\bigcup_{i,j\in\omega} E^i_j$ is dense in X. We claim that each E^i_j is a closed discrete subset of X, which will conclude our proof.

From this point on, let $i, j \in \omega$ be fixed.

To see that E^i_j is discrete, pick an arbitrary $x \in E^i_j$, and let this be witnessed by the pair $(F, p) \in P^i_j$. Note that the choice of φ ensures that this (F, p) is unique. Pick any $\eta \in \lambda \setminus (F \cup \mathrm{dom}(p))$ and let

$$V = [x \restriction (\mathrm{dom}(p) \cup F \cup \{\eta\})].$$

Then V is an open neighbourhood of x in X satisfying $E^i_j \cap V = \{x\}$.

It remains to show that E^i_j is closed in X. Let then $y \in X \setminus E^i_j$; we must find an open neighbourhood V of y in X such that $V \cap E^i_j = \emptyset$. We shall do so by considering several cases.

· *Case 1.* $G = \{\xi \in \lambda : y(\xi) \geq \kappa\}$ has more than i elements.

Then we may take any $H \in [G]^{i+1}$ and define $V = [y \restriction H]$.

· *Case 2.* $G = \{\xi \in \lambda : y(\xi) \geq \kappa\}$ has cardinality at most i.

We will split this case in two:

· *Case 2.1.* $\mathrm{ran}(y) \cap \kappa$ is infinite.

Then we can take $A \in [\kappa]^{j+2}$ such that $y''A \in [\kappa]^{j+2}$ and define $V = [y \restriction A]$.

· *Case 2.2.* $\mathrm{ran}(y) \cap \kappa$ is finite.

Let $\mu = \max(\mathrm{ran}(y) \cap \kappa)$ and $H = \{\xi \in \lambda : y(\xi) < \mu\}$. We divide this case into three subcases:

· *Case 2.2.1.* $|H| > j$.

Pick $H' \in [H]^{j+1}$ and $\beta \in \lambda$ such that $y(\beta) = \mu$. Now take $V = [g \restriction H' \cup \{\beta\}]$.

· *Case 2.2.2.* $|H| \leq j$ and $\mu \notin \mathrm{ran}(\varphi)$.

Let $B \in [\lambda]^{j+1-|H|}$ be such that $y''B = \{\mu\}$ and consider $[g \restriction H \cup B]$.

· *Case 2.2.3.* $|H| \leq j$ and $\mu \in \operatorname{ran}(\varphi)$.

Let $(F, p) \in P^i_j$ be such that $\varphi(F, p) = \mu$ and, as in the previous case, take $B \in [\lambda]^{j+1-|H|}$ satisfying $y''B = \{\mu\}$. Now define

$$V = [g \restriction G \cup H \cup B \cup F \cup \operatorname{dom}(p)].$$

Suppose, in order to get a contradiction, that there is $x \in V \cap E^i_j$ and let $(F', p') \in P^i_j$ witness that $x \in E^i_j$. Since $|H \cup B| = j + 1$ and $x''(H \cup B) = y''(H \cup B) \subseteq \kappa$, we have that $\varphi(F', p') = \max(x''(H \cup B)) = \max(y''(H \cup B)) = \mu$. Hence $(F', p') = (F, p)$ by injectivity of φ. Now, since $F = \{\xi \in \lambda : x(\xi) \geq \kappa\}$ and $G = \{\xi \in \lambda : y(\xi) \geq \kappa\}$, it follows from $x \restriction (F \cup G) = y \restriction (F \cup G)$ that $F = G$. Similarly, as $H = \{\xi \in \lambda : y(\xi) < \mu\}$ and $\operatorname{dom}(p) = \{\xi \in \lambda : x(\xi) < \mu\}$, it follows from $x \restriction (H \cup \operatorname{dom}(p)) = y \restriction (H \cup \operatorname{dom}(p))$ that $H = \operatorname{dom}(p)$. Thus the pair $(G, y \restriction H) = (F, p) \in P^i_j$ witnesses that $y \in E^i_j$, a contradiction. □

Finally, we are ready to present

Proof of Theorem 5.9. Suppose that $\lambda \leq \mathfrak{c}$ is minimal so that there are e-separable spaces X_α such that $X = \prod\{X_\alpha : \alpha < \lambda\}$ is not e-separable. By Lemma 5.11, we can suppose that each X_α is discrete.

Note that

$$X = \prod\{X_\alpha : \alpha < \lambda, |X_\alpha| < \lambda\} \times \prod\{X_\alpha : \alpha < \lambda, |X_\alpha| \geq \lambda\};$$

we know that the second term on the right side is e-separable by Lemma 5.13. So if X is not e-separable then $\prod\{X_\alpha : \alpha < \lambda, |X_\alpha| < \lambda\}$ is not e-separable either by Corollary 5.12.

Now, we define $Y_\nu = \prod\{X_\alpha : \alpha < \lambda, |X_\alpha| = \nu\}$ for $\nu \in \lambda \cap \mathrm{Card}$. Note that Y_ν is e-separable by Theorem 5.1. Hence, the minimality of λ implies that $I = \{\nu \in \lambda \cap \mathrm{Card} : Y_\nu \neq \emptyset\}$ has size λ; otherwise $X = \prod\{Y_\nu : \nu \in I\}$ is a smaller non e-separable product of e-separable spaces. Note that this already shows that $\lambda = \aleph_\lambda$.

Let us suppose that $\lambda \in \mathcal{M}^*$; we will arrive at a contradiction shortly. Take a decreasing sequence of sets $\{I_n : n \in \omega\}$ in I so that $\lambda = |I_n| = |I \setminus I_n|$. Note that the density of $\prod\{Y_\nu : \nu \in I_n\}$ is $\sup I_n = \lambda$ by Lemma 5.6.

Observation 5.14. $\prod\{Y_\nu : \nu \in I \setminus I_n\}$ *contains a closed discrete set of size* λ.

Proof. $\lambda \in \mathcal{M}^*$ implies that $Z = \prod_{\nu \in \lambda \cap \mathrm{Card}} D(\nu)^\lambda$ contains a closed discrete subset of size λ. Hence, it suffices to show that Z embeds into $\prod\{Y_\nu : \nu \in I \setminus I_n\}$ as a closed subspace. Now note that the set $\{\nu \in I \setminus I_n : \nu > \nu_0\}$ has size λ for every $\nu_0 \in \lambda \cap \mathrm{Card}$. Now it is routine to construct the embedding of Z.

□

Finally, we can apply Lemma 3.8 to see that the product $X = \prod\{Y_\nu : \nu \in I\}$ must be e-separable. This contradicts our initial assumption on X.

□

6. Final remarks

Recall that if $D(\lambda)$ is the discrete space of size $\lambda \geq \kappa$ then $D(\lambda)^\kappa$ is e-separable by Lemma 5.13. Actually, we can say a bit more in this case:

Lemma 6.1. *Let $(\kappa_i)_{i\in I}$ be a sequence of cardinals and consider the product space $X = \prod_{i\in I} D(\kappa_i)$. Suppose that the set $\{i \in I : \kappa_i = \kappa\}$ is infinite, where $\kappa = \sum_{i\in I} \kappa_i$. Then X has a σ-discrete π-base.*

Proof. Let J be a countable infinite subset of $\{i \in I : \kappa_i = \kappa\}$. Note that $\kappa_j = \sum_{i\in I\setminus\{j\}} \kappa_i$ for all $j \in J$. Now let $\{p^j_n(\alpha) : \alpha \in \kappa_j\}$ be an enumeration of the set

$$\{p \subseteq \bigcup_{i\in I\setminus\{j\}} (\{i\} \times \kappa_i) : p \text{ is a function and } |p| = n\}$$

for every $j \in J$ and $n \in \omega$. Consider

$$A^j_n = \{p^j_n(\alpha) \cup \{(j, \alpha)\} : \alpha \in \kappa_j\};$$

finally, define $\mathcal{V}^j_n = \{[q] : q \in A^j_n\}$.

Note that each $\mathcal{V}^j_n$ is a discrete family: if $a = (a_i)_{i\in I}$ is any point of X then

$$U = \{(x_i)_{i\in I} \in X : x_j = a_j\}$$

is an open neighbourhood of a in X such that

$$\{V \in \mathcal{V}^j_n : V \cap U \neq \emptyset\} = \{V_{p^j_n(a_j)\cup\{(j,a_j)\}}\}.$$

Moreover, $\mathcal{V} = \bigcup_{j\in J} \bigcup_{n\in\omega} \mathcal{V}^j_n$ is a π-base for X, since any nonempty open subset of X is determined by a finite number of coordinates which constitutes a finite subset of $I \setminus \{j\}$ for some $j \in J$. □

Corollary 6.2. *If $\lambda \geq \kappa$ then $D(\lambda)^\kappa$ has a σ-discrete π-base.*

Finally, let us introduce the selective version of e-separability:

Definition 6.3. *A topological space X is* E-separable *if for every sequence of dense sets $(D_n)_{n\in\omega}$ of X we can select $E_n \subseteq D_n$ so that E_n is closed discrete in X and $\bigcup_{n\in\omega} E_n$ is dense in X.*

Note that every space with a σ-discrete π-base is E-separable as well. Let us point out that the example of Theorem 4.2 is an e-separable space which is not E-separable.

We ask the following questions:

Problem 6.4. *Suppose that X is an e-separable space which is the product of discrete spaces. Is X E-separable as well?*

Problem 6.5. *How does E-separability behave under powers and products?*

7. Acknoledgements

A significant portion of the above presented research was done at the University of Toronto. The authors would like to thank F. Tall and W. Weiss for their helpful comments. We thank the Set Theory and Topology group of the Alfréd Rényi Institute of Mathematics (Hungarian Academy of Sciences) for further help in preparing this paper.

References

[1] A. V. Arhangel'skiĭ, *On d-separable spaces*, Seminar on General Topology, Moscow State University Publishing House, Moscow, 1981, 3–8.

[2] L. F. Aurichi, *D-spaces, topological games, and selection principles*, Topology Proceedings **36** (2010), 107–122.

[3] L. F. Aurichi, R. R. Dias and L. R. Junqueira, *On d- and D-separability*, Topology Appl. 159 (2012), no. 16, 3445–3452.

[4] H. R. Bennett, *Quasi-developable spaces*, Topology Conference, Arizona State University, Tempe, 1968, 314–317.

[5] W. Boos, Infinitary Compactness without Strong Inaccessibility, JSL vol. 41, pp. 33–38, 1976.

[6] Cudnovskii, D. V., Topological properties of products of discrete spaces, and set theory. (Russian) Dokl. Akad. Nauk SSSR 204 (1972), 298–301.

[7] Engelking, Ryszard. "General topology, Sigma series in pure mathematics, vol. 6." (1989).

[8] Isaac Gorelic, The G_δ-topology and incompactness of ω^α. Commentationes Mathematicae Universitatis Carolinae, Vol. 37 (1996), No. 3, 613–616

[9] Jech, Set Theory, 2nd edition, Springer-Verlag 1997.

[10] Juhász, I.; Szentmiklóssy, Z. On d-separability of powers and $C_p(X)$. Topology Appl. 155 (2008), no. 4, 277–281.

[11] K. Kuratowski, Topology, vol. I, Academic Press, New York, 1966.

[12] D. Kurepa, Le probleme de Souslin et les espaces abstraits, Comptes Rendus 203 (1936) 1049–1052

[13] . J. Lós, Linear equations and pure subgroups, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 7 (1959), 13–18.

[14] Moore, Justin Tatch. A solution to the L-space problem. J. Amer. Math. Soc. 19 (2006), no. 3, 717–736

[15] Moore, Justin Tatch An L space with a d-separable square. Topology Appl. 155 (2008), no. 4, 304–307.

[16] S. Mrowka, On E-compact spaces II, Bull. Acad. Polon. Sci. 14 (1966), 597 - 605.

[17] Mrowka, Some strengthenings of the Ulam nonmeasurability condition, Proc. Amer. Math. Soc. 25 (1970), 704–711.

[18] Mycielski, J. α-incompactness of N^α Bull. Acad. Pol., vol. XII, no. 8, 1964, 437.

[19] Y.-Q. Qiao, *On non-Archimedean spaces*, Ph.D. thesis, University of Toronto, 1992.

[20] Qiao, Yuan-Qing, and Franklin Tall. "Perfectly normal non-metrizable non-Archimedean spaces are generalized Souslin lines." Proceedings of the American Mathematical Society 131.12 (2003): 3929-3936.

[21] Tkachuk, Vladimir V. Function spaces and d-separability. Quaest. Math. 28 (2005), no. 4, 409–424.

[22] Watson, Stephen. "Problems I wish I could solve." Open Problems in Topology, North-Holland, Amsterdam (1990): 37-76.

Universidade de Sao Paulo
E-mail address: roque@ime.usp.br

Universität Wien Kurt Gödel Research Center for Mathematical Logic Währinger Strasse 25 1090 WIEN AUSTRIA
E-mail address, Corresponding author: daniel.soukup@univie.ac.at