TREES, LADDERS AND GRAPHS

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ABSTRACT. We introduce a new method to construct uncountably chromatic graphs from non special trees and ladder systems. Answering a question of P. Erdős and A. Hajnal [3], we construct ω_1-chromatic graphs without uncountable ω-connected subgraph. Second, we build ω_1-chromatic triangle free graphs without subgraphs isomorphic to $H_{\omega,\omega+2}$; the existence of such graphs was known only under the Continuum Hypothesis so far [5].

1. INTRODUCTION

The *chromatic number* of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that the vertices of G can be covered by κ many independent sets. A fundamental problem of graph theory asks how large chromatic number affects structural properties of a graph and in particular, is it true that a graph with large chromatic number has certain obligatory subgraphs?

The first result in this area is due to W. Tutte, alias Blanche Descartes [16] (and independently A. Zykov [17]): a construction of triangle free graphs of arbitrary large finite chromatic number. This result was significantly extended by Paul Erdős: there are graphs of arbitrary large finite chromatic number with arbitrary large girth (i.e. no small cycles) [1]. Tutte's result extends to arbitrary *infinite* chromatic graphs [4] but can we generalize the second result?

In 1966 in their seminal paper, P. Erdős and András Hajnal proved, to the great surprise of many, that every uncountably chromatic graph G contains a copy of the complete bipartite graph K_{n,ω_1} for each $n \in \omega$ [2] and hence a copy of a four cycle which is in striking contrast with the finite case and Tutte's construction. Now, we are interested in the corollary that every uncountably chromatic graph G contains an n-connected subgraph for each finite n. They conjectured that this result can be extended to ω-connectivity and asked in [2] if every graph with chromatic number and size ω_1 contains a subgraph of chromatic number ω_1 which is ω-connected i.e. any two points are connected by infinitely many pairwise disjoint paths. In 1985, Erdős and Hajnal asked if every ω_1-chromatic graph contains an ω_1-chromatic ω-connected subgraph [3]; these problems were highly popularized and are included in the recent surveys [9] and [11] as well. The main purpose of this paper is to solve the problem from [3] using a new and rather flexible construction.

Most advances on these questions are due to Péter Komjáth: in [7], he proves that every uncountably chromatic graph contains uncountably chromatic n-connected subgraphs (with minimal degree ω) for each $n \in \omega$. Regarding the original Erdős-Hajnal question, he shows in [8] that under PFA[1] every graph of size and chromatic number ω_1 contains an uncountably chromatic ω-connected subgraph. Furthermore, in the same paper, he presents a forcing which introduces a graph of size and chromatic number ω_1 *without* uncountably chromatic ω-connected subgraphs. His proof is slightly

[1]PFA is the well known Proper Forcing Axiom.

flawed[2], but the error is corrected in the recent [10] where he forces an example of a graph of size and chromatic number ω_1 without *uncountable* ω-connected subgraphs. In particular, the original question from [2] is independent of ZFC and the answer to the second question [3] is consistently no.

Our main result is an answer to the above problem from [3] in Theorem 3.5: we construct a graph of chromatic number ω_1 and size continuum which contains no uncountable ω-connected subgraphs. Our proof is purely within the usual axioms of set theory, i.e. we use no extra set theoretical assumptions nor forcing arguments. In Section 4, we refine our methods in order to get an uncountably chromatic graph G such that any uncountable set A contains two points which are connected by only finitely many pairwise disjoint path *even in G*.

In the second part of the paper, we present a different example using the same framework. In light of the fact that finite bipartite graphs embed into every uncountably chromatic graph, it is reasonable to ask if $K_{\omega,\omega}$ embeds into every uncountably chromatic graph. This is *not* the case, however we have the following results by Hajnal and Komjáth [5]: the graph $H_{\omega,\omega+1}$ embeds into every uncountably chromatic graph, however under CH (the Continuum Hypothesis) there is an uncountably chromatic graph without subgraphs isomorphic to $H_{\omega,\omega+2}$. Recall that $H_{\omega,\omega+2}$ is the graph on vertices $\{x_i, y_i, z, z' : i \in \mathbb{N}\}$ with edges

$$\big\{\{x_i, y_j\}, \{x_i, z\}, \{x_i, z'\} : i \leqslant j \in \mathbb{N}\big\}$$

and $H_{\omega,\omega+1}$ is the subgraph spanned by $\{x_i, y_i, z : i \in \mathbb{N}\}$.

In Section 5 we construct a graph of chromatic number ω_1 and size continuum which is triangle free and has no subgraphs isomorphic to $H_{\omega,\omega+2}$. Thus the assumption of CH can be eliminated from the above result of Hajnal and Komjáth.

Finally, in Section 6 we close with some remarks on our constructions.

Our belief is that this framework for constructing uncountably chromatic graphs will find diverse applications in infinite combinatorics. I would like to thank Chris Eagle for enlightening conversations in the early stages of this project, as well as William Weiss for his help in clarifying my arguments.

2. Preliminaries

A set theoretic *tree* $(T, \leqslant)$ is a partially ordered set such that

$$t^{\downarrow} = \{s \in T : s < t\}$$

is well ordered for all $t \in T$; by a tree we will always mean such an object. Every tree admits a *height* function: $ht(t)$ denotes the order type of $t^{\downarrow}$ for $t \in T$. The *height of the tree T* is $\sup\{ht(t) : t \in T\}$.

Definition 2.1. Let $G(T)$ denote the *comparability graph* of a tree $(T, \leqslant)$ i.e. the set of vertices of $G(T)$ is T and $\{x, y\} \in [T]^2$ is an edge iff $x \leqslant y$ or $y \leqslant x$.

Note that $A \subseteq T$ is independent in $G(T)$ iff it is an antichain in T thus $Chr(G(T)) \leqslant \omega$ iff T is a special tree (i.e. the union of countable many antichains). Throughout this paper, we will be interested in constructing subgraphs of a graph $G(T)$.

We investigate connectivity properties of infinite graphs:

Definition 2.2. We say that a set of vertices F in a graph *separates* two vertices s and t iff every path from s to t passes through F. A graph G is *ω-connected* iff no finite set separates two points of G.

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[2]See the introduction of [10] for details.

Note that a subgraph A of a graph G is ω-connected iff any two points $s, t \in A$ are connected by infinitely many pairwise disjoint paths in A.

Observation 2.3. *Suppose that the tree T does not contain uncountable chains. Then every uncountable subset of $G(T)$ can be separated by a countable set.*

Proof. Take any uncountable $A \subseteq T$ and find two incomparable elements $s, t \in A$. We claim that the countable set $s^{\downarrow}$ separates s and t. Indeed, it is straightforward to see that the connected component of s in $T \backslash s^{\downarrow}$ contains only elements r with $r \geqslant s$. $\square$

Corollary 2.4. *If T is a non special tree with no uncountable chains then the graph $G = G(T)$ satisfies $Chr(G) > \omega$ while every uncountable set of vertices can be separated by a countable set.*

The above simple corollary indicates that it is reasonable to investigate non special trees T without uncountable chains and the corresponding graphs $G(T)$ regarding the Erdős-Hajnal question.

Classical examples of non special trees with no uncountable chains are

(1) $\sigma\mathbb{Q} = \{t \subset \mathbb{Q} : t$ is well ordered$\}$ with $s \leqslant t$ iff s is an initial segment of t,
(2) $T(S) = \{t \subset S : t$ is closed$\}$ with $s \leqslant t$ iff s is an initial segment of t where $S \subseteq \omega_1$ is stationary, co-stationary.

The above trees both have size continuum and height ω_1 and hence the corresponding compatibility graphs have chromatic number exactly ω_1. $\sigma\mathbb{Q}$ was introduced and studied first by D. Kurepa in connection with Souslin's problem while the importance of $T(S)$ was realized by S. Todorcevic [15]. We will use trees of the second kind i.e. of the form $T(S)$ throughout this paper.

Trees of the form $T(S)$ also have the following nice property: $T(S)$ has *no branching at limit levels* i.e. $t^{\downarrow} = s^{\downarrow}$ implies $t = s$ for all limit elements $t, s \in T(S)$. This is proved using that every element of $T(S)$ is a *closed* subset of ω_1.

We will use the following notation: if $T = T(S)$ for some $S \subset \omega_1$ then let

$$T_{\delta} = \{t \in T : \max(t) = \delta\}$$

for $\delta \in \omega_1$ and similarly define $T_{<\delta}, T_{\leqslant \delta}$.

In general, we use standard set theoretical notation and we refer the reader to [12] for basic facts in set theory.

3. Connectivity and chromatic number

Our aim is to construct a subgraph X of the comparability graph of $T(S)$ which has chromatic number ω_1 and does not contain uncountable ω-connected subsets i.e. every uncountable set of vertices A contains two points $s, t \in A$ and a finite set $F \subset A$ such that any path $P \subset A$ betwen s and t passes through F.

Definition 3.1. Suppose that T is a tree. A *ladder system on T* is a family $\underline{C} = \{C_t : t \in T\}$ so that $C_t \subset t^{\downarrow}$ is either finite or a cofinal sequence of type ω.

Each ladder system $\underline{C}$ defines a subgraph $X_{\underline{C}}$ of $G(T)$ with vertices T and edges

$$\{\{s, t\} : s \in C_t, t \in T\}.$$

This is in direct analogy with the Hajnal-Máté graphs introduced in [6] i.e. the case where the tree T is simply the cardinal ω_1. We remark that it is independent of ZFC whether there is a ladder system $\underline{C}$ on ω_1 so that the corresponding Hajnal-Máté graph is uncountably chromatic. Also, ladder systems on trees were invetigated in [14] in a different context.

Definition 3.2. A ladder system $\underline{C}$ on T is *transitive* iff

$$C_t \cap s^{\downarrow} \subseteq C_s$$

for all $t \in T$ and $s \in C_t$.

Note that $\underline{C}$ is transitive iff C_t spans a complete graph in $X_{\underline{C}}$ for all $t \in T$. The next two lemmas explain why we introduced the notion of a transitive ladder system.

Lemma 3.3. *Suppose that T is a tree and $\underline{C}$ is a transitive ladder system on T. If $s, t \in T$ and P is a finite path in $X_{\underline{C}}$ from s to t then there is a path $Q \subseteq P$ which is the union of two monotone paths.*

A *monotone path* in $X_{\underline{C}}$ is a path which is a chain in the tree ordering.

Proof. Let $Q \subseteq P$ be a path of *minimal length* from s to t. Let $\{q_i : i < n\}$ enumerate Q by its path ordering. Note that we cannot have $q_{i-1}, q_{i+1} < q_i$ for any $1 \leqslant i \leqslant n - 1$; indeed, this would imply that $q_{i-1}, q_{i+1} \in C_{q_i}$ and hence, by transitivity, q_{i-1} and q_{i+1} are connected by an edge which contradicts the minimality of Q. This means that if $q_{i-1} < q_i$ then $q_i < q_{i+1}$ for any $1 \leqslant i \leqslant n - 1$ i.e. Q is monotone increasing from the first step up (in the tree ordering). In other words, Q is the union of a monotone decreasing and a monotone increasing path. $\qquad\square$

Lemma 3.4. *Suppose that T is a tree and $\underline{C}$ is a transitive ladder system on T. If T has no branching at limit levels and contains no uncountable chains then $X_{\underline{C}}$ contains no uncountable ω-connected subsets.*

Proof. Fix an uncountable $A \subseteq T$ and find two incomparable elements $s, t \in A$. Let r denote the maximal common initial part of s and t; this exists and $r < s, t$ as T does not branch at limits. Find $s' \in A$ so that $r < s' \leqslant s$ and

$$s'^{\downarrow} \cap A \subseteq r^{\downarrow} \cup \{r\}.$$

We claim that s' and t are separated by the finite set $F = \{r\} \cup (r^{\downarrow} \cap C_{s'})$.

Suppose that $P = \{p_i : i < n\} \subset A$ is a finite path from $p_0 = s'$ to t. By Lemma 3.3, we can suppose that P is the union of two monotone paths. Note that $p_1 \in A \cap C_{s'}$ as $p_1 < p_0 = s'$ and hence $p_1 \in r^{\downarrow} \cup \{r\}$ by the choice of s'. In particular, $p_1 \in F$; this shows that F separates s' and t. $\qquad\square$

We are ready now to prove our main result:

Theorem 3.5. *Fix a stationary, costationary $S \subset \omega_1$ and let $T = T(S)$. Then there is a subgraph X of $G(T)$ such that $Chr(X) = \omega_1$ and X contains no uncountable ω-connected subsets.*

Proof. It suffices to show that there is a transitive ladder system $\underline{C}$ on T such that $Chr(X_{\underline{C}}) = \omega_1$; indeed, T has no uncountable chains nor branches at limit levels hence Lemma 3.4 implies that $X_{\underline{C}}$ has no uncountable ω-connected subsets. Furthermore, as the tree T has height ω_1 we have that $Chr(X_{\underline{C}}) \leqslant Chr(G(T)) \leqslant \omega_1$ for any $\underline{C}$. Thus we will have to show that $Chr(X_{\underline{C}}) > \omega$ in the end.

By induction on $\delta \in S'$ (where S' denotes the accumulation points of S), we define the transitive ladder system $\underline{C}$ on $T_{<\delta}$ and hence the corresponding part of $X_{\underline{C}}$ on $T_{<\delta}$.

Fix $\delta \in S'$ and suppose that we already defined a transitive ladder system $(C_t : t \in T_{<\delta})$ on $T_{<\delta}$. We extend this to $T_{<\delta^+}$ while preserving transitivity where δ^+ is the minimum of $S' \backslash (\delta + 1)$. Note that transitivity is necessarily preserved at limit steps. If $\delta \notin S$ then we let $C_t = \varnothing$ for $t \in T_{<\delta^+} \backslash T_{<\delta}$.

Suppose that $\delta \in S$ and let us define C_t for $t \in T_\delta$ first. Let $\{(A_\xi, f_\xi) : \xi < \mathfrak{c}\}$ denote a 1-1 enumeration of all the pairs (A, f) so that $A \in [T_{<\delta}]^\omega$, $f : A \to \omega$ and A satisfies

(1) for every $t \in A$ and $\varepsilon < \delta$ there are incomparable $s^0, s^1 \in A$ so that $s^i \geqslant t$ and $\max(s^i) > \varepsilon$ for $i < 2$.

By induction on $\xi < \mathfrak{c}$ we will find $t_\xi \in T_\delta \backslash \{t_\zeta : \zeta < \xi\}$ and sets $C_{t_\xi} \subset t_\xi^\downarrow$ so that

$$\left(C_t : t \in T_{<\delta} \cup \{t_\xi : \xi < \mathfrak{c}\}\right)$$

is still a transitive ladder system. We will let $C_t = \varnothing$ for $t \in T_\delta \backslash \{t_\xi : \xi < \mathfrak{c}\}$.

Fix a cofinal ω-type sequence $\{\delta_n : n \in \omega\}$ in δ. Suppose we defined $t_\zeta \in T_\delta$ and $C_{t_\zeta} \subseteq t_\zeta^\downarrow$ for $\zeta < \xi$.

Define a map $\psi : 2^{<\omega} \to A_\xi$ and a partial map $\varphi : 2^{<\omega} \to A_\xi$ so that

(i) ψ and φ are order preserving injections and

$$\psi(x) \leqslant \varphi(x) \leqslant \psi(x{}^\frown i)$$

for $i < 2$ provided that $x \in \mathrm{dom}(\varphi)$,
(ii) $\{\varphi(x \restriction k) : k \leqslant |x|, x \restriction k \in \mathrm{dom}(\varphi)\}$ is a complete graph in $T_{<\delta}$,
(iii) $\psi(x{}^\frown 0)$ and $\psi(x{}^\frown 1)$ are incomparable and contained in $A_\xi \backslash T_{<\delta_n}$,
(iv) <u>if</u> there is $t \in A_\xi$ such that $t \geqslant \psi(x)$, $f_\xi(t) = n$ and $\{\varphi(x \restriction k) : k < |x|, x \restriction k \in \mathrm{dom}(\varphi)\} \cup \{t\}$ is complete <u>then</u> $x \in \mathrm{dom}(\varphi)$ and $f_\xi(\varphi(x)) = n$ as well

for all $n \in \omega$ and $x \in 2^n$.

We define $\psi(x)$ and $\varphi(x)$ for $x \in 2^n$ by induction on $n \in \omega$. We set $\psi(\varnothing) \in A_\xi$ arbitrarily. If $\psi(x)$ is defined for some $x \in 2^n$ then check if

$$R_x = \{t \in A_\xi : t \geqslant \psi(x), f_\xi(t) = n \text{ and } \{\varphi(x \restriction k) : k < |x|, x \restriction k \in \mathrm{dom}(\varphi)\} \cup \{t\} \text{ is complete}\}$$

is empty or not. If $R_x \neq \varnothing$ then we put x into $\mathrm{dom}(\varphi)$ and pick $\varphi(x)$ from R_x arbitrarily; otherwise $x \notin \mathrm{dom}(\varphi)$. Now find incomparable $\psi(x{}^\frown 0), \psi(x{}^\frown 1) \in A_\xi$ above $\psi(x)$ so that $\max(\psi(x{}^\frown i)) \geqslant \delta_n$ and $\psi(x{}^\frown i) \geqslant \varphi(x)$ if $x \in \mathrm{dom}(\varphi)$; this can be done as A_ξ satisfies (1) above. This finishes the construction of ψ and φ.
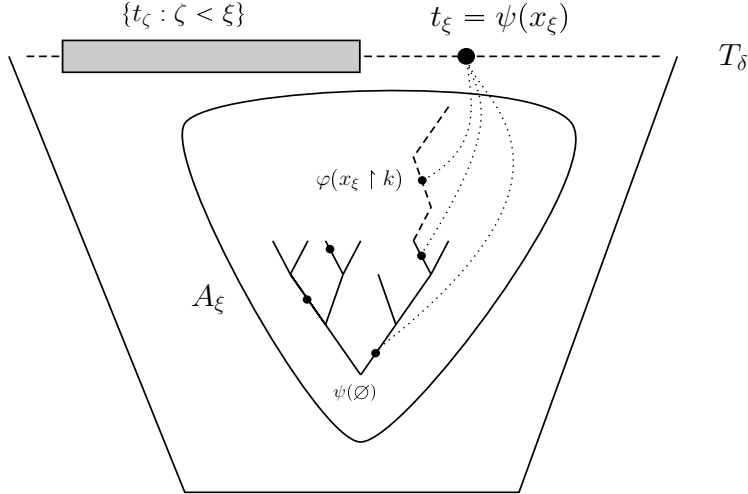

FIGURE 1. Step ξ in the induction.

We extend ψ to 2^ω in the obvious way:

$$\psi(x) = \bigcup \{\psi(x \restriction k) : k < \omega\} \cup \{\delta\}$$

for $x \in 2^\omega$; note that $\psi(x)$ is a closed subset of S by condition (iii) and hence $\psi(x) \in T_\delta$ for all $x \in 2^\omega$. Also, ψ remains 1-1 on 2^ω by condition (iii). Hence, we can find an $x_\xi \in 2^\omega$ such that $\psi(x_\xi) \in T_\delta \backslash \{t_\zeta : \zeta < \xi\}$ and we let $t_\xi = \psi(x_\xi)$. Finally, let

$$C_{t_\xi} = \{\varphi(x_\xi \restriction k) : k < \omega, x_\xi \restriction k \in \mathrm{dom}(\varphi)\}.$$

Transitivity of this extension is assured by condition (ii). This finishes the induction on $\xi < \mathfrak{c}$ and we have a transitive ladder system $\underline{C} = (C_t : t \in T_{\leqslant \delta})$ on $T_{\leqslant \delta}$. We now simply let $C_t = \varnothing$ for $t \in T_{<\delta^+} \backslash T_{\leqslant \delta}$. This finishes step δ of the main induction and hence, in the end, we have a transitive ladder system $\underline{C}$ on T.

We are left to prove:

Claim 3.5.1. $Chr(X_{\underline{C}}) > \omega$.

Proof. Fix a colouring $f : T \to \omega$; we will find $s, t \in T$ so that $f(s) = f(t)$ and $s \in C_t$. Take a countable elementary submodel $M \prec H(\Theta)$ so that $S, \underline{C}, f \in M$ and $\delta = M \cap \omega_1 \in S$; this can be done as S is stationary.

Consider the construction of $\{C_t : t \in T_\delta\}$. If we set $A = M \cap T$ then there must be a $\xi < \mathfrak{c}$ so that $(A, f \restriction A) = (A_\xi, f_\xi)$; M being an elementary submodel ensures that A satisfies property (1) as the branching of T reflects to M.

Our goal now is to prove that there is an $s \in C_{t_\xi}$ so that $f(s) = f(t_\xi)$; we let $n = f(t_\xi)$. Recall that in the definition of t_ξ we had two maps ψ and φ and t_ξ was of the form $\psi(x_\xi)$ for an $x_\xi \in 2^\omega$. C_{t_ξ} was defined to be $\{\varphi(x_\xi \restriction k) : k < \omega, x_\xi \restriction k \in \mathrm{dom}(\varphi)\}$.

Now, recall the definition of $\varphi(x_\xi \restriction n)$: we looked at the set

$$R_{x_\xi \restriction n} = \{s \in A_\xi : s \geqslant \psi(x_\xi \restriction n), f_\xi(s) = n \text{ and}$$
$$\{\varphi(x_\xi \restriction k) : k < n, x_\xi \restriction k \in \mathrm{dom}(\varphi)\} \cup \{s\} \text{ is complete}\} \quad (3.1)$$

and if $R_{x_\xi \restriction n}$ was not empty the we chose $\varphi(x_\xi \restriction n) \in R_{x_\xi \restriction n}$; in particular, $f(s) = n$ for $s = \varphi(x_\xi \restriction n) \in C_{t_\xi}$ which would finish the proof.

Let us show that $R_{x_\xi \restriction n}$ is not empty. Let

$$R = \{s \in T : s \geqslant \psi(x_\xi \restriction n), f(s) = n \text{ and}$$
$$\{\varphi(x_\xi \restriction k) : k < n, x_\xi \restriction k \in \mathrm{dom}(\varphi)\} \cup \{s\} \text{ is complete}\} \quad (3.2)$$

and note that R is in the model M and $R_{x_\xi \restriction n} = R \cap M$. Hence, by elementarity, it suffices to show that $R \neq \varnothing$. However, this is clear as $t_\xi \in R$.

$\square$

This finishes the proof of the theorem. $\square$

4. A VARIATION WITH STRONGER PROPERTIES

The aim of this section is to modify the construction of Theorem 3.5 and produce a ladder system $\underline{C}$ on a tree T such that any two $<_T$-incomparable vertices are separated by a finite set in the graph $X_{\underline{C}}$ i.e. if $s, t \in T$ are incomparable then there is a finite set $F \subset T$ such that any path P from s to t intersects F.

We will say that a sequence $\underline{\eta} = (\eta_t : t \in T)$ is a *true ladder system* on T iff $\eta_t = \{t\}$ for all successor $t \in T$ and η_t is a cofinal ω-sequence if t is limit. Let $supp(\underline{C}) = \{t \in T : |C_t| = \omega\}$ for any ladder system $\underline{C}$ on T.

We use the following notation: if T is a tree, $s \in T$ and $\varepsilon < ht(s)$ then $s \restriction \varepsilon$ denotes the unique element $r \in s^{\downarrow}$ with $ht(r) = \varepsilon$.

Definition 4.1. Suppose that $\underline{C}$ is a ladder systems on a tree T. We say that $\underline{C}$ is *coherent* iff

(1) $C_s = C_t \cap s^{\downarrow}$

for all $t \in supp(\underline{C})$ and $s \in C_t$ with $|C_s| < \omega$ and there is a true ladder system $\underline{\eta}$ on T such that

(2) $\eta_t \cap s^{\downarrow} \sqsubseteq \eta_s$,
(3) $C_t \cap r^{\downarrow} = C_s \cap r^{\downarrow}$ for $r = s \restriction ht(\max(\eta_t \cap s^{\downarrow})) + 1$

for every $s, t \in supp(\underline{C})$ with $s \in C_t$.

The next lemma explains why we introduced this somewhat technical coherence property.

Lemma 4.2. *Suppose that T is a tree with no branching at limits. If $\underline{C}$ is a transitive and coherent ladder system on T then any two $<_T$-incomparable points are separated by a finite set in $X_{\underline{C}}$.*

We use the following notation in the proof: if $s, t \in T$ then $\Delta(s, t)$ denotes the maximal common initial segment of s and t.

Proof. Fix a point $t' \in T$ and we prove that for any $t \in T$ which is incomparable with t' there is a finite set F_t which separates t and t'. The proof is done by induction on $ht(t)$.

We will actually prove that the following choice works: let $F_t = C_t$ if $t \notin supp(C)$ and let

$$F_t = C_t \cap r_t^{\downarrow} \text{ where } r_t = t \restriction ht(\min\{r \in \eta_t : r > \Delta(t, t')\}) + 1$$

if $t \in supp(\underline{C})$.

It is clear that if $t \notin supp(\underline{C})$ then F_t separates t and t' by Lemma 3.3.

Suppose that $t \in supp(\underline{C})$; we note that $r_t < t$ as η_t is nontrivial and $\Delta(t, t') < t$ and hence F_t is finite. Now, suppose that $P = \{p_0 \ldots p_n\}$ is a path from t to t' avoiding F_t which has minimal length and is the union of two monotone paths. We have $p_0 = t > p_1 \in C_t$ and $p_1 > \Delta(t, t')$ and hence $\Delta(t, t') = \Delta(p_1, t')$. We let $s = p_1$ and note that $p_2 \in C_s \backslash C_t$ as P has minimal length and $\underline{C}$ is transitive. If $|C_s| < \omega$ then by assumption $C_t \cap s^{\downarrow} = C_s$ which contradicts $p_2 \in C_s \backslash C_t$.

We conclude that $s \in supp(\underline{C})$. As F_s separates s and t' (by the inductive hypothesis) it suffices to prove that $F_s = F_t$. By definition

$$r_s = s \restriction ht(\min\{r \in \eta_s : r > \Delta(s, t')\}) + 1.$$

As $t, s \in supp(\underline{C})$ we have $\eta_t \cap s^{\downarrow} \sqsubseteq \eta_s$ by coherence. Hence, by $r_t < s$ and $\Delta(t, t') = \Delta(s, t')$ we must have $r_s = r_t$. Finally, by coherence again, we have $F_t = C_t \cap r_t^{\downarrow} = C_s \cap r_s^{\downarrow} = F_s$. $\qquad\square$

Theorem 4.3. *Fix a stationary, costationary $S \subset \omega$ and let $T = T(S)$. Then there is a subgraph X of $G(T)$ such that $Chr(X) = \omega_1$ and any two $<_T$-incomparable points are separated by a finite set in X. In particular, every uncountable set $A \subset T$ contains two vertices which are separated by a finite set in X.*

Note that this new separation property is *stronger* than the lack of uncountable ω-conected sets.

Proof. It suffices to find a transitive and coherent ladder system $\underline{C}$ on T such that $Chr(X_{\underline{C}}) > \omega$; indeed, T does not branch at limits (and every uncountable $A \subset T$ contains two incomparable points) hence Lemma 4.2 can be applied.

We define a true ladder system $\underline{\eta}$ on T first: fix a true ladder system $\{\nu_\delta : \delta \in \omega_1\}$ on ω_1 and let

$$\eta_t = \{t \cap (\varepsilon + 1) : \varepsilon \in \nu_\delta\} \text{ where } \delta = \max(t)$$

for any limit $t \in T$ and let $\eta_t = \{t\}$ for any successor $t \in T$.

Let $D = (S \cap S')'$ and note that D is a club. Let $\delta^+ = \min D \backslash (\delta + 1)$ for $\delta \in D$. By induction on $\delta \in D$, we define a transitive ladder system $\underline{C}$ on $T_{<\delta}$, and hence the corresponding graph on $T_{<\delta}$, so that $\underline{C}$ is coherent and its coherence is witnessed by $\underline{\eta}$.

Suppose we defined $(C_t)_{t \in T_{<\delta}}$. We define $\underline{C}$ on $T_{<\delta^+}$ in two steps: first we define C_t for $t \in T_\delta$ so that $(C_t)_{t \in T_{\leqslant \delta}}$ is still transitive and coherent and then extend to $T_{<\delta^+}$ in the trivial way: $C_t = \varnothing$ for $t \in T_{<\delta^+} \backslash T_{\leqslant \delta}$.

We can suppose $\delta \in S$ otherwise $T_\delta = \varnothing$. Let $\{((A_n^\xi)_{n \in \omega}, f_\xi) : \xi < \mathfrak{c}\}$ denote a 1-1 enumeration of all pairs $((A_n)_{n \in \omega}, f)$ where

(1) $A_n \subset A_{n+1} \in [T_{<\delta}]^\omega$,
(2) for every $t \in A_n$ and $\varepsilon < \delta_n = \sup\{\max(s) : s \in A_n\}$ there are incomparable $s^0, s^1 \in A_n$ so that $s^i \geqslant t$ and $\max(s^i) > \varepsilon$ for $i < 2$,
(3) $f_\xi : A \to \omega$ where $A = \cup\{A_n : n \in \omega\}$,
(4) $(\delta_n)_{n \in \omega}$ is a strictly increasing cofinal sequence in δ.

By induction on $\xi < \mathfrak{c}$, we define $t_\xi \in T_\delta \backslash \{t_\zeta : \zeta < \xi\}$ and C_{t_ξ} while preserving transitivity and coherence. Suppose we defined $t_\zeta \in T_\delta$ and $C_{t_\zeta} \subset t_\zeta^\downarrow$ for $\zeta < \xi$.

We let $A_\xi = \cup\{A_n^\xi : n \in \omega\}$ and $\delta_n^\xi = \sup\{\max(t) : t \in A_n^\xi\}$. Also, let

$$\varepsilon_n = \max(\{\delta_n^\xi\} \cup (\nu_\delta \cap \delta_{n+1}^\xi))$$

for $n \in \omega$ and $\varepsilon_{-1} = \max(\nu_\delta \cap \delta_0^\xi)$. Finally, let $\{l_n : n \in \omega\}$ enumerate each natural number infinitely many times.

Define a map $\psi : 2^{<\omega} \to A_\xi$ and a partial map $\varphi : 2^{<\omega} \to A_\xi$ so that

(i) ψ and φ are order preserving injections and $\psi(x^\frown i) \geqslant \varphi(x)$ for $i < 2$ provided that $x \in \mathrm{dom}(\varphi)$,
(ii) $\{\varphi(x \upharpoonright k) : k \leqslant |x|, x \upharpoonright k \in \mathrm{dom}(\varphi)\}$ is a complete graph in $T_{<\delta}$,
(iii) $\psi(x^\frown 0)$ and $\psi(x^\frown 1)$ are incomparable and contained in $A_{n+1}^\xi \backslash T_{<\varepsilon_n}$,
(iv) <u>if</u> there is $s \in A_n^\xi$ such that
    (a) $s \geqslant \psi(x)$, $f_\xi(s) = l_n$,
    (b) $\{\varphi(x \upharpoonright k) : k < |x|, x \upharpoonright k \in \mathrm{dom}(\varphi)\} \cup \{s\}$ is complete,
    (c) $\nu_\delta \cap \varepsilon_{n-1} \sqsubseteq \nu_{\max(s)}$ and

$$C_s \cap r^\downarrow = \{\varphi(x \upharpoonright k) : k < n, x \upharpoonright k \in \mathrm{dom}(\varphi)\}$$

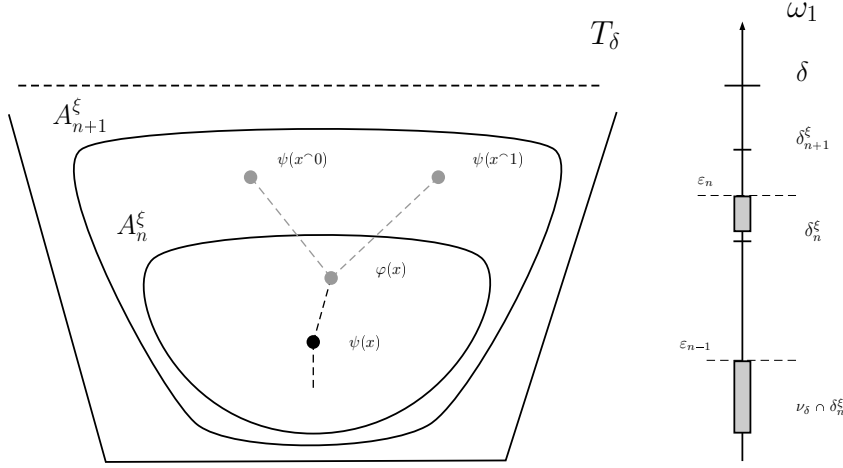   for $r = s \cap (\varepsilon_{n-1} + 1)$ if C_s is infinite,
    (d) $C_s = \{\varphi(x \upharpoonright k) : k < n, x \upharpoonright k \in \mathrm{dom}(\varphi)\}$ if C_s is finite
   <u>then</u> $x \in \mathrm{dom}(\varphi)$ and $s = \varphi(x)$ satisfies (a)-(d) as well,
(v) if (iv) fails then $x \notin \mathrm{dom}(\varphi)$

for all $n \in \omega$ and $x \in 2^n$.

FIGURE 2. Extending the maps φ and ψ.

We define $\psi(x)$ and $\varphi(x)$ for $x \in 2^n$ by induction on $n \in \omega$. We let $\psi(\varnothing) \in A_0^\xi \setminus T_{\leqslant \varepsilon - 1}$ arbitrarily. Given $\psi(x)$, we consider the set R_x^ξ of all elements $s \in A_n^\xi$ such that

(a') $s \geqslant \psi(x)$, $f_\xi(s) = l_n$,
(b') $\{\varphi(x \upharpoonright k) : k < |x|, x \upharpoonright k \in \text{dom}(\varphi)\} \cup \{s\}$ is complete,
(c') $\nu_\delta \cap \varepsilon_{n-1} \sqsubseteq \nu_{\max(s)}$ and

$$C_s \cap r^\downarrow = \{\varphi(x \upharpoonright k) : k < n, x \upharpoonright k \in \text{dom}(\varphi)\}$$

for $r = s \cap (\varepsilon_{n-1} + 1)$ if C_s is infinite,
(d') $C_s = \{\varphi(x \upharpoonright k) : k < n, x \upharpoonright k \in \text{dom}(\varphi)\}$ if C_s is finite.

If R_x^ξ is not empty then we set $x \in \text{dom}(\varphi)$ and let $\varphi(x) \in R_x^\xi$. Otherwise, $x \notin \text{dom}(\varphi)$. Now, we simply pick $\psi(x \cap i)$ for $i < 2$ satisfying conditions (i) and (iii) by applying condition (2) for A_{n+1}^ξ. This finishes the construction of ψ and φ.

We extend ψ to 2^ω in the obvious way:

$$\psi(x) = \bigcup \{\psi(x \upharpoonright k) : k < \omega\} \cup \{\delta\}$$

for $x \in 2^\omega$; note that $\psi(x)$ is a closed subset of S and hence $\psi(x) \in T_\delta$ for all $x \in 2^\omega$ by condition (iii). Also, ψ remains 1-1 on 2^ω by condition (iii). Hence, we can find an $x_\xi \in 2^\omega$ such that $\psi(x_\xi) \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and we let $t_\xi = \psi(x_\xi)$. Finally, let

$$C_{t_\xi} = \left\{\varphi(x_\xi \upharpoonright k) : k < \omega, x_\xi \upharpoonright k \in \text{dom}(\varphi)\right\}.$$

This finishes the induction on $\xi < \mathfrak{c}$ and we define $C_t = \varnothing$ for $t \in T_\delta \setminus \{t_\xi : \xi < \mathfrak{c}\}$.

Claim 4.3.1. $\{C_t : t \in T_{\leqslant \delta}\}$ *is transitive and coherent.*

Proof. Transitivity is assured by condition (ii). We check that $\underline{\eta}$ witnesses that $\{C_t : t \in T_{\leqslant \delta}\}$ is coherent. We fix $\xi < \mathfrak{c}$ and $n < \omega$ such that $x_\xi \upharpoonright n \in \text{dom}(\varphi)$ i.e. $\overline{s} = \varphi(x_\xi \upharpoonright n) \in C_{t_\xi}$.

If C_s is finite then we need that $C_s = C_{t_\xi} \cap s^\downarrow$; this is assured by condition (iv)-(d) above. Suppose that C_s is infinite; we need to check that

$$\eta_{t_\xi} \cap s^\downarrow \sqsubseteq \eta_s \text{ and } C_{t_\xi} \cap r^\downarrow = C_s \cap r^\downarrow$$

for $r = s \restriction ht(\max(\eta_{t_\xi} \cap s^\downarrow)) + 1$. Recall that

$$\eta_{t_\xi} = \{t_\xi \cap (\varepsilon + 1) : \varepsilon \in \nu_\delta\} \text{ and } \eta_s = \{s \cap (\varepsilon + 1) : \varepsilon \in \nu_{\max(s)}\}.$$

Hence $\eta_{t_\xi} \cap s^\downarrow = \{t_\xi \cap (\varepsilon + 1) : \varepsilon \in \nu_\delta \cap \varepsilon_{n-1}\} = \{s \cap (\varepsilon + 1) : \varepsilon \in \nu_\delta \cap \varepsilon_{n-1}\} \sqsubseteq \eta_s$; in the first equality, we use the fact that if $\varepsilon < \max(s)$ and $\varepsilon \in \nu_\delta$ then $\nu_\delta \in \varepsilon_{n-1}$, the second equality follows from $s \sqsubseteq t_\xi$ and $\max(s) > \varepsilon_{n-1}$, and the last relation holds by $\nu_\delta \cap \varepsilon_{n-1} \sqsubseteq \nu_{\max(s)}$ which is the first half of condition (iv)-(c).

Finally, note that $\max(\eta_{t_\xi} \cap s^\downarrow) \leqslant s \cap \varepsilon_{n-1}$ and condition (iv)-(c) says that C_s and C_{t_ξ} agree below $s \cap (\varepsilon_{n-1} + 1)$. This proves $C_{t_\xi} \cap r^\downarrow = C_s \cap r^\downarrow$. $\hfill\square$

This finishes our main induction and hence we constructed a transitive and coherent ladder system $\underline{C}$ on T. It is left to prove

Claim 4.3.2. $Chr(X_{\underline{C}}) > \omega$.

Proof. Fix a colouring $f : T \to \omega$; we will find $s, t \in T$ such that $f(s) = f(t)$ and $s \in C_t$. We fix an increasing sequence $(M_n : n \in \omega)$ of countable elementary submodels of $H(\Theta)$ so that $S, \underline{C}, f \in M_n$ for all $n \in \omega$ and $\delta = M \cap \omega_1 \in S$ with $M = \bigcup\{M_n : n \in \omega\}$.

We consider the construction of $\{C_t : t \in T_\delta\}$. There is a $\xi < \mathfrak{c}$ so that $A_n^\xi = T \cap M_n$ for all $n \in \omega$ and $f_\xi = f \restriction (M \cap T)$. Our goal is to show that there is an $s \in C_{t_\xi}$ such that $f(s) = f(t_\xi)$; we let $l = f(t_\xi)$. Recall that there is an $x = x_\xi \in 2^\omega$ so that $t_\xi = \psi(x)$ and $C_{t_\xi} = \{\varphi(x \restriction k) : k < \omega, x \restriction k \in dom(\varphi)\}$; we will show that there is an $n < \omega$ such that $x \restriction n \in dom(\varphi)$ and $f(\varphi(x \restriction n)) = l$, which finishes the proof.

We first show that C_{t_ξ} is infinite, equivalently that there are infinitely many $n \in \omega$ such that $x \restriction n \in dom(\varphi)$. Suppose otherwise i.e. there is an $n_0 < \omega$ such that $x \restriction n \notin dom(\varphi)$ for all $n \in \omega \backslash n_0$. Let n be the minimal element of $\omega \backslash n_0$ such that $l = l_n$.

Now, recall the definition of $\varphi(x \restriction n)$: we looked at the set $R_{x \restriction n}^\xi$ and if $R_{x \restriction n}^\xi$ was not empty then we put $x \restriction n \in dom(\varphi)$ and chose $\varphi(x_\xi \restriction (n + 1)) \in R_{x \restriction n}^\xi$. Let us show that $R_{x \restriction n}^\xi$ is not empty. Let

$$R = \{s \in T : s \geqslant \psi(x \restriction n), f(s) = l_n \text{ and } C_s = \{\varphi(x \restriction k) : k < n, x \restriction k \in dom(\varphi)\}\}$$

and note that R is in the model M_n and $R \cap M_n \subseteq R_{x \restriction n}^\xi$. Hence, by elementarity, it suffices to show that $R \neq \varnothing$. However, this is clear as $t_\xi \in R$.

Now, we know that C_{t_ξ} is infinite and, with a quite similar argument, we will show that $x \restriction n \in dom(\varphi)$ for the minimal $n \in \omega$ such that $l = l_n$; this finishes the proof of the claim as $s = \varphi(x \restriction n) \in C_{t_\xi}$ and $f(s) = l = f(t_\xi)$. Again, we look at $R_{x \restriction n}^\xi$ and prove that $R_{x \restriction n}^\xi \neq \varnothing$. Consider

$$R = \{s \in T \cap supp(\underline{C}) : s \geqslant \psi(x \restriction n), f(s) = l_n,$$
$$\{\varphi(x \restriction k) : k < n, x \restriction k \in dom(\varphi)\} \cup \{s\} \text{ is complete,}$$
$$\nu_\delta \cap \varepsilon_{n-1} \sqsubseteq \nu_{\max(s)} \text{ and } C_s \cap r^\downarrow = \{\varphi(x \restriction k) : k < n, x \restriction k \in dom(\varphi)\} \text{ for } r = s \cap (\varepsilon_{n-1} + 1)\}.$$
$$\text{(4.1)}$$

It is clear that $R \in M_n$ and $R \cap M_n \subseteq R_{x \restriction n}^\xi$ hence it suffices to show, by elementarity, that $R \neq \varnothing$. However, $t_\xi \in R$.

$\hfill\square$

This finishes the proof of the theorem. $\hfill\square$

5. A TRIANGLE FREE EXAMPLE

In this section, we adapt ideas of A. Hajnal and P. Komjáth [5] to our setting and construct a ladder system $\underline{C}$ on $T = T(S)$ (with $S \subset \omega_1$ stationary and costationary) so that $X_{\underline{C}}$ is uncountably chromatic, triangle free and contains no copies of the graph $H_{\omega,\omega+2}$. A graph with these particular properties is constructed in [5] but using the Continuum Hypothesis; this assumption is no longer used in our construction.

Definition 5.1. Suppose that X is a subgraph of $G(T)$. A cycle $x_0, x_1, ... x_n = x_0$ is *special* if it is the union of two $<_T$-monotone paths.

Note that every triangle is a special cycle; our aim is to construct the graph $X_{\underline{C}}$ without special cycles.

Definition 5.2. Suppose that X is a subgraph of $G(T)$ and $\gamma < \omega_1$. We say that a vertex $v \in T$ is γ-*covered* iff there exists a point $w \in T_{\leqslant \gamma}$ and a monotone path from w to v.

Definition 5.3. A ladder system $\underline{C}$ on T is *sparse* iff s is not $\max(r)$-covered for each $t \in T$ and $r, s \in C_t$ with $r < s$.

Note that if $\underline{C}$ is sparse then C_t is independent in $X_{\underline{C}}$ for all $t \in T$. The following was essentially proved in [5]:

Lemma 5.4. *Suppose that $\underline{C}$ is a ladder system on T. Then*
   (1) if $\underline{C}$ is sparse then $X_{\underline{C}}$ contains no special cycles,
   (2) if $X_{\underline{C}}$ contains no special cycles then $H_{\omega,\omega+2} \not\hookrightarrow X_{\underline{C}}$

Proof. (1) Suppose that $x_0, x_1, \ldots x_n = x_0$ is a special cycle with $\max(x_i) = \alpha_i$. Hence, there is $i < n$ so that $\alpha_j < \alpha_i$ if $i \neq j \leqslant n$. In particular, $x_{i-1}, x_{i+1} \in C_{x_i}$ and without loss of generality $\alpha_{i-1} < \alpha_{i+1} < \alpha_i$. However, this implies that x_{i+1} is α_{i-1}-covered, witnessed by the path $x_n, x_{n-1}, \ldots x_{i+1}$, which contradicts that $\underline{C}$ is sparse.

(2) Suppose that $\{x_i, y_i, z, z' : i < \omega\}$ is a subgraph of $X_{\underline{C}}$ isomorphic to $H_{\omega,\omega+2}$ i.e. the following pairs of points are edges
$$\left\{ \{x_i, y_j\}, \{x_i, z\}, \{x_i, z'\} : i \leqslant j \in \mathbb{N} \right\}.$$
First, as x_i and x_j has infinitely many common neighbors (for $i < j < \omega$) they must be $<_T$-comparable; hence we can suppose that $x_0 < x_1 < ....$. Second, either z or z' is $<_T$-below infinitely many x_i so we might as well suppose that $z < x_i$ for all $i < \omega$. Finally, we have $z <_T x_0 <_T x_1$ and x_0, x_1 have infinitely many common neighbors of the form y_j with $\max(y_j) > \alpha_1$. In particular, we can find special cycles of length 4 which contradicts our assumption. $\square$

Lemma 5.5. *Fix a stationary, costationary $S \subset \omega_1$ and let $T = T(S)$. Suppose that X is a subgraph of $G(T)$ and $f : T \to \omega$. Then there is $\delta \in \omega_1$ and $t \in T_\delta$ so that for every $n \in \omega$ either:*
   (1) for every $s \geqslant t$ and every $\gamma \in \omega_1$ there is a non γ-covered $r \geqslant s$ with $f(r) = n$, or
   (2) every $r \geqslant t$ with $f(r) = n$ is δ-covered.

Proof. Take a countable elementary submodel $M \prec H(\Theta)$ with $f, X, S \in M$ so that $M \cap \omega_1 = \delta \in S$. Fix a cofinal sequence $\{\delta_n : n \in \omega\}$ of type ω in δ.

Now, construct a sequence $t_0 \leqslant .... \leqslant t_n \leqslant ...$ in $M \cap T$ so that $\max(t_n) \geqslant \delta_n$ and for every $n \in \omega$ either
   (i) for every $s \geqslant t_{n+1}$ and every $\gamma \in \omega_1$ there is a *non* γ-covered $r \geqslant s$ with $f(r) = n$, or
   (ii) there is a $\gamma_n \in \delta$ so that every $r \geqslant t_{n+1}$ with $f(r) = n$ is γ_n-covered.

We can pick $t_0 \in M \cap T$ with $\max(t_0) \geqslant \delta_0$ arbitrarily. Given $t_n \in M$ we select $t'_{n+1} \in M$ above t_n so that $\max(t'_{n+1}) \geqslant \delta_{n+1}$. If the choice $t_{n+1} = t'_{n+1}$ satisfies (i) from above then we are done; otherwise, there is $s = t_{n+1} \geqslant t'_{n+1}$ and $\gamma = \gamma_n$ so that every $r \geqslant t_{n+1}$ with $f(r) = n$ is γ_n-covered. As t_{n+1} and γ_n can be chosen in M, by elementarity, we are done as (ii) is satisfied.

Now, let $t = \bigcup \{t_n : n \in \omega\} \cup \{\delta\}$ and note that $t \in T_\delta$; we claim that this t decides f. Indeed, as $t \geqslant t_n$ for all $n \in \omega$ and t_n satisfies (i) or (ii), t must satisfy either (1) or (2) respectively.

$\square$

Theorem 5.6. *Fix a stationary, costationary $S \subset \omega_1$ and let $T = T(S)$. Then there is subgraph X of $G(T)$ with $Chr(X) = \omega_1$ such that X contains no special cycles; in particular, X contains no triangles or copies of $H_{\omega, \omega+2}$.*

Proof. It suffices to construct a sparse ladder system $\underline{C}$ on T so that $Chr(X_{\underline{C}}) = \omega_1$; indeed, by Lemma 5.4, a sparse ladder system $\underline{C}$ induces a graph $X_{\underline{C}}$ on T with no special cycles and hence no triangles or copies of $H_{\omega, \omega+2}$.

We define a sparse ladder system $(C_t : t \in T_{<\delta})$ by induction on $\delta \in Lim(\omega_1)$ and so that $C_t = \varnothing$ if $t \in T$ is a successor. Suppose we constructed C_t for $t \in T_{<\delta}$; we only need to define C_t for $t \in T_\delta$ hence we can suppose that $\delta \in S$.

We let $\{(A_\xi, f_\xi, t_{0\xi}) : \xi < \mathfrak{c}\}$ be an 1-1 enumeration of all triples (A, f, t_0) with $A \in [T_{<\delta}]^\omega$, $f : A \to \omega$ and $t_0 \in A$ so that

(1) for every $t \in A$ and $\varepsilon < \delta$ there are incomparable $s^0, s^1 \in A$ so that $s^i \geqslant t$ and $\max(s^i) \geqslant \varepsilon$ for $i < 2$, and

By induction on $\xi < \mathfrak{c}$ we define $t_\xi \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and sparse C_{t_ξ}. Suppose we have $\{t_\zeta : \zeta < \xi\}$ defined and consider the triple $(A_\xi, f_\xi, t_{0\xi})$. Fix a cofinal increasing sequence $\{\delta_n : n \in \omega\}$ of type ω in δ.

We define a map $\psi : 2^{<\omega} \to A_\xi$ and a partial map $\varphi : 2^{<\omega} \to A_\xi$ so that

(i) ψ and φ are order preserving injections and $\psi(x^\frown i) \geqslant \varphi(x) \geqslant \psi(x) > t_{0\xi}$ for $i < 2$ provided that $x \in \mathrm{dom}(\varphi)$,
(ii) $\psi(x^\frown 0)$ and $\psi(x^\frown 1)$ are incomparable and contained in $A_\xi \setminus T_{<\delta_n}$,
(iii) $\varphi(x)$ is not $\max(\psi(x))$-covered,
(iv) <u>if</u> there is $s \in A_\xi$ such that
    (a) $s \geqslant \psi(x), f_\xi(s) = n$,
    (b) s is not $\max(\psi(x))$-covered
    <u>then</u> $x \in \mathrm{dom}(\varphi)$ and $\varphi(x)$ satisfies (a)-(b) as well

for all $x \in 2^n$ and $n \in \omega$.

We define $\psi(x)$ and $\varphi(x)$ for $x \in 2^n$ by induction on $n \in \omega$. We select $\psi(\varnothing) > t_{0\xi}$ arbitrarily in A_ξ. Given $\psi(x)$ for $x \in 2^n$ we look at the set

$$R_x^\xi = \{s \in A_\xi : s \geqslant \psi(x), f_\xi(s) = n \text{ and } s \text{ is not } \max(\psi(x))\text{-covered}\}.$$

If R_x^ξ is not empty then let $x \in \mathrm{dom}(\varphi)$ and pick any $\varphi(x) \in R_x^\xi$; otherwise $x \notin \mathrm{dom}(\varphi)$. Now, using condition (1) of A_ξ, select incomparable $\psi(x^\frown 0)$ and $\psi(x^\frown 1)$ so that conditions (i)-(ii) are satisfied. This finishes the construction of ψ and φ.

Now extend ψ to 2^ω in the obvious way:

$$\psi(x) = \bigcup \{\psi(x \upharpoonright k) : k < \omega\} \cup \{\delta\}$$

for $x \in 2^\omega$; note that $\psi(x)$ is a closed subset of S and hence $\psi(x) \in T_\delta$ for all $x \in 2^\omega$ by condition (ii). Also, ψ remains 1-1 on 2^ω by condition (ii). Hence, we can find an $x_\xi \in 2^\omega$ such that $\psi(x_\xi) \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and we let $t_\xi = \psi(x_\xi)$. Finally, let

$$C_{t_\xi} = \big\{\varphi(x_\xi \upharpoonright k) : k < \omega, x_\xi \upharpoonright k \in \operatorname{dom}(\varphi)\big\}.$$

Note that condition (iii) ensures that C_{t_ξ} is sparse. This finishes the induction on $\xi < \mathfrak{c}$ and in turn the induction on $\delta \in Lim(\omega_1)$.

We are left to prove

Claim 5.6.1. $Chr(X_{\underline{C}}) > \omega$.

Proof. Fix a colouring $f : T \to \omega$; we will find $s, t \in T$ so that $f(s) = f(t)$ and $s \in C_t$. Take a countable elementary submodel $M \prec H(\Theta)$ so that $S, \underline{C}, f \in M$ and $\delta = M \cap \omega_1 \in S$. By Lemma 5.5, we can find $t_0 \in M \cap T$ so that t_0 decides f.

Now, consider the construction of $\{C_t : t \in T_\delta\}$; note that there is a $\xi < \mathfrak{c}$ so that $(A_\xi, f_\xi, t_{0\xi}) = (A, f \upharpoonright A, t_0)$ where $A = T \cap M$. We will show that there is $s \in C_{t_\xi}$ with $f(s) = f(t_\xi)$.

Let $f(t_\xi) = n$.

Observation 5.7. *For every $s \geqslant t_0$ and every $\gamma \in \omega_1$ there is a* non γ-covered $r \geqslant s$ with $f(r) = n$.

Proof. Recall that t_0 decides f, so if the above statement fails then every $s \geqslant t_0$ with $f(s) = n$ is $\max(t_0)$-covered. In particular, t_ξ is $\max(t_0)$-covered. However, this implies that C_{t_ξ} is not empty and there is $s \in C_{t_\xi}$ which is $\max(t_0)$-covered (possibly $s \leqslant t_0$). However, every $s \in C_{t_\xi}$ is not $\max(t_{0\xi})$-covered by conditions (i) and (iii) above; this contradiction finishes the proof. $\square$

Recall that there is an $x \in 2^\omega$ such that $t_\xi = \psi(x)$ and $C_{t_\xi} = \{\varphi(x \upharpoonright k) : k < \omega, x \upharpoonright k \in \operatorname{dom}(\varphi)\}$. Our aim is to show that $x \upharpoonright n \in \operatorname{dom}(\varphi)$ and hence $f(s) = f(t_\xi)$ where $s = \varphi(x \upharpoonright n)$. Hence, we need to show that

$$R^\xi_{x \upharpoonright n} = \{s \in A_\xi : s \geqslant \psi(x \upharpoonright n), f_\xi(s) = n \text{ and } s \text{ is not } \max(\psi(x \upharpoonright n))\text{-covered}\}$$

not empty.

Note that $R^\xi_{x \upharpoonright n} = R \cap M$ and $R \in M$ where

$$R = \{s \in T : s \geqslant \psi(x \upharpoonright n), f(s) = n \text{ and } s \text{ is not } \max(\psi(x \upharpoonright n))\text{-covered}\}.$$

Hence, by elementarity, it suffices to show that $R \neq \varnothing$. This clearly follows from Observation 5.7.
$\square$

This finishes the proof of the theorem. $\square$

Let us remark that sparse and transitive ladder systems represent two extremes in the spectrum of subgraphs of $G(T)$; if $\underline{C}$ is sparse then C_t is an independent set while if $\underline{C}$ is transitive then C_t is a complete subgraph.

6. REMARKS

We would like to point out that some of the graphs defined in our paper satisfy strong *partition properties*. If T is a tree and X is a subgraph of $G(T)$ then we write

$$X \to (K_{\omega+1})^1_\omega$$

iff for every colouring $f : T \to \omega$ there is an $n \in \omega$ and a set $A \subset T \cap f^{-1}(n)$ of $<_T$-ordertype $\omega + 1$ such that A spans a complete graph (i.e. A is a monochromatic copy of $K_{\omega+1}$). Clearly,

$X \to (K_{\omega+1})^1_\omega$ implies that $Chr(X) > \omega$ but not necessarily the other way; indeed, as seen in Theorem 5.6, there are even triangle free subgraphs of $G(T)$ (for some T) which are uncountably chromatic.

Let us first show that satisfying the above partition property or having large chromatic number are equivalent for *transitive* ladder systems.

Proposition 6.1. *Suppose that T is a tree and $\underline{C}$ is a ladder system on T. If $\underline{C}$ is transitive and $Chr(X_{\underline{C}}) > \omega$ then*

$$X_{\underline{C}} \to (K_{\omega+1})^1_\omega.$$

Proof. Fix an $f : T \to \omega$; we will show that there is an $n \in \omega$ and $t \in f^{-1}(n)$ so that $A = C_t \cap f^{-1}(n)$ is infinite hence, by transitivity, $A \cup \{t\}$ gives a monochromatic copy of $K_{\omega+1}$ in $X_{\underline{C}}$.

Suppose otherwise i.e. $C_t \cap f^{-1}(n)$ is finite for every $t \in T$ with $f(t) = n$. We can define a new colouring $g : T \to \omega \times \omega$ using induction on the height so that $g(t) = (f(t), g_1(t))$ where $g_1(t) = \max\{g_1(s) : s \in C_t \cap f^{-1}(n)\} + 1$ with $n = f(t)$. It is easy to see that g witnesses $Chr(X_{\underline{C}}) \leqslant \omega$ which is a contradiction. $\qquad\square$

The above proposition is nicely complemented by

Observation 6.2. *If T is a tree of height ω_1 and $\underline{C}$ is a ladder system on T then there is no complete subgraph in $X_{\underline{C}}$ of $<_T$-order type $\omega + 2$.*

Recall that in the proof of Theorem 4.3, we used a ladder system on ω_1 (denoted by $\underline{\nu}$ there) to define another ladder system (denoted by $\underline{\eta}$) on $T = T(S)$ for $S \subset \omega_1$ in a very natural way. We did not consider the subgraph of $G(T)$ corresponding to $\underline{\eta}$ at that point so let us present a result here.

Proposition 6.3. *Suppose that $S \subset \omega_1$ is stationary and let $T = T(S)$. Fix a ladder system $\underline{\nu} = \{\nu_\delta : \delta \in Lim(\omega_1)\}$ on ω_1. Let*

$$C_t = \{t \cap (\varepsilon + 1) : \varepsilon \in \nu_\delta\}$$

for any limit $t \in T_\delta$ and $\delta \in S$ and let $C_t = \varnothing$ otherwise. Then

$$X_{\underline{C}} \to (K_{\omega+1})^1_\omega.$$

Proof. Let $f : T \to \omega$. We say that $D \subseteq T$ is *dense above* $t \in T$ iff for every $s \geqslant t$ there is $r \in D$ such that $r \geqslant s$. D is *empty above* t iff $s \notin D$ for every $s \geqslant t$.

Claim 6.3.1. *There is t_0 in T so that either $f^{-1}(k)$ is empty or dense above t_0 for every $k \in \omega$.*

Proof. The proof is very similar to the argument seen in Lemma 5.5 so we will be brief here. Take a countable elementary submodel $M \prec H(\Theta)$ with $f, S \in M$ so that $\delta = M \cap \omega_1 \in S$. Build a sequence $s_0 \leqslant s_1 \leqslant \ldots$ in $T \cap M$ so that $(\max(s_k) : k \in \omega)$ is a cofinal ω-type sequence in δ and $f^{-1}(k)$ is either empty or dense above s_k. It is easy to see that

$$t_0 = \bigcup \{s_k : k \in \omega\} \cup \{\delta\}$$

is in T and satisfies the claim. $\qquad\square$

Take a countable elementary submodel $M \prec H(\Theta)$ with $t_0, f, S, \underline{\nu} \in M$ and $\delta = M \cap \omega_1 \in S$. Let $\{k_n : n \in \omega\}$ enumerate those $k \in \omega$ so that $f^{-1}(k)$ is dense above t_0 (or equivalently, not empty above t_0), each ω times. Let $(\delta_n : n \in \omega)$ be an arbitrary cofinal ω-type sequence in δ.

We construct $t_0 \leqslant t_1 \leqslant \ldots. \leqslant t_n \leqslant \ldots$ in $T \cap M$ so that

(1) $\max(t_{n+1}) \geqslant \delta_n$,

(2) $t_{n+1} \cap (\varepsilon_n + 1) = t_n$ for $\varepsilon_n = \min \nu_\delta \backslash (\max(t_n) + 1)$,

(3) <u>if</u> there is $t \geqslant t_n$ in $T \cap M$ such that

     (a) $\max(t) \geqslant \delta_n$, $t \cap (\varepsilon_n + 1) = t_n$,

     (b) $\{t_i : i \leqslant n\} \subseteq C_t$ and

     (c) $f(t) = k_n$

     <u>then</u> $t = t_{n+1}$ satisfies (a)-(c) as well.

Let $t = \bigcup \{t_n : n \in \omega\} \cup \{\delta\}$; note that $t \in T_\delta$ and $t \geqslant t_n$ for all $n \in \omega$. Also, (2) ensures that $t_n \in C_t$ for $n \in \omega$ as $t_n = t \cap (\varepsilon_n + 1)$ and $\varepsilon_n \in \nu_\delta$.

Let $k = f(t)$; as $t \geqslant t_0$ we know that $f^{-1}(k)$ is dense above t_0. We claim that

$$A = \{t_{n+1} : k = k_n, n \in \omega\} \cup \{t\}$$

is a complete subgraph $X_{\underline{C}}$ and A is coloured with k which finishes the proof. It suffice to show that whenever $k_n = k$ and t_{n+1} is constructed then condition (3) is satisfied. Fix an $n \in \omega$ such that $k_n = k$. Consider the set

$$R_n = \{t \in T \cap M : \max(t) \geqslant \delta_n, t \cap (\varepsilon_n + 1) = t_n, \{t_i : i \leqslant n\} \subseteq C_t \text{ and } f(t) = k_n\}$$

and we wish to show that $R_n \neq \varnothing$.

It suffices to show that

$$R = \{t \in T : \max(t) \geqslant \delta_n, t \cap (\varepsilon_n + 1) = t_n, \{t_i : i \leqslant n\} \subseteq C_t \text{ and } f(t) = k_n\}$$

is not empty as $R_n = R \cap M$ and $R \in M$. However $t \in R$ which finishes the proof.

$\square$

Finally, we mention that the following most general form of the Erdős-Hajnal problem is still open:

Problem 6.4. *Does every uncountably chromatic (or ω_1-chromatic) graph contain an ω-connected subset?*

We know, by Theorem 3.5, that this ω-connected set can only be countable in some cases however excluding countable ω-connected subsets seems to be a very hard problem.

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