UNCOUNTABLE STRONGLY SURJECTIVE LINEAR ORDERS

DÁNIEL T. SOUKUP

Abstract. A linear order L is *strongly surjective* if L can be mapped onto any of its suborders in an order preserving way. We prove various results on the existence and non-existence of uncountable strongly surjective linear orders answering questions of Camerlo, Carroy and Marcone. In particular, $\diamondsuit^+$ implies the existence of a lexicographically ordered Suslin-tree which is strongly surjective and minimal; every strongly surjective linear order must be an Aronszajn type under $2^{\aleph_0} < 2^{\aleph_1}$ or in the Cohen and other canonical models (where $2^{\aleph_0} = 2^{\aleph_1}$); finally, we prove that it is consistent with CH that there are no uncountable strongly surjective linear orders at all.

1. Introduction

Our paper deals with a simple, natural yet newly introduced notion due to R. Camerlo, R. Carroy and A. Marcone [12]: a linear order L is *strongly surjective* iff $L \twoheadrightarrow K$ for all $K \subseteq L$. That is, L maps onto any of its suborders in an order preserving way. The reader is invited to show that, for example, ω and $\mathbb{Q}$ are strongly surjective but $\omega + 1$ is not. Strongly surjective ordinals were characterized in [12, Corollary 29] and then the authors proceeded by studying strong surjectivity in more generality [13].

The main question we tackle in our paper is as simple as this: are there any strongly surjective linear orders which are not countable? If yes, what are the possible order types of these linear orders? It was observed in [13] already that any strongly surjective linear order L must be *short* i.e. contains no copies of ω_1 or $-\omega_1$; this implies that $|L| \leq \mathfrak{c} = 2^{\aleph_0}$ (see [14, Theorem 3.4]). Now, an uncountable short linear order either contains an uncountable real order type, or it is a so called *Aronszajn-line* (by definition).

Our first goal is to look at separable and real order types with regards to strong surjectivity. It was proved in [13] that if the Proper Forcing Axiom holds then any $\aleph_1$-dense set of real numbers is strongly surjective; on the other hand, the Continuum Hypothesis implies that every separable, strongly surjective linear order is countable. In particular, uncountable separable strongly surjective linear orders may or may not exist. Now, how crucial is CH in eliminating uncountable, separable, strongly surjective linear orders? Firstly, we prove that CH actually implies that if L is strongly surjective then any countable subset has countable closure in the interval topology. Second, we prove that $2^{\aleph_0} = \aleph_2$ is also consistent with every separable, strongly surjective linear order being countable; this can be achieved using Cohen-forcing or other canonical models (e.g. Sacks or Miller) where we can use the technique of parametrized weak diamonds. These topics are discussed in Section 2 and 3.

Now, looking at Aronszajn-types our main result is the following: the guessing principle $\diamondsuit^+$ implies that there is a ccc, non separable strongly surjective linear order L. In [13], the

Date: June 30, 2017.

2010 *Mathematics Subject Classification.* 03E04, 03E35.

Key words and phrases. strongly surjective, Suslin tree, weak diamond, axiom A, $\aleph_1$-dense.

authors outlined a plan to achieve this result building on a theorem of J. Baumgartner, who constructs a *doubly ordered Suslin-tree* that is doubly isomoprhic to all its large subsets. However, Baugartner's original proof (his crucial [4, Lemma 4.14]) has a serious gap as pointed out by Hossein Lamei Ramandi; we intend to present a complete proof of these results in Section 4.

At this point, we achieved that there could be uncountable real or Aronszajn linear orders which are strongly surjective, but CH only allows the second type. Is it possible that there is a model of CH where there are no uncountable strongly surjective linear orders at all? Our final main result is a positive answer to this question: we show, in Section 5, that CH together with J. Moore's axiom (A) implies that every strongly surjective linear order is countable. Our paper closes with a healthy list of open problems.

Notations. Given a linear order L with order $<$, we use the notation $[x,y]_<$ to denote the closed interval between x and y i.e. $\{z \in L : x \leq z \leq y\}$. We say that D is dense in L iff $x < y$ implies that $x < d < y$ for some $d \in D$; D is *κ-dense* if $x < y$ implies that $x < d < y$ for κ-many $d \in D$.

Given a linear order X and linear orders K_x for each $x \in X$ we can define $L = \sum_{x\in X} K_x$ as the disjoint union $\sqcup_{x\in X} K_x$ so that $a <_L b$ iff $a \in K_x, b \in K_y$ and $x <_X y$ or $x = y$ and $a <_{K_x} b$. Note that $\sum_{x\in X} K_x \twoheadrightarrow X$ by the order-preserving $K_x \mapsto \{x\}$. On the other hand, if $f : L \twoheadrightarrow X$ then $L = \sum_{x\in X} f^{-1}(x)$.

In Section 5, we work with Aronszajn trees (or A-trees, for short) i.e. $\aleph_1$-trees without uncountable branches. We say that a tree is *Hausdorff* if there are no branching at limit levels; in contrast to the literature, we will work with non Hausdorff trees mostly. We refer the reader for more on trees and their relation to linear orders to [14].

2. On strongly surjective real suborders

We mentioned already that it is independent of ZFC whether there is an uncountable, strongly surjective suborder of $\mathbb{R}$ [13]. A notion which seems closely tied to strong surjectivity is being *minimal*: an uncountable linear order L is minimal if L embeds into all of its suborders. Minimal linear orders have been studied extensively in the last few decades (see [4, 2, 9] for example).

Our first goal is to further clarify when uncountable strongly surjective suborders of $\mathbb{R}$ exist and if so, how are they related to minimal orders. Let us recall a theorem from [13] first.

Theorem 2.1. [13, Proposition 5.9] *Suppose that $X \subseteq \mathbb{R}$ is the unique κ-dense suborder of the reals. Then X is strongly surjective (and minimal if $\kappa = \aleph_1$).*

The existence of a unique $\aleph_1$-dense suborder of $\mathbb{R}$ was proved consistent by J. Baumgartner [3] in 1973. Let BA_κ denote the statement that any two κ-dense subsets of $\mathbb{R}$ are isomorphic. $\mathrm{BA}_{\aleph_1}$ is a consequence of PFA (but requires no large cardinals) and $\mathrm{BA}_{\mathfrak{c}}$ fails in ZFC. The consistency of $\mathrm{BA}_{\aleph_2}$ was only recently announced by I. Neeman and his proof uses large cardinals. However, the consistency of BA_κ is still open for $\kappa > \aleph_2$; in turn, Theorem 2.1 at the moment only implies the existence of strongly surjective, real suborders of size $\leq \aleph_2$.

Now, under $\mathrm{BA}_{\aleph_1}$, any $\aleph_1$-dense suborder L of $\mathbb{R}$ is strongly surjective and minimal. Our first goal is to show the following.

Theorem 2.2. *Consistently, there is an $\aleph_1$-dense strongly surjective suborder L of $\mathbb{R}$ which is not minimal.*

We start by a definition: a set $A \subseteq \mathbb{R}$ is *increasing* if whenever $\{a_\xi : \xi < \omega_1\} \subseteq [A]^n$ for some $n \in \omega$ then there is $\alpha < \beta$ so that $a_\alpha(i) \leq a_\beta(i)$ for all $i < n$.

Lemma 2.3. [1] *If A is increasing then A and $-A$ has no common uncountable suborder.*

In particular, $BA_{\aleph_1}$ fails if there is an uncountable, increasing $A \subseteq \mathbb{R}$.

Theorem 2.4. [2, Theorem 3.1, 6.2] *Consistently, $MA_{\aleph_1} + OCA$ holds and there is an $\aleph_1$-dense, increasing $A \subseteq \mathbb{R}$. Furthermore, if B is $\aleph_1$-dense and does not contain a copy of $-A$ then B is isomorphic to A.*

Suppose that A is as in Theorem 2.4; note that Lemma 2.3 implies that A is minimal.

Proposition 2.5. *A is strongly surjective.*

Proof. If $X \subseteq A$ then there is a countable $Y \subseteq X$ so that $X \setminus Y$ is $\aleph_1$-dense. Since $X \setminus Y$ cannot contain a copy of $-A$, $X \setminus Y$ must be isomorphic to A. Now let $K = \sum_{x \in X} K_x$ where

$$K_x = \begin{cases} \{x\}, & \text{for } x \in X \setminus Y, \text{ and} \\ A, & \text{for } x \in Y. \end{cases}$$

K is still a suborder of $\mathbb{R}$ and contains no copies of $-A$ being a countable union of copies of A. So K is again isomoprhic to A. However, K trivially maps onto X (by $K_x \mapsto \{x\}$) so $A \twoheadrightarrow X$ as desired. □

Hence, $\mathrm{BA}_{\aleph_1}$ can fail while there are uncountable, strongly surjective suborders of $\mathbb{R}$. However, A was minimal so this is not a big surprise.

Now, note that if L is strongly surjective then so does $L \times 2$. Hence, if $L \subseteq \mathbb{R}$ is strongly surjective then $L \times 2$ is also strongly surjective but cannot be minimal (since $L \times 2$ does not embed into L). But $L \times 2$ is not a suborder of $\mathbb{R}$ so we need to finish the proof of Theorem 2.2 slightly differently.

Proof of Theorem 2.2. Suppose that A is the increasing set from the model of Theorem 2.4. It suffices to find a strongly surjective $L \subseteq \mathbb{R}$ which contains both A and $-A$. Then L is clearly not minimal.

Lemma 2.6. *Suppose that L_i is a strongly surjective linear orders for $i < \omega$. Then there is a single strongly surjective L so that $L_i \subseteq L$.*

Proof. We define $L = \sum_{q \in \mathbb{Q}} K_q$ so that each K_q is a copy of some L_i and $\{q \in \mathbb{Q} : K_q = L_i\}$ is dense in $\mathbb{Q}$ for all $i < \omega$.

Now let $X \subseteq L$ and let $g : \mathbb{Q} \twoheadrightarrow A$ so that $g^{-1}(q)$ is an open interval in $\mathbb{Q}$ for all $q \in A$ where $A = \{r : X \cap K_r \neq \varnothing\}$. We will define $f(x) \in K_g(q) \cap X$ for $x \in K_q$ so that $f : L \twoheadrightarrow X$. Note that if $r \in A$ then $X \cap K_r$ is some nonempty subset of L_i.

So fix $r \in A$ and suppose that $X \cap K_r \subseteq L_i$. First, we can map $\sum_{q \in g^{-1}(r)} K_q$ onto $\sum_{q \in g^{-1}(r)} \tilde{K}_q$ where we let $\tilde{K}_q = L_i$ if $K_q = L_i$ and $\tilde{K}_q$ singleton otherwise. It suffices to show that $\sum_{q \in g^{-1}(r)} \tilde{K}_q$ can be mapped onto L_i; since $L_i \twoheadrightarrow X \cap K_r$ holds as L_i is strongly surjective, we clearly have $\sum_{q \in g^{-1}(r)} K_q \twoheadrightarrow X \cap K_r$ as desired.

To see that there is $h : \sum_{q \in g^{-1}(r)} \tilde{K}_q \twoheadrightarrow L_i$ let us pick $(q_k)_{k \in \mathbb{Z}} \subseteq g^{-1}(r)$ cofinal, coinitial of type $\mathbb{Z}$ so that $\tilde{K}_{q_k} = L_i$ and cofinal, coinitial $(z_k)_{k \in \mathbb{Z}}$ in L_i. Define h so that $h \restriction \sum_{q \in (q_{k-1}, q_k)_{\mathbb{Q}}} \tilde{K}_q$ is constant z_k and $h \restriction \tilde{K}_{q_k} : \tilde{K}_{q_k} \twoheadrightarrow (z_k, z_{k+1})_{L_i}$. □

This finishes the proof of Theorem 2.2.

□

At this point, it is unclear if a minimal uncountable $L \subseteq \mathbb{R}$ is necessary strongly surjective.

A well-studied strengthening of being increasing is the following: we say that A is k-*entangled* iff for any uncountable set $\{(a_0^\xi \dots a_{k-1}^\xi) : \xi < \omega_1\}$ of k-tuples from A and any $\varepsilon : k \to 2$ there are $\xi < \zeta < \omega_1$ so that

$$a_i^\xi < a_i^\zeta \text{ iff } \varepsilon(i) = 0$$

for all $i < k$. Note that if L is strongly surjective then L is short and being short is equivalent to being 1-entangled.

Proposition 2.7. *Suppose that L is 2-entangled. If $f : L \twoheadrightarrow X$ and X is uncountable then there is a countable $A \subseteq X$ so that $f^{-1}(x) = \{x\}$ for all $x \in X \setminus A$.*

Proof. Inductively define $a_\alpha, b_\alpha = f(a_\alpha)$ for $\alpha < \omega_1$ so that $b_\beta \in X \setminus \{f(b_\alpha), b_\alpha, a_\alpha : \alpha < \beta\}$ and $a_\beta \in f^{-1}(b_\beta) \setminus \{b_\beta\}$. It is straightforward to check that $a_\beta, b_\beta \notin \{a_\alpha, b_\alpha : \alpha < \beta\}$.

If the induction can go on for ω_1 steps then $\{(a_\alpha, b_\alpha) : \alpha < \omega_1\}$ violates that L is 2-entangled. Hence, $f^{-1}(b) = \{b\}$ for all $b \in X \setminus A$ for some $\beta < \omega_1$ where $A = \{f(b_\alpha), b_\alpha, a_\alpha : \alpha < \beta\}$. □

Corollary 2.8. *If L is uncountable and strongly surjective then L is not 2-entangled.*

Proof. Suppose that L is also 2-entangled and pick an $\aleph_1$-dense $X \subseteq L$ so that $\overline{X} \cap L \setminus X$ is uncountable. Suppose that $f : L \twoheadrightarrow X$. Let A be countable so that $f^{-1}(x) = \{x\}$ for all $x \in X \setminus A$. In particular $f(x) = x$ for all $x \in X \setminus A$. Also, $f[L \setminus X] \subseteq A$ and so there is $c < d \in \overline{X} \cap L \setminus X$ so that $f(c) = f(d) \in A$. Now, there must be some $x \in X \setminus A$ so that $c < x < d$; however, $f(x) = x \in A$ is a contradiction.

□

I don't know if a strongly surjective linear order can contain a 2-entangled set. [2] claims that $\mathrm{MA}_{\aleph_1}$ is consistent with the statement that every uncountable set of reals contains an $\aleph_1$-dense, 2-entangled set. So we conjecture that there are no uncountable real suborders which are strongly surjective in this model of $\mathrm{MA}_{\aleph_1}$.

Also, we don't know how to produce strongly surjective orders of size $> \aleph_2$, or if every strongly surjective (real) order necessarily contains a minimal suborder.

3. When all uncountable, strongly surjective linear orders are Aronszajn

Now, we look at various models where every uncountable, strongly surjective linear order must be Aronszajn. It was proved in [13] already that $\mathfrak{c} < 2^\kappa$ implies that there are no separable, strongly surjective linear orders of size κ. In particular, CH implies that all uncountable, strongly surjective linear orders are Aronszajn.

First, we strengthen the above by proving the next result.

Lemma 3.1. *Suppose that L is strongly surjective, $K \subseteq L$ is arbitrary and $D \subseteq K$ is dense in K. Then $2^{|K|} \leq 2^{|D|}$.*

Proof. If L is strongly surjective then each convex subset C of L is a G_δ so the family of all convex subsets $\mathcal{C}(L)$ has size $\leq |L|^\omega \leq \mathfrak{c}$.

Suppose that D is dense in $K \subseteq L$. Consider two order preserving maps $f, g : L \to K$ such that $f^{-1}(q) = g^{-1}(q) \neq \emptyset$ for all $q \in D$. We claim that $\mathrm{ran}(f) = \mathrm{ran}(g)$. Suppose that

$f(x) = y \in \mathrm{ran}(f) \setminus D$. Hence $q \lhd y$ implies $f^{-1}(q) \lhd x$ (let us denote these q with D^-) and $y \lhd q$ implies $x \lhd f^{-1}(q)$ (let us denote these q with D^+) for all $q \in D$. As D is dense, if $z \in K$ satisfies $D^- \lhd z \lhd D^+$ then $z = y$. Now, what could be $g(x)$? $f^{-1}(q) \lhd x$ implies that $\{q\} = g(f^{-1}(q)) \lhd g(x)$ for all $q \in D^-$ and $x \lhd f^{-1}(q)$ implies $g(x) \lhd \{q\} = g(f^{-1}(q))$ for all $q \in D^-$. Hence, by the previous observation, $y = g(x) \in \mathrm{ran}(g)$.

Now, the number of choices for a sequence $(C_q)_{q \in D}$ where $C_q \in \mathcal{C}(L)$ is at most $|\mathcal{C}(L)|^{|D|} \leq \mathfrak{c}^{|D|} = 2^{|D|}$. So, $2^{|D|}$ is an upper bound for the number of possible ranges for an order preserving map $f : L \to K$ with $D \subseteq \mathrm{ran}(f)$. Since L is strongly surjective, this number is $2^{|K|}$ and so $2^{|K|} \leq 2^{|D|}$ as desired.

□

The following two corrollaries immediately follow:

Corollary 3.2. *A strongly surjective linear order cannot contain real suborders of size* $\mathfrak{c}$*.*

We will later see that there could be strongly surjective linear orders of size $\mathfrak{c}$ i.e. the assumption of being a real suborder cannot be omitted from the previous corollary (even if one assumes ccc).

Corollary 3.3. $\mathfrak{c} < 2^{\aleph_1}$ *implies that any uncountable, strongly surjective linear order is Aronszajn.*

P. Schlicht independently proved the above corollary using the stronger assumption of $\diamondsuit$ (personal communication).

Corollary 3.4. $\mathfrak{c} = \aleph_2 = 2^{\aleph_1}$ *implies that any uncountable, strongly surjective linear order has size* $< \mathfrak{c}$*.*

Proof. Suppose that $|L| = \mathfrak{c}$ and build a 2-branching partition tree T for L (see [14] for details on partition trees). The height of T is at most ω_1 and so there is a minimal $\beta < \omega_1$ such that $|T_\beta| = \aleph_2$ since $|T| = \aleph_2$. Pick one point from each convex set in T_β to find a $K \subseteq L$ of size $\mathfrak{c} = \aleph_2$ which has density $\aleph_1 = |\bigcup_{\alpha<\beta} T_\alpha|$. Now, $2^{\mathfrak{c}} \leq 2^{\aleph_1}$ should hold by Lemma 3.1 however this is clearly not possible since $2^{\aleph_1} = \mathfrak{c}$. □

Finally, let us show that $\mathfrak{c} = 2^{\aleph_1}$ is also consistent with the statement that "any uncountable, strongly surjective linear order is Aronszajn". Let $\mathbb{C}_\kappa$ be the forcing adding κ-many Cohen reals i.e. $\mathbb{C}_\kappa = Fn(\kappa, 2)$ the set of finite partial functions from κ to 2.

Theorem 3.5. *Suppose GCH holds in* V*. Then* $V^{\mathbb{C}_{\aleph_2}}$ *is a model of* $\mathfrak{c} = \aleph_2 = 2^{\aleph_1}$ *and every uncountable, strongly surjective linear order is Aronszajn.*

Proof. It is standard to show that $V^{\mathbb{C}_{\aleph_2}} \models \mathfrak{c} = \aleph_2 = 2^{\aleph_1}$. So, our goal is to show that a strongly surjective linear order contains no uncountable separable suborders.

First, any strongly surjective linear order L in $V^{\mathbb{C}_{\aleph_2}}$ has size $< \aleph_2$ by Corollary 3.4 and in turn, L appears in an intermediate model. Since $\mathbb{C}_{\aleph_2} = \mathbb{C}_{\aleph_2} * \mathbb{C}_{\aleph_1}$, it suffices to prove the following.

Lemma 3.5.1. *Suppose that* $L = (\aleph_1, \lhd)$ *is a linear order from the ground model* V *so that* ω *is dense in some uncountable* $L_0 \subseteq L$*. Then* L *is not strongly surjective in* $V^{\mathbb{C}_{\aleph_1}}$*.*

Proof. Let $G \subseteq \mathbb{C}_{\aleph_1}$ be a V-generic filter; this gives a generic map $g : L_0 \to 2$ by $g = \cup G \restriction L_0$. Now, in $V[G]$, we define

$$K = \omega \cup g^{-1}(1) \subseteq L_0.$$

We will show that L cannot be mapped onto K. To this end, suppose that $f : L \twoheadrightarrow K$ is an order preserving surjection in $V[G]$. Find an appropriate countable $\nu \in \omega_1$ so that $V[G \restriction \nu] \models f(\xi_\ell) = \ell$ for an appropriate sequence $(\xi_\ell)_{\ell\in\omega} \in \mu^\omega \cap V[G \restriction \nu]$. Now, note that $O_\alpha = \{\ell \in \omega : \ell \triangleleft \alpha\} \in V$ for all $\alpha \in \omega_1$ since $L \in V$; hence

$$A_\alpha = \{\xi_\ell : \ell \in O_\alpha\}, B_\alpha = \{\xi_\ell : \ell \in \omega \setminus O_\alpha\} \in V[G \restriction \nu].$$

In particular, for any $\alpha \in \omega_1$, it is decided in $V[G \restriction \nu]$ whether there is some $\xi \in \omega_1$ such that $A_\alpha \triangleleft \xi \triangleleft B_\alpha$.

Claim. *For any $\alpha < \omega_1$, $g(\alpha) = 1$ iff there is some $\xi \in \omega_1$ such that $A_\alpha \triangleleft \xi \triangleleft B_\alpha$.*

Proof. If $g(\alpha) = 1$ then $\alpha \in K$ and hence any $\xi \in f^{-1}(\alpha)$ has to satisfy $A_\alpha \triangleleft \xi \triangleleft B_\alpha$. On the other hand, if $A_\alpha \triangleleft \xi \triangleleft B_\alpha$ for some $\xi \in \omega_1$ then $\alpha' = f(\xi) \in L_0$ must satisfy $O_\alpha \triangleleft \alpha' \triangleleft \omega \setminus O_\alpha$. Since ω is dense in L_0, this clearly implies that $\alpha = \alpha' \in K$ and so $g(\alpha) = 1$.

□

Note that the claim implies that g can be defined in $V[G \restriction \nu]$ which is clearly not possible. This contradiction finishes the proof of the lemma.

□

In turn, we proved the theorem.

□

Now, a natural question is whether other classical models (Sacks, Miller, etc.) allow non Aronszajn strongly surjective linear orders. We claim that these models behave as the Cohen-model i.e. if there is a strongly surjective linear order, it has to be Aronszajn or countable. To prove this result, we employ the technique of parametrized weak diamonds [11, 6].

Definition 3.6. Let $\diamondsuit^{\omega_1}(2,=)$ denote the following statement: let X be an ω_1 set of ordinals and $F : \bigcup_{\delta<\omega_1} \delta^\delta \to 2$ so that $F \restriction \delta^\delta \in L(\mathbb{R})[X]$ for all $\delta < \omega_1$. Then there is a $g : \omega_1 \to 2$ so that for all $f : \omega_1 \to \omega_1$ the set

$$\{\delta < \omega_1 : f \restriction \delta \in \delta^\delta \text{ and } F(f \restriction \delta) = g(\delta)\}$$

is stationary.

Recall that $L(\mathbb{R})$ is the class of sets constructible from $\mathbb{R}$ (in the sense of Gödel) and $L(\mathbb{R})[X]$ is the minimal model extending $L(\mathbb{R})$ which contains X. See [8, Chapter 13] for more details on constructibility.

In Corollary 3.3, we saw that $\mathfrak{c} < 2^{\aleph_1}$ was used to deduce that all strongly surjective linear orders are Aronszajn. So, it is not a surprise that we turn to use weak diamonds: a celebrated result of K. Devlin and S. Shelah [5] states that $\mathfrak{c} < 2^{\aleph_1}$ is equivalent to the above weak diamond statement if one drops the requirement of $F \restriction \delta^\delta \in L(\mathbb{R})[X]$ i.e. that we only guess constructible functions F. However, $\diamondsuit^{\omega_1}(2,=)$ suffices to prove many classical consequences of $\mathfrak{c} < 2^{\aleph_1}$ e.g. the failure of $\mathrm{BA}_{\aleph_1}$.

Now, the advantage of $\diamondsuit^{\omega_1}(2,=)$ over $\mathfrak{c} < 2^{\aleph_1}$ is clear from the following theorem.

Theorem 3.7. [6] $\diamondsuit^{\omega_1}(2,=)$ *holds in all models resulting from a length* $\aleph_2$ *countable support iteration of a single Suslin-definable, homogeneous (i.e.* $\mathbb{P} \equiv \{0,1\} \times \mathbb{P}$*) and proper poset* $\mathbb{P}$.

The classical tree forcings (Sacks, Miller, etc.) satisfy all these requirements. Also, in any such canonical model $\mathfrak{c} = 2^{\aleph_1} = \aleph_2$. In turn, strongly surjective linear orders must have size $\leq \aleph_1$ by Corollary 3.4. So, our final aim in this section is to prove the following theorem.

Theorem 3.8. $\diamondsuit^{\omega_1}(2,=)$ *implies that all strongly surjective linear orders of size* $\aleph_1$ *are Aronszajn.*

Proof. Suppose that $L = (\omega_1, \lhd)$ is a linear order and $A = \{a_\alpha : \alpha < \omega_1\} \subseteq L$ is an uncountable suborder with a countable set D which is dense in A. We aim to find $B \subseteq A$ so that there is no $\lhd$-preserving $f : \omega_1 \twoheadrightarrow B$.

Let us define a function $F : \bigcup_{\delta<\omega_1} \delta^\delta \to 2$ using $\lhd$ and A as parameters and make sure that $F \restriction \delta^\delta \in L(\mathbb{R})[A, \lhd]$ as desired; it is standard to code A and $\lhd$ into a single set of ordinals of size $\aleph_1$ so we omit these details.

Fix $\delta < \omega_1$ and $f : \delta \to \delta$. Also, let $\alpha_n = \alpha_n^\delta \lhd \beta_n = \beta_n^\delta \in D$ so that

$$\{a_\delta\} = A \cap \bigcap_{n\in\omega} [\alpha_n, \beta_n]_\lhd.$$

We let $F(f) = 1$ iff

(a) $D \subseteq \operatorname{ran}(f)$,
(b) f is $\lhd$-preserving and
(c) the set
$$\bigcap_{n\in\omega} \{[\xi, \zeta]_\lhd : \xi \in f^{-1}(\alpha_n), \zeta \in f^{-1}(\beta_n)\}$$
is empty.

In any other case, we let $F(f) = 0$. Note that (a) and (b) are clearly Borel conditions but we do use $A, \lhd$ as parameters in condition (c).

Now, $\diamondsuit^{\omega_1}(2,=)$ hands us some function $g : \omega_1 \to 2$. We use g to define a subset of A as follows:

$$B = D \cup \{a_\delta : \delta \in \omega_1 \text{ and } g(\delta) = 1\}.$$

We claim that there is no order preserving map $f : \omega_1 \twoheadrightarrow B$. Otherwise, the set

$$S = \{\delta < \omega_1 : f \restriction \delta \in \delta^\delta \text{ and } F(f \restriction \delta) = g(\delta)\}$$

is stationary. Pick a large enough $\delta \in S \setminus \omega$ so that $D \subseteq \operatorname{ran}(f \restriction \delta)$.

Now, there are two possible cases: first, suppose that $g(\delta) = 1$. Then $a_\delta \in B$ so there is some $\nu \in f^{-1}(a_\delta)$. Note that $\alpha_n \lhd a_\delta \lhd \beta_n$ implies that $\xi \lhd \nu \lhd \zeta$ for all $\xi \in f^{-1}(\alpha_n), \zeta \in f^{-1}(\beta_n)$ since f is $\lhd$-preserving. So $\nu \in \bigcap_{n\in\omega}\{[\xi,\zeta]_\lhd : \xi \in f^{-1}(\alpha_n), \zeta \in f^{-1}(\beta_n)\}$ which means that $F(f \restriction \delta) = 0$. However, $\delta \in S$ implies that $0 = F(f \restriction \delta) = g(\delta) = 1$ a contradiction.

Second, suppose that $g(\delta) = 0 = F(f \restriction \delta)$. In particular $a_\delta \notin B$. However, $0 = F(f \restriction \delta)$ implies that

$$\bigcap_{n\in\omega} \{[\xi, \zeta]_\lhd : \xi \in f^{-1}(\alpha_n), \zeta \in f^{-1}(\beta_n)\} \neq \emptyset$$

since conditions (a) and (b) are satisfied. Pick any $\nu \in \bigcap_{n\in\omega}\{[\xi,\zeta]_\lhd : \xi \in f^{-1}(\alpha_n), \zeta \in f^{-1}(\beta_n)\}$ and look at the image $f(\nu) \in B$. We must have $\alpha_n \lhd f(\nu) \lhd \beta_n$ for all $n \in \omega$

however the only element of A which can satisfy this is a_δ (by the choice of α_n, β_n). So $f(\nu) = a_\delta \in B$, a contradiction again.

$\square$

It is not clear at this point if there are any uncountable strongly surjective linear orders in the Cohen-or the above canonical models.

4. A minimal, strongly surjective Suslin-tree

As the title suggests, our goal is to construct a lexicographically ordered Suslin-tree which is strongly surjective and minimal. A construction for a minimal Suslin-tree was presented first by J. Baumgartner [4]. However, recently Hossein Lamei Ramandi pointed out that Baumgartner's crucial [4, Lemma 4.14] is unrepairably flawed. We hope to present a correct and complete proof now.

We will start by a few necessary definitions: a tree T is *ω-branching* iff

(1) $|\{s \in T : s^\downarrow \setminus \{s\} = t^\downarrow \setminus \{t\}\}| = \omega$, and
(2) $|\{s \in T_{\alpha+1} : t < s\}| = \omega$

for all $t \in T_\alpha$. We say that a tree T is *doubly ordered* iff there is a lexicographic order on T so that $\{s \in T : s^\downarrow \setminus \{s\} = t^\downarrow \setminus \{t\}\}$ is isomorphic to $\mathbb{Q}$ for all $t \in T$. In particular, $\min T$ is ordered as $\mathbb{Q}$ as well. A *double isomorphism* between two doubly ordere trees is a bijection which preserves both the tree and lexicographic order.

We say that $Y \subseteq T$ is *large* if Y is cofinal in $T[Y] = \{t \in T : t \leq y$ or $y \leq t$ for some $y \in Y\}$; note that a large subset is not necessarily downward closed. Finally recall that $\diamondsuit^+$ is the following statement: there is a sequence $(\mathcal{S}_\alpha)_{\alpha\in\omega_1}$ so that $\mathcal{S}_\alpha$ is countable and there is a club C so that $Y \cap \alpha, C \cap \alpha \in \mathcal{S}_\alpha$ for all $\alpha \in C$. $\diamondsuit^+$ is a well known consequence of $V = L$.

We will prove:

Theorem 4.1. *Under $\diamondsuit^+$, there is an ω-branching doubly ordered Suslin-tree T so that T and $T \restriction Y$ are doubly isomorphic whenever $Y \subseteq T$ is large and* $\min Y$ *is isomorphic to* $\min T$.

At the end of the section (in Corollary 4.6), we show that the above tree is strongly surjective; this argument is presented in [13] already but we include it as well for the sake of completeness.

Our construction will be a modification of [7, Theorem 2.3] where a Suslin tree T is constructed such that T and $T \restriction Y$ are tree isomorphic whenever $Y \subseteq T$ is large. Actually, we believe that the tree constructed in [7, Theorem 2.3] can be doubly ordered to satisfy our requirements but we repeat the construction anyways. The proof that large subsets of T are actually doubly isomorphic to T and not just isomorphic as trees requires extra thought anyways. It seems that the authors of [7] were not aware of Baumgartner's [4] at the time of publishing.

We start by stating the relevant main lemma from [7] in a strong form. Given an ω-branching tree T and a set B of unbounded branches in T, one can naturally define a tree $T \oplus B$ which is an end extension of T as follows: $T \oplus B$ is defined on the set $T \cup \{(b,i) : b \in B, i < \omega\}$ and we require that $s < (b,i)$ for all $s \in b \in B$ and $i < \omega$. Note that $T \oplus B$ is ω-branching if T was ω-branching and $\mathrm{ht}(T \oplus B) = \mathrm{ht}(T) + 1$.

Lemma 4.2. *Suppose that T is an ω-branching tree of limit height $\alpha \in \omega_1$ and $\mathcal{N}$ is a countable set. Then there is a countable set $B = B(T, \mathcal{N})$ of unbounded branches in T so that the ω-branching end extension $T \oplus B$ of T of height $\alpha + 1$ satisfies the following:*

(1) if $A \in \mathcal{N}$ is a maximal antichain in T then A is still maximal in $T \oplus B$, and
(2) if $f \in \mathcal{N}$ is a tree isomorphism between large subsets of T then
  (a) $b \in B$ implies $f[b] \in B$, and
  (b) the assignment $(b, i) \mapsto (f[b], i)$ extends f to an isomorphism between large subsets of $T \oplus B$.

The authors of [7] actually state their Lemma 2.10 with less details about the nature of the extension but they prove exactly what we wrote above; the proof is rather delicate and requires careful thought. We omit reproducing this argument here but highly recommend that the interested reader studies the details.

Suppose we are in the setting of Lemma 4.2 and T is actually doubly ordered. We can extend this double order to $T \oplus B$ as follows: order $\{(b, i) : i < \omega\}$ as $\mathbb{Q}$ for each $b \in B$ by a fixed bijection $\varphi : \omega \to \mathbb{Q}$. The particular way in which the trees and isomorphisms in Lemma 4.2 were extended allows double isomorphism to extend too; this will be the content of the next lemma. Let $\mathcal{L}(T, S)$ be the set of double isomorphisms between large subsets of T and S.

Lemma 4.3. *Suppose that $T, \mathcal{N}$ and B are as in Lemma 4.2, T is doubly ordered and T^* is a doubly ordered end extension of $T \oplus B$.*

If $f \in \mathcal{N} \cap \mathcal{L}(T, T)$ and g is a bijection between large subsets of $T \oplus B$ and T^* so that
(1) $f \subseteq g$,
(2) $f[b] \leq_{T^*} g(b, i)$, and
(3) $g \restriction \{(b, i) : i < \omega\}$ is $<_{\mathrm{lex}}$-preserving for each $b \in B$

then $g \in \mathcal{L}(T \oplus B, T^*)$ i.e. g is a double isomorphism as well.

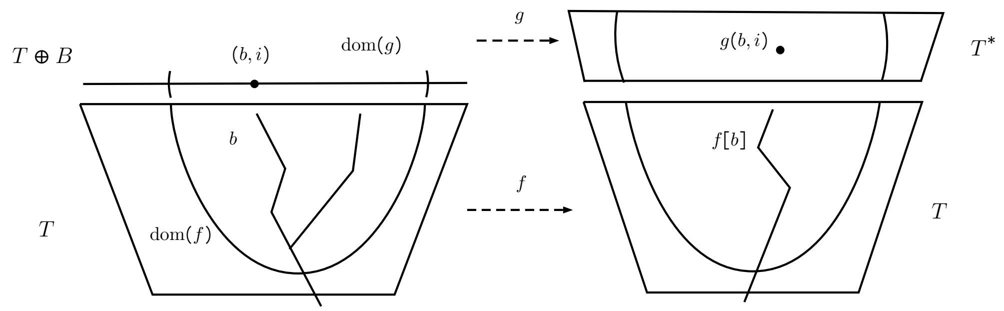

FIGURE 1. The setting of Lemma 4.3

Proof. Suppose that g is as above and $s <_{T[B]} t \in \mathrm{dom}(g)$. Note that g must be a tree isomorphism by (1) and (2) and Lemma 4.2.

First, if $s <_{T[B]} t$ and $t = (b, i)$ then $f(s) = g(s) \in f[b] <_{T^*} g(b, i)$ by (2). Second, suppose that $s \restriction \xi <_{\mathrm{lex}} t \restriction \xi$ where ξ is minimal so that $s \restriction \xi \neq t \restriction \xi$; if $\xi = \alpha$ then (3) implies that $g(s) <_{\mathrm{lex}} g(t)$ as well.

If $\xi < \alpha$ then $f(s \restriction \zeta) = f(t \restriction \zeta)$ for all $\zeta < \xi$ and $f(s \restriction \xi) <_{\mathrm{lex}} f(t \restriction \xi)$ since f was assumed to be a double isomorphism. Now, $f(s \restriction \xi) = g(s \restriction \xi) <_{T^*} g(s)$ and $f(t \restriction \xi) = g(t \restriction \xi) <_{T^*} g(t)$ so $g(s) <_{\mathrm{lex}} g(t)$ as desired.

□

Finally, we need:

Lemma 4.4. *Any two countable, ω-branching doubly ordered trees S, T of the same countable limit height are doubly isomorphic.*

Proof. This is standard back-and-forth argument. Alternatively, consider the partial order $Q = Q_{S,T}$ of finite partial double-isomorphisms and show that the set of $q \in Q$ so that $t \in \mathrm{dom}(q), s \in \mathrm{ran}(q)$ is dense open for any $s \in S, t \in T$. Now apply the Baire-category theorem (or the Rasiowa-Sikorski lemma) to find a filter $G \subseteq Q$ meeting all these countably many dense sets. The map $\cup G : S \to T$ is the desired double isomorphism.

□

We are ready to construct our Suslin-tree now:

Proof of Theorem 4.1. Let $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$ denote the $\diamondsuit^+$ sequence. Take an increasing sequence of elementary submodels $(N_\alpha)_{\alpha<\omega_1}$ of $(H(\aleph_2), \in, \prec)$ so that $\mathcal{S}_\beta, (N_\alpha)_{\alpha<\beta} \in N_\beta$ for all $\beta < \omega_1$. Here $\prec$ denotes an arbitrary well order of $H(\aleph_2)$.

We construct an increasing sequence of ω-branching, doubly ordered trees $(T^\alpha)_{\alpha\leq\omega_1}$ on subsets of ω_1 so that

$(i)_\beta$ $(T^\alpha)_{\alpha\leq\beta} \in N_\beta$,
$(ii)_\beta$ $\mathrm{ht}(T^\beta) = \beta$. and
$(iii)_\beta$ the sequence of trees $(T^\alpha)_{\alpha\leq\beta}$ can be uniquely recovered from the sequence $(N_\alpha)_{\alpha<\beta}$,
$(iv)_\beta$ T^β is an end extension of T^α for all $\alpha < \beta \leq \omega_1$.

These properties will be ensured by making uniform choices when extending the trees using the well order $\prec$.

Suppose $(T^\alpha)_{\alpha<\beta}$ is constructed so that $(i)_{\beta'} - (iv)_{\beta'}$ holds for all $\beta' < \beta$. If β is a limit then we let $T^\beta = \bigcup\{T^\alpha : \alpha < \beta\}$. If $\beta = \alpha + 1$ and α is a successor then we take the $\prec$-minimal end extension of T^α in N_β that satisfies our requirements (i) and (ii).

Finally, if $\beta = \alpha + 1$ and α is a limit then we apply Lemma 4.2 to T^α and $\mathcal{N} = N_\beta$ and define $T^\beta = T^\alpha \oplus B(T^\alpha, N_\alpha)$. The set of branches $B_\alpha = B(T^\alpha, N_\alpha)$ is chosen $\prec$-minimal in $N_{\beta+1}$ which again ensures $(i)_\beta$.

The tree $T = T^{\omega_1}$ we constructed is Suslin. Indeed, suppose that A is a maximal antichain. Then there is an $\beta < \omega_1$ so that $A \cap \beta \in \mathcal{S}_\beta \subseteq N_\beta$ and $A \cap \beta$ is maximal in $T_{<\beta} = T^\beta$. Recall that we applied Lemma 4.2 with $\mathcal{N} = N_\beta$ to construct $T_{\leq\beta} = T^{\beta+1}$. So we preserved $A \cap \beta$ as a maximal antichain in $T_{\leq\beta}$ and hence in T. So $A = A \cap \beta$ is countable.

Now suppose that Y is large. Let C be a club subset of ω_1 so that $\gamma \in C$ implies that $C \cap \gamma, Y \cap \gamma \in N_\gamma$ and $T_{<\gamma} = \gamma$ and $Y \cap \gamma = Y_{<\gamma}$ is large in $T_{<\gamma}$. Let $C = \{\gamma_\nu : \nu < \omega_1\}$ denote the increasing enumeration of C.

We inductively construct maps $(\pi_\nu)_{\nu\leq\omega_1}$ along the club C so that

(1) $\pi_\nu : T_{<\gamma_\nu} \to Y \cap T_{<\gamma_\nu}$ is a double isomorphism,
(2) the sequence of trees $(\pi_\mu)_{\mu\leq\nu}$ can be uniquely recovered from the sequence $(N_{\gamma_\mu})_{\mu<\nu}$,

At limit steps, we take unions and our construction guarantees that the double isomorphism $\pi_{<\nu} = \cup_{\mu<\nu}\pi_\mu : T_{<\gamma_\nu} \to Y_{<\gamma_\nu}$ is in N_{γ_ν}.

Now, at successors of limit stages we do the following: recall that in our construction of $T_{\leq\gamma_\nu}$ from $T_{<\gamma_\nu}$ we added nodes $\{(b,i) : b \in B_{\gamma_\nu}, i \in \omega\}$ for a contable set of unbounded chains B_{γ_ν}. So, we will define $\rho : T_{\leq\gamma_\nu} \to Y_{\leq\gamma_\nu}$ in N_{γ_ν} such that

(1) $\pi_{<\nu} \subseteq \rho$,
(2) $\pi_{<\nu}[b] \leq_T \rho(b,i)$, and
(3) $\rho \restriction \{(b,i) : i < \omega\}$ is $<_{\mathrm{lex}}$-preserving for each $b \in B_{\gamma_\nu}$.

If we can do this then ρ must be a double isomorphism by Lemma 4.3. Next, we can extend ρ to π_ν in $N_{\gamma_{\nu+1}}$ further by an isomorphism $T_{<\gamma_{\nu+1}} \setminus T_{\le\gamma_\nu} \to Y_{<\gamma_{\nu+1}} \setminus Y_{\le\gamma_\nu}$; this is possible by Lemma 4.4.

Lets see how to construct ρ: first, recall that $(\pi[b], i)$ is a node of T_{γ_ν} which is above the set $\pi_{<\nu}[b] = \{\pi_{<\nu}(t) : t \in b\}$ for all $b \in B_{\gamma_\nu}$. Now, consider $R_b = \{y \in Y_{\gamma_\nu} : \pi_{<\nu}[b] < y\}$.

Observation 4.5. *R_b is isomorphic to $\mathbb{Q}$ in the lexicographic order.*

Proof. First, note that $R_b = \bigcup\{R_{b,i} : i < \omega\}$ where $R_{b,i} = \{y \in Y_{\gamma_\nu} : (\pi_{<\nu}[b], i) \le y\}$. If $(\pi_{<\nu}[b], i) \in Y$ then $R_{b,i} = \{(\pi_{<\nu}[b], i)\}$; otherwise, $R_{b,i}$ is isomorphic to $\mathbb{Q}$ since Y was large so for each successor x of $(\pi_{<\nu}[b], i)$ there is $x \le y \in Y_{\gamma_\nu}$. This clearly implies that R_b is isomorphic to $\mathbb{Q}$ as well. □

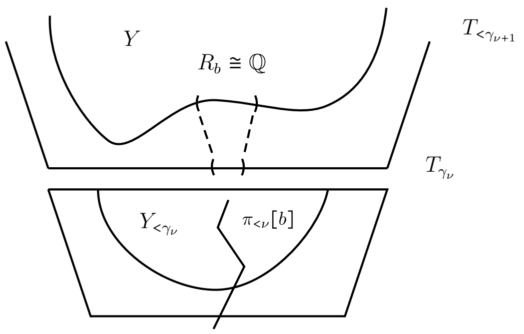

FIGURE 2. Extending $\pi_{<\nu}$

Hence, we can take a sequence $(\rho_b)_{b\in B_{\gamma_\nu}}$ so that $\rho_b : \{(b,i) : i < \omega\} \to R_b$ is an order isomorphism; we make sure to choose this sequence $\prec$-minimal and in N_{γ_ν}. We let $\rho = \pi_{<\nu} \cup \{\rho_b : b \in B_{\gamma_\nu}\}$ and $\rho \in N_{\gamma_\nu}$ is as desired.

At successors of successors, we simply apply Lemma 4.4. Finally, $\pi = \pi_{\omega_1}$ is a double isomorphism between T and Y.

□

Corollary 4.6. *Under $\diamondsuit^+$, there is a lexicographically ordered Suslin-tree which is strongly surjective and minimal.*

Proof. We will show that the doubly ordered Suslin tree T from Theorem 4.1 with the lexicographic order is strongly surjective; T is clearly minimal.

Claim 4.7. *(1) T and $T \times \mathbb{Q}$ are isomoprhic.*
(2) If $X \subseteq T$ then there is a countable $A \subseteq T$ so that $X = \sum_{a\in A} K_a$ and either
  (a) K_a is a singleton, or
  (b) K_a isomoprhic to T and then there is a dense interval I of A around a so that $a' \in I$ implies $K_{a'}$ is a copy of T as well.
(3) Any linear order T with properties (1) and (2) is strongly surjective.

Proof. (1) First, write $\mathbb{Q}$ as $\sum_{q\in\mathbb{Q}} I_q$ so that $I_q \subseteq \mathbb{Q}$ is isomoprhic to $\mathbb{Q}$. Take an isomorphism $\pi : \mathbb{Q} \to \min T$. Note that $T[\pi[I_q]]$ and T are isomoprhic since $T[\pi[I_q]]$ is a large subset of T. Hence, the decomposition $T = \sum_{q\in\mathbb{Q}} T[\pi[I_q]]$ witnesses that T and $T\times\mathbb{Q}$ are isomoprhic.

(2) We start by a standard observation:

Observation 4.8. *If $X \subseteq T$ is uncountable then $X \cap a^{\uparrow}$ is large for some $a \in X$.*

Proof. This is elementary Suslin-tree combinatorics: we can show that $X \cap a^{\uparrow}$ is cofinal in $a^{\uparrow}$. Otherwise, for all $a \in X$ we can find $t_a \geq a$ so that $\{t \in X : t > t_a\}$ is empty . Select a maximal antichain W from $\{t_a : a \in X\}$ and pick any $b \in X$ so that $\mathrm{ht}(b) > \mathrm{ht}[W]$. Then $b \leq t_b$ and $t_a \leq t_b$ for some $t_a \in W$. This, however, implies that $t_a < b \in X$ which contradicts the choice of t_a. □

Now, we find a countable A_0 so that $X = \sum_{a \in A_0} K_a$ where either K_a is a singleton or has order type T. (2) clearly follows, since each copy of T is actually isomorphic to $T \times \mathbb{Q}$ by (1).

In order to find A_0, let S denote the minimal elements of the set

$$\{a \in T : X \cap a^{\uparrow} \text{ is large}\}.$$

S is an antichain so $|S| \leq \omega$. Furthermore, the set $R = X \setminus \bigcup_{a \in S} X \cap a^{\uparrow}$ is countable. Indeed, this follows from Observation 4.8.

Finally, let $A_0 = S \cup R$ and

$$K_a = \begin{cases} \{a\}, & \text{for } a \in R, \text{ and} \\ X \cap a^{\uparrow}, & \text{for } a \in S. \end{cases}$$

Now, if $K_a = X \cap a^{\uparrow}$ then K_a is isomorphic to T so $X = \sum_{a \in A_0} K_a$ as desired.

(3) Fix $X \subseteq T$ and write $X = \sum_{a \in A} K_a$ as in (2). Our aim is to find an $f : T \times \mathbb{Q} \twoheadrightarrow X$; since $T \times \mathbb{Q}$ and t are isomorphic by (1), this will finish the proof.

Let

$$Q_a = \begin{cases} \{a\}, & \text{if } K_a \text{ is a copy of } T, \text{ and} \\ \mathbb{Q}, & \text{otherwise.} \end{cases}$$

The choice of A guarantees that $\sum_{a \in A} Q_a$ is countable and dense without endpoints, so it is isomorphic to $\mathbb{Q}$. The function

$$g : \mathbb{Q} \cong \sum_{a \in A} Q_a \twoheadrightarrow A$$

that maps Q_a onto $\{a\}$ is order preserving and satisfies that $|g^{-1}(a)| = 1$ iff K_a is a copy of T.

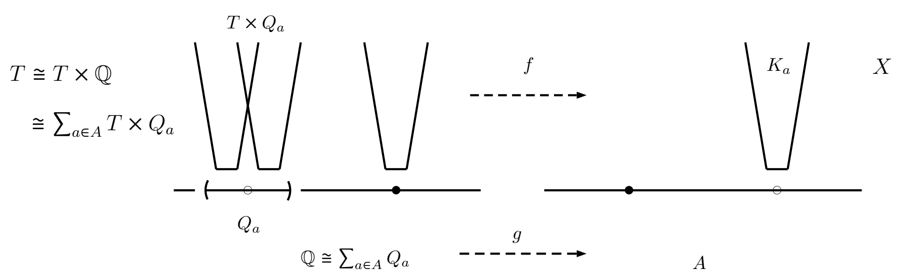

FIGURE 3. Defining $f : T \twoheadrightarrow X$

Next, define $f : T \times \mathbb{Q} \twoheadrightarrow X$ so that $f \restriction T \times \{q\}$ is a map $T \times \{q\} \twoheadrightarrow K_{g(q)}$ which is either constant or an isomorphism depending on whether $K_{g(q)}$ is a singleton or a copy of T. We prove that f is order preserving: the only non trivial thing to check is the case when

$s \in T \times \{q\}$, and $t \in T \times \{q'\}$ with $q < q'$ and $a = g(q) = g(q')$. However, then K_a must be a singleton by the choice of g so $f(s) = f(t)$.

□

The above claim proves that T is strongly surjective.

□

5. A model without uncountable strongly surjective linear orders

We aim to show next that, in some models of ZFC, all strongly surjective linear orders must be countable. We have seen already that strong surjectivity is closely related to minimal orders, so it is very natural to look at models of ZFC where the only uncountable minimal orders are ω_1 and $-\omega_1$: J. Moore showed that if CH and *axiom (A)* holds then this is true [9]. Our goal will be to prove that the same assumptions imply that there are no uncountable strongly surjective linear orders either.

First, recall that axiom (A) says that given a ladder system $(C_\alpha)_{\alpha\in\lim(\omega_1)}$, functions $f_\alpha : C_\alpha \to \omega$ and a Hausdorff A-tree T, we can find a downward closed, pruned subtree $S \subseteq T$ and $f : S \to \omega$ so that if $u \in S$ is of limit height α then $f(u \restriction \xi) = f_\alpha(\xi)$ for almost all $\xi \in C_\alpha$. Such an f is called a T-uniformization. It was proved in [9] that models of CH + (A) can be produced by starting from CH and forcing with a countable support iteration of proper posets; the individual posets introduce the uniformizations carefully so that no new reals are added in the process (not even when iterating). Therefore CH can be preserved.

Our goal is to prove the following result.

Theorem 5.1. *CH + (A) implies that no lexicographically ordered A-tree is strongly surjective. In particular, it is consistent that CH holds and there are no uncountable, strongly surjective linear orders.*

The crux of Moore's result on minimal linear orders is [9, Lemma 3.3]: CH + (A) implies that no (Hausdorff) A-tree T is club-embeddable into all its downward closed, pruned subtrees. Our Lemma 5.2 is the surjective counterpart of [9, Lemma 3.3] and essentially the content of Theorem 5.1.

Main Lemma 5.2. *Suppose CH + (A). Then there is no lexicographically ordered A-tree T which can be mapped onto all of its downward closed, pruned subtrees S in a lex-order preserving way.*

The main reason we need this new lemma is the following: we do not know whether a lexicographic embedding is actually a tree embedding restricted to a club subset; otherwise, we could have simply applied [9, Lemma 3.3] to prove our Theorem 5.1. Furthermore, [9] deals with Hausdorff trees only while we cannot make this assumption now.

First, let us show why Lemma 5.2 implies Theorem 5.1:

Proof of Theorem 5.1. Take any model of CH + (A) e.g. [9, Theorem]. Any uncountable, strongly surjective linear order L must be an Aronszajn line by Corollary 3.3 and CH. Any A-line is isomorphic to a lexicographically ordered A-tree by [4, Theorem 4.2]. In turn, Lemma 5.2 implies that L cannot be strongly surjective if CH together with (A) holds. □

Now, we proceed to prove Lemma 5.2 which will be done through a sequence of claims. We will prove the following through Claim 5.2.1, 5.2.2 and 5.2.3: given a countable elementary submodel $M \prec H(\aleph_2)$ and a map $f \in M$ so that $f : T \twoheadrightarrow S$ and $S \subseteq T$ are lexicographically ordered A-trees, one can define *an unbounded branch* $b(f,M)$ of $S \cap M$ which *has an upper bound in* S using solely $f^M = f \cap M$.

Claim 5.2.1. *Suppose that* $S, T \in M \prec H(\aleph_2)$ *where* $S \subseteq T$ *are a lexicographically ordered A-trees. If* $s \in S \cap M$ *and* $w \in S \setminus M$ *then there is* $w' \in S \cap M$ *so that* $s \lhd w' \lhd w$ *if* $s \lhd w$ *and* $w \lhd w' \lhd s$ *if* $w \lhd s$.

Proof. We prove when $s \lhd w$; the other case is completely symmetric. First, note that $s \lhd w$ implies $s \lhd w \restriction \varepsilon_0$ for some $\varepsilon_0 \in \omega_1 \cap M$. Now, let

$$Z_r = \{t \in S : w \restriction \varepsilon_0 <_{\text{lex}} t <_{\text{lex}} r\}.$$

Note that $Z_r \in M$ whenever $r \in M$. Furthermore, if there is some $\varepsilon \in M \cap \omega_1$ above ε_0 so that $Z_{w \restriction \varepsilon}$ is uncountable then there is some $w' \in Z_{w \restriction \varepsilon}$ so that $w' \lhd w \restriction \varepsilon$; indeed, $w \restriction \varepsilon$ has only countably many $<_T$-predecessors. In turn, $s \lhd w' \lhd w$ as desired.

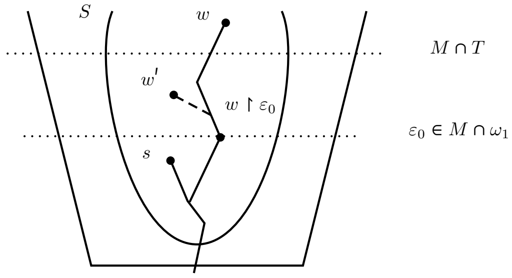

FIGURE 4. Finding w' between s and w

So, suppose that $Z_{w \restriction \varepsilon}$ is countable for all $\varepsilon \in M \cap \omega_1$ above ε_0. Hence, the set

$$Z = \{r \in T : |Z_r| \leq \omega\}$$

must be uncountable. However, $r <_{\text{lex}} r' \in Z$ implies that $r \in Z_{r'}$ and so Z contains a copy of ω_1. This is a contradiction.

□

Claim 5.2.2. *Suppose* $f \in M \prec H(\aleph_2)$ *where* $f : T \twoheadrightarrow S$ *and* $S \subseteq T$ *are lexicographically ordered A-trees; let* $\delta = M \cap \omega_1$ *and* $h = f \restriction T_{<\delta}$. *Then the following holds:*

1. *there is* $u \in T_\delta$ *so that the downward closure of* $h^{-1}(s)$ *is* $<_T$*-bounded below* u *for every* $s \in S_{<\delta}$, *and*
2. $f(u) \notin S_{<\delta}$ *for any* $u \in T_\delta$ *that satisfies (1).*

The downward closure of a set $A \subseteq T$ will be denoted by $A^\downarrow$, that is: $A^\downarrow = \{t \in T : t \leq s$ for some $s \in A\}$.

Proof. (1) Take $w \in S_\delta$ and $t \in f^{-1}(w)$; note that $t \notin T_{<\delta}$. Let $u \in T_\delta \cap t^\downarrow$; we will show that u works i.e. $h^{-1}(s)^\downarrow$ is $<_T$-bounded below u for every $s \in S_{<\delta}$.

Fix $s \in S_{<\delta}$ and suppose that $v_n \in T_{<\delta} \cap h^{-1}(s)$ and $u_n \in v_n^\downarrow \cap u^\downarrow$ so that $\sup\{\text{ht}(u_n) : n \in \omega\} = \delta$. We will reach a contradiction.

First, if $v_n <_{\text{lex}} t <_{\text{lex}} v_k$ for some $n \neq k < \omega$ then $f(t) = s \in S_{<\delta}$ which is a contradiction.

So, let us suppose that $t <_{\text{lex}} v_n$ for all n. Hence $t \lhd v_n$ so $f(t) = w <_{\text{lex}} f(v_n) = s$ and so $w \lhd s$ also holds. Let us pick some $w' \in S_{<\delta}$ so that $w <_{\text{lex}} w' <_{\text{lex}} s$; this can be done by Claim 5.2.1.

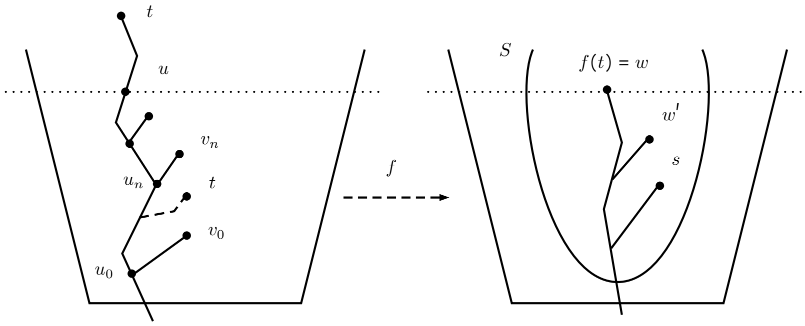

FIGURE 5. Proof of Claim 5.2.2

Let $t' \in h^{-1}(w') \cap T_{<\delta}$ and note that $t <_{\text{lex}} t'$ and so $t \lhd t'$ since $t \notin T_{<\delta}$. So $u_n <_{\text{lex}} t'$ and hence $v_n <_{\text{lex}} t'$ for some n. But this implies that $f(v_n) = s \leq_{\text{lex}} f(t') = w'$ which is a contradiction.

Hence, $v_n <_{\text{lex}} t$ for all $n < \omega$ so $s <_{\text{lex}} w$. Again, by Claim 5.2.1, we can find $w' \in S_{<\delta}$ so that $s <_{\text{lex}} w' <_{\text{lex}} w$. Pick any $t' \in h^{-1}(w') \cap T_{<\delta}$ and note that $t' \lhd t$. In turn, there must be some n so that $t' <_{\text{lex}} v_n$. However, this implies that $w' \leq_{\text{lex}} s$, a contradiction.

(2) This is standard using elementarity: suppose that $s = f(u) \in S_{<\delta}$ and $\varepsilon < \delta$. Then $H(\aleph_2) \models$ "$f(v) = s$ for some $v \in T$ with $u \restriction \varepsilon <_T v$". So M must also satisfy this sentence i.e. there is $v \in T_{<\delta}$ so that $f(v) = s$ and $u \restriction \varepsilon <_T v$. However, this contradicts the assumption that the downward closure of $h^{-1}(s)$ is $<_T$-bounded below u.

□

At this point, we can define $u \in T_\delta$ with $f(u) \in S \setminus S_{<\delta}$ only using f^M and a well older $\prec$ of $H(\aleph_1)$. That is, given $f \in M$ as above, we let $u = u(f, M)$ be the $\prec$-minimal element of $T = \text{dom}(f)$ which satisfies the requirements of Claim 5.2.2 (1) i.e. $(f \restriction M)^{-1}(s)$ is $<_T$-bounded below u for every $s \in \text{ran}(f) \cap M = \text{ran}(f \cap M)$.

Claim 5.2.3. *Let $f \in M \prec H(\aleph_2)$ where $f : T \twoheadrightarrow S$ and $S \subseteq T$ are lexicographically ordered A-trees; let $\delta = M \cap \omega_1$ and $h = f \restriction T_{<\delta}$. Suppose that $u \in T_\delta$ and $f(u) \in S \setminus S_{<\delta}$. Then*

$$f(u)^\downarrow \cap S_{<\delta} = \bigcap \{([h(u \restriction \varepsilon), h(v)]_{<_{\text{lex}}} \cap S_{<\delta})^\downarrow : \varepsilon < \delta, u <_{\text{lex}} v \in T_{<\delta}\}.$$

Here, $[h(u \restriction \varepsilon), h(v)]_{<_{\text{lex}}}$ stands for all the $t \in T$ such that $h(u \restriction \varepsilon) \leq_{\text{lex}} t \leq_{\text{lex}} h(v)$.

Proof. First, suppose that $w \in f(u)^\downarrow \cap S_{<\delta}$. Take $\varepsilon < \delta$ and $u <_{\text{lex}} v \in T_{<\delta}$. $u \restriction \varepsilon <_{\text{lex}} u$ so $f(u \restriction \varepsilon) <_{\text{lex}} f(u) <_{\text{lex}} f(v)$. So $H(\aleph_2) \models$ "$w \in z^\downarrow$ for some $z \in [f(u \restriction \varepsilon), f(v)]_{<_{\text{lex}}} \cap S$"; indeed $z = f(u)$ satisfies this. So there must be some $z \in [f(u \restriction \varepsilon), f(v)]_{<_{\text{lex}}} \cap S \cap M$ such that $w \in z^\downarrow$. In turn, $w \in ([h(u \restriction \varepsilon), h(v)]_{<_{\text{lex}}} \cap S_{<\delta})^\downarrow$ as desired.

Second, suppose that $w \in S_{<\delta} \setminus f(u)^\downarrow$. We distinguish two cases: first, we consider if $w <_{\text{lex}} f(u)$ and so $w \lhd f(u)$.

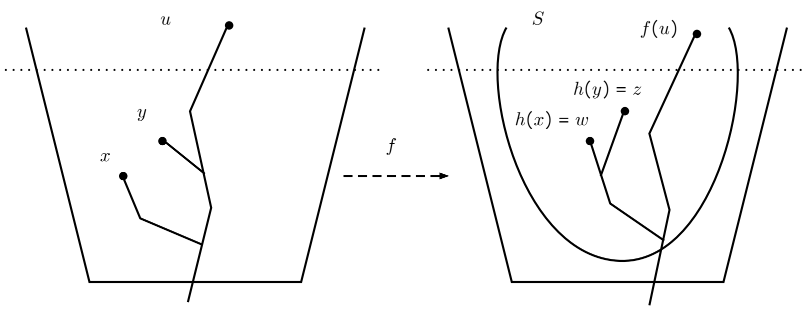

FIGURE 6. Proof of Claim 5.2.3

Find some $z \in S_{<\delta}$ so that $w \lhd z \lhd f(u)$ holds (this can be done by Claim 5.2.1). If $h(x) = w$ and $h(y) = z$ for some $x, y \in T_{<\delta}$ then $x <_{\text{lex}} y <_{\text{lex}} u$ so $y <_{\text{lex}} u \restriction \varepsilon$ for some large enough $\varepsilon < \delta$. The main point is that $w \notin t^{\downarrow}$ if $f(u \restriction \varepsilon) \leq_{\text{lex}} t$. Indeed, $y <_{\text{lex}} u \restriction \varepsilon$ implies that $z \leq_{\text{lex}} f(u \restriction \varepsilon) \leq_{\text{lex}} t$ so $w \lhd t$ i.e. $w \notin t^{\downarrow}$. Hence, we found an ε so that $w \notin ([h(u \restriction \varepsilon), h(v)]_{<_{\text{lex}}} \cap S_{<\delta})^{\downarrow}$ for any $u <_{\text{lex}} v \in T_{<\delta}$.

The second case, when $f(u) <_{\text{lex}} w$ is rather similar: find $z \in S_{<\delta}$ so that $f(u) \lhd z \lhd w$ and let $x, y \in T_{<\delta}$ so that $h(x) = w, h(y) = z$. Then $u \lhd y <_{\text{lex}} x$. Observe that $w \notin t^{\downarrow}$ for any $t \leq_{\text{lex}} z = h(y)$. Hence $w \notin ([h(u \restriction \varepsilon), h(y)]_{<_{\text{lex}}} \cap S_{<\delta})^{\downarrow}$ for any $\varepsilon < \delta$.

□

In summary, given $f \in M$ as above, we can define an unbounded branch $b(f, M)$ of $S_{<\delta}$ with an upper bound in S using only f^M as follows: we set

$$b(f, M) = f(u(f, M))^{\downarrow} \cap S_{<\delta}.$$

Claim 5.2.2 (2) says that this is really unbounded in $S_{<\delta}$ while Claim 5.2.3 shows that $b(f, M)$ is definable from f^M.

Finally, we need the following

Claim 5.2.4. *(A) implies that any ladder system colouring $f_\alpha : C_\alpha \to 2$ for $\alpha \in \lim(\omega_1)$ has a T-uniformization for any A-tree T.*

That is, the assumption of T being Hausdorff can be dropped from the definition of axiom (A).

Proof. Suppose that $f_\alpha : C_\alpha \to 2$ ($\alpha \in \lim(\omega_1)$) is a ladder system colouring and T is an A-tree. Construct a Hausdorff tree $\tilde{T}$ from T by inserting new, unique smallest upper bounds for bounded chains of T. Note that T and $\tilde{T}$ can be uniquely recovered from one another. Let $D_\alpha = \{\xi + 1 : \xi \in C_\alpha\}$ and $g_\alpha : D_\alpha \to 2$ by $g_\alpha(\xi + 1) = f_\alpha(\xi)$.

Now, there is a uniformization $\tilde{\varphi} : \tilde{S} \to 2$ of $(g_\alpha)_{\alpha \in \omega_1}$ where $\tilde{S} \subseteq \tilde{T}$ is downward closed and pruned. Let $S = \tilde{S} \cap T$ and define $\varphi : S \to 2$ by $\varphi = \tilde{\varphi} \restriction S$.

If $\delta \in \lim(\omega_1)$ and $u \in S_\delta$ then there is $\tilde{u} \in \tilde{S}_\delta$ so that $\{v <_T u\} <_{\tilde{T}} \tilde{u} <_{\tilde{T}} u$. So $\tilde{\varphi}[\tilde{u}] \restriction D_\delta =^* g_\delta$ i.e. $\tilde{\varphi}(\tilde{u} \restriction \xi + 1) = f_\delta(\xi)$ for almost all $\xi \in C_\delta$. However, note that $(\tilde{u} \restriction \xi + 1)_{\tilde{T}} = (u \restriction \xi)_T$ for all $\xi < \delta$. So $\varphi(u \restriction \xi) = f_\delta(\xi)$ for almost all $\xi \in C_\delta$.

□

We are ready to prove our Lemma 5.2 which, given the above work, will be very similar to the original proof of [9, Lemma 3.3].

Proof of Lemma 5.2. Assume that T is a lexicographically ordered A-tree. We will find a subtree S of T so that there is no $f : T \twoheadrightarrow S$ which preserves the lexicographic order.

Our first step is to define a map $F : H(\aleph_1) \to 2$. Fix an arbitrary ladder system $(C_\alpha)_{\alpha\in\lim(\omega_1)}$. Suppose that $\mathcal{U} = (f, \varphi)$ where $f : T \twoheadrightarrow S$ and $\varphi : S \to 2$; furthermore, let $\mathcal{U} \in M \prec H(\aleph_2)$. Now, we define $F(\mathcal{U}^M) = i$ iff $\varphi[b] \restriction C_\delta =^* i$ where $b = b(f, M)$ is the cofinal branch in $S_{<\delta}$ with an upper bound in S defined above. We set F to be 0 on all other elements of $H(\aleph_1)$.

Now, [9, Theorem 3.2] says that there is a $g : \omega_1 \to 2$ so that for every $\mathcal{U} \in H(\aleph_2)$ there is a countable elementary $M \prec H(\aleph_2)$ so that $g({\omega_1}^M) \neq F(\mathcal{U}^M)$. Let us define $f_\alpha : C_\alpha \to 2$ constant $g(\alpha)$. By (A) and Claim 5.2.4, there is some (pruned, downward closed) subtree S of T and uniformization $\varphi : S \to 2$ of $(f_\alpha)_{\alpha\in\lim(\omega_1)}$.

We claim that there is no $f : T \twoheadrightarrow S$. Otherwise, we set $\mathcal{U} = (f, \varphi)$ and claim that $F(\mathcal{U}^M) = g(\delta)$ for all $\mathcal{U} \in M \prec H(\aleph_2)$ and $\delta = \omega_1 \cap M$; this would contradict the choice of g. So fix M. The fact that the branch $b = b(f, M)$ of $S_{<\delta}$ has an upper bound in S implies that $\varphi[b] \restriction C_\delta =^* f_\delta = g(\delta)$ since φ was a uniformization. In turn, we defined $F(\mathcal{U}^M) = g(\delta)$. □

6. Open problems

Regarding suborders of $\mathbb{R}$ and the results of Section 2, the following is very natural:

Conjecture 6.1. *$MA_{\aleph_1}$ does not imply the existence of uncountable, strongly surjective real suborders.*

Problem 6.2. *Does every uncountable (real) strongly surjective order contain a minimal suborder?*

Section 3 motivated the next question:

Problem 6.3. *Are there any uncountable strongly surjective linear orders in the Cohen-or other canonical models (Sacks, Miller, etc.)?*

Problem 6.4. *Could there be a strongly surjective L of size $\mathfrak{c} > \aleph_1$?*

In particular, we ask for a short linear order of size $\mathfrak{c} > \aleph_1$ without real suborders of size $\mathfrak{c}$ (by Corollary 3.2). Such linear orders can be constructed, say with $\mathfrak{c} = \aleph_{\omega_1}$; we only sketch the proof.

Proposition 6.5. *Suppose that $V \models CH$. Then $V^{\mathbb{C}_{\aleph_{\omega_1}}} \models \mathfrak{c} = \aleph_{\omega_1}$ and there is a short linear order of size $\mathfrak{c}$ without real suborders of size $\mathfrak{c}$.*

Proof. L is defined as the lexicographic order on a tree T with the following properties: T has height ω_1, size $\mathfrak{c}$ and levels of size $< \mathfrak{c}$. Furthermore, each level T_α is separable and T has no $\aleph_1$ chains; these properties ensure that the linear order is short and has no separable suborders of size $\mathfrak{c}$.

Now, to construct T using a generic $G \subseteq \mathbb{C}_{\aleph_{\omega_1}}$, we inductively define T_α for $\alpha < \omega_1$. Given $T_{<\alpha}$ we select a cofinal copy S_α of $2^{<\omega}$ from $T_{<\alpha}$ and use $G \restriction [\omega_\alpha, \omega_\alpha + \omega_\alpha)$ to find $\aleph_\alpha$-many generic branches through S. These branches give T_α.

The only non trivial property to check is that there are no $\aleph_1$ chains in T. However, note that there is a club $C \subseteq \omega_1$ so that if $b \in V[G \restriction \omega_\alpha]$ is a cofinal branch through $T_{<\alpha}$ then b has no upper bound in T_α; this follows from genericity. In turn, there could be no $\aleph_1$-chains. □

On the other hand, the following holds.

Proposition 6.6. *Suppose that* $\aleph_1 < \mathfrak{c}$ *and*

$$\mathrm{cf}([\lambda]^\omega, \subseteq) < \mathrm{cf}(\mathfrak{c})$$

for all $\lambda < \mathfrak{c}$. *Then any short linear order* L *of size* $\mathfrak{c}$ *contains a real suborder of size* $\mathfrak{c}$.

The assumptions of the Proposition are satisfied if $\aleph_1 < \mathfrak{c} < \aleph_\omega$ since $\mathrm{cf}([\aleph_n]^\omega, \subseteq) = \aleph_n$.

Proof. First, note that $\aleph_1 < \mathrm{cf}(\mathfrak{c})$. Let T be an everywhere 2-branching partition tree for L. Then T has height $\leq \omega_1$ so there is a minimal $\alpha < \omega_1$ such that T_α has size $\mathfrak{c}$. So $\lambda = |T_{<\alpha}| < \mathfrak{c}$ and hence there is cofinal family in $[T_{<\alpha}]^\omega$ of size $< \mathrm{cf}(\mathfrak{c})$. Also, any element of T_α is given by a branch of a countable subset of $T_{<\alpha}$. In particular, there is a single countable $S \in [T_{<\alpha}]^\omega$ so that there are $\mathfrak{c}$ many branches through S with upper bounds in T_α. This gives a real suborder of size $\mathfrak{c}$. □

Also, the following question is open although it could be as hard as proving the consistency of Baumgartner's axiom for $\aleph_3$-dense sets of reals:

Problem 6.7. *Construct strongly surjective orders of size* $> \aleph_2$. *Can a strongly surjective linear order have size* $2^{\aleph_1}$*?*

⋆

We would like to restate the following question which came up in Section 5:

Problem 6.8. *Suppose that* T *is a lexicographically ordered* A*-tree,* $f : T \to T$ *preserves the lexicographic order and* $f[T]$ *is a large. Is it true that* f *must preserve the tree order on a club?*

⋆

J. Moore showed that, under PFA, uncountable linear orders have a 5 element basis: any uncountable linear order must embed $\omega_1, -\omega_1$, or an uncountable real order type, or a fixed Countryman line C or its reverse $-C$. Also, already under $\mathrm{MA}_{\aleph_1}$, there are minimal Countryman lines [15, Theorem 2.2.5]. So the next question is rather natural:

Problem 6.9. *Does* $MA_{\aleph_1}$ *imply that there is an uncountable strongly surjective linear order?*

Problem 6.10. *Can a strongly surjective linear order* L *be a Countryman line i.e. is it possible that* L^2 *is the union of countably many chains?*

⋆

Now, one can easily refine the notion of being strongly surjective by requiring the existence of maps for only a restricted class of suborders. Let us say that L is *surjective for the class* $\mathcal{K}$ iff $L \twoheadrightarrow K$ for any $K \in \mathcal{K}$ such that $K \subseteq L$. So L is strongly surjective iff it is surjective for the class of all linear orders. We say that L is κ-surjective iff $|L| \geq \kappa$ and L is surjective for the class of all linear orders of size κ.

Note that every cardinal κ with its usual well order is κ-surjective but not λ-surjective for $\lambda < cf(\kappa)$. In particular, ω_1 is $\aleph_1$-surjective but not strongly surjective.

Problem 6.11. *Is there a short,* $\aleph_1$*-surjective linear order* L *which is not strongly surjective i.e.* $L \not\twoheadrightarrow \mathbb{Q}$*?*

It is easy to see that a set of reals of size $\aleph_1$ is strongly surjective iff it is $\aleph_1$-surjective.

Problem 6.12. *Suppose that $L \subseteq \mathbb{R}$ is $\aleph_2$-surjective of size $\aleph_2$. Is L strongly surjective?*

⋆

The following problems concern the question if strong surjectivity reflects:

Problem 6.13. *Suppose that L is strongly surjective and $x \in L$. Is $L \setminus \{x\}$ strongly surjective?*

Problem 6.14. *Suppose that $\omega \leq \lambda < \kappa$ and L is a strongly surjective linear order of size κ. Is there a strongly surjective suborder of L of size λ?*

Yes, for $\lambda = \omega$ trivially (either ω or $-\omega$ embeds into L, and also $\mathbb{Q}$ embeds into any uncountable, short linear order by an old result of Hausdorff).

⋆

Finally, about mixing the order types of strongly surjective orders, we ask:

Problem 6.15. *Is it consistent that there are real and Aronszajn strongly surjective linear orders at the same time?*

Problem 6.16. *Is it consistent that there are uncountable, strongly surjective linear orders but each such order is separable?*

7. Acknowledgments

We thank S. Friedman, M. Hrusak, J. Moore and A. Rinot for helpful discussions and remarks. We are especially grateful to R. Carroy for stimulating conversations throughout this project.

The author was supported in part by the FWF Grant I1921.

References

[1] Abraham, U., Shelah, A. "Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic" Israel J Math 38 (1981) 161-176

[2] Abraham, Uri, Matatyahu Rubin, and Saharon Shelah. "On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types." Annals of Pure and Applied Logic 29.2 (1985): 123-206.

[3] Baumgartner, James E. "All $\aleph_1$-dense sets of reals can be isomorphic." Fundamenta Mathematicae 79.2 (1973): 101-106.

[4] Baumgartner, James E. "Order types of real numbers and other uncountable orderings." Ordered sets. Springer Netherlands, 1982. 239-277.

[5] Devlin, Keith J.; Shelah, Saharon A weak version of $\diamond$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. Israel J. Math. 29 (1978), no. 2-3, 239–247.

[6] O. Guzman, M. Hrusak "Parametrized $\diamond$-principles and canonical models." Slides from Retrospective workshop on Forcing and its applications, Fields Institute, 2015.

[7] Hajnal, A., Zs Nagy, and L. Soukup. "On the number of certain subgraphs of graphs without large cliques and independent subsets." A Tribute to Paul Erdos (1990): 223.

[8] Jech, T. "Set theory. The third millennium edition, revised and expanded." Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. xiv+769 pp. ISBN: 3-540-44085-2

[9] Moore, Justin Tatch. "ω_1 and $-\omega_1$ may be the only minimal uncountable linear orders." Michigan Math. J 55.2 (2007): 437-457.

[10] Moore, Justin Tatch. "A five element basis for the uncountable linear orders." Annals of Mathematics (2006): 669-688.

[11] Moore, Justin, Michael Hrušák, and Mirna Džamonja. "Parametrized $\diamond$ principles." Transactions of the American Mathematical Society 356.6 (2004): 2281-2306.

[12] Camerlo, Riccardo; Carroy, Raphael; Marcone, Alberto. "Epimorphisms between linear orders." Order 32 (2015), 387-400. Erratum, Order 33 (2016), 187.

[13] Camerlo, Riccardo; Carroy, Raphael; Marcone, Alberto; "Linear orders: when embedding and epimorphism coincide." preprint, arXiv:1701.02020

[14] Todorcevic, Stevo. "Trees and linearly ordered sets." Handbook of set-theoretic topology (1984): 235-293.

[15] Todorcevic, Stevo. "Walks on ordinals and their characteristics." Vol. 263. Springer Science & Business Media, 2007.

(D.T. Soukup) Universität Wien, Kurt Gödel Research Center for Mathematical Logic, Austria

E-mail address, Corresponding author: `daniel.soukup@univie.ac.at`

URL: `http://www.logic.univie.ac.at/~soukupd73/`