

TWO INFINITE CARDINALS AND THEIR SURPRISING RELATIONSHIP

DÁNIEL T. SOUKUP

As early as the 17th century, Galileo Galilei wondered how to compare the sizes of infinite sets. Fast forward almost four hundred years, and in the summer of 2017, at the 6th European Set Theory Conference, a young model theorist, Maryanthe Malliaris, and the well-known polymath, Saharon Shelah, received the Hausdorff Medal for the most influential work in set theory published in the last five years. Malliaris and Shelah made significant breakthroughs both regarding a model theoretic classification problem, and proved that two well-studied infinite cardinalities, \mathfrak{p} and \mathfrak{t} , are in fact the same. This latter result is the focus of our expository paper.



S. Shelah and M. Malliaris in the center
(Photo by Joan Bagaria)

Galilei considered the set of natural numbers \mathbb{N} and the set of perfect squares $\{1, 4, 9, 16 \dots\}$. The argument, that these two sets have the same size, goes as follows: since any perfect square has exactly one positive root, and any positive natural number is the root of some perfect square, there should be the same amount of perfect squares and positive natural numbers. On the other hand, there are many natural numbers which are not perfect squares; indeed, if one looks at the ratio of perfect squares to all natural numbers in larger and larger intervals, this quantity tends to zero rather fast. Galilei thought of these observations as a paradox, which prevents to distinguish between the sizes of infinite sets.

In the 1870s, Georg Cantor came forward with the following, now generally accepted definition: *two sets are equinumerous* or *have the same cardinality* exactly if there is a one-to-one correspondence between the elements of the two sets. So, Galilei's first argument proves that \mathbb{N} and the set of perfect squares have the same cardinality. Those sets which are equinumerous with \mathbb{N} are called *countably infinite* and we use \aleph_0 (in plain words 'aleph zero') to denote their size; the \aleph_0 notation refers to the fact that this is the smallest possible infinite size, or cardinality in other words.¹

One of Cantor's great contributions to logic was that he did not consider Galileo's argument as an irresolvable paradox, but instead, he started to develop a rich theory of infinities. First, he proved that even the set of rational numbers \mathbb{Q} is countably infinite; now, how about the set of all real number \mathbb{R} ? Cantor claims, that no matter how we produce a list x_1, x_2, x_3, \dots of real numbers, there is always a real number y which is missing from our enumeration. Indeed, if y differs from x_1 at the first decimal place, and y and x_2 differ at the second decimal place, and so on, then y cannot possibly appear on the list. Hence, there is no one-to-one correspondence between the natural numbers and the set \mathbb{R} , and so \mathbb{R} must be *uncountable*. We use the notation 2^{\aleph_0} (the 'continuum') for the cardinality of \mathbb{R} , and in turn, we proved the following inequality:

$$\text{the cardinality of } \mathbb{N} = \aleph_0 < 2^{\aleph_0} = \text{the cardinality of } \mathbb{R}.$$

Based on Cantor's theory, one can compare the cardinalities of any two sets, and in any non empty collection of sets, there is a smallest in size. So, it makes sense to define the first infinite cardinality which is uncountable, and we use \aleph_1 to denote this. The next larger cardinality will be denoted by \aleph_2 , then \aleph_3 , and so on.²

At this point, we defined a strictly increasing sequence $\aleph_0 < \aleph_1 < \aleph_2 < \dots$ of infinite cardinalities. Where does 2^{\aleph_0} sit in this list? Interestingly, this question is *undecidable* using the generally accepted ZFC axioms of mathematics.³ Put it differently, in some models of mathematics the equation

Is there a largest infinite cardinality? Not according to Cantor's definition. Any set X has more subsets than elements, in other words, $\kappa < 2^\kappa$ for any cardinality κ .

¹We say that a set X has cardinality *at most the cardinality of* Y , if X and some subset of Y are equinumerous.

²The list of \aleph 's does not stop here: the \aleph_n cardinalities indexed by the natural number have a supremum, denoted by \aleph_ω , the next strictly larger cardinality is $\aleph_{\omega+1}$, then $\aleph_{\omega+2} \dots$

³ZFC stands for the Zermelo-Fraenkel axiom system with the 'Axiom of Choice'.

$2^{\aleph_0} = \aleph_1$ holds, i.e. \mathbb{R} has the smallest possible uncountable cardinality, and in this case, we say that the *Continuum Hypothesis* holds. In many other interesting and important models, the Continuum Hypothesis fails and the gap between \aleph_0 and 2^{\aleph_0} can be arbitrary large: some models satisfy ad hoc equations, such as $2^{\aleph_0} = \aleph_{16}$.

A simple analogy might shed more light on this undecidability business: \mathbb{R} and \mathbb{Q} , as algebraic structures, both satisfy certain axioms of addition and multiplication,⁴ however the equation $x^2 = 2$ has two roots in \mathbb{R} , but none in \mathbb{Q} . In other words, whether the statement ' $x^2 = 2$ has a solution' is true or false, is not decided by the axioms of addition and multiplication. Similarly, while the ZFC axiom system decides that 'the inner angles of a plane triangle sum to 180 degrees', the axioms do not decide whether $2^{\aleph_0} = \aleph_1$ or $2^{\aleph_0} > \aleph_1$ holds; one of these statements holds in any model, but it depends on the particular model which.⁵

How does one make new models? Kurt Gödel, in 1938, discovered the so-called *constructible universe*, and proved that the Continuum Hypothesis holds in this model of mathematics. Then, in the 1960s, Paul Cohen genuinely surprised the set theoretic community: he was the first to show that, in some other models of ZFC, the Continuum Hypothesis fails i.e., $\aleph_1 < 2^{\aleph_0}$. Cohen's technique is called *forcing*, and its underlying idea is fairly simple: given a set theoretic model M , we can construct a larger model N by adding a generic object G to M . We can use the generic G to increase the value of 2^{\aleph_0} , which leads to the failure of the Continuum Hypothesis. Cohen received a Fields Medal for his work in set theory in 1966.

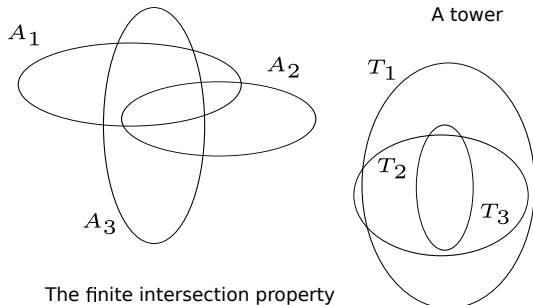
In the last fifty years, people constructed various different models of mathematics, which have plenty of cardinalities between \aleph_0 and 2^{\aleph_0} , and a significant portion of modern set theory focuses on the analysis of these models. Now, two models of the ZFC axioms which both satisfy $2^{\aleph_0} = \aleph_{16}$ can behave very differently, even if one only considers theorems from algebra, measure theory or topology. To the surprise of many specialists, the Whitehead-problem in group theory or the existence of outer automorphisms on the Calkin-algebra are both undecidable using the usual axioms.⁶

On the bright side, the study of the real line and its fundamental topological and measure theoretic properties is possible through looking at combinatorial properties of families of subsets of the natural numbers; indeed, there is a one-to-one corre-

spondence between the real numbers and subsets of \mathbb{N} .⁷ The *cardinal characteristics of the continuum* is exactly the study of such matters.

What attracts many people to this area of research, is that one can understand a breakthrough result such as Malliaris and Shelah's with minimal background. From now on, we will only talk about subsets of the natural numbers, and our concern is the following relation: if A and B are sets of natural numbers, then we write $A \subseteq^* B$ (in plain words, ' A is almost contained in B ') if all but finitely many elements of A are elements of B as well. For example, the set $A = \{1, 3, 5, 7, \dots\}$ is almost contained in $B = \{4, 5, 6, 7, \dots\}$, since each element of A , apart from 1 and 3, is an element of B too.

What is the advantage of working with such a weak relation instead of the real containment? Let's take the sets A_n of positive natural numbers which are divisible by n : so A_1 is the set of all positive natural numbers, A_2 collects the even natural numbers, $A_3 = \{3, 6, 9, \dots\}$, and so on.



It is easy to see that if we take finitely many sets A_1, A_2, \dots, A_n until a fixed n , then these sets have infinite intersection.⁸ We usually say in this case that the system of sets $\{A_n\}$ has the *finite intersection property*, a rather counterintuitive name for sets with infinite intersection...

Of course we cannot find any positive natural number which is divisible by all the numbers, that is, there is no real intersection to the whole family $\{A_n\}$. On the other hand, we can easily find infinite sets B so that $B \subseteq^* A_n$ for all n ; it suffices to make sure that the n th element of B is selected from $A_1 \cap A_2 \cap \dots \cap A_n$, so for

example $B = \{n! : n = 1, 2, 3, \dots\}$ works. Since B behaves somewhat like an intersection to the family

⁴Think about commutativity, distributivity or the *field axioms* in general.

⁵There is a similar parallel with the discovery of non-standard geometries in the 19th century.

⁶However, in light of present techniques, it is highly unlikely that the Riemann-hypothesis or the existence of general solutions for the Navier-Stokes equations is undecidable. Anyone who solves one of these problems, will be awarded the Millenium Prize and its one million dollar prize purse.

⁷In other words, the cardinality of all subsets of \mathbb{N} is 2^{\aleph_0} .

⁸Indeed, the multiples of $n! = 1 \cdot 2 \cdot \dots \cdot n$ are contained in A_1, A_2, \dots, A_n .

$\{A_n\}$, we call B a *pseudo-intersection* of $\{A_n\}$. It is also easily checked, that one can add B to the family $\{A_n\}$, and the finite intersection property is still preserved.

Now, given an arbitrary family of sets \mathcal{A} with the finite intersection property, we can extend \mathcal{A} to a maximal family \mathcal{A}_{\max} with the finite intersection property.⁹ This maximal family \mathcal{A}_{\max} however cannot have a pseudo-intersection anymore.¹⁰ This leads to our first main definition:

The *pseudo-intersection number* \mathfrak{p} is the cardinality of the smallest family of sets \mathcal{A} with the finite intersection property, such that \mathcal{A} does not have a pseudo-intersection.

In the above example, we essentially showed that any family *indexed by the natural numbers*, which has the finite intersection property, also has a pseudo-intersection, and in turn, \mathfrak{p} must be uncountable. Hence, the following inequality holds:

$$\aleph_0 < \mathfrak{p} \leq 2^{\aleph_0}.$$

There are numerous models of mathematics, where $\mathfrak{p} = 2^{\aleph_0}$ and this common value can be essentially any \aleph . On the other hand, $\aleph_1 = \mathfrak{p} < 2^{\aleph_0} = \aleph_2$ can also be true in other models. So the usual axioms do not decide where the pseudo-intersection number \mathfrak{p} sits on the list $\aleph_1, \aleph_2, \dots$, nor if $\mathfrak{p} = 2^{\aleph_0}$ or $\mathfrak{p} < 2^{\aleph_0}$ holds.

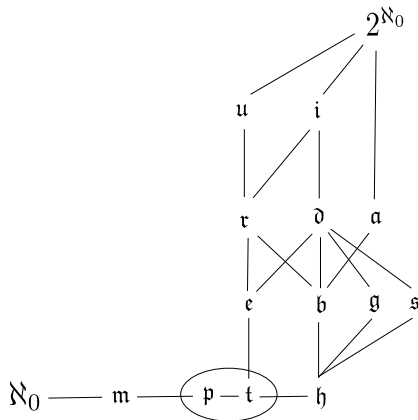
We need another definition. A typical family with the finite intersection property is not ordered in any sense: in our original example about divisibility, if one only considers the sets A_p for primes p , no two of these sets are in \subseteq^* relation. Now, call a family of sets \mathcal{T} a *tower* if for any two sets X, Y from \mathcal{T} , either $X \subseteq^* Y$ or $Y \subseteq^* X$ holds. In other words, the relation \subseteq^* linearly orders \mathcal{T} .

The so called *tower number* \mathfrak{t} is the cardinality of the smallest tower \mathcal{T} without pseudo-intersection.

Since any tower has the finite intersection property, any witness for the invariant \mathfrak{p} must have at most the size of a tower witnessing \mathfrak{t} . This shows the next equation:

$$\aleph_0 < \mathfrak{p} \leq \mathfrak{t} \leq 2^{\aleph_0}.$$

The value of the invariant \mathfrak{t} can be manipulated similarly to \mathfrak{p} . Actually, since the 1940s, more than a dozen cardinal invariants between \aleph_0 and 2^{\aleph_0} have been studied. Moreover, we know that apart from certain simple inequalities that were known from the mid 20th century, there is no provable relationship between the invariants. That is, with various versions and combinations of forcing techniques, one can not only set the value of 2^{\aleph_0} but also the values of the invariants to any fixed alephs, which do not violate the known inequalities.¹¹



A few cardinal invariants and their provable relationships

Large towers. Surprisingly, even from the subsets of \mathbb{N} , one can construct towers of size 2^{\aleph_0} . First of all, list all the rational numbers as q_1, q_2, q_3, \dots . Then, for each real number r , we define X_r to be the set of all those indexes n so that $q_n < r$. So X_r essentially collects the rational numbers smaller than r . Now, if $r < t$ are two real numbers then X_r is fully contained in X_t , moreover, X_t has infinitely many extra elements. Can we find a pseudo-intersection for this family?

Indeed, the most advanced techniques can independently manipulate the values of more than five invariants at the same time.

Despite all these investigations in the last sixty years, no one constructed a model, where \mathfrak{p} and \mathfrak{t} are not equal. The generally accepted conjecture was that $\mathfrak{p} < \mathfrak{t}$ is possible, however, we knew that a proof could not be too easy: it was proved a long time ago that if $\mathfrak{p} = \aleph_1$, then $\mathfrak{t} = \aleph_1$ holds as well. So, $\mathfrak{p} < \mathfrak{t}$ can only hold in models where 2^{\aleph_0} is bigger than \aleph_2 , however fine tuning such models is significantly harder than controlling models with $2^{\aleph_0} \leq \aleph_2$.

Malliaris and Shelah's new, unexpected result states that

$$\mathfrak{p} = \mathfrak{t},$$

⁹This is a standard application of Zorn's lemma.

¹⁰Otherwise, we could add this pseudo-intersection to \mathcal{A}_{\max} while preserving the finite intersection property, and so violate maximality.

¹¹We recommend A. Blass' classical text on cardinal invariants for an excellent overview.

no matter which model we look at. What is it exactly that the authors prove? Since $\mathfrak{p} \leq \mathfrak{t}$ was known, they needed to show $\mathfrak{t} \leq \mathfrak{p}$: given an arbitrary family \mathcal{A} with the finite intersection property but no pseudo-intersection, no matter how random or involved the overlays between the elements of \mathcal{A} are, one can construct a tower \mathcal{T} of size at most the size of \mathcal{A} which has no pseudo-intersection. The naive approach to construct the tower from the elements \mathcal{A} fails quickly, since \mathcal{A} might have no two elements which are related by \subseteq^* , but \mathcal{T} must be totally ordered by \subseteq^* .

One can compare Malliaris and Shelah's result to showing that two equations have the same solutions, but without actually determining the value of these solutions. The ZFC axioms do not decide whether $\mathfrak{p} = \aleph_1$ or $\mathfrak{p} = \aleph_2$ or $\mathfrak{p} = \aleph_3$, and similarly the value of \mathfrak{t} is undecidable. However, Malliaris and Shelah could prove that, no matter what the value of \mathfrak{p} is, it has to be the same as \mathfrak{t} .

We should mention that not only the authors solved the sixty year old mystery surrounding \mathfrak{p} and \mathfrak{t} , but they uncovered a novel connection between a model theoretic complexity hierarchy, the Keisler-order, and the theory of cardinal characteristics [6, 7, 8]; unfortunately, it is not in the scope of our paper to sketch these results. While the original proof of $\mathfrak{p} = \mathfrak{t}$ employs serious tools from model theory and modern set theory, there are now new versions which only require basic knowledge and some perseverance [9]¹²

An open problem. We say that a family \mathcal{R} is *unsplit* if there is no Y which splits all elements of \mathcal{R} at the same time, i.e., no Y so that the intersection $X \cap Y$ and the difference $X - Y$ are both infinite for any X from \mathcal{R} . Let \mathfrak{r} denote the cardinality of the smallest unsplit family. Moreover, let \mathfrak{r}_σ denote the size of the smallest family which cannot be split by even countably many sets Y_0, Y_1, \dots . It is easy to see that $\aleph_0 < \mathfrak{r} \leq \mathfrak{r}_\sigma \leq 2^{\aleph_0}$, however, it remains unknown if $\mathfrak{r} < \mathfrak{r}_\sigma$ is possible in some model. The conjecture is that $\mathfrak{r} = \mathfrak{r}_\sigma$ holds, and this is certainly true in all known models [3].

The results of Malliaris and Shelah are far from the last of cardinal characteristics, these results will more likely spark a renewed interest in the field. So what problems does a regular set theorist work on? On one hand, the relationship of some classical invariants are still unknown, and we mention a problem of this sort in the side note [3]. On the other hand, people are defining new, interesting invariants to this day, and it is often a hard task to determine the position of these new invariants relative to the classical ones [2]. Finally, a rich theory is growing out of the study of cardinal characteristics which are defined using families of uncountable sets, rather than the subsets of \mathbb{N} , showing striking differences with the classical studies [4].

We close by a few words about the awardees: Maryanthe Malliaris graduated from Berkeley in 2009, currently a professor at the University of Chicago, she is the recipient of multiple, prestigious awards, and is invited to present at the International Congress of Mathematicians in 2018.

The name Saharon Shelah might ring a bell for a lot of the readers: the 72 years old mathematician is the author of 1023 published papers (!), on groundbreaking results from combinatorics and model theory, to logic and group theory. He still works 6 days a week, splitting the year between Rutgers and the Hebrew University of Jerusalem.

The author was supported in part by the FWF Grant I1921. The original Hungarian version was prepared for the journal Matematikai Lapok, and we thank Emese Bottyán, Lajos Soukup, Zoltán Vidnyánszky, and Zita Zádorvölgyi for their careful reading. The English version significantly improved thanks to the help of Vera Fischer.

REFERENCES

- [1] A. Blass. Combinatorial cardinal characteristics of the continuum. *Handbook of set theory*, pages 395–489, 2010.
- [2] A. Blass, J. Brendle, W. Brian, J. D. Hamkins, M. Hardy, and P. B. Larson. The rearrangement number. *arXiv preprint:1612.07830*, 2016.
- [3] J. Brendle. Around splitting and reaping. *Commentationes Mathematicae Universitatis Carolinae*, 39(2):269–279, 1998.
- [4] J. Brendle, A. Brooke-Taylor, S.-D. Friedman, and D. Montoya. Cichon's diagram for uncountable cardinals. *to appear in Israel Journal of Mathematics*, *arXiv:1611.08140*, 2016.
- [5] G. Galilei. *Dialogue concerning the two chief world systems, Ptolemaic & Copernican*. Univ of California Press, 1967.
- [6] M. Malliaris and S. Shelah. General topology meets model theory, on \mathfrak{p} and \mathfrak{t} . *Proceedings of the National Academy of Sciences*, 110(33):13300–13305, 2013.
- [7] M. Malliaris and S. Shelah. Cofinality spectrum theorems in model theory, set theory, and general topology. *Journal of the American Mathematical Society*, 29(1):237–297, 2016.

¹²The Fields Medalist, Timothy Gowers' blog also looks at this problem, [link here](#).

- [8] J. T. Moore. Model theory and the cardinal numbers \mathfrak{p} and \mathfrak{t} . *Proceedings of the National Academy of Sciences*, 110(33):13238–13239, 2013.
- [9] G. M. Roccasalvo. Ultraproducts of finite partial orders and some of their applications in model theory and set theory. Master's thesis, University of Torino, 2014.

(D.T. Soukup) UNIVERSITÄT WIEN, KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC, AUSTRIA
E-mail address, Corresponding author: `daniel.soukup@univie.ac.at`
URL: `http://www.logic.univie.ac.at/~soukupd73/`