PARTITIONING BASES OF TOPOLOGICAL SPACES

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Abstract. We investigate whether an arbitrary base for a topological space can be partitioned into two bases. We prove that every base for a T_3 Lindelöf topology can be partitioned into two bases while there exists a consistent example of a first countable, 0-dimensional, Hausdorff space of size and weight ω_1 which admits a base without a partition to two bases.

1. Introduction

At the Trends in Set Theory conference in Warsaw, Barnabás Farkas[1] raised the natural question whether one can partition any given base for a topological space into two bases; we will call this property being *base resolvable*. The aim of this paper is to present two streams of results: in the first part of the article, we will show that certain natural classes of spaces are base resolvable. In the second part, we present a method to construct non base resolvable spaces.

The paper is structured as follows: in Section 2, we will start with general observations about bases and we prove that metric spaces and left-or right-separated spaces are base resolvable. This section also serves as an introduction to the methods that will be applied in Section 3 where we prove one of our main results in Theorem 3.6: every T_3 (locally) Lindelöf space is base resolvable.

In Section 4, we investigate base resolvability from a purely combinatorial viewpoint which leads to further results: every hereditarily Lindelöf space (without any separation axioms) is base resolvable and any base for a T_1 topology which is closed to finite unions can be partitioned into two bases, see Theorem 4.6 and 4.7.

In Section 5, we prove, in Theorem 5.6, that every base $\mathbb{B}$ for a space X (resolvable or not) contains a large *negligable* portion, i.e. there is $\mathcal{U} \in [\mathbb{B}]^{|\mathbb{B}|}$ such that $\mathbb{B} \setminus \mathcal{U}$ is still a base for X.

The second part of the paper starts with Section 6; here, we isolate a partition property, denoted by $\mathbb{P} \to (I_\omega)^1_2$, of the partial order $\mathbb{P} = (\mathbb{B}, \supseteq)$ associated to a base $\mathbb{B}$ which is closely related to resolvability.

[1] personal communication

We will construct a partial order $\mathbb{P}$ with this property in Theorem 6.5 and deduce the existence of a T_0 non base resolvable topology (in ZFC) in Corollary 6.13.

Next, in Section 7 we present a ccc forcing (of size ω_1) which introduces a first countable, 0-dimensional, Hausdorff space X of size and weight ω_1 such that X is not base resolvable. The main ideas of the construction already appear in Section 6 however the details here are much more subtle and the proofs are more technical.

The paper finishes with a list of open problems in Section 8. We remark that Section 7 was prepared by the second author and the rest of the paper is the work of the first author.

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2. General results

In this section, we prove some basic results concerning partitions of bases; these proofs will introduce us to the more involved techniques of the upcoming sections.

Definition 2.1. *A base $\mathbb{B}$ for a space X is **resolvable** iff it can be decomposed into two bases. A space X is **base resolvable** if every base of X is resolvable.*

Note that every space X with an isolated point admits an irresolvable base; to avoid this triviality in the future, we suppose that by *space* we mean a dense-in-itself topological space.

Partitioning sets with additional structure is a highly investigated theme in mathematics; let us cite a classical result of A. H. Stone which is relevant to our case:

Theorem 2.2 (A. H. Stone, [2])**.** *Every partially ordered set $(\mathbb{P}, \leq)$ without maximal elements can be partitioned into two cofinal subsets.*

Proposition 2.3.
(1) Every base can be partitioned to a cover and a base.
(2) Every π-base can be partitioned to two π-bases.
(3) Every neighborhood base can be partiotioned to two neighborhood bases.

Proof. To prove (1), note that every cover contains a well founded (with respect to $\subset$) subcover. Also, well founded families of open sets cannot form neighborhood bases in dense-in-itself spaces; thus, if $\mathcal{U}$ is a well founded cover of X and $\mathbb{B}$ is a base then $\mathbb{B} \setminus \mathcal{U}$ is still a base of X.

Note that (2) and (3) follows from Theorem 2.2. □

Now we prove our first general result.

Proposition 2.4. *Every space with a σ-disjoint base is base resolvable; in particular, every metrizable space is base resolvable.*

Proof. Fix a space X with a base $\cup\mathbb{E}_n$ where each $\mathbb{E}_n$ is a disjoint family; fix an arbitrary base $\mathbb{B}$ as well which we aim to partition.

By induction on $n \in \omega$, construct $\mathbb{B}_{i,n} \subseteq \mathbb{B}$ for $i < 2$ such that

(1) $\mathbb{B}_{i,n}$ is well founded for $i < 2$, $n \in \omega$,
(2) $\mathbb{B}_{i,n} \cap \mathbb{B}_{j,m} = \emptyset$ if $i, j < 2$, $n, m \in \omega$ and $(i, n) \neq (j, m)$,
(3) for every $V \in \mathbb{E}_n$ and $i < 2$ there is $\mathcal{U} \subseteq \mathbb{B}_{i,n}$ such that $\cup\mathcal{U} = V$.

Note that property (1) assures that $\mathbb{B} \setminus \cup\{\mathbb{B}_{i,k} : i < 2, k < n\}$ is still a base of X for each $n < \omega$ thus the induction can be carried out. Let $\mathbb{B}_i = \cup\{\mathbb{B}_{i,n} : n \in \omega\}$ for $i < 2$; it is easy to see that these disjoint families will form a base by property (3). □

A somewhat similar technique, which will be used later as well, gives the following result:

Proposition 2.5. *Suppose that a regular space X satisfies $L(X) < \kappa = w(X) = \min\{\chi(X, x) : x \in X\}$. Then X is base resolvable.*

Proof. Fix a base $\mathbb{B}$ for X and an enumeration $\{(U_\alpha, V_\alpha) : \alpha < \kappa\}$ of all pairs of elements $U, V \in \mathbb{B}$ such that $\overline{U} \subseteq V$; without loss of generality, we can suppose that $\mathbb{B}$ has size κ.

Inductively construct increasing $\mathbb{B}_{0,\alpha}, \mathbb{B}_{1,\alpha} \subseteq \mathbb{B}$ such that

(1) $\mathbb{B}_{0,\alpha} \cap \mathbb{B}_{1,\alpha} = \emptyset$,
(2) there is $\mathcal{U} \subseteq \mathbb{B}_{i,\alpha}$ such that $\overline{U_\alpha} \subseteq \cup\mathcal{U} \subseteq V_\alpha$ for every $i < 2$,
(3) $|\mathbb{B}_{i,\alpha}| \leq L(X) \cdot |\alpha|$ for $i < 2$.

Note that our assumptions on the space and the inductive hypothesis (3) implies that

$$\mathbb{B} \setminus \bigcup\{\mathbb{B}_{i,\beta} : \beta < \alpha, i < 2\}$$

is still a base for X for every $\alpha < \kappa$. It follows that the induction can be carried out and the disjoint families $\mathbb{B}_i = \cup\{\mathbb{B}_{i,\alpha} : \alpha < \kappa\}$ form a base for X; thus X is base resolvable. □

We end this section by giving further classes of spaces which are base resolvable.

Observation 2.6. *Every right or left separated space is base resolvable. Furthermore, the Sorgenfrey line or the Double Arrow space is base resolvable.*

Proof. Recall that every neighborhood base can be partitioned into two neighborhood bases by Proposition 2.3. Thus, if $\mathbb{B}$ is a base of X and there is a map $f : \mathbb{B} \to X$ such that $f^{-1}(x)$ is a base at x for any $x \in X$ then by partitioning $f^{-1}(x)$ for each $x \in X$ into two neighborhood bases of x we get a partition of $\mathbb{B}$ into two bases of X. Now, it is a fairly easy excersice to finish the proof. □

3. Lindelöf spaces are base resolvable

Our aim in this section is to prove that T_3 Lindelöf spaces are base resolvable; we start with a definition and some observations while the most important part of the work is done in the proof of Lemma 3.3.

Definition 3.1. *Let $\mathcal{A}, \mathcal{B}$ families of open sets in a space X. We say that $\mathcal{A}$ **weakly fills** $\mathcal{B}$ iff for every $U, V \in \mathcal{B}$ such that $\overline{U} \subset V$ there is $\mathcal{W} \subseteq \mathcal{A}$ such that*

$$\overline{U} \subseteq \cup\mathcal{W} \subset V.$$

*$\mathcal{A}, \mathcal{B}$ is called a **weakly good pair** iff $\mathcal{A}, \mathcal{B}$ are disjoint, $\mathcal{A}$ weakly fills $\mathcal{B}$ and $\mathcal{B}$ weakly fills $\mathcal{A}$.*

We remark that in the next section we introduce stronger notions called *filling* and *good pairs*. The following observations summarize the importance of weakly good pairs:

Observation 3.2. *Suppose that X is a regular space.*

(1) If $(\mathcal{A}, \mathcal{B})$ is a weakly good pair in X then $\mathcal{A}$ contains a neighborhood base at x iff $\mathcal{B}$ contains a neighborhood base at x, for any $x \in X$.
(2) If a family of open sets $\mathcal{A}$ weakly fills a base $\mathbb{B}$ of X then $\mathcal{A}$ is a base as well.
(3) If $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ are increasing and $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ is a weakly good pair in X then $(\cup\mathcal{A}_\alpha, \cup\mathcal{B}_\alpha)$ is a weakly good pair as well.

We encourage the reader to compare these observations with the proof of Proposition 2.5.

We say that the weakly good pair $(\mathcal{A}', \mathcal{B}')$ **extends** the weakly good pair $(\mathcal{A}, \mathcal{B})$ iff $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{B} \subset \mathcal{B}'$. A family of weakly good pairs $\{(\mathcal{A}_\xi, \mathcal{B}_\xi) : \xi < \Theta\}$ is **pairwise disjoint** iff $\mathcal{A}_\xi \cap \mathcal{B}_\zeta = \emptyset$ for each $\xi, \zeta < \Theta$.

Next, we prove that weakly good pairs can be nicely extended in Lindelöf spaces.

Lemma 3.3. *Suppose that X is a T_3 Lindelöf space. Given a weakly good pair of open sets $\mathcal{A}, \mathcal{B}$ and a single pair of open sets (U, V) such that $\overline{U} \subset V$ there is a weakly good pair $\mathcal{A}', \mathcal{B}'$ extending $\mathcal{A}, \mathcal{B}$ such that both $\mathcal{A}'$ and $\mathcal{B}'$ fills $\{U, V\}$.*

Proof. We will show this essentially by induction on the size of $\mathcal{A}$ and $\mathcal{B}$ however we need to prove something significantly stronger (and more technical) then the statement of the lemma itself.

Let $\triangle_\kappa$ stand for the following statement: for every pairwise disjoint family of weakly good pairs $\{(\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$ such that $|\mathcal{A}_i|, |\mathcal{B}_i| \leq \kappa$ and arbitrary open family $\mathcal{E}$ of size at most κ there is a weakly good pair $(\mathcal{A}, \mathcal{B})$ of size at most κ such that

1. $\cup_{i<n}\mathcal{A}_i \subset \mathcal{A}$ and $\cup_{i<n}\mathcal{B}_i \subset \mathcal{B}$,
2. $\mathcal{A}$ and $\mathcal{B}$ weakly fills $\mathcal{E}$,
3. $\{(\mathcal{A}, \mathcal{B}), (\mathcal{C}_j, \mathcal{D}_j) : j < k\}$ is still pairwise disjoint.

We prove that $\triangle_\kappa$ holds for every infinite κ.

Claim 3.4. *$\triangle_\omega$ holds.*

Proof. Fix $\{(\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$ and $\mathcal{E}$. We inductively build disjoint, increasing chains $\mathcal{A}^m$ and $\mathcal{B}^m$ such that

1. $\mathcal{A}^0 = \cup_{i<n}\mathcal{A}_i$, $\mathcal{B}^0 = \cup_{i<n}\mathcal{B}_i$,
2. $\mathcal{A}^{m+1} \setminus \mathcal{A}^m$ and $\mathcal{B}^{m+1} \setminus \mathcal{B}^m$ are countable well-founded families for each $n \in \omega$,
3. $\mathcal{A}^m \cap \mathcal{B}_i = \emptyset, \mathcal{A}^m \cap \mathcal{D}_j = \emptyset$ and $\mathcal{B}^m \cap \mathcal{A}_i = \emptyset, \mathcal{B}^m \cap \mathcal{C}_j = \emptyset$ for $i < n, j < k, m < \omega$.

Furthermore, we will make sure that $\mathcal{A} = \cup_{m\in\omega}\mathcal{A}^m$ and $\mathcal{B} = \cup_{m\in\omega}\mathcal{B}^m$ forms a weakly good pair and they both weakly fill $\mathcal{E}$. Therefore, we partition ω into infinite sets $\omega = \cup\{D_m : m \in \omega\}$ and at each step we define a surjective map $f_m : D_m \setminus (m+1) \to \{(U, V) \in (\mathcal{A}^m \cup \mathcal{B}^m \cup \mathcal{E})^2 : \overline{U} \subset V\}$; if $m \in D_l \setminus (l + 1)$ and $f_l(m) = (U, V)$ then at step m we extend so that $\mathcal{A}^m$ and $\mathcal{B}^m$ weakly fills $\{U, V\}$.

Now our goal is reduced to construct $\mathcal{A}^{m+1}$ and $\mathcal{B}^{m+1}$ from $\mathcal{A}^m$ and $\mathcal{B}^m$ such that they satisfy (2)-(3) above while they both weakly fill a given $\{U, V\}$. We construct $\mathcal{A}^{m+1}$, the proof for $\mathcal{B}^{m+1}$ is analoguous. Define

$$\begin{aligned} F_i &= \{x \in X : \mathcal{A}_i \text{ contains a neighborhood base at } x\} \\ &= \{x \in X : \mathcal{B}_i \text{ contains a neighborhood base at } x\} \end{aligned}$$

and

$$\begin{aligned} G_j &= \{x \in X : \mathcal{C}_j \text{ contains a neighborhood base at } x\} \\ &= \{x \in X : \mathcal{D}_j \text{ contains a neighborhood base at } x\}. \end{aligned}$$

For every $i < 2$ and $x \in F_i \cap \overline{U}$ pick $U_{x,i} \in \mathcal{A}_i$ such that $x \in U_{x,i} \subset V$; let $\mathcal{U} = \{U_{x,i} : i < 2, x \in F_i \cap \overline{U}\}$. For $j < k$ and $x \in G_j \cap \overline{U}$ pick $V_{x,j} \in \mathcal{C}_i$ such that $x \in V_{x,j} \subset V$; let $\mathcal{V} = \{V_{x,j} : j < k, x \in F_j \cap \overline{U}\}$. Now note that for every $x \in \overline{U} \setminus \bigcup(\mathcal{V} \cup \mathcal{U})$ there is a neighborhood base for x in $\mathbb{B} \setminus \bigcup_{i<2,j<k}(\mathcal{B}_i \cup \mathcal{D}_j)$; hence for every $x \in \overline{U} \setminus \bigcup(\mathcal{V} \cup \mathcal{U})$ we can pick $W_x \in \mathbb{B} \setminus \bigcup_{i<2,j<k}(\mathcal{B}_i \cup \mathcal{D}_j)$ such that $x \in W_x \subset V$; let $\mathcal{W} = \{W_x : x \in \overline{U} \setminus \bigcup(\mathcal{V} \cup \mathcal{U})\}$. Select a countable well-founded subcover $\mathcal{Q} \subset \mathcal{U} \cup \mathcal{V} \cup \mathcal{W}$ of $\overline{U}$ and define $\mathcal{A}^{m+1} = \mathcal{A}^m \cup \mathcal{Q}$. □

Claim 3.5. *Suppose that $\triangle_\lambda$ holds for every $\omega \leq \lambda < \kappa$. Then $\triangle_\kappa$ holds.*

Proof. Fix $\{(\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$ and $\mathcal{E}$, let $cf(\kappa) = \mu$ and fix a cofinal sequence of ordinals $(\kappa_\xi)_{\xi<\mu}$ in κ. Take a chain of elementary submodels $(M_\xi)_{\xi<\mu}$ such that everything relevant is in M_0, $\kappa_\xi \subset M_\xi$ and $|M_\xi| = |\kappa_\xi|$ for $\xi < \mu$. The following is an easy consequence of M_ξ being elementary and X being Lindelöf:

Subclaim 3.5.1. *$(\mathcal{A}_i \cap M_\xi, \mathcal{B}_i \cap M_\xi)$ are weakly good pairs of size at most $|\kappa_\xi|$ for all $i < n$.*

By induction on $\xi < \mu$ construct an increasing sequence of weakly good pairs $\{(\mathcal{A}^\xi, \mathcal{B}^\xi) : \xi < \mu\}$ such that

- (i) $\cup_{i<n}(A_i \cap M_\xi) \subset \mathcal{A}^\xi$ and $\cup_{i<n}(B_i \cap M_\xi) \subset \mathcal{B}^\xi$,
- (ii) $\mathcal{A}^\xi, \mathcal{B}^\xi$ has size $\leq |\kappa_\xi|$,
- (iii) $\mathcal{A}^\xi, \mathcal{B}^\xi$ weakly fills $\mathcal{E} \cap M_\xi$,
- (iv) $\mathcal{A}^\xi \cap \mathcal{B}_i = \emptyset, \mathcal{A}^\xi \cap \mathcal{D}_j = \emptyset$ and $\mathcal{B}^\xi \cap \mathcal{A}_i = \emptyset, \mathcal{B}^\xi \cap \mathcal{C}_j = \emptyset$.

This can be done using $\triangle_{|\kappa_\xi|}$ at stage ξ. First note that $\mathcal{A}^{<\xi} = \cup\{\mathcal{A}^\zeta : \zeta < \xi\}$ and $\mathcal{B}^{<\xi} = \cup\{\mathcal{B}^\zeta : \zeta < \xi\}$ are of size at most $|\kappa_\xi|$ and $(\mathcal{A}^{<\xi}, \mathcal{B}^{<\xi})$ is a weakly good pair. Also, the family

$$\{(\mathcal{A}^{<\xi}, \mathcal{B}^{<\xi}), (\mathcal{A}_i \cap M_\xi, \mathcal{B}_i \cap M_\xi); (\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$$

is pairwise disjoint. Hence $\triangle_{|\kappa_\xi|}$ implies that there is a weakly good pair $(\mathcal{A}^\xi, \mathcal{B}^\xi)$ of size at most $|\kappa_\xi|$ which fills $\mathcal{E} \cap M_\xi$ and is pairwise disjoint from $\{(\mathcal{A}_i, \mathcal{B}_i), (\mathcal{C}_j, \mathcal{D}_j) : i < n, j < k\}$ while

$$\mathcal{A}^{<\xi} \cup \bigcup_{i<n} (A_i \cap M_\xi) \subset \mathcal{A}^\xi$$

and

$$\mathcal{B}^{<\xi} \cup \bigcup_{i<n} (B_i \cap M_\xi) \subset \mathcal{B}^\xi.$$

Note that $\triangle_{|\kappa_\xi|}$ was used to find the common extension of $n+1$ weakly good pairs such that this extension is disjoint from $n+k$ given weakly good pairs. Now define $\mathcal{A} = \cup\{\mathcal{A}^\xi : \xi < \zeta\}$ and $\mathcal{B} = \cup\{\mathcal{B}^\xi : \xi < \zeta\}$; $(\mathcal{A}, \mathcal{B})$ is the desired extension. $\square$

This finishes the proof the lemma.

$\square$

Corollary 3.6. *Every T_3 (locally) Lindelöf space is base resolvable. In particular, every T_3 locally countable or locally compact space is base resolvable.*

Proof. Fix a base $\mathbb{B}$ for a T_3 Lindelöf space X and consider the set $\mathbb{P}$ of all weakly good pairs $\mathcal{A}, \mathcal{B} \subseteq \mathbb{B}$ partially ordered by extension. Note that we can apply Zorn's lemma to $\mathbb{P}$ by Observation 3.2; pick a maximal weakly good pair $\mathcal{A}, \mathcal{B}$. Lemma 3.3 implies that a maximal weakly good pair must weakly fill every $\overline{U} \subset V$ pair, hence both $\mathcal{A}$ and $\mathcal{B}$ are bases of X.

Given a T_3 locally Lindelöf space X with a base $\mathbb{B}$ consider it's one-point Lindelöfization $X^* = X \cup \{x^*\}$ with the base $\mathbb{B}^* = \mathbb{B} \cup \{U \subseteq X^* : U \text{ is open in } X^*, x^* \in U\}$. X^* is T_3 Lindelöf hence base resolvable; thus $\mathbb{B}^*$ can be partitioned to two bases which clearly gives a partition of $\mathbb{B}$. $\square$

4. Combinatorics of resolvability

In this section, we will prove a combinatorial lemma which will be our next tool in showing that further classes of space are base resolvable.

Definition 4.1. *Let $\mathcal{A}, \mathcal{B} \subseteq P(X)$. We say that $\mathcal{A}$ **fills** $\mathcal{B}$ iff*

$$U = \cup\{V \in \mathcal{A} : V \subsetneq U\}$$

*for every $U \in \mathcal{B}$. $\mathcal{A}, \mathcal{B}$ is called a **good pair** iff $\mathcal{A}, \mathcal{B}$ are disjoint, $\mathcal{A}$ fills $\mathcal{B}$ and $\mathcal{B}$ fills $\mathcal{A}$. $\mathcal{A}$ is **self-filling** if $\mathcal{A}$ fills $\mathcal{A}$.*

Note that $\mathcal{A} \subseteq P(X)$ generates a topology on X iff $\mathcal{A}$ fills $\{\cap\mathcal{B} : \mathcal{B} \in [\mathcal{A}]^{<\omega}\}$.

Definition 4.2. *A self-filling family $\mathcal{A}$ is **resolvable** iff there is a partition $\mathcal{A}_0, \mathcal{A}_1$ of A such that $\mathcal{A}_i$ fills $\mathcal{A}$ for $i < 2$.*

Lemma 4.3. *Suppose that $\mathbb{B} \subseteq P(X)$ fills itself. Then the following our equivalent:*

(1) *for every $U \in \mathbb{B}$ there is a good pair $\mathbb{B}_0^U, \mathbb{B}_1^U \subseteq \mathbb{B}$ such that $U = \cup\mathbb{B}_0^U = \cup\mathbb{B}_1^U$,*
(2) *$\mathbb{B}$ is resolvable.*

Proof. (2) implies (1) is trivial.

Without loss of generality $\mathbb{B} = \{U_\xi : \xi < \kappa\}$ where $\kappa = w(X)$. We construct $\mathbb{B}_0^\alpha, \mathbb{B}_1^\alpha \subseteq \mathbb{B}$ by induction on $\alpha < \kappa$ such that

(i) $\mathbb{B}_0^\alpha, \mathbb{B}_1^\alpha$ is a good pair,
(ii) $\mathbb{B}_i^\alpha$ fills U_α for $i < 2$.

Note that $\mathbb{B}_i = \cup\{\mathbb{B}_i^\alpha : \alpha < \kappa\}$ will be disjoint bases.

Suppose we constructed $\mathbb{B}_0^\beta, \mathbb{B}_1^\beta$ for $\beta < \alpha$ as above, let $\mathbb{B}_i^{<\alpha} = \cup\{\mathbb{B}_i^\beta : \beta < \alpha\}$. $\mathbb{B}_0^{<\alpha}, \mathbb{B}_1^{<\alpha}$ is still a good pair. Let

$$\xi = \min\{\zeta : \mathbb{B}_0^{<\alpha} \text{ does not fill } U_\zeta\}.$$

Note that $\xi \geq \alpha$.

Let

$$\mathbb{B}_i^\alpha = \mathbb{B}_i^{<\alpha} \cup (\mathbb{B}_i^{U_\xi} \setminus \mathbb{B}_{1-i}^{<\alpha}).$$

It is clear that $\mathbb{B}_i^\alpha$ fills U_α for $i < 2$ and that $\mathbb{B}_0^\alpha, \mathbb{B}_1^\alpha$ are disjoint. Pick $U \in \mathbb{B}_i^\alpha$ and $x \in U$, wlog $U \notin \mathbb{B}_i^{<\alpha}$ so $U \in \mathbb{B}_i^{U_\xi}$. There is a $V \in \mathbb{B}_{1-i}^{U_\xi}$ such that $x \in V \subsetneq U$. If $V \in \mathbb{B}_{1-i}^{U_\xi} \setminus \mathbb{B}_i^{<\alpha}$ then we are done as $V \in \mathbb{B}_{1-i}^\alpha$. Otherwise $V \in \mathbb{B}_i^{<\alpha}$; $\mathbb{B}_i^{<\alpha}$ is filled by $\mathbb{B}_{1-i}^{<\alpha}$ so there is a $W \in \mathbb{B}_{1-i}^{<\alpha}$ such that $x \in W \subsetneq V$. Thus $\mathbb{B}_{1-i}^\alpha$ fills $\mathbb{B}_i^\alpha$. $\square$

The first corollary is a direct application and shows that resolvability is preserved by unions.

Corollary 4.4. *Suppose that $\mathbb{B}_\alpha$ is a resolvable self-filling family for each $\alpha < \kappa$. Then $\cup\{\mathbb{B}_\alpha : \alpha < \kappa\}$ is a resolvable self-filling family as well.*

Corollary 4.5. *Suppose that a self-filling family $\mathbb{B}$ has the property that for every $U \in \mathbb{B}$ there is $\mathcal{U} \in [\mathbb{B} \setminus \{U\}]^{\leq\omega}$ such that $U = \cup\mathcal{U}$. Then $\mathbb{B}$ is resolvable.*

Proof. We shall apply Lemma 4.3: fix a $U \in \mathbb{B}$ and we build the good pair $\mathbb{B}_0^U, \mathbb{B}_1^U \subseteq \mathbb{B}$ covering U by induction of lenght ω. First pick disjoint well founded, countable covers of U denoted by $\mathbb{B}_0^0, \mathbb{B}_1^0$. Then in each step $n \in \omega$ pick countable well founded subfamilies $\mathbb{B}_0^n, \mathbb{B}_1^n$ from $\mathbb{B} \setminus \cup\{\mathbb{B}_i^j : i < 2, j < n\}$ such that they are disjoint and they both fill in a previously chosen member of $\cup\{\mathbb{B}_i^j : i < 2, j < n\}$. By a straightforward bookkeeping we can have a good pair $\mathbb{B}_i^U = \cup\{\mathbb{B}_i^n : n \in \omega\}$ (both covering U) in ω steps. $\square$

Corollary 4.6. *Locally countable or hereditarily Lindelöf spaces are base resolvable without assuming any separation axioms.*

Our next corollary establishes that every reasonable space admits a resolvable base.

Corollary 4.7. *Suppose that $\mathbb{B}$ is a base closed to finite unions in a T_1 topological space. Then $\mathbb{B}$ can be partitioned into two disjoint bases.*

Proof. We shall apply Lemma 4.3 again: fix $U \in \mathbb{B}$ and we construct a good pair covering U. Fix an arbitrary strictly decreasing sequence $\{U_n : n \in \omega\} \subseteq \mathbb{B}$ such that $U_0 \subseteq U$. Let

$$\mathbb{B}_i^U = \{V \in \mathbb{B} \cap \mathcal{P}(U) : \exists k \in \omega : U_{2k+i} \subseteq V \text{ but } U_{2k-1+i} \not\subseteq V\}$$

for $i < 2$. $\mathbb{B}_0^U \cap \mathbb{B}_1^U = \emptyset$ and it is easy to see that the assumption on the base guarantees that $(\mathbb{B}_0^U, \mathbb{B}_1^U)$ is a good pair. □

Corollary 4.8. *The set of all open sets in a T_1 topological space can always be partitioned into two disjoint bases.*

Corollary 4.9. *Under Martin's Axiom every space X of local size $< 2^\omega$ is base resolvable without assuming any separation axioms.*

Proof. We shall apply the good pair lemma: fix $U \in \mathbb{B}$ and we construct a good pair covering U. Note that we can suppose that $|U| = \kappa < 2^\omega$ without loss fo generality. Select $\mathbb{B}^U \in [\mathbb{B}]^\kappa$ which fills itself and $\cup\mathbb{B}_U = U$. Now consider the ccc partial order $\mathbb{P} = Fn(\mathbb{B}_U, 2, \omega)$, i.e. the set of all finite partial functions from $\mathbb{B}_U$ to 2; it is an easy exercise to see that $\mathbb{P}$ forces a partition of $\mathbb{B}_U$ into the desired good pair. □

5. Thinning self filling families

Let $\mathbb{B}$ be a self filling family; note that $\mathbb{B}$ is *redundant* in the sense that $\mathbb{B} \setminus \mathcal{U}$ still fills $\mathbb{B}$ for a finite or more generally, a well founded family $\mathcal{U}$.

Definition 5.1. *We say that $\mathcal{U} \subseteq \mathbb{B}$ is* negligible *iff $\mathbb{B} \setminus \mathcal{U}$ still fills $\mathbb{B}$.*

Our aim in this section is to show that every self filling family $\mathbb{B}$ contains a negligible subfamily of size $|\mathbb{B}|$. Note that a base $\mathbb{B}$ for a space X is resolvable iff it contains a negligible subfamily $\mathcal{U} \subseteq \mathbb{B}$ such that $\mathcal{U}$ is a base of X as well. We will make use of the following definitions:

Definition 5.2. $\mathcal{U} \subseteq \mathcal{P}(X)$ *is* weak increasing *iff there is a well order $\prec$ of $\mathcal{U}$ such that $A \prec B$ implies that $B \setminus A \neq \emptyset$.*

Definition 5.3. *If $\mathbb{B}$ fills itself then let*

$$L(U, \mathbb{B}) = \min\{|\mathcal{V}| : V \subseteq \mathbb{B} \setminus \{U\}, U = \cup\mathcal{V}\}$$

for $U \in \mathbb{B}$.

Observation 5.4. *Suppose that $\mathbb{B}$ fills itself and $\mathcal{U} \subseteq \mathbb{B}$.*

(1) There is weak increasing $\mathcal{U}' \subseteq \mathcal{U}$ such that $\cup\mathcal{U} = \cup\mathcal{U}'$.

(2) *If $\mathcal{U}$ is weak increasing then $\mathcal{U}$ contains no infinite decreasing sequences with respect to inclusion; in particular, $\mathcal{U}$ is negligible.*
(3) *If $\mathbb{B} \setminus \mathcal{U}$ fills $\mathcal{U}$ then $\mathcal{U}$ is negligible.*

Our first proposition establishes the main result for regular $|\mathbb{B}|$.

Proposition 5.5. *Suppose that $\mathbb{B}$ fills itself, and $\kappa = |\mathbb{B}|$ is regular. Then $\mathbb{B}$ contains a negligible family of size κ.*

Proof. We can suppose that $L(U, \mathbb{B}) < \kappa$ for every $U \in \mathbb{B}$; otherwise we can find a weak increasing subfamily of size κ which is negligible by (2) of Observation 5.4. It suffices to define an increasing sequence of disjoint subsets $\mathcal{U}_\xi, \mathcal{V}_\xi \in [\mathbb{B}]^{<\kappa}$ for $\xi < \kappa$ such that $\mathcal{V}_\xi$ fills $\mathcal{U}_\xi$ and $\mathcal{U}_{\xi+1} \setminus \mathcal{U}_\xi \neq \emptyset$; clearly, $\mathcal{U} = \cup\{U_\xi : \xi < \kappa\}$ is a negligible set of size κ in $\mathbb{B}$ by (3) of Observation 5.4. Suppose we have $\mathcal{U}_\xi, \mathcal{V}_\xi \in [\mathbb{B}]^{<\kappa}$ for $\xi < \zeta$ as above for some $\zeta < \kappa$; then $\mathbb{B} \setminus \cup\{\mathcal{U}_\xi, \mathcal{V}_\xi : \xi < \zeta\} \neq \emptyset$ by κ being regular hence we can select $U_\zeta \in \mathbb{B} \setminus \cup\{\mathcal{U}_\xi, \mathcal{V}_\xi : \xi < \zeta\}$ and define

$$\mathcal{U}_\zeta = \{U_\zeta\} \cup \bigcup\{\mathcal{U}_\xi : \xi < \zeta\}.$$

Find $\mathcal{W} \subseteq \mathbb{B} \setminus \{U_\zeta\}$ of size $< \kappa$ such $\cup\mathcal{W} = U_\zeta$; define

$$\mathcal{V}_\zeta = \bigcup\{\mathcal{V}_\xi : \xi < \zeta\} \cup (\mathcal{W} \setminus \mathcal{U}_\zeta).$$

It is easy to show that $\mathcal{V}_\zeta$ fills $\mathcal{U}_\zeta$; see the proof of Lemma 4.3. $\square$

Theorem 5.6. *Suppose that $\mathbb{B}$ fills itself. Then $\mathbb{B}$ contains a negligible family of size $|\mathbb{B}|$.*

Proof. We can suppose that $\mu = cf(k) < \kappa = |\mathbb{B}|$ and that every weak increasing sequence in $\mathbb{B}$ is of size less than κ. Fix a cofinal strictly increasing sequence of regular cardinals κ_ξ in κ such that $\mu < \kappa_0$ and define

$$\mathbb{B}_\xi = \{U \in \mathbb{B} : L(U, \mathbb{B}) \leq \kappa_\xi\}.$$

If there is a ξ such that every weak increasing sequence is of size less than κ_ξ then $\mathbb{B} = \mathbb{B}_\xi$; define a set mapping $F : \mathbb{B} \to [\mathbb{B}]^{<\kappa_\xi^+}$ such that $U = \cup F(U)$ where $F(U) \subseteq \mathbb{B} \setminus \{U\}$. As $\kappa_\xi^+ < \kappa$ we can apply the Hajnal's Set Mapping theorem (see Theorem 19.2 in [1]): there is an F-free set $\mathcal{U}$ of size κ in $\mathbb{B}$, i.e. $F(U) \cap \mathcal{U} = \emptyset$ for all $U \in \mathcal{U}$; observe that $\mathcal{U}$ is negligible as $\cup\{F(U) : U \in \mathcal{U}\} \subseteq \mathbb{B} \setminus \mathcal{U}$ fills $\mathcal{U}$.

From now on we suppose that there are arbitrarily large weak increasing sequences in $\mathbb{B}$. It suffices to define increasing sequnces $\mathcal{U}_\xi, \mathcal{V}_\xi \in [\mathbb{B}]^{<\kappa}$ for $\xi < \mu$ such that

(i) $\mathcal{U}_\xi, \mathcal{V}_\xi$ are disjoint and $\kappa_\xi \leq |U_\xi|$,
(ii) $\mathcal{V}_\xi$ fills $\mathcal{U}_\xi$.

Indeed, the union $\cup\{\mathcal{U}_\xi : \xi < \mu\}$ is negligible in $\mathbb{B}$ of size κ. Suppose we defined $\mathcal{U}_\xi, \mathcal{V}_\xi \in [\mathbb{B}]^{<\kappa}$ for $\xi < \zeta$; let

$$\lambda = \left(|\bigcup\{\mathcal{U}_\xi \cup \mathcal{V}_\xi : \xi < \zeta\}| \cdot \kappa_\zeta\right)^+.$$

Note that $\lambda < \kappa$ thus we can pick a weak increasing $\mathcal{W} \in [\mathbb{B}]^\lambda$; without loss of generality, we can suppose that $\mathcal{W}$ is disjoint from $\bigcup\{\mathcal{U}_\xi \cup \mathcal{V}_\xi : \xi < \zeta\}$. Note that

$$\mathcal{W} = \cup\{\mathbb{B}_\delta \cap \mathcal{W} : \delta < \mu\}$$

and that $\mu < cf(\lambda) = \lambda$, hence there is $\delta < \mu$ such that $\mathcal{W}' = \mathcal{W} \cap \mathbb{B}_\delta$ has size λ. Define $U_\zeta = \mathcal{W}' \cup \bigcup\{\mathcal{U}_\xi : \xi < \zeta\}$.

Now, for every $U \in \mathcal{W}'$ select $F(U) \in [\mathbb{B} \setminus \{U\}]^{\kappa_\delta}$ such that $U = \cup F(U)$. Define

$$\mathcal{V}_\zeta = \bigcup\{\mathcal{V}_\xi : \xi < \zeta\} \cup \bigcup\{F(U) : U \in \mathcal{W}'\} \setminus \mathcal{U}_\zeta.$$

Note that $\kappa_\zeta \le |\mathcal{U}_\zeta| = \lambda$ and $|\mathcal{V}_\zeta| \le \lambda \cdot \kappa_\delta < \kappa$. It is only left to prove that $\mathcal{V}_\zeta$ fills $\mathcal{U}_\zeta$; in fact, it suffices to show that $\mathcal{V}_\zeta$ fills $\mathcal{W}'$. Suppose that $\prec$ is the well ordering witnessing that $\mathcal{W}'$ is weak increasing and suppose that there is a $U \in \mathcal{W}'$ which is not filled by $\mathcal{V}_\xi$; we can suppose that U is $\prec$-minimal. Fix an $x \in U$ witnessing that $\mathcal{V}_\zeta$ does not fill U. Pick $V \in F(U)$ such that $x \in V \subset U$; if $V \in \mathcal{W}'$ then $V \prec U$, thus V is filled by $\mathcal{V}_\zeta$ by the minimality of U. This contradicts the choice of x, hence $V \notin \mathcal{W}'$. Thus $V \in \mathcal{V}_\zeta \cup \bigcup\{\mathcal{U}_\xi : \xi < \zeta\}$ which is filled by $\mathcal{V}_\zeta$ by the inductional hypothesis; this again contradicts the choice of x, which finishes the proof.

□

6. Irresolvable self filling families

The aim of this section is to construct an irresolvable self filling family and deduce the existence of a non base resolvable T_0 topological space.

Given a partial order $(\mathbb{P}, \le)$ and $p, q \in \mathbb{P}$ let

$$[p, q] = \{r \in \mathbb{P} : p \le r \le q\}.$$

The key to our construction is the following definition:

Definition 6.1. *We say that a poset $\mathbb{P}$ without maximal elements satisfies*

$$\mathbb{P} \to (I_\omega)^1_2$$

iff for every partition $\mathbb{P} = \cup_{i<2} D_i$ there is $i < 2$ and strictly increasing $\{p_n : n \in \omega\} \subseteq D_i$ such that $[p_0, p_n] \subseteq D_i$ for every $n \in \omega$. The negation is denoted by $\mathbb{P} \nrightarrow (I_\omega)^1_2$.

The above definition is motivated by the following:

Observation 6.2. *For any irresolvable self filling family* $\mathbb{B} \subseteq \mathcal{P}(X)$ *the partial order* $\mathbb{P} = (\mathbb{B}, \supseteq)$ *satisfies* $\mathbb{P} \to (I_\omega)^1_2$.

Proof. Consider a partition of $\mathbb{P} = (\mathbb{B}, \supseteq)$ into sets D_0, D_1; as $\mathbb{B}$ is irresolvable, there is $i < 2$, $x \in X$ and $U \in D_i$ such $V \in D_i$ for every $V \in \mathbb{B}$ with $x \in V \subseteq U$. Pick a strictly decreasing sequence $\{V_n : n \in \omega\} \subseteq \mathbb{B}$ such that $x \in V_n \subseteq U$ for every $n \in \omega$; clearly, $[V_0, V_n] \subseteq D_i$ for every $n \in \omega$. □

Our next aim is to find a partial order $\mathbb{P}$ first with $\mathbb{P} \to (I_\omega)^1_2$; note that trees or $Fn(\kappa, 2)$ cannot satisfy $\mathbb{P} \to (I_\omega)^1_2$. Moreover:

Proposition 6.3. *For every countable poset* $\mathbb{P}$ *without maximal elements we have* $\mathbb{P} \not\to (I_\omega)^1_2$.

Proof. Define a rank function by induction on a well founded subset of $U_p = \{q \in \mathbb{P} : p \leq q\}$ (for each $p \in \mathbb{P}$) as follows:

$$\begin{aligned} & rk_p(p) = 0, \\ (1) \qquad & rk_p(t) = \sup\{rk_p(s) + 1 : s \in U_p, s < t\} \\ & \text{ if } rk_p(s) \text{ is defined for all } s \in U_p, s < t. \end{aligned}$$

We will refer to rk_p as the p-rank. Also, let $\{I_n : n \in \omega\}$ enumerate all intervals $I = [p', p]$ in $\mathbb{P}$ which contain an infinite chain and let $\mathbb{P} = \{p_n : n \in \omega\}$ denote a 1-1 enumeration.

We inductively construct disjoint $P_{0,n}, P_{0,n} \subseteq \mathbb{P}$ for $n \in \omega$ such that

(i) $P_{i,n}$ is a finite union of antichains for $i < 2$,
(ii) $p_n \in \cup_{i<2} P_{i,n}$ and there is $q \in P_{i,n}$ such that $p_n \leq q$ for each $i < 2$,
(iii) $I_n \cap P_{i,n} \neq \emptyset$ for $i < 2$,
(iv) for every strictly increasing chain $C = \{c_n : n \in \omega\} \subseteq P$ containing only finite intervals such that $p_n \in C$ we have
$$\bigcup_{n \in \omega} [c_0, c_n] \cap P_{i,n} \neq \emptyset$$
for each $i < 2$.

It is easy to see that such a construction yields a partition $P_i = \cup\{P_{i,n} : n \in \omega\}$ witnessing $\mathbb{P} \not\to (I_\omega)^1_2$.

Suppose we constructed $P_{i,n-1}$ satisfying the above conditions; note that finitely many elements can be added to both $P_{0,n-1}$ and $P_{1,n-1}$ without violating (i), thus (ii) and (iii) are easy to satisfy; note that $I_n \setminus \cup_{i<2} P_{i,n-1}$ is infinite as I_n contains an infinite chain.

It suffices to show the following to finish our proof:

Claim 6.4. *Fix $p \in \mathbb{P}$ and $A \subseteq \mathbb{P}$ which is covered by finitely many antichains. Then there is an antichain $B \subseteq \mathbb{P} \setminus A$ such that for every increasing chain $C = \{c_n : n \in \omega\} \subseteq P$ containing only finite intervals with $p \in C$ we have*

$$\bigcup_{n\in\omega} [c_0, c_n] \cap B \neq \emptyset.$$

Proof. Let $Q = \{q \in \mathbb{P} \setminus A : p < q, q \text{ has a } p\text{-rank}\}$ and define q^+ to be the element minimizing rk_p on $[p, q] \setminus A$ for $q \in Q$; let $B = \{q^+ : q \in Q\}$. First note that B is an antichain. Now fix a strictly increasing chain $C = \{c_n : n \in \omega\} \subseteq P$ containing only finite intervals with $p \in C$; note that there is $q \in C \setminus A$ such that $p < q$; also, $q \in Q$ by $[p, q]$ being finite. Thus $q^+ \in \bigcup_{n\in\omega}[c_0, c_n] \cap B$. □

To finish the proof of the theorem, apply the claim twice: to $A = \cup P_{i,n-1}$ and define $P_{0,n} = P_{0,n-1} \cup B$ and next to $A = P_{0,n} \cup P_{1,n-1}$ similarly.

□

We will call a countable strictly increasing sequence of elements of $\mathbb{P}$ a *branch*; we say that a branch $x = (x_n)_{n\in\omega}$ goes above an element $p \in \mathbb{P}$ iff $p \leq x_n$ for some $n \in \omega$.

Theorem 6.5. *There is a partial order $\mathbb{P}$ of size ω_1 without minimal element such that $\mathbb{P} \to (I_\omega)^1_2$. Furthermore,*

(1) every $p \in \mathbb{P}$ has finitely many predecessors,
(2) if $p \not\leq q$ in $\mathbb{P}$ then there is a branch x in $\mathbb{P}$ which goes above q but not p.

Proof. Let us fix a function $c : [\omega_1]^2 \to \omega$ such that $c(\cdot, \zeta) : \zeta \to \omega$ is 1-1 for every $\zeta \in \omega_1$. It is easy to see that such functions satisfy the following:

Fact 6.6. *If $c(\cdot, \zeta) : \zeta \to \omega$ is 1-1 for every $\zeta \in \omega_1$ for some $c : [\omega_1]^2 \to \omega$ then for every uncountable, disjoint family $\mathcal{A} \subseteq [\omega_1]^{<\omega}$ and $N \in \omega$ there are $a < b$[1] in $\mathcal{A}$ such that $c(\xi, \zeta) > N$ for every $\xi \in a, \zeta \in b$.*

Also, fix an enumeration $\{(y_\alpha, w_\alpha) : \omega \leq \alpha < \omega_1\}$ of all pairs of elements of $\omega_1 \times \omega$ such that $y_\alpha, w_\alpha \in \alpha \times \omega$.

We define $\mathbb{P} = (\omega_1 \times \omega, \leq)$ as follows: by induction on $\alpha \in L_1$ (where L_1 stands for the limit ordinals in ω_1) we construct a poset $\mathbb{P}_\alpha = ((\alpha + \omega) \times \omega, \leq_\alpha)$ with properties:

(i) $\mathbb{P}_\alpha$ has no maximal elements and every $p \in \mathbb{P}_\alpha$ has finitely many predecessors,

[1] $a < b$ iff $\xi < \zeta$ for all $\xi \in a, \zeta \in b$

(ii) $\leq_\alpha \restriction \beta = \leq_\beta$ for all $\beta < \alpha$,
(iii) $(\xi, n) <_\alpha (\zeta, m)$ implies that $\xi < \zeta$ and $\max(n, c(\xi, \zeta)) < m$,
(iv) there is $t_\alpha \in \mathbb{P}_\alpha$ such that $t <_\alpha t_\alpha$ implies that $t \leq_\alpha y_\alpha$ or $t \leq_\alpha w_\alpha$,
(v) if $p \not\leq q$ in $\mathbb{P}_\alpha$ then there is a branch x in $\mathbb{P}_\alpha$ which goes above q but not p.

We only sketch the inductive step: suppose that $y_\alpha = (\xi, n)$ and $w_\alpha = (\zeta, m)$. Now find $k \in \omega$ larger than n, m and $c(\nu, \alpha)$ for every $\nu \in \omega_1$ such that there is $s \leq y_\alpha$ or $s \leq w_\alpha$ with $s = (\nu, l)$ for some $l \in \omega$; this can be done by (i). Now define $t_\alpha = (\alpha, k)$ and $\leq_\alpha$ so that $t <_\alpha t_\alpha$ implies that $t \leq_\alpha y_\alpha$ or $t \leq_\alpha w_\alpha$. Extend $\leq_\alpha$ further so that $\mathbb{P}_\alpha$ has no maximal elements and satisfies (v); this can be done by "placing" copies of $2^{<\omega}$ above elements of $\mathbb{P}_\alpha \setminus \cup\{\mathbb{P}_\beta : \beta < \alpha\}$.

Let us define $\mathbb{P} = \cup\{\mathbb{P}_\alpha : \alpha < \omega_1\}$ and $\leq = \cup\{\leq_\alpha : \alpha < \omega_1\}$; observe that $(\mathbb{P}, \leq)$ is well defined and trivially satisfies (1) and (2). In what follows, π_{ω_1} and π_ω denotes the projections from $\omega_1 \times \omega$ to the first and second coordinates respectively.

Claim 6.7. $\mathbb{P} \to (I_\omega)^1_2$.

Proof. Suppose that $\mathbb{P} = D_0 \cup D_1$; we can assume that D_0 and D_1 are both cofinal. Now suppose that there is no increasing chain with each interval in one of the D_i and reach a contradiction as follows. We will say that an interval $[s, t]$ in $\mathbb{P}$ is *i-maximal* for some $i < 2$ if $[s, t] \subseteq D_i$ but $[s, t'] \nsubseteq D_i$ for every $t < t'$. Observe that for every $s \in D_i$ there is $t \in D_i$ such that $[s, t]$ is i-maximal; otherwise we can construct an increasing chain starting from s with each interval in D_i. Now construct increasing 4-element sequences $R_\alpha = \{\tilde{x}_\alpha \leq \tilde{y}_\alpha \leq \tilde{z}_\alpha \leq \tilde{w}_\alpha\} \subseteq \mathbb{P}$ for $\alpha < \omega_1$ such that

(a) $[\tilde{x}_\alpha, \tilde{y}_\alpha] \subseteq \mathbb{P}_0$ is a 0-max interval,
(b) $[\tilde{z}_\alpha, \tilde{w}_\alpha] \subseteq \mathbb{P}_1$ is a 1-max interval,
(c) $\pi_{\omega_1} R_\alpha < \pi_{\omega_1} R_\beta$ if $\alpha < \beta$.

By passing to a subsequence of $\{R_\alpha : \alpha < \omega_1\}$ we can suppose that $\pi_\omega R_\alpha$ is independent of α; let $N = \max \pi_\omega R_\alpha$. Find $\alpha < \beta$, using Fact 6.6, such that

$$c \restriction [\pi_{\omega_1} R_\alpha, \pi_{\omega_1} R_\beta] > N.$$

Observe that $\tilde{x}_\alpha \not\leq \tilde{w}_\beta$ by $\pi_\omega w_\beta = N < c(\pi_{\omega_1} \tilde{x}_\alpha, \pi_{\omega_1} \tilde{w}_\beta)$ and (iii). Now find $\gamma < \omega_1$ such that $(y_\gamma, w_\gamma) = (\tilde{y}_\alpha, \tilde{w}_\beta)$ and consider $t_\gamma \in \mathbb{P}_\gamma$. We claim that t_γ is a minimal extension of $\tilde{y}_\alpha$ and $\tilde{w}_\beta$ in the following sense:

(1) $[\tilde{x}_\alpha, t_\gamma] = [\tilde{x}_\alpha, \tilde{y}_\alpha] \cup \{t_\gamma\}$,
(2) $[\tilde{z}_\beta, t_\gamma] = [\tilde{z}_\beta, \tilde{w}_\beta] \cup \{t_\gamma\}$.

Indeed, if $\tilde{x}_\alpha \leq t' < t_\gamma$ then $t' \leq \tilde{y}_\alpha$ or $t' \leq \tilde{w}_\beta$; $\tilde{x}_\alpha \not\leq \tilde{w}_\beta$ implies that $t' \not\leq w_\beta$ hence $t' \in [\tilde{x}_\alpha, \tilde{y}_\alpha]$. Similarly, if $\tilde{z}_\beta \leq t' < t_\gamma$ then $t' \leq \tilde{y}_\alpha$ or $t' \leq \tilde{w}_\beta$; however, $t' \not\leq \tilde{y}_\alpha$ by $\pi_\omega t' > \pi_\omega \tilde{y}_\alpha$ so $t' \in [\tilde{z}_\beta, \tilde{w}_\beta]$.

Note that $t \in \mathbb{P}_0$ contradicts the 0-maximality of $[x_\alpha, y_\alpha]$ and (1) while $t \in \mathbb{P}_1$ contradicts the 1-maximality of $[z_\beta, w_\beta]$ and (2). □

The above claim finishes the proof. □

Using the previous theorem, we construct an irresolvable self-filling family; we can actually realize this family as a system of open sets in a first countable compact space. We remark that this space is base resolvable, as every compact space, by Corollary 3.6.

Theorem 6.8. *There is a first countable Corson compact space (X, τ) and $\mathcal{U} \subseteq \tau$ such that $\mathcal{U}$ fills $\{\cap \mathcal{V} : \mathcal{V} \in [\mathcal{U}]^{<\omega}\}$ and $\mathcal{U}$ is irresolvable.*

Proof. Consider the poset $\mathbb{P}$ in Theorem 6.5. We say that $x \in \mathbb{P}^\omega$ is a *maximal chain* iff $(x(n))$ is a branch in $\mathbb{P}$, $x(0)$ is a minimal element of $\mathbb{P}$ and $[x(n), x(n+1)] = \{x(n), x(n+1)\}$. Note that there are no increasing chains of order type $\omega + 1$ in $\mathbb{P}$. Furthermore

Observation 6.9. *(1) Any branch $y \in \mathbb{P}^\omega$ can be extended to a maximal chain $\bar{y} \in \mathbb{P}^\omega$,*
(2) there is a $n_0 \in \omega$ such that $\cup_{n_0 \leq n}[\bar{y}(n_0), \bar{y}(n)] \subseteq \cup_{n \in \omega}[y(0), y(n)]$.

Note that (2) implies that if $y \in \mathbb{P}^\omega$ had homogeneous intervals with respect to some coloring of $\mathbb{P}$ then the an end-segment of the maximal extension $\bar{y}$ had the same property.

Now consider $X = \{x \in \mathbb{P}^\omega : x \text{ is a maximal chain}\}$ as a subspace of $2^{\mathbb{P}}$; here $2^{\mathbb{P}}$ is equipped with the usual product topology.

Claim 6.10. *X is a compact subspace of $\Sigma(2^{\mathbb{P}}) = \Sigma(2^{\omega_1})$.*

Proof. $\Sigma(2^{\mathbb{P}}) = \Sigma(2^{\omega_1})$ follows from $|\mathbb{P}| = \omega_1$ and clearly every chain is countable so $X \subseteq \Sigma(2^{\mathbb{P}})$.

We prove that X is a closed subset of $2^{\mathbb{P}}$. Suppose that $y \in 2^{\mathbb{P}} \setminus X$; clearly, if y is not a chain then y can be separated from X. Suppose that y is a chain, then either $y(0)$ is not minimal in $\mathbb{P}_c$ or there is $n \in \omega$ such that $(y(n), y(n+1)) \neq \emptyset$. In the first case let $\varepsilon \in Fn(\mathbb{P}, 2)$ be defined to be 1 on $y(0)$ and $\varepsilon(p) = 0$ for $p \geq y(0)$, $p \in \mathbb{P}_c$ (note that each element in $\mathbb{P}$ has only finitely many predecessors); then $y \in [\varepsilon]$ and $[\varepsilon] \cap X = \emptyset$. In the second case let $\varepsilon \in Fn(\mathbb{P}, 2)$ such that $1 = \varepsilon(y(n)) = \varepsilon(y(n+1))$ and $\varepsilon \restriction (y(n), y(n+1)) = 0$; then $y \in [\varepsilon]$ and $[\varepsilon] \cap X = \emptyset$. □

Claim 6.11. *$\{x\} = \cap\{[\chi_{x(n)}] \cap X : n \in \omega\}$ for every $x \in X$. Hence every point in X has countable Ψ-character; in particular, X is first countable.*

Proof. Suppose that $y \in \cap\{[\chi_{x(n)}] \cap X : n \in \omega\}$, that is $\{x(n) : n \in \omega\} \subset \{y(n) : n \in \omega\}$. We prove that $x(n) = y(n)$ by induction on $n \in \omega$. $y(0) = x(0)$ as they are both maximal elements in $\mathbb{P}_c$. Suppose that $x(i) = y(i)$ for $i < n$; if $x(n) \neq y(n)$ then $x(n) = y(k)$ for some $n < k$, thus $y(n) \in (x(n), x(n-1)) = (y(k), y(n-1))$ which contradicts the maximality of the chain x. $\square$

Now define

$$V_p = \{x \in X : \exists n \in \omega : x(n) \geq p\} \text{ for } p \in \mathbb{P},$$

and note that V_p is open since $V_p = \cup\{[\chi_{\{q\}}] \cap X : q \leq p\}$. We define $\mathcal{U} = \{V_p : p \in \mathbb{P}_c\}$.

Claim 6.12. *$\mathcal{U}$ is an irresolvable self filling family.*

Proof. Note that $p < q$ in $\mathbb{P}_c$ if and only if $V_q \subsetneq V_p$; the nontrivial direction is implied by $\mathbb{P}$ being separative. Now it is easy to see that $\mathcal{U}$ fills itself.

We show that $\mathcal{U}$ is irresolvable; suppose that we partitioned $\mathcal{U}$, equivalently $\mathbb{P}$ into two parts $\mathbb{P}_0, \mathbb{P}_1$. Applying $\mathbb{P} \to (I_\omega)^1_2$ we that there is a chain $y \in \mathbb{P}^\omega$ and $i < 2$ such that $[y(0), y(n)] \subseteq \mathbb{P}_i$ for every $n \in \omega$. By our previous Observation 6.9 there is $\bar{y} \in X$ such that $[\bar{y}(n_0), \bar{y}(n)] \subseteq D_i$ for some $n_0 \in \omega$ and every $n \geq n_0$. We claim that there is no $V \in \{V_p : p \in \mathbb{P}_{1-i}\}$ such that $\bar{y} \in V \subseteq V_{\bar{y}(n_0)}$. Indeed, if $\bar{y} \in V_p \subseteq V_{\bar{y}(n_0)}$ for some $p \in \mathbb{P}$ then $\bar{y}(n_0) \leq p$ and there is $n \in \omega \setminus n_0$ such that $\bar{y}(n) \leq p$; that is $p \in [\bar{y}(n), \bar{y}(n_0)] \subseteq \mathbb{P}_i$. $\square$

The last claim finishes the proof of the theorem. $\square$

Let us finish this section with the following:

Corollary 6.13. *There is a non base resolvable, T_0 topological space.*

Proof. There is an irresolvable self filling family $\mathcal{U} \subseteq \mathcal{P}(X)$ (on some set X) such that $\mathcal{U}$ fills $\{\cap\mathcal{V} : \mathcal{V} \in [\mathcal{U}]^{<\omega}\}$ by Theorem 6.8. Define a relation $\sim$ on X by $x \sim y$ iff $\{U \in \mathcal{U} : x \in U\} = \{U \in \mathcal{U} : y \in U\}$; clearly, $\sim$ is an equivalence relation on X. Let $[x]$ stand for the $\sim$-class of $x \in X$; let $[U] = \{[x] : x \in U\}$ and note that $[\mathbb{B}] = \{[U] : U \in \mathcal{U}\}$ is a base for a T_0 topology on $[X]$. It is easy to see that $[\mathbb{B}]$ is an irresolvable base. $\square$

7. A 0-dimensional, Hausdorff space with an irresolvable base

In this section, we significantly strengthen Corollary 6.13 by showing

Theorem 7.1. *It is consistent that there is a 0-dimensional T_2 space which has an irresolvable base.*

Proof. For $\langle \alpha, n\rangle, \langle \beta, m\rangle \in \omega_1 \times \omega$ write $\langle \alpha, n\rangle \lhd \langle \beta, m\rangle \in \omega_1 \times \omega$ iff $\langle \alpha, n\rangle = \langle \beta, m\rangle$ or ($\alpha < \beta$ and $n < m$).

Definition 7.2. *If $\mathcal{A} = \langle \omega_1 \times \omega, \preceq\rangle$ is a poset with $\preceq \subset \lhd$, and for each $\alpha \in L_1$ we have a set T_α such that*
(C) $T_\alpha \subset \alpha \times \omega$ and $\langle T_\alpha, \preceq\rangle$ is an everywhere ω-branching tree,
then we say that the pair $\langle \mathcal{A}, \langle T_\alpha : \alpha \in L_1\rangle\rangle$ is a candidate.

Denote $T_\alpha(n)$ the n^{th} level of the tree $\langle T_\alpha, \preceq\rangle$.

Definition 7.3. *Fix a candidate $\mathbb{A} = \langle \mathcal{A}, \langle T_\alpha : \alpha \in L_1\rangle\rangle$. We will define a topological space $X(\mathbb{A})$ as follows.*

For $\alpha \in L_1$ let $B(T_\alpha)$ be the collection of the cofinal branches of T_α, and let

$$\mathcal{B}(\mathbb{A}) = \bigcup\{\mathcal{B}(T_\alpha) : \alpha \in L_1\}.$$

The underlying set of the space $X(\mathbb{A})$ is $\mathcal{B}(\mathbb{A})$.
For $x \in \omega_1 \times \omega$ let $U(x) = \{y \in \omega_1 \times \omega : x \preceq y\}$ and

$$V(x) = \{b \in \mathcal{B}(\mathbb{A}) : \exists y \in b\ (x \preceq y)\}.$$

Clearly $V(x) = \{b \in \mathcal{B}(\mathbb{A}) : b \subseteq^* U(x)\}$.
We declare that the family

$$\mathcal{V} = \{V(x) : x \in \omega_1 \times \omega\}$$

is the base of $X(\mathbb{A})$.

Lemma 7.4. *$\mathcal{V}$ is a base, and so $X(\mathbb{A})$ is a topological space.*

Proof. Assume that $b \in V(x) \cap V(y)$. Then there is $z \in b$ such that $x \preceq z$ and $y \preceq z$. Then $b \in V(z) \subset V(x) \cap V(y)$. □

For $x, y \in \omega_1 \times \omega$ with $x \preceq y$ let

$$[x, y] = \{t \in \omega_1 \times \omega : x \preceq t \preceq y\}.$$

Definition 7.5. *We say that a candidate $\mathbb{A} = \langle \mathcal{A}, \langle T_\alpha : \alpha \in L_1\rangle\rangle$ is* good *iff*
(G1) $V(u) \subset V(v)$ iff $u \preceq v$.
(G2) $\forall \alpha \in L_1\ \forall \zeta < \alpha\ (T_\alpha \setminus (\zeta \times \omega)) \neq \emptyset$,
(G3) (a) $\forall \alpha \in L_1\ (\forall x, y \in T_\alpha)\ U(x) \cap U(y) \neq \emptyset$ iff x and y are $\preceq$-comparable.
(b) for each $\{\alpha, \beta\} \in [L_1]^2$ there is $f(\alpha, \beta) \in \omega$ such that

$$\forall x \in T_\alpha(f(\alpha, \beta))\ \forall y \in T_\beta(f(\alpha, \beta))\ U(x) \cap U(y) = \emptyset$$

(G4) *For each $x \in \omega_1 \times \omega$ and $\alpha \in L_1$ there is $g(x,\alpha) \in \omega$ such that for each $y \in T_\alpha(g(x,\alpha))$*
$$U(y) \subset U(x) \text{ or } U(y) \cap U(x) = \emptyset.$$

(G5) *If for all $\alpha \in L_1$ and $\zeta < \alpha$ we choose a four element $\prec$-increasing sequence*
$$\left\langle x^\alpha_\zeta, y^\alpha_\zeta, z^\alpha_\zeta, w^\alpha_\zeta \right\rangle \subset T_\alpha \setminus (\zeta \times \omega)$$
then there are $\{\alpha,\beta\} \in \left[L_1\right]^2$, $\zeta < \alpha$, $\xi < \beta$, and $t \in T_\alpha \cap T_\beta$ such that
(i) $y^\alpha_\zeta \prec t$ and $[x^\alpha_\zeta, t] = [x^\alpha_\zeta, y^\alpha_\zeta] \cup \{t\}$,
(ii) $w^\beta_\xi \prec t$ and $[z^\beta_\xi, t] = [z^\beta_\xi, w^\beta_\xi] \cup \{t\}$.

Lemma 7.6. *If $\mathbb{A}$ is a good candidate, then $X(\mathbb{A})$ is a crowded 0-dimensional T_2 space such that the base $\{V(x) : x \in \omega_1 \times \omega\}$ is irresolvable.*

Proof.

Claim 7.7. *$X(\mathbb{A})$ is crowded*

Indeed, assume that $b \in B(T_\alpha)$ and $V(x)$ is an open neighbourhood of b. Then there is $y \in b$ with $x \preceq y$ and so $b \in V(y) \subset V(x)$. Thus $V(x) \supset V(y) \supset \{b' \in B(T_\alpha) : y \in b'\}$, and so $V(x)$ has 2^ω many elements. So b is not isolated.

Claim 7.8. *$X(\mathbb{A})$ is T_2.*

Indeed, let $b \in B(T_\alpha)$ and $c \in B(T_\beta)$.

If $\alpha = \beta$ then pick n such that x, the n^{th} element of b, and y, the n^{th} element of c, are different. Then $b \in V(x)$, $c \in V(y)$ and $V(x) \cap V(y) = \emptyset$ by (G3)(a).

If $\alpha \neq \beta$ then $n = f(\alpha,\beta)$ (see G3)(b)), and let x be the the n^{th} element of b, and let y be the n^{th} element of c. Then $b \in V(x)$, $c \in V(y)$ and $V(x) \cap V(y) = \emptyset$ by (G3)(b).

Claim 7.9. *$X(\mathbb{A})$ is 0-dimensional.*

Indeed, assume that $x \in \omega_1 \times \omega$, $b \in \mathcal{B}(T_\alpha)$ and $b \notin V(x)$. Let $y \in b \cap T_\alpha(g(\alpha,x))$. Then $y \notin U(x)$ because $b \notin V(x)$, so $V(x) \cap v(y) = \emptyset$ by (G5).

Claim 7.10. *The base $\{V(x) : x \in \omega_1 \times \omega\}$ is irresolvable.*

Assume on the contrary that there is a partition (K_0, K_1) of $\omega_1 \times \omega$ such that both $\mathcal{V}_0 = \{V(x) : x \in K_0\}$ and $\mathcal{V}_1 = \{V(x) : x \in K_1\}$ are bases.

Assume that $\alpha \in L_1$, $x, y \in T_\alpha$ with $x \preceq y$ and $i \in 2$. We say that interval $[x,y]$ is *i-maximal in T_α* iff

(i) $[x, y] \subset K_i$, but $[x, z] \not\subset K_i$ for any $y \prec z \in T_\alpha$.

Subclaim 7.10.1. *If $\alpha \in L_1$ and $x \in T_\alpha \cap K_i$, then there is $x \preceq y \in T_\alpha$ such that the interval $[x, y]$ is K_i-maximal.*

Proof of the Claim. Assume on the contrary that there is no such y. Then we can construct a strictly increasing sequence $\langle x, y_0, y_1, \ldots \rangle$ in T_α such that $[x, y_n] \subset K_i$ for all $n < \omega$.

Then $b = \{y \in T_\alpha : \exists n \in \omega \; y \preceq y_n\} \in \mathcal{B}(T_\alpha)$.

Since $b \in V(x)$, and we assumed that $\{V(z) : z \in K_{1-i}\}$ is a base, there is $z \in K_1$ with $b \in V(z) \subset V(x)$. Then $x \preceq z$ by (G1). Moreover, there is $y \in b$ with $z \prec y$ because $b \in V(z)$. Thus $z \in [x, y] \cap K_{1-i}$, so $[x, y] \not\subset K_i$. Contradiction, the subclaim is proved. $\square$

Using the subclaim for all $\alpha \in L_1$ and $\forall \zeta < \alpha$ we will construct a four element $\prec$-increasing sequence

$$\left\langle x^\alpha_\zeta, y^\alpha_\zeta, z^\alpha_\zeta, w^\alpha_\zeta \right\rangle \subset T_\alpha \setminus (\zeta \times \omega)$$

as follows.

First, using (G2) pick $s^\alpha_\zeta \in T_\alpha \setminus (\zeta \times \omega)$.

If $K_0 \cap U(s^\alpha_\zeta) \cap T_\alpha = \emptyset$, then let $x^\alpha_\zeta = y^\alpha_\zeta = s^\alpha_\zeta$.

Otherwise pick

$$x^\alpha_\zeta \in K_0 \cap U(s^\alpha_\zeta) \cap T_\alpha,$$

and then, using the Subclaim above, pick

$$y^\alpha_\zeta \in U(x^\alpha_\zeta) \cap T_\alpha$$

such that

$$[x^\alpha_\zeta, y^\alpha_\zeta] \text{ is 0-maximal in } T_\alpha. \tag{2}$$

If $K_1 \cap U(y^\alpha_\zeta) \cap T_\alpha = \emptyset$, then let $z^\alpha_\zeta = w^\alpha_\zeta = y^\alpha_\zeta$.

Otherwise pick

$$z^\alpha_\zeta \in K_1 \cap U(y^\alpha_\zeta) \cap T_\alpha,$$

and then, using the Subclaim above, pick

$$w^\alpha_\zeta \in U(z^\alpha_\zeta) \cap T_\alpha \tag{3}$$

such that

$$[z^\alpha_\zeta, w^\alpha_\zeta] \text{ is 1-maximal in } T_\alpha. \tag{4}$$

By (G5), there are $\{\alpha, \beta\} \in \left[L_1\right]^2$, $\zeta < \alpha$, $\xi < \beta$, and $t \in T_\alpha \cap T_\beta$ such that

(i) $y^\alpha_\zeta \prec t$ and $[x^\alpha_\zeta, t] = [x^\alpha_\zeta, y^\alpha_\zeta] \cup \{t\}$,
(ii) $w^\beta_\xi \prec t$ and $[z^\beta_\xi, t] = [z^\beta_\xi, w^\beta_\xi] \cup \{t\}$.

Assume first that $t \in K_0$. Then $t \in K_0 \cap U(s^\alpha_\zeta) \cap T_\alpha$, so

$$[x^\alpha_\zeta, y^\alpha_\zeta] \text{ is 0-maximal in } T_\alpha. \tag{5}$$

But $t \in K_0$ and $[x^\alpha_\zeta, t] = [x^\alpha_\zeta, y^\alpha_\zeta] \cup \{t\}$, so $[x^\alpha_\zeta, t] \subset K_0$, i.e. $[x^\alpha_\zeta, y^\alpha_\zeta]$ was not 0-maximal in T_α. Contradiction. If $t \in K_1$, then a similar argument works using the interval $[z^\beta_\xi, w^\beta_\xi]$ and K_1.

So in both cases the obtained contradiction, so the base $\{V(x) : x \in \omega_1 \times \omega\}$ should be irresolvable, which proves the lemma. □

Next we show that some c.c.c. forcing introduces a good candidate.

Define the poset $\mathcal{P} = \langle P, \leq \rangle$ as follows. The underlying set consists of 6-tuples

$$\langle A, \preceq, I, \{T_\alpha : \alpha \in I\}, f, g \rangle , \tag{6}$$

where

(P1) $A \in \left[\omega_1 \times \omega\right]^{<\omega}$, $\langle A, \preceq \rangle$ is a poset, $\preceq \subset \lhd$, $I \in \left[\omega_1\right]^{<\omega}$,
(P2) $T_\alpha \subset (A \cap \alpha) \times \omega$ and $\langle T_\alpha, \preceq \rangle$ is a tree for $\alpha \in I$,
(P3) f and g are functions, $\mathrm{dom}(f) \subset \left[I\right]^2$ $\mathrm{dom}(g) \subset U^p \times I^p$, $\mathrm{ran}(f) \cup \mathrm{ran}(g) \subset \omega$
(P4) (a) if $\alpha \in I$ and $x, y \in T_\alpha$ then $U(x) \cap U(y) \neq \emptyset$ iff x and y are $\leq$-comparable.
(b) if $\{\alpha, \beta\} \in \mathrm{dom}(f)$ and $n = f(\alpha, \beta)$, then

$$U[T_\alpha(n)] \cap U[T_\beta(n)] = \emptyset \text{ and } U[T_\alpha(n)] \cap T_\beta(< n) = \emptyset \tag{7}$$

(P5) if $\langle x, \alpha \rangle \in \mathrm{dom}(g)$ then for all $y \in T_\alpha(g(x, \alpha))$ we have $U(y) \subset U(x)$ or $U(y) \cap U(x) = \emptyset$

For $p, q \in P$ let $p \leq q$ iff

(O1) $A^p \supset A^q$, and $\leq^q = \leq_p \restriction A_q$
(O2) $I^p \supset I^q$ and $T^q_\alpha = T^p_\alpha \cap A^q$ for $\alpha \in I^q$,
(O3) if $x \in A^p \setminus A^q$, then $U^p(x) \cap A^q = \emptyset$,
(O4) $f^p \supset f^q$,
(O5) if $U^q(x) \cap U^q(y) = \emptyset$ then $U^p(x) \cap U^p(y) = \emptyset$.

Clearly $\leq$ is a partial order on P.

For $p \in P$ let $\mathrm{supp}(p) = A^p \cup I^p$.

If $\mathcal{G}$ be a $\mathcal{P}$-generic filter, then let

$$A = \bigcup\{A^p : p \in \mathcal{G}\},$$
$$\preceq = \bigcup\{\preceq^p : p \in \mathcal{G}\},$$
$$I = \bigcup\{I^p : p \in \mathcal{G}\},$$
$$T_\alpha = \bigcup\{T^p_\alpha : \alpha \in p \in \mathcal{G}\} \text{ for } \in L_1,$$
$$f = \bigcup\{f^p : p \in \mathcal{G}\},$$
$$g = \bigcup\{g^p : p \in \mathcal{G}\}.$$

We will show that $\mathbb{A} = \langle\langle\omega_1 \times \omega, \preceq\rangle, \{T_\alpha : \alpha \in L_1\}\rangle$ is a good candidate.

Lemma 7.11. *$A = \omega_1 \times \omega$, $I = L_1$ and $T_\gamma(0) \setminus \zeta \times \omega$ is infinite for all $\gamma \in L_1$ and $\zeta < \gamma$. Especially (G2) holds.*

Proof. If $p \in P$, $y \in (\omega_1 \times \omega \setminus A^p) \setminus (\zeta \times \omega)$ and $\gamma \in L_1$ with $y \in \gamma \times \omega$, define $p \uplus \{x\}_\alpha$ as follows:

(8)
$$p \uplus \{y\}_\gamma = \left\langle A^p \cup \{y\}, \leq^p, I^p \cup \{\gamma\}, \{A^p_\gamma \cup \{y\}, A^p_\alpha : \alpha \in I^p \setminus \{\gamma\}\}, f^p, g^p \right\rangle$$

Then $q = p \uplus \{y\}_\gamma \in P$ and $p \uplus \{y\}_\gamma \leq p$.

Since $\gamma \in I^q$ and $y \in T^q_\gamma(0)$, we are done. □

Lemma 7.12. *(a) Assume that $p \in P$, $a \in T^p_\gamma$, and $b \in (\gamma \times \omega) \setminus A^p$ with $a \lhd b$. Let*

(9)
$$p \uplus_a \{b\}_\gamma = \left\langle A^p \cup \{b\}, \leq^p \trianglelefteq \langle a, b\rangle, \{A^p_\gamma \cup \{b\}, A^p_\alpha : \alpha \in I^p \setminus \{\gamma\}\}, f^p, g^p \right\rangle$$

Then $p \uplus_a \{b\}_\alpha \in P$ and $p \uplus_a \{b\}_\alpha \leq p$.

(b) The structure $\mathbb{A}$ is a candidate.

Proof. First we check $q = p \uplus_a \{b\}_\alpha \in P$.

(P1)-(P3) are straightforward.

(P4)(a): Since $U^q(b) = \{b\}$, we can assume that $x, y \neq b$. If $U^p(x) \cap U^p(y) \neq \emptyset$ then x and y are $\leq^p$-comparable. So we can assume that $b \in U^q(x) \cap U^q(y)$. But then $a \in U^p(x) \cap U^p(y)$, so we are done.

(P4)(b): Assume that $x \in T^q_\alpha(n)$, $y \in T^q_\beta(n)$ and $z \in U^q(x) \cap U^q(y)$. If $z \neq b$, then $z \in U^p(x) \cap U^p(y)$ which is not possible. So $z = b$.

If $x, y \neq b$, then $a \in U^p(x) \cap U^p(y)$ which is not possible. So we can assume that $x = b$ and $\alpha = \gamma$. So $b \in T^q_\alpha(n)$ and so $a \in T^p_\alpha(n-1)$. Thus $T^p_\alpha(n-1) \cap U^p(y) \neq \emptyset$ which is not possible because (P4)(b) holds for p.

Assume that $x \in T^q_\alpha(n)$, $y \in T^q_\beta(<n)$ and $y \in U^q(x)$. If $y \neq b$, then $y \in U^p(x) \cap T^p_\beta(<n)$ which is not possible. So $y = b$ and $\beta = \gamma$. Thus $a \in T^p_\beta(<n) \cap U^p_\alpha(x)$ which is not possible because (P4)(b) holds for p.

(P5) Since $U(b) = \{b\}$, we can assume that $y \in A^p$. Since $b \in U^q(z)$ iff $a \in U^q(z)$ for $z \in A^p$, if $U^p(y) \subset U^p(x)$ then $U^q(y) \subset U^q(x)$, and if $U^p(y) \cap U^p(x) = \emptyset$ then $U^q(y) \cap U^q(x) = \emptyset$.

Thus we proved $q \in P$. Since $q \le p$ is straightforward, we are donw

(b) is clear from (a) by standard density arguments. □

Lemma 7.13. *$\mathbb{A}$ has property (G1).*

Proof. Assume that $p \in P$, $u, v \in A^p$, $v \notin U^p(u)$ Pick $\gamma \in L_1 \setminus I^p$ with $\operatorname{supp}(p) \subset \gamma$, and pick $b \in \gamma \times \omega$ with $u \lhd b$.

Consider the condition

$$p \uplus_u \{b\}_\gamma = \left\langle A^p \cup \{b\}, \le^p \sqcup \langle u, b\rangle, \{A_\gamma = \{b\}, A^p_\alpha : \alpha \in I^p\}, f^p, g^p \right\rangle$$

Then $p \uplus_x \{b\}_\gamma \in P$.

(P1)-(P3) are straightforward.

(P4)(a): Since $U^q(b) = \{b\}$, we can assume that $x, y \neq b$. If $U^p(x) \cap U^p(y) \neq \emptyset$ then x and y are $\le^p$-comparable. So we can assume that $b \in U^q(x) \cap U^q(y)$. But then $u \in U^p(x) \cap U^p(y)$, so we are done.

(P4)(b): Assume that $x \in T^q_\alpha(n)$, $y \in T^q_\beta(n)$ and $z \in U^q(x) \cap U^q(y)$. If $z \neq b$, then $z \in U^p(x) \cap U^p(y)$ which is not possible. So $z = b$.

Since $\gamma \notin I^p$, we have $x, y \in I^p$. So $a \in U^p(x) \cap U^p(y)$ which is not possible.

Assume that $x \in T^q_\alpha(n)$, $y \in T^q_\beta(<n)$ and $y \in U^q(x)$. Since $\gamma \notin I^p$, we have $y \neq b$ and so $y \in U^p(x) \cap T^p_\beta(<n)$ which is not possible.

(P5) Since $U(b) = \{b\}$, we can assume that $y \in A^p$. Since $b \in U^q(z)$ iff $a \in U^q(z)$ for $z \in A^p$, if $U^p(y) \subset U^p(x)$ then $U^q(y) \subset U^q(x)$, and if $U^p(y) \cap U^p(x) = \emptyset$ then $U^q(y) \cap U^q(x) = \emptyset$.

Thus $q \in P$. It is clear that $q \le p$. Since $b \in T^q_\gamma$, we have $V(b) \cap \mathcal{B}(T_\gamma) \neq \emptyset$, so $V(b) \neq \emptyset$. Since $U^p(y) \cap U^q(b) = \emptyset$ we have $U(y) \cap U(b) = \emptyset$, and so $V(y) \cap V(b) = \emptyset$, and so $\emptyset \neq V(v) \subset V(y) \setminus V(x)$. □

Lemma 7.14. $\operatorname{dom}(f) = \left[L1\right]^2$ *and* $\operatorname{dom}(g) = \omega_1 \times \omega \times L_1$. *Hence (G3) and (G4) hold.*

Proof. Assume that $\gamma, \delta \in \left[I^p\right]^2 \setminus \operatorname{dom}(f^p)$

Pick m such that $T^p_\alpha(m) = \emptyset$ for all $\alpha \in I^p$.

Extends f^p to f^q as follows: $\operatorname{dom}(f^q) = \operatorname{dom}(f^p) \cup \{\gamma, \delta\}$ and $f^q(\gamma, \delta) = m$.

Let

$$q = \langle A^p, \leq^p, I^p, \{A^p_\alpha : \alpha \in I^p, f^q, g^p\}\rangle .$$

Then $q \in P$ and $q \leq p$.

Similar argument works for g. □

Definition 7.15. *We say that the conditions p and q are* twins *iff*

(T1) $|\operatorname{supp}(p)| = |\operatorname{supp}(q)$ *and* $\operatorname{supp}(p) \cap \operatorname{supp}(q) < \operatorname{supp}(p) \triangle \operatorname{supp}(q)$*,*
Denote ρ the unique order preserving bijection between $\operatorname{supp}(p)$ *and* $\operatorname{supp}(q)$*. Denote $\underline{\rho}$ the function defined by the formula $\underline{\rho}(\langle \alpha, n\rangle = (\langle \rho(\alpha), n\rangle$.*
(T2) $\underline{\rho}'' A^p = A^q$
(T3) $x \leq^p y$ *iff* $\underline{\rho}(x) \leq^q \underline{\rho}(y)$
(T4) $\rho'' I^p = I^q$
(T5) $T^q_{\rho(\alpha)} = \underline{\rho}'' T_\alpha$.
(T6) $f^q = \{\langle \{\underline{\rho}(x), \underline{\rho}(y)\}, m\rangle : \langle \{x, y\}, m\rangle \in f^p\}$.
(T7) $g^q = \{\langle \{\underline{\rho}(x), \rho(\alpha)\}, m\rangle : \langle \langle x, \alpha\rangle, m\rangle \in g^p\}$.

Lemma 7.16. *If p and q are twins then*
(10)
$$p \oplus q = \langle A^p \cup A^q, \leq^p \cup \leq^q, I^p \cup I^q, \{T^p_\alpha \cup T^q_\alpha : \alpha \in I^p \cup I^q\}, f^p \cup f^q, g^p \cup g^q\rangle$$
is a common extension of p and q, where $T^p(\alpha) = \emptyset$ for $\alpha \notin I^p$.

Proof. Straightforward. □

So $\mathcal{P}$ satisfies c.c.c

Lemma 7.17. *There is a function $\varphi : P \to \omega$ such that if $\varphi(p) = \varphi(q)$ and* $\operatorname{supp}(p) \cap \operatorname{supp}(q) < \operatorname{supp}(p) \triangle \operatorname{supp}(q)$*, then p and q are twins.*

Proof. Let $\varphi(p)$ be the type of the first order structure

(11) $$\langle \operatorname{supp}(p) \times \omega, A^p, \preceq^p, I^p, \{T^p_\alpha : \alpha \in I^p\}, f^p\rangle$$

□

Finally we are ready to verify that (G5) also holds.

Assume that

(12) $$V^P \models \forall \alpha \in L_1\ \forall \zeta < \alpha$$
$$\langle x^\alpha_\zeta, y^\alpha_\zeta, z^\alpha_\zeta, w^\alpha_\zeta\rangle \subset T_\alpha \setminus (\zeta \times \omega) \text{ is } \prec\text{-increasing}$$

For all $\alpha \in L_1$ and $\zeta < \alpha$ pick a condition $p^\alpha_\zeta = \langle A^\alpha_\zeta, \preceq^\alpha_\zeta, \dots\rangle$ which decides the sequence $\langle x^\alpha_\zeta, y^\alpha_\zeta, z^\alpha_\zeta, w^\alpha_\zeta\rangle$ and $\{x^\alpha_\zeta, y^\alpha_\zeta, z^\alpha_\zeta, w^\alpha_\zeta\} \subset T^\alpha_\zeta$.

Using the Fodor lemma, for each $\zeta \in \omega_1$ find $m_\zeta < \omega$ and $I_\zeta \in [L_1]^{\omega_1}$ such that

(i) $\varphi(p^\alpha_\zeta)=m_\zeta$ for all $\alpha\in I_\zeta$
(ii) $\{\operatorname{supp}(p^\alpha_\zeta):\alpha\in I_\zeta\}$ forms a nice Δ-system with kernel S_ζ, moreover $\alpha\in\operatorname{supp}(p^\alpha_\zeta)\setminus S_\zeta$.
(iii) $\langle x^\alpha_\zeta,y^\alpha_\zeta,z^\alpha_\zeta,w^\alpha_\zeta\rangle=\langle x_\zeta,y_\zeta,z_\zeta,w_\zeta\rangle$, and so $\{x^\alpha_\zeta,y^\alpha_\zeta,z^\alpha_\zeta,w^\alpha_\zeta\}\subset S_\zeta$ for $\alpha\in I_\zeta$.

Find $m\in\omega$ and $I\in\left[\omega_1\right]^{\omega_1}$ such that
(iv) $m_\zeta=m$ for all $\zeta\in I$, and so

$$\forall\zeta\in I\ \forall\alpha\in I_\zeta\ \varphi(p^\alpha_\zeta)=m. \tag{13}$$

(v) $\{S_\zeta:\zeta\in I\}$ forms a nice Δ-system with kernel S.

Pick $\{\xi,\zeta\}\in\left[I\right]^2$. Then pick $\alpha\in I_\zeta$ such that $S_\xi\cup S_\zeta<\operatorname{supp}(p^\alpha_\zeta)\setminus S_\zeta$. So

$$S<(S_\xi\cup S_\zeta)\setminus S<\operatorname{supp}(p^\alpha_\zeta)\setminus S_\zeta. \tag{14}$$

Now pick $\beta\in I_\xi$ such that $\operatorname{supp}(p^\alpha_\zeta)<\operatorname{supp}(p^\beta_\xi)\setminus S_\xi$. So

$$S<(S_\xi\cup S_\zeta)\setminus S<\operatorname{supp}(p^\alpha_\zeta)\setminus S_\zeta<\operatorname{supp}(p^\beta_\xi)\setminus S_\xi. \tag{15}$$

Thus $\operatorname{supp}(p^\alpha_\zeta)\cap\operatorname{supp}(p^\beta_\xi))=S$, $\alpha\in\operatorname{supp}(p^\alpha_\zeta)\setminus S_\zeta$ and $\beta\in\operatorname{supp}(p^\beta_\xi)\setminus S_\xi$.

Since $\varphi(p^\alpha_\zeta)=\varphi(p^\beta_\xi)$, the conditions $\varphi(p^\alpha_\zeta)$ and $\varphi(p^\beta_\xi))$ are twins, and

$$q=p^\alpha_\zeta\oplus p^\beta_\xi \tag{16}$$

is their common extension. Pick $t\in(\alpha\times\omega)\setminus(A^\alpha_\zeta\cup A^\beta_\zeta)$ with $y_\zeta\lhd t$ and $w_\xi\lhd t$.

Define $r\in P$ as follows

$$\left\langle A^q,\leq_q\cup\langle y_\zeta,t\rangle\cup\langle w_\xi,t\rangle,I^q,\{A^q_\alpha\cup\{t\},A^q_\beta\cup\{t\},A^\gamma:\gamma\in I^q\setminus\{\alpha,\beta\}\},f^q\right\rangle. \tag{17}$$

We should check $r\in P$.

Key observation:

$$r\restriction(\operatorname{supp}(p^\alpha_\zeta)\cup\{t\})=p^\alpha_\zeta\uplus_{y^\alpha_\zeta}\{t\}_\alpha \text{ and } r\restriction(\operatorname{supp}(p^\beta_\xi)\cup\{t\})=p^\beta_\xi\uplus_{w^\beta_\xi}\{t\}_\beta \tag{18}$$

(P1) is trivial.

(P2). Let $\gamma\in I^q$. If $\gamma\ne\alpha,\beta$, then $T^q_\gamma=T^p_\gamma$, so we are done.

Moreover, $T^r_\alpha=T^q_\alpha\cup\{t\}$ and $t\in\alpha\times\omega$. and $\langle T^r_\alpha,\preceq\rangle$ is a tree by the key observation.

The same argument works for T^r_β.

(P3) is trivial.

(P4)(a).

Assume that $\gamma \in I^r$, $x, y \in T^r_(\alpha)$ with $U^r(x) \cap U^r(y) \neq \emptyset$. Since $U^r(t) = \{t\}$ we can assume $x, y \in A^q$. Assume that $\gamma \in I^{p^\alpha_\zeta}$. Then $T^q_\gamma \subset A^{p^\alpha_\zeta}$, so $x, y \in A^{p^\alpha_\zeta}$. So $t \in U^r(x) \cap U^r(y)$ implies $y^\alpha_\zeta \in U^r(x) \cap U^r(y)$. Thus $U^q(x) \cap U^q(y) \neq \emptyset$ and so x and y are $\preceq^q$ comparable.

Similar argument works for $\gamma \in I^{p^\beta_\xi}$.

(P4)(b). Assume that $\{\alpha', \beta'\} \in dom(f^r) = \mathrm{dom}(f^q) = \mathrm{dom}(p^\alpha_\zeta)) \cup \mathrm{dom}(p^\beta_\xi)$. We can assume that $\{\alpha', \beta'\} \in \mathrm{dom}(p^\beta_\xi)$.

Write $n = f^r(\{\alpha', \beta'\})$.

(i)Assume on the contrary that there are $a \in T^r_{\alpha'}(n)$ and $b \in T^r_{\beta'}(n)$ with $U^r(a) \cap U^r(b) \neq \emptyset$.

First assume that $\{a, b\} \in \left[A^q\right]^2$ The only possible case is when $U^q(a) \cap U^q(b) = \emptyset$, but $t \in U^r(a) \cap U^q(b)$.

Then we can assume that $a \in A^{p^\alpha_\zeta} \setminus A^{p^\alpha_\zeta}$ with $y^\alpha_\zeta \in U^q(a)$, and $b \in A^{p^\alpha_\zeta} \setminus A^{p^\alpha_\zeta}$ with $w^\beta_\xi \in U^q(b)$.

But then $\alpha' \in \mathrm{supp}(p^\alpha_\zeta) \setminus S$ and $\beta' \in \mathrm{supp}(p^\beta_\xi) \setminus S$, so $f(\alpha', \beta')$ is undefined.

So we can assume that $t = a$. If $b \in A^{p^\alpha_\zeta}$, then we can use the first part of the key observation.

If $b \in A^{p^\beta_\xi}$, then we can use the second part of the key observation.

(ii) (i)Assume on the contrary that there are $a \in T^r_{\alpha'}(n)$ and $b \in T^r_{\beta'}(< n) \cap U^r(a)$.

Clearly $a \neq t$ If $b \neq t$, then $a \in T^q_{\alpha'}(n)$ and $b \in T^q_{\beta'}(< n) \cap U^q(a)$.

Assume that $b = t$ If $b \in A^{p^\alpha_\zeta}$, then we can use the first part of the key observation.

If $b \in A^{p^\beta_\xi}$, then we can use the second part of the key observation.

(P5). Let $\langle x, \gamma \rangle \in \mathrm{dom}(g^r)$ and $y \in T^r_\gamma(g(x, \gamma))$

Since $U^r(t) = \{t\}$, we can assume that $x, y \neq t$.

So $x, y \in A^q$. If $U^q(y) \subset U^q(x)$, then $x \leq^q y$ and so $U^r(y) \subset U^r(x)$.

Assume on the contrary that $U^q(x) \cap U^q(y) = \emptyset$, but $t \in U^q(x) \cap U^q(y)$.

We can assume that $\langle x, \gamma \rangle \in g^{p^\alpha_\zeta}$. Thus $x \in A^\alpha_\nu$ and $\gamma \in A^\alpha_\nu$.

However $T^q_\gamma \subset A^\alpha_\zeta$, so $y \in A^\alpha_\zeta$. However for $z \in A^\alpha_\zeta$ we have $t \in U^r(z)$ iff $y^\alpha_\zeta \in U^q(z)$, so $y^\alpha_\zeta \in U^q(x) \cap U^q(y)$. Contradiction.

So

$$r \in P. \tag{19}$$

Next we show that $r \leq p^\alpha_\zeta, p^\beta_\xi$. (O1)–(O4) are trivial. To check (O5), assume on the contrary that $U^{p^\alpha_\zeta}(a) \cap U^{p^\alpha_\zeta}(b) = \emptyset$, but $U^r \cap U^r(b) = \emptyset$.

Then $t \in U^r(a) \cap U^r(b)$, and so $y^\alpha_\zeta \in U^{p^\alpha_\zeta}(a) \cap U^{p^\alpha_\zeta}(b)$ which is a contradiction.

Finally, it is also straightforward that

$$r \Vdash \text{ (G5)(i)–(ii) holds for } \alpha, \beta, \zeta, \xi, t \tag{20}$$

So we proved the theorem. □

8. Open problems

In this section, we present a list open problems which could be of further interest and are closely connected to our results.

Problem 8.1. *Is every linearly ordered space base resolvable?*

Problem 8.2. *Is every T_3 (hereditarily) separable space base resolvable?*

Problem 8.3. *Is every paracompact space base resolvable?*

Note that under PFA, every T_3 hereditarily separable space is Lindelöf hence base resolvable by Corollary 3.6. Also, we conjecture that our forcing construction can be modified to produce a separable non base resolvable space.

Problem 8.4. *Is every power of $\mathbb{R}$ base resolvable? Is it true that base resolvability is preserved by products?*

We know that every π-base is the union of two disjoint π-bases by Proposition 2.3. However:

Problem 8.5. *Does every base contain a disjoint base and π-base?*

Bases closed to finite unions are resolvable by Corollary 4.7 which raises to following question:

Problem 8.6. *Is it true that every base which is closed to finite intersections is base resolvable?*

We do not know whether the following generalization of Corollary 3.6 and 4.9 holds:

Problem 8.7. *Does MA implies that every space X with (local) Lindelöf-number $< 2^\omega$ is base resolvable?*

Concerning negligible subsets we ask the following:

Problem 8.8. *Is there a base $\mathbb{B}$ for some space X such that every $\mathcal{U} \in [\mathbb{B}]^{|\mathbb{B}|}$ contains a neighborhood base at some point?*

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