UNBOUNDED AND MAD FAMILIES OF FUNCTIONS ON UNCOUNTABLE CARDINALS

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Abstract. Motivated by a recent result of D. Raghavan and S. Shelah, we present ZFC results about the bounding number $\mathfrak{b}(\kappa)$ and the almost disjointness invariants $\mathfrak{a}_e(\kappa), \mathfrak{a}_p(\kappa)$ and $\mathfrak{a}_g(\kappa)$ for uncountable, successor κ. We show that if $\kappa = \lambda^+$ for some $\lambda \geq \omega$ and $\mathfrak{b}(\kappa) = \kappa^+$ then $\mathfrak{a}_e(\kappa) = \mathfrak{a}_p(\kappa) = \kappa^+$. If additionally $2^{<\lambda} = \lambda$ then $\mathfrak{a}_g(\kappa) = \kappa^+$ as well.

Let us start by recalling some well known definitions. $\mathfrak{b}(\kappa)$ is the minimal size of a family $\mathcal{F} \subset \kappa^\kappa$ so that there is no single function $g \in \kappa^\kappa$ so that $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ has size $< \kappa$ for all $f \in \mathcal{F}$. We use the fact that $\mathfrak{b}_{cl}(\kappa) = \mathfrak{b}(\kappa)$ for any uncountable, regular κ [2]: there is $\mathcal{F} \subset \kappa^\kappa$ of size $\mathfrak{b}(\kappa)$ so that for any $g \in \kappa^\kappa$ there is some $f \in \mathcal{F}$ with $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ stationary.

We also remind the reader of the usual almost disjointness numbers; in our context almost disjoint (eventually different) means that the intersection of the sets (functions) has size $< \kappa$.

(i) $\mathfrak{a}(\kappa)$ is the minimal size of a maximal almost disjoint family $\mathcal{A} \subset [\kappa]^\kappa$ that is of size $\geq \kappa$ (the latter rules out trivialities like $\mathcal{A} = \{\kappa\}$).
(ii) $\mathfrak{a}_e(\kappa)$ is the minimal size of a maximal, eventually different family of functions in κ^κ.
(iii) $\mathfrak{a}_p(\kappa)$ is the minimal size of a maximal, eventually different family of functions in $S(\kappa)$, the set of bijective members of κ^κ.
(iv) $\mathfrak{a}_g(\kappa)$ is the minimal size of an almost disjoint subgroup of $S(\kappa)$, that is maximal among such subgroups.

D. Raghavan and S. Shelah [4] recently proved that $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$ for any regular, uncountable κ, by a surprisingly simple application of Fodor's pressing down lemma. Building on their momentum, we extend this result to related cardinal invariants on maximal families of eventually different functions and permutations (see [1, 3] for a detailed background).

Theorem 0.1. *Suppose that $\kappa = \lambda^+$ for some $\lambda \geq \omega$ and $\mathfrak{b}(\kappa) = \kappa^+$. Then $\mathfrak{a}_e(\kappa) = \mathfrak{a}_p(\kappa) = \kappa^+$. If additionally $2^{<\lambda} = \lambda$ then $\mathfrak{a}_g(\kappa) = \kappa^+$ as well.*

This is a strengthening of [1, Theorem 2.2], where $\mathfrak{d}(\kappa) = \kappa^+$ implies $\mathfrak{a}_e(\kappa) = \kappa^+$ was proved for successor κ, and also of [3, Theorem 4] where $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}_e(\kappa) = \kappa^+$ was proved using additional assumptions.

Proof. Let $\{f_\delta : \delta < \kappa^+\}$ witness $\mathfrak{b}_{cl}(\kappa) = \kappa^+$. Also, fix bijections $e_\delta : \delta \to \kappa$ for $\delta < \kappa^+$ and bijections $d_\alpha : \lambda \to \alpha$ for $\alpha < \kappa$. The latter will allow us, given some $H \subseteq \alpha < \kappa$ with

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$|H|=\lambda$ and $\zeta<\lambda$, to select the ζ^{th} element of H with respect to d_α; that is, to pick $\beta\in H$ so that $d_\alpha(\beta)\cap d_\alpha[H]$ has order type ζ.

Let us start with $\mathfrak{a}_e(\kappa)=\kappa^+$. We will define functions $h_{\delta,\zeta}\in\kappa^\kappa$ for $\delta<\kappa^+,\zeta<\lambda$ that will form our maximal eventually different family.

We go by induction on $\delta<\kappa^+$. Let $\mathbb{H}_\delta(\mu)=\{h_{\delta',\zeta'}:\delta'\in\mathrm{ran}(e_\delta\restriction\mu),\zeta'<\lambda\}$, a good portion of the previously constructed functions. Note that

$$H_\delta(\mu)=\{h(\mu):h\in\mathbb{H}_\delta(\mu)\}$$

has size $<\kappa$, so we can define

$$f^*_\delta(\mu)=\max\{f_\delta(\mu),\min\{\alpha<\kappa:|\alpha\setminus H_\delta(\mu)|=\lambda\}\}.$$

We define $h_{\delta,\zeta}(\mu)$ to be the ζ^{th} element of $f^*_\delta(\mu)\setminus H_\delta(\mu)$ with respect to $d_{f^*_\delta(\mu)}$.

Claim 0.2. $\mathbb{H}=\{h_{\delta,\zeta}:\delta<\kappa^+,\zeta<\lambda\}\subset\kappa^\kappa$ *is eventually different.*

Proof. For a fixed δ and $\zeta<\zeta'<\lambda$, $h_{\delta,\zeta}(\mu)\neq h_{\delta,\zeta'}(\mu)$ by definition for all $\mu<\kappa$.

Given $\delta<\delta'$, and $\zeta,\zeta'<\lambda$, whenever $\delta'\in\mathrm{ran}(e_\delta\restriction\mu)$, then $h_{\delta',\zeta'}\in\mathbb{H}_\delta(\mu)$ and so $h_{\delta',\zeta'}(\mu)\neq h_{\delta,\zeta}(\mu)\notin H_\delta(\mu)$. □

Claim 0.3. $\mathbb{H}$ *is maximal.*

Proof. Fix some $h\in\kappa^\kappa$, and find $\delta<\kappa^+$ so that

$$S=\{\mu<\kappa:h(\mu)<f_\delta(\mu)\}$$

is stationary. Now, there is a stationary $S_0\subset S$ so that either

(1) $h(\mu)\in H_\delta(\mu)$ for all $\mu\in S_0$, or
(2) $h(\mu)\notin H_\delta(\mu)$ for all $\mu\in S_0$.

In the first case, for each μ, we can find some $\delta'=e_\delta(\eta_\mu)$ with $\eta_\mu<\mu$ and $\zeta'=\zeta'_\mu<\lambda$ so that $h(\mu)=h_{\delta',\zeta'}(\mu)$. In turn, by Fodor's lemma, we can find a stationary $S_1\subset S_0$ and single $\delta'=e_\delta(\eta)$ and $\zeta'<\lambda$ so that $h(\mu)=h_{\delta',\zeta'}(\mu)$ for all $\mu\in S_1$; hence, $h\cap h_{\delta',\zeta'}$ has size κ.

In the second case, $h(\mu)\in f^*_\delta(\mu)\setminus H_\delta(\mu)$ must hold too, and so there is a $\zeta_\mu<\lambda$ so that $h(\mu)$ is the ζ_μ^{th} element of $f^*_\delta(\mu)\setminus H_\delta(\mu)$ with respect to $d_{f^*_\delta(\mu)}$. Again, we can find a single $\zeta<\lambda$ and stationary $S_1\subset S_0$ so that $\zeta_\mu=\zeta$ for all $\mu\in S_1$ and so $h\cap h_{\delta,\zeta}$ has size κ. □

This shows that $\mathbb{H}$ is the desired maximal eventually different family.

Now, we proceed with $\mathfrak{a}_p(\kappa)=\kappa^+$. We will modify the previous argument to ensure $h_{\delta,\zeta}\in S(\kappa)$ and to keep the family maximal in $S(\kappa)$. We will need some elementary submodels: for each $\delta<\kappa^+$, we fix a continuous, increasing sequence of elementary submodels $\bar{N}^\delta=(N^\delta_\eta)_{\eta<\kappa}$ of some $H(\theta)$ so that

(i) $|N^\delta_\eta|=\lambda$, and $N^\delta_\eta\cap\kappa\in\kappa$,
(ii) $\delta,\bar{e},\bar{d},\bar{f}\in N^\delta_\eta$,
(iii) $\delta'\in N^\delta_\eta$ implies $\bar{N}^{\delta'}\in N^\delta_\eta$.

Let $E_\delta = \{N^\delta_\eta \cap \kappa : \zeta < \kappa\} \cup \{0\}$ which is a club in κ.

Again, we proceed by induction on δ, but use the notation $\mathbb{H}_\delta(\nu)$ and $H_\delta(\nu)$ with minor modifications: $\mathbb{H}_\delta(\nu) = \{h_{\delta',\zeta'} : \delta' \in \operatorname{ran}(e_\delta \upharpoonright \mu), \zeta' < \lambda\}$ where $\mu = \sup(E_\delta \cap \nu) \leq \nu$, and

$$H_\delta(\nu) = \{h(\nu) : h \in \mathbb{H}_\delta(\nu)\}.$$

So $\mathbb{H}_\delta(\nu) = \mathbb{H}_\delta(\mu)$ but $H_\delta(\nu)$ and $H_\delta(\mu)$ are typically different (where $\mu = \sup(E_\delta \cap \nu)$).

We construct $h_{\delta,\zeta}$ for $\zeta < \lambda$ so that

(1) $h_{\delta,\zeta} \upharpoonright [\mu, \mu^+) \in S([\mu, \mu^+))$ for any successive elements μ, μ^+ of E_δ,
(2) $h_{\delta,\zeta} \cap h_{\delta,\zeta'} = \emptyset$ for $\zeta' < \zeta$,
(3) $h_{\delta,\zeta}(\nu) \in \kappa \setminus H_\delta(\nu)$,
(4) $h_{\delta,\zeta}(\mu)$ is the ζ^{th} element of $f^*_\delta(\mu) \setminus (H_\delta(\mu) \cup \mu)$ with respect to $d_{f^*_\delta(\mu)}$, where

$$f^*_\delta(\mu) = \max\{f_\delta(\mu), \min\{\alpha < \kappa : |\alpha \setminus (H_\delta(\mu) \cup \mu)| = \lambda\}\},$$

(5) $(h_{\delta,\zeta})_{\zeta<\lambda}$ is uniquely definable from $\bar{N}^\delta$.

These conditions clearly ensure that $h_{\delta,\zeta} \in S(\kappa)$, and as before, the family $\{h_{\delta',\zeta'} : \delta' \leq \delta, \zeta' < \lambda\}$ remains eventually different by conditions (2) and (3). Maximality, just as before, follows from condition (4) and Fodor's lemma.

Let us show that we can actually construct functions with the above properties. Fix successive elements $\mu < \mu^+$ of E_δ, and we define $h_{\delta,\zeta} \upharpoonright [\mu, \mu^+) \in S([\mu, \mu^+))$ by an induction in λ steps. We list all triples from $\lambda \times [\mu, \mu^+) \times 2$ as $(\zeta_\xi, \nu_\xi, i_\xi)$ for $\xi < \lambda$.

First of all, let $\mu = \kappa \cap N^\delta_\eta$ and $\mu^+ = \kappa \cap N^\delta_{\eta+1}$; we will write N for $N^\delta_{\eta+1}$ temporarily.

Claim 0.4. *$\mu^+ \setminus (H_\delta(\nu) \cup \mu)$ has size λ for all $\nu \in \mu^+ \setminus \mu$.*

Proof. Note that $e_\delta \upharpoonright \mu \in N$ and so $\operatorname{ran}(e_\delta \upharpoonright \mu)$ is an element and subset of N. Furthermore, we can apply (5) to see that $\mathbb{H}_\delta(\mu) \in N$ and so $H_\delta(\nu) \in N$ for any $\nu < \mu^+$. Moreover, $N \models |H_\delta(\nu)| < \kappa$ so $\mu^+ \setminus (H_\delta(\nu) \cup \mu)$ has size λ. □

In turn, since $f_\delta(\mu) \in N$ as well, the value $f^*_\delta(\mu)$ in condition (4) is well defined and $< \mu^+$.

Now, we can start our induction on $\xi < \lambda$ by partial functions $h_{\delta,\zeta}$, each defined only at μ to satisfy condition (4). At step ξ, we do the following. Let $\zeta = \zeta_\xi, \nu = \nu_\xi$; if $i_\xi = 0$ then we make sure that ν gets into the domain of $h_{\delta,\zeta}$, and if $i_\xi = 1$ then we make sure that ν is in the range of $h_{\delta,\zeta}$.

Suppose $i_\xi = 0$. We need to find a value for $h_{\delta,\zeta}(\nu)$ which is in $\mu^+ \setminus (H_\delta(\nu) \cup \mu)$ and which also avoids $h_{\delta,\zeta'}(\nu)$ where $\zeta' = \zeta_{\xi'}$ for some $\xi' < \xi$. The set $\mu^+ \setminus (H_\delta(\nu) \cup \mu)$ has size λ (using that $H_\delta(\nu) \in N$ as before), and we only defined $< \lambda$ many functions so far, hence we can find a (minimal) good choice.

Next, if $i_\xi = 1$ then we need to find some $\vartheta \in \mu^+ \setminus \mu$ so that $h(\vartheta) \neq \nu$ for $h \in \mathbb{H}_\delta(\mu)$ and $h_{\delta,\zeta'}(\vartheta) \neq \nu$ for all $\zeta' = \zeta_{\xi'}$ for some $\xi' < \xi$. First, $\mathbb{H}_\delta(\mu) \in N$ and has size $< \kappa$ so the set of good choices

$$\mu^+ \setminus (\mu \cup \{\vartheta < \kappa : h(\vartheta) = \nu, h \in \mathbb{H}_\delta(\mu)\}$$

still has size λ by elementarity. Each $h_{\delta,\zeta'}$ introduces ≤ 1 bad ϑ, and we have $\leq |\xi| < \lambda$ many of these, so we can find a good (minimal) ϑ.

If we carry out all this work in $N^\delta_{\eta+2}$, always taking minimal choices, then in the end condition (5) is preserved as well.

Finally, we turn to the proof of $\mathfrak{a}_g(\kappa) = \kappa^+$. We use the additional assumption that $2^{<\lambda} = \lambda$. We keep the notations $\mathbb{H}_\delta(\nu), H_\delta(\nu)$ from the previous section, as well as the elementary submodels. However, we can now assume that each successor model $N^\delta_{\eta+1}$ is $< \lambda$-closed. This will help us when we are constructing the functions $h_{\delta,\zeta}$ in the induction of length λ, because at each intermediate step ξ, the model $N^\delta_{\xi+1}$ will contain all the functions which we constructed so far (there was no reason for this hold before).

So, our aim now is to construct $\mathbb{H} = \{h_{\delta,\zeta} : \delta < \kappa^+, \zeta < \lambda\} \subset S(\kappa)$, so that in the generated subgroup $\mathbb{G} = \langle \mathbb{H} \rangle$, only the identity has κ fixed points and $\mathbb{G}$ is maximal. We use the notation

$$\mathbb{G}_\delta(\nu) = \langle \mathbb{H}_\delta(\nu) \rangle \text{ and } G_\delta(\nu) = \{g(\nu) : g \in \mathbb{G}_\delta(\nu)\}$$

for $\delta < \kappa^+$ and $\mu < \kappa$.

We go by induction on δ as before, and construct $h_{\delta,\zeta}$ so that

(1) $h_{\delta,\zeta} \restriction [\mu, \mu^+) \in S([\mu, \mu^+))$ for any successive elements μ, μ^+ of E_δ,
(2) any fixed point of a non identity function $h \in \langle \mathbb{H}_\delta(\mu) \cup \{h_{\delta,\zeta} \restriction \mu^+ : \zeta < \lambda\} \rangle$ is below μ,
(3) $(h_{\delta,\zeta})_{\zeta<\lambda}$ is uniquely definable from $\bar{N}^\delta$.

These conditions ensure that only the identity in $\mathbb{G}$ has κ fixed points. Indeed, suppose $g \in \mathbb{G}$ is not the identity and write it as a finite product of $h_{\delta,\zeta}$ functions. Let δ_1 be the maximal δ that occurs; if no other δ is in this product then g has no fixed points by (2). If δ_0 is the maximum of all other δ's that occur then we can find a $\mu < \kappa$ so that $\delta_0 \in \operatorname{ran}(e_\delta \restriction \mu)$ and so (2) implies that all fixed points of g are below μ.

As before, we fixed some $\mu < \mu^+$, and $h_{\delta,\zeta}$ is constructed by an induction of length λ, using an enumeration of all triples from $\lambda \times [\mu, \mu^+) \times 2$ as $(\zeta_\xi, \nu_\xi, i_\xi)$ for $\xi < \lambda$.

We start by empty functions now, and at step ξ, we either need to put $\nu = \nu_\xi$ into the domain of $h_{\delta,\zeta}$ or into the range of $h_{\delta,\zeta}$ (where $\zeta = \zeta_\xi$).

Lets look at the first case: in order to preserve (2), it suffices to ensure that $h_{\delta,\zeta}(\nu) \neq h(\nu)$ for any

$$h \in Z = \langle \mathbb{H}_\delta(\mu) \cup \{h_{\delta,\zeta'} : \zeta' = \zeta_{\xi'} \xi' < \xi\} \rangle$$

whenever $h(\nu)$ can be computed.

The maps $h_{\delta,\zeta'}$ are some partial functions on μ^+ that extend $h_{\delta,\zeta'} \restriction \mu$ by $< \lambda$ many new values. Since $N = N^\delta_{\xi+1}$ now contains these functions as well as the set $\{h_{\delta,\zeta'} : \zeta' = \zeta_{\xi'} \xi' < \xi\}$, it also contains the set Z (we applied that N is $< \lambda$-closed and the inductive hypothesis (3)). So, since

$$N \models |\{h(\nu) : h \in Z\}| < \kappa,$$

we can take $h_{\delta,\zeta}(\nu) = \min \mu^+ \setminus (\{h(\nu) : h \in Z\} \cup \mu)$.

To ensure maximality in the end, we consider the case $\nu = \mu$ separately. Now, we don't just take a minimal good choice but look at the minimal $\alpha \geq f_\delta(\mu)$ so that $\alpha \setminus (\{h(\mu) : h \in Z\} \cup \mu)$ has size λ. Since $Z \in N$ and $N \models |Z| < \kappa$, $\alpha \in N$ too. Now, we define $h_{\delta,\zeta}(\mu)$ to be the ζ^{th} element of $\alpha \setminus \{h(\mu) : h \in Z\}$ with respect to d_α.

Second, to put ν in the range of $h_{\delta,\zeta}$: we need some $\vartheta \in \mu^+ \setminus \mu$ so that $h(\vartheta) \neq \nu$ for any $h \in Z$ (and then we can set $h_{\delta,\zeta}(\theta) = \nu$). Again, $N \models |Z| < \kappa$ and each $h \in Z$ contributes with at most one bad ϑ so we can pick a minimal ϑ that works.

It is left to check that we constructed a maximal $\mathbb{G}$. Fix any $g \in S(\kappa) \setminus \mathbb{G}$ and find $\delta < \kappa^+$ so that $S = \{\mu < \kappa : g(\mu) < f_\delta(\mu)\}$ is stationary. Now, there is a stationary $S_0 \subset S$ so that either

(1) $g(\mu) = h(\mu)$ for some $\langle H_\delta(\mu) \cup \{h_{\delta,\zeta} : \zeta < \lambda\} \rangle$ for all $\mu \in S_0$, or

(2) $g(\mu) \neq h(\mu)$ for all $\langle H_\delta(\mu) \cup \{h_{\delta,\zeta} : \zeta < \lambda\}\rangle$ for all $\mu \in S_0$.

In the first case, we can use Fodor's theorem to fix a single $h \in \mathbb{G}$ so that $g \cap h$ has size κ. In the latter, there is some $\zeta < \lambda$ so that $g \cap h_{\delta,\zeta}$ has size κ (just as in the previous proofs). $\square$

We do not know at this point if our theorem is true without the assumption of κ being successor, nor how to remove $2^{<\lambda} = \lambda$ from the last part of the result.

References

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