PARTIAL INDEPENDENT TRANSVERSALS IN GRAPHS AVOIDING LARGE CLIQUES

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Abstract. Our goal is to investigate the independent transversal problem in the class of K_n-free graphs: we show that for any infinite K_n-free graph $G = (V, E)$ and $m \in \mathbb{N}$ there is a minimal $r = r(G, m)$ so that for any balanced r-colouring of the vertices of G one can find an independent set which meets at least m colour classes in a set of size $|V|$. Answering a conjecture of S. Thomassé, we express the exact value of $r(H_n, m)$ using certain Ramsey numbers for directed graphs where H_n is Henson's countable universal K_n-free graph. Using our results, we deduce a new partition property of H_n regarding balanced embeddings of bipartite graphs into H_n.

1. Introduction

The goal of our paper is to investigate the following natural problem: given a sparse graph with the vertices partitioned into equally large classes, can we find an independent set which meets a certain number of these classes in a large set? The well known independent transversal problem is a rather close relative of this question; as presented by P. Haxell [10]: imagine that the dean at your university is looking to form a committee so that each faculty is represented however, for the sake of reaching decisions in reasonable times, no two members of the committee hold strictly opposing opinions on certain topics. We model this problem by forming a graph with vertices corresponding to faculty members and edges connecting colleagues who can't be put on the same committee. Now, we are looking for an independent set which meets each faculty.

The above question goes back to papers of B. Bollobás, P. Erdős, E. G. Strauss and E. Szemerédi in the 1970s [3, 4] and still is an active area of research (let us refer to [10] again). While the strongest results for the independent transversal problem come from assumptions on the maximum degree versus the number of classes (see e.g. [11]), we set out to investigate infinite graphs avoiding cliques of a fixed finite size. This is a rather natural assumption: the Erdős-Dushnik-Miller theorem says that any infinite graph on κ vertices either contains an infinite complete subgraph or an independent set of size κ.

The first main result of our paper is presented in Section 2:

Theorem 2.2. If G is an infinite K_n-free graph (for some $n \in \mathbb{N}$) and $m \in \mathbb{N}$ then there is a finite r so that whenever the vertices of G are partitioned into r sets of equal size then there is an independent set A which meets at least m classes in a set of size of G.

The minimal such r will be denoted by $r(G, m)$; the above result says that given such a balanced partition, we are able to find a large partial independent transversal which meets several classes in a large set. In the proof of Theorem 2.2, we bound $r(G, m)$ with a known

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Ramsey-number of directed graphs (denoted by $\mathrm{dr}(n,m)$); this bound is shown to be tight for certain graphs.

The finite counterpart is as follows:

Theorem 2.5. Suppose that $n,m \geqslant 2$ and $\ell \geqslant 1$. Then there is a finite $N = N(n,m,\ell)$ so that for every K_n-free graph G and pairwise disjoint sets of vertices $V_i \subseteq V$ with $|V_i| \geqslant N$ for $i < r = \mathrm{dr}(n,m)$ there is an independent set A so that

$$|\{i < r : |V_i \cap A| \geqslant \ell\}| \geqslant m.$$

At this point, we don't have any information on the size of $N = N(n,m,\ell)$.

After proving general properties of the function $m \mapsto r(G,m)$ in Section 3, the rest of our paper deals with calculating $r(G,m)$ for specific graphs G. In particular, in Section 4, we first focus on Henson's countable, universal K_n-free graphs H_n and we show that

$$r(H_n,m) = \mathrm{dr}(n,m-1)+1$$

in Theorem 4.3. In particular, $r(H_n,2) = 2$ for all $2 \leqslant n \in \mathbb{N}$ which answers Conjecture 46 of S. Thomassé [17].

H_n is known to satisfy a very nice Ramsey property: $H_n \to (H_n)^1_r$ for any $r \in \mathbb{N}$ [13, 2], that is whenever the vertices of H_n are partitioned into r classes then one can find a monochromatic copy of H_n. Our techniques and the fact $r(H_n,2) = 2$ can be applied to show a new partition (or univesality) property:

Theorem 5.3. Fix a finite bipartite graph G with bipartition A, B. Whenever the vertices of H_n are partitioned into two classes then there is an induced copy of G in H_n such that the images of A and B are contained in different classes.

This topic is explored in Section 5.

Finally, in Section 6, we look at various well-known graphs G (e.g. unit distance graphs, orthogonality graphs on $\mathbb{R}^n$, and shift graphs) and aim to calculate the exact values of $r(G,m)$. Our paper ends with a healthy list of open problems in Section 7.

1.1. **Notations.** In what follows, r will always denote a non zero natural number which we also identify with the set $\{0,1\dots r-1\}$. κ will always stand for an infinite cardinal. $[X]^k$ denotes the set of k-element subsets of X. Let $\subseteq^*$ stand for 'contained in mod finite' and let $=^*$ stand for 'equals to mod finite'.

We say that a partition (or colouring) $\{V_i : i < r\}$ of a set V is *balanced* iff every colour class has size $|V|$ if V is infinite, and $||V_i| - |V_j|| \leqslant 1$ for all $i < j < r$ if V is finite.

Let $H_{\omega,\omega}$ and $K_{\omega,\omega}$ denote the *half graph* and complete bipartite graph on $V = \mathbb{N} \times 2$ i.e. $E(H_{\omega,\omega}) = \{(0,k)(1,\ell) : k < \ell \in \mathbb{N}\}$ and $E(K_{\omega,\omega}) = \{(0,k)(1,\ell) : k,\ell \in \mathbb{N}\}$. Let $E_{\omega,\omega}$ denote the empty bipartite graph on $\mathbb{N} \times 2$.

If G is any graph then let $G[A,B]$ denote the graph on $A \cup B$ with edges $\{uv : u \in A, v \in B, uv \in E(G)\}$. We denote $G[A,A]$ by simply $G[A]$. Suppose that G and H are graphs and $A,B \subseteq V(G)$ and $A',B' \subseteq V(H)$. We write

$$H[A',B'] \hookrightarrow G[A,B]$$

if there is a 1-1 graph homomorphism which maps A' into A and B' into B. If this homomorphism can be chosen surjective as well, we write $H[A',B'] \hookrightarrow\!\!\!\!\rightarrow G[A,B]$

Let G,H be two graphs. Define $G \otimes H$ on vertices $V(G) \times V(H)$ and let $(u,v)(u',v') \in E(G \otimes H)$ iff $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$. E.g. $G \otimes E_\omega$ is the graph we get by blowing up the vertices of G into infinite independent sets, in particular $K_2 \otimes E_\omega = K_{\omega,\omega}$. Here, K_n denotes the complete graph on n vertices.

We will write $G \to (H)^1_r$ if for every r-colouring of the vertices of G one can find a monochromatic copy of H.

2. Finding partial independent transversals in general

Our first goal is to show that whenever G is an infinite K_n-free graph for some $n \geqslant 2$ and $m \geqslant 1$ then there is a minimal number $r = r(G, m)$ so that for every balanced r-partition of $V(G)$ there is an independent set A so that $\{i < r : |A \cap V_i| = |V|\}$ has at least m elements. Recall, that whenever G is an infinite K_n-free graph then G contains an independent set of the size of G; indeed, this is an easy consequence of the famous Erdős-Dushnik-Miller theorem [14].

We need a few definitions first:

Definition 2.1. Let $\mathrm{dr}(n, m)$ denote the minimal r so that any directed graph on r vertices contains either a transitive set of size n i.e. $\{v_i : i < n\}$ so that $v_i v_j \in E$ whenever $i < j < n$, or an independent subset of size m.

Note that $R(n, m) \leqslant \mathrm{dr}(n, m) \leqslant R(n, n, m)$ where $R(n_0 \dots n_{k-1})$ denotes the minimal r so that for any colouring of the pairs of r with k colours, one can find a j-homogeneous set of size n_j for some $j < k$.

$\mathrm{dr}(n, m)$ was introduced by A. Gyárfás [7] (denoted by $R^*(n, m)$ there). In [7], certain general bounds and values of $dr(n, m)$ for small n, m are calculated. In particular

1. $\mathrm{dr}(n, m) \leqslant 2\,\mathrm{dr}(n-1, m) + \mathrm{dr}(n, m-1) - 1$,
2. $\mathrm{dr}(3, 3) = 9, \mathrm{dr}(3, 4) = 15$,
3. $2^{(n-1)/2} \leqslant \mathrm{dr}(n, 2) \leqslant 2^{n-1}$,
4. $3^{(n-1)/2} \leqslant \mathrm{dr}(n, n) \leqslant 3^{2n-2}$, and
5. $c_1 \frac{m^2}{\log m} \leqslant \mathrm{dr}(3, m) \leqslant c_2 \frac{m^2}{\log m}$.

Our first goal is to prove

Theorem 2.2. *Let $n, m \geqslant 2$ and suppose that G is an infinite K_n-free graph. Then $r(G, m) \leqslant \mathrm{dr}(n, m)$.*

We start the proof by introducing a notion of largeness:

Definition 2.3. Let $G = (V, E)$ be a graph and suppose that $A, B \subseteq V$ are of size $|V|$. We say that A, B is a *fat pair* (in G) iff $G[A', B']$ is not empty whenever $A' \subseteq A$ and $B' \subseteq B$ are of size $|V|$.

For example, the two canonical classes of $H_{\omega,\omega}$ form a fat pair in $H_{\omega,\omega}$. Now, we establish a few simple properties of fat pairs:

Lemma 2.4. *Suppose that G is a graph on κ vertices and $A_0, A_1 \in [V]^\kappa$.*

1. *either $E_{\kappa,\kappa} \hookrightarrow G[A_0, A_1]$ or A_0, A_1 is a fat pair;*
2. *if A_0, A_1 is a fat pair then so is A'_0, A'_1 where $A'_i \subseteq A_i$ are of size κ;*
3. *if A_0, A_1 is a fat pair then there is $i^* < 2$ so that $|\{v \in A_{i^*} : |N(v) \cap A_{1-i^*}| < \kappa\}| < \kappa$ and we call A_{i^*}* essential *in the pair A_0, A_1;*
4. *if A_0, A_1 is a fat pair then there is $i^* < 2$ and $A'_i \subseteq A_i$ of size κ for $i < 2$ so that A''_{i^*} is essential in any pair A''_0, A''_1 for any $A''_i \subseteq A'_i$ of size κ; we say that A'_{i^*} is the* strongly essential *part of the pair A'_0, A'_1.*

(5) if A_0, A_1 is a fat pair, A_i is strongly essential and $A'_i \subseteq A_i$ of size κ then A'_i is strongly essential in the fat pair A'_0, A'_1;

(6) K_n embeds into G if there are sets of vertices $\{A_i : i < n\}$ of size κ so that A_i, A_j is a fat pair with A_i being strongly essential for all $i < j < n$.

Proof. (1) and (2) are trivially true.

Now, we prove (3): suppose the statement fails and we will find $A' \in A_0, B' \in A_1$ of size κ so that $G[A', B']$ is empty. By (2), we can suppose that $|N(v) \cap A_{1-i}| < \kappa$ for all $v \in A_i$ and $i < 2$. We distinguish to cases: if κ is regular then by a straightforward transfinite induction, one picks vertices $a_\xi \in A_0$ and $b_\xi \in A_1$ so that $a_\zeta, b_\zeta \notin N(a_\xi) \cup N(b_\xi)$ if $\xi < \zeta < \kappa$. Clearly, $A' = \{a_\xi : \xi < \kappa\}$ and $B' = \{b_\xi : \xi < \kappa\}$ satisfies the requirements.

If κ is singular then note that for every $\lambda < \kappa$ and $i < 2$ there is $X \in [A_i]^\lambda$ so that $\sup\{|N(v) \cap A_{1-i}| : v \in X\} < \kappa$. Now, by induction on $\xi < \mathrm{cf}(\kappa)$ we select $X_\xi \subseteq A_0$ and $Y_\xi \subseteq A_1$ so that $\sup\{|X_\xi| : \xi < \mathrm{cf}(\kappa)\} = \sup\{|Y_\xi| : \xi < \mathrm{cf}(\kappa)\} = \kappa$ and $(X_\zeta \cup Y_\zeta) \cap (N(X_\xi) \cup N(Y_\xi)) = \varnothing$ for all $\xi < \zeta < \mathrm{cf}(\kappa)$. We let $A' = \bigcup\{X_\xi : \xi < \kappa\}$ and $B' = \{Y_\xi : \xi < \kappa\}$.

Next, we prove (4). Suppose that the choice of $i^* = 0$ and $A'_i = A_i$ fails the assumption i.e. we can find $B_i \subseteq A_i$ so that B_0 is not essential; so without loss of generality $|N(v) \cap B_1| < \kappa$ for all $v \in B_0$. Now, if the choice $i^* = 1$ and the $A'_i = B_i$ fails the assumption as well then we can find $C_i \subseteq B_i$ so that C_1 is not essential in C_0, C_1; so by further shrinking C_1, we can suppose that $|N(v) \cap C_0| < \kappa$ for all $v \in C_1$. However, now $|N(v) \cap C_i| < \kappa$ for all $v \in C_{1-i}$ for both $i = 0, 1$ which contradicts (3) as C_0, C_1 is a fat pair.

Note that (5) easily follows from the definition of being strongly essential.

Finally, we show (6) by induction on n: the case $n = 2$ is trivial. Suppose that $\{A_i : i < n\}$ satisfies the assumptions above and $n \geqslant 3$. Using the fact that A_0 is strongly essential in the pair A_0, A_i for $1 \leqslant i < n$, we find a $v_0 \in A_0$ so that $A'_i = N(v) \cap A_i$ has size κ for $1 \leqslant i < n$. Note that A'_i, A'_j is still a fat pair with A'_i being strongly essential for all $1 \leqslant i < j < n$ by (5). Now, apply the inductive hypothesis for $\{A'_i : 1 \leqslant i < n\}$ to find $v_i \in A'_i$ so that $\{v_i : 1 \leqslant i < n\}$ induces K_{n-1}. Hence, $\{v_i : i < n\}$ induces K_n. □

Proof of Theorem 2.2. Suppose that $r = \mathrm{dr}(n, m)$ and fix a balanced partition $\{V_i : i < r\}$ of a K_n-free graph G of size κ.

List $[r]^2$ as $\{\{i_k, j_k\} : k < N\}$ so that $i_k < j_k$. Now, define a sequence $W_i^{-1} \supseteq W_i^0 \supseteq \dots \supseteq W_i^k$ for $k < N$ and $i < r$ and a function $f : [r]^2 \to 3$ simultaneously as follows: let W_i^{-1} be an infinite independent subset of V_i of size κ (this exists by the Erdős-Dushnik-Miller theorem) and either

(a) $G[W_{i_k}^k, W_{j_k}^k]$ is empty and we let $f(\{i_k, j_k\}) = 2$, or
(b) $W_{i_k}^k, W_{j_k}^k$ is a fat pair, $W_{i_k}^k$ is strongly essential and we let $f(\{i_k, j_k\}) = 1$, or
(c) $W_{i_k}^k, W_{j_k}^k$ is a fat pair, $W_{j_k}^k$ is strongly essential and we let $f(\{i_k, j_k\}) = 0$.

We apply Lemma 2.4 repeatedly, so that at step $k+1$ we only shrink $W_{i_k}^k$ and $W_{j_k}^k$ to $W_{i_k}^{k+1}$ and $W_{j_k}^{k+1}$ respectively. We let $W_i = W_i^{N-1}$ for all $i < r$.

Now that the map $f : [r]^2 \to 3$ is defined, we construct a directed graph D on vertices r as follows: let $ij \in E$ if $i < j$ and $f(i, j) = 1$ and $ji \in E$ if $i < j$ and $f(i, j) = 0$. Otherwise, if $f(i, j) = 2$ then ij is not an edge. We claim that there are no transitive sets of size n in D. Indeed, if $\{i_k : k \in I\}$ is the increasing enumeration of a transitive set then simply apply Lemma 2.4 (6) to $\{W_{i_k} : k \in I\}$ to find a copy of K_n in G.

Now, apply $r = \mathrm{dr}(n, m)$: there must be an independent set $\{i_k : k \in J\}$ of size m in D which means that $\bigcup\{W_{i_k} : k \in J\}$ is the desired independent set in G.

□

Let us show the finite counterpart of Theorem 2.2:

Theorem 2.5. *Suppose that $n, m \geqslant 2$ and $\ell \geqslant 1$. Then there is a finite $N = N(n, m, \ell)$ so that for every K_n-free graph G and pairwise disjoint sets of vertices $V_i \subseteq V$ with $|V_i| \geqslant N$ for $i < r = \mathrm{dr}(n, m)$ there is an independent set A so that*

$$|\{i < r : |V_i \cap A| \geqslant \ell\}| \geqslant m.$$

In other words, if G is a K_n-free graph on at least $\mathrm{dr}(n, m) \cdot N(n, m, \ell)$ vertices and $\{V_i : i < \mathrm{dr}(n, m)\}$ is a balanced partition of $V(G)$ then we can find an independent set A so that $\{i < r : |V_i \cap A| \geqslant \ell\}$ has at least m elements.

The proof follows a standard compactness argument.

Proof. Fix $n, m \geqslant 2$ and $\ell \geqslant 1$. Suppose that the statement fails i.e. for every N there is a K_n-free graph G_N and pairwise disjoint sets of vertices $V_i^N \subseteq V(G_N)$ with $|V_i^N| \geqslant N$ for $i < r = \mathrm{dr}(n, m)$ so that

$$|\{i < r : |V_i^N \cap A| \geqslant \ell\}| < m$$

for any independent set A in G_N. We will reach a contradiction.

We can suppose that $|V_i^N| = N$ and moreover that $V(G_N) = N \cdot r$ and $V_i^N = \{t \cdot r + i : t < N\}$. Take a nonprinciple ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define a graph G with $V(G) = \mathbb{N}$ as follows: $uv \in E(G)$ iff

$$I_{uv} = \{N \in \mathbb{N} : uv \in E(G_N)\} \in \mathcal{U}.$$

Claim 2.5.1. *G is K_n-free.*

Proof. Suppose that a set of vertices X induces a copy of K_n in G. Then $I_{uv} \in \mathcal{U}$ for all $u \neq v \in X$ so

$$I = \bigcap\{I_{uv} : u \neq v \in X\} \in \mathcal{U}$$

as well; in particular, $I \neq \varnothing$. Clearly, X induces a copy of K_n in G_N whenever $N \in I$. □

Let $V_i = \{t \cdot r + i : t \in \mathbb{N}\}$ for $i < r$. By Theorem 2.2, there is an independent A^* and distinct $i_0 \dots i_{m-1}$ so that

$$|V_{i_j} \cap A^*| = \omega$$

for $j < m$. Select $A_j \in [V_{i_j} \cap A^*]^\ell$ for each $j < m$. Let $A = \bigcup\{A_j : j < m\}$.

It suffices to show the following claim in order to reach a contradiction and hence to finish the proof of the theorem.

Claim 2.5.2. *There is an N so that A is independent in G_N and*

$$|\{i < r : |V_i^N \cap A| \geqslant \ell\}| \geqslant m.$$

Proof. Note that $\mathbb{N} \setminus I_{uv} \in \mathcal{U}$ for all $u \neq v \in A$ and hence

$$J = \bigcap\{\mathbb{N} \setminus I_{uv} : u \neq v \in A\} \in \mathcal{U}$$

as well; in particular, $J \neq \varnothing$. Clearly, A is independent in G_N whenever $N \in J$. Also, $A_j \subseteq V_{i_j}$ and $N \in J$ implies that $A_j \subseteq V_{i_j}^N$ and so $\ell = |A_j| \leqslant |V_{i_j}^N \cap A|$ must hold for $j < m$. □

□

Finally, we mention that the bound $\mathrm{dr}(n, m)$ can be attained for K_n-free graphs G:

Proposition 2.6. *Suppose that $n, m \geqslant 2$. Then there is a K_n-free graph G so that $r(G, m) = \mathrm{dr}(n, m)$.*

Proof. Let $r = \mathrm{dr}(n, m) - 1$ and let D be a digraph on vertice r without transitive sets of size n and independent sets of size m. Define an r-partite graph G on classes $V_i = \{i\} \times \omega$ for $i < r$ so that $H_{\omega,\omega} \hookrightarrow\!\!\!\!\to G[V_i, V_j]$ if $ij \in E(D)$ for $i \neq j < r$ and $G[V_i, V_j]$ is empty if ij, ji are not edges. Note that if A is independent and meets both V_i and V_j in infinitely many points then $G[V_i, V_j]$ is empty and hence ij, ji are note edges in D. In turn, as D has no independent sets of size m, we cannot find an independent set A which meets m classes in infinitely many points. Hence $r(G, m) > r$.

Finally, let us prove that G is K_n-free: suppose that $v_i = (i, k_i) \in V_i$ and $\{v_i : i \in I\}$ induces K_n in G. Note that $k_i \neq k_j$ if $i \neq j \in I$. Furthermore, $k_i < k_j$ and $(i, k_i)(j, k_j) \in E(G)$ implies that $ij \in E(D)$. However this contradicts the fact that D has no transitive sets of size n.

□

3. General properties of $r(G, \cdot)$

We begin by a few simple properties of the map $m \mapsto r(G, m)$.

Observation 3.1. *If $r(G, 2)$ exists then every set of $|V|$ vertices contains an independent set of size $|V|$.*

In particular, G is $K_{|V|}$-free if $r(G, 2)$ exists. Let us proceed with a few observations on monotonicity:

Observation 3.2. *Suppose that G is any graph and $2 \leqslant m \in \mathbb{N}$.*

(1) $r(G, m) \leqslant r(G, m + 1)$;
(2) If G and H are isomorphic modulo a set of size $< \kappa = |G| = |H|$ then $r(G, m) = r(H, m)$;
(3) If H is a subgraph of G and $|G| = |H|$ then either $r(G, m) \geqslant r(H, m)$ or $r(G, m+1) \geqslant r(H, m) + 1$; in any case, $r(G, m + 1) \geqslant r(H, m)$.

Proof. (1) and (2) are trivial.

To prove (3) suppose that H is a subgraph of G and $\{V_i\}_{i<r}$ is a balanced partition of $V(H)$ for $r = r(H, m) - 1$ so that any independent set in H meets at most $m - 1$ classes in a set of size $|H|$. If $V_r = V(G) \backslash V(H)$ has size $< |V(G)|$ then $V_0 \dots V_{r-2}, V_{r-1} \cup V_r$ is a balanced partition of G witnessing $r(G, m) > r$. Hence, by (1), $r(G, m + 1) > r$ as well. If V_r has size $|V|$ then $V_0 \dots V_{r-2}, V_{r-1}, V_r$ is a balanced partition of G witnessing $r(G, m + 1) > r + 1$. □

Now, we present a simple idea to bound $r(G, m)$ from above:

Lemma 3.3. *Suppose that G is a graph on κ vertices and $V \backslash F \subseteq \bigcup_{j<t} W_j$ where $|W_j| = \kappa$ and $|F| < \kappa$, $t \in \mathbb{N}$. Then*

$$r(G, m) \leqslant \sum_{j<t} (r(G[W_j], m) - 1) + 1 \text{ for all } 2 \leqslant m \in \mathbb{N}.$$

Proof. Let $V \backslash F = \bigcup_{j<t} W_j$ as above. Let $r = \sum_{j<t} (r(G[W_j], m) - 1) + 1$ and take any balanced r-partition of $V = \bigcup\{V_i : i < r\}$. We claim that there is a $j < t$ such that $I_j = \{i < r : |V_i \cap W_j| = \kappa\}$ has at least $r(G[W_j], m)$ elements. Indeed, given $i < r$, $|F| < |V_i| = \kappa$ and $V_i \subseteq \bigcup_{j<t} W_j \cup F$ implies that there is $j < t$ so that $i \in I_j$. So

$r \subseteq \bigcup_{j<t} I_j$. In turn, if $|I_j| \leqslant r(G[W_j], m) - 1$ for all $j < t$ then $r \leqslant \sum_{j<t}(r(G[W_j], m) - 1)$ which is a contradiction.

Now suppose that $I_j = \{i_k : k < \ell\}$ has at least $r(G[W_j], m)$ elements. Let $W_{j,i} = W_j \cap V_i$ and $X = W \setminus \bigcup_{k<\ell} W_{j,i_k}$. Note that $|X| < \kappa$ and

$$W_j = W_{j,i_0} \cup \ldots W_{j,i_{\ell-2}} \cup (W_{j,i_{\ell-1}} \cup X)$$

is a balanced partition of W_j into ℓ pieces so there must be an independent set $A \subseteq W_j$ which meets at least m pieces in a set of size κ. As X has size $< \kappa$ then A must meet at least m many of the sets $W_{j,i_k} \subseteq V_{i_k}$. □

Corollary 3.4. *Suppose that G is a graph on κ many vertices. Then*

$$r(G, m) \leqslant \chi(G)(m - 1) + 1.$$

Proof. Simply note that $r(E_\kappa, m) = m$ where E_κ is the empty graph on κ vertices and apply Lemma 3.3. □

Note that the above argument actually gives

$$r(G, m) \leqslant \min\{\chi(G[V\setminus F]) : F \in [V]^{<\kappa}\}(m - 1) + 1.$$

Corollary 3.5. *Suppose that G is finite and H is countably infinite. Then*

$$r(G \otimes H, m) \leqslant |G|(r(H, m) - 1) + 1.$$

Corollary 3.6. $r(K_n \otimes E_\omega, m) = n(m - 1) + 1$ *and* $r(H_{\omega,\omega}, m) = 2(m - 1) + 1$ *for all* $n \geqslant 1, m \geqslant 2$. *Moreover, if A, B is a fat pair in a graph G then*

$$r(G[A, B], m) = 2(m - 1) + 1$$

.

Proof. $r(K_n \otimes E_\omega, m) \leqslant n(m - 1) + 1$ follows from Corollary 3.5. On the other hand, if we partition each canonical class of $K_n \otimes E_\omega$ into $m-1$ infinite pieces then we get a partition of $K_n \otimes E_\omega$ into $n(m-1)$ independent sets so that no independent set A intersects m different pieces. The same argument works for $H_{\omega,\omega}$ and the fat pair. □

Next, we show a somewhat surprising property of the function $m \mapsto r(G, m)$:

Theorem 3.7. *Suppose that G is an arbitrary infinite graph. If $r(G, m) = m$ for any $m \geqslant 3$ then $r(G, m) = m$ for all $m \geqslant 2$.*

Proof. Let us start with a lemma about fat pairs:

Lemma 3.8. *If G has no fat pairs and $r(G, 2)$ exists then $r(G, m) = m$ for all $m \geqslant 2$.*

Proof. Suppose that $\{V_i\}_{i<m}$ is a balanced partition. Find independent $V_i' \in [V_i]^{|V|}$ for each $i < m$; this can be done by Observation 3.1. Apply the fact that G has no fat pairs $\binom{m}{2}$-times to find $W_i \subseteq V_i'$ so that there is no edge between W_i and W_j if $i < j < m$. Now, $\bigcup_{i<m} W_i$ is the desired independent set. □

Finally suppose, that $r(G, m) = m$ for some $m \geqslant 3$. We claim that G cannot have any fat pairs and hence we are done by Lemma 3.8. Indeed, suppose that A, B is a fat pair; then

$$m = r(G, m) \geqslant r(G[A, B], m - 1) = 2(m - 2) + 1 = 2m - 3$$

by Observation 3.2 (3) and hence $m = 3$. If $V\setminus(A \cup B)$ has size κ then $A, B, V\setminus(A \cup B)$ is a balanced partition witnessing $r(G, 3) > 3$; a contradiction. If $V\setminus(A \cup B)$ has size $< \kappa$ then $3 = r(G, 3) = r(G[A, B], 3) = 5$ (a contradiction again). □

Hence, we showed that the reason $r(G,m)$ is bigger than m for any m is because there is a fat pair in G in which case $r(G,m) \geqslant 2m-3$ for all $2 \leqslant m \in \mathbb{N}$.

Finally, let us prove that fat pairs in countable graphs are rather easily detected; this result will be applied in the next section as well.

Lemma 3.9. *If A, B is a fat pair in a countable graph G then $H_{\omega,\omega} \hookrightarrow G[A,B]$, or $H_{\omega,\omega} \hookrightarrow G[B,A]$.*

Proof. We define disjoint finite $E_0, E_1 \dots \subseteq A$ and $F_0, F_1 \dots \subseteq B$ along with infinite $A = A_{-1} \supseteq A_0 \supseteq A_1 \dots$ and $B = B_{-1} \supseteq B_0 \supseteq B_1 \dots$ as follows:

(i) $G[E_n, F_n]$ is independent and $E_n \cup F_n \neq \varnothing$,
(ii) $E_n \cap A_n = \varnothing$, $F_n \cap B_n = \varnothing$,
(iii) if $E_n \neq \varnothing$ then there is $u_n \in E_n$ such that $B_n \subseteq N(u_n) \cap B_{n-1}$,
(iv) if $F_n \neq \varnothing$ then there is $v_n \in F_n$ such that $A_n \subseteq N(v_n) \cap A_{n-1}$.

Given A_n and B_n we inductively select distinct $x_0 \in A_n$, $y_0 \in B_n$, $x_1 \in A_n$, $y_1 \in B_n \dots$ so that $G[\{x_k : k < n\}\{y_k : k < n\}]$ is the empty graph and $A_n \backslash \bigcup\{N(y_k) : k < n\}$ and $B_n \backslash \bigcup\{N(x_k) : k < n\}$ are both infinite. This process must stop at some point as A, B is a fat pair. If n is minimal so that we can't choose x_n then $N(x) \cap B_n$ is infinite for any $x \in A_n \backslash \{x_k : k < n\}$. We pick any $u_{n+1} \in A_n \backslash \{x_k : k < n\}$ and let $E_{n+1} = \{u_{n+1}\}$ and $F_{n+1} = \varnothing$, and we define $B_{n+1} = N(u_{n+1}) \cap B_n$ and $A_{n+1} = A_n \backslash E_{n+1}$.

If we can choose x_n but n is minimal so that we can't choose y_n then $N(y) \cap A_n$ must be infinite for any $y \in B_n \backslash \{x_k : k < n\}$. We finish the proof as before but now $F_{n+1} \neq \varnothing$.

Suppose we defined these sequences. If $E_n \neq \varnothing$ for infinitely many n then $H_{\omega,\omega} \hookrightarrow G[A,B]$ and if $F_n \neq \varnothing$ for infinitely many n then $H_{\omega,\omega} \hookrightarrow G[B,A]$. □

The case for uncountable graphs is much more subtle: let $f : [\omega_1]^2 \to 2$ be J. Moore's L-space colouring [15] and define a bipartite graph G by letting $V(G) = \omega_1 \times 2$ with $(\alpha, i)(\beta, j) \in E(G)$ iff $\alpha < \beta$, $i = 0, j = 1$ and $f(\alpha, \beta) = 1$. Recall that f has the property that whenever X, Y are uncountable subsets of ω_1 and $i < 2$ then there is $\alpha \in X, \beta \in Y$ so that $\alpha < \beta$ and $f(\alpha, \beta) = i$. Hence, the half graph on ω_1 does not embed into G while A, B is still a fat pair.

4. Henson's K_n-free graphs

The first graphs we look at in detail are H_n: the countable, universal K_n-free graphs defined by Henson [12]. Aside from H_n being homogenous, we mention the following properties for further reference:

(1) for any finite set of vertices A and finite K_{n-1}-free B in H_n there is a vertex v so that $A \cap N(v) = \varnothing$ and $B \subseteq N(v)$ [12];
(2) $H_n \to (H_n)^1_r$ for any $r < \omega$ [13, 2];
(3) if $vw \notin E(H_n)$ then v and w has infinitely many common neighbours;
(4) if $v \in V(H_{n+1})$ then the graph induced by $N_{H_{n+1}}(v)$ in H_{n+1} is isomorphic to H_n.

We remark that H_n is the unique countable graph satisfying property (1) [12]. Let us state a simple lemma as well:

Lemma 4.1. *H_{n+1} is not covered by finitely many subgraphs of H_n.*

Proof. Suppose that $f : V(H_{n+1}) \to r$ is a colouring so that $H_{n+1}[f^{-1}(i)]$ is a subgraph of H_n. Now, by the partition property (2), there is an $i < r$ so that H_{n+1} embeds into

$H_{n+1}[f^{-1}(i)]$. However, H_{n+1} is not a subgraph of H_n since K_n embeds into H_{n+1} but not into H_n. □

Observation 4.2. $r(G,m) \leqslant r(H_n, m+1) - 1$ *for any countably infinite K_n-free graph G.*

Proof. Suppose that G is an infinite K_n-free graph with a balanced partition $\{V_i : i < r\}$ where $r = r(H_n, m+1) - 1$. Using the universality of H_n, embed G into H_n with a map f so that $V(H_n)\backslash\mathrm{ran}(f)$ is infinite. Let $W_i = f[V_i]$ for $i < r$ and $W_r = V(H_n)\backslash\mathrm{ran}(f)$. Now, $\{W_i : i \leqslant r\}$ is a balanced $r+1 = r(H_n, m+1)$-partition of H_n so there is an independent set A such that $\{i \leqslant r : |W_i \cap A| = \omega\}$ has at least $m+1$ elements. Hence

$$|\{i < r : |W_i \cap A| = \omega\}| \geqslant m.$$

Now $B = \bigcup\{f^{-1}(A \cap W_i) : i < r\}$ is the independent set of G which meets at least m classes of the orginal partition $\{V_i : i < r\}$. □

Next, we determine the exact values $r(H_n, m)$ for all m using the Ramsey numbers $\mathrm{dr}(n,m)$. In particular, we show $r(H_n, 2) = 2$ for all $n \geqslant 3$ which answers Conjecture 46 of Thomassé [17].

Theorem 4.3. $r(H_n, m) = \mathrm{dr}(n, m-1) + 1$ *for all $n, m \geqslant 2$.*

Now, using that $\mathrm{dr}(n,1) = 1$, we get:

Corollary 4.4. $r(H_n, 2) = 2$ *for all $n \geqslant 2$.*

We mention here that Theorem 2.2 for countable graphs G is now an easy corollary of Theorem 4.3: $r(H_n, m) = \mathrm{dr}(n, m-1) + 1$ together with the above observation on $r(G,m) \leqslant r(H_n, m+1) - 1$ yields $r(G,m) \leqslant \mathrm{dr}(n,m)$.

Now, we prove Theorem 4.3; recall that $\alpha(D) = \sup\{|A| : A \subseteq V(D) \text{ is independent}\}$.

Lemma 4.5. *Suppose $n \geqslant 3$ and D is a finite digraph on r vertices. If D has no transitive sets of size n then there is a balanced partition of H_n into $r+1$ classes so that every independent set is contained in the union of $\leqslant \alpha(D) + 1$ classes modulo finite.*

Proof. Suppose that D has vertices $\{w_i : i < r\}$ and consider the graph G on vertices $\bigcup\{\{w_i\} \times \mathbb{N} : i < r\}$ so that $(w_i, k)(w_j, \ell) \in E(G)$ iff $ij \in E(D)$ and $k < \ell \in \mathbb{N}$. Note that $G[\{w_i\} \times \mathbb{N}]$ is empty for all $i < r$.

Claim 4.5.1. *G is K_n-free.*

Indeed, suppose that $\{(w_i, k_i) : i \in I\}$ is a copy of K_n for some $I \subseteq r$ and $k_i \in \mathbb{N}$. Note that $k_i \neq k_j$ if $i \neq j \in I$. Furthermore, $k_i < k_j$ and $(w_i, k_i)(w_j, k_j) \in E(G)$ implies that $ij \in E(D)$. However this contradicts the fact that D has no transitive sets of size n.

Claim 4.5.2. *Given a countable K_n-free G with an arbitrary partition A, B there is a balanced partition V_0, V_1 of H_n so that $G[A : B] \hookrightarrow H_n[V_0 : V_1]$ as an induced subgraph (for any $n \geqslant 3$).*

Proof. Extend $V(G)$ by an infinite set of new vertices $W = \{v_\ell : \ell \in \mathbb{N}\}$. List all pairs (a, b) of finite subsets of $V = W \cup V(G)$ as $\{(a_k, b_k) : k \in \mathbb{N}\}$. Inductively add edges as follows: at step k, if b_k is K_{n-1}-free then take a so far isolated vertex $v_{l_k} \in W$ and connect with all points in b_k. This process guarantees that the graph spanned by $a_k \cup b_k \cup \{v_{l_k}\}$ does not change after step k and, after ω steps, we have a graph on vertices V so that the extension property (1) of H_n is satisfied. Hence the graph constructed on V is isomorphic to H_n. Finally, we let $V_0 = W \cup A$ and $V_1 = B$. □

We only use the fact now that G embeds into H_n and we identify G and its copy in H_n. Consider the $r+1$-partition $V(H_n)\setminus V(G), \{w_0\} \times \mathbb{N} \ldots \{w_{r-1}\} \times \mathbb{N}$ of $V(H_n)$. If A is an independent set then $J = \{i < r : |A \cap (\{w_i\} \times \mathbb{N})| = \omega\}$ has size $\leqslant \alpha(D)$. Indeed, if $i \neq j \in J$ then $ij, ji \notin E(D)$, and hence $\{w_i : i \in J\}$ is an independent set in D.

So A meets at most $\alpha(D) + 1$ members of the partition in an infinite set. □

Proof of Theorem 4.3. First, we show $r(H_n, m) \geqslant \mathrm{dr}(n, m-1) + 1$. Let D be a digraph on $r = \mathrm{dr}(n, m-1) - 1$ vertices without transitive sets of size n and independent sets of size $m-1$ i.e. $\alpha(D) \leqslant m-2$. Now apply Lemma 4.5 to find a partition of H_n into $r+1 = \mathrm{dr}(n, m-1)$ classes so that evey independent set is contained in $\leqslant \alpha(D)+1 \leqslant m-1$ classes. This partition witnesses $r(H_n, m) \geqslant \mathrm{dr}(n, m-1) + 1$.

Now, we prove that $r(H_n, m) \leqslant \mathrm{dr}(n, m-1) + 1$. Let $\{V_i : i \leqslant r\}$ denote a balanced partition of H_n where $r = \mathrm{dr}(n, m-1)$. We can suppose that there is $W \subseteq V_r$ so that $H_n[W]$ is isomorphic to H_n by property (2). Now, find infinite $W_r \subseteq W$ and $W_i \subseteq V_i$ for $i < r$ so that there are no edges from W_i to W_r; this can be done by successively picking vertices and applying

Claim 4.6. *$W \setminus \bigcup\{N(v) : v \in F\}$ is infinite for all finite $F \subseteq V(H_n)$.*

Proof. Indeed, this follows from Lemma 4.1 and the fact that $N(v) \cap W$ is a subgraph of H_{n-1} by property (4).

□

By shrinking each W_i, we can suppose that $H_n[W_i]$ is empty for $i \leqslant r$. By successively applying Lemma 3.9 and shrinking W_i for $i < r$, we can suppose that either $H_n[W_i, W_j]$ is empty or $H_{\omega,\omega} \hookrightarrow\!\!\!\!\rightarrow H_n[W_i, W_j]$ or $H_{\omega,\omega} \hookrightarrow\!\!\!\!\rightarrow H_n[W_j, W_i]$ for all $i < j < r$.

Next, define a digraph D on vertice r so that $\vec{ij} \in E(D)$ iff $H_{\omega,\omega} \hookrightarrow\!\!\!\!\rightarrow H_n[W_i, W_j]$ for all $i \neq j < r$. As D has $\mathrm{dr}(n, m-1)$ many vertices, we can either find an independent set of size $m-1$ or a transitive set of size n. As the second alternative must fail by Lemma 2.4 (6), there is $I \subseteq r$ of size $m-1$ so that $H_n[W_i, W_j]$ is empty if $i \neq j \in I$. Hence, $\bigcup\{W_i : i \in \{r\} \cup I\}$ is the desired independent set. □

Next, we prove a result, one that implies $r(H_n, 2) = 2$, which will be applied in the proof of Theorem 5.3 later:

Theorem 4.7. *Fix $n \geqslant 3$ and let $V_0 \cup V_1$ be a balanced partition of the vertices of H_n. Then there is an induced copy of H_{n-1} intersecting both V_0 and V_1 in an infinite set*

It is clear that $r(H_n, 2) = 2$ follows by induction on n.

Proof of Theorem 4.7. Let $\{V_i : i < 2\}$ be a balanced partition of the vertices of $G = H_{n+1}$ for some $n \geqslant 2$. Recall that $N(v)$ induces a subgraph isomorphic to H_n for every vertex v. So, without loss of generality, we can suppose that there is $j_v < 2$ so that $N(v) \subseteq^* V_{j_v}$ for every vertex v.

Claim 4.7.1. *If $uv \notin E$ then $j_v = j_u$.*

Indeed, if $uv \notin E$ then $N(u) \cap N(v) \subseteq^* V_{j_v}$ is infinite by (3) and hence $N(u) \cap V_{j_v}$ is infinite as well. This case, $N(u) \subseteq^* V_{j_u}$ so $j_v = j_u$.

Now, let $i_v < 2$ denote the class of v i.e. $v \in V_{i_v}$. The map $v \mapsto (i_v, j_v)$ is a 4-colouring of the vertices of H_{n+1} and so by (2) we can find a set of vertices W_0 and $(i, j) \in 2 \times 2$ so that $(i_v, j_v) = (i, j)$ for all $v \in W_0$ and W_0 induces a copy of H_{n+1}. Note that $i \neq j$ would imply that every vertex in W_0 has finite degree in $H_{n+1}[W_0]$ which contradicts that $H_{n+1}[W_0]$ is H_{n+1}. Hence $i = j$ and let us suppose that this common value is 0.

Claim 4.7.2. *$N(u) \subseteq^* V_0$ for almost every vertex $u \in V_1$ and every $u \in V_0$.*

Proof. We would like to show first that $j_u = 0$ for almost every $u \in V_1$. Fix an arbitrary $v \in W$. Then $j_v = 0$ so $uv \notin E(H_n)$ for almost every $u \in V_1$. Hence, $j_u = 0$ for almost every $u \in V_1$ by Claim 4.7.1.

Now, take $u \in V_0$ and suppose that $N(u) \subseteq^* V_1$. As $j_v = 0$ for all $v \in W$, we must have $W \subseteq N(u)$ by Claim 4.7.1. However, W induces a copy of H_{n+1} so it cannot be covered by a copy of H_n.

□

Without loss of generality, we can assume that $N(u) \subseteq^* V_0$ for every vertex $u \in V_1$.

Claim 4.7.3. *(1) $V_1 \setminus \bigcup\{N(v) : v \in F\}$ is infinite for any finite set of vertices F.*
(2) $V_0 \cap \bigcap\{N(v) : v \in F_0\} \setminus \bigcup\{N(v) : v \in F_1\}$ is infinite for any finite, nonempty F_0 which induces a K_{n-1}-free subgraph and any finite F_1.

Proof. (1) $N(v) \subseteq^* V_0$ implies that $V_1 \cap N(v)$ is finite so $V_1 \setminus \bigcup\{N(v) : v \in F\}$ is infinite for any finite set of vertices F.

(2) Note that $\bigcap\{N(v) : v \in F_0\} \setminus \bigcup\{N(v) : v \in F_1\}$ is infinite by property (1) and that

$$\bigcap\{N(v) : v \in F_0\} \setminus \bigcup\{N(v) : v \in F_1\} \subseteq N(u) \subseteq^* V_0$$

for any $u \in F_0$. This proves that $V_0 \cap \bigcap\{N(v) : v \in F_0\} \setminus \bigcup\{N(v) : v \in F_1\}$ is infinite. □

Now, take an enumeration $x_0, x_1 \ldots$ of the vertices of H_n so that $I = \{i \in \mathbb{N} : x_j \notin N(x_i)$ for all $j < i\}$ is infinite. It suffices to construct an embedding $f : H_n \to H_{n+1}$ as $x_i \mapsto y_i$ so that $i \in I$ if and only if $y_i \in V_1$.

Let $f(x_0) = y_0 \in V_1$ arbitrary. Now, given y_i for $i < k$, we consider two cases: if $k \in I$ then simply find $y_k \in V_1 \setminus \bigcup\{N(y_i) : i < k\}$ so that $y_k \neq y_i$ for $i < k$ by applying Claim 4.7.3 (1). If $k \in \mathbb{N} \setminus I$ then let $F_0 = \{y_i : i < k, x_i \in N(x_k)\}$ and $F_1 = \{y_i : i < k, x_i \notin N(x_k)\}$. Now, find $y_k \in V_0 \cap \bigcap\{N(v) : v \in F_0\} \setminus \bigcup\{N(v) : v \in F_1\}$ so that $y_k \neq y_i$ for $i < k$ by applying Claim 4.7.3 (2).

□

Finally, let us mention that any kind of analogue statement for the Rado graph R fails, and in particular $r(R, 2)$ does not exist.

Proposition 4.8. *There is a balanced 2-partition of the vertices of the Rado graph R so that any infinite independent or infinite complete subgraph is modulo finite contained in one class.*

Proof. We start from the infinite half graph $G_0 = (\omega \times \{0\} \cup \omega \times \{1\}, E)$ where

$$\{(n, i), (m, j)\} \in E \iff i = 0, j = 1 \text{ and } n < m < \omega.$$

Let $V = \omega \times \{0\} \cup \omega \times \{1\}$ and $V_i = \omega \times \{i\}$. Let us enumerate all pairs (a, b) of finite subsets of V as $\{(a_n, b_n) : n < \omega\}$. Let

$$m_n = \max\{k : (k, i) \in a_n \cup b_n \text{ for some } i\} + m_{n-1}.$$

Now define G on vertices V so that

$$E(G) = E(G_0) \cup \{\{(m_n, 0), v\} : v \in a_n, n < \omega\}.$$

First, note that G satisfies Rado's extension property and so G is the Rado graph. Second, the partition $V_0 \cup V_1$ witnesses the theorem; indeed, $G[V_1]$ is independent so every complete subgraph intersects V_1 in at most 1 vertex. If A is an independent set and $(n, 0) \in A$ then $A \cap V_1 \subseteq \{(k, 1) : k < n\}$.

□

5. Balanced embeddings in H_n

Recall, that in Claim 4.5.2 we showed that whenever G was a countable bipartite graph on classes A, B then then we could find a balanced partition V_0, V_1 of H_n so that $G[A, B] \hookrightarrow H_n[V_0, V_1]$.

Now, we are interested if the following stronger property is satisfied: given G on classes A, B and an *arbitrary balanced partition* V_0, V_1 of H_n is there an $i < 2$ so that $G[A : B] \hookrightarrow H_n[V_i : V_{1-i}]$? If the answer is yes, then we will write write

$$H_n \xrightarrow[ind]{bal} (G)^1_2$$

while the negation will be denoted by $H_n \overset{bal}{\underset{ind}{\not\longrightarrow}} (G)^1_2$.

For example, $r(H_n, 2) = 2$ is equivalent to the above statement for $H_n \xrightarrow[ind]{bal} (E_{\omega,\omega})^1_2$. In Theorem 4.7, we strengthened this by proving that

$$H_n \xrightarrow[ind]{bal} (H_{n-1})^1_2$$

for all $n \geqslant 3$.

There are some easy limitation on the type of results we can hope to prove concerning the partition relation $H_n \xrightarrow[ind]{bal} (G)^1_2$.

Observation 5.1. $H_n \overset{bal}{\underset{ind}{\not\longrightarrow}} (K_{\omega,\omega})^1_2$.

Proof. Apply the proof of Claim 4.5.2 starting with $G = E_{\omega,\omega}$. The inductive construction carried out there gives a balanced partition of H_n with V_0, V_1 so that $N(v_0) \cap V_1$ is finite for all $v \in V_0$. Hence, any copy of $K_{\omega,\omega}$ is modulo finite contained in V_0. □

Let us also remark that the above partition shows why we allow embeddings into $H_n[V_0 : V_1]$ and $H_n[V_1 : V_0]$ at the same time. Indeed, $H_{\omega,\omega} \not\hookrightarrow H_n[V_0 : V_1]$ in the previous example but $H_{\omega,\omega} \hookrightarrow H_n[V_1 : V_0]$. In fact, we have the following:

Theorem 5.2. *If $n \geqslant 3$ then*

$$H_n \xrightarrow[ind]{bal} (H_{\omega,\omega})^1_2.$$

Proof. Fix a balanced partition V_0, V_1 of H_n. By $r(H_n, 2) = 2$, there are infinite $X = \{x_k : k \in \mathbb{N}\} \subseteq V_0$, $Y = \{y_k : k \in \mathbb{N}\} \subseteq V_1$ so that $X \cup Y$ is independent. Let $F_k = \{x_\ell, y_\ell : \ell \leqslant k\}$ and note that for every $k \in \mathbb{N}$ there is a $j_k \in 2$ so that H_n embeds into $N[F_k] \cap V_{j_k}$; here, $N[F] = \bigcap\{N(v) : v \in F\}$. In particular, there is a single $j \in 2$ and infinite $I \subseteq \mathbb{N}$ so that $j_k = j$ whenever $j \in I$. Without loss of generality, we suppose that $j = 1$.

Select a decreasing sequence $W_k \subseteq N[F_k] \cap V_k$ so that $H_n[W_k]$ is isomorphic to H_n. First, try to select $k_0, k_1 \dots \in I$ and $w_0 \in W_{k_0}, w_1 \in W_{k_1} \dots$ so that $\{x_{k_i}, w_i : i \in \mathbb{N}\}$ induces a copy of $H_{\omega,\omega}$. We do this while making sure that $\{x_k : k \in I\} \setminus \bigcup\{N(w_{i'}) : i' < i\}$ is infinite which ensures that the next x_{k_i} can be selected.

Given $x_{k_0}, w_0 \dots x_{k_i}$ note that $W_{k_i} \setminus \bigcup\{N(w_{i'}) : i' < i\}$ still contains a copy of H_n by Lemma 4.1. So, if we can find $w_i \in W_{k_i} \setminus \bigcup\{N(w_{i'}) : i' < i\}$ so that $\{x_k : k \in I\} \setminus \bigcup\{N(w_{i'}) : i' \leqslant i\}$ is still infinite then we can continue to select $x_{k_{i+1}}$ and we construct the desired copy of $H_{\omega,\omega} \hookrightarrow H_n[V_0, V_1]$.

Otherwise, there is some i and a copy $W \subseteq V_1$ of H_n so that the infinite independent set $A = \{x_k : k \in I\} \setminus \bigcup\{N(w_{i'}) : i' < i\}$ is modulo finite covered by $N(w)$ whenever $w \in W$. We claim that $H_{\omega,\omega} \hookrightarrow H_n[V_1, V_0]$ holds in this case.

Indeed, start selecting distinct $w_0 \in W, v_0 \in A, w_1 \in W, v_1 \in A \dots$ so that

$$v_k \in \bigcap\{N(w_\ell) : \ell \leqslant k\} \setminus \{v_\ell : \ell < k\} \tag{5.1}$$

and

$$w_{k+1} \in W \setminus \bigcup\{N(w_\ell), N(v_\ell), \{w_\ell\} : \ell \leqslant k\}. \tag{5.2}$$

Note that (5.1) is possible as $A \subseteq^* N(w_\ell)$ and (5.2) is possible by Lemma 4.1 and the fact that W is a copy of H_n. Now, $\{w_k, v_k : k \in \mathbb{N}\}$ is the desired copy of $H_{\omega,\omega}$.

□

The main result of this section is

Theorem 5.3. *Suppose that G is a bipartite graph on classes A, B and A is finite. If $n \geqslant 3$ then*

$$H_n \xrightarrow[ind]{bal} (G)^1_2.$$

Our proof will make use of Theorem 4.7 i.e. the strong form of $r(H_n, 2) = 2$ as well as the multi-dimensional Hales-Jewett theorem [9] (with dimension n, and size of alphabet and number of colours 2) which we state here:

Lemma 5.3.1. *Given $\ell \in \mathbb{N}$ there is $N \in \mathbb{N}$ so that if the set of functions from N to 2 is partitioned as $\mathcal{F}_0 \cup \mathcal{F}_1$ then there is $i < 2$, a set $T = \{t_k : k < \ell\} \subseteq N$ of size ℓ and function $h : N \setminus T \to 2$ so that $h \cup g \in \mathcal{F}_i$ for any $g : T \to 2$.*

Proof of Theorem 5.3. Fix G on classes A, B. We will show $H_3 \xrightarrow[ind]{bal} (G)^1_2$. Then, using Theorem 4.7 and induction on n, the general result follows.

Suppose that G is on classes $A = \{0\} \times \ell$ and $B = \{1\} \times \mathbb{N}$ where $\ell \in \mathbb{N}$. Fix a balanced partition V_0, V_1 of H_3 as well. Our goal is to find $i < 2$ and independent $A' \subseteq V_i, B' \subseteq V_{1-i}$ so that $G[A, B] \hookrightarrow\!\!\!\!\rightarrow H_3[A', B']$.

First, given the number ℓ, the Hales-Jewett theorem provides $N \in \mathbb{N}$ as in Lemma 5.3.1. Now, by $r(H_3, 2) = 2$, there is $X = \{x_0 \dots x_{N-1}\} \in [V_0]^N$, $Y = \{y_0 \dots y_{N-1}\} \in [V_1]^N$ and $v^* \in V(H_n) \setminus (X \cup Y)$ so that $X \cup Y$ is independent and there are no edges between v^* and $X \cup Y$.

Let us define a partition $\mathcal{F}_0, \mathcal{F}_1$ of all functions $f : N \to 2$. We let $f \in \mathcal{F}_i$ if $i < 2$ is minimal so that

$$\{v \in V_i : v^* \in N(v), x_k, y_k \in N(v) \text{ if } f(k) = 0, x_k, y_k \notin N(v) \text{ if } f(k) = 1\}$$

is infinite. Let us denote this set by $Z(f)$. This partition is well defined by the extension property of H_n.

Claim 5.3.1. *$Z(f) \cup Z(f')$ is an independent set for any $f, f' \in 2^N$.*

Indeed, if $v \in Z(f)$ and $v' \in N(v) \cap Z(f')$ then $\{v^*, v, v'\}$ would induce a copy of K_n in H_n.

By the choice of N, we can find $i < 2$, a set $T = \{t_k : k < \ell\} \subseteq N$ of size ℓ and $h : N \setminus T \to 2$ so that $h \cup g \in \mathcal{F}_i$ for all $g : T \to 2$.

We are ready to define A' and B'. Let $A' = \{u_k : k < \ell\}$ where

$$u_k = \begin{cases} x_{t_k}, & \text{if } i = 1, \\ y_{t_k}, & \text{if } i = 0. \end{cases}$$

Clearly, A' is an independent subset of V_{1-i}. To define $B' \subseteq V_i$, we first define functions $g_m : T \to 2$ by letting $g_m(t_k) = 0$ iff $(0,k)(1,m) \in E(G)$ for all $m \in \mathbb{N}$. Let us pick $v_m \in Z(h \cup g_m)$ so that $v_m \neq v_{m'}$ for $m' < m \in \mathbb{N}$; this can be done as each $Z(h \cup g_m)$ is infinite. We let $B' = \{v_m : m \in \mathbb{N}\}$ and note that B' is independent by Claim 5.3.1. We remark that the only role of v^* was to force B' independent via Claim 5.3.1.

We finish the proof of the theorem by

Claim 5.3.2. *The map $(0,k) \mapsto u_k$ (for $k < \ell$) and $(1,m) \mapsto v_m$ (for $m \in \mathbb{N}$) witnesses $G[A,B] \hookrightarrow\!\!\!\!\rightarrow H_n[A',B']$.*

Indeed, $v_m u_k$ is an edge in H_n iff $(h \cup g_m)(t_k) = 0$ iff $g_m(t_k) = 0$ iff $(0,k)(1,m)$ is an edge in G.

□

At this point, it would be natural to look at embeddings of graphs G into H_n where $\chi(G) \geqslant 3$ i.e. to prove

$$H_n \xrightarrow[ind]{bal} (G)^1_{\chi(G)}.$$

The following observation shows that we will not succeed in proving any positive results in this direction:

Observation 5.4. *Let $n \geqslant 3$ and $2 \leqslant r \in \mathbb{N}$. Then there is a balanced r-colouring of H_n so that any 3-cycle of H_n is coloured with at most 2 colours.*

Indeed, we can find a balanced r-partition $\{V_i : i < r\}$ of H_n so that $H_n[\bigcup\{V_i : 1 \leqslant i < r\}]$ is empty; for details, see Lemma 4.5. However, it would still be interesting to see what graphs H can satisfy the property:

$$H \xrightarrow[ind]{bal} (C_3)^1_r$$

for some $2 \leqslant r \in \mathbb{N}$.

6. Finding the exact value of $r(G, \cdot)$ for specific graphs

Next, we present a few further results (and attempts) on finding the exact values of the function $m \mapsto r(G,m)$ for specific K_n-free graphs G. These examples include shift graphs, unit distance graphs and orthogonality graphs. We begin by a new definition

Definition 6.1. Let $r^*(G,m) = \min\{r :$ if $V_i \in [V]^{|V|}$ for $i < r$ then there is an independent set A so that $|\{i < r : |A \cap V_i| = |V|\}| \geqslant m\}$.

In general, the following holds:

Observation 6.2. *Fix any graph G.*

(1) *$r(G,m) \leqslant r^*(G,m) \leqslant r(G,m+1) - 1$ for any $m \geqslant 2$;*
(2) *$r^*(G,2) = 2$ implies that $r^*(G,m) = r(G,m) = m$ for all $m \geqslant 2$.*

6.1. **Shift graphs.** Recall that $Sh_n(\kappa)$ denotes the graph on vertices $[\kappa]^n$ so that $pq \in E$ iff $p = \{\xi_0 \dots \xi_{n-1}\}$ and $q = \{\xi_1 \dots \xi_n\}$ for some increasing sequence $\xi_0 < \xi_1 < \dots < \xi_n$ from κ. Our main result on shift graphs is the following:

Theorem 6.3. *$r^*(Sh_n(\kappa), m) = m$ for all $n < \omega$, infinite κ and $m < \mathrm{cf}(\kappa)$.*

We need the following well known Δ-system lemma [14]:

Lemma 6.3.1. *Suppose that κ is a regular infinite cardinal and $n \in \mathbb{N}$. If $V \subseteq [\kappa]^n$ is of size κ then there is a Δ-system $W \subseteq V$ of size κ i.e. there is some r (called the* root *of W) so that $a \cap b = r$ for all $a \neq b \in W$.*

We say that $p, q \subseteq \kappa$ are *strongly disjoint* if $\max(p) < \min(q)$ or $\max(q) < \min(p)$. We prove the theorem now:

Proof of Theorem 6.3. Fix $m < \mathrm{cf}(\kappa), 2 \leqslant n \in \mathbb{N}$ and $V_i \subseteq [\kappa]^n$ of size κ for $i < m$.

First, suppose that κ is regular and pick Δ-systems $W^i \in [V_i]^\kappa$ with root r^i for $i < m$. By shrinking W_i appropriately, we can suppose that there is a $\delta < \kappa$ so that

(1) $\sup\{\max r^i : i < m\} \subseteq \delta$, and
(2) $\{p \backslash r^i : p \in W^i, i < m\}$ is strongly pairwise disjoint and contained in $\kappa \backslash \delta$.

We claim that $A = \bigcup\{W^i : i < m\}$ is the desired independent set. Indeed, if $p \neq q \in A$ then $p \cap q \subseteq \delta$ and $p \backslash \delta < q \backslash \delta$ or $q \backslash \delta < p \backslash \delta$. In any case, pq cannot be an edge.

Now, suppose that κ is singular. Apply Lemma 6.3.1 to find Δ-systems $W^i_\varepsilon \subseteq V_i$ of size κ_ε with root r^i_ε for each $i < m$ where $(\kappa_\varepsilon)_{\varepsilon < \mathrm{cf}(\kappa)}$ is cofinal in κ. Let

$$I = \{i < m : \sup_{\varepsilon < \mathrm{cf}(\kappa)} (\max r^i_\varepsilon) < \kappa\}.$$

We can suppose that $\sup \bigcup W^i_\varepsilon < \kappa$ for all $(\varepsilon, i) \in \mathrm{cf}(\kappa) \times m$. Finally, let $\delta < \kappa$ be an upper bound for all $\sup_{\varepsilon < \mathrm{cf}(\kappa)}(\max r^i_\varepsilon)$ where $i \in I$.

Our goal is to define $U^i_\varepsilon \subseteq V_i$ for $(\varepsilon, i) \in \mathrm{cf}(\kappa) \times m$ by induction on the lexicographical order $<_{lex}$ so that

(1) $|U^i_\varepsilon| = \kappa_\varepsilon$,
(2) $\sup(\bigcup U^i_\varepsilon) < \kappa$,
(3) $a \backslash \delta \neq \emptyset$ for all $a \in U^i_\varepsilon$ and
(4) $ab \notin E(Sh_n(\kappa))$ if $a \in U^i_\varepsilon, b \in U^j_{\varepsilon'}$ for $i < j < m$ and $\varepsilon, \varepsilon' < \mathrm{cf}(\kappa)$.

If we succeed then we can find $A_i \subseteq \bigcup\{U^i_\varepsilon : \varepsilon < \mathrm{cf}(\kappa)\}$ of size κ which is independent (using the Erdős-Dushnik-Miller theorem again) and then $A = \bigcup\{A_i : i < m\}$ is the desired independent set which meets each V_i in a set of size κ.

Suppose that U^i_ε is defined already for $(\varepsilon, i) <_{lex} (\varepsilon^*, j)$. Let

$$\lambda = \sup\{\delta, \sup(\bigcup U^i_\varepsilon) : (\varepsilon, i) <_{lex} (\varepsilon^*, j)\}$$

and note that $\lambda < \kappa$.

If $j \in I$ then find $\varepsilon^* \leqslant \gamma < \mathrm{cf}(\kappa)$ and $U^j_{\varepsilon^*} \subseteq W^j_\gamma$ of size κ_{ε^*} so that $b \backslash r^j_\gamma \cap \lambda = \emptyset$ for all $b \in U^j_{\varepsilon^*}$. Note that if $a \in U^i_\varepsilon, b \in U^j_{\varepsilon^*}$ then $a \cap b \subseteq \delta$ and both $a \backslash \delta$ and $b \backslash \max a$ are not empty. Hence ab is not an edge.

If $j \in m \backslash I$ then find $\varepsilon^* \leqslant \gamma < \mathrm{cf}(\kappa)$ and $U^j_{\varepsilon^*} \subseteq W^j_\gamma$ of size κ_{ε^*} so that both $r^j_\gamma \backslash \lambda$ and $b \backslash (r^j_\gamma \cup \lambda)$ are non empty for all $b \in U^j_{\varepsilon^*}$. Note that if $a \in U^i_\varepsilon, b \in U^j_{\varepsilon^*}$ then $2 \leqslant |b \backslash \max a|$ and hence ab is not an edge.

□

6.2. **Unit distance graphs.** Given a metric space (V, d) one defines the *unit distance graph* G corresponding to (V, d) on vertices V and $xy \in E(G)$ iff $d(x, y) = 1$.

Proposition 6.4. *$r^*(G, m) = m$ for all m where G is the unit distance graph on $\mathbb{R}^n$ with the usual Euclidean metric.*

Let $\mathfrak{c}$ denote the cardinality of $\mathbb{R}$. We say that $x \in \mathbb{R}^n$ is a complete accumulation point of of a set $W \subseteq \mathbb{R}^n$ if $B \cap W$ has size $\mathfrak{c}$ for any open neighbourhood of x.

Proof. It suffices to show that $r^*(G, 2) = 2$ by Observation 6.2. Suppose that $V_i \subseteq \mathbb{R}^n$ are of size $\mathfrak{c}$ for $i < 2$. Note that $r^*(G, 2) = 2$ follows from

Claim 6.4.1. *There are complete accumulation points u_i of V_i such that $|u_0 - u_1| \neq 1$.*

Indeed, if B_i is a small enough ball with radius less than 1 around u_i then $|x - y| \neq 1$ for all $x \in B_i, y \in B_j$ and $i \leqslant j < 2$; hence $A = \bigcup\{B_i \cap V_i : i < 2\}$ is the desired independent set.

We prove the claim now: let W be a maximal set of points so that $V_1' = V_1 \cap \bigcap\{N_G(u) : u \in W\}$ still has size $\mathfrak{c}$; note that W is finite. Select a complete accumulation point $u_0 \in V_0 \setminus W$ of V_0. We claim that $|V_1' \cap N_G(u_0)| < \mathfrak{c}$. Indeed, otherwise $W' = W \cup \{u_0\}$ still satisfies

$$|V_1 \cap \bigcap\{N_G(u) : u \in W'\}| = \mathfrak{c}$$

however W was already maximal.

Hence, we can select a complete accumulation point u_1 of $V_1' \setminus N_G(u_0)$. Now $|u_0 - u_1| \neq 1$ so u_0, u_1 are as desired. □

Note that $r^*(G, m) = m$ or even $r(G, m) = m$ can easily fail for other metrics which still induce the Euclidean topology; indeed, if $d(x, y) = \min\{1, |x - y|\}$ then d induces the usual topology while $K_\omega \otimes E_\omega$ embeds into the corresponding unit distance graph. In particular, already $r(G, 3)$ and $r^*(G, 2)$ does not exist. However, we still have the following:

Proposition 6.5. *For any metric that induces the usual topology on $\mathbb{R}^n$ for $n \geqslant 2$, the corresponding unit distance graph G will satisfy $r(G, 2) = 2$.*

The above proposition will be a corollary of the following more general fact.

Lemma 6.6. *Suppose that G is a graph on a separable metric space (V, d) of size $\mathfrak{c}$. If V has an open cover by G-independent sets then either*

(1) $r(G, 2) = 2$, or

(2) there is $Y \subseteq V$ of size $< |V|$ so that $V \setminus Y$ is not connected.

Proof. Suppose that (1) fails and this is witnessed by the balanced partition V_0, V_1 of V. For every $x \in V$ there is an open neighbourhood B_x of x so that B_x is independent. As B_x is independent, there must be a set Y_x of size $< |V|$ and $i_x < 2$ so that $B_x \setminus Y_x \subseteq V_{i_x}$ for every $x \in V$. Now, there is a countable set W so that $\{B_x : x \in W\}$ covers V so

$$V \setminus Y = \bigcup\{B_x \setminus Y : x \in W\}$$

where $Y = \bigcup\{Y_x : x \in W\}$. Now note that $V_i \setminus Y$ is open in $V \setminus Y$; indeed, if $z \in V_i \setminus Y$ then there is $x \in W$ so that $z \in B_x \setminus Y \subseteq V_{i_x}$ and hence $i = i_x$ and $B_x \setminus Y$ is an open neighbourhood of z in $V_i \setminus Y$. Note that $V_i \setminus Y \neq \emptyset$ as $|Y| < |V_i| = \mathfrak{c}$. Now, the clopen partition $V \setminus Y = (V_0 \setminus Y) \cup (V_1 \setminus Y)$ witnesses that $V \setminus Y$ is not connected.

□

We do need some connectivity assumption, as demonstrated by the following:

Observation 6.7. *Suppose that $X \subseteq \mathbb{R}^n$ and $\{X_k : k < \ell\}$ is a clopen partition of X into sets of size $|X|$. Then there is a metric d inducing the usual topology on X so that $r(G, 2) > \ell$ where G is the unit distance graph on (X, d).*

In particular, $r(G, 2)$ might not exists if X has infinitely many connected components of size $|X|$.

Proof. Simply find a metric d so that the diameter of each X_k is less than 1 while $d(x, y) = 1$ if $x \in X_k, y \in X_{k'}$ for some $k < k' < \ell$. The partition $\{X_k : k < \ell\}$ witnesses $r(G, 2) > \ell$. □

On the other hand, if $X \subseteq \mathbb{R}^n$ and $\{X_k : k < \ell\}$ is a cover by sets of size $|X|$ which are connected even after the removal of $< \mathfrak{c}$ points (e.g. X_k is connected and open) then $r(G, 2) \leqslant \ell + 1$ by Lemma 6.6.

6.3. **Orthogonality graphs.** Finally, let us take a look at another class of geometric graphs: let $G_{\mathbb{R}^n}$ be defined on vertices $\mathbb{R}^n \backslash \{0\}$ so that $uv \in E(G_{\mathbb{R}^n})$ iff $u \cdot v = 0$ i.e. u and v are orthogonal vectors. It is clear that $G_{\mathbb{R}^n}$ is K_{n+1}-free so $r(G_{\mathbb{R}^n}, m)$ exists for all $2 \leqslant m \in \mathbb{N}$.

Proposition 6.8. $r(G_{\mathbb{R}^n}, 2) = 2$ *for all* $n \geqslant 2$.

Proof. Recall that $\mathbb{R}^n \backslash Y$ is connected whenever Y has size $< \mathfrak{c}$. Also, for any $x \neq 0$ there is small enough open ball around x which is independent in $G_{\mathbb{R}^n}$. Hence, Lemma 6.6 can be applied. □

Unfortunately, finding $r(G_{\mathbb{R}^n}, m)$ will be much more difficult in general. First, note the following:

Proposition 6.9. $r(G_{\mathbb{R}^n}, m) = r^*(G_{\mathbb{R}^n}, m-1) + 1$ *for all* $n, m \geqslant 2$.

Proof. Let us prove first that $r(G_{\mathbb{R}^n}, m) \leqslant r = r^*(G_{\mathbb{R}^n}, m-1) + 1$. Take a balanced r-partition $\{V_i : i < r\}$; we can suppose that V_0 is dense in some n-dimensional ball by the Baire category theorem. Select $A_i \subseteq V_i$ of size $\mathfrak{c}$ for $1 \leqslant i < m$ so that $\bigcup\{A_i : 1 \leqslant i < r\}$ is independent and let $x_i \in A_i$ be complete accumulation points of A_i. Now, it is easy to see that if we take small enough balls B_i around x_i then $V_0 \backslash \bigcup\{B_i^{\perp} : i = 1 \dots m-1\}$ has size $\mathfrak{c}$. Hence, if $A_0 \subseteq V_0 \backslash \bigcup\{B_i^{\perp} : i = 1 \dots m-1\}$ is of size $\mathfrak{c}$ and independent then $A_0 \cup (A_1 \cap B_1) \cup \dots \cup (A_{m-1} \cap B_{m-1})$ is the desired independent set.

Equality now follows from Observation 6.2 (1). □

Hence finding $r(G_{\mathbb{R}^n}, m)$ and $r^*(G_{\mathbb{R}^n}, m)$ will be equally hard.

Observation 6.10. *$r^*(G_{\mathbb{R}^n}, m)$ is the minimal number $\hat{r} = \hat{r}(n, m)$ so that any $\hat{r}$ non zero vectors of $\mathbb{R}^n$ contain m pairwise non orthogonal points ($2 \leqslant n, m \in \mathbb{N}$).*

Proof. Let us show $r^*(G_{\mathbb{R}^n}, m) \leqslant \hat{r}$ first: let $V_i \subseteq \mathbb{R}^n$ be of size $\mathfrak{c}$ and pick a complete accumulation point $x_i \in V_i$ for each $i < \hat{r}$. By the definition of $\hat{r}$, $\{x_i : i \in I\}$ is pairwise non orthogonal for some set $I \subseteq \hat{r}$ of size m. If B_i is a small enough ball around x_i then $\bigcup\{V_i \cap B_i : i \in I\}$ is the desired independent set.

On the other hand, take $r^*(G_{\mathbb{R}^n}, m)$ points x_i and let V_i denote the set of nonzero scalar multiples of x_i. Now, if $A \subseteq \bigcup\{V_i : i < r^*(G_{\mathbb{R}^n}, m)\}$ is the independent set which intersects m of the sets V_i then $\{x_i : |A \cap V_i| \neq \emptyset\}$ must be pairwise non orthogonal. Hence $r^*(G_{\mathbb{R}^n}, m) \geqslant \hat{r}$ holds as well. □

In other words, the largest set A in $\mathbb{R}^n$ so that any $B \in [A]^{m+1}$ contains two perpendicular vectors has size $r^*(G_{\mathbb{R}^n}, m+1) - 1$. This number, denoted by $\alpha(n,m) = r^*(G_{\mathbb{R}^n}, m+1) - 1$ was introduced by P. Erdős and investigated by several people [16, 5, 1]. Let us summarize the known results. Erdős conjectured that $\alpha(n,m) = nm$ for all n, m (see [16, 5]) i.e $r^*(G_{\mathbb{R}^n}, m) = n(m-1) + 1$. Note that this is true if the points are in general position i.e. any k of them spans a k dimensional subspace (for $k \leqslant n$). Indeed, if A is general then we can actually extend any k-element pairwise non orthogonal set into an m element pairwise non orthogonal set. Indeed, fix n and prove by induction on m. Fix k points $x_1 \dots x_k$ which are pairwise non orthogonal; remove x_i and $x_i^\perp \cap A$ from A. Note that $|x^\perp \cap A| \leqslant n-1$ for all x in A hence we still have $n(m-k-1)+1$ points. So, we can select $m-k$ additional vectors which are pairwise non orthogonal using the inductive hypothesis.

The conjecture in general was disproved by Z. Füredi and R. Stanley [5] by showing that there are 24 vectors in $\mathbb{R}^4$ without 6 vectors being pairwise non orthogonal. The currently known best lower bound is due to N. Alon and M. Szegedy [1]: they show that there is a constant $\delta > 0$ so that

$$r^*(G_{\mathbb{R}^n}, m) > n^{\frac{\delta \log(m+1)}{\log\log(m+1)}}$$

for every large enough m and $n \geqslant 2\log m$.

On the other hand, the following upper bound is presented in [5]:

$$r^*(G_{\mathbb{R}^n}, m) \leqslant (1 + \mathrm{o}(1))\sqrt{\frac{n\pi}{8}}2^{n/2}(m-1) + 1$$

for any n, m

Hence, by Proposition 6.9, we get

Corollary 6.11.

$$n^{\frac{\delta \log(m)}{\log\log(m)}} + 1 < r(G_{\mathbb{R}^n}, m) \leqslant (1 + o(1))\sqrt{\frac{n\pi}{8}}2^{n/2}(m-2) + 2$$

Here, the lower bound holds for all large enough m and $n \geqslant 2\log(m)$; the upper bound holds for all n, m.

There is very little known about the exact values of $\alpha(n,m)$ or, equivalently, the values of $r^*(G_{\mathbb{R}^n}, m)$. Clearly, $r^*(G_{\mathbb{R}^n}, 2) = n+1$. It is easy to see that $r^*(G_{\mathbb{R}^2}, m) = 2(m-1)+1$ and it was proved by M. Rosenfeld [16] that $r^*(G_{\mathbb{R}^n}, 3) = 2n+1$; in particular, the conjecture of Erdős still holds for these cases. Now, Proposition 6.9 yields

Corollary 6.12. *$r(G_{\mathbb{R}^2}, m) = 2(m-1)$, $r(G_{\mathbb{R}^n}, 3) = n+2$ and $r(G_{\mathbb{R}^n}, 4) = 2n+2$ for all n, m.*

The smallest unknown value to us is $r(G_{\mathbb{R}^3}, 6)$ or equivalently $r^*(G_{\mathbb{R}^3}, 5)$.

Finally, we mention that the chromatic number of $G_{\mathbb{R}^3}$ is 4 while, somewhat surprisingly, $\chi(G_{\mathbb{R}^3}[\mathbb{Q}^3]) = 3$ [6]. As Corollary 3.4 can be easily extended to $r^*(G, m)$, we have $r^*(G_{\mathbb{R}^3}[\mathbb{Q}^3], m) = 3(m-1) + 1$ as predicted by the conjecture of Erdős.

7. Open problems

We close our paper with a list of open problems that we found the most interesting:

Problem 7.1. *Is there a single K_n-free graph G so that $r(G, m) = \mathrm{dr}(n, m)$ for all $2 \leqslant m \in \mathbb{N}$?*

We are not sure how fast $r(G,m)$ might grow for a fixed graph G:

Problem 7.2. *Suppose that $g:\omega \to \omega$ is monotone increasing. Is there a single graph G so that $g(m) < r(G,m) < \infty$ for all $2 \leqslant m \in \mathbb{N}$?*

Note that if $g(m) > \mathrm{dr}(n,m)$ for some $m \in \mathbb{N}$ then G cannot be K_n-free.

Now, regarding finite graphs and Theorem 2.5:

Problem 7.3. *Estimate the function $N = N(n,m,\ell)$ from Theorem 2.5.*

A natural way to strengthen Theorem 5.3 would be answering the following:

Problem 7.4. *Does $H_n \xrightarrow[ind]{bal} (G)^1_2$ hold if $n \geqslant 3$ and G is an arbitrary subgraph of $H_{\omega,\omega}$?*

We also ask if H_n is the only graph satisfying Theorem 5.3:

Problem 7.5. *Suppose that $H \xrightarrow[ind]{bal} (G)^1_2$ holds for all finite bipartite G. Is $H = H_n$ for some $n \geqslant 3$?*

Next, we mention a question of set theoretical flavour. The existence of the numbers $r(G,m)$ for a graph G of size κ clearly implies that G contains independent sets of size κ. The same conclusion follows from Hajnal's Set Mapping Theorem [8]: if $\lambda < \kappa$ and each vertex v of a graph G has degree $< \lambda$ then G has an independent set of size κ. Hence, our question is if one can strengthen Hajnal's theorem as follows:

Problem 7.6. *Suppose that $\lambda < \kappa$ and each vertex v of a graph G has degree $< \lambda$. Does $r(G,m) = m$ or even $r^*(G,m) = m$ hold for all/some $2 \leqslant m \in \mathbb{N}$?*

Finally, it would be interesting to see the exact values of $r(G,m)$ determined for any particular K_n-free graphs.

Problem 7.7. *Let $G_{\mathbb{R}^3}$ denote the orthogonality graph on $\mathbb{R}^3 \backslash \{0\}$. Is $r^*(G_{\mathbb{R}^3}, m) = 3(m-1)+1$ for all $m \geqslant 2$?*

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