π

Football cheer : Rensselaer Polytechnic Institute (RPI)

e^x, $\frac{dy}{dx}$,

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3.14159,

RPI, Hold That Line !

Calculate π with a bicycle wheel

C = circumference = 206.0 cm

$D = 2r$ = diameter = 66.4 cm

$\frac{C}{D} = 3.10$

Or try

$C = 6'$ 9 1/4"

$D = 2'$ 2 5/16"

Aren't you glad you live in Canada!

Measurements of a Circle

$$C = 2\pi r = \pi D \qquad A = \pi r^2$$

Quantum mechanics : Planck's constant

$$\hbar = \frac{h}{2\pi}$$

Electromagnetism : Maxwell's equations

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \quad \operatorname{div} \mathbf{D} = 4\pi\rho$$

Statistics : normal distribution

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Einstein field equations :

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}$$
$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

How about π^2 ?

Let S_r^n be a sphere of dimension n and radius r. Its hypervolume equals :

$$V(S_r^1) = \pi r^2,\ V(S_r^2) = \frac{4}{3}\pi r^3,\ V(S_r^3) = \frac{1}{2}\pi^2 r^4$$

Euler (1734) :

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}\ ,$$

$$\zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}\ .$$

Evaluating π

Ancient Egypt, Babylonia, China, Biblical :

$$\frac{19}{6} = 3.17, \quad \frac{22}{7} = 3.1429, \quad \frac{25}{8} = 3.125, \quad 3$$

Ancient Greece - Archimedes :

$$\frac{223}{71} < \pi < \frac{22}{7}, \quad \text{average} = 3.1419$$

Ancient China - Liu Hui, Tsu Ch'ung-chih and Tsu Keng-chih (about 300AD)

$$\frac{355}{113} = 3.14159292$$

Formula of Liu : Let a regular polygon of n sides be inscribed in a circle of radius 1 and call its side x. Let the side of a regular polygon of $2n$ sides be y. Then

$$y = \sqrt{2 - \sqrt{4 - x^2}}\ .$$

France - Viète (1593) : $\pi = 3.141592653589793..$

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \times ...$$

equivalent to

$$\pi = \frac{2}{\sqrt{2}} \times \frac{2}{\sqrt{2+\sqrt{2}}} \times \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \times \ldots = \prod_{n=1}^{\infty} \frac{2}{r_n} .$$

Proof outline : Start with a square inscribed in the unit circle, its side equals $\sqrt{2} := x_1$. Next keep doubling the number of sides and use Liu's formula to obtain, by induction, $x_n = \sqrt{2 - r_{n-1}}$. A little algebra now shows that

$$\frac{x_{n+1}}{x_n} = \frac{1}{r_n},$$

from which the second version of Viète's formula follows (recall that the n'th approximation to π will be $C/D = 2^n \cdot x_n$).

Machin's formula (1706)

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

$$= 4\left(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \dots\right) - \left(\frac{1}{239} - \frac{1}{3\cdot 239^3} + \dots\right)$$

William Brown (1706) : Gives π its name.

R. K. Guy, Y. Matiyasevich (1986) : Let $u_0 = u_1 = 1,\ u_{n+2} = u_n + u_{n+1}$ be the Fibonbacci numbers. Then

$$\pi = \lim_{n\to\infty} \sqrt{\frac{6\log(u_1 \cdot u_2 \cdots u_n)}{\log(\mathrm{lcm}(u_1 \cdot u_2 \cdots u_n))}}\ .$$

Bailey, Borwein, Plouffe formula (1996)

$$\pi = \sum_{n=1}^{\infty} \frac{1}{16^n}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right)$$