A MODEL WITH SUSLIN TREES BUT NO MINIMAL UNCOUNTABLE LINEAR ORDERS OTHER THAN ω_1 AND $-\omega_1$

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Abstract. We show that the existence of a Suslin tree does not necessarily imply that there are uncountable minimal linear orders other than ω_1 and $-\omega_1$, answering a question of J. Baumgartner. This is done by a Jensen-type iteration, proving that one can force CH together with a weak form of ladder system uniformization on trees, all while preserving a rigid Suslin tree.

One can quickly see that any infinite linear order either contains a copy of ω (the order type of $\mathbb{N}$) or its reverse $-\omega$. In other words, $\pm\omega$ forms a 2-element basis for infinite linear orders. Also, ω and $-\omega$ are the only *minimal infinite linear orders* in the sense that they embed into each of their infinite suborders.

In 1982, James Baumgartner published a seminal paper [2] on the analysis and classification of uncountable order types. Let us briefly summarize the main points relevant to our paper: the ordinal ω_1 and its reverse $-\omega_1$ are *minimal uncountable order types* in the sense that they embed into each of their uncountable suborders. Now, a suborder L of the real line can not contain copies of $\pm\omega_1$ and, interestingly, there may or may not be minimal uncountable real order types: the Continuum Hypothesis (CH) implies that no uncountable suborder of the reals is minimal, while the Proper Forcing Axiom (PFA) implies that any two $\aleph_1$-dense suborder of the reals are isomorphic. In turn, any $\aleph_1$-dense suborder of the reals is minimal if PFA holds.

J. Baumgartner in Oberwolfach, 1975
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The order types which embed no copies of $\pm\omega_1$ nor uncountable real order types are called *Aronszajn orders*. Baumgartner sketched the construction of a minimal Aronszajn type using $\diamondsuit^+$,[1] and he emphasized multiple (now mostly solved) problems that motivated research in combinatorial set theory in the following 30 years.

Soon after, Stevo Todorcevic showed that under $\mathrm{MA}_{\aleph_1}$, there is a minimal Aronszajn linear order $C(\rho_0)$ which is also a Countryman-line: the square of $C(\rho_0)$ is the union of countably many chains [14, Theorem 2.1.12]. This line of research culminated in the work

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[1] For a presentation in full detail see [12, Theorem 18].

of Justin Moore, who proved that any Countryman line together with its reverse forms a 2-element basis for Aronszajn linear orders under PFA; in turn, if we add $\pm\omega_1$ and an $\aleph_1$-dense set of reals, we have a 5-element basis for all uncountable linear orders [11].

Complementing the above results, Moore went on to show that consistently, the only minimal uncountable linear orders are $\pm\omega_1$ [9], and this theorem is where our interest lies. Embeddings of Aronszajn linear orders are closely tied to tree embeddings of certain associated Aronszajn trees, and indeed, Moore's result is built on finding a model of CH where a weak version of the ladder system uniformization property holds for trees.

Baumgartner [2] also asked if, rather than using $\diamondsuit^+$, the existence of a single Suslin tree suffice to find a minimal Aronszajn type. The main result of our paper is a negative answer to this question: we construct a model of CH with a rigid Suslin tree R, so that any Aronszajn tree T either embeds a derived subtree of R, or T satisfies Moore's uniformization property. All together, we will see that this implies that the only minimal uncountable linear orders are $\pm\omega_1$ in our model, yet a Suslin-tree exists.

In order to achieve this result, we will apply Ronald Jensen's technique for constructing a ccc forcing iteration of length ω_2 such that each initial segment of this iteration is a Suslin tree itself [3]. Originally, this method was established to produce a model of CH without any Suslin trees, however, and lucky for us, Uri Abraham and Saharon Shelah [1] developed an analogous iteration theorem which does preserve a fixed Suslin tree R given that each successor step of the iteration preserves R in a strong sense. Let us mention that our framework also provides an alternative way to show Moore's result on minimal linear orders [9], but using a ccc forcing.

First, in Section 1, we will define the various tree-and uniformization properties that we need, and explain how an appropriate combination of these can be used to achieve our result on Suslin trees and minimal linear orders. In Section 2, we state our main result that solves the problem of uniformizing a ladder system colouring on a certain tree A while preserving another Suslin tree R. We end this section by showing how to piece together our result (used in successor stages of the iteration), and the Abraham-Shelah iteration theorem (for limit stages of our forcing) to produce the desired model. Finally, in Section 3, we prove our main theorem. We close our paper with a few remarks on our approach and directions for future research.

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1. Preliminaries on trees and uniformization

Notation. We use fairly standard notation consistent with classical textbooks (e.g. [6]), however in our forcing arguments the stronger conditions are larger (in this aspect, we follow [1] because we build on the forcing iteration framework there). For a set X, we let $\mathrm{Fn}(X, n)$ denote the set of finite partial functions from X to n.

Now, let us review the basic notions we will use, and state the most important facts that will help us prove our main result.

Trees. By *tree*, we mean a partially ordered set $(T, \leq_T)$ so that $t^\downarrow = \{s \in T : s < t\}$ is well ordered by $\leq_T$.[2] We write T_α for the set of $t \in T$ such that $t^\downarrow$ has order type α; these sets

[2]We usually omit the subscript from $\leq_T$ if it leads to no confusion.

are the levels of T, and the height of T is the minimal α so that $T_\alpha = \emptyset$. If $t \in T_\alpha$ and $\xi < \alpha$ then we let $t \restriction \xi$ denote the unique element of $t^{\downarrow} \cap T_\xi$.

An $\aleph_1$-tree is a tree T of height ω_1 with countable levels. By a *subtree* S of T, we will always mean a downward closed subset $S \subset T$ which is also pruned: for any $s \in S_\alpha$ and $\alpha < \beta$ less than the height of T, there is $t \in S_\beta$ with $s \leq t$. Given $s \in T$, we will write T_s for the subtree $\{t \in T : s \leq t\}$.

Now, an *Aronszajn tree* is an $\aleph_1$-tree with no uncountable chains, and a *Suslin tree* is an Aronszajn tree with no uncountable antichains (see [6] on the existence of such objects). The correspondence between trees and linearly ordered sets are described in detail in [13], but we will not really use that analysis in our work.

We call a tree R *full Suslin* if all its derived trees $R' = R_{a_1} \otimes \ldots \otimes R_{a_n}$ are Suslin where $a_i \neq a_j \in R_\delta$ for $1 \leq i < j \leq n$ and a fixed $\delta < \omega_1$.[3] We write ∂R for the set of all derived trees R' of R. The original rigid Suslin tree constructed by Jensen using $\diamondsuit$ is full Suslin actually [3, Theorem V.1].

We will use two crucial facts on full Suslin trees.

Fact 1.1. *Suppose that R is a full Suslin tree, $D \subseteq \omega_1$ is a club, and $s \neq t \in S$ are of the same height. Then there is no order preserving injection from $R_s \restriction D$ to $R_t \restriction D$. In particular, any order preserving injections $R_s \restriction D \to R_s \restriction D$ must be the identity.*

This result can be extracted from [13, Lemma 6.7] or [3, Theorem V.1] and the proofs there, but let us present an argument for completeness.

Proof. Suppose that $f : R_s \restriction D \to R_t \restriction D$ is order preserving but not the indentity. Take a countable elementary submodel $M \prec H(\Theta)$ so that $f, D, R, s, t \in M$. Note that $\delta = M \cap \omega_1 \in D$. Pick any $s \leq s^* \in R_\delta$ and let $t^* = f(s^*) \in R_t$. Now, since $R_s \otimes R_t$ is Suslin, the branch determined by (s^*, t^*) should be M-generic for $R_s \otimes R_t$. By the so called product lemma (see [3, Lemma I.8]), t^* should be $M[s^*]$-generic for R_t. However, $s^*, f \in M[s^*]$ implies that $t^* = f(s^*) \in M[s^*]$, a contradiction. $\square$

Given two trees S, T, a *club-embedding of T into S* is an order preserving map f into S defined on $T \restriction C = \bigcup\{T_\alpha : \alpha \in C\}$ where $C \subset \omega_1$ is a club.

Fact 1.2. [1, Lemma 3.2] *For any full Suslin tree R and Aronszajn tree A, either*

(1) $\Vdash_{R'} A$ is Aronszajn for any $R' \in \partial R$, or

(2) some $R' \in \partial R$ can be club-embedded into A.

By the following observation, we can always suppose that club-embeddings are level preserving:

Observation 1.3. *Suppose that $f : T \restriction C \to S$ is a club-embedding. Then there is some club $D \subset C$ and order and level preserving $\hat{f} : T \restriction D \to S \restriction D$ that satisfies $\hat{f}(t) \leq f(t)$.*

Proof. Given $f : T \restriction C \to S$, we can find a club $D \subset C$ so that $\delta \in D$ implies that $f[T \restriction \delta \cap C] \subset S_{<\delta}$. Now, note that if $t \in T_\delta$ for some $\delta \in D$ then $f(t) \in T \restriction [\delta, \min D \setminus (\delta+1))$, so we can let $\hat{f}(t) = f(t) \restriction \delta$. Now $\hat{f}(t) < \hat{f}(t')$ for any $t < t' \in T \restriction D$ and $\hat{f}$ is as desired. $\square$

We also need some lemmas from Moore's framework:

[3] $R_{a_1} \otimes \ldots \otimes R_{a_n} = \{(t_i)_{1 \leq i \leq n} : a_i \leq_R t_i \in T_\varepsilon$ for some fixed $\varepsilon < \omega_1\}$ with the coordinatewise partial order.

Lemma 1.4. [9, Lemma 2.9] *If there is a minimal Aronszajn type then there is an Aronszajn tree A which is* club-minimal *i.e. A can be club-embedded into any subtree S of A.*

Uniformization. A ladder system $\underline{\eta}$ is a sequence $(\eta_\alpha)_{\alpha\in\lim(\omega_1)}$ so that η_α is a type ω cofinal subset in α. An n-coloring of $\underline{\eta}$ is a sequence $\underline{h} = (h_\alpha)_{\alpha\in\lim(\omega_1)}$ so that $h_\alpha : \eta_\alpha \to n$. We say that $\underline{h}$ is a constant colouring if all the h_α are constant, in which case we can code $\underline{h}$ by an element of $n^{\lim(\omega_1)}$.

The main definition is the following:

Definition 1.5. *Given some $\aleph_1$-tree A and n-colouring $\underline{h}$ of a ladder system $\underline{\eta}$, we say that f is an A-uniformization of h if $\operatorname{dom} f = S$ is a subtree of A and for any $\alpha \in \lim(\omega_1)$, $t \in S_\alpha$ and for almost all $\xi \in \eta_\alpha$, $f(t \restriction \xi) = h_\alpha(\xi)$.*

The main use of this definition is the following:

Lemma 1.6. [9, Lemma 3.3] *If CH holds and for a fixed ladder system $\underline{\eta}$, any constant 2-colouring of $\underline{\eta}$ has an A-uniformization then A is not club-minimal.*

Moore [9] showed that CH is consistent with the statement that for any ladder system $\underline{\eta}$, any ω-colouring of $\underline{\eta}$ has an A-uniformization for any Aronszajn tree A.[4] In turn, such a model cannot contain minimal Aronszajn lines.

Let us mention a simple, somewhat technical result for later reference.

Lemma 1.7. *Suppose that A is a tree of countable height and countable levels, $h : A \to 2$ and $\underline{\eta}$ is a ladder system on $\operatorname{ht}(A)$. Then for any $\psi \in \operatorname{Fn}(A, 2)$, there is some $f : A \to 2$ extending ψ so that $f(t \restriction \xi) =^* h(t)$ for any $t \in A_\alpha$, limit $\alpha < \operatorname{ht}(A)$ and almost all $\xi \in \eta_\alpha$.*

Proof. First, if A is isomorphic to an ordinal, then this result is well known.

Now, in general, we force: let $\mathcal{Q}$ bet the poset of functions g where there is some $a \in [A]^{<\omega}$ so that $g : a^\downarrow \to 2$ and g uniformizes h on $\underline{\eta}$. Extension is containment.

The aforementioned special case for ordinals implies that the set $D_t = \{g \in \mathcal{Q} : t \in \operatorname{dom} g\}$ is dense in $\mathcal{Q}$ for any $t \in A$. So, we can take a filter $G \subseteq \mathcal{Q}$ which meets all D_t, and so $f = \cup G$ is as desired. $\square$

The main lemma. To summarize the above cited results, we have the following:

Main Lemma 1.8. *Suppose that CH holds, $\underline{\eta}$ is a ladder system and R is full Suslin. Suppose further that for any Aronszajn tree A either*

(1) $\Vdash_{R'} A$ is not Aronszajn for some $R' \in \partial R$, or
(2) any constant 2-colouring of $\underline{\eta}$ has an A-uniformization.

Then there are no minimal uncountable linear orders other than $\pm\omega_1$.

Proof. If there is a minimal uncountable linear order other than $\pm\omega_1$, then it has to be Aronszajn by the CH, and so there is an Aronszajn tree A which is club-minimal by Lemma 1.4. For this particular tree, condition (2) must fail by Lemma 1.6. So $\Vdash_{R'} A$ is not Aronszajn for some $R' \in \partial R$, which implies that there is a club D_0, $R' \in \partial R$ and an order preserving embedding $R' \restriction D_0 \to A$ (by Fact 1.2).

In particular, there is $s \in R$ with a level preserving club embedding $f_s : R_s \restriction D_0 \to A$. Let S denote the downward closure of $f_s[R_s]$ and note that $S \restriction D_0 = f_s[R_s] \restriction D_0$. Since A is

[4] This is rather surprising given the fact that CH implies that for any $\underline{\eta}$, there is a constant 2-colouring without an ω_1-uniformization [4].

club-minimal, we can find $D_1 \subset D_0$ with a level preserving embedding $g : A \restriction D_1 \to S \restriction D_1$. Now, g must fix each point of $f_s[R_s] \restriction D_1$ by Fact 1.1, and so $A \restriction D_1$ must be equal $f_s[R_s] \restriction D_1$, which is of course isomorphic to $R_s \restriction D_1$. This is a contradiction, since $A \restriction D_1$ must have non-trivial club embeddings (A being club-minimal), while $R_s \restriction D_1$ has no such embeddings by Fact 1.1.

□

Our goal for the rest of the paper is to show that the assumptions of this lemma are consistent with ZFC (assuming that ZFC itself is consistent).

2. The outline of the forcing construction

Our aim is to construct a sequence of partial orders $\langle T^\tau : \tau < \omega_2 \rangle$, that will serve as our iteration, so that

(1) T^τ is a Suslin tree (which will ensure the ccc and that no new reals are added), and
(2) T^τ forces an A-uniformization for some colouring and Aronszajn tree A

for all $\tau < \omega_2$. In order to make this sequence $\langle T^\tau : \tau < \omega_2 \rangle$ a forcing iteration, we will ensure that

(3) T^τ is a *refinement* of T^ν for $\nu < \tau < \omega_2$ i.e., there is a club $C \subset \omega_1$ and a so-called projection $\pi : T^\tau \to T^\nu \restriction C$, that is
   (i) π is an order preserving surjection, and
   (ii) if $t \in T^\nu \restriction C$ and $t > \pi(s)$ for some $s \in T^\tau$ then $\pi(s') = t$ for some $s' > s$.

Finally, we will have a full Suslin tree R, so that

(4) $\Vdash_{T^\tau} R$ is full Suslin for any $\tau < \omega_2$.

All of this boils down to two separate goals: we need to specify how to construct $T^{\tau+1}$ from T^τ (that is, the successor stages of the iteration), and how to build limits for such sequences (while preserving R full Suslin).

First, our main theorem handles the successor stages. We let $\mathcal{C}$ denote the *club forcing*

$$\mathcal{C} = \{(\nu, A) : \nu < \omega_1, A \subset \omega_1 \text{ is a club}\}$$

ordered by $(\nu, A) \leq (\nu', A')$ if $\nu \leq \nu'$, $A' \subseteq A$ and $\nu \cap A = \nu \cap A'$; from now on, we reserve the letter C for a V-generic club with canonical name $\dot{C}$. $\mathcal{C}$ is a countably closed forcing with the property that the generic club C is contained mod countable in any ground model club D.

Main Theorem 2.1. *Suppose that*

(1) T is a Suslin tree in V, $\underline{\eta}$ is a ladder system,
(2) $V^T \models$"R is full Suslin", and
(3) $\dot{A}, \dot{h}$ are T-names so that $V^T \models \dot{h} \in 2^{\omega_1}$ and

$$V^{T \times R'} \models \dot{A} \textit{ is an Aronszajn-tree}$$

for any $R' \in \partial R$.

Then, in $V[C]$, there is a refinement $\tilde{T}$ of T so that

(a) $V[C]^{\tilde{T}} \models R$ is full Suslin, and
(b) $V[C]^{\tilde{T}} \models$ the colouring $\dot{h}$ on $\underline{\eta}$ has an $\dot{A}$-uniformization.

The proof will be presented in the next section, but let us show the reader how this theorem can be applied to prove our main result on linear orders.

The Main Theorem above is complemented by Abraham and Shelah's iteration theorem that we include here for ease of reference; let us say that σ is an *s-operator*[5] if σ is defined on $<\omega_2$ sequences of Suslin trees $\mathcal{T} = \langle T^\nu : \nu \leq \tau \rangle$ so that $\sigma(\mathcal{T})$ is a refinement of T^τ. We say that σ is R-preserving if whenever $\Vdash_{T^\tau}$ "R is full Suslin" implies that $\Vdash_{\sigma(\mathcal{T})}$ "R is full Suslin".

Theorem 2.2. [1, Theorem 4.14] *Suppose that a model W satisfies $\diamondsuit$ and $\square$, and R is a full Suslin tree. Given any R-preserving s-operator σ, there is a sequence $\langle T^\tau : \tau < \omega_2 \rangle$ so that*

(i) $\Vdash_{T^\tau} R$ *is full Suslin,*
(ii) $T^{\tau+1} = \sigma(\langle T^\nu : \nu \leq \tau \rangle)$, *and*
(iii) T^τ *is a refinement of* T^ν *for all* $\nu < \tau$

for any $\tau < \omega_2$.

Putting these together yields the following corollary.

Main Corollary 2.3. *Consistently, CH holds and there is a rigid Suslin tree, while there are no minimal uncountable linear orders other than $\pm\omega_1$.*

Proof. Our goal is to find a model where the assumptions of Lemma 1.8 are satisfied. We start from a model $W = W_0$ that satisfies $\diamondsuit$ and $\square$, and first iterate $\mathcal{C}$ with countable support in length ω_2. The resulting model W_{ω_2} still satisfies $\diamondsuit$ and $\square$ (see [1] for why we can do this). Let R be a full Suslin tree in W_{ω_2}, and $\underline{\eta}$ a ladder system on ω_1. Now, we construct an R-preserving s-operator in W_{ω_2} using our Main Theorem, which in turn gives an iteration sequence of Suslin trees $\langle T^\tau : \tau < \omega_2 \rangle$ by the Abraham-Shelah theorem.

To define σ, suppose we are given some $\mathcal{T} = \langle T^\nu : \nu \leq \tau \rangle$ so that $V^{T^\tau} \models$ "R is full Suslin". An appropriate bookkeeping hands us a T^τ-name $\dot{A}$ for an Aronszajn tree and $\dot{h}$ coding a 2-colouring of $\underline{\eta}$. Find the minimal $\alpha < \omega_2$ so that $T^\tau, \dot{A} \in V = W_\alpha$.

Now, if $V^{T^\tau \times R'} \models$ "$\dot{A}$ is Aronszajn" for all $R' \in \partial R$ then we apply our Main Theorem in $V[\dot{C}_\alpha]$ to find a refinement $\sigma(\mathcal{T})$ of T^τ so that $V[C_\alpha]^{\sigma(\mathcal{T})} \models$ "the colouring $\dot{h}$ on $\underline{\eta}$ has an $\dot{A}$-uniformization". Otherwise, we just let $\sigma(\mathcal{T}) = T^\tau$.

Now $\langle T^\tau : \tau < \omega_2 \rangle$ is given to us by Theorem 2.2, and we let T be the direct limit of this sequence. We claim that the model $W^T_{\omega_2}$ is as desired. Indeed, note that T is ccc of size $\aleph_2$, and T adds no new reals (since any new set of size at most $\aleph_1$ must be introduced by T^τ for some $\tau < \omega_2$, and T^τ is a Suslin tree so adds no new reals). Furthermore, for any tree A in $W^T_{\omega_2}$, either A embeds some derived tree of R on a club or A remains Aronszajn after forcing with any $R' \in \partial R$. In the latter case, for any colouring h of $\underline{\eta}$, there was an intermediate stage when we uniformized h on A. $\square$

This leaves us to prove the Main Theorem.

3. The proof of the main theorem

The current section is devoted entirely to show our Main Theorem, which we break down into a few reasonable segments.

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[5]Short for successor-operator.

Let us recall the setting first: we have a model V, a Suslin tree T considered as a forcing notion, and names $\dot{A}$ for an Aronszajn tree and $\dot{h}$ for an element of 2^{ω_1}.

Some preparations. First, working in V, find a club $F \subset \omega_1$ and $A(x), h(x)$ for $x \in T_\gamma$ with $\gamma \in F$ so that

1. $x \Vdash_T$ "$\dot{A} \restriction \gamma = A(x)$ and $\dot{h} \restriction \gamma = h(x)$",
2. there is a maximal antichain $\mathcal{T}_x \subseteq T$ above x, countable sets B_z and $i_z \in 2$ for $z \in \mathcal{T}_x$ so that, for any $z \in \mathcal{T}_x$,
   (a) $\mathrm{ht}(z) < \min F \setminus (\gamma + 1)$,
   (b) $b \in B_z$ iff $z \Vdash_T$ "b is a branch in $A(x)$ which has an upper bound in $\dot{A}_\gamma$", and
   (c) $z \Vdash_T \dot{h}(\gamma) = i_z$.

In other words, $(B_z)_{z \in \mathcal{T}_x}$ collects the countably many possibilities that can be forced (above x) for the γth level of $\dot{A}$. We set $B(x) = \cup\{B_z : z \in \mathcal{T}_x\}$. Note that for any $z \in \mathcal{T}_x$, and any node $t \in A(x)$, some $b \in B_z$ extends t (which can be written concisely as $\bigcup B_z = A(x)$).

Also in V, take a $\diamondsuit^*$ sequence W which remains a $\diamondsuit^*$ sequence after forcing with $T \times R'$ for any $R' \in \partial R$, and let $W^*_\delta \supseteq W_\delta$ in $V[C]$ (where C is the generic club added by the forcing $\mathcal{C}$) so that W^* is still a $\diamondsuit^*$ sequence after forcing with R' for any $R' \in \partial R$ (let us refer the reader to [1] for details on why this is possible).

Recall that $\mathcal{C}$ was the club forcing and C is the V-generic club. Working in $V[C]$, the generic club C is mod countable contained in F, so we let $\gamma_0 = 0$ and let $\{\gamma_\alpha : 1 \leq \alpha < \omega_1\}$ enumerate an end-segment of C that is contained in F.

How will the elements of $\tilde{T}$ look like? The αth level $\tilde{T}_\alpha$ of the tree $\tilde{T}$ will consist of pairs (x, f) so that $x \in T_{\gamma_\alpha}$ and $f : S \to 2$ so that $S \subseteq A(x)$ is downward closed and pruned, and f is a uniformization of the coloring coded by $h(x) \in 2^{\gamma_\alpha}$ on the ladder system $\underline{\eta} \restriction \gamma_\alpha$.[6]

The extension is defined as follows: $(x, f) \leq (x', f')$ if $x \leq x'$ in T and $f \subseteq f'$. We would like the second coordinates to introduce the desired uniformization, and hence we need that for any (x, f) and $\delta < \omega_1$, there is some (x', f') above (x, f) so that $\mathrm{dom}\, f'$ has height at least δ.

In turn, we require the following *richness property* (RP):

> for any $(x, f) \in \tilde{T}$ and $z \in \mathcal{T}_x$, each $t \in \mathrm{dom}\, f$ is extended by some branch $b \in B_z$ so that $b \subseteq \mathrm{dom}\, f$ and $f(b \restriction \xi) = i_z$ for almost all $\xi \in \eta_{\gamma_\alpha}$.

Claim 3.1. *Any T of the above described form with the (RP) will introduce an $\dot{A}$-uniformization for the colouring $\dot{h}$.*

Proof. We would like to show that the set of conditions (x', f') such that $\mathrm{dom}\, f'$ has height at least δ is dense for any $\delta < \omega_1$. Given (x, f) and $\delta < \omega_1$, we first find $x < x' \in T_{\gamma_\beta}$ with $\gamma_\beta > \delta$. Then, we can take the unique $z \in \mathcal{T}_x$ which is compatible with x'; note that

$$x' \Vdash_T \dot{h} \restriction \gamma_\alpha + 1 = h(x) \cup \{(\gamma_\alpha, i_z)\}.$$

Now, select a subset $\{b_t : t \in \mathrm{dom}\, f\}$ from $\{b \in B_z : f(b \restriction \xi) = i_z \text{ for almost all } \xi \in \eta_{\gamma_\alpha}\}$ so that $t \in b_t$. Extend $\mathrm{dom}\, f$ by addding unique upper bounds for the branches b_t and

[6] I.e., for any $t \in S_\delta$, $f(t \restriction \xi) = h(x)(\delta)$ for almost all $\xi \in \eta_\delta$.

let this set be $S_0 \supseteq \operatorname{dom} f$. Note that any function $f_0 : S_0 \to 2$ that extends f is still a uniformization for $h(x) \cup \{(\gamma_\alpha, i_z)\}$. So, using Lemma 1.7, we can find $f' \supseteq f_0$ such that $(x, f) \le (x', f')$.

□

The extension property. Finally, we have to ensure that $\tilde{T}$ is a refinement of T, which will be witnessed by the projection $(x, f) \mapsto x$ that maps $\tilde{T}$ onto $T \restriction \{\gamma_\alpha : \alpha < \omega_1\}$.[7] So, our goal will be to ensure that if $(x, f) \in \tilde{T}_\alpha$ and $x < x' \in T_{\gamma_\beta}$ for $\alpha < \beta$, then there is some f' so that $(x, f) \le (x', f') \in \tilde{T}_\beta$. In fact, we need a stronger property to carry out the inductive construction of $\tilde{T}$.

Suppose $x < x' \in T$ and $b \in B(x')$ a cofinal branch through $A(x')$. If $f : S \to 2$ for some pruned, downward closed $S \subseteq A(x)$ then we say that b is x'*-compatible* with f if

$$b \restriction \gamma_\alpha \subseteq S \text{ and } f(b \restriction \xi) =^* i \text{ for almost all } \xi \in \eta_{\gamma_\alpha}$$

where $i = h(x')(\gamma_\alpha)$. This simply means that if x forced f to be a uniformization of $\dot{h}$ so far, then we have the possibility to add an upper bound to b to $\operatorname{dom} f$ without running into trouble at the level γ_α with the uniformization (in the universe forced by x'). We will say that a finite function $p \in \operatorname{Fn}(B(x'), 2)$ is x'-compatible with f if each $b \in \operatorname{dom} p$ is x'-compatible with f.

We will assume inductively and preserve the following *extension property* (EP) along the construction of $\tilde{T}$:

for any $\alpha < \beta < \omega_1$, $(x, f) \in \tilde{T}_\alpha$ and $x < x' \in T_{\gamma_\beta}$, if $p \in \operatorname{Fn}(B(x'), 2)$ is x'-compatible with f and $\psi \in \operatorname{Fn}(A(x') \setminus A(x), 2)$ then there is f' so that $(x, f) \le (x', f') \in \tilde{T}_\beta$ and

(1) $b \subseteq \operatorname{dom} f'$ and $f'(b \restriction \xi) = p(i)$ for almost all $\xi \in \eta_{\gamma_\beta}$ and all $b \in \operatorname{dom} p$, and
(2) $\psi(t) = f'(t)$ whenever both are defined.

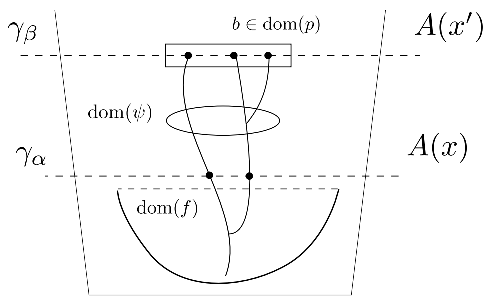

FIGURE 1. The (EP)

Note that only those points $t \in \operatorname{dom} \psi$ matter where the branch $t^{\downarrow}$ is compatible with f, and by extending p we can make sure f' is defined on these t as well. In other words, we can suppose that $\operatorname{dom} \psi \subset \cup \operatorname{dom} p$ (see Figure 1 for the setting).

The construction of $\tilde{T}$. The levels $\tilde{T}_\alpha$ of $\tilde{T}$ will be constructed by induction on $\alpha < \omega_1$ preserving the (RP) and (EP). In fact, $\tilde{T}_\alpha$ will be a result of taking an N_α-generic filters for appropriate posets (approximating the conditions in $\tilde{T}_\alpha$), where $(N_\alpha)_{\alpha < \omega_1}$ is a canonically chosen sequence of countable elementary submodels. In fact, we let N_α be

$$L_\delta[T \restriction \gamma_\alpha + 1, A \restriction \gamma_\alpha + 1, R \restriction \alpha + 1, C \cap \gamma_\alpha, W^* \restriction \gamma_\alpha + 1]$$

for the unique minimal $\delta > \alpha$ which makes this a model of ZF^-. This ensures that any model N with the parameters $T \restriction \gamma_\alpha + 1, A \restriction \gamma_\alpha + 1, \dots$ actually contains N_α as an element (and subset).

[7] To remind the reader, we need a refinement because our main theorem provides the successor steps of an iteration.

Successor steps - $\tilde{T}_{\alpha+1}$ from $\tilde{T}_\alpha$. Given $\tilde{T}_\alpha$, we will construct $\tilde{T}_{\alpha+1}$ while preserving the (RP) and (EP). Fix some $(x, f) \in \tilde{T}_\alpha$ and $x < x' \in T_{\gamma_{\alpha+1}}$. Define the poset

$$\mathcal{P}_{x',f} = \{p \in \mathrm{Fn}(B(x'), 2) : p \text{ is } x'\text{-compatible with } f\}$$

where extension is simply containment. For each $p_0 \in \mathcal{P}_{x',f}$, we take a minimal[8] $H \in \mathrm{Gen}(N_{\alpha+1}, \mathcal{P}_{x',f}) \cap N_{\alpha+2}$ so that $p_0 \in H$. Let $p^H = \cup H$, and let $S = \mathrm{dom}\, f$.

Claim 3.2. *The set $S' = \cup\, \mathrm{dom}\, p^H$ is a pruned and downward closed subtree of $A(x')$, and $S = S' \cap A(x)$.*

Proof. Clearly, S' is pruned and downward closed as a union of branches through $A(x')$.

The fact that any $p \in H$ is compatible with f implies that $b' \upharpoonright \gamma_\alpha \subset S$ for $b' \in \mathrm{dom}\, p$ and so $S \subset S' \cap A(x)$. Note that if $z \in \mathcal{T}_x$ is compatible with x' then for any $s \in S$, there are infinitely many branches $b \in B_z$ through $A(x)$ so that $s \in b$ and $f(b \upharpoonright \xi) = i_z$ for almost all $\xi \in \eta_{\gamma_\alpha}$; let B_s^* denote these branches. Any $b \in B_z$ has an upper bound in $A(x')_{\gamma_\alpha}$ which is extended to branches $b' \in B(x')$. So, any $b \in B_s^*$ has an extension in S' by genericity and so $S = S' \cap A(x)$.

□

Now, we use p^H to form an $f' \supseteq f$ defined on S' that satisfies the (RP).

Claim 3.3. *For any ψ compatible with f, there is an $f' : S' \to 2$ so that*

(1) *$f \subseteq f' \in N_{\alpha+2}$ and f' uniformizes $h(x')$ on $\underline{\eta} \upharpoonright \gamma_{\alpha+1}$,*
(2) *$\psi \subset f'$, and*
(3) *$f'(b \upharpoonright \xi) = p^H(b)$ for all $b \in \mathrm{dom}\, p^H$, and almost all $\xi \in \eta_{\gamma_{\alpha+1}}$.*

Proof. This is simply done by Lemma 1.7. □

Claim 3.4. *f' satisfies the (RP).*

Proof. We again use the genericity of H: given $z' \in \mathcal{T}_{x'}$, we need to show that for any $s \in S'$ there is some $b \in B_{z'}$ extending s so that $f'(b \upharpoonright \xi) = i_{z'}$ for almost all $\xi \in \eta_{\gamma_{\alpha+1}}$. It suffices to show, by condition (3) of f', that there is some $b \in \mathrm{dom}\, p^H \cap B_{z'}$ extending s so that $p^H(b) = i_{z'}$. By genericity of H (and since $A(x'), \mathcal{T}_{x'} \in N_{\alpha+1}$), it suffice that there are infinitely many $b \in B_{z'}$ that extend s; however, this clearly holds since $\Vdash_T \dot{A}$ is pruned and $B_{z'}$ just collects the branches in $A(x')$ that are forced to be bounded in $\dot{A}$. □

Now, we put $(x', f') \in \tilde{T}_{\alpha+1}$. The function f' depended on the initial choice of $p_0 \in \mathcal{P}_{x',f}$ and on ψ, and we do this for all countably many possible choices. This in turn defines $\tilde{T}_{\alpha+1}$ (in $N_{\alpha+2}$), and the (EP) is preserved.

Limit steps - $\tilde{T}_\beta$ from $\tilde{T}_{<\beta}$. Suppose that $\tilde{T}_{<\beta} = \bigcup\{\tilde{T}_\alpha : \alpha < \beta\}$ is already constructed, and fix some $x' \in T_{\gamma_\beta}$. We will now force with the poset $\mathcal{P}_{x'}$ of all pairs (p, f) so that

(1) $(x, f) \in \tilde{T}_{<\beta}$ for some $x < x'$,
(2) $b \upharpoonright \mathrm{ht}(f) \neq b' \upharpoonright \mathrm{ht}(f)$ for any $b \neq b' \in \mathrm{dom}\, p$, and
(3) $p \in \mathrm{Fn}(B(x'), 2)$ is x'-compatible with f.

Extension is defined by $(p, f) \leq (\bar{p}, \bar{f})$ if $p \subseteq \bar{p}$, $f \subseteq \bar{f}$ and for any $b \in \mathrm{dom}\, p$,

(4) $\bar{f}(b \upharpoonright \xi) = p(b)$ for any $\xi \in \eta_{\gamma_\beta} \cap \mathrm{ht}\, \bar{f} \setminus \mathrm{ht}\, f$.

[8]Minimal with respect to a fixed well-order of L_{ω_1}.

Given some $(p_0, f_0) \in \mathcal{P}_{x'}$, we take a minimal $H \in \mathrm{Gen}(N_\beta, \mathcal{P}_{x'}) \cap N_{\beta+1}$ with $(p_0, f_0) \in H$. Let

$$f' = \bigcup \{f : (p, f) \in H\}$$

and $p^H = \bigcup \{p : (p, f) \in H\}$.

Claim 3.5. *$S' = \operatorname{dom} f' = \cup \operatorname{dom} p^H$ is a pruned, downward closed subtree of $A(x')$. Furthermore, for any $b \in \operatorname{dom} p^H$, $f'[b] \restriction \eta_{\gamma_\beta} =^* p^H(i)$.*

Proof. First, $\operatorname{dom} f' \supseteq \cup \operatorname{dom} p^H$ holds since $b \restriction \mathrm{ht}(\bar{f}) \subset \operatorname{dom} \bar{f}$ for any $b \in \operatorname{dom} p^H$. To see the reverse inclusion, just note that for any $(p, f) \in \mathcal{P}_{x'}$ and $s \in \operatorname{dom} f$, there are infinitely many $b \in B(x')$ that are x'-compatible with f and extend s. So, by genericity, we included some of these in $\operatorname{dom} p^H$.

The latter statement is clear from the way we extend conditions in $\mathcal{P}_{x'}$ (see condition (4) above). $\square$

Claim 3.6. *f' satisfies the (RP).*

Proof. Let $z' \in \mathcal{T}_{x'}$, and we need to show that for any $s \in S'$ there is some $b \in B_{z'}$ extending s so that $f(b \restriction \xi) = i_{z'}$ for almost all $\xi \in \eta_{\gamma_\beta}$. It suffices that there is some $b \in \operatorname{dom} p^H \cap B_{z'}$ extending s so that $p^H(b) = i_{z'}$. By genericity of H (and since $A(x'), \mathcal{T}_{x'} \in N_\beta$), we need that there are infinitely many $b \in B_{z'}$ that extend s; however, this clearly holds since $\Vdash_T \dot{A}$ is pruned. $\square$

Now, we put $(x', f') \in \tilde{T}_\beta$, and we repeat this for all possible choices of $(p_0, f_0) \in \mathcal{P}_{x'}$ (again, we only have countably many such), which in turn defines $\tilde{T}_\beta$.

Claim 3.7. *The (EP) is preserved.*

Proof. Indeed, given some (x, f), p_0 and ψ we can first use the (EP) for $\tilde{T}_{<\beta}$ to find $(x, f) \leq (x_0, f_0)$ that is still compatible with p_0 and $\psi \subseteq f_0$. Now, take f' that corresponds to the filter H that we chose for (p_0, f_0). Then (x', f') witnesses the (EP). $\square$

This finishes the construction of $\tilde{T} = \cup_{\alpha<\omega_1} \tilde{T}_\alpha$, which is at least an $\aleph_1$-tree (by the coordinatewise ordering) and a certainly a refinement of T by the (EP).

$\tilde{T}$ forces an $\dot{A}$-uniformization for $\dot{h}$. Let us show first, that $\tilde{T}$ introduces an $\dot{A}$-uniformization for $\dot{h}$ on $\underline{\eta}$. A $V[C]$-generic filter $G \subset \tilde{T}$ defines a generic branch $x \subset T$ that evaluates $\dot{A}$ to be $\dot{A}[x] = \cup\{A(x \restriction \gamma_\alpha) : \alpha < \omega_1)\}$, and $\dot{h}$ is evaluated as $\dot{h}[x] = \cup\{h(x \restriction \gamma_\alpha) : \alpha < \omega_1\}$. The union f of the second coordinates in G defines a function on a subset of $\dot{A}[x]$ that uniformizes $\dot{h}[x]$. Finally, $\operatorname{dom} f$ is really a subtree: by the (EP), any condition in $\tilde{T}$ has arbitrary high extensions, so by genericity, $\operatorname{dom} f$ must be pruned.[9]

[9] At this point, using the $\diamondsuit$ sequence W^*, it would be standard to show that $\tilde{T}$ is Suslin (see [3, Chapter IV, Lemma 2]). Hence, together with Jensen's iteration framework, we arrive to an alternative proof to Moore's result: the consistency of CH with no minimal uncountable linear orders other than $\pm\omega_1$, but now forced by a ccc iteration.

Why is R still full Suslin after forcing with $\tilde{T}$? This will be the crux of the proof, where we simultaneously show that $\tilde{T}$ is Suslin, and that $\tilde{T}$ preserves R full Suslin. In order to do this, we will actually argue that, in $V[C]$, $\tilde{T} \times R'$ is Suslin for any derived tree R' of R. If this is not the case, and we let $\dot{R}'$ denote a $V[C]$-generic branch for R', then

$$V[C][\dot{R}'] \models \tilde{T} \text{ is not Suslin.}$$

Let $\dot{X}$ be a name for a maximal antichain of $\tilde{T}$; as usual, we would like to find an $\alpha < \omega_1$ so that $\dot{X} \restriction \alpha$ is maximal already in $\tilde{T}$. To do this, we need that any $(x', f') \in \tilde{T}_\alpha$ extends some element of $\dot{X} \restriction \alpha$.

Observation 3.8. *$V[C][\dot{R}'] = V[\dot{R}'][C]$, and the club C is also $V[\dot{R}']$-generic.*

Proof. Indeed, on one hand R' is ccc so any club of ω_1 in $V^{R'}$ contains a club from V. Also, R' introduces no new ω-sequences, so the poset $\mathcal{C}$ is the same in V and $V^{R'}$. □

Now, working in $V[\dot{R}']$, we take countable elementary submodels $N \in M \prec H_{\omega_3}^{V[\dot{R}']}$ with $T, R, A, W, \mathcal{C}, \dot{X} \ldots \in N$. Let π_N, π_M denote the collapsing functions for N and M, and let let $\bar{N}$ and $\bar{M}$ denote the transitive collapses of N and M, respectively. Let $\alpha = N \cap \omega_1$.

Let us cite two results [3, Chapter IX, Lemma 2 and 3]:

Claim 3.9. *$\alpha \cap C$ is $\bar{M}$-generic over $\pi_N(\mathcal{C}) = \{(\nu, B \cap \alpha) : (\nu, B) \in \mathcal{C} \cap N\}$.*

Claim 3.10. *For any formula φ with constants from $\{\check{x} : x \in N\} \cup \{C\}$, the following are equivalent:*

$$\bar{N}[\alpha \cap C] \models \pi_N(\varphi) \text{ if and only if } H_{\omega_3}^{V[\dot{R}'][C]} \models \varphi.$$

So π_N^{-1} extends to an elementary embedding

$$\pi_N^{-1} : \bar{N}[\alpha \cap C] \to H_{\omega_3}^{V[\dot{R}'][C]}$$

that maps $\alpha \cap C$ to C.

In the construction of $\tilde{T}$, we worked with the model sequence $(N_\beta)_{\beta<\omega_1}$.

Claim 3.11. *The set $\{\beta \leq \alpha : \dot{X} \restriction \beta \in N_\beta\}$ is closed and unbounded in $\alpha + 1$ and in particular, $\dot{X} \restriction \alpha \in N_\alpha$.*

Proof. Since $\bar{N}$ contains the relevant parameters, we can carry out the construction of $\tilde{T}$ in $\bar{N}[\alpha \cap C]$, and hence, $\bar{N}[\alpha \cap C]$ contains the tree $\tilde{T} \restriction \alpha$ (this is the point we use that throughout the construction of $\tilde{T}$, we made canonical choices for the filters that defined the levels).

Furthermore, recall that W^* was a $\diamondsuit^*$ sequence in $V[\dot{R}'][C]$, so there is a club D in ω_1 so that $\beta \in D$ implies $\dot{X} \restriction \beta \in W^*_\beta \subseteq N_\beta$. By elementarity, there is $E \in \bar{N}[\alpha \cap C]$ so that $D = \pi_N^{-1}(E)$ satisfies the above requirements. Since $D \cap \alpha = E \cap \alpha$ and D was closed ubounded, the claim follows.

□

By elementarity, $\dot{X} \restriction \alpha$ is a maximal antichain in $\tilde{T} \restriction \alpha$.

We will also need the next claim, where, given $x' \in T_\alpha$, we let $\dot{x}'$ denote the branch $\{x \in T : x < x'\}$ of $T \restriction \alpha$.

Observation 3.12. *For any $x' \in T_\alpha$, $\bar{N}[C \cap \alpha][\dot{x}'] \models$ "$\dot{A} \restriction \alpha$ is Aronszajn".*

Proof. Since $V[R'] \models$ "T is Suslin", we also have $\bar{N} \models$"$T \restriction \alpha$ is Suslin". This is preserved by σ-closed forcing, so $\bar{N}[\dot{C} \cap \alpha] \models$ "$T \restriction \alpha$ is Suslin". In turn, $\dot{x}'$ is an $\bar{N}[\dot{C} \cap \alpha]$-generic branch. As $\dot{A}$ was a name for a tree that remains Aronszajn after forcing with any R', $\bar{N}[\dot{C} \cap \alpha][\dot{x}'] \models$ "$\dot{A} \restriction \alpha$ is Aronszajn". □

After all this preparation, lets show that whenever $(x', f') \in \tilde{T}_\alpha$ then (x', f') is above some element of $X \restriction \alpha$. Recall that f' was constructed using an N_α-generic filter for the poset $\mathcal{P}_{x'}$. In turn, we will aim for a density argument, and it suffices to show that the following claim holds.

Main Claim 3.13. *Let $\mathcal{D}$ be the set of all $(p, f) \in \mathcal{P}_{x'}$ so that $(x, f) \in \tilde{T} \restriction \alpha$ for some $x < x'$ and (x, f) is above an element of $X \restriction \alpha$. Then $\mathcal{D} \in N_\alpha$ and $\mathcal{D}$ is dense in $\mathcal{P}_{x'}$.*

Proof. Suppose that this is not the case, and we reach a contradiction. That is, we assume that some $(p_0, f_0) \in \mathcal{P}_{x'}$ has no extension in $\mathcal{D}$. There is some $x_0 < x'$ so that $(x_0, f_0) \in \tilde{T} \restriction \alpha$, and let $\gamma_\tau < \alpha$ so that $x_0 \in T_{\gamma_\tau}$.

In this proof, we will say that q is *bad*, if $q \subset A(x')_\beta$ for some $\gamma_\tau \leq \beta < \alpha$ and

(1) $|q| = |p_0|$,
(2) $\cup \operatorname{dom} q \restriction \gamma_\tau = \cup \operatorname{dom} p_0 \restriction \gamma_\tau$ (which implies that q is x'-compatible with f_0),
(3) if $(x, f) \in \tilde{T}$ is of height $\leq \beta$ so that
  (a) $(x_0, f_0) \leq (x, f)$ and $x < x'$,
  (b) f is x'-compatible with q, and
  (c) $f(b \restriction \xi) = p_0(b)$ for any $b \in \operatorname{dom} p_0$ and $\xi \in \eta_{\gamma_\alpha} \cap \beta \setminus \gamma_\tau$,
  then (x, f) does not extend any element of $X \restriction \alpha$.

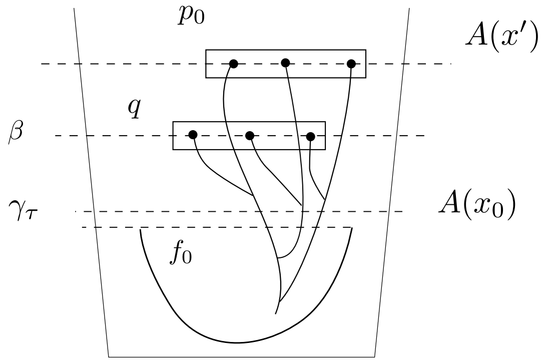

FIGURE 2. The position of bad q's

Observe that any q of the form $(\cup \operatorname{dom} p_0) \restriction \beta$ is bad where $\gamma_\tau \leq \beta < \alpha$. Moreover, to define the set of bad q, we used only parameters in $\bar{M}[\dot{C} \cap \alpha]$ (e.g. x', p_0). So $\bar{M}[\dot{C} \cap \alpha] \models$ $q = \cup \operatorname{dom} p_0 \restriction \beta$ is bad for $\gamma_\tau \leq \beta < \alpha$, and hence there is a single $c \in \pi_N(\mathcal{C})$ that forces this. In turn, for any $\gamma_\tau \leq \beta < \alpha$,

$$\bar{M} \models c \Vdash_{\pi_N(\mathcal{C})} (\cup \operatorname{dom} p_0) \restriction \beta \text{ is bad.}$$

So, as $\bar{N} \prec \bar{M}$, this must hold in $\bar{N}[\dot{x}'] \prec \bar{M}[\dot{x}'] = \bar{M}$ as well:

$$\bar{N}[\dot{x}'] \models c \Vdash_{\pi_N(\mathcal{C})} (\cup \operatorname{dom} p_0) \restriction \beta \text{ is bad.}$$

In turn, in $\bar{N}[\dot{x}']$, the tree

$$S_0 = \{q \in (A \restriction \alpha)^{|p_0|} : c \Vdash q \text{ is bad}\}$$

is uncountable.

Let $S \subset S_0$ be the set of those $q \in S_0$ which have uncountably many extensions in S_0. Since ($\bar{N}[\dot{x}']$ thinks) S is an uncountable subset of the Aronszajn tree $(\dot{A} \restriction \alpha)^{|p_0|}$, the next claim follows (see [3, Chapter VI, Lemma 7]):

Claim 3.14. *There is a club $B \subset \alpha$ in $\bar{N}[\dot{x}']$ so that for any $\eta < \beta \in B$ and $q \in S_\eta$, the set $\{q' \in S_\beta : q \leq q'\}$ contains infinitely many pairwise disjoint elements.*

Now, still working in $\bar{N}[\dot{x}']$, the sequence $\pi_N(W) = W \restriction \alpha$ is still a $\diamondsuit^*$ sequence, so there is a club $B' \subset B$ so that $\beta \in B'$ implies $S \restriction \beta \in W_\beta$. Now, the generic club $C \cap \alpha$ is eventually contained in B' i.e., in $\bar{N}[\dot{x}'][C \cap \alpha]$, there is some $\eta \geq \tau$ so that $C \setminus \eta \subset B'$. So, for any $\beta \geq \eta$, $S \restriction \gamma_\beta \in N_\beta$. We can suppose that $X \restriction \beta \in N_\beta$ as well by Claim 3.11.

We fix now such a β, and find a large enough $\beta_0 < \beta$ which is above $\eta_{\gamma_\alpha} \cap \beta$. Fix any $q_0 \in S_{\beta_0}$, and let ψ code the values of p_0 on the finite set $(\cup \operatorname{dom} p_0) \restriction \eta_{\gamma_\alpha} \cap \beta \setminus \gamma_\tau$. Use the (EP) to find $(x, f) \in \tilde{T}_{\beta_0} \cap N_\beta$ with $x < x'$ so $(x_0, f_0) \leq (x, f)$ and f is compatible with q_0 and $\psi \subset f$.

Now, we can extend (x, f) further above some element of $\dot{X} \restriction \alpha$ by [1, Lemma 3.5]:

Claim 3.15. *There is an extension $(z, g) \in \tilde{T}_\beta$ of (x, f) with $z < x'$ so that (z, g) extends some element of $\dot{X} \restriction \alpha$ as well.*

The next final claim will yield the desired contradiction:

Claim 3.16. *There is some $q \in S_{\gamma_\beta}$ such that q is compatible with g.*

Proof. We proceed by induction on $\beta \geq \beta_0$, showing that any extension $(z, g) \in \tilde{T}_\beta$ of (x, f) is compatible with some $q \in S_{\gamma_\beta}$. First, if $\beta = \beta_0$ then $q = q_0$ works.

In the successor step, we are given $(z, g) \in \tilde{T}_{\beta+1}$. Take $(z', g') \in \tilde{T}_\beta$ below (z, g), and find $q' \in S_{\gamma_\beta}$ that is compatible with g'. Recall that g was constructed using an $N_{\beta+1}$-generic filter for the poset $\mathcal{P}_{z,g'}$. Also, there are infinitely many pairwise disjoint extensions of q' in $S_{\gamma_{\beta+1}}$ (see Claim 3.14) and this set is in $N_{\beta+1}$. So, by genericity, we must have some $q \in S_{\gamma_\beta+1}$ exntending q' that is compatible with g.

Finally, suppose that $(z, g) \in \tilde{T}_\beta$ for some limit β. Again, recall how g was constructed using an N_β-generic filter for the poset $\mathcal{P}_z$. In order to show that the generic function satisfies our property, we need that a dense set of conditions does; so fix $(p_1, g_1) \in \mathcal{P}_z$ (without loss of generality, $(z_1, g_1) \geq (x, f)$ with $z_1 < z$). First, by the inductive hypothesis, we can find $q_1 \in S_{\gamma_{\beta_1}}$ compatible with g_1. Then, we can go one level higher to $\beta_1 + 1 < \beta$ and find $q_2 \in S_{\gamma_{\beta_1+1}}$ above q_1 that is disjoint from $(\cup \operatorname{dom} p_1) \restriction \gamma_{\beta_1+1}$.

We can use the (EP) to find (z_2, g_2) extending (z_1, g_1) so that $z_2 < z$ and

(i) (z_2, g_2) is still compatible with $(\cup \operatorname{dom} p_1) \restriction \gamma_{\beta_1+1} \cup q_2$, and
(ii) $g_2(b \restriction \xi) = p_1(b)$ for all $b \in \operatorname{dom} p_1$ and $\xi \in \eta_{\gamma_\beta} \cap \gamma_{\beta_1+1} \setminus \gamma_{\beta_1}$.

Now pick any $q \in S_{\gamma_\beta}$ that extends q_2, and note that $(p_1 \cup q, g_2) \in \mathcal{P}_z$ is an extension of (p_1, g_1) so that q is compatible with g_2; see Figure 3 for a summary. $\square$

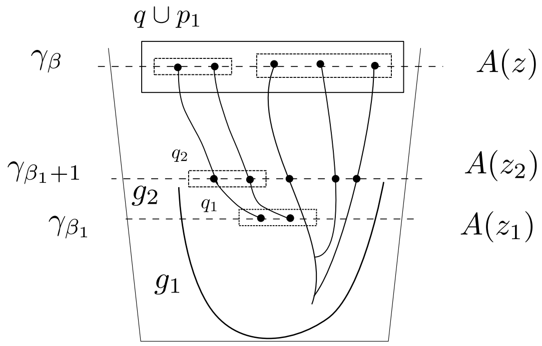

FIGURE 3. Finding q compatible with g

The latter claim clearly constradicts that q was bad, which in turn finishes the proof of the Main Claim 3.13.

□

We ended the proof now that shows that $X \restriction \alpha$ is a maximal antichain in $\tilde{T}$, and so $X = X \restriction \alpha$ is countable. In turn,

$$V[C][\dot{R}'] \models \tilde{T} \text{ is Suslin}$$

and so $\tilde{T}$ must preserve each derived tree $R' \in \partial R$ Suslin i.e., R remains full Suslin after forcing with $\tilde{T}$.

This finishes the proof of the Main Theorem.

4. Closing remarks

Once the reader is familiar with the above proof, it should take no time to realize that one can actually prove the following:

Theorem 4.1. *Consistently, CH holds, there exists a full Suslin tree R, and for any Aronszajn tree A, either*

(1) a derived tree of R club-embeds into A, or
(2) for any ladder system $\underline{\eta}$, any ω-colouring of $\underline{\eta}$ has an A-uniformization.

The point being that so far we aimed to uniformize only constant 2-colourings on a fixed ladder system, and now we are allowed to use ω colours and vary the ladder system. We decided to prove the special case only (which was enough to yield our corollary on minimal linear orders and Suslin trees) to simplify notation and since we had no further application for this stronger result in mind.

We can also mix-in the Abraham-Shelah forcings with our iteration to achieve that there is a single special Aronszajn tree U so that any Aronszajn tree A that fails (1), actually embdeds into U on a club. In this case, any Suslin tree in the resulting model is a countable union of derived trees of R when restricted to an appropriate club.

Second, given all the developments in countable support iteration and preservation theorems since the 1980s, we should address why we chose Jensen's iteration framework to prove our result. Especially so that Tadatoshi Miyamoto [8] proved that a countable support iteration of proper forcings will preserve a Suslin tree R given that each successor step preserves

R (also see [7]). So what does prevent us from repeating Moore's countable support iteration to force the uniformizations for certain Aronszajn trees while preserving some fixed tree R being Suslin?[10]

We are unsure at this point if this can be done, but Jensen's framework certainly has a great advantage over everyday CS-iterations, even ones adding no new reals: being ccc, we can guess a countable object in the extension (e.g., a level of an Aronszajn tree $\dot{A}$) by countably many ground model sets, and then find a single function that is a uniformization no matter which one of the guesses turns out to be true in the extension. This is not the case for a general CSI of proper forcings, however, the proof we presented certainly resembles some completeness arguments ubiquitous in no-new-real iterations,[11] and the fusion argument in Moore's work [9]. Despite the similarities, we have not succeeded so far in working out our result using a countable support iteration. At this point, we are even unsure whether Moore's original model can contain any Suslin trees (say if one starts forcing from L).

A question still remains open from [2]:

Question 4.2. *Does $\diamondsuit$ imply the existence of a minimal Aronszajn order?*[12]

Finally, there is a quite reasonable but rather different approach to find a model with a Suslin tree but no minimal uncountable linear orders other than $\pm\omega_1$: start with Moore's model and add a single Cohen real (this approach was recommended by Moore and Sy Friedman, independently, in personal communication). CH will still hold, and we do have a Suslin tree in the resulting model [14], but it is not hard to see that the uniformization property for trees will fail. At this point, we could not verify that the extension has no minimal Aronszajn types.

Yet another angle would be to look at the Sacks model which does have Suslin trees but no minimal real order types; this can be deduced from certain parametrized diamonds [10]. If there are no minimal Aronszajn types here, then this model would also provide the first example showing that CH is not necessary to achieve Moore's result.

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[10] It would suffice to prove that a single step from the iteration keeps R full Suslin by the Miyamoto preservation theorem. The sufficient conditions detailed in [7] for example (which apply to e.g. the Sacks forcing) does not apply to our case unfortunately.

[11] In particular, see the comments on p. 2686 and 2688 on "countably many guesses" [5]

[12] Keep in mind that $\diamondsuit^+$ does imply the existence of a minimal Aronszajn order [2, 12].

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