# University of Toronto
## Faculty of Arts and Science
## Term Test
## MAT135H1F - Calculus I (A)
## Duration - 2 hours
## No Aids Allowed

**Family Name:** ______________________________

**Given Name:** ______________________________

**Student Number:** ______________________________

**Lecture and Tutorial section:**

| **L0101 TR10-1** | **L5101 TR6-9** | **T0101 TR1** | **T0102 TR1** | **T5101 TR5** |
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This exam contains 11 pages (including this cover page) and 7 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work!** Write your answers in the spaces provided. Work scattered over the page without clear ordering will receive very little credit.
- **Justify your answers!** A correct answer without explanation or algebraic work will receive **no credit**; an incorrect answer supported by correct calculations and explanations **may** still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

| Problem | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 10 | |
| 3 | 30 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 25 | |
| Total: | 120 | |

1. (a) (5 points) Define what it means for a function $f(x)$ to be continuous at $a = 1$.

**Solution:** $\lim_{x\to 1} f(x) = f(a)$.

(b) (5 points) Give an example of a function $f(x)$ and point $a$ such that both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ do not exist. Sketch its graph.

**Solution:** $f(x) = \frac{1}{x}$ or $\sin(\frac{1}{x})$ at $a = 0$.

(c) (5 points) Let $g(x) = x^2$ and let $f(x)$ be defined by

$$f(x) = \begin{cases} -3 & \text{when } x \geq 0, \\ 3 - x & \text{when } x < 0. \end{cases}$$

Is $f \circ g$ continuous? Is $g \circ f$ continuous?

**Solution:**

$$g \circ f(x) = \begin{cases} (-3)^2 = 9 & \text{if } x \geq 0, \\ (3 - x)^2 & \text{if } x < 0 \end{cases}$$

is not continuous as $\lim_{x\to 0^+} (3 - x)^2 = 3 \neq 9$.

$f \circ g(x) = f(x^2) = -3$ hence continuous.

(d) (10 points) What are the three main types of discontinuity discussed in class? For each one, give its name, a function with that type of discontinuity, and a sketch of its graph.

**Solution:** Removable:

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Jump:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Infinite: $f(x) = 1/x$.

2. (a) (4 points) State the Intermediate Value Theorem.

**Solution:** Intermediate Value Theorem: Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then for every $y$ between $f(a)$ and $f(b)$ there is $x \in [a, b]$ so that $f(x) = y$.

(b) (6 points) Show that the equation $\sin(t) = t^2 - 1$ has at least two solutions.

**Solution:** Let $f(t) = \sin(t) - (t^2 - 1)$. Every root $t$ of $f$ solves the above equation.
The function $f$ is continuous and $f(0) > 0$ while $f(\pm 2) < 0$ (as $\sin(\pm 2) < 3 = (\pm 2)^2 - 1$).
So IVT implies that $f(t) = 0$ for some $t \in [-2, 0]$ and $t \in [0, -2]$.

3. (a) (5 points) Evaluate $\lim_{s \to 1} \ln(\cos(2\pi s))$.
Here "ln" denotes the natural logarithm, which is sometimes also denoted by "log".

**Solution:** ln is continuous at $\cos(2\pi 1) = 1$ so we have $\lim_{s \to 1} \ln(\cos(2\pi s)) = \ln(\cos(2\pi 1)) = \ln(1) = 0$.

(b) (5 points) Find the horizontal asymptotes of the function $f(x) = \dfrac{1 - 8x + 2x^2}{3x^2 + 9}$.

**Solution:** The horizontal asymptote is $y = 2/3$ by theorem covered in class.
Alternatively, consider

$$\frac{1 - 8x + 2x^2}{3x^2 + 9} = \frac{1/x^2 - 8/x + 2}{3 + 9/x^2} \to \frac{0 - 0 + 2}{3 + 0}$$

as $x \to \pm\infty$.

(c) (5 points) Suppose that $\lim_{x \to 0} \dfrac{f(x)}{x} = -22$. What is $\lim_{x \to 0} f(x)$?

**Solution:** Write $f(x) = x\frac{f(x)}{x}$ so by the limit laws:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x\frac{f(x)}{x} = (\lim_{x \to 0} x)(\lim_{x \to 0} \frac{f(x)}{x}) = 0 \cdot (-22) = 0.$$

(d) (5 points) Evaluate $\lim_{s\to\infty}(2^s - 3^s)$.

**Solution:** $2^s - 3^s = \frac{2^s-3^s}{3^s}3^s = ((2/3)^s - 1)\cdot 3^s \to (0-1)\cdot\infty = -\infty$ as $s \to \infty$.

(e) (5 points) Find a value for $c$ such that $f(x)$ is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$, where

$$f(x) = \begin{cases} \tan(x) & \text{when } x \geq \pi/4, \\ c\cdot x & \text{when } x < \pi/4. \end{cases}$$

**Solution:** We need that $\lim_{x\to\pi/4^-} f(x) = \lim_{x\to\pi/4^+} f(x)$, in other words $c\cdot\pi/4 = \tan(\pi/4)$. As $\tan(\pi/4) = 1$ this means $c = 4/\pi$.

(f) (5 points) Evaluate $\lim_{t\to 0}\dfrac{\tan(2t)}{\tan(3t)}$.

**Solution:** We use the fact that $\frac{\sin(x)}{x} \to 1$ when $x \to 0$.

$$\frac{\tan(2t)}{\tan(3t)} = \frac{\sin(2t)}{\sin(3t)}\frac{\cos(3t)}{\cos(2t)} = \frac{\sin(2t)}{2t}\frac{3t}{\sin(3t)}\frac{2}{3}\frac{\cos(3t)}{\cos(2t)} \to 1\cdot 1\cdot\frac{2}{3}\cdot\frac{1}{1} = \frac{2}{3}$$

as $t \to 0$.

4. (10 points) Evaluate $\lim_{x\to\infty} \frac{\cos(2x) - e^x}{e^x + \arctan(x)}$.

**Solution:** We see that the fastest growing term is $e^x$ so we do the following

$$\frac{\cos(2x) - e^x}{e^x + \arctan(x)} = \frac{\frac{\cos(2x)}{e^x} - 1}{1 + \frac{\arctan(x)}{e^x}}.$$

Now

$$\frac{\cos(2x)}{e^x} \to 0$$

as $x \to \infty$ by the Squeeze Theorem as $-1 \leq \cos(2x) \leq 1$ and $e^x$ goes to infinity.

$$\frac{\arctan(x)}{e^x} \to 0$$

as $x \to \infty$ by the Squeeze Theorem as $\arctan(x) \to \pi/4$ and $e^x \to \infty$ as $x \to \infty$.

Hence

$$\lim_{x\to\infty} \frac{\cos(2x) - e^x}{e^x + \arctan(x)} = \frac{0-1}{0+1} = -1.$$

5. (10 points) Find the derivative of $f(x) = \dfrac{1}{\sqrt{1+x}}$ at $x = 0$ using the definition of the derivative as a limit.

**Solution:**

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1+0}}}{x - 0} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1+x}} - 1}{x}$$

$$= \lim_{x\to 0} \frac{1 - \sqrt{1+x}}{x\sqrt{1+x}} = \lim_{x\to 0} \frac{1 - \sqrt{1+x}}{x\sqrt{1+x}} \cdot \frac{1 + \sqrt{1+x}}{1 + \sqrt{1+x}}$$

$$= \lim_{x\to 0} \frac{1^2 - (\sqrt{1+x})^2}{x\sqrt{1+x}(1+\sqrt{1+x})} = \lim_{x\to 0} \frac{-x}{x\sqrt{1+x}(1+\sqrt{1+x})} =$$

$$= \lim_{x\to 0} \frac{-1}{\sqrt{1+x}(1+\sqrt{1+x})} = -\frac{1}{2}$$

6. Let $g(x) = xe^{-x^2}$.

(a) (5 points) Find all points where the graph of $g(x)$ has horizontal tangent lines.

**Solution:** The slope of the tangent is $g'(x) = 1 \cdot e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2}(1 - 2x^2)$ by the product rule and chain rule.
So $g'(x) = 0$ if and only if $1 - 2x^2 = 0$ that is

$$x = \pm\frac{1}{\sqrt{2}}$$

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(b) (5 points) What is the equation of the tangent line to $g(x)$ at $x = 1$?

**Solution:** The slope is $g'(1) = -e^{-1} = -\frac{1}{e}$ and $g(1) = \frac{1}{e}$. This gives the equation:

$$y - \frac{1}{e} = -\frac{1}{e}(x - 1)$$

or

$$y = -\frac{1}{e}x + \frac{2}{e}$$

7. Answer the following questions about derivatives.

(a) (5 points) Find the 3rd derivative of $x^3 + x^2 + x + 1$ at $x = -1$.

**Solution:** The 3rd derivative is $6x$ so the answer is $-6$.

(b) (5 points) Find the derivative of $f(x) = \dfrac{\sin(x)}{\cos(x) + 1}$. Simplify your answer as much as possible.

**Solution:** We use the quotient rule:

$$f'(x) = \frac{\cos(x)(\cos(x) + 1) - \sin(x)(-\sin(x))}{(\cos(x) + 1)^2}$$

$$= \frac{\cos^2(x) + \sin^2(x) + \cos(x)}{(\cos(x) + 1)^2} = \frac{1 + \cos(x)}{(\cos(x) + 1)^2} = \frac{1}{\cos(x) + 1}.$$

(c) (5 points) Find the derivative of $g(x) = \ln(x \ln(x^2))$.

**Solution:** We use the chain rule multiple times:

$$g'(x) = \frac{1}{x \ln(x^2)}(1 \cdot \ln(x^2) + x\frac{1}{x^2}2x)$$

$$= \frac{\ln(x^2) + 2}{x \ln(x^2)}$$

(d) (5 points) You are falling into a black hole along a straight line. Your distance from the spaceship from which you fell is given by $e^{\sqrt{t}}$ meters after $t$ seconds. What is your *acceleration* after 4 seconds?

**Solution:** Acceleration is the second derivative of the position hence we compute $\frac{d^2}{(dx)^2}e^{\sqrt{t}}$ at $t = 4$.
$(e^{\sqrt{t}})' = e^{\sqrt{t}}\frac{1}{2\sqrt{t}}$ and so

$$(e^{\sqrt{t}})'' = e^{\sqrt{t}}\frac{1}{2\sqrt{t}}\frac{1}{2\sqrt{t}} + e^{\sqrt{t}}\frac{1}{2}(-\frac{1}{2}t^{-3/2})$$

$$= e^{\sqrt{t}}(\frac{1}{4t} - \frac{1}{4t^{3/2}}).$$

At $t = 4$ we have

$$e^2(\frac{1}{16} - \frac{1}{4 \cdot 8}) = \frac{e^2}{32}.$$

(e) (5 points) Suppose that $h(x)$ is a differentiable function and that $h'(x) = 2x + 1$. Further assume that $h(1) = 2$ and $h(2) = 3$. What is $\frac{d}{dx}(h \circ h \circ h)(x)$ at $x = 1$?

**Solution:** Applying the chain rule twice gives that

$$\frac{d}{dx}(h \circ h \circ h)(x) = h'(h(h(x))) \cdot h'(h(x)) \cdot h'(x).$$

As $h(h(1)) = h(2) = 3$ so

$$\frac{d}{dx}(h \circ h \circ h)(1) = h'(3) \cdot h'(2) \cdot h'(1).$$

This gives

$$\frac{d}{dx}(h \circ h \circ h)(1) = (2 \cdot 3 + 1)(2 \cdot 2 + 1)(2 \cdot 1 + 1) = 7 \cdot 5 \cdot 3 = 105.$$