University of Toronto

Faculty of Arts and Science
Term Test
MAT135H1F - Calculus I (A)
Duration - 2 hours
No Aids Allowed

Family Name: ______________________________

Given Name: ______________________________

Student Number: ______________________________

Lecture and Tutorial section:

L0101 TR10-1	L5101 TR6-9	T0101 TR1	T0102 TR1	T5101 TR5

This exam contains 11 pages (including this cover page) and 7 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work!** Write your answers in the space provided. Work scattered over the page without clear ordering will receive very little credit.
- **Justify your answers!** A correct answer without explanation or algebraic work will receive no credit; an incorrect answer supported by correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	10	
3	30	
4	10	
5	10	
6	10	
7	25	
Total:	120	

1. Answer the following short question.

   (a) (5 points) Define what it means that "the function f is continuous at $a = 1$".

   Solution: $\lim_{x\to 1} f(x) = f(a)$.

   (b) (5 points) Provide an example of a function f and point a so that no 1-sided limit of f exists at a.

   Solution: $f(x) = \frac{1}{x}$ or $\sin(\frac{1}{x})$ at $a = 0$.

   (c) (5 points) Let $g(x) = x^2$ and let f be defined by

$$f(x) = \begin{cases} -3 & \text{if } x \geq 0, \\ 3 - x & \text{if } x < 0. \end{cases}$$

   Is $f \circ g$ continuous? Is $g \circ f$ continuous?

   Solution:

$$g \circ f(x) = \begin{cases} (-3)^2 = 9 & \text{if } x \geq 0, \\ (3 - x)^2 & \text{if } x < 0 \end{cases}$$

   is not continuous as $\lim_{x\to 0^+} (3 - x)^2 = 3 \neq 9$.

   $f \circ g(x) = f(x^2) = -3$ hence continuous.

(d) (10 points) What are the 3 main types of discontinuities? Provide the name, a function and a sketch of its graph for each.

Solution: Removable:

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Jump:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Infinite: $f(x) = 1/x$.

2. (a) (4 points) State the Intermediate Value Theorem.

Solution: Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then for every y between $f(a)$ and $f(b)$ there is $x \in [a, b]$ so that $f(x) = y$.

(b) (6 points) Show that the equation $\sin(t) = t^2 - 1$ has at least 2 roots.

Solution: Let $f(t) = \sin(t) - (t^2 - 1)$. This function is continuous and $f(0) > 0$ while $f(\pm 2) < 0$ (as $\sin(\pm 2) < 3 = (\pm 2)^2 - 1$). So IVT implies that $f(t) = 0$ for some $t \in [-2, 0]$ and $t \in [0, -2]$.

3. Answer the following questions regarding limits.

(a) (5 points) Find $\lim_{s\to 1} \log(\cos(2\pi s))$.

Solution: log is continuous at $\cos(2\pi 1) = 1$ so we have $\lim_{s\to 1} \log(\cos(2\pi s)) = \log(\cos(2\pi 1)) = \log(1) = 0$.

(b) (5 points) Find the horizontal asymptotes of $\frac{1-8x+2x^2}{3x^2+9}$.

Solution: The horizontal asymptote is $y = 2/3$ by theorem covered in class. Alternatively, consider $\frac{1-8x+2x^2}{3x^2+9} = \frac{1/x^2-8/x+2}{3+9/x^2} \to \frac{0-0+2}{3+0}$ as $x \to \infty$.

(c) (5 points) Suppose that $\lim_{x\to 0} \dfrac{f(x)}{x} = -22$. What is $\lim_{x\to 0} f(x)$?

Solution: Write $f(x) = x\frac{f(x)}{x}$ so by the limit laws:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\frac{f(x)}{x} = (\lim_{x\to 0} x)(\lim_{x\to 0} \frac{f(x)}{x}) = 0 \cdot (-22) = 0.$$

(d) (5 points) Find $\lim_{s\to\infty}(2^s - 3^s)$.

Solution: $2^s - 3^s = \frac{2^s-3^s}{3^s}3^s = ((2/3)^s - 1)\cdot 3^s \to (0-1)\cdot\infty = -\infty$ as $s \to \infty$.

(e) (5 points) Find c such that f is continuous on $(-\pi/2, \pi/2)$ where

$$f(x) = \begin{cases} \tan(x) & \text{if } x \geq \pi/4, \\ c\cdot x & \text{if } x < \pi/4. \end{cases}$$

Solution: We need that $\lim_{x\to\pi/4^-} f(x) = \lim_{x\to\pi/4^+} f(x)$, in other words $c\cdot\pi/4 = \tan(\pi/4)$. As $\tan(\pi/4) = 1$ this means $c = 4/\pi$.

(f) (5 points) Find $\lim_{t\to 0}\frac{\tan(2t)}{\tan(3t)}$.

Solution: $\frac{\tan(2t)}{\tan(3t)} = \frac{\sin(2t)}{\sin(3t)}\frac{\cos(3t)}{\cos(2t)} = \frac{\sin(2t)}{2t}\frac{3t}{\sin(3t)}\frac{2}{3}\frac{\cos(3t)}{\cos(2t)} \to 1\cdot 1\cdot\frac{2}{3}\cdot\frac{1}{1} = \frac{2}{3}$

4. (10 points) Find $\lim_{x\to\infty} \dfrac{\cos(2x) - e^x}{e^x + \arctan(x)}$.

Solution: We see that the fastest term is e^x so we do the following

$$\frac{\cos(2x) - e^x}{e^x + \arctan(x)} = \frac{\frac{\cos(2x)}{e^x} - 1}{1 + \frac{\arctan(x)}{e^x}}.$$

$\frac{\cos(2x)}{e^x} \to 0$ as $x \to \infty$ by the Squeeze Theorem as $-1 \leq \cos(2x) \leq 1$ and e^x goes to infinity.
$\frac{\arctan(x)}{e^x} \to 0$ as $x \to \infty$ by the Squeeze Theorem as $\arctan(x) \to \pi/4$ and $e^x \to \infty$ as $x \to \infty$.
Hence

$$\lim_{x\to\infty} \frac{\cos(2x) - e^x}{e^x + \arctan(x)} = \frac{0-1}{0+1} = -1.$$

5. (10 points) Find the derivative of $\frac{1}{\sqrt{1+x}}$ at $x = 0$ using the definition.

6. Let $g(x) = xe^{-x^2}$.

   (a) (5 points) Find all the points where the graph of g has horizontal tangent lines.

   (b) (5 points) What is the equation for the tangent line of g when $x = 1$?

7. Answer the following questions about derivatives.

   (a) (5 points) Find the 3rd derivative of $x^3 + x^2 + x + 1$ at $x = 1$.

   Solution: The 3rd derivative is $6x$ so the answer is 6.

   (b) (5 points) Find the derivative of $\frac{\sin(u)}{\cos(u)+1}$.

   Solution:

   (c) (5 points) Find the derivative of $\log(x \log(x))$.

(d) (5 points) A particle is falling into a black hole along a straight line. Its position from the start is $e^{\sqrt{t}}$ centimeters after t seconds. What is the *acceleration* after 4 seconds?

(e) (5 points) Suppose that h is differentiable at 0, $h(0) = 0$ and $h'(0) = 3$. What is $\frac{d}{dx}(h \circ h \circ h)$ at 0?