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Communication Theory (5ETB0) 2024-2025

Assignment 1

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Table of contents

Title
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1 Problem 1: 4-PAM Modulation

1.1 Q1.1

The set of transmitted symbols is given as:

$$s_m \in \{-3, -1, 1, 3\}.$$

Here, the symbols s_m represent the possible signal levels in the 4-PAM modulation scheme.

Since there are $M = 4$ distinct symbols, the number of bits b that can be represented by each symbol is calculated using the formula:

$$b = \log_2(M), \quad \text{(where } M \text{ is the number of symbols in the modulation scheme).} \tag{1.1}$$

Substituting $M = 4$ into the equation:

$$\begin{aligned} b &= \log_2(4), \quad \text{(logarithm base 2)} \\ &= 2 \quad \text{bits per symbol.} \end{aligned} \tag{1.2}$$

This means that each transmitted symbol in the 4-PAM modulation can encode exactly 2 bits of information.

1.2 Q1.2

1. Symbols

The set of possible transmitted symbols in the 4-PAM modulation scheme is:

$$s_m \in \{-3, -1, 1, 3\}.$$

2. Maximum Likelihood (ML) Decision Rule

For equiprobable symbols, the ML decision rule minimizes the squared Euclidean distance between the received signal r and the possible transmitted symbols s_m:

$$f(r) = \arg\min_{s_m} |r - s_m|^2, \quad \text{(select the } s_m \text{ closest to } r\text{).} \tag{1.3}$$

3. Decision Thresholds

The thresholds for deciding between adjacent symbols are derived as the midpoints:

- **Threshold 1 (between -3 and -1):**

$$\begin{aligned} \text{Threshold 1} &= \frac{-3 + (-1)}{2}, \quad \text{(average of } -3 \text{ and } -1\text{)} \\ &= -2. \end{aligned} \tag{1.4}$$

- **Threshold 2 (between -1 and 1):**

$$\begin{aligned} \text{Threshold 2} &= \frac{-1 + 1}{2}, \quad \text{(average of } -1 \text{ and } 1\text{)} \\ &= 0. \end{aligned} \tag{1.5}$$

- **Threshold 3 (between** 1 **and** 3**):**

$$\begin{aligned}\text{Threshold } 3 &= \frac{1+3}{2}, \quad \text{(average of 1 and 3)} \\ &= 2. \end{aligned} \tag{1.6}$$

4. Decision Boundaries

The derived thresholds divide the range of r into decision regions corresponding to each symbol s_m:

- $r \leq -2$: Decide $s_m = -3$,
- $-2 < r \leq 0$: Decide $s_m = -1$,
- $0 < r \leq 2$: Decide $s_m = 1$,
- $r > 2$: Decide $s_m = 3$.

5. Final Decision Rule

$$f(r) = \begin{cases} -3 & \text{if } r \leq -2, \\ -1 & \text{if } -2 < r \leq 0, \\ 1 & \text{if } 0 < r \leq 2, \\ 3 & \text{if } r > 2. \end{cases}$$

This decision rule ensures the transmitted symbol s_m is accurately decoded based on the received signal r.

1.3 Q1.3

To minimize the error probability in the presence of Gaussian noise, the **Maximum A Posteriori (MAP)** decision rule is used.

1. Decision Rule

$$f(r) = \arg\max_{s_m} P(S = s_m) p_R(r|S = s_m), \quad \text{(choose the symbol } s_m \text{ with the highest posterior probability).} \tag{1.7}$$

2. Conditional PDF for Gaussian Noise

$$p_R(r|S = s_m) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r - s_m)^2}{2\sigma^2}\right), \quad \text{(Gaussian PDF with mean } s_m\text{).} \tag{1.8}$$

3. Simplification Using Logarithms

$$\ln\left(P(S = s_m) p_R(r|S = s_m)\right) = \ln P(S = s_m) - \frac{(r - s_m)^2}{2\sigma^2}. \tag{1.9}$$

Thus, the decision rule becomes:

$$f(r) = \arg\max_{s_m} \left(\ln P(S = s_m) - \frac{(r - s_m)^2}{2\sigma^2}\right) \tag{1.10}$$

4. Decision Boundaries

The decision boundaries b_1, b_2, b_3 are the points where the posterior probabilities of adjacent symbols are equal. This happens when the likelihoods of two adjacent symbols s_m and s_{m+1} intersect.

(a) Between $s_1 = -3$ and $s_2 = -1$:

$$\ln P(S = -3) - \frac{(b_1 + 3)^2}{2\sigma^2} = \ln P(S = -1) - \frac{(b_1 + 1)^2}{2\sigma^2}.$$

Rearranging terms:

$$b_1 = -2 - 0.42\sigma^2, \quad \text{(threshold depends on noise variance } \sigma^2\text{)}. \tag{1.11}$$

(b) Between $s_2 = -1$ and $s_3 = 1$:

$$\ln P(S = -1) - \frac{(b_2 + 1)^2}{2\sigma^2} = \ln P(S = 1) - \frac{(b_2 - 1)^2}{2\sigma^2}.$$

Rearranging terms:

$$b_2 = 0, \quad \text{(this threshold is symmetric and does not depend on } \sigma^2\text{)}. \tag{1.12}$$

(c) Between $s_3 = 1$ and $s_4 = 3$:

$$\ln P(S = 1) - \frac{(b_3 - 1)^2}{2\sigma^2} = \ln P(S = 3) - \frac{(b_3 - 3)^2}{2\sigma^2}.$$

Rearranging terms:

$$b_3 = 2 + 0.42\sigma^2, \quad \text{(threshold depends on noise variance } \sigma^2\text{)}. \tag{1.13}$$

5. Final Decision Thresholds

$$b_1 = -2 - 0.42\sigma^2, \quad b_2 = 0, \quad b_3 = 2 + 0.42\sigma^2.$$

1.4 Q1.4

1. Known Details

We consider variance $\sigma^2 = 1$ and the decision boundaries are given as:

$$b_1 = -2.42, \quad b_2 = 0, \quad b_3 = 2.42.$$

The **average symbol error probability** is calculated using:

$$P_e = \sum_{m=1}^{4} P(s_m \neq f(r) \mid S = s_m) P(S = s_m) \tag{1.14}$$

2. Decision Boundaries and Error Regions

For each transmitted symbol s_m, the distances x_{ij} from the i-th symbol to the j-th decision boundary determine the probability of error. The Gaussian **Q-function** is used to compute the error probabilities:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$$

The error probability contributions are:

$$P(s_1 \neq f(r) \mid S = s_1) = Q\left(\frac{x_{11}}{\sigma}\right), \tag{1.15}$$

$$P(s_2 \neq f(r) \mid S = s_2) = Q\left(\frac{x_{21}}{\sigma}\right) + Q\left(\frac{x_{22}}{\sigma}\right), \tag{1.16}$$

$$P(s_3 \neq f(r) \mid S = s_3) = Q\left(\frac{x_{32}}{\sigma}\right) + Q\left(\frac{x_{33}}{\sigma}\right), \tag{1.17}$$

$$P(s_4 \neq f(r) \mid S = s_4) = Q\left(\frac{x_{43}}{\sigma}\right). \tag{1.18}$$

3. Distance Calculations

The distances x_{ij} are calculated as:

- $x_{11} = |b_1 - (-3)| = 0.58,$
- $x_{21} = |b_1 - (-1)| = 1.42, \quad x_{22} = |b_2 - (-1)| = 1,$
- $x_{32} = |b_2 - 1| = 1, \quad x_{33} = |b_3 - 1| = 1.42,$
- $x_{43} = |b_3 - 3| = 0.58.$

4. Symbol Probabilities

- **Gaussian distribution**: $P(s_1) = 0.15,\ P(s_2) = 0.35,\ P(s_3) = 0.35,\ P(s_4) = 0.15,$
- **Equiprobable distribution**: $P(s_m) = 0.25$ for all m.

5. Average Symbol Error Probability

(a) Gaussian Symbol Distribution Substituting the values of x_{ij}, $Q(x)$, and $P(S = s_m)$, the average symbol error probability is:

$$P_e = (0.15 \cdot Q(0.58)) + (0.35 \cdot (Q(1.42) + Q(1))) + (0.35 \cdot (Q(1) + Q(1.42))) + (0.15 \cdot Q(0.58)). \tag{1.19}$$

Using approximations for the Q-function:

$$Q(0.58) \approx 0.28, \quad Q(1.42) \approx 0.077, \quad Q(1) \approx 0.16,$$

we calculate:

$$\begin{aligned} P_e &= (0.15 \cdot 0.28) + (0.35 \cdot (0.077 + 0.16)) + (0.35 \cdot (0.077 + 0.16)) + (0.15 \cdot 0.28) \\ &= 0.042 + 0.083 + 0.083 + 0.042 \\ &= 0.25. \end{aligned} \tag{1.20}$$

(b) Equiprobable Symbol Distribution For the equiprobable case, all distances are equal ($x_{ij} = 1$), and $Q(1) = 0.16$. Substituting:

$$P_e = (0.25 \cdot Q(1)) + (0.25 \cdot (Q(1) + Q(1))) + (0.25 \cdot (Q(1) + Q(1))) + (0.25 \cdot Q(1)). \tag{1.21}$$

Simplifying:

$$\begin{aligned} P_e &= (0.25 \cdot 0.16) + (0.25 \cdot 2 \cdot 0.16) + (0.25 \cdot 2 \cdot 0.16) + (0.25 \cdot 0.16) \\ &= 0.04 + 0.08 + 0.08 + 0.04 \\ &= 0.24. \end{aligned} \tag{1.22}$$

1.5 Q1.5

1. Problem Setup

The average symbol energy E_s:

$$E_s = \sum_{m=1}^{4} P(S = s_m)(s_m)^2 \tag{1.23}$$

Here:

- $P(S = s_m)$ is the probability of symbol s_m,
- $s_m \in \{-3, -1, 1, 3\}$ are the transmitted symbols.

2. Case 1: Equiprobable Distribution

For the equiprobable case, all symbols have the same probability:

$$P(S = s_m) = 0.25 \quad \text{for all } m.$$

The energy contributions:

$$\text{Symbol } s_1 = -3: \; P(S = -3)(-3)^2 = 0.25 \cdot 9 = 2.25, \tag{1.24}$$
$$\text{Symbol } s_2 = -1: \; P(S = -1)(-1)^2 = 0.25 \cdot 1 = 0.25, \tag{1.25}$$
$$\text{Symbol } s_3 = 1: \; P(S = 1)(1)^2 = 0.25 \cdot 1 = 0.25, \tag{1.26}$$
$$\text{Symbol } s_4 = 3: \; P(S = 3)(3)^2 = 0.25 \cdot 9 = 2.25. \tag{1.27}$$

$$E_s = 2.25 + 0.25 + 0.25 + 2.25 = 5. \tag{1.28}$$

3. Case 2: Gaussian Distribution

For the Gaussian case, the symbol probabilities are:

$$P(S = -3) = 0.15, \quad P(S = -1) = 0.35, \quad P(S = 1) = 0.35, \quad P(S = 3) = 0.15.$$

The energy contributions:

$$\text{Symbol } s_1 = -3: \; P(S=-3)(-3)^2 = 0.15 \cdot 9 = 1.35, \tag{1.29}$$

$$\text{Symbol } s_2 = -1: \; P(S=-1)(-1)^2 = 0.35 \cdot 1 = 0.35, \tag{1.30}$$

$$\text{Symbol } s_3 = 1: \; P(S=1)(1)^2 = 0.35 \cdot 1 = 0.35, \tag{1.31}$$

$$\text{Symbol } s_4 = 3: \; P(S=3)(3)^2 = 0.15 \cdot 9 = 1.35. \tag{1.32}$$

$$E_s = 1.35 + 0.35 + 0.35 + 1.35 = 3.4. \tag{1.33}$$

4. Summary of Results

These results show that the Gaussian distribution results in a lower average symbol energy because the lower-energy symbols (-1 and 1) have higher probabilities, which reduces the overall energy contribution.

2 Problem 2: 8-PSK Modulation and Repetition Coding

2.1 Q2.1

1. Signal Representation

In 8-PSK, the transmitted symbols:

$$s_i = e^{j\theta_i}, \quad \theta_i = \frac{(i-1)\pi}{4}, \quad i = 1, 2, \ldots, 8, \quad \text{(equally spaced on the unit circle in the complex plane).} \tag{2.1}$$

In Cartesian coordinates:

$$s_{i,x} = \cos\left(\frac{(i-1)\pi}{4}\right), \quad \text{(real component of } s_i\text{),} \tag{2.2}$$

$$s_{i,y} = \sin\left(\frac{(i-1)\pi}{4}\right), \quad \text{(imaginary component of } s_i\text{).} \tag{2.3}$$

$$|s_i| = \sqrt{s_{i,x}^2 + s_{i,y}^2} = 1. \tag{2.4}$$

2. Decision Rule

The optimum receiver chooses s_i that minimizes the Euclidean distance between the received signal r and s_i:

$$\hat{s}_m = \arg\min_i \left(|r - s_i|^2\right) \tag{2.5}$$

Expanding the Euclidean distance:

$$|r - s_i|^2 = |r|^2 - 2\,\mathrm{Re}(r \cdot s_i^*) + |s_i|^2, \tag{2.6}$$

$$= |r|^2 - 2\,\mathrm{Re}(r \cdot s_i^*) + 1, \quad \text{(since } |s_i|^2 = 1\text{).} \tag{2.7}$$

Since $|r|^2 + 1$ are constants and do not affect the decision rule, the criterion simplifies to:

$$\hat{s}_m = \arg\max_i \mathrm{Re}(r \cdot s_i^*), \quad \text{(maximize the real part of the correlation between } r \text{ and } s_i\text{).} \tag{2.8}$$

3. Angle-Based Decision Rule

$$\hat{s}_m = \arg\max_i \cos(\theta_{rs_i}), \quad \text{(select the } s_i \text{ with the smallest angle to } r\text{).} \tag{2.9}$$

Maximizing real part of $r \cdot s_i^*$ is same as to minimizing the angular separation between r and s_i.

4. Symmetry in Decision Regions

Due to the symmetry of the 8-PSK constellation, the decision regions are identical and evenly spaced around the unit circle. Each decision region I_i corresponds to one symbol s_i, and adjacent regions are separated by angles of $\pi/4$.

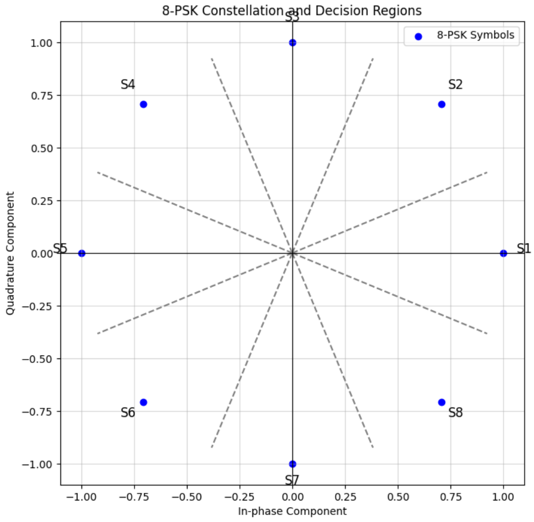

Figure 2.1: 8-PSK Constellation

2.2 Q2.2

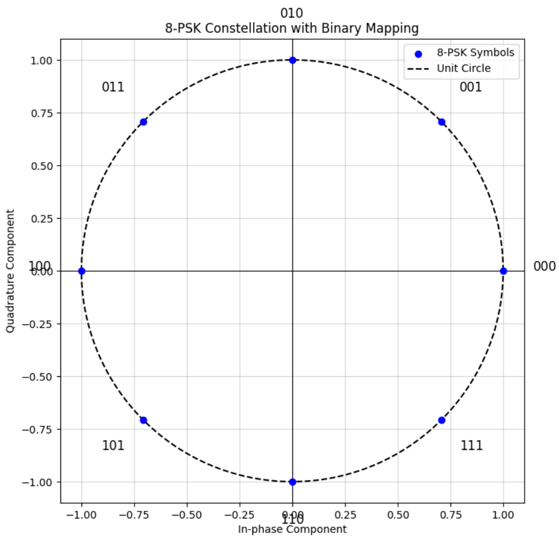

Figure 2.2: 8-PSK Binary Encoded Constellation

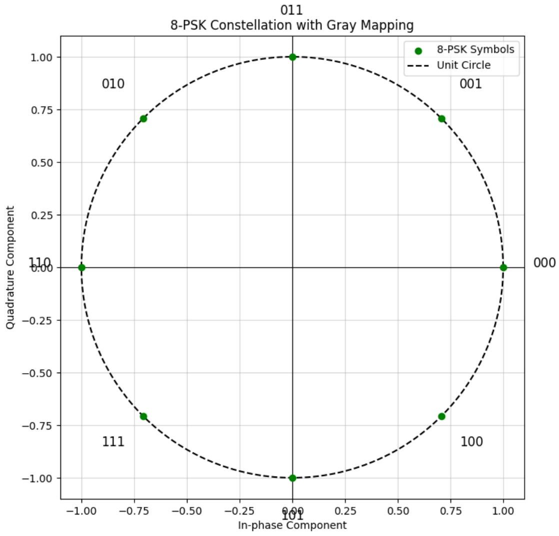

Figure 2.3: 8-PSK Gray Encoded Constellation

2.3 Q2.3

1. Binary Mapping

In binary mapping, adjacent symbol errors can flip 1 to 3 bits. For example:

- 000 $\leftrightarrow$ 001: 1-bit flip,
- 000 $\leftrightarrow$ 111: 3-bit flip.

The bit error probability is proportional to:

$$P_b^{\text{binary}} \propto P_s \cdot (\text{average bits flipped per error}). \tag{2.10}$$

2. Gray Mapping

In Gray mapping, adjacent symbol errors always flip exactly 1 bit. For example:

- 000 $\leftrightarrow$ 001: 1-bit flip.

The bit error probability is proportional to:

$$P_b^{\text{gray}} \propto P_s. \tag{2.11}$$

3. Conclusion

Gray mapping minimizes P_b because adjacent errors flip only 1 bit:

$$P_b^{\text{gray}} < P_b^{\text{binary}}. \tag{2.12}$$

2.4 Q2.4

1. Given Information

Each bit error is independent with probability b_e. The received sequence $r = [r_1, r_2, r_3]$ corresponds to one of the two codewords:

- Codeword $0 \to 000$,
- Codeword $1 \to 111$.

2. Likelihood of a Codeword

The probability of receiving r given a transmitted codeword is:

- If the true codeword is 000, the likelihood is:
$$P(r|000) = (1 - b_e)^{3-a} \cdot b_e^a, \quad \text{(where } a \text{ is the number of ones in } r\text{)}. \tag{2.13}$$
- If the true codeword is 111, the likelihood is:
$$P(r|111) = (1 - b_e)^{a} \cdot b_e^{3-a}, \quad \text{(where } a \text{ is the number of ones in } r\text{)}. \tag{2.14}$$

3. Optimal Decision Rule

The optimal decision rule selects the codeword $\hat{m}$ that maximizes the posterior probability:

$$\hat{m} = \arg\max_m P(r|m). \tag{2.15}$$

- If $P(r|000) > P(r|111)$, decide $\hat{m} = 0$,
- If $P(r|111) > P(r|000)$, decide $\hat{m} = 1$.

4. Simplification: Majority Voting

The decision depends on number of ones a in the received sequence:

- If $a \geq 2$, decide $\hat{m} = 1$ (majority ones),
- If $a \leq 1$, decide $\hat{m} = 0$ (majority zeros).

2.5 Q2.5

1. Conditions for Error

An error occurs if the received sequence r is wrongly identified:

- $m = 0$ (true codeword is 000):
$$\text{Error if } a = 2 \text{ or } a = 3. \tag{2.16}$$
- $m = 1$ (true codeword is 111):
$$\text{Error if } a = 0 \text{ or } a = 1. \tag{2.17}$$

2. Error Probabilities

The probabilities of these error events are:

$$P(a=2|m=0) = \binom{3}{2} b_e^2 (1-b_e)^1 = 3 \cdot 0.25^2 \cdot 0.75 = \frac{3}{64}, \tag{2.18}$$

$$P(a=3|m=0) = \binom{3}{3} b_e^3 (1-b_e)^0 = 1 \cdot 0.25^3 = \frac{1}{64}, \tag{2.19}$$

$$P(a=0|m=1) = \binom{3}{0} b_e^3 (1-b_e)^0 = 1 \cdot 0.25^3 = \frac{1}{64}, \tag{2.20}$$

$$P(a=1|m=1) = \binom{3}{1} b_e^1 (1-b_e)^2 = 3 \cdot 0.25^1 \cdot 0.75^2 = \frac{3}{64}. \tag{2.21}$$

3. Total Bit Error Probability

Average over the two codewords:

$$P_e = \frac{1}{2}\big(P(a=2|m=0) + P(a=3|m=0)\big) + \frac{1}{2}\big(P(a=0|m=1) + P(a=1|m=1)\big). \tag{2.22}$$

Substituting the values:

$$\begin{aligned} P_e &= \frac{1}{2}\left(\frac{3}{64} + \frac{1}{64}\right) + \frac{1}{2}\left(\frac{1}{64} + \frac{3}{64}\right), \\ &= \frac{1}{2} \cdot \frac{4}{64} + \frac{1}{2} \cdot \frac{4}{64}, \\ &= \frac{8}{64} = \frac{1}{8}. \end{aligned} \tag{2.23}$$

Shows effectiveness of repetition coding in lowering errors through redundancy.

3 Problem 3: Error Probability of 16-QAM

3.1 Q3.3

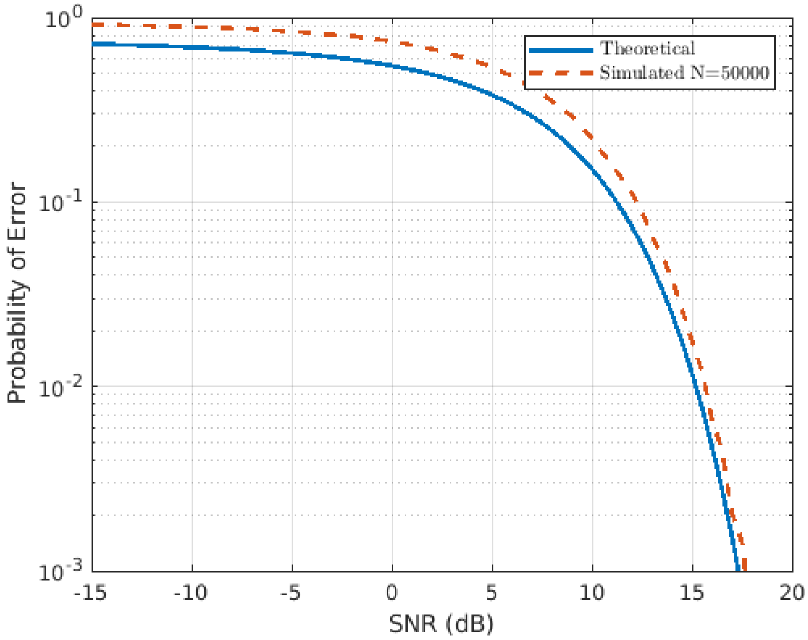

Figure 3.1: Probability of Error Simulation for 50000 Symbols

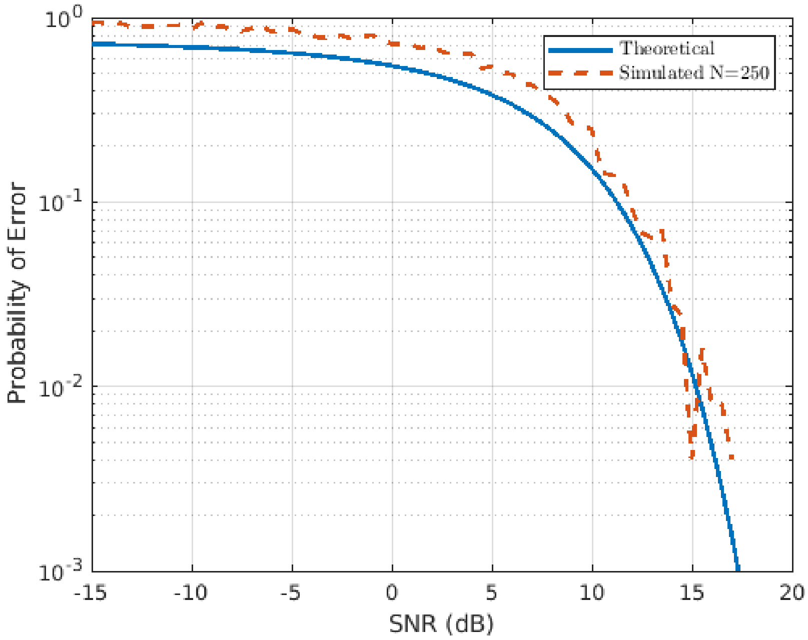

Figure 3.2: Probability of Error Simulation for 250 Symbols

The simulation with 50000 symbols took 6.06 seconds and the one with 250 symbols took 0.12 seconds.

3.2 Q3.4

1. Comparison: $N = 250$ vs. $N = 50000$

$N = 50000$ yield results closer to predictions due to reduced variance. However, requires more computational time. For $N = 250$, the higher variance causes deviations, but shorter compuation time.

2. Nearest-Neighbor Approximation Validity

The nearest-neighbor approximation assumes errors occur only with adjacent symbols. This is valid in high SNR regimes where noise more rarely pushes a symbol beyond the nearest decision boundary.

$$\text{Pr}\{n_x \geq d/2\} = Q\left(\sqrt{\frac{2P}{10\sigma^2}}\right).$$

Calculation: SNR at 10 dB The signal-to-noise ratio (SNR) in linear scale is:

$$\text{Linear SNR} = 10^{10/10} = 10. \tag{3.1}$$

Substituting into the Q-function argument:

$$x = \sqrt{\frac{2 \times P}{10 \times \sigma^2}} = \sqrt{\frac{2 \times 10}{10}} = \sqrt{2}.$$

The $Q(\sqrt{2})$ value confirms that the nearest-neighbor approximation remains valid.

For lower SNR, e.g., SNR $= 5\,\text{dB}$, the $Q(x)$ value increases a lot, indicating a higher likelihood of wrong symbols beyond the nearest neighbor.