Department of Electrical Engineering

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Communication Theory (5ETB0) 2024-2025

Assignment 2

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Where innovation starts

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Communication Theory (5ETB0)
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Where innovation starts

1 Problem 1: Gram-Schmidt Orthogonalization and Signal Energy Considerations

1.1 Q1.1

The signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ are piecewise defined over the time intervals $[0, \frac{T}{3}]$, $[\frac{T}{3}, \frac{2T}{3}]$, $[\frac{2T}{3}, T]$.

Energy of $s_1(t)$: The energy of $s_1(t)$ is calculated as:

$$\|s_1(t)\|^2 = \int_0^T s_1^2(t)\,dt \tag{1.1}$$

$$= \int_0^{T/3} (2A)^2\,dt + \int_{T/3}^{2T/3} (-A)^2\,dt + \int_{2T/3}^{T} A^2\,dt$$

$$= \frac{4A^2T}{3} + \frac{A^2T}{3} + \frac{A^2T}{3} = 2A^2T. \tag{1.2}$$

Normalization of $s_1(t)$:

$$\phi_1(t) = \frac{s_1(t)}{\|s_1(t)\|} = \begin{cases} \sqrt{\frac{2}{T}}, & 0 < t \leq \frac{T}{3}, \\ -\sqrt{\frac{1}{2T}}, & \frac{T}{3} < t \leq \frac{2T}{3}, \\ \sqrt{\frac{1}{2T}}, & \frac{2T}{3} < t \leq T. \end{cases} \tag{1.3}$$

Orthogonalization of $s_2(t)$: The projection of $s_2(t)$ onto $\phi_1(t)$:

$$\int_0^T s_2(t)\phi_1(t)\,dt = A\sqrt{2T}. \tag{1.4}$$

$$\theta_2(t) = s_2(t) - \left(\int_0^T s_2(t)\phi_1(t)\,dt\right)\phi_1(t). \tag{1.5}$$

Normalization of $\theta_2(t)$:

$$\|\theta_2(t)\| = A\sqrt{\frac{2T}{3}}. \tag{1.6}$$

The normalized function $\phi_2(t)$:

$$\phi_2(t) = \frac{\theta_2(t)}{\|\theta_2(t)\|} = \begin{cases} 0, & 0 < t \leq \frac{T}{3}, \\ -\sqrt{\frac{3}{2T}}, & \frac{T}{3} < t \leq \frac{2T}{3}, \\ -\sqrt{\frac{3}{2T}}, & \frac{2T}{3} < t \leq T. \end{cases} \tag{1.7}$$

To orthogonalize $s_4(t)$:

$$\theta_4(t) = s_4(t) - \left(\int_0^T s_4(t)\phi_1(t)\,dt\right)\phi_1(t) - \left(\int_0^T s_4(t)\phi_2(t)\,dt\right)\phi_2(t). \tag{1.8}$$

The projections are:

$$\int_0^T s_4(t)\phi_1(t)\,dt = -2A\sqrt{2T}, \tag{1.9}$$

$$\int_0^T s_4(t)\phi_2(t)\,dt = \frac{2A}{3}\sqrt{\frac{3T}{2}}. \tag{1.10}$$

After subtracting the projections:

$$\theta_4(t) = 0, \quad \text{for all } t \in [0, T]. \tag{1.11}$$

No other orthogonal basis function $\phi_4(t)$ needed.

Since $\phi_1(t)$ and $\phi_2(t)$ span the signal spacce:

$$\text{Dimensionality} = 2. \tag{1.12}$$

1.2 Q1.2

Energy of $s_1(t)$ is:

$$\|s_1(t)\|^2 = E_1 = 2A^2T. \tag{1.13}$$

Vector Representations:

1. **For $s_1(t)$:**

$$s_1 \cdot \phi_1 = \int_0^T s_1(t)\phi_1(t)\,dt = A\sqrt{2T}, \tag{1.14}$$

$$s_1 \cdot \phi_2 = 0. \tag{1.15}$$

Vector:

$$s_1 = (A\sqrt{2T}, 0) = (\sqrt{E_1}, 0). \tag{1.16}$$

2. **For $s_2(t)$:**

$$s_2 \cdot \phi_1 = \int_0^T s_2(t)\phi_1(t)\,dt = A\sqrt{2T}, \tag{1.17}$$

$$s_2 \cdot \phi_2 = \|\theta_2\| = A\sqrt{\frac{2T}{3}}. \tag{1.18}$$

Vector:

$$s_2 = (A\sqrt{2T}, A\sqrt{\frac{2T}{3}}) = (\sqrt{E_1}, \sqrt{\frac{E_1}{3}}). \tag{1.19}$$

3. **For $s_3(t)$:** Since $s_3(t) = -2s_1(t)$:

$$s_3 = -2s_1 = (-2\sqrt{E_1}, 0). \tag{1.20}$$

4. **For $s_4(t)$:**

$$s_4 \cdot \phi_1 = \int_0^T s_4(t)\phi_1(t)\,dt = -2A\sqrt{2T}, \tag{1.21}$$

$$s_4 \cdot \phi_2 = \int_0^T s_4(t)\phi_2(t)\,dt = \frac{2A}{3}\sqrt{\frac{3T}{2}}. \tag{1.22}$$

Vector:

$$s_4 = (-2A\sqrt{2T}, \frac{2A}{3}\sqrt{\frac{3T}{2}}) = (-2\sqrt{E_1}, \sqrt{\frac{E_1}{3}}). \tag{1.23}$$

Signal Constellation:

Assume $E_1 = 1$. Then:

$$s_1 = (1, 0), \tag{1.24}$$

$$s_2 = (1, \sqrt{\frac{1}{3}}), \tag{1.25}$$

$$s_3 = (-2, 0), \tag{1.26}$$

$$s_4 = (-2, \sqrt{\frac{1}{3}}). \tag{1.27}$$

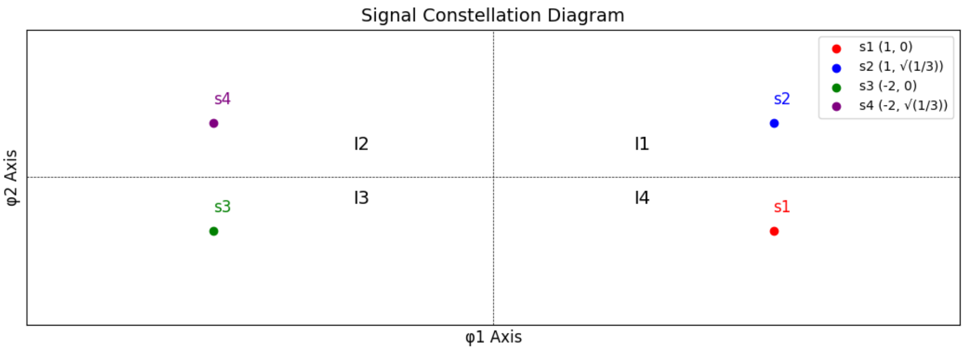

Figure 1.1: Signal Constellation Diagram

1.3 Q1.3

The average signal energy E_s:

$$E_s = \frac{1}{4}\sum_{i=1}^{4} E_i, \tag{1.28}$$

$$E_1 = 2A^2T, \tag{1.29}$$

$$E_2 = E_1 + \frac{E_1}{3} = 2A^2T + \frac{2A^2T}{3} = \frac{8A^2T}{3}, \tag{1.30}$$

$$E_3 = 4E_1 = 4 \cdot 2A^2T = 8A^2T, \tag{1.31}$$

$$E_4 = 4E_1 + \frac{E_1}{3} = 8A^2T + \frac{2A^2T}{3} = \frac{26A^2T}{3}. \tag{1.32}$$

Substitute into (1.28):

$$\begin{aligned} E_s &= \frac{1}{4}(E_1 + E_2 + E_3 + E_4) \\ &= \frac{1}{4}\left(2A^2T + \frac{8A^2T}{3} + 8A^2T + \frac{26A^2T}{3}\right) \\ &= \frac{1}{4} \cdot \frac{116A^2T}{3} = \frac{16A^2T}{3}. \end{aligned} \tag{1.33}$$

Average bit energy E_b:

$$E_b = \frac{E_s}{\log_2(4)} = \frac{E_s}{2}, \tag{1.34}$$

$$E_b = \frac{\frac{16A^2T}{3}}{2} = \frac{8A^2T}{3}. \tag{1.35}$$

1.4 Q1.4

Probability error for optimal detector calculated using the distances between points and the Q-function.

Symbol Distances

1. Distance between s_1 and s_2:

$$\begin{aligned} d_{12}^2 &= \|s_1 - s_2\|^2 = (\sqrt{E_1} - \sqrt{E_1})^2 + (0 - \sqrt{\frac{E_1}{3}})^2 \\ d_{12}^2 &= \frac{E_1}{3}, \quad d_{12} = \sqrt{\frac{E_1}{3}}. \end{aligned} \tag{1.36}$$

2. Distance between s_1 and s_3:

$$\begin{aligned} d_{13}^2 &= \|s_1 - s_3\|^2 = (\sqrt{E_1} - (-2\sqrt{E_1}))^2 + (0 - 0)^2 \\ d_{13}^2 &= (3\sqrt{E_1})^2 = 9E_1, \quad d_{13} = 3\sqrt{E_1}. \end{aligned} \tag{1.37}$$

Union Bound

$$P_e \leq Q\left(\frac{d_{12}}{2\sigma}\right) + Q\left(\frac{d_{13}}{2\sigma}\right), \tag{1.38}$$

$$P_e \leq Q\left(\sqrt{\frac{E_1}{6N_0}}\right) + Q\left(3\sqrt{\frac{E_1}{2N_0}}\right). \tag{1.39}$$

Exact Error Probability

Complement of correct detection probability P_c:

$$P_c = P(R_1 > -0.5\sqrt{E_1}) \cdot P(R_2 < 0.5\sqrt{\frac{E_1}{3}}). \tag{1.40}$$

$$\begin{aligned} P(R_1 > -0.5\sqrt{E_1}) &= 1 - Q\left(\frac{d_{13}}{2\sigma}\right) \\ &= 1 - Q\left(3\sqrt{\frac{E_1}{2N_0}}\right). \end{aligned} \tag{1.41}$$

$$\begin{aligned} P(R_2 < 0.5\sqrt{\frac{E_1}{3}}) &= 1 - Q\left(\frac{d_{12}}{2\sigma}\right) \\ &= 1 - Q\left(\sqrt{\frac{E_1}{6N_0}}\right). \end{aligned} \tag{1.42}$$

The exact error probability:

$$P_e = 1 - P_c$$
$$P_e = 1 - \left(1 - Q\left(3\sqrt{\frac{E_1}{2N_0}}\right)\right)\left(1 - Q\left(\sqrt{\frac{E_1}{6N_0}}\right)\right). \tag{1.43}$$

1.5 Q1.5

Bits Per Symbol

$$\text{Bits per Symbol} = \log_2(M), \quad M = 4 \implies \log_2(4) = 2. \tag{1.44}$$

Transmission Rate

$$\text{Transmission Rate (bps)} = \frac{\text{Bits per Symbol}}{\text{Time per Symbol}}. \tag{1.45}$$

$$\text{Time per Symbol} = T = 2 \times 10^{-8} \text{ seconds}, \tag{1.46}$$
$$\text{Transmission Rate} = \frac{2}{2 \times 10^{-8}} \text{ bps}. \tag{1.47}$$

$$\text{Transmission Rate} = 10^8 \text{ bps}. \tag{1.48}$$

2 Problem 2: Wireless Transmission of Recorded Temperature Data

2.1 Q2.1

Transmitted bits: 1101001110010001

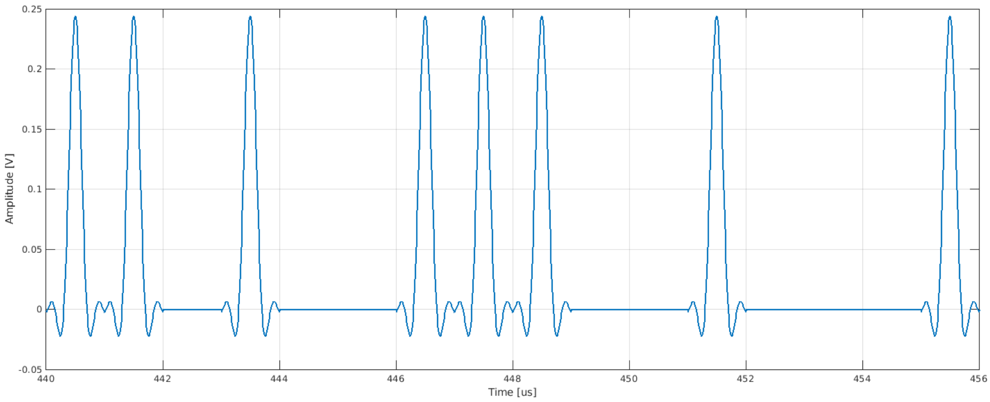

Figure 2.1: Transmitted symbols Plot

2.2 Q2.2

1. **Highest SNR:** For $r^{(2)}(t)$, the modes separated with minimal overlap:

$$\mathrm{SNR}_{r^{(2)}} \gg \mathrm{SNR}_{r^{(1)}}, \mathrm{SNR}_{r^{(3)}}. \tag{2.1}$$

2. **Lowest SNR:** For $r^{(3)}(t)$, significant overlap observed:

$$\mathrm{SNR}_{r^{(3)}} \ll \mathrm{SNR}_{r^{(1)}}, \mathrm{SNR}_{r^{(2)}}. \tag{2.2}$$

$r^{(2)}(t)$ is least affected by noise, while $r^{(3)}(t)$ has highest noise variance.

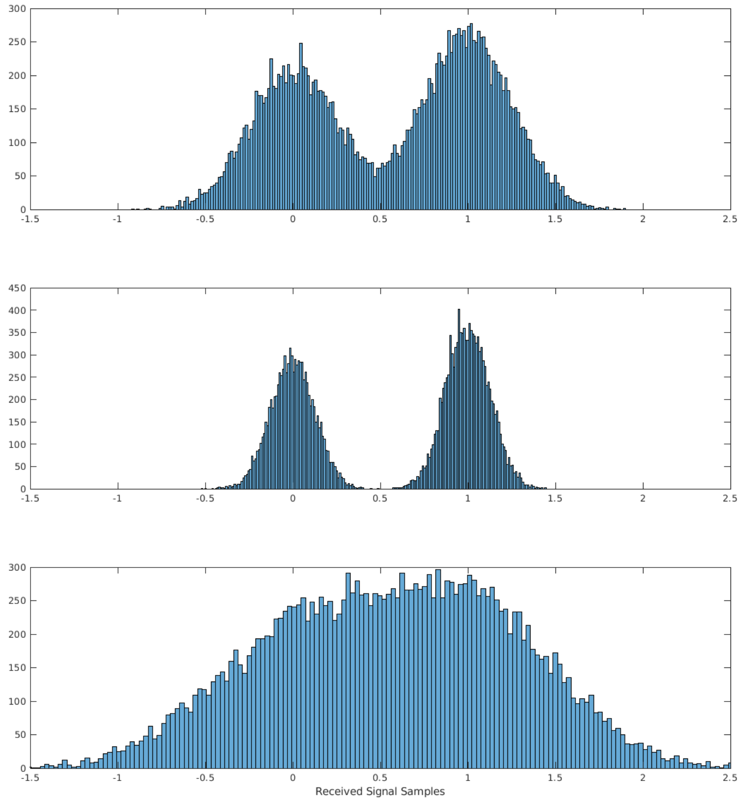

Figure 2.2: Histograms of Symbolic Vectors

2.3 Q2.4

1. **For** $r^{(1)}(t)$**:**

$$\mathrm{BER}_1 = 0.02, \tag{2.3}$$

2. **For** $r^{(2)}(t)$**:**

$$\mathrm{BER}_2 = 0, \tag{2.4}$$

error free transmission

3. **For** $r^{(3)}(t)$**:**

$$\mathrm{BER}_3 = 0.2, \tag{2.5}$$

2.4 Q2.5

1. **For** $r^{(1)}(t)$**:**

$$\mathrm{SNR}_1 = 9\,[\mathrm{dB}], \tag{2.6}$$

2. **For** $r^{(2)}(t)$**:**

$$\mathrm{SNR}_2 = 35\,[\mathrm{dB}], \tag{2.7}$$

reliable transmission.

3. **For** $r^{(3)}(t)$**:**

$$\mathrm{SNR}_3 = 2\,[\mathrm{dB}], \tag{2.8}$$

degradatimg transmission.

2.5 Q2.6

r_2 aligns with transmitted reading, r_1 sometimes shows large deviations due to flipped significant bits, r_3 displays oscillations.

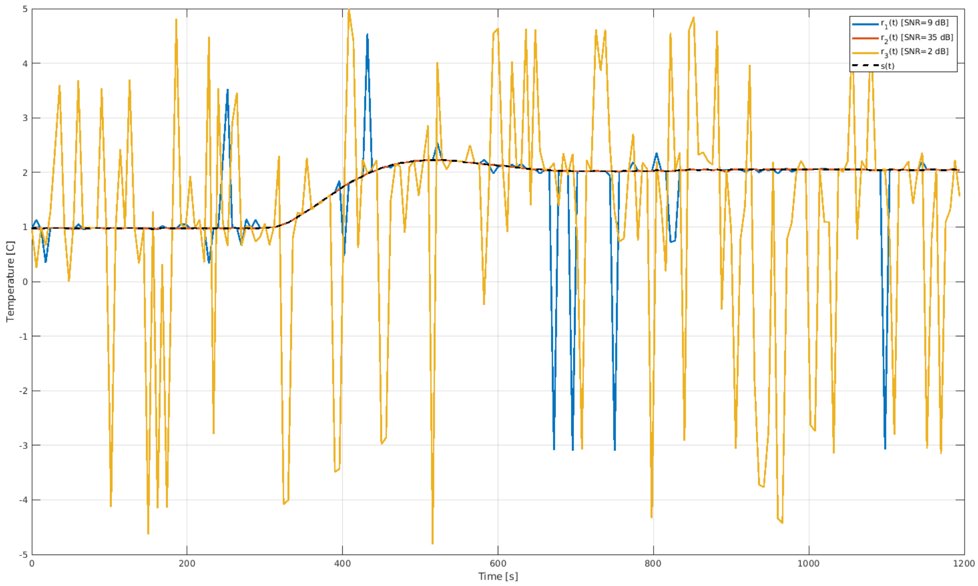

Figure 2.3: Decoded Temperaturee Readings