Communication Theory (5ETB0)
Module 7.2

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Module 7.2

Presentation Outline

Orthogonal Signal Structures: Definition

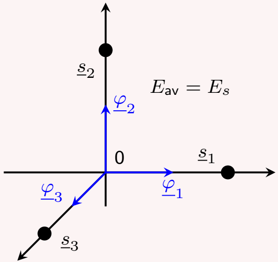
Orthogonal Signal set: Definition

Consider $|\mathcal{M}|$ signals $s_m(t)$ with a-priori probabilities $1/|\mathcal{M}|$ for $m \in \mathcal{M}$. All signals in an orthogonal set are assumed to have equal energy and are orthogonal i.e.,

$$\underline{s}_m \triangleq \sqrt{E_s}\underline{\varphi}_m \text{ for } m \in \mathcal{M},$$

where $\underline{\varphi}_m$ is the unit-vector corresponding to dimension m. There are as many building-block waveforms $\varphi_m(t)$ and dimensions in the signal space as there are messages.

Example: $|\mathcal{M}| = 3$, $\underline{\varphi}_1 = (1, 0, 0)$, $\underline{\varphi}_2 = (0, 1, 0)$, $\underline{\varphi}_3 = (0, 0, 1)$



Orthogonal Signal Structures: Optimum Receiver

Optimum Receiver

Given $\underline{r} = (r_1, r_2, \ldots, r_{|\mathcal{M}|})$, the optimum receiver is:

$$\begin{aligned}
\hat{m} &= \operatorname*{argmin}_{m\in\mathcal{M}} \{\|\underline{r} - \underline{s}_m\|^2\} \\
&= \operatorname*{argmin}_{m\in\mathcal{M}} \{\|\underline{r}\|^2 + \|\underline{s}_m\|^2 - 2(\underline{r} \cdot \underline{s}_m)\} \\
&= \operatorname*{argmax}_{m\in\mathcal{M}} \{(\underline{r} \cdot \underline{s}_m)\} \\
&= \operatorname*{argmax}_{m\in\mathcal{M}} \{(\underline{r} \cdot \sqrt{E_s}\underline{\varphi}_m)\} \\
&= \operatorname*{argmax}_{m\in\mathcal{M}} \{r_m\}
\end{aligned}$$

Chose $\hat{m} = i$, where i is the index of the largest component in $\underline{r}$.

Error Probability (1/2)

Model:

$$r_1 = \sqrt{E_s} + n_1, \qquad r_m = n_m, \text{ for } m = 2, 3, \ldots, |\mathcal{M}|,$$

$$p_{\underline{N}}(\underline{n}) = \prod_{m=1}^{|\mathcal{M}|} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{n_m^2}{N_0}\right).$$

Correct Prob.:

$$P_{\mathsf{c}} = \int_{-\infty}^{\infty} P_{R_1}(\alpha|M=1) \Pr\{\hat{M}=1|M=1, R_1=\alpha\} d\alpha$$

$$P_{R_1}(\alpha|M=1) = p_N\left(\alpha - \sqrt{E_s}\right) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(\alpha-\sqrt{E_s})^2}{N_0}\right)$$

$$\Pr\{\hat{M}=1|M=1, R_1=\alpha\} = \left(\int_{-\infty}^{\alpha} p_N(\beta) d\beta\right)^{|\mathcal{M}|-1}$$

Error Probability (2/2)

We use $\alpha = \mu\sqrt{N_0/2}$, and thus, the correct probability is

$$P_c = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\mu - \sqrt{2E_s/N_0})^2}{2}\right)$$
$$\left(\int_{-\infty}^{\mu\sqrt{N_0/2}} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{\beta^2}{N_0}\right) d\beta\right)^{|\mathcal{M}|-1} d\mu$$

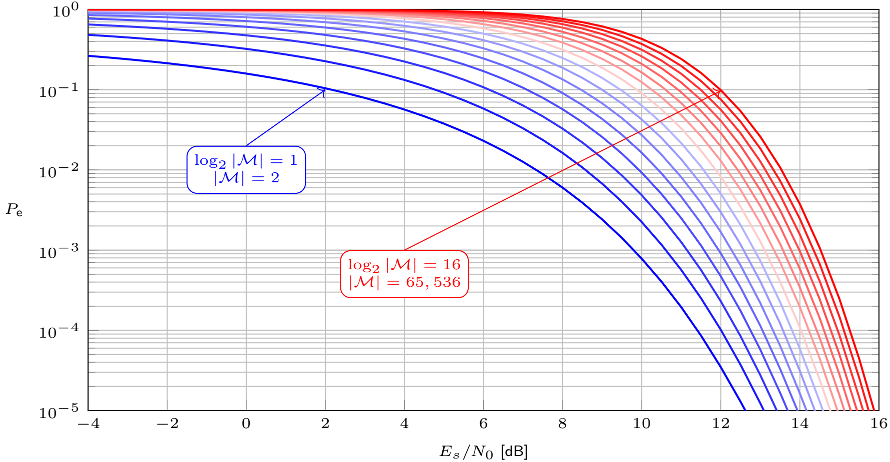
But the inner integral looks familiar... (use $\beta = \lambda\sqrt{N_0/2}$)

$$\int_{-\infty}^{\mu\sqrt{N_0/2}} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{\beta^2}{N_0}\right) d\beta = \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\lambda^2}{2}\right) d\lambda$$

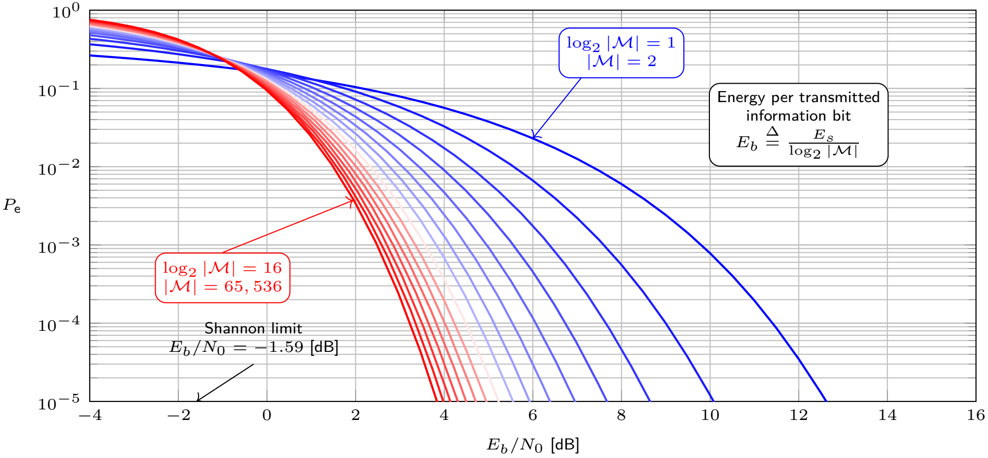
The correct probability is then:

$$P_c = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\mu - \sqrt{2E_s/N_0})^2}{2}\right) \big(Q(-\mu)\big)^{|\mathcal{M}|-1} d\mu$$

Error Probability Orthogonal Signals



Error Probability Orthogonal Signals



Module 7.2

Presentation Outline

A Channel Capacity Result

Error Probability and a Capacity Result

The error probability for orthogonal signaling satisfies

$$P_{\mathrm{e}} \leq \begin{cases} 2\exp(-[\sqrt{E_b/N_0} - \sqrt{\ln 2}]^2 \log_2 |\mathcal{M}|) & \ln 2 \leq E_b/N_0 \leq 4\ln 2, \\ 2\exp(-[E_b/(2N_0) - \ln 2] \log_2 |\mathcal{M}|) & 4\ln 2 \leq E_b/N_0. \end{cases}$$

If $E_b > N_0 \ln 2$: (i) Both arguments of the exponentials are negative, and (ii) reliable communication (arbitrarily low error probability) is therefore possible if bit energy is higher than a threshold and $|\mathcal{M}| \to \infty$

Wideband Capacity (Chapter 9)

Reliable transmission of a bit requires at least energy $N_0 \ln 2$, and thus,

$$R = \frac{P_s}{E_b} \leq \frac{P_s}{N_0 \ln 2} \left[\frac{\text{bits}}{\text{seconds}}\right] = C$$

where P_s is the transmitter power.

Energy of Orthogonal Signals

Are Orthogonal Signals Optimal?

Average Energy:

$$E_{\mathsf{av}} = E[\|\underline{S}\|^2] = \sum_{m \in \mathcal{M}} \Pr\{M = m\} \|\underline{s}_m\|^2 = E_s$$

Center of gravity:

$$E[\underline{S}] = \left(\frac{1}{|\mathcal{M}|}, \frac{1}{|\mathcal{M}|}, \cdots, \frac{1}{|\mathcal{M}|} \right) \sqrt{E_s}$$

But in the limit

$$\lim_{|\mathcal{M}| \to \infty} E[\underline{S}] = \mathbf{0}$$

Which means:

- Suboptimal in general
- Asymptotically zero loss

Summary Module 7.2

Take Home Messages

- Nonbinary orthogonal signaling: detecion and error probability
- Bit energy vs. symbol energy
- Nonbinary orthogonal signaling leads to a capacity result

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