Formulasheet Electromagnetics I (5EPA0)

1 Analysis

1.1 Vector calculus

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A},$$
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}),$$
$$\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A}),$$
$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}.$$

1.2 Vector operators

$$\nabla\Phi = \vec{a}_x\frac{\partial}{\partial x}\Phi + \vec{a}_y\frac{\partial}{\partial y}\Phi + \vec{a}_z\frac{\partial}{\partial z}\Phi,$$
$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z,$$
$$\begin{aligned}\nabla\times\vec{A} &= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \vec{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\ &\quad + \vec{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right),\end{aligned}$$
$$\Delta\Phi = \nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi.$$

1.3 Vector identities

$$\nabla(\Phi+\Psi) = \nabla\Phi + \nabla\Psi,$$
$$\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B},$$
$$\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}.$$
$$\nabla(\Phi\Psi) = \Phi\,\nabla\Psi + \Psi\nabla\Phi,$$
$$\nabla\cdot(\Phi\vec{A}) = \Phi\,\nabla\cdot\vec{A} + \vec{A}\cdot\nabla\Phi,$$
$$\nabla\times(\Phi\vec{A}) = \Phi\,\nabla\times\vec{A} - \vec{A}\times\nabla\Phi,$$
$$\begin{aligned}\nabla(\vec{A}\cdot\vec{B}) &= (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} \\ &\quad + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B}),\end{aligned}$$
$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}),$$
$$\begin{aligned}\nabla\times(\vec{A}\times\vec{B}) &= (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} \\ &\quad - \vec{B}(\nabla\cdot\vec{A}) + \vec{A}(\nabla\cdot\vec{B}).\end{aligned}$$
$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A},$$
$$\nabla\cdot(\nabla\times\vec{A}) = 0,$$
$$\nabla\times(\nabla\Phi) = \vec{0}.$$

2 Integral theorems

Divergence/Gauss's theorem:

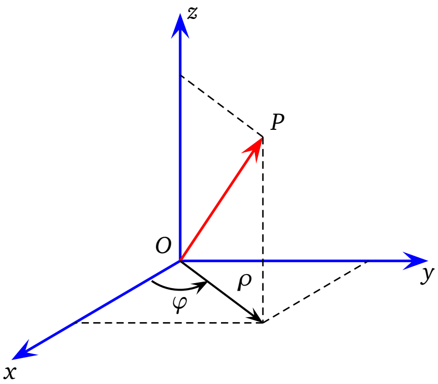
$$\iiint_V \nabla\cdot\vec{A}\,\mathrm{d}V = \oiint_S \vec{A}\cdot\mathrm{d}\vec{S},$$

Stokes' Theorem:

$$\iint_S \nabla\times\vec{A}\cdot\mathrm{d}\vec{S} = \oint_C \vec{A}\cdot\mathrm{d}\vec{\ell},$$

where $\vec{A}$ is a vector field.

3 Circular cylindrical coordinates



Coordinates: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = z$.
Ranges: $\rho \geq 0$, $0 \leq \varphi \leq 2\pi$.
Line elements: $\mathrm{d}\rho$, $\rho\mathrm{d}\varphi$, $\mathrm{d}z$.
Unit vectors expressed in cartesian components:

$$\left.\begin{aligned}\vec{a}_\rho &= \cos\varphi\,\vec{a}_x + \sin\varphi\,\vec{a}_y \\ \vec{a}_\varphi &= -\sin\varphi\,\vec{a}_x + \cos\varphi\,\vec{a}_y\end{aligned}\right\}\ \cos^2\varphi + \sin^2\varphi = 1.$$

The other way around: $\rho = \sqrt{x^2+y^2}$, $\varphi = \mathrm{atan2}(y,x)$. (note: atan2(y,x) is similar to $\arctan(y/x)$, but changed to work on the whole circle)
Unit vectors expressed in cilindrical components:

$$\begin{aligned}\vec{a}_x &= \cos\varphi\,\vec{a}_\rho - \sin\varphi\,\vec{a}_\varphi \\ \vec{a}_y &= \sin\varphi\,\vec{a}_\rho + \cos\varphi\,\vec{a}_\varphi\end{aligned}$$

Vector operators:

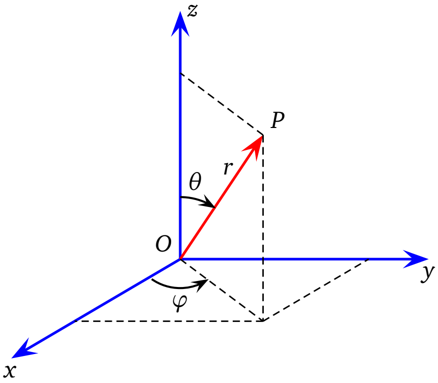
$$\nabla\Psi = \vec{a}_\rho \frac{\partial \Psi}{\partial \rho} + \vec{a}_\varphi \frac{1}{\rho}\frac{\partial \Psi}{\partial \varphi} + \vec{a}_z \frac{\partial \Psi}{\partial z},$$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z},$$

$$\nabla\times\vec{A} = \vec{a}_\rho\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right] + \vec{a}_\varphi\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \vec{a}_z \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi}\right],$$

$$\nabla^2\Psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Psi}{\partial \varphi^2} + \frac{\partial^2 \Psi}{\partial z^2}.$$

4 Spherical coordinates



Coordinates: $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$.
Ranges: $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$.
Line elements: $\mathrm{d}r$, $r\mathrm{d}\theta$, $r\sin\theta\,\mathrm{d}\varphi$.
Unit vectors expressed in cartesian components:

$$\begin{aligned}
\vec{a}_r &= \sin\theta\cos\varphi\,\vec{a}_x + \sin\theta\sin\varphi\,\vec{a}_y + \cos\theta\,\vec{a}_z,\\
\vec{a}_\theta &= \cos\theta\cos\varphi\,\vec{a}_x + \cos\theta\sin\varphi\,\vec{a}_y - \sin\theta\,\vec{a}_z,\\
\vec{a}_\varphi &= -\sin\varphi\,\vec{a}_x + \cos\varphi\,\vec{a}_y.
\end{aligned}$$

And vice versa: $r = \sqrt{x^2+y^2+z^2}$, $\theta = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$, and $\varphi = \frac{y}{|y|}\arccos\frac{x}{\sqrt{x^2+y^2}}$.

$$\begin{aligned}
\vec{a}_x &= \sin\theta\cos\varphi\vec{a}_r + \cos\theta\cos\varphi\vec{a}_\theta - \sin\varphi\vec{a}_\varphi\\
\vec{a}_y &= \sin\theta\sin\varphi\vec{a}_r + \cos\theta\sin\varphi\vec{a}_\theta + \cos\varphi\vec{a}_\varphi\\
\vec{a}_z &= \cos\theta\vec{a}_r - \sin\theta\vec{a}_\theta
\end{aligned}$$

Vector operators:

$$\nabla\Psi = \vec{a}_r\frac{\partial \Psi}{\partial r} + \vec{a}_\theta\frac{1}{r}\frac{\partial \Psi}{\partial \theta} + \vec{a}_\varphi\frac{1}{r\sin(\theta)}\frac{\partial \Psi}{\partial \varphi},$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}[\sin(\theta)A_\theta] + \frac{1}{r\sin(\theta)}\frac{\partial A_\varphi}{\partial \varphi},$$

$$\begin{aligned}
\nabla\times\vec{A} = {}&\vec{a}_r\frac{1}{r\sin(\theta)}\left\{\frac{\partial}{\partial \theta}[\sin(\theta)A_\varphi] - \frac{\partial A_\theta}{\partial \varphi}\right\}\\
&+ \vec{a}_\theta\left[\frac{1}{r\sin(\theta)}\frac{\partial A_r}{\partial \varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right]\\
&+ \vec{a}_\varphi\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right],
\end{aligned}$$

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 \Psi}{\partial \varphi^2}.$$

5 Electrostatics

Coulomb's law:

$$\vec{E} = \frac{Q(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|^3}, \quad \vec{E} = \iiint_V \frac{\rho_V(\vec{r}\,')(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|^3}\mathrm{d}V.$$

Gauss's law for $\vec{D} = \varepsilon\vec{E}$:

$$\oiint_S \vec{D}\cdot\mathrm{d}\vec{S} = Q_{\mathrm{encl}}, \quad \oiint_S \vec{D}\cdot\mathrm{d}\vec{S} = \iiint_V \rho_V(\vec{r}\,')\mathrm{d}V.$$

$$\nabla\cdot\vec{D} = \rho_V.$$

Scalar potential:

$$V_{\mathrm{final}} - V_{\mathrm{initial/ref}} = \int_{\mathrm{initial/ref}}^{\mathrm{final}} -\vec{E}\cdot\mathrm{d}\vec{\ell}$$

$$\vec{E} = -\nabla V, \qquad \oint_C \vec{E}\cdot\mathrm{d}\vec{\ell} = 0.$$

$$V = \frac{Q}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|}, \qquad V = \int_C \frac{\rho_L(\vec{r}\,')}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|}\mathrm{d}\ell,$$

$$V = \iint_S \frac{\rho_S(\vec{r}\,')}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|}\mathrm{d}\vec{S}, \quad V = \iiint_V \frac{\rho_V(\vec{r}\,')}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|}\mathrm{d}V.$$

Poisson's equation, Laplace's equation:

$$\nabla^2 V = -\frac{\rho_V}{\varepsilon_0}, \qquad \nabla^2 V = 0.$$

Capacitance: $C = \dfrac{Q}{\Delta V}$ (ΔV is voltage difference)

Ohm's law: $\vec{J} = \sigma \vec{E}$

Energy:

$$W_E = \frac{1}{2}\iiint_V \rho_V V \mathrm{d}V, \qquad W_E = \frac{1}{2}\iiint_V \vec{D}\cdot\vec{E}\mathrm{d}V,$$

$$W_E = Q\Delta V.$$

Boundary conditions:

conductor	dielectric
$\vec{n}\times\vec{E} = \vec{0}$	$\vec{n}\times(\vec{E}_1 - \vec{E}_2) = \vec{0}$
$\vec{n}\cdot\vec{D} = \rho_S$	$\vec{n}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_S$
$\vec{n}$ away from conductor	$\vec{n}$ from medium 2 to 1

6 Magnetostatics

The Biot-Savart law:

$$\vec{H} = \iiint_V \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|^3}\mathrm{d}V' = \oint_{\text{1D loop}} \frac{I\,\mathrm{d}\vec{\ell}\,'\times(\vec{r}-\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|^3}$$

Ampère's circuital law:

$$\oint_C \vec{H}\cdot\mathrm{d}\vec{\ell} = I_{\text{encl}}, \qquad \underbrace{\oint_{C=\partial S} \vec{H}\cdot\mathrm{d}\vec{\ell} = \iint_S \vec{J}\cdot\mathrm{d}\vec{S}}_{\forall \text{ open } S:\ \nabla\times\vec{H} = \vec{J}}.$$

Gauss's law for $\vec{B} = \mu\vec{H}$:

$$\oiint_S \vec{B}\cdot\mathrm{d}\vec{S} = 0 \quad \Leftrightarrow \quad \nabla\cdot\vec{B} = 0.$$

Vector potential (in vacuum):

$$\vec{B} = \mu_0\vec{H} = \nabla\times\vec{A}.$$

$$\vec{A} = \iiint_V \frac{\mu_0\vec{J}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}\mathrm{d}V' = \oint_{\text{1D loop}} \frac{\mu_0 I\,\mathrm{d}\vec{\ell}\,'}{4\pi|\vec{r}-\vec{r}\,'|}$$

Poisson's equation, Laplace's equation:

$$\nabla^2\vec{A} = -\mu_0\vec{J}, \qquad \nabla^2\vec{A} = 0.$$

Magnetic flux and flux linkage

$$\Phi = \iint_S \vec{B}\cdot\mathrm{d}\vec{S}, \qquad \Lambda = N\Phi.$$

Inductance and mutual inductance

$$L = \frac{\Lambda}{I}, \qquad M_{12} = \frac{\Lambda_{12}}{I_1}$$

Lorentz force on charge:

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

Lorentz force for volume, surface and line-current:

$$\vec{F} = \iiint_V \vec{J}\times\vec{B}dV; \vec{F}\iint_S \vec{J}_s\times\vec{B}dS; \vec{F} = \oint Id\vec{\ell}\times\vec{B}$$

Energy:

$$W_H = \frac{1}{2}\iiint_V \vec{B}\cdot\vec{H}\mathrm{d}V.$$

Boundary conditions:

conductor	magnetic material
$\vec{n}\times\vec{H} = \vec{J}_S$	$\vec{n}\times(\vec{H}_1 - \vec{H}_2) = \vec{J}_s$
$\vec{n}\cdot\vec{B} = 0$	$\vec{n}\cdot(\vec{B}_1 - \vec{B}_2) = 0$
$\vec{n}$ away from conductor	$\vec{n}$ from medium 2 to 1

7 Electromagnetism

Faraday's induction law:

$$\oint_{C=\partial S} \vec{E}\cdot\mathrm{d}\vec{\ell} = \text{emf} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \vec{B}\cdot\mathrm{d}\vec{S}$$

Lenz's law: induced emf acts to oppose a change of flux

Maxwell's equations, global formulation:

$$\oint_{C=\partial S} \vec{E}\cdot\mathrm{d}\vec{\ell} = \overbrace{-\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \vec{B}\cdot\mathrm{d}\vec{S}}^{\underbrace{-\iint_S \frac{\partial\vec{B}}{\partial t}\cdot\mathrm{d}\vec{S}}_{\text{transformer emf}} + \underbrace{\oint_{\partial S}\vec{v}\times\vec{B}\cdot\mathrm{d}\vec{\ell}}_{\text{motional emf}}}$$

$$\oint_{C=\partial S} \vec{H}\cdot\mathrm{d}\vec{\ell} = \underbrace{\iint_S \vec{J}\cdot\mathrm{d}\vec{S}}_{\text{conduction current}} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \vec{D}\cdot\mathrm{d}\vec{S}}_{\text{displacement current}}$$

$$\oiint_{S=\partial V} \vec{D}\cdot\mathrm{d}\vec{S} = \iiint_V \rho_V\,\mathrm{d}V \qquad \oiint_S \vec{B}\cdot\mathrm{d}\vec{S} = 0$$

Charge conservation: $\displaystyle \oiint_{S=\partial V} \vec{J}\cdot \mathrm{d}\vec{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho_V \,\mathrm{d}V$

Continuity equation: $\displaystyle \nabla\cdot\vec{J} = -\frac{\partial \rho_V}{\partial t}$

Maxwell's equations, local formulation:

$$\begin{aligned} \nabla\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} & &\overset{\text{causality}}{\Rightarrow} & &\nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} & &\overset{\text{causality}}{\Rightarrow} & &\nabla\cdot\vec{D} = \rho_V \end{aligned}$$

Time-harmonic fields: phasor $\vec{E}_s$

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_s(\vec{r})\, e^{j\omega t}\right] \quad \Rightarrow \quad \frac{\partial}{\partial t} \to j\omega$$

Helmholtz's equation:

$$\nabla^2 \vec{E}_s + k^2 \vec{E}_s = 0, \quad k^2 = \omega^2 \mu\varepsilon \left(1 - \frac{j\sigma}{\omega\varepsilon}\right)$$

A plane-wave solution: $\vec{E}_s(z) = \left(A_-\, e^{jkz} + A_+\, e^{-jkz}\right)\vec{a}_x$

$\sigma \ll \omega\varepsilon \quad \Rightarrow \quad$ lossy plane waves, $c = \dfrac{1}{\sqrt{\varepsilon\mu}}$, $Z = \sqrt{\dfrac{\mu}{\varepsilon}}$

$\sigma \gg \omega\varepsilon \quad \Rightarrow \quad$ skin effect, $\delta = \sqrt{\dfrac{2}{\omega\mu\sigma}}$

Poynting's theorem (for space-time fields)

$$-\oiint_{S=\partial V} (\vec{E}\times\vec{H})\cdot \mathrm{d}\vec{S} = \iiint_V \vec{E}\cdot\vec{J}\,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\iiint_V \varepsilon\vec{E}\cdot\vec{E} + \mu\vec{H}\cdot\vec{H}\,\mathrm{d}V$$

$\vec{S} = \vec{E}\times\vec{H}$: Poynting vector (instantaneous power density)

Wednesday 18th October, 2023, 4:44pm