1. **Loop-shaping:**
Consider a system with the transfer function

$$G(s) = \frac{5}{(s+1)(s+10)}.$$

(a) Draw the Bode plot of $G(s)$.

(b) Design a controller $D(s)$ by hand, that satisfies the following requirements:

- zero steady-state error when the reference is a step function;
- phase margin PM $\geq 60°$;
- closed-loop bandwidth $\omega_{\text{bw}} \geq 12$ rad/sec.

(c) Calculate and plot using Matlab, the Bode plot for the sensitivity function $S(s)$ for the compensated system. Determine in which frequency domain disturbances are reduced by at least 1/2.

Solution:
Following the guideines in the Appendix in Exercise Set 5, a Lead compensator of the following form is required.

$$D_{\text{Lead}}(s) = K\frac{1+Ts}{1+\alpha Ts},$$

(a) Since we want zero steady state error for step inputs, we require one integrator in our controller. This gives

$$L(s) = D(s)G(s) = \frac{1}{s}\frac{5}{(s+1)(s+10)}.$$

(b) For the second step, we must determine the open-loop gain, K_1. Note that for $\omega_{bw} \geq 12$ rad/s we have $|L(j\omega_{bw})| = \frac{1}{j12}\frac{5}{(j12+1)(j12+10)} \approx -53.09$ dB. Thus, we shall increase the gain K_1 such that $K_1 = +53$ dB $\simeq 451$. This results in the following controller

$$L(s) = D(s)G(s) = \frac{451}{s}\frac{5}{(s+1)(s+10)}$$

(c) As can be seen in Figure 1, the resulting system has a phase margin of $PM = -45°$. Thus according to step 3 and 4 of the guidelines, we should aim to add approximately $\phi_{max} = 45+60+10 = 115°$ of phase lead, adding an additional $10°$ for extra margin.

(d) Using this information, we can now follow steps 5 and 6 to compute the parameters for our lead filter. Using the required phase lead, we can compute

$$\alpha = \frac{1-\sin(\phi_{max})}{1+\sin(\phi_{max})} = 0.0491 \quad \text{and } T \quad = \frac{1}{\omega_{max}\sqrt{\alpha}} = 0.3759$$

resulting in the following controller, where $D_L(s)$ is the lead compensator

$$D(s) = \frac{451}{s}D_L(s) = \frac{451}{s}K_2\frac{1+0.3759s}{1+0.0491\cdot 0.3759s}$$

(e) Finally, we find a K_2 such that $|K_2 D(j\omega_{wc})G(j\omega_{wc})| = 1$. To find this K_2, we compute the magnitude

$$|L(j\omega_{wc})| = D(s)G(s) = \frac{451}{j12}\frac{1+0.3759j12}{1+0.0491\cdot 0.3759j12}\frac{5}{(j12+1)(j12+10)} \approx 13.08 \text{ dB}.$$

This reveals that we need an attenuation of -13.08dB, which results in $K_2 \simeq 0.2217$. Thus, the final compensated system is given by

$$L(s) = D(s)G(s) = \frac{451}{s}\cdot 0.2217\cdot\frac{1+0.3759s}{1+0.0491\cdot 0.3759s}\frac{5}{(s+1)(s+10)}$$

which gives the bode plot in Figure 2.

(f) However, note that we still have not fulfilled our phase margin requirement of $PM \geq 60°$. Thus we add another lead filter based on the results of $L(s)$ in Figure 2. From the bode plot, we can estimate the required phase lead to fulfill the requirement, $\phi_{max} = 40 + 10 = 50°$, which can be used to compute new values for the additional lead filter. Namely,

$$\alpha_2 = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})} = 0.1325$$

and thus

$$T_2 = \frac{1}{\omega_{max}\sqrt{\alpha}} = 0.2290$$

resulting in the following lead compensator

$$\begin{aligned} D(s) &= \frac{451}{s} \cdot 0.2217 \cdot \frac{1 + 0.3759s}{1 + 0.0491 \cdot 0.3759s} D_{L2}(s) \\ &= \frac{451}{s} \cdot 0.2217 \cdot \frac{1 + 0.3759s}{1 + 0.0491 \cdot 0.3759s} K_3 \frac{1 + 0.2290s}{1 + 0.1325 \cdot 0.2290s} \end{aligned}$$

Once more, we must compute the required gain K_3 to achieve $|K_3 D(j\omega_{wc})G(j\omega_{wc})| = 1$. Thus, we compute the magnitude

$$\begin{aligned} |L(j\omega_{wc})| &= D(s)G(s) \\ &= \frac{451}{j12} \cdot 0.2217 \cdot \frac{1 + 0.3759j12}{1 + 0.0491 \cdot 0.3759j12} \frac{1 + 0.2290j12}{1 + 0.1325 \cdot 0.2290j12} \frac{5}{(j12+1)(j12+10)} \\ &\approx 8.77 \text{ dB.} \end{aligned}$$

This reveals that we need an attenuation of -8.77dB, resulting in a gain $K_3 \simeq 0.3640$. Thus, the final open-loop transfer function reads as

$$L_{\text{final}}(s) = D(s)G(s) = \frac{451}{s} \cdot 0.2217 \cdot \frac{1 + 0.3759s}{1 + 0.0491 \cdot 0.3759s} \cdot 0.3640 \cdot \frac{1 + 0.2290s}{1 + 0.1325 \cdot 0.2290s} \frac{5}{(s+1)(s+10)}$$

The final design of the fully compensated system is plotted in Figure 3, where it can be seen that all design requirements are now met.

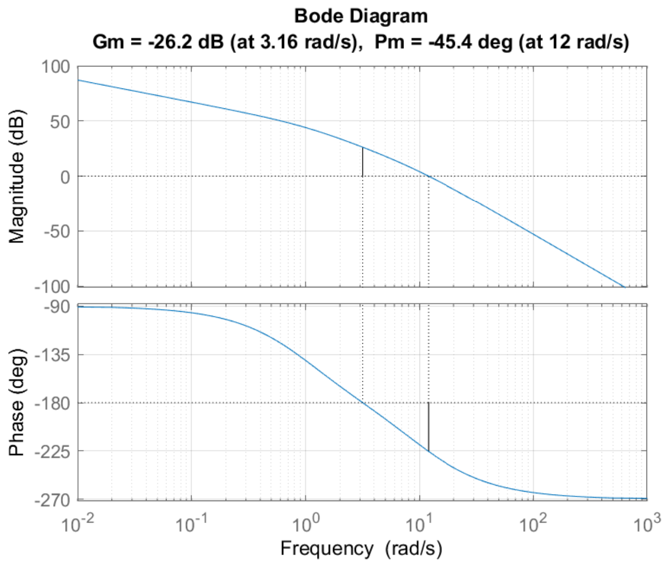


Figure 1: Bode plot for $L(s) = \frac{451}{s}\frac{5}{(s+1)(s+10)}$

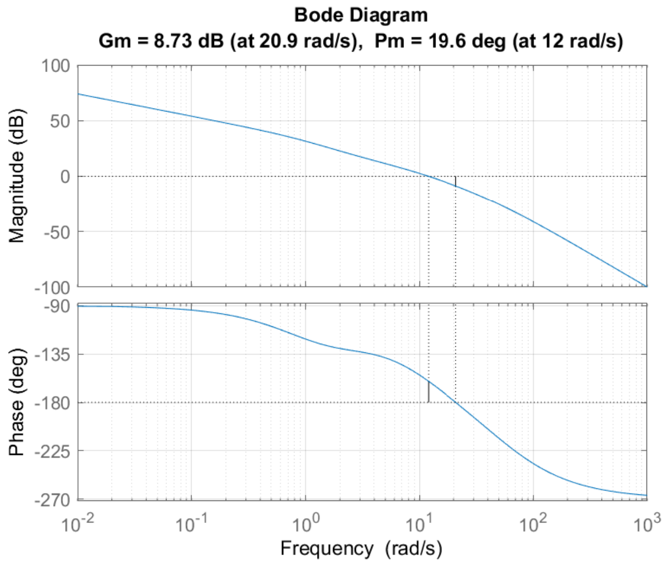


Figure 2: Bode plot for $L(s) = \frac{451}{s}\frac{1+0.3759s}{1+0.0491\cdot 0.3759s}\frac{5}{(s+1)(s+10)}$

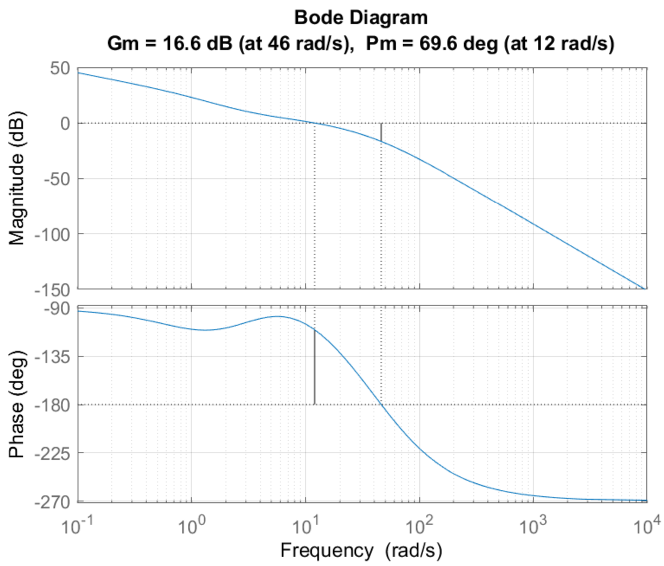


Figure 3: Bode plot for $L_{\text{final}}(s)$

2. **Problem 6.59 in the Franklin book:**
Golden Nugget Airlines had great success with their free bar near the tail of the airplane. However, when they purchased a much larger airplane to handle the passenger demand, they discovered that there was some flexibility in the fuselage that caused a lot of unpleasant yawing motion at the rear of the airplane when in turbulence and was causing the travelers to spill their drinks. The approximate transfer function for the Dutch roll mode (see Section 10.3.1) is

$$\frac{r(s)}{\delta_{\mathrm{r}}(s)} = \frac{8.75\,(4s^2 + 0.4s + 1)}{(100s + 1)(s^2 + 0.24s + 1)},$$

where r is the airplane yaw rate and δ_{r} is the rudder angle.

In performing a Finite Element Analysis (FEA) of the fuselage structure and adding those dynamics to the Dutch roll motion, they found that the transfer function needed additional terms that reflected the fuselage lateral bending that occurred due to excitation from the rudder and turbulence. The revised transfer function is

$$G(s) = \frac{r(s)}{\delta_{\mathrm{r}}(s)} = \frac{8.75\,(4s^2 + 0.4s + 1)}{(100s + 1)(s^2 + 0.24s + 1)} \cdot \frac{1}{(s^2/\omega_{\mathrm{b}}^2 + 2\zeta s/\omega_{\mathrm{b}} + 1)},$$

where $\omega_{\mathrm{b}} = 10$ rad/sec is the frequency of the bending mode and $\zeta = 0.02$ is the bending mode damping ratio.

Most swept wing airplanes have a "yaw damper" which essentially feeds back the measured yaw rate by a rate gyro to the rudder with a simple proportional control law. For the new Golden Nugget airplane, the proportional feedback gain is $K = 1$.

Remark: You may use Matlab for this exercise.

(a) Draw the Bode plot for the open-loop transfer function, determine the PM and GM for the given controller design ($D(s) = K = 1$).
Plot the step response and the Bode magnitude of the closed-loop system. Which is the frequency of the lightly damped mode that is causing troubles?

(b) Investigate remedies to quiet down the oscillations, but maintain the same low frequency gain in order not to affect the quality of the Dutch roll damping provided by the yaw-rate feedback.
Specifically, investigate one at a time:

i. Increasing the damping of the bending mode from $\zeta = 0.02$ to $\hat{\zeta} = 0.04$ (which would require adding energy absorbing material in the fuselage structure)

ii. increasing the frequency of the bending mode from $\omega_{\mathrm{b}} = 10$ rad/sec to $\hat{\omega}_{\mathrm{b}} = 20$ rad/sec (which would require stronger and heavier structural elements)

iii. adding a low-pass filter in the feedback loop, that is, replace the linear controller $u(e) = K\,e$ with some controller
$$D_{lpf}(s) = K\frac{1}{s\tau + 1}$$
pick τ so that the objectionable features of the bending mode are reduced while maintaining PM $\geq 60°$;

iv. adding a notch filter in the feedback loop, that is, replace the linear controller $u(e) = K\,e$ with some controller
$$D_{nf}(s) = \frac{\left(s^2/\omega_b^2 + \zeta s/\omega_b + 1\right)}{(1 + s/100)^2}$$
pick the frequency of the notch zero to be at ω_{b} with a damping of $\zeta = 0.04$ and pick the denominator poles to be $(s/100 + 1)^2$ keeping the DC gain of the filter equal to 1.

(c) Investigate the sensitivity of the two compensated designs above (iii, iv) by determining the effect of a reduction in the bending mode frequency of 10%. Specifically, re-examine the two designs by looking at the GM, PM, closed-loop bending mode damping ratio and resonant peak amplitude, and qualitatively describe the differences in the step response.

(d) What would you recommend to Golden Nugget to help their customers stop spilling their drinks? (Make the recommendation in terms of improvements to the yaw damper.)

Solution:

(a) The Bode plot of the open-loop transfer function is shown in Figure 4. Note that GM $\simeq 1$ dB is quite small due to the resonance effect at $\omega = 10$ rad/sec which almost leading to instability.

The step response of the closed loop system $G_{\text{cl}}(s) = \frac{G(s)}{1+G(s)}$ is shown in Figure 5. From the Bode plot of G_{cl} shown in Figure 6 we note that the frequency of the poorly damped mode is $\omega = 10$ rad/sec.

(b) i. The Bode plot for the open-loop transfer function with the bending mode damping increased from $\zeta = 0.02$ to $\hat{\zeta} = 0.04$ is shown in Figure 7. We can see that the GM has increased because the resonant peak is well below magnitude 1; thus the system would be much better behaved.

ii. The Bode plot of the open-loop transfer function with $\omega_{\text{b}} = 10$ rad/sec replaced by $\hat{\omega}_{\text{b}} = 20$ rad/sec is shown in Figure 8. Again, we note that the GM has increased and the resonant peak has reduced.

iii. We select $\tau = 1$ as parameter of the low-pass filter and obtain the Bode plot shown in Figure 9. Stability margins are satisfactory.

iv. The Bode plot of the open-loop transfer function with the given notch filter is shown in Figure 10. We obtain PM = 97.5 (at 0.085 rad/sec) and GM = 55.1 (at 99.7 rad/sec). These margins are the largest among all the considered control designs. The notch filter has essentially cancelled the bending mode resonant peak out.

(c) Note that the notch filter is very sensitive to where to place the notch zeros to reduce the lightly damped resonant peak. So if you want to use the notch filter, you must have a good estimation of the location of the bending mode poles and the poles must remain at that location for all aircraft conditions. On the other hand, the low pass filter is relatively robust to where to place its break point.

Evaluation of the margins with the bending mode frequency lowered by 10%, that is, to $\tilde{\omega}_{\text{b}} = 9$ rad/sec, will show a drastic reduction in the margins for the notch filter and very little reduction for the low pass filter.

Let $\tilde{G}(s)$ be the system transfer function with bending mode frequency $\tilde{\omega}_{\text{b}}$. Figures 11, 12 show the Bode plots of the two corresponding open-loop transfer functions.

(d) While increasing the natural damping of the system would be the best solution, it might be difficult and expensive to carry out. Likewise, increasing the frequency typically is expensive and makes the structure heavier, not a good idea in an aircraft.

Of the remaining two options, it might be better to go with the low pass filter since the exact location of the bending mode might not be known but rather an estimate, choosing the low pass filter is the more robust approach.

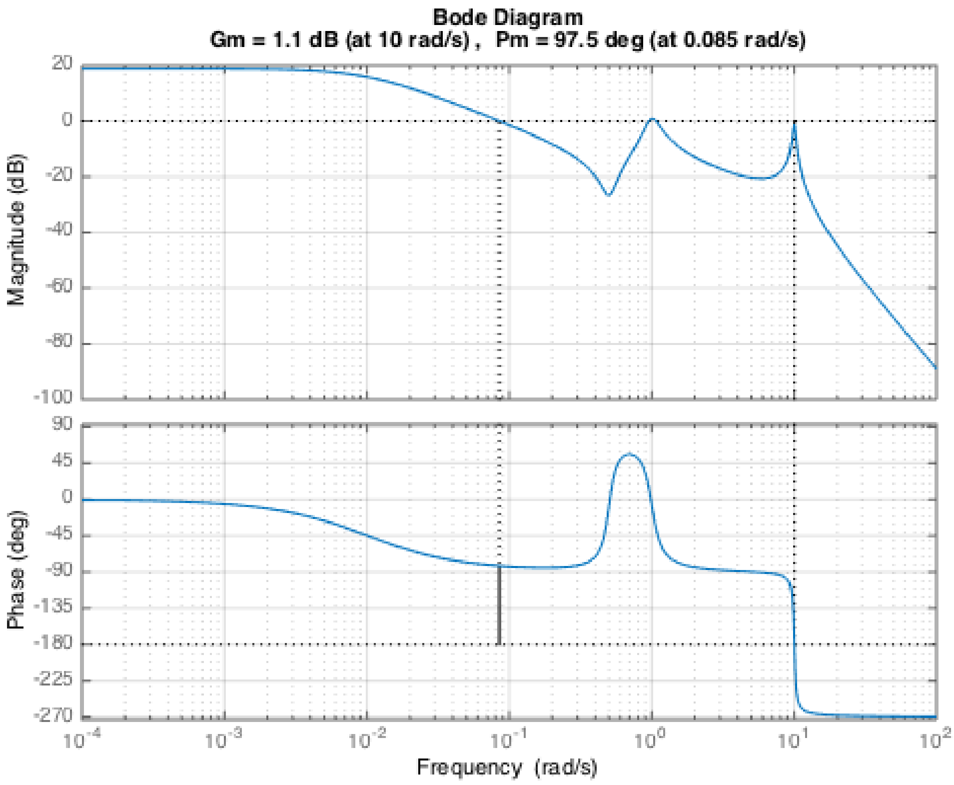


Figure 4: Bode plot for the open-loop transfer function in Problem 2.

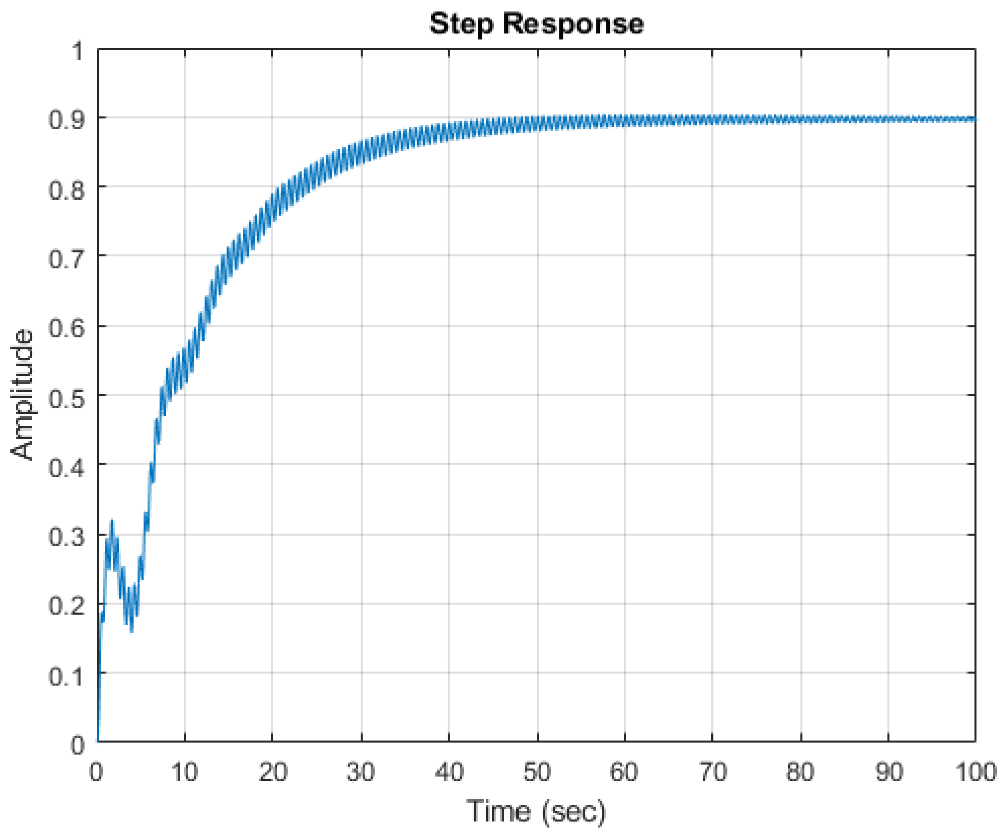


Figure 5: Step response for the closed-loop system with linear control law.

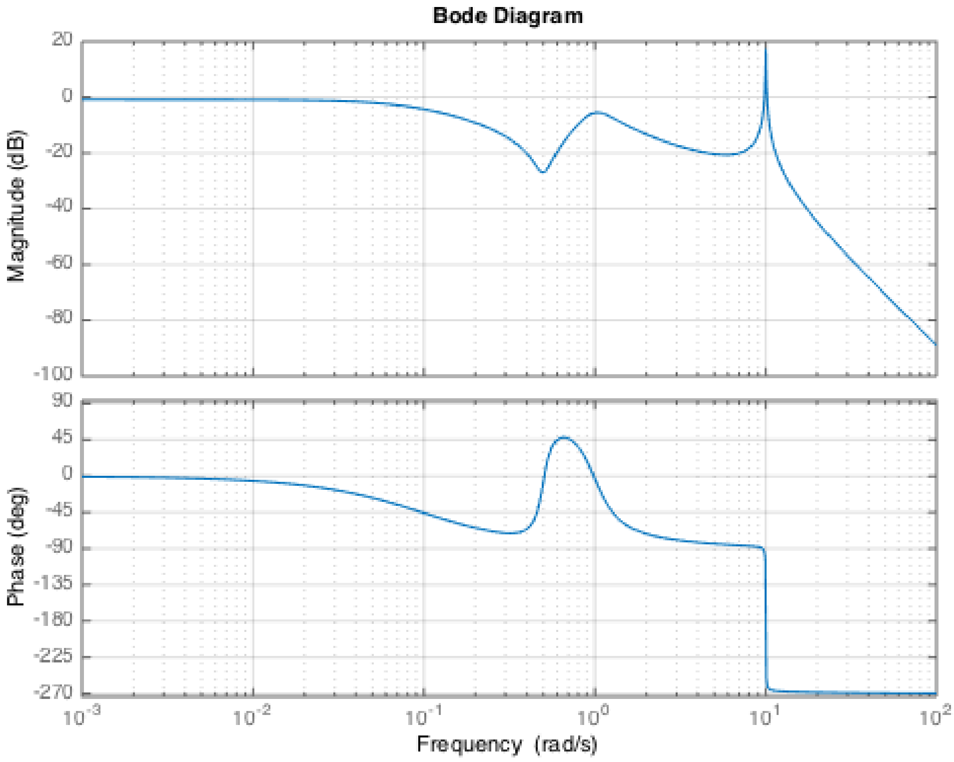


Figure 6: Bode plot for the closed-loop transfer function in Problem 2 with control law $D(s) = K = 1$.

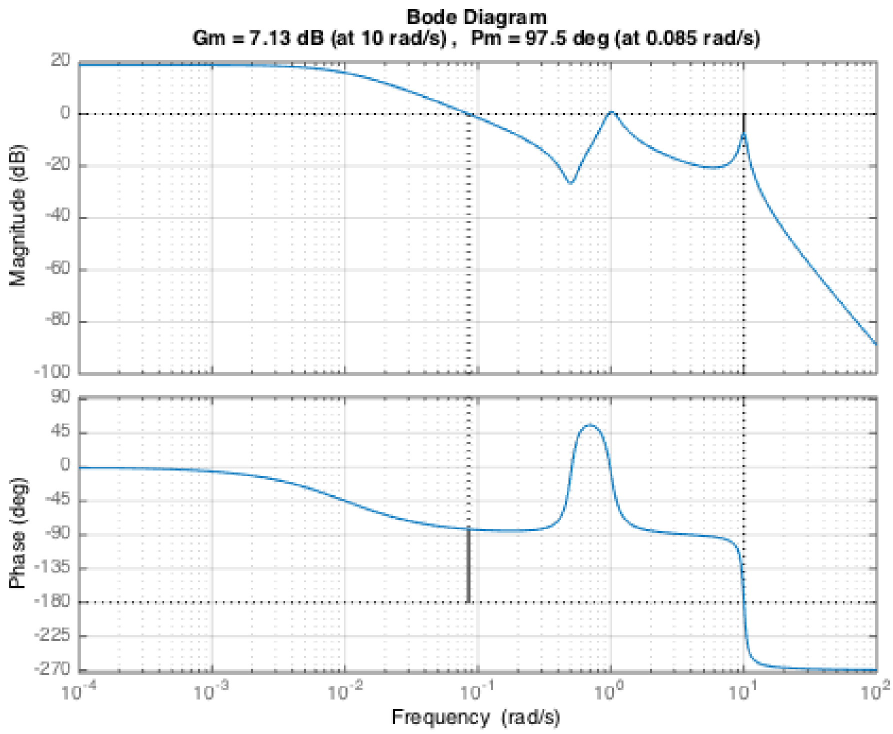


Figure 7: Bode plot for the open-loop transfer function in Problem 2 with damping parameter $\hat{\zeta} = 0.04$ (rather than $\zeta = 0.02$).

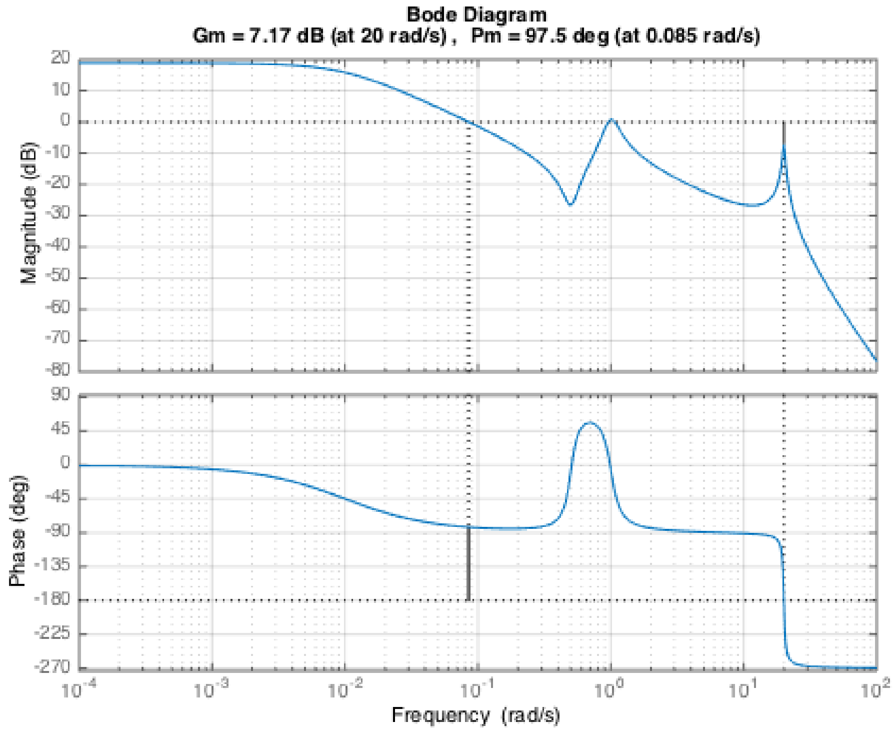


Figure 8: Bode plot for the open-loop transfer function in Problem 2 with resonance frequency $\hat{\omega}_{\mathrm{b}} = 20$ (rather than $\omega_{\mathrm{b}} = 10$).

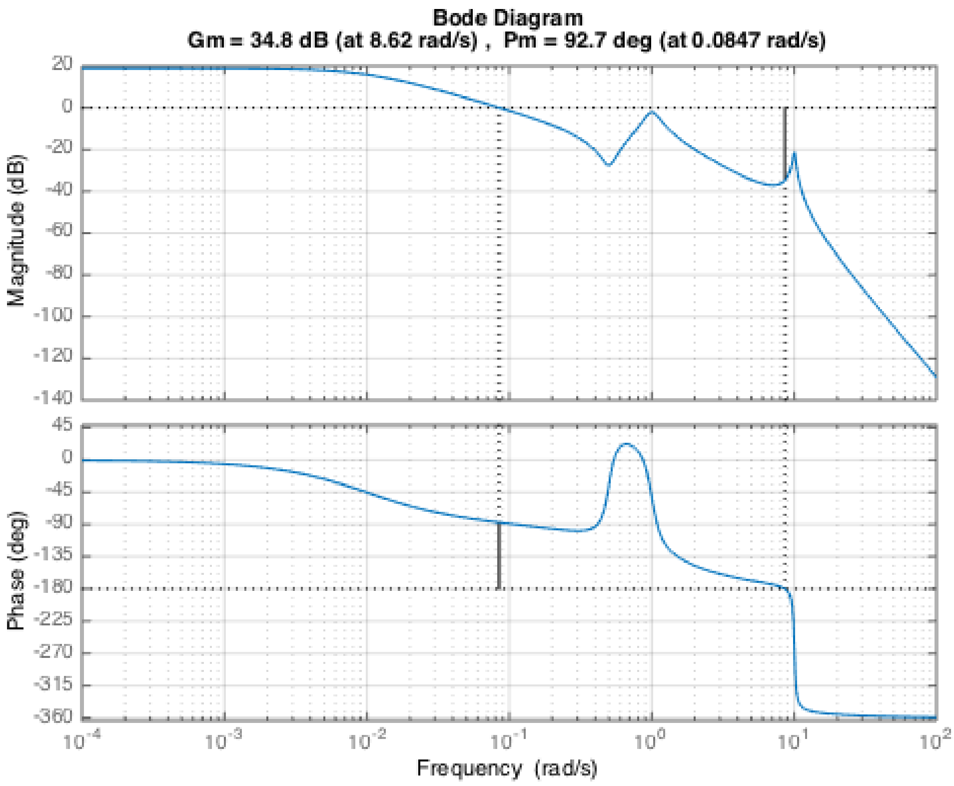


Figure 9: Bode plot for the open-loop transfer function $D_{lpf}(s)\,G(s) = \frac{1}{1+s}\,G(s)$ in Problem 2.

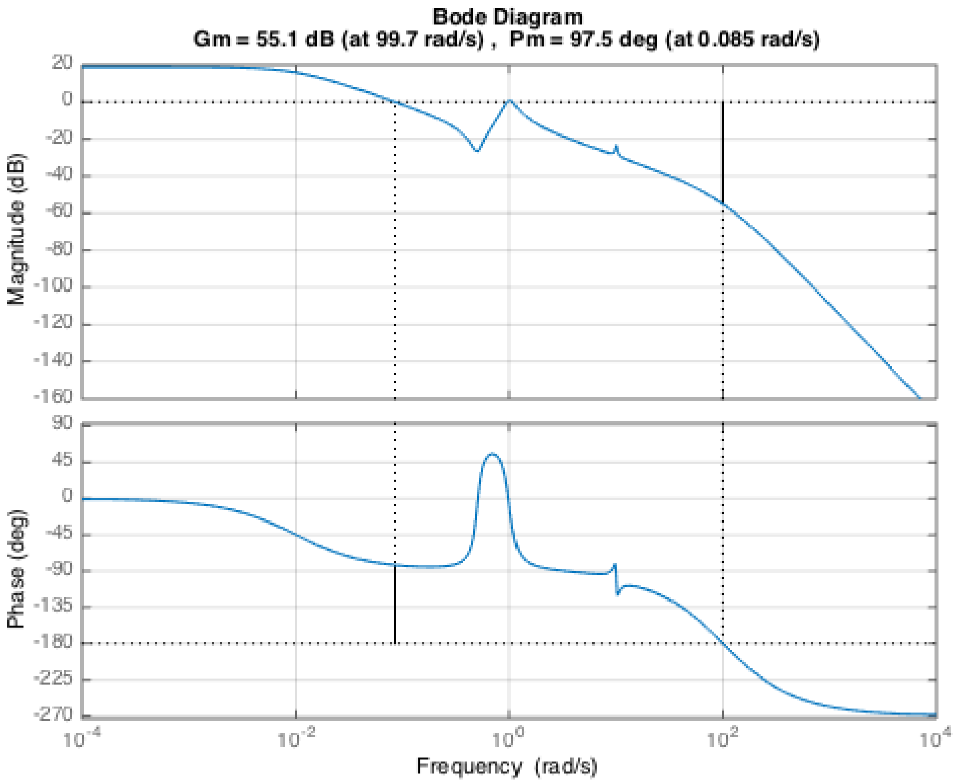


Figure 10: Bode plot for the open-loop transfer function $D_{\mathrm{nf}}(s)\,G(s)$ in Problem 2.

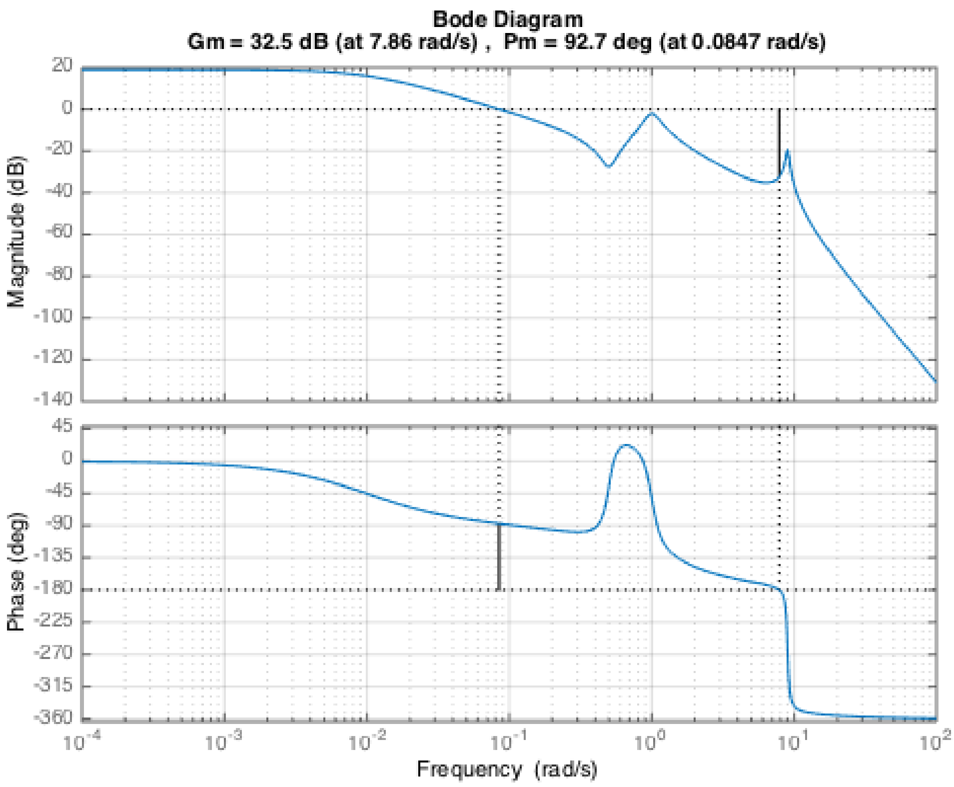


Figure 11: Bode plot for the open-loop transfer function $D_{lpf}(s)\,\tilde{G}(s)$ in Problem 2.

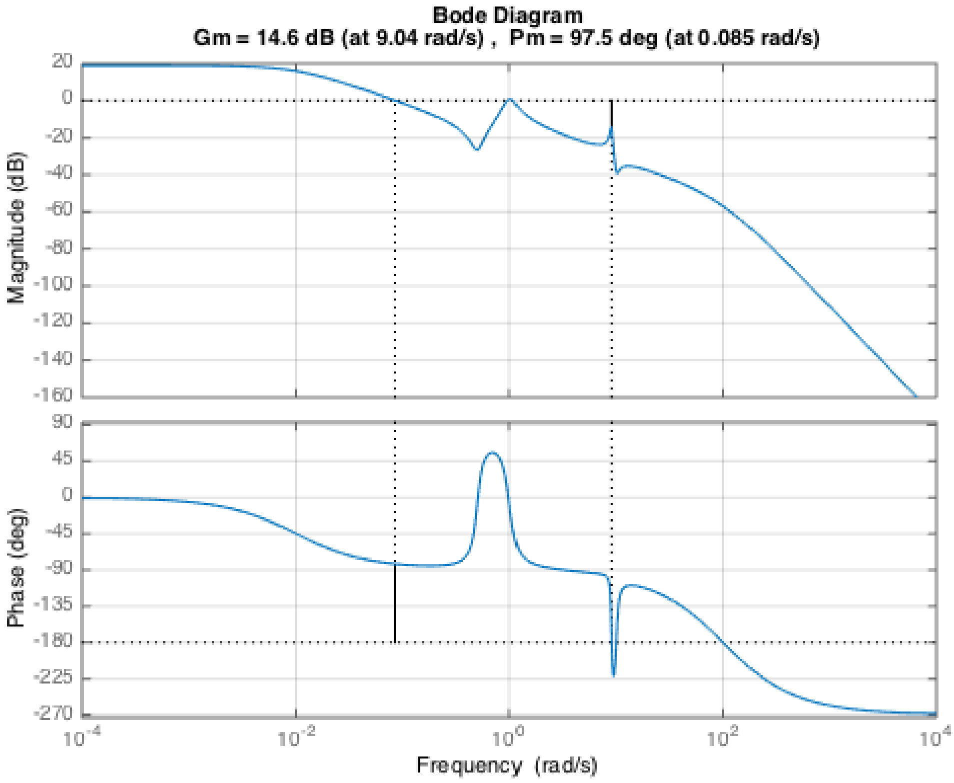


Figure 12: Bode plot for the open-loop transfer function $D_{\mathrm{nf}}(s)\,\tilde{G}(s)$ in Problem 2.

3. **Sensor noise and output disturbance:**
   Consider the block diagram in Figure 13, where

$$G(s) = \frac{1}{s\,\left(1+\frac{s}{5}\right)\left(1+\frac{s}{20}\right)}.$$

Design a compensator $D(s)$ such that the closed-loop system satisfies the following specifications:

(a) the noise n introduced with the sensor signal at frequencies higher than 200 rad/s are to be attenuated at the output by at least a factor of 100 (-40 dB);

(b) the disturbance d at the plant output acting at up to 10 rad/s shall be attenuated of at least a factor 2 ($1/2 \simeq -6$ dB).

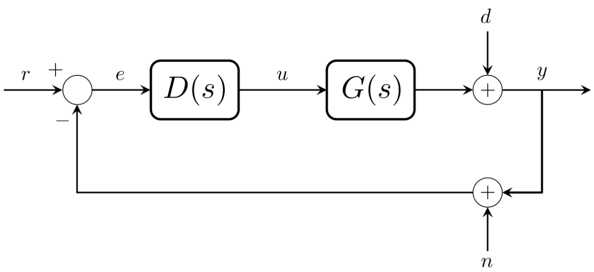


Figure 13: Block scheme for Problem 2.

Solution:
We look at the Bode plot of the open-loop transfer function $G(s)$, see Figure 14, and at the sensitivity plots for $D(s) = 1$, see Figure 15. From the Bode plot, we note that PM $\simeq 76°$ at $\omega_c \simeq 1$ rad/s.

Since disturbances d up to 10 rad/s must be attenuated, we shall increase the bandwidth of the system. One possible way of doing so is canceling the stable pole at -5 with the compensator

$$D(s) = K\frac{1 + s/5}{1 + s/50},$$

where we tune $K = 40$ for achieving PM $\simeq 15°$ at $\omega_c = 23.5$ rad/s, see Figure 16. Next we plot the sensitivity functions for the compensated open-loop transfer function $D(s)G(s)$ in Figure 17. We note that $|S(j\omega)| \leq -9$ dB for all $\omega \leq 10$ rad/s, and that $|T(j\omega)| \leq -45$ dB for all $\omega \geq 200$ rad/s.

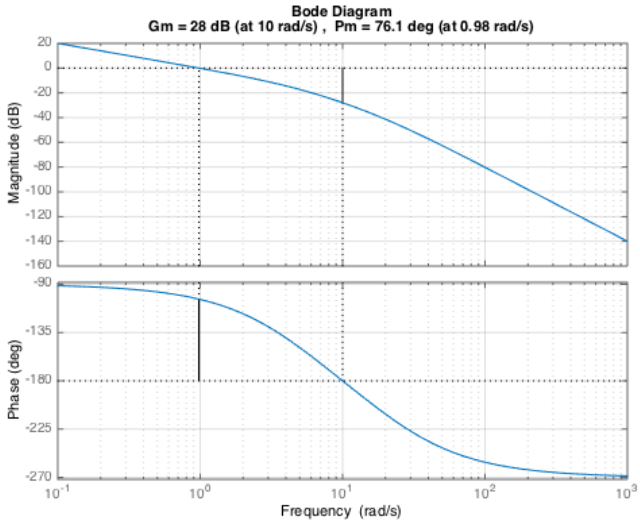


Figure 14: Problem 3: Bode plot for $G(s)$.

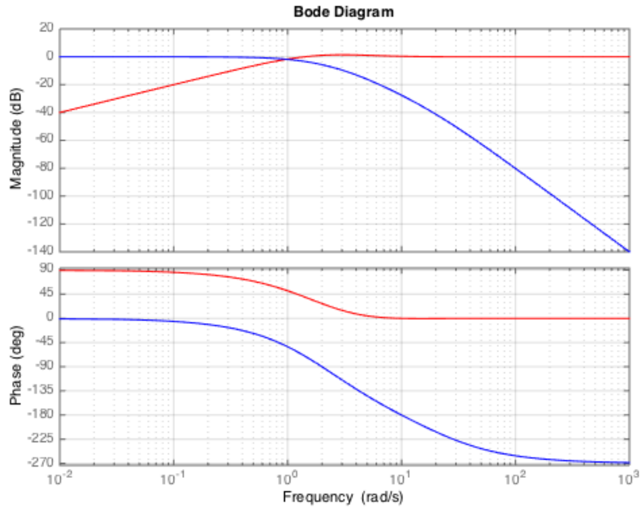


Figure 15: Problem 3: Sensitivity functions $S(s) = \frac{1}{1+G(s)}$, $T(s) = \frac{G(s)}{1+G(s)}$.

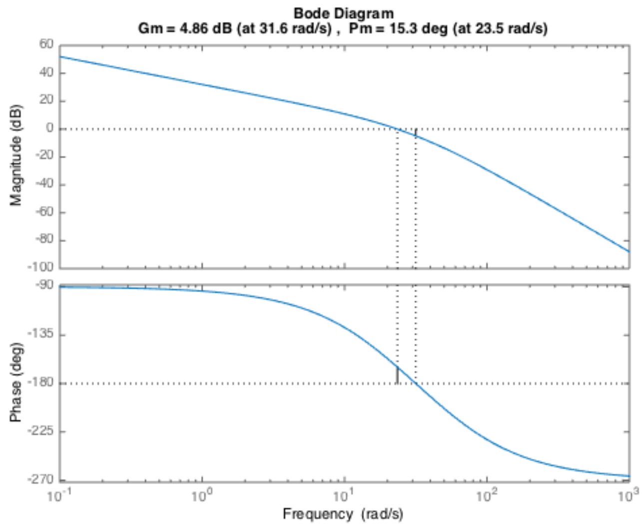


Figure 16: Problem 3: Bode plot for $D(s)G(s)$.

(Another) Solution:
We compute the transfer function from d to y. $y(s) = d(s) - D(s)G(s)y(s)$, hence $y(s)/d(s) = \frac{1}{1+D(s)G(s)}$. From the low-frequency specification, we need that $|1+D(j\omega)G(j\omega)| > 2$ for all $\omega \leq 10$ rad/s. The worst case situation is characterized by $D(j\omega)G(j\omega) = -3$. Thus, we shall design $|D(j\omega)G(j\omega)| > 3$ for all $\omega \leq 10$ rad/s. First, we note that $|D(j\omega)G(j\omega)| > 3$ for all $\omega \leq 10$ rad/s requires $\omega_c > 10$. The bode diagram for $G(s)$ in Figure 14 shows us that we need to increase ω_c, but also that we need to add phase in this frequency region in order to achieve a stable closed-loop interconnection. For this reason a lead compensator will be designed.

Let us then consider a lead compensator $D(s) = K\frac{Ts+1}{\alpha Ts+1}$, for adding some phase (margin) at $\omega_{\max} = 10$ rad/s. Let us choose $\alpha = 1/8$ (this is a free choice, since no requirement on the PM is specified). By the formula $\omega_{\max}T = 1/\sqrt{\alpha}$, we obtain $T = \frac{1}{\omega_{\max}}\frac{1}{\sqrt{\alpha}} = 0.2828$. The last step for the lead compensator design is to calculate K, which will be used to achieve $|D(j\omega)G(j\omega)| > 3$ for all $\omega \leq 10$ rad/s. Observe that $|D(j\omega)G(j\omega)|$ is decreasing for increase ω and therefore we require that $|D(j10)G(j10)| > 3$, which gives us $K > 26.52$.

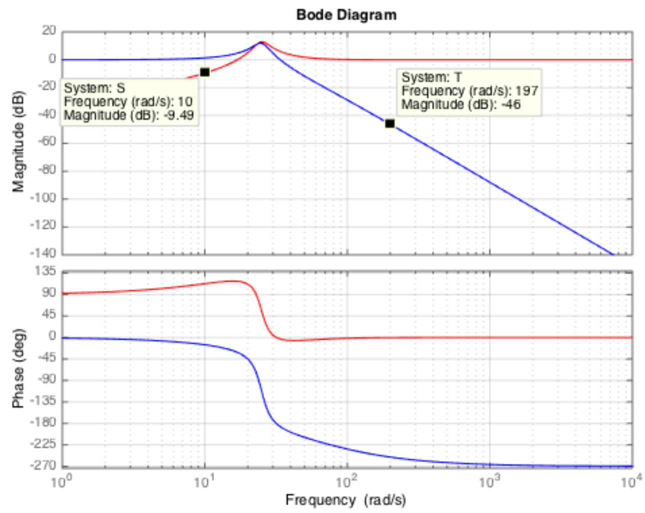


Figure 17: Problem 3: Sensitivity functions $S(s) = \frac{1}{1+D(s)G(s)}$, $T(s) = \frac{D(s)G(s)}{1+D(s)G(s)}$.

We can compute the transfer function from n to y, which is $y(s)/n(s) = \frac{-D(s)G(s)}{1+D(s)G(s)} = -T(s)$. The requirement specifies that $|T(j\omega)| < \frac{1}{100}$ for all $\omega \geq 200$ rad/s. Observe that we must have that $|D(j\omega)G(j\omega)| \ll 1$. Therefore $T(j\omega)$ can be approximated by $|D(j\omega)G(\omega)| < \frac{1}{100}$ for all $\omega \geq 200$ rad/s. Using this approximation we can transform our closed-loop noise requirement to a open-loop requirement on $|D(s)G(s)|$ again.

By choosing $K = 26.52$ the low-frequency specification is met and at the same time also the high-frequency one, see Figure 18. For this reason a single lead compensator is sufficient to meet al requirements. If the high-frequency specification was not met, then we could for example add a pole, or a lag compensator.

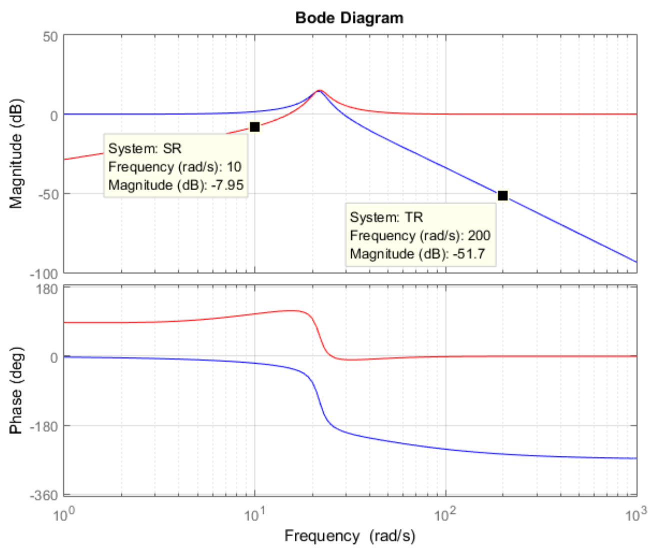


Figure 18: Problem 3 (2nd solution): Sensitivity functions $S(s) = \frac{1}{1+D(s)G(s)}$, $T(s) = \frac{D(s)G(s)}{1+D(s)G(s)}$.

4. **Non-minimum phase system:**
Consider the system with transfer function

$$G(s) = \frac{s-2}{(s+1)(s+3)}$$

Design via SisoTool a controller $D(s)$ such that the closed-loop system is stable and has "as high as possible" bandwidth. Also describe what happens to the sensitivity function as you increase the bandwidth.

Hint: Try (1) a proportional controller with positive gain, (2) a proportional controller with negative gain, and (3) a proportional-integral controller.

Solution:
(1) $D(s) = K_{pos}$ (proportional controller)

The root locus shows that $K_{pos} \in (0, 1.49)$ are the positive values that stabilize the closed-loop system, see Figure 19. We select $K_{pos} = 1.49$ for the almost maximum bandwidth achievable with proportional control, see the Bode plots in Figure 20.

(2) $D(s) = K_{neg}$ (proportional controller)

On the other hand, a negative gain, e.g. $K_{neg} = -3.99$, would cancel the negative static gain in $G(s)$, ensure closed-loop stability and high bandwidth, see Figures 21 and 22.

(3) $D(s) = K\frac{(s+p)}{s}$ (proportional-integral controller)

We choose a negative gain K for the controller to cancel out the negative gain in $G(s)$, and a zero at $s = -1$ near the stable pole (e.g. $p = 1$). One possible design is then $D(s) = -2.99\frac{(s+1)}{s}$. See Figure 23 for the corresponding root locus, and Figure 24 for the compensated Bode plot.

Observing the sensitivity plots for controllers (2), Figure 25, and (3), Figure 26 ,we note that as we increase the bandwidth the peak in the sensitivity function increases in magnitude which can severely effect robustness of our system. For controller (1), Figure 27, we see the low frequent part of our sensitivity function increases which is also not desirable.

As we note from this exercise, RHP zeros pose a fundamental limitation on the bandwidth of the system since the locations of our closed-loop poles move to the open-loop zeros. Since we cannot cancel these zeros, we will always have a limitation on our gain and thus our bandwidth.

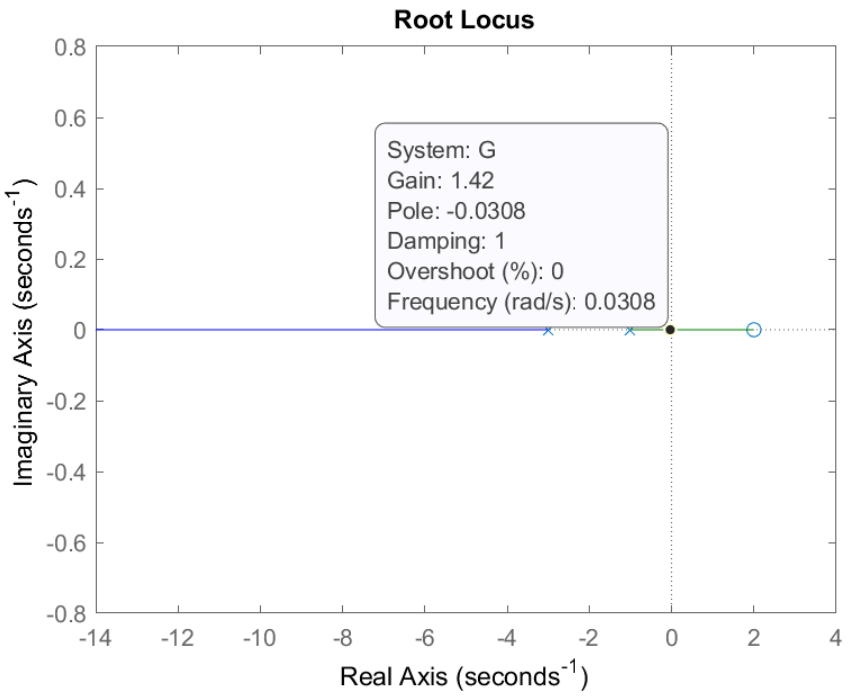


Figure 19: Problem 4: Root Locus for $G(s)$.

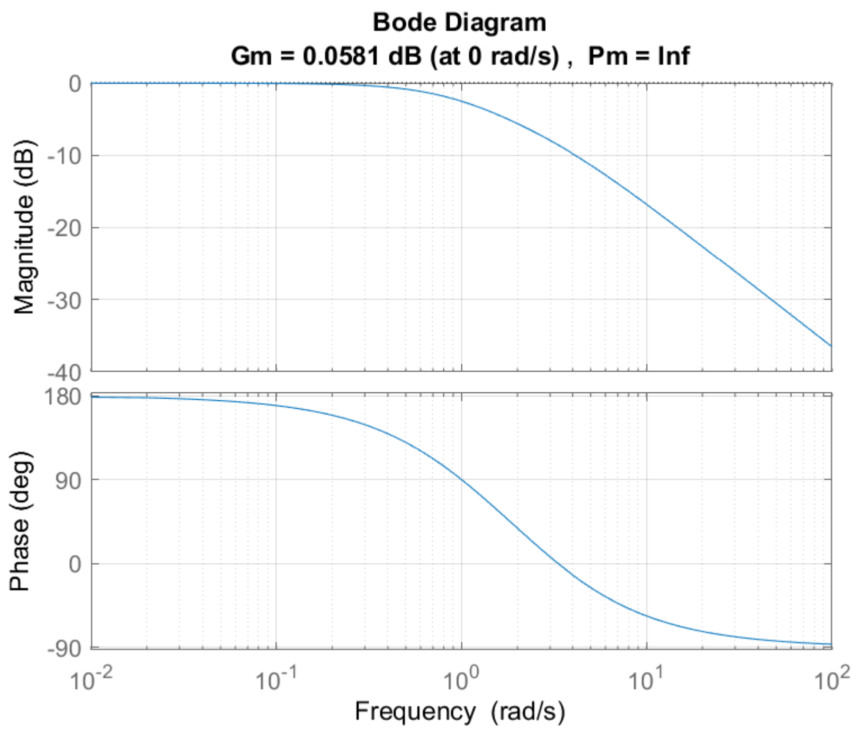


Figure 20: Problem 4: Bode plot for $K_{pos}G(s)$.

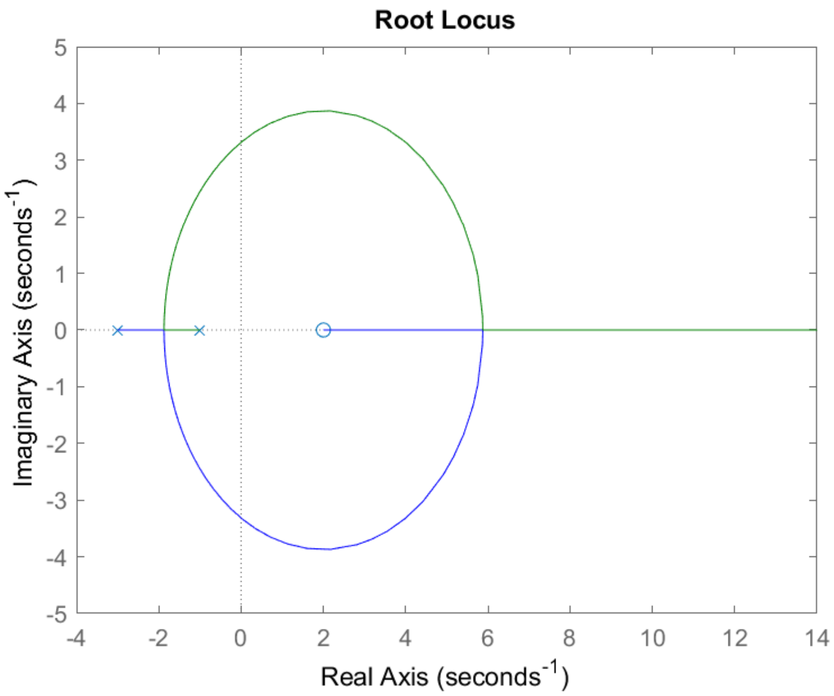


Figure 21: Problem 4: Root Locus for $-G(s)$.

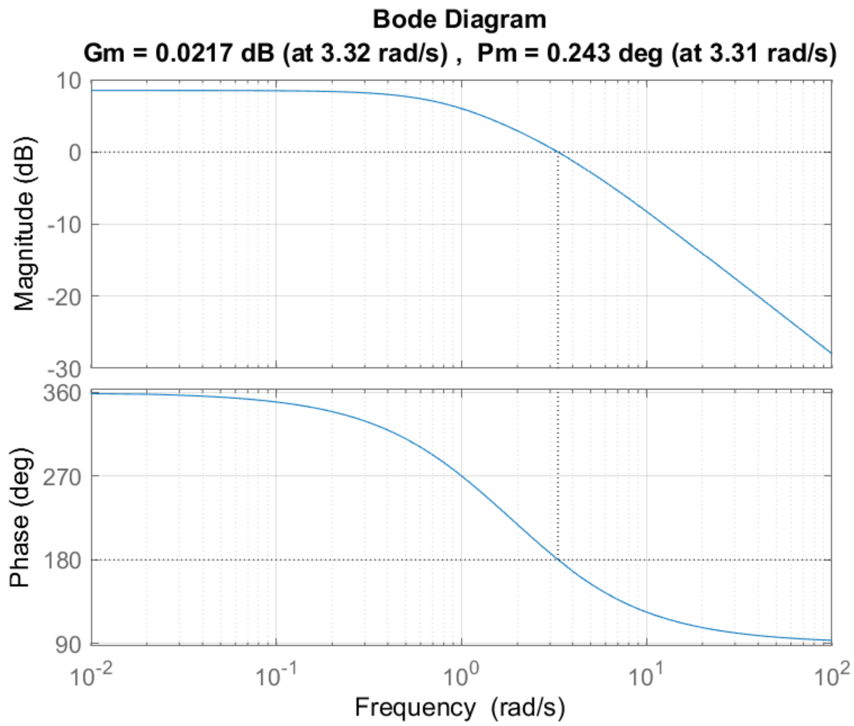


Figure 22: Problem 4: Bode plot for $K_{neg}G(s)$.

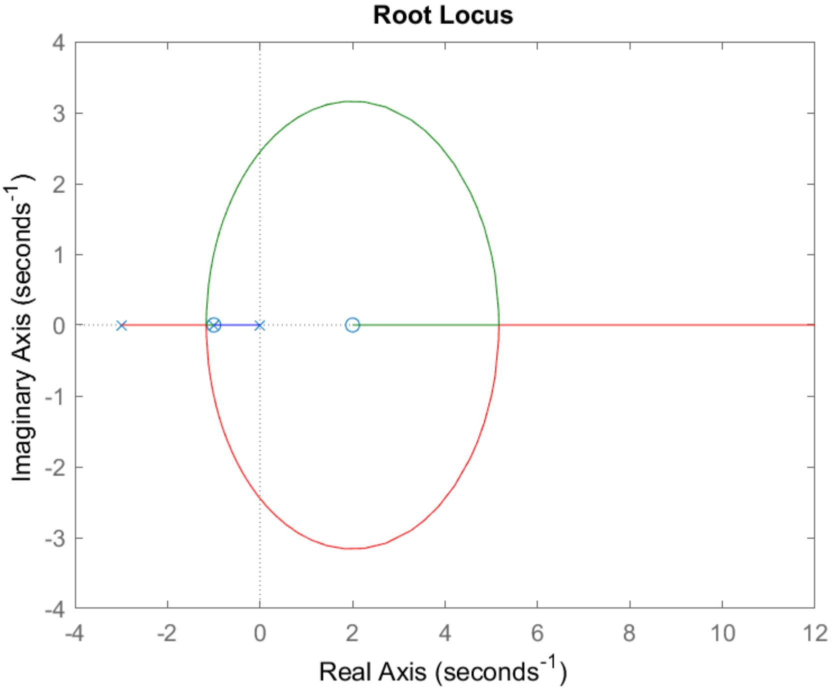


Figure 23: Problem 4: Root Locus for $\frac{s+1}{s}G(s)$.

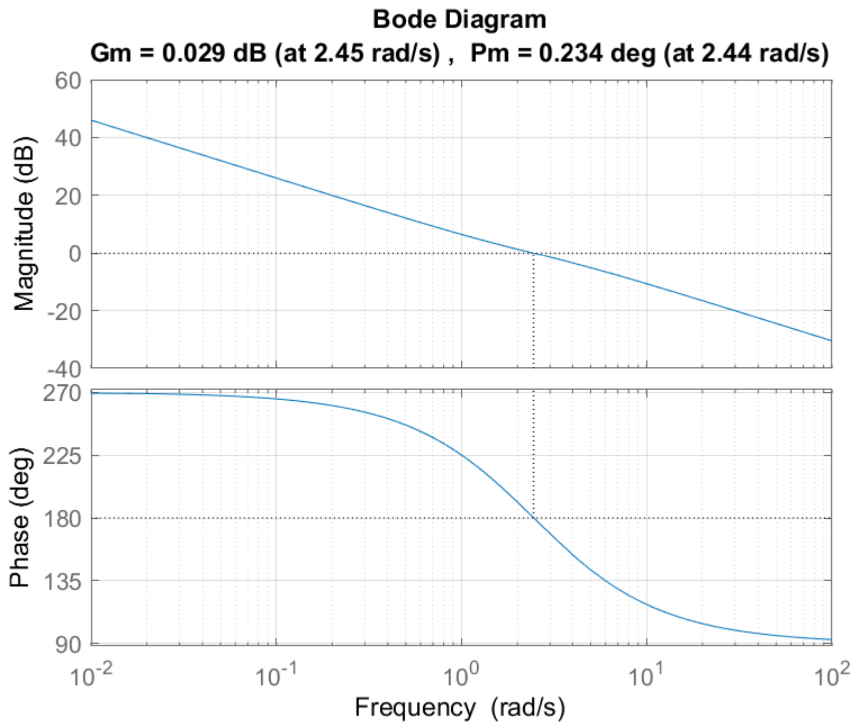


Figure 24: Problem 4: Bode plot for $D_{PI}(s)G(s)$.

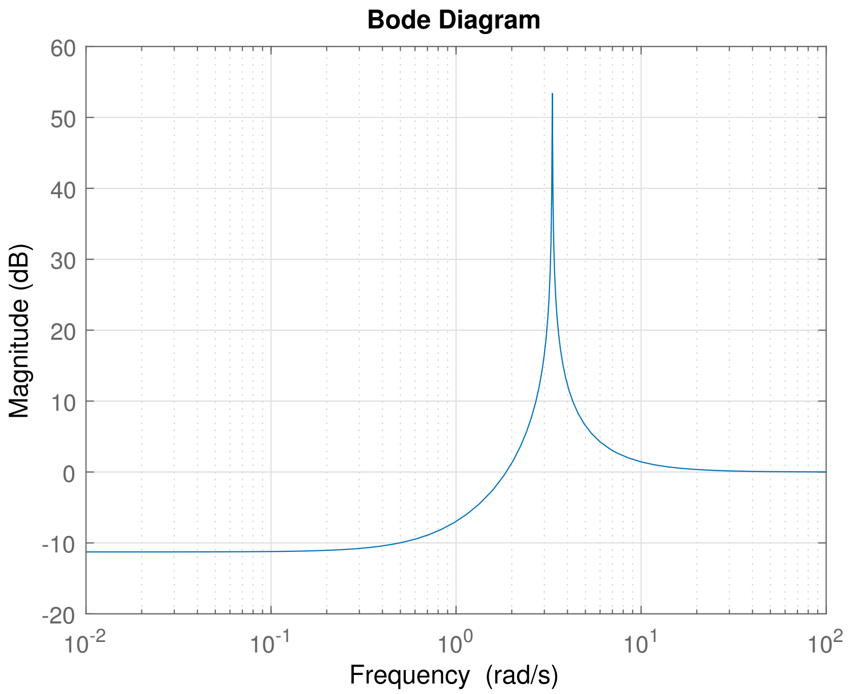


Figure 25: Problem 4: Sensitivity function for $K_{neg}G(s)$.

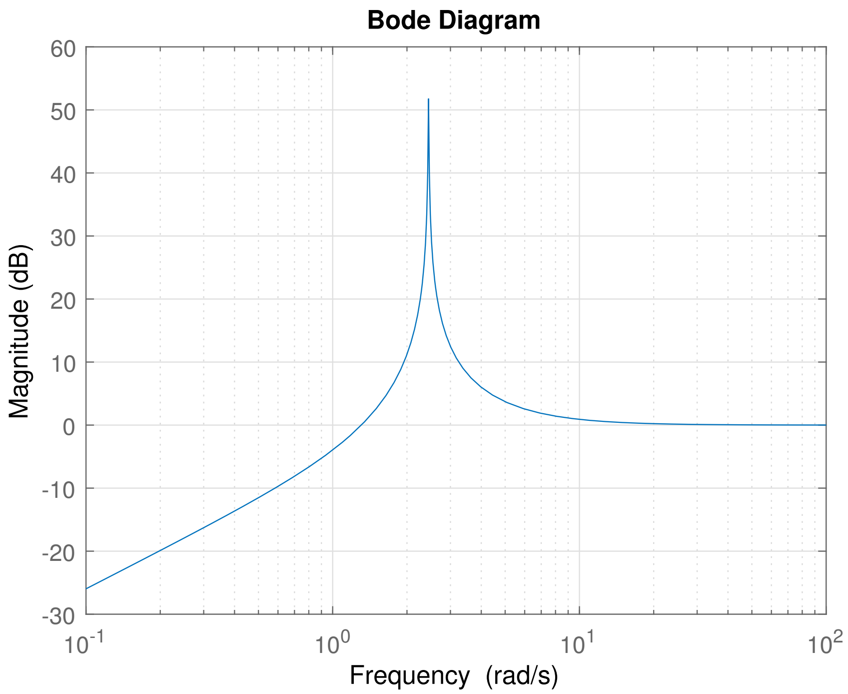


Figure 26: Problem 4: Sensitivity function for $D_{PI}(s)G(s)$.

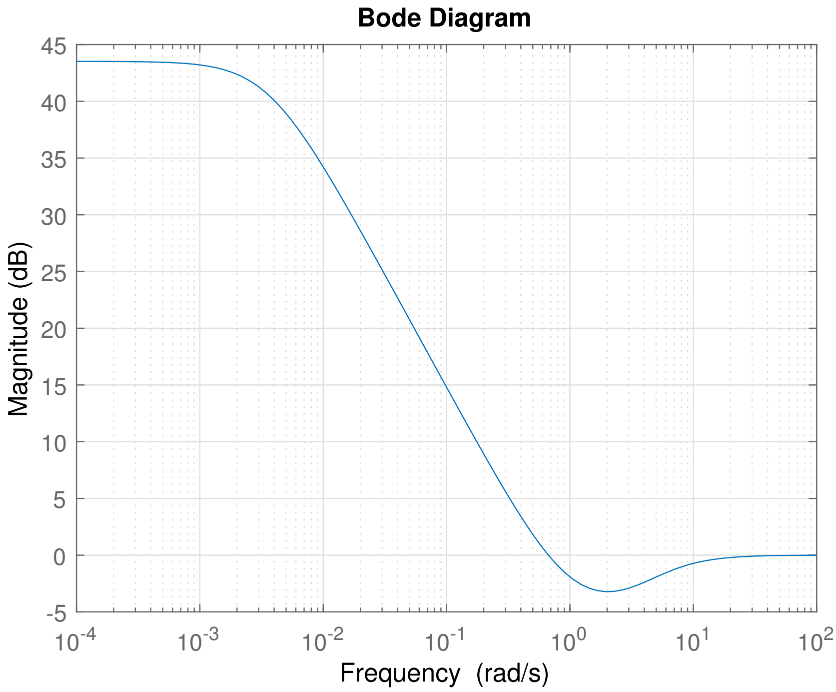


Figure 27: Problem 4: Sensitivity function for $K_{pos}G(s)$.

5. **Robustness margin:**
Consider the block diagram in Figure 13, where

$$G(s) = \frac{1000}{s^2\left(s^2 + 10s + 300^2\right)}.$$

(a) Design a stabilizing controller $D(s)$ such that the crossover frequency $\omega_c \geq 40$rad/s and disturbances are never amplified more than 6dB for all frequencies.

(b) What is the response of the system if

- $d(t) = \sin(10t)$
- $d(t) = \sin(50t)$

(c) What is the worst case response if $d(t) = \sin(ft)$ with $f \in \mathcal{R}^+$

Solution:

(a) We start by evaluating the bode plot of $G(s)$, Figure 28. We note that at $\omega = 40$ rad/s the magnitude is -103dB, hence we increase the gain with 103 dB, e.g. $K = 1.4125 \cdot 10^5$, resulting in the bode plot given in Figure 29.

To render the system stable we design a lead filter in the usual matter with $\omega_{max} = 40$ rad/s, $\phi_{max} = 35°$. This gives a stable system with a crossover frequency $\omega_c \geq 40$ rad/s, Figure 30.

Disturbances may never be amplified more than 6dB, hence the sensitivity function cannot go over 6dB. Looking at the peak of the sensitivity function, e.g. the worst case amplification, we see that with the designed controller suffices the criteria, Figure 31.

(b) We see from the sensitivity function that for a 10 rad/s disturbance the amplification is approximately -18 dB which is an amplification of 0.1259. For a disturbance with frequency of 50 rad/s the amplification is approximately 5 dB which is an amplification of 1.7783.

(c) We designed our controller such that the sensitivity function is always below 6 dB, in fact, the highest amplification for the designed system is 5.45 dB at 291 rad/s. Hence our worst case amplification is 1.8728.

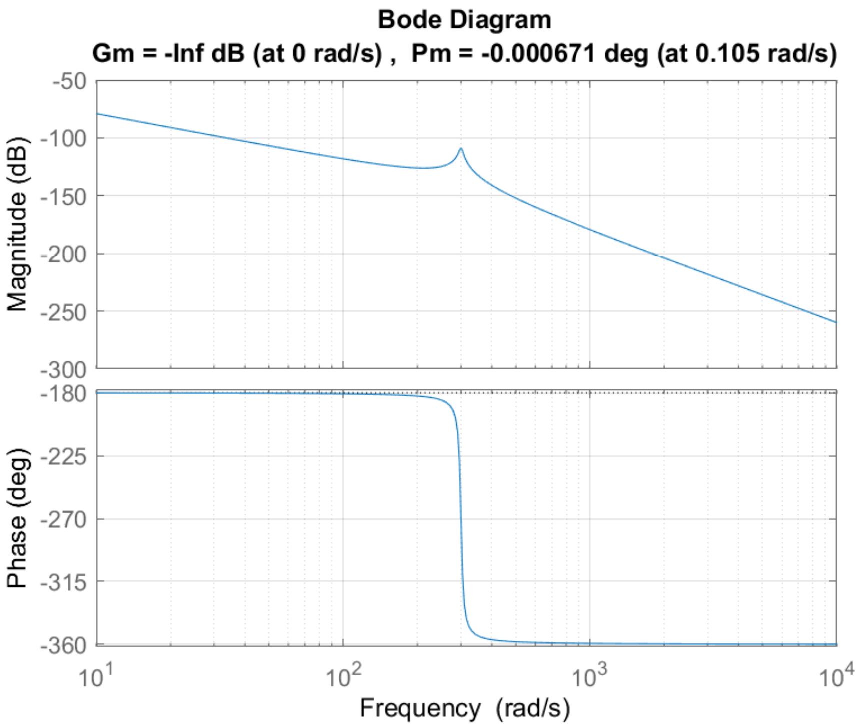


Figure 28: Problem 5: Bode plot for $G(s)$.

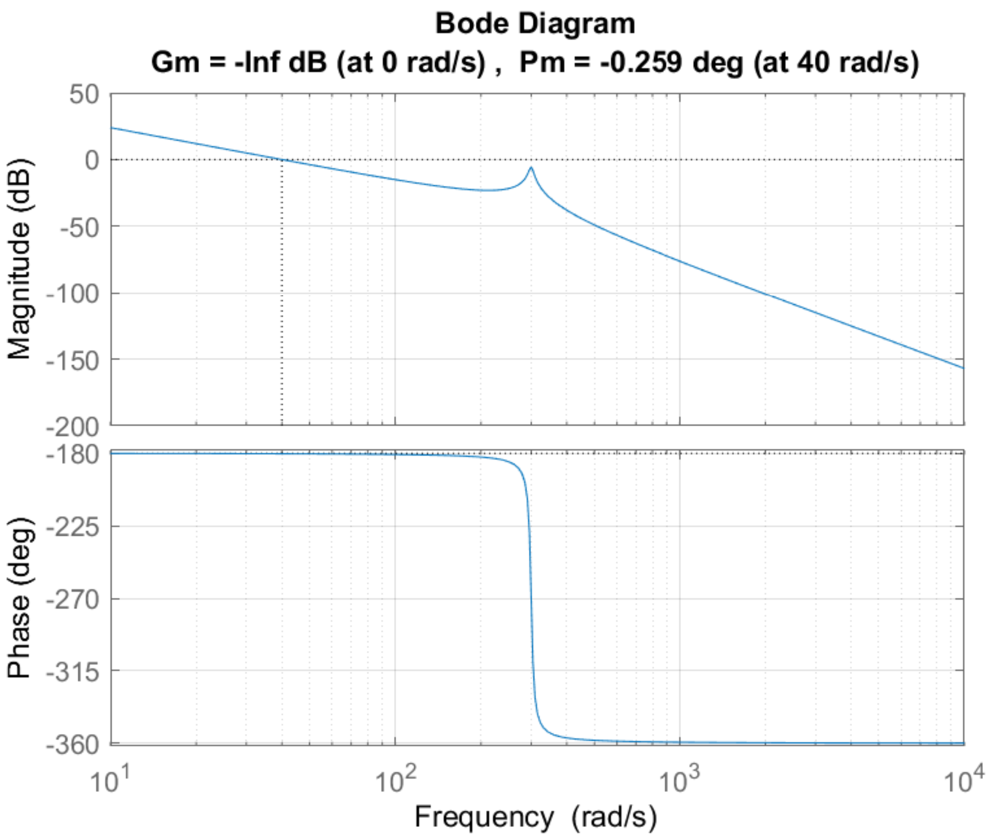


Figure 29: Problem 5: Bode plot for $K\,G(s)$.

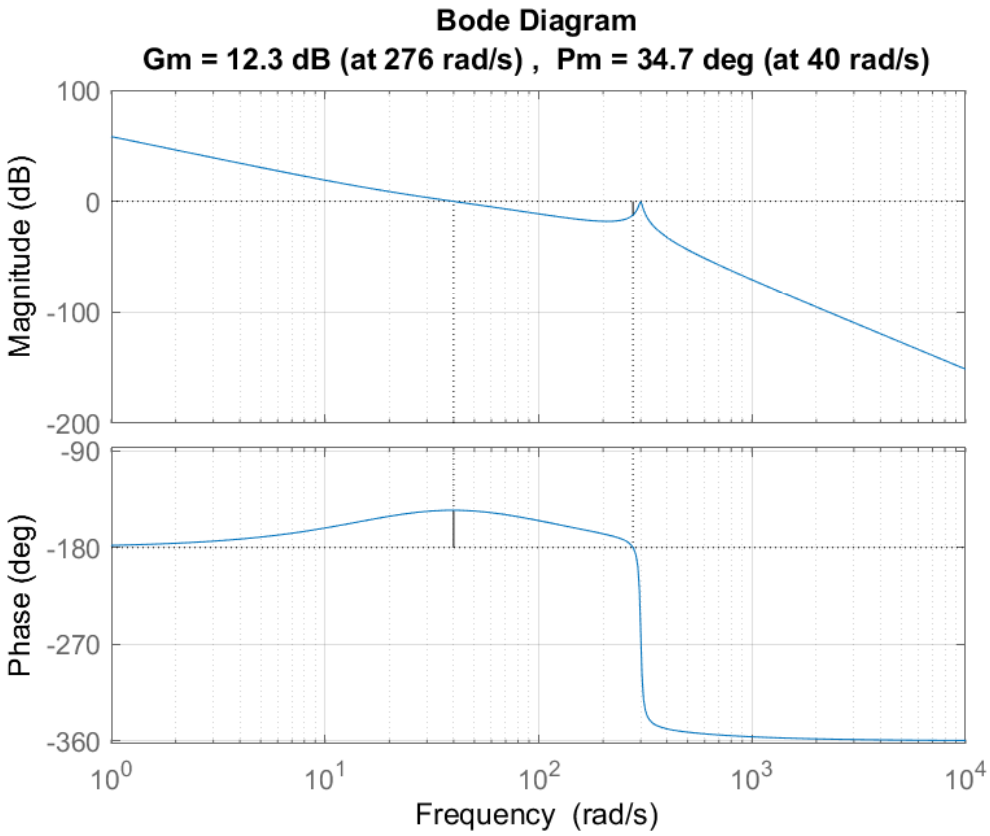


Figure 30: Problem 5: Bode plot for $K\, D_{lead}\, G(s)$.

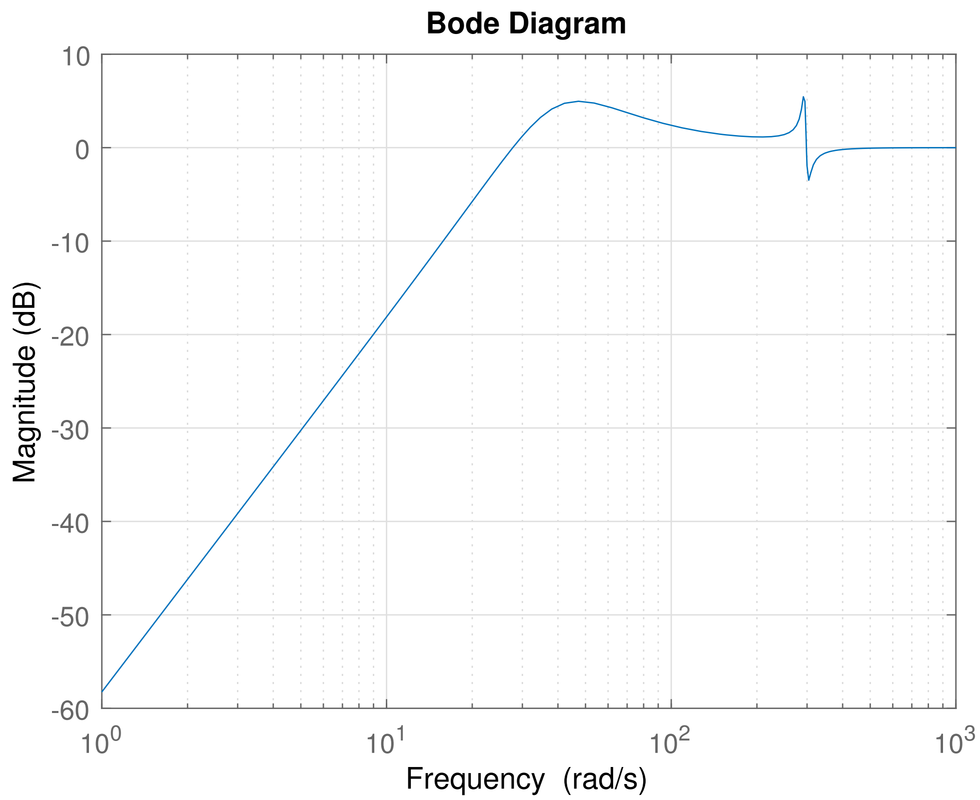


Figure 31: Problem 5: Sensitivity plot for $K\, D_{lead}\, G(s)$.

6. **Disturbance reduction**
Given the following loopgains

- $L_1(s) = \frac{2}{s^2+3s}$
- $L_2(s) = \frac{2-2s}{s^2+3s}$
- $L_3(s) = \frac{2+2s}{s^2+3s}$
- $L_4(s) = \frac{4+4s}{s^2-3s}$

For every loopgain

(a) Plot the sensitivity function and verify the Bode sensitivity integral

(b) Predict what the response of the system will be if it is excited by a disturbance signal

$$d(t) = \sin(2t) + \sin(0.1t)$$

(c) Verify your prediction using Matlab (Hint: you can use the function `lsim`.m)

Solution:
The sensitivity function is given by

$$S(s) = \frac{1}{1 + L(s)}$$

Plotting the bode diagram of the sensitivity function gives Figure 32.

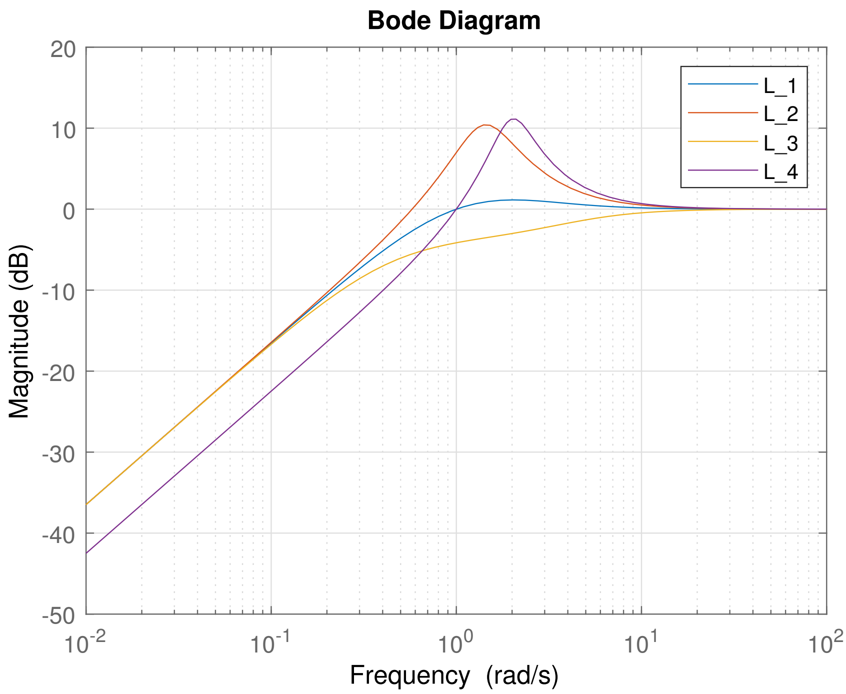


Figure 32: Sensitivity function for Loopgains L_1 to L_4.

The bode sensitivity integral is given by

$$\int_0^\infty \ln |S(j\omega)| d\omega = \pi \sum Re\,(p_k) - \frac{\pi}{2} \lim_{s\to\infty} sL(s)$$

where p_k are the open loop RHP poles. Starting with L_1, we see this transfer function has no open loop RHP poles and that it has two more poles than zeros, e.g. relative degree 2. We expect the integral to equal zero. Computing the integral for L_1 gives

$$\begin{aligned}\int_0^\infty \ln|S(j\omega)|d\omega &= \pi\sum Re\,(p_k) - \frac{\pi}{2}\lim_{s\to\infty} sL(s)\\ &= 0 - \frac{\pi}{2}\lim_{s\to\infty}\frac{2s}{s^2+3s}\\ &= -\frac{\pi}{2}\lim_{s\to\infty}\frac{2}{s+3}\\ &= 0\end{aligned}$$

L_2 has a non-minimum phase zero and no RHP poles, we expect this function to equal a positive constant indicating a peak in the Sensitivity function. Computing the integral for L_2 gives

$$\begin{aligned}\int_0^\infty \ln|S(j\omega)|d\omega &= \pi\sum Re\,(p_k) - \frac{\pi}{2}\lim_{s\to\infty} sL(s)\\ &= -\frac{\pi}{2}\lim_{s\to\infty}\frac{-2s(s-1)}{s(s+3)}\\ &= -\frac{\pi}{2}\lim_{s\to\infty}\frac{-2s+2}{s+3}\\ &= -\frac{\pi}{2}\lim_{s\to\infty}\frac{-2s}{s}\\ &= 2\frac{\pi}{2}\\ &= \pi\end{aligned}$$

L_3 has no RHP poles or zeros but is of relative degree 1, hence we expect it to equal a negative constant indicating the sensitivity function does not peak.

$$\begin{aligned}\int_0^\infty \ln|S(j\omega)|d\omega &= \pi\sum Re\,(p_k) - \frac{\pi}{2}\lim_{s\to\infty} sL(s)\\ &= -\frac{\pi}{2}\lim_{s\to\infty}\frac{2s(s+1)}{s(s+3)}\\ &= -\frac{\pi}{2}\lim_{s\to\infty}\frac{2(s+1)}{s+3}\\ &= -\frac{\pi}{2}\lim_{s\to\infty}\frac{2s}{s}\\ &= -2\frac{\pi}{2}\\ &= -\pi\end{aligned}$$

L_4 has one RHP pole and relative degree 1, hence we expect the Bode sensitivity integral to equal a positive constant again indicating a peak in the sensitivity function.

$$\begin{aligned}\int_0^\infty \ln|S(j\omega)|d\omega &= \pi\sum Re\,(p_k) - \frac{\pi}{2}\lim_{s\to\infty} sL(s)\\ &= 3\pi - \frac{\pi}{2}\lim_{s\to\infty}\frac{4s(s+1)}{s(s-3)}\\ &= 3\pi - \frac{\pi}{2}\lim_{s\to\infty}\frac{4(s+1)}{s-3}\\ &= 3\pi - 4\frac{\pi}{2}\\ &= \pi\end{aligned}$$

To predict the response we look at the magnitude of the sensitivity function at the excited frequencies. Starting with L_1 we see that signals with a frequency of 0.1 rad/s are severely reduced, namely by -16.5 dB, and we see a slight amplification for signals with a frequency of 2 rad/s. Plotting the response confirms this, see Figure 33.

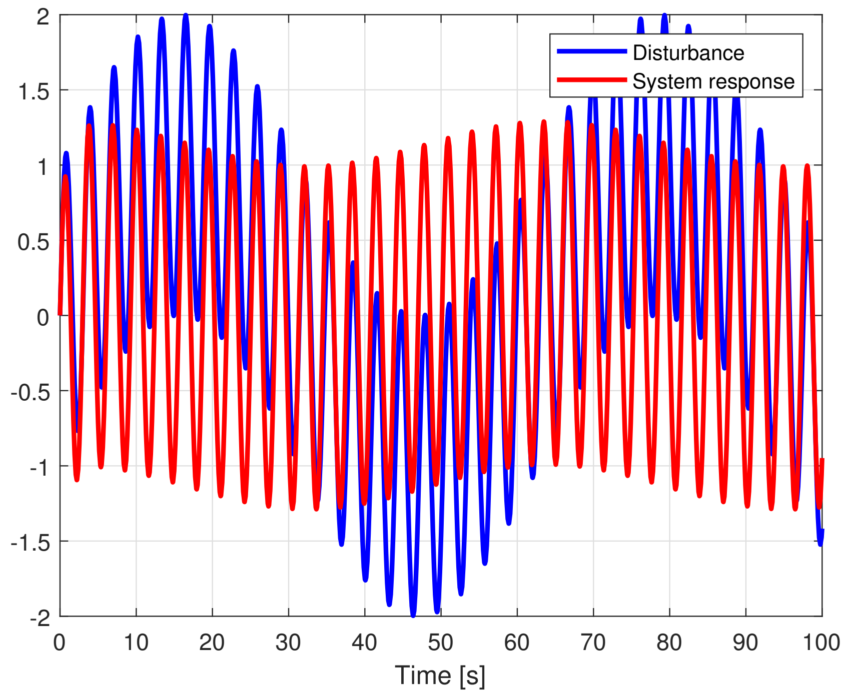


Figure 33: System response with loopgain L_1 to the given disturbance signal.

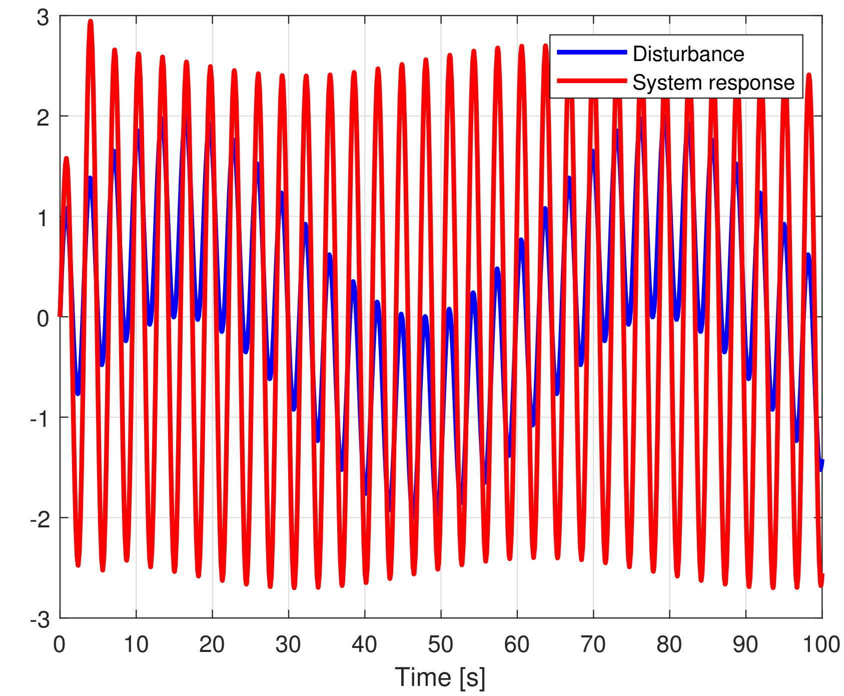


Figure 34: System response with loopgain L_2 to the given disturbance signal.

Observing the sensitivity function of L_2, we see again that signals with a frequency of 0.1 rad/s are reduced, but signals of 2 rad/s are amplified by 8 dB. This again can be verified by the response given in Figure 34

Now taking a look at the sensitivity function for L_3 we see that both excited frequencies are reduced. This can again be verified by the system response shown in Figure 35

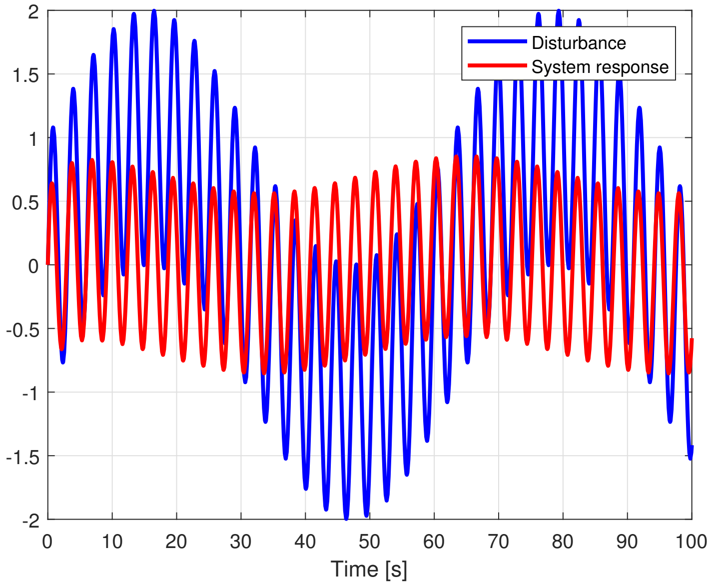


Figure 35: System response with loopgain L_3 to the given disturbance signal.

Lastly, looking at the sensitivity function for L_4, we see a reduction for signals with frequency of 0.1 rad/s. The sensitivity function peaks at 2 rad/s, hence signals with a frequency of 2 rad/s are heavily amplified with a gain of 11 dB. This also shows from the response given in Figure 36.

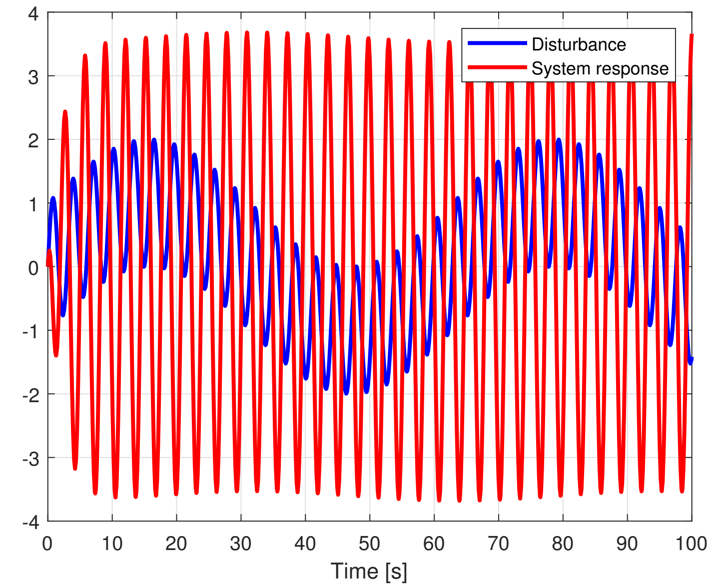


Figure 36: System response with loopgain L_4 to the given disturbance signal.

7. **Design - Lag compensator (Exam Level Question):**
Consider the feedback control scheme in Figure 37, with a plant that has the following transfer function:

$$G(s) = \frac{1}{s(s+40)} \tag{1}$$

and consider the following performance specifications:

i) A steady-state tracking error $\lim_{t\to\infty} e(t) \leq 4 \cdot 10^{-4}$ with $e(t) = r(t) - y(t)$ for a ramp input given by

$$r(t) = \begin{cases} t & \text{for } t > 0, \\ 0 & \text{for } t \leqslant 0, \end{cases} \tag{2}$$

for the situation that $d(t) = n(t) = 0$,

ii) Measurement disturbances $n(t) = \sin(\omega_d t)$ for $\omega_d > 600$ rad/sec are amplified by at most a factor 0.01 at the output $y(t)$ (in other words: they should be attenuated at least by a factor 100).

iii) A phase margin of at least 45 degrees.

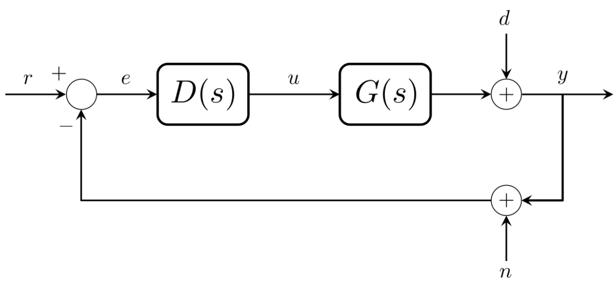


Figure 37: Feedback interconnection for Question 3.

Design a controller $D(s)$ by taking the following steps:

a) Consider $D(s) = k_1$ for some constant gain $k_1 \in \mathbb{R}$. Determine the range of values for k_1 such that performance requirement i) is satisfied.

b) Consider $D(s) = k_2$ for some constant $k_2 \in \mathbb{R}$. Determine the range of values for k_2 such that performance requirement ii) is satisfied.

c) Consider $D(s) = k_3$ for some constant $k_3 \in \mathbb{R}$. Determine the range of values for k_3 such that performance requirement iii) is satisfied.

d) Realise a *lag compensator* that will ensure satisfaction of all performance requirements i)- iii). Provide the numerical values of this controller/compensator $D(s)$ and motivate your design choices.

Solution:

a) System type $n = 1$ and input type is $k = 1$. Apply final value theorem of sensitivity function times reference to conclude that steady-state error is $\frac{40}{k}$, which needs to be smaller than 4×10^4, so $k_1 > 10^5$.

b) Relevant transfer function from n to y is $\frac{DG}{1+DG}$. Above the crossover frequency, this can be approximated by DG, so magnitude of DG needs to be smaller than 0.01 at $\omega = 600$. This gives $\left|\frac{k}{600j(600j+40)}\right| < 0.01$, which means that $k_2 < 3.6 \times 10^3$.

c) i) PM means that angle should be at least -135 degrees at crossover frequency. Since the system has 2 poles and no zeros, the angle starts at -90 for low frequencies and moves to -180 for high frequencies. This puts a maximum on the achievable crossover frequency.

ii) To find this crossover frequency, compute at which frequency the angle is -135 degrees, which can be done by solving angle $\left(\frac{1}{j\omega(j\omega+40)}\right) = -135$, which corresponds to the angle $(j\omega + 40) = 45$ degrees, so $\frac{\omega}{40} = \tan(45) = 1$, so maximum crossover is $\omega = 40\,\text{rad/sec}$.

iii) We want to have that the magnitude $\frac{k_3}{40j(40j+40)} < 1$ (maximum crossover), which means that $k_3 < 2.2 \times 10^3$.

d) i) Realize that $k_3 < k_2$ so if you satisfy performance requirement 2, you satisfy performance requirement 3. So the lag filter should have a low frequency gain of k_1 and a high frequency gain of k_3.

ii) Gain $K = k_3$.

iii) Alpha needs to be such that $K\alpha = k_1$ (to have high frequency attenuation).

iv) T_I needs to be placed such that the zero is a decade lower than the crossover frequency.

8. **Peak of sensitivity function (Exam Level Question):**
Which of the following loop gains $L(s)$, which all have a crossover frequency around 0.7rad/sec, has the largest peak in the sensitivity function? (Hint: you should be able to answer this question without computing the sensitivity function.) Check your result using Matlab.

i) $L(s) = \frac{2-2s}{s^2+3s}$

ii) $L(s) = \frac{2}{s^2+3s}$

iii) $L(s) = \frac{2+2s}{s^2+3s}$

Solution:

9. **Gang of four sensitivity functions:**

Consider the Bode magnitude diagram of a loop-gain $L(s)$ and the closed-loop system configuration in Figure 39. The controller D is given a single gain $K = 10$. For the following tasks, use the Bode plot of $L(s)$ to approximate the gain and use the asymptotic approximation of the sensitivity functions.

(a) Under the assumption that $r = d = 0$, what would be the amplitude of the plant output y due to the presence of noise described by $n_1 = \sin(10t)$? And when $n_2 = \sin(500t)$? If n_1 and n_2 come from different sensors, which sensor would you recommend for the given system?

(b) Under the assumption that $r = n = 0$, what would be the amplitude of the plant output y due to the presence of a disturbance following $d_1 = \sin(10t)$? And when $d_2 = \sin(500t)$? Which disturbance has more impact on the output?

(c) If our actuator has an amplitude limit of 5 (so the control input u cannot be higher than 5), under the assumption that $d = n = 0$, determine the control amplitude for the references $r_{eff1} = \sin(10t)$ or $r_{eff2} = \sin(500t)$. For which reference will the actuator limit be exceeded?

(d) Consider again $R_{eff1} = \sin(10t)$ or $R_{eff2} = \sin(500t)$. Determine the amplitude of the error for the two references. Which reference will result in a lower amplitude of the error?

Solution:

a) The amplitude of the plant output y due to the noise can be determined from $\frac{Y}{N} = \frac{1}{1+DG}$. For $n_1 = \sin(10t)$, $\omega << \omega_c$, hence $y \approx \frac{1}{DG} = 0.1$ for 10 rad/s. For $n_2 = \sin(500t)$, $y = \frac{1}{1+DG} \approx 1$. Therefore, the sensor corresponding to n_1 should be recommended.

b) For $d_1 = \sin(10t)$, where $\omega \ll \omega_c$, the amplitude is approximately $\frac{1}{D} \approx 0.1$. For $d_2 = \sin(500t)$, where $\omega \gg \omega_c$, the gain G is around -25 dB for DG, resulting in an amplitude, $y \approx G$, of 0.0056 (-45 dB). Therefore, $d_1 = \sin(10t)$ has a greater impact on the output.

c) For $r_{eff1} = \sin(10t)$, u is approximated as $\approx \frac{1}{G}$. For $r_{eff2} = \sin(500t)$, $u \approx D = 10$. Therefore, the actuator limit will be exceeded for R_{eff2}.

d) For $R_{eff1} = \sin(10t)$, the error $e \approx \frac{1}{DG}$. For $R_{eff2} = \sin(500t)$, $u \approx 1$. Therefore, R_{eff1} will result in better reference tracking and a lower amplitude of the error.

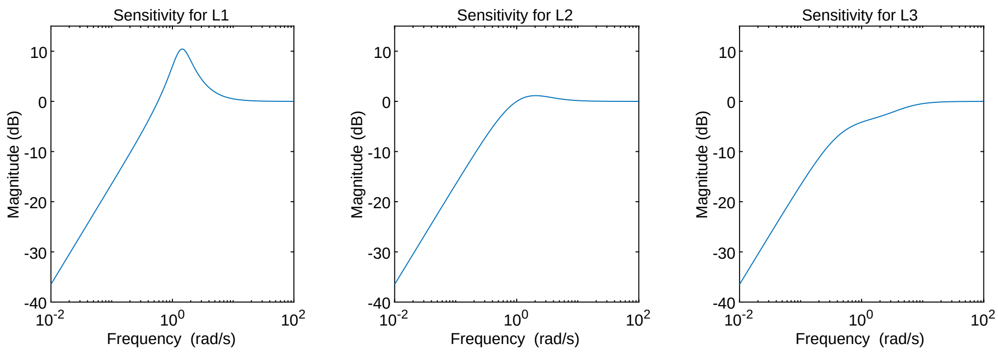
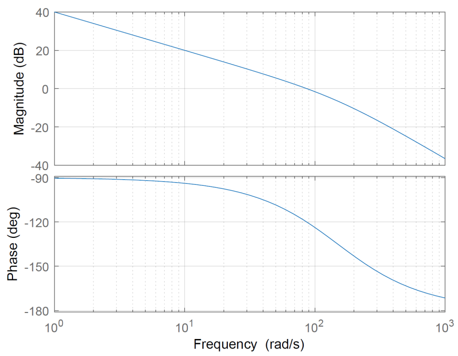
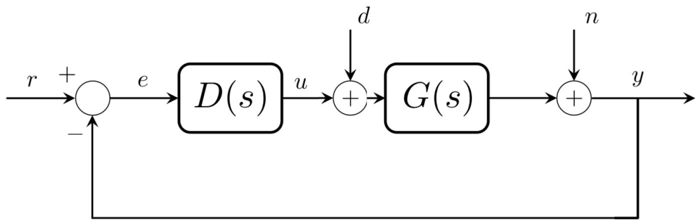


Figure 38: Sensitivity functions for $L1$, $L2$ and $L3$



(a) Bode magnitude plot of loop-gain $L(s)$



(b) Closed-loop system configuration

Figure 39: Bode magnitude plot of loop-gain $L(s)$ and the closed-loop system configuration under consideration.