5ESD0 Summary - Samenvatting Control systems

Control systems (Technische Universiteit Eindhoven)

5ESD0 Summary

Ramon Boon

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1 Systems Recap

1.1 Laplace transform

The Laplace transform of a time-signal $f(t)$ is defined as

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{1}$$

It maps time-signals to a complex function $F(s)$.

$f(t)$	$F(s)$
$a \cdot 1(t)$	$\frac{a}{s}$
$\delta(t)$	1
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$

Differentiation:

$$L(\frac{f^{(m)}(t)}{dt}) = s^m F(s) - s^{m-1} f(0) - s^{m-2} \dot{f}(0)... \tag{2}$$

Integration in the time domain leads to a multiplication by $\frac{1}{s}$ in the s-domain: $\frac{1}{s}F(s)$

Convolution in the time domain corresponds to a multiplication in the s-domain: $F_1(s)F_2(s)$ and vice versa for multiplication on the time domain.

Multiplication by time $tf(t)$ leads to differentiation in the s-domain: $-\frac{d}{ds}F(s)$

Frequency response $H(j\omega) = H_(s)$ for $s = j\omega$.

If you want to go from $F(s)$ to $f(t)$, decompose the first one into functions with known inverse Laplace transforms (see table 1).

1.2 Poles and zeros

$$F(s) = K \frac{\Pi_{i=1}^{m}(s - z_i)}{\Pi_{i=1}^{n}(s - p_i)} \tag{3}$$

Where z_i are the zeros and p_i are the poles. K is the gain.
Under zero initial conditions:

$$\Pi(s - p_i)Y(s) = K\Pi(s - z_i)U(s) \tag{4}$$

For zero input (U(s)=0) the output is $e^{p_i t}$ and for zero output (Y(s)=0) the input is $e^{z_i t}$.

Poles determine the natural (unforced) response of the system, zeros determine the input signals that do not reach through to the output.

For a system to be realizable and causal, there should be less zeros than poles. If not, then the transfer function also has zeros in infinity.

Impulse response $h(t) = e^{-\sigma t} \cdot 1(t)$ with pole σ decays for growing t if the pole is in the left half plane ($\sigma > 0$, system is stable). It grows for if it is in the right half plane (unstable system)

If all poles are distinct, then

$$F(s) = \frac{C_1}{s - p_1} + \cdots + \frac{C_n}{s - p_n} \tag{5}$$

with

$$C_i = [(s - p_i)F(s)]_{s=p_i} \tag{6}$$

and from this follows:

$$f(t) = \sum C_i e^{p_i t} \cdot 1(t) \tag{7}$$

If some poles are repeated, so not all distinct, the repeated pole looks like this:

$$F(s) = \frac{C_1}{s - p_1} + \frac{C_2}{(s - p_1)^2} + \frac{C_3}{(s - p_1)^3} \tag{8}$$

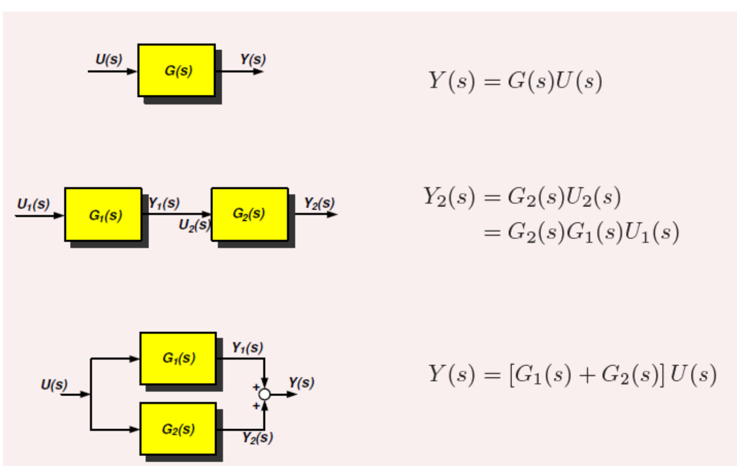
Coefficients are determined like this:

$$C_3 = [(s - p_1)^3 F(s)]_{s=p_1} \tag{9}$$

$$C_2 = \frac{d}{ds}[(s - p_1)^3 F(s)]_{s=p_1} \tag{10}$$

$$C_1 = \frac{1}{2}\frac{d^2}{ds^2}[(s - p_1)^3 F(s)]_{s=p_1} \tag{11}$$

1.3 Transfer functions and block diagrams



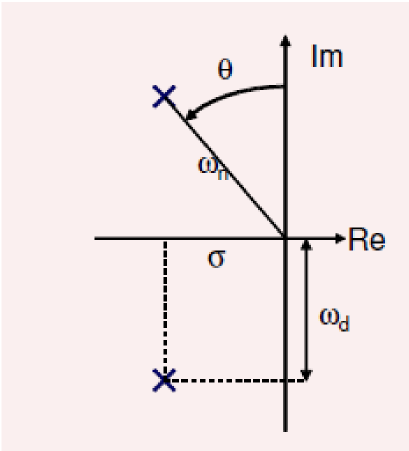
1.4 Second order dynamic responses

Poles and zeros of H come in complex conjugate pairs, these relate to particular oscillatory time responses. Complex pole pair:

$$s = -\sigma \pm j\omega_d \tag{12}$$

Denominator of transform:

$$s^2 + 2\sigma s + \sigma^2 + \omega_d^2 \tag{13}$$



or

$$s = \omega_n e^{\pm j(\frac{\pi}{2}+\theta} \tag{14}$$

with denominator

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{15}$$

with undamped natural frequency ω_n and damping $\zeta = \sin\theta$

1.5 Final Value theorem

If all poles of $sY(s)$ are in the left half-plane, then $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$ With this, the steady-state gain of a transfer function can simply be determined.

1.6 Step response characterization

Consider 2nd order system with transfer function

$$H(s) = \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2} = \frac{\sigma^2\omega_d^2}{s^2 + 2\sigma s + \sigma^2\omega_d^2} \tag{16}$$

evaluate its step response:

- Rise time t_r: time it takes to move from 10% to 90% of the steady-state value
- Peak time t_p: time it takes the step response to reach max. amplitude
- Settling time t_s: time it takes to reach and stay withing $\pm 1\%$ of the steady-state value
- Overshoot M_p: max. relative amount the step response overshoots the final value

$$t_r \approx \frac{1.8}{\omega_n} \tag{17}$$

$$t_p = \frac{\pi}{\omega_d} \tag{18}$$

$$t_s = \frac{4.6}{\sigma} \tag{19}$$

$$M_p = y(t_p) - 1 = e^{-\sigma t_p} = e^{-\frac{\sigma\pi}{\omega_d}} \tag{20}$$

with

$$\frac{\sigma}{\omega_d} = \tan\theta = \frac{\zeta}{\sqrt{1-\zeta^2}} \tag{21}$$

For second order systems:

- $M_p = 5\% \rightarrow \zeta = 0.7$

- $M_p = 16\% \rightarrow \zeta = 0.5$
- $M_p = 35\% \rightarrow \zeta = 0.3$

If an additional zero is added to the system: M_p increases and t_r decreases if zero is in neighbourhood of poles (LHP) and has an initial negative response if zero in RHP.

An additional pole causes t_r to increase if the pole is near other poles.

1.7 Stability

An LTI system is stable if all poles of the transfer function are in the open left half plane, i.e. have negative real parts. In this case, the solutions to the homogeneous part of the diff. equation decay to zero.

1.8 Feedback & system types

See lecture 7 of 5ESB0 (Systems)

2 Root Locus

2.1 Root Locus Design Method

Analyse changes in the system parameters that modify the roots of the characteristic equation With this method, closed loop behaviour can be analysed using the open loop plant.

2.1.1 Poles and zeros basics

A transfer function is

- **Monic** if the highest power has a coefficient of 1. We use a monic transfer function for the root locus method to make the notation and math more simple.
- **Strictly proper** if it has more poles than zeros
- **(Bi-) proper** if the number of poles are equal to the number of zeros
- **Not proper** if it has less poles than zeros

When a pole is on the real axis in the LHP, the function will decay (stable), and when it is on the real axis in the RHP, it will grow (unstable).
If it is on the imaginary axis, it will show sinusoidal behavior. If it is not on one of the axis, it will show a combination of these behaviours.

Now, consider a controller D(s) connected to a plant G(s). The transfer function when closing the loop is

$$\frac{D(s)G(s)}{1 + D(s)G(s)} \tag{22}$$

The denominator of this transfer function is called the **closed loop system characteristic equation**, and can be rewritten as

$$1 + kL(s) \tag{23}$$

where k is a scalar gain. The roots of this characteristic equation are thus the closed loop poles of this system, and can be found by setting the equation to zero, or in any of the following equivalent forms:

- $1 + kL(s) = 0$
- $1 + k\frac{b(s)}{a(s)} = 0$
- $a(s) + kb(s) = 0$

- $L(s) = -\frac{1}{k}$

Note that $b(s)$ represent the zeros and $a(s)$ the poles.

2.1.2 Root locus

Locus: position or place where something occurs or is situated → curve or figure formed by points satisfying an equation.

The root locus of a system displays the roots of the characteristic equation ($1 + kL(s) \Leftrightarrow 1 + D(s)G(s)$), by varying scalar k from 0 to ∞, which is the plot of the closed loop poles. This is very powerful since we can use the open loop plant to analyse the dynamic behaviour of the closed loop plant.

Solving the characteristic equation for every possible k is obviously not doable. Instead, drawing rules exist to quickly sketch the root locus of a system.

2.1.3 Example

Consider the following transfer function:

$$G(s) = \frac{s + 2c}{s(s + c)} \tag{24}$$

For the controller, we take a unity gain ($D(s) = 1$). This gives the following closed loop equation:

$$1 + D(s)G(s) = 1 + \frac{s + 2c}{s(s + c)} \tag{25}$$

with characteristic equation

$$s^2 + s + c(s + 2) = 0 \tag{26}$$

and the following terms:

- $L(s) = \frac{s+2}{s(s+1)}$
- $k = c$
- $b(s) = s + 2$, $m = 1$, $z_i = -2$
- $a(s) = s(s + 1)$, $n = 2$, $p_i = 0, -1$

The solution to the characteristic equation is

$$r_1, r_2 = \frac{-(k + 1) \pm \sqrt{k^2 - 6k + 1}}{2} \tag{27}$$

2.2 Root Locus Drawing Rules

Key definitions

- Definition I: $1 + kL(s) = 0$ must be satisfied under varying k, $k \in [0, \infty)$. Looking at the rewritten form $L(s) = \frac{-1}{k}$, $D(s)G(s)$ must be real negative, as k is real positive. When writing this in polar form $Re^{st} = Re^{j\omega t}$, its phase should be 180°.
- Definition II: Root locus of $L(s)$ (or $D(s)G(s)$) is the set of points in the s-plane where the phase of $L(s)$ is 180 deg

To test if a point in the s-plane is on the locus, a degree test can be performed:

$$\sum \psi_i - \sum \phi_i = 180\,\text{deg} + 360\,\text{deg}(l - 1) \tag{28}$$

where l is an integer value.

2.2.1 Drawing rules

- Rule 1: n branches start at the poles of $L(s)$, m branches end at the zeros of $L(s)$
- Rule 2: The loci are on the real axis to the left of an odd number of poles and zeros.
- Rule 3: For large s and k, the loci that go to infinity are asymptotic lines radiating at a fixed angle from a central point, called the centre of gravity of the poles. The nr. of loci going to infinity is given by $n - m$. The angles at which the asymptotes branch out is given by

$$\phi_l = \frac{180\deg + 360\deg(l-1)}{n-m}, l = 1, 2, ..., n-m \tag{29}$$

  the centre of gravity is given by

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} \tag{30}$$

  where only the real parts need to be considered, as the imaginary parts cancel each other.

- Rule 4: The departure angle of a branch of the locus from a single pole is given by the following expression:

$$\phi_{dep} = \sum \psi_i - \sum \phi_i - 180\deg \tag{31}$$

  where the first term is the sum of zeros and the second the sum of poles not considered

- Rule 5: The locus can have multiple roots at points on the locus and the branches approach a point of q roots with angles separated by

$$\frac{180\deg + 360\deg(l-1)}{q} \tag{32}$$

  Angles of departure and arrival: **Continuation Locus**

3 Frequency Response Functions

3.1 Frequency Response Design Method

Main objective: Design controllers/compensators for closed loop system, using the open-loop frequency response function $D(j\omega)G(j\omega)$.

- We specify what DG should look like
- We (approximately know) G
- We want to design D

This works if we can easily manipulate the frequency response functions (FRF):

- The Bode Diagram is a convenient representation of the frequency response
- Composition of FRF's: addition of log curves in the frequency domain

3.2 Frequency response functions

- A sinusoidal input applied to a linear time-invariant (LTI) system leads (in steady-state conditions) to a sinusoidal output with the same frequency
- Bode diagram plots amplification and phase shift of this frequency
- The FRF can be obtained from the transfer function by evaluating at $s = j\omega$

Magnitude/gain of the FRF:

$$|G(j\omega)| = \sqrt{Re(G(j\omega))^2 + Im(G(j\omega))^2} \tag{33}$$

Phase:

$$\angle G(j\omega) = \arctan(\frac{Re(G(j\omega))}{Im(G(j\omega))}) \tag{34}$$

Decibel conversion: $20 \log |G(j\omega)|$

3.3 Bode Diagrams

- Bode form of FRF:
$$KG(j\omega) = K_0(j\omega)^n \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)...}{(j\omega\tau_a + 1)(j\omega\tau_b + 1)...} \tag{35}$$

- Composition is adding the magnitude plot and the phase plot:
$$\log|KG(j\omega)| = \log K_0 + n\log|j\omega| + \log|j\omega\tau_1 + 1| - \log|j\omega\tau_a + 1| + ... \tag{36}$$
$$\angle KG(j\omega) = n\angle j\omega + \angle(j\omega\tau_1 + 1) - \angle(j\omega\tau_a + 1) + ... \tag{37}$$

- This allows for sketching Bode diagrams by considering
  - Class 1: $K_0(j\omega)^n$ for poles/zeros in the origin
  - Class 2: $(j\omega\tau + 1)^{\pm 1}$ for real poles/zeros. +1 for a zero and -1 for a pole.
  - Class 3: $((\frac{j\omega}{\omega_n})^2 + 2\zeta\frac{j\omega}{\omega_n} + 1)^{\pm 1}$ for complex conjugate poles/zeros

3.3.1 Class 1: $K_0(j\omega)^n$

$$20\log|K_0(j\omega)^2| = 20\log K_0 + 20n\log|\omega| \tag{38}$$
$$\angle K_0(j\omega)^n = \angle K_0 + \angle(j\omega)^n = 0 + n\cdot 90^\circ \tag{39}$$

3.3.2 Class 2: $(j\omega\tau + 1)^{\pm 1}$

$j\omega\tau + 1 \approx$

- 1 (0 dB) for $\omega\tau << 1$, then phase is 0°, gain is 0 dB
- $\sqrt{2}$ (3 dB) for $\omega\tau = 1$, then phase is 45°
- $(j\omega\tau)^{\pm 1}$ for $\omega\tau >> 1$, phase is 90°, gain is $\pm$20dB/decade

3.3.3 Class 3: $((\frac{j\omega}{\omega_n})^2 + 2\zeta\frac{j\omega}{\omega_n} + 1)^{\pm 1}$

Behaviour:

- $\frac{\omega}{\omega_n} >> 1$, ± 40 dB/decade
- $\frac{\omega}{\omega_n} \approx 1$, resonant peak M_r
- Phase lead/lag 180°, 90° at ω_n

Three ranges:

- $\omega << \omega_n$: Gain = 0 dB, phase = 0
- $\omega >> \omega_n$: Gain = $20\log(\frac{\omega}{\omega_n})$, phase = $\pm 180^\circ$
- $\omega = \omega_n$: Gain = $\pm 20\log(2\zeta)$ dB, phase = ∞. $\log(2\zeta)$ will be 0 when $\zeta = 1/2$, negative when $\zeta < 1/2$ and positive when $\zeta > 1/2$

Important takeaways about sketching Bode diagrams:

- Asymptote:
$$\log|\frac{1}{1 + j\omega\tau_a}| \approx \log(\omega) - \log(\tau_a) \tag{40}$$

- Linear function with slope -1 on log-log scale: 20 db/decade
- Bode phase-gain relation: if asymptote gain has slope n ($20n$ dB/decade), phase has asymptote of $n \times 90^\circ$

3.4 Systems with RHP zero/pole

The gain rules still apply, but the phase change is larger than for systems with poles and zeros in the LHP: non-minimum phase systems.

3.5 Summary Bode Sketching

1. Write transfer function in Bode form $KG(j\omega) = K_0(j\omega)^n \frac{(j\omega\tau_1+1)(j\omega\tau_2+1)...}{(j\omega\tau_a+1)(j\omega\tau_b+1)...}$
2. Draw the asymptote through K_0 at $\omega = 1$ with slope n
3. Compute break points $\omega = 1/\tau_i$ and order them from small to large. Start at low frequencies and change the slope with $\pm n$ for n^{th}-order terms
4. At break points,
   - For first order terms, the magnitude is 3 dB above/below the asymptote
   - For second order terms, the magnitude is $\pm 20 \log(2\zeta)$ in dB
5. For the phase plot, start low-frequency phase $n \times 90°$
6. The asymptotic phase is $n \times 90°$
7. For first order terms: create additional points at 5 times and 1/5 times the break points
8. Break points far apart: the phase reaches the asymptotes. Break points close: they don't

(See lecture slides for 2 examples on sketching)

4 Nyquist Stability & Stability Margins

4.1 Frequency-Response Design Method

As discussed in the previous section, the main objective is to design controllers/compensators for a CL system using OL FRF of $D(j\omega)G(j\omega)$.

To do this, Bode diagrams and Nyquist diagrams are graphical methods that use FR info:

- Bode diagram: magnitude and phase on y-axis, frequency on x-axis. Time-domain specifications (next section)
- Nyquist diagram: imaginary part on y-axis, real part on x-axis (over the range of frequencies). But handles poles in the origin and the contour at infinity differently

4.2 The Nyquist stability criterion

$$Z = N + P \tag{41}$$

where Z is the number of RHP zeros of $1 + KG(s)$ (i.e. the closed-loop poles), N is the number of clockwise encirclements of the point -1. P is the number of RHP (unstable) poles of $KG(s)$.

Stability:

- CL system stable, given that OL system is stable: **-1 point should not be encircled**
- CL system stable, given that OL system has 1 unstable pole: **-1 encircled once in counter-clockwise direction**

The Nyquist stability criterion is for stability, but does not say anything about the robustness of the stability.

- Robustness means that we can handle uncertainty, in this case process variations
- Phase and gain margin. (section 6)

4.3 Using the Nyquist stability criterion

1. Plot $KG(j\omega)$ for $-\infty < \omega < \infty$. To do this:
   - First plot for $0 < \omega < \omega_h$ for a huge ω_h
   - Mirror the plot with respect to the real axis
2. Evaluate the number of clockwise encirclements of -1 and call that number N
   - Do this by counting the crossings of the curve with the line segment $(-\infty, -1)$
   - A crossing in the upward direction is +1 and a crossing in the downward direction is -1.
3. Let P denote the number of RHP poles of $KG(s)$
4. The number of closed-loop RHP roots/unstable poles is given by $Z = N + P$

To plot a simple transfer function you only need 4 points:

- $\omega = 0$
- $\omega = \infty$
- Intercepts with imaginary axis
- Intercepts with real axis

To get the intercepts, multiply the transfer function, multiply the transfer function by $\frac{\text{complex conj. of denominator}}{\text{complex conj. of denominator}}$ and separate into real and imaginary components. To get the imaginary intercepts, set the real component to 0 and plug the obtained value for ω into the imaginary parts, and do this vice versa for the real intercepts.

4.4 Stability margins

The gain margin (GM) is the factor by which the gain can be increased (or decreased) before instability results.
The gain margin is given by:

$$GM = \frac{1}{G(j\omega_{180}} \tag{42}$$

so where the phase of $G(j\omega)$ crosses $-180°$. The phase is $-180°$ when $G(j\omega)$ is real valued, a.k.a. you have to calculate when the real part of the Bode function is 0.

The phase margin is the amount by which the phase of the plant exceeds $-180°$. A positive PM is required for stability.
The phase margin can be calculated using

$$|KG(j\omega)| = 1 \text{ (or 0 dB)} \tag{43}$$

Solve this for ω and calculate the phase in this point. Then:

$$PM = 180 - \omega \tag{44}$$

5 Frequency Response Controller Design

Main objective is to design a controller/compensator for a CL system using the OL FRF of DG

Feedback is good for:

- Stabilize an unstable system
- Track a reference signal r (given an imprecise model of G)
- Attenuate unknown disturbances

Feedback leads to new challenges:

- Give unstable CL system even when plant is stable (stable in simulations, unstable in experiments, model uncertainty)
- Amplification of sensor noise n

From previous sections:

- Bode diagram: visualize and manipulate the frequency response
- Nyquist diagram: assess stability

Two step design philospohy, given a plant model $G(s)$:

1. Specify what $D(s)G(s)$ should look like
   - Mapping closed loop requirements to open loop FRF
2. Design $D(s) = KD_c(s)$, by adding integrators, selecting gain K and adding lead/lag compensators.

5.1 Performance specifications

5.1.1 Stability

Use Nyquist: $Z = N + P$ and get a decent GM/PM

- For $P = 0$ (open-loop stable), gain needs to be ¡1 where phase is $-180°$

Bode phase-gain relation:

- $\angle L(j\omega) \approx n \cdot 90°$ with n the slope of the Bode magnitude
- Slope $n = -1$ is needed at crossover frequency ω_c

5.1.2 Steady-state reference tracking errors

- Input type k: Reference signal $R(s) = \frac{1}{s^{k+1}}$
- System type n: Loop-gain in Bode form $L(s) = D(s)G(s) = K_0 s^{-n} \frac{(\tau_1 s+1)(\tau_2 s+1)\ldots}{(\tau_a s+1)(\tau_b s+1)\ldots}$
- Final value theorem $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+L(s)}R(s) \approx \lim_{s\to 0} \frac{1}{(1+K_0 s^{-n} s^k}$
- Reference tracking is ensured with sufficient n:
  - Slope of bode at $\omega \to 0$
  - Nr. of integrators at 0

5.1.3 Damping and overshoot

Assume:

$$L(s) \approx \frac{\omega_n^2}{s(s+2\zeta\omega_n} \tag{45}$$

with closed-loop transfer function:

$$T = \frac{L(s)}{1+L(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{46}$$

Relation between damping and overshoot:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \tag{47}$$

So graphically (see slides and book), the overshoot decays exponentially with increasing damping. For the relation between damping and phase margin we have the following:

$$PM \approx 100 \cdot \zeta \tag{48}$$

The actual equation is rather complex and can be seen in the slides, but graphically speaking, the damping increases slightly exponential (nearly linear) with increasing phase margin.

5.1.4 Crossover frequency

Again assume the equations of $L(s)$ and $T(s)$ from the previous subsection.

Relation crossover frequency ω_c and bandwidth ω_{BW}:

- $\omega_{BW} = \omega_c$ for $PM = 90°$ and $\omega_{BW} = 2\omega_c$ for $PM = 45°$
- $\omega_{BW} \in [0.5\omega_n, 2\omega_n]$

Time-domain specification:

- Natural frequency $\omega_n \in [0.5\omega_c, \omega_c]$
- Rise time $t_r \approx \frac{1.8}{\omega_n}$, settling time $t_s \approx \frac{4.6}{\zeta\omega_n}$

5.2 Compensators

Lead compensator:

- $D_c(s) = \frac{T_D s+1}{\alpha T_D s+1}, \alpha < 1$
- (+) Increases PM (phase stabilization)
- (-) Increases gain at high frequencies

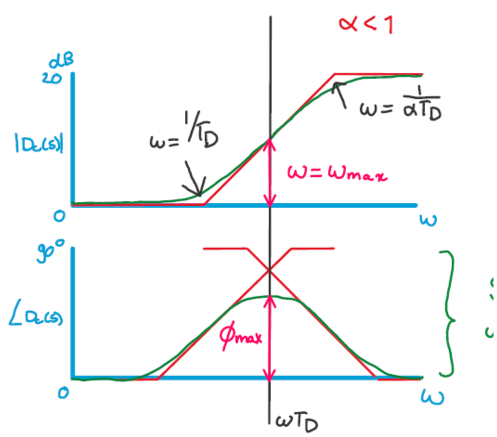
Lag compensator:

- $D_c(s) = \alpha\frac{T_I s+1}{\alpha T_I s+1}, \alpha > 1$
- (+) Decreases gain (gain stabilization)
- (-) Adds phase lag

Comparison to PID controllers:

- PD = lead compensator with $\alpha \to 0$ (Never used in practice)
- PI = lag compensator with $\alpha \to \infty$ (Reduces steady-state error)

5.2.1 Lead filter design approach

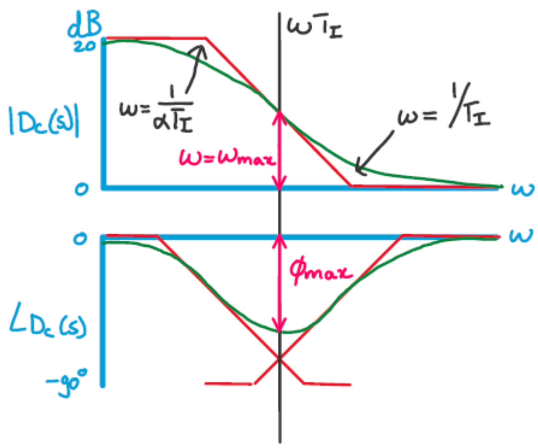


1. Select K to satisfy steady-state error or bandwidth requirements:
   - Error requirements: Choose K and add integrators (if needed) to satisfy e_{ss}
   - Bandwidth requirements: Choose K so crossover frequency $\omega_c = 0.5\omega_{BW}$
2. Evaluate PM for chosen K
3. Determine phase lead needed (and add 10° for safety): ϕ_{max}

4. Determine:
$$\alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}} \Leftrightarrow \sin\phi_{max} = \frac{1-\alpha}{1+\alpha} \tag{49}$$

5. Select $T_d = \frac{1}{\omega_c\sqrt{\alpha}}$ to ensure maximum phase lead at crossover frequency
6. Draw the open FRF and check the PM
7. Iterate until all specifications are met

5.2.2 Lag filter design approach



1. Add integrators (if needed) to get desired steady-state errors
2. Select K that will meet PM requirements without compensation
3. Evaluate low frequency gain (Bode diagram)
4. Choose α as the additional low-frequency gain needed to achieve desired e_{ss}
5. Choose $T_I = \frac{1}{\omega_z}$ so the zero of the compensator is a decade lower than ω_c
6. Pole of the compensator is then at $\frac{1}{\alpha T_I}$
7. Iterate until all specifications are met

6 Fundamental Limitations in Control Design

6.1 Closed-Loop Specifications

Typical design considerations (for stable minimum-phase systems):

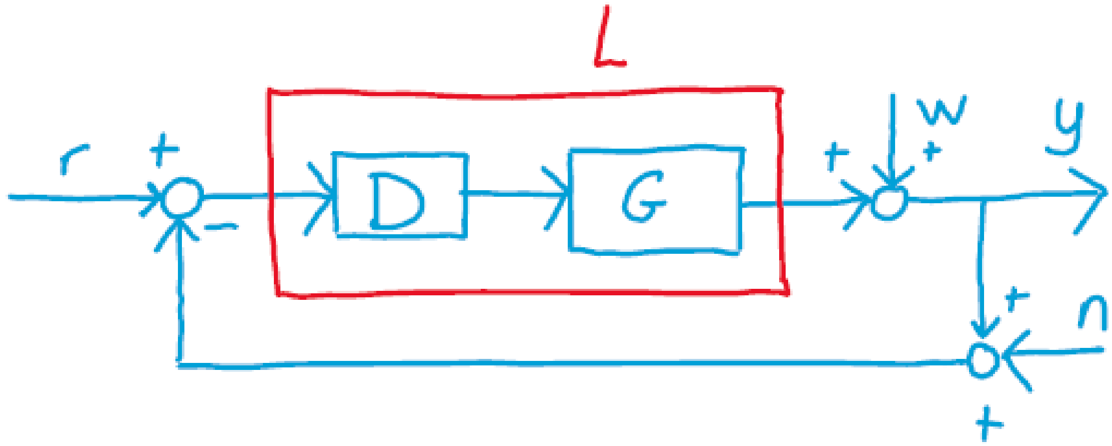
- High gain at low frequencies for disturbance attenuation and reference tracking
- Low gain at high frequencies against sensor noise and model uncertainty
- Slope of -1 around ω_c ($-90°$ according to Bode's phase/gain relation)

Take a simple system: Then we obtain the following function for y:

$$y = \frac{1}{1+L}w + \frac{L}{1+L}(r-n) \tag{50}$$

The first term is the sensitivity: $S = \frac{1}{1+L}$ and the second term is the complementary sensitivity:

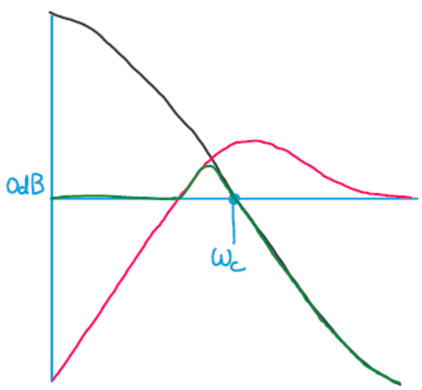
$$T = \frac{L}{1+L} \tag{51}$$



It is not possible to make both these functions as low as possible (which is desirable) and thus the following equation poses a fundamental limitation:

$$S + T = 1 \tag{52}$$

In other words, reducing the effects of disturbances **AND** measurement noise is not possible!



As can be seen from the figure (black is L, red is S and green is T):

- $|L| >> 1$ for $\omega \to 0$
- $|L| \to 0$ for $\omega \to \infty$
- $S \approx \frac{1}{L}$ for $\omega \to 0$
- $S \approx 1$ for $\omega \to \infty$
- $T \approx 1$ for $\omega \to 0$
- $T \approx L$ for $\omega \to \infty$

6.2 Bode Sensitivity Integral

The peak of the sensitivity function is the reciprocal of the minimum distance to -1 in the Nyquist diagram (robustness):

$$\min |L(j\omega) + 1| = \max |\frac{1}{L(j\omega) + 1}| \tag{53}$$

Waterbed effect:

- $S > 1$ is good for disturbance suppression
- $S < 1$ is bad for disturbance suppression

6.2.1 Cauchy's Integral Formula

For stable minimum-phase systems (no RHP poles or zeros) and for systems with two more poles than zeros, the sensitivity integral is given by:

$$\int_0^\infty \ln|S(j\omega)|d\omega = 0 \tag{54}$$

For minimum-phase systems (with two more poles than zeros) with unstable poles p_i:

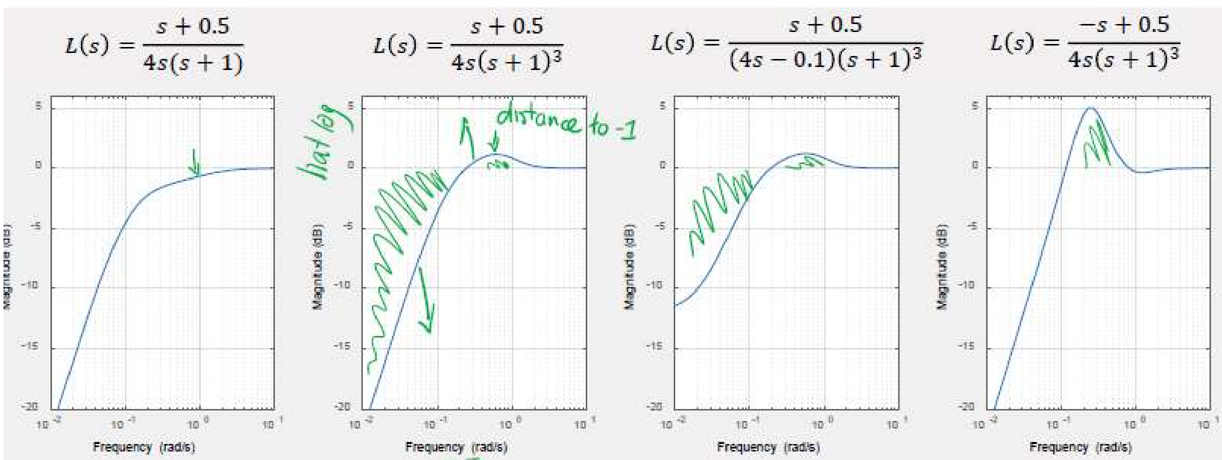
$$\int_0^\infty \ln|S(j\omega)|d\omega = \pi \sum_i Re\{p_i\} \tag{55}$$

For non-minimum-phase systems (with two more poles than zeros) with RHP zero at $z_0 = \sigma_0 + j\omega_0$:

$$\int_{-\infty}^\infty \ln|S(j\omega)| \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega = \pi \sum_i \ln|\frac{p_i + z_0}{p_i - z_0}| \tag{56}$$

For stable minimum-phase systems with less than two more poles than zeros:

$$\int_0^\infty \ln|S(j\omega)|d\omega = \frac{\pi}{2} \lim_{s\to\infty} sL(s) \tag{57}$$



The peak of the sensitivity function can be used to measure the robustness, where the GM and PM are important.

6.3 RHP Poles & Zeros

Performance limitations, RHP poles & zeros

- Root Locus method → Shows how feedback changes closed-loop behaviour
- $S + T = 1$ → Asymptotic behaviour of sensitivity and complementary sensitivity
- Bode sensitivity integral & waterbed effect → $\oint_\Gamma \ln(|S(s)|)ds = 0$

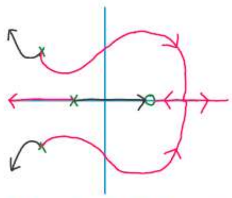
7 State-Space Models

7.1 Introduction

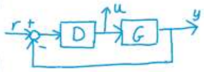
Example from circuit theory:

- Capacitor: $C\frac{dU_C}{dt} = I_C$

Performance Limitations of RHP zeros



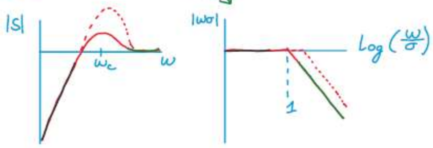
Eventually all closed-loop poles move to open-loop zeros



$$u = \underbrace{\frac{D}{1+DG}}_{\text{Control sensitivity}} r \approx \begin{cases} \frac{1}{G} r & \text{For } \omega_c > \omega \;\; X \\ D r & \text{For } \omega_c < \omega \;\; V \end{cases}$$

We don't want this approximation to hold for $\omega \approx$ RHP zero because $\frac{1}{G}$ is unstable

$$\int_{-\infty}^{\infty} \ln(|S(j\omega)|) \frac{\sigma}{\sigma^2+\omega^2} d\omega \qquad \text{RHP zero at } s=\sigma$$

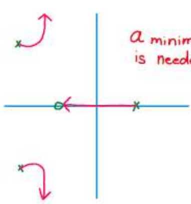


To minimize impact of the unavoidable bump

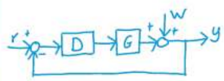
$\frac{\omega_c}{\sigma} > 1 \iff \omega_c > \sigma$

$\Rightarrow$ The bandwidth of a closed-loop system is constrained from above by the open-loop RHP zero

Performance limitations of RHP poles



a minimal gain is needed

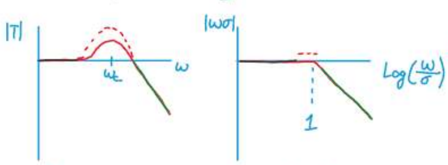


$$y = \underbrace{\frac{G}{1+DG}}_{\text{Process sensitivity}} w \approx \begin{cases} G w & \text{For } \omega > \omega_c \;\; X \\ \frac{1}{D} w & \text{For } \omega < \omega_c \;\; V \end{cases}$$

unstable, we don't want this approximation to hold for $\omega =$ RHP pole so high bandwidth is needed

$T = \frac{L}{1+L} = \frac{1}{1+1/L}$ Sensitivity of the inverse loop gain

$$\int_{-\infty}^{\infty} \ln(|T(j\omega)|) \frac{\sigma}{\sigma^2+\omega^2} d\omega = \text{constant} \qquad \text{For RHP pole at } s=\sigma$$



$\Rightarrow$ The bandwidth of a closed-loop system is constrained by an open-loop RHP pole

To minimize the effect of the bump $\omega_c > \sigma$

Summary

- Non minimum phase systems have a maximum bandwidth
- Unstable systems have a minimum bandwidth

- Inductor: $L\frac{dI_L}{dt} = U_L$
- Resistor: $U_R = RI_R$
- Interconnected using Kirchhoff's laws

State-space model with state $x = \begin{bmatrix} U_C \\ I_L \end{bmatrix}$
States represent physical/internal variables.

7.2 Deriving State-Space Models

$y(s) = G(s)u(s) \rightarrow \dot{x}(t) = Ax(t) + Bu(t)$ & $y(t) = Cx(t) + Du(t)$ So the Laplace transform of impulse response, which is a differential equation, gets rewritten to a set of first order differential equations.

How to represent a differential equation as a state-space model?

1. Simulation diagram
2. Modal decomposition
3. Control canonical form

7.2.1 Simulation diagram appproach

1. Group highest derivatives (initially)
2. Integrate (not differentiate!)
3. Complete block diagram / Construct the equality
4. Output of every integrator is a state

$\dddot{y} = -7.5\ddot{y} - 13\dot{y} - 6.5y + 7u$ with $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix}$

Results in:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} -7.5 & -13 & -6.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} u$$

And

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

7.2.2 Using model decomposition

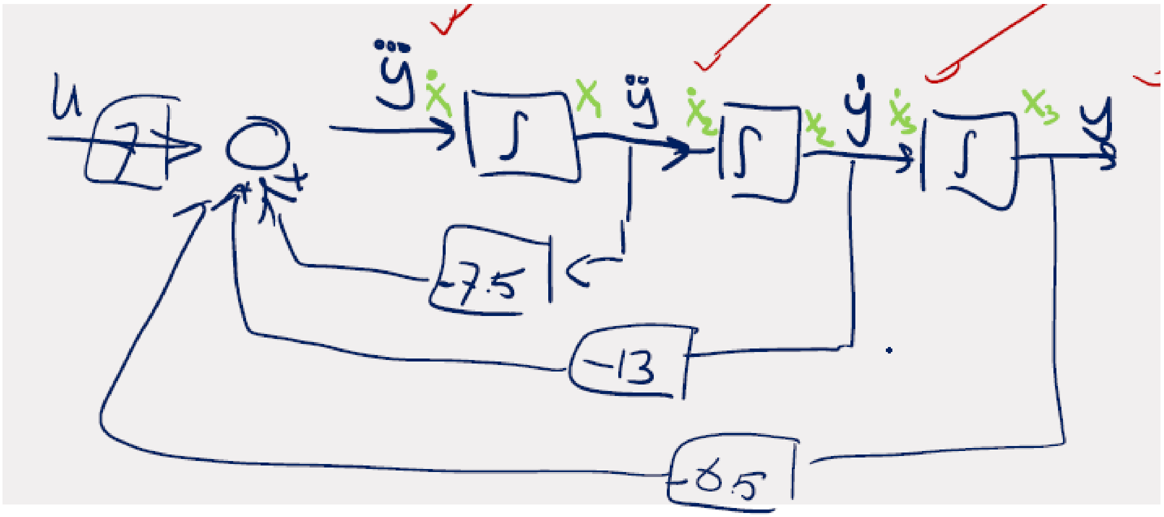
$G(s) = \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + ... + c_0$

$$A = \begin{bmatrix} -p_1 & 0 & 0 \\ 0 & -p_2 & 0 \\ 0 & 1 & -p_3 \end{bmatrix} B = \begin{bmatrix} c_1 \\ c2 \\ c_3 \end{bmatrix} C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} D = c_0$$

7.2.3 Control canonical form

$G(s) = \frac{b_1 s^{n-1} + ... + b_n}{s^n + a_1 s^{n-1} + ... + a_n} + b_0$

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_n \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} C = \begin{bmatrix} b_1 & b_2 & b_n \end{bmatrix} D = b_0$$



7.3 State-Space Analysis

- Transfer function of state-space models:

$$G(s) = C(sI - A)^{-1}B + D \tag{58}$$

- Two state-space models with matrices (A, B, C, D) and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ are equivalent if they have the same transfer function. This is true when there exists a matrix T that satisfies:

$$\bar{A} = T^{-1}AT,\ \bar{B} = T^{-1}B,\ \bar{C} = CT,\ \bar{D} = D \tag{59}$$

- Matrix T can be constructed using the so-called controllability matrix:
  1. Compute $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = T \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \dots & \bar{A}^{n-1}\bar{B} \end{bmatrix}$
  2. Check $\bar{C} = CT$
- Similarity transform that brings state-space model into modal form:
  - Matrix $\bar{A} = T^{-1}AT$ should be diagonal
  - $T = \begin{bmatrix} t_1 & \dots & t_n \end{bmatrix}$ is the matrix of eigenvectors
- Similarity transform that brings state-space model into control canonical form:
  - Compute last row of transformation matrix $t_n = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}^{-1}$
  - The inverse of the transformation matrix is:

$$T^{-1} = \begin{bmatrix} t_n A^{n-1} \\ \dots \\ t_n A \\ t_n \end{bmatrix} \tag{60}$$

Poles:

- Values of s for which $\det(sI - A) = 0$, eigenvalues of matrix A

Zeros:

- Values of s for which $\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0$

7.4 Linearisation

- Linearisation of nonlinear models:

$$\dot{x} = f(x, u) \tag{61}$$

- Write $x = x_0 + x_\delta$ and $u = u_0 + u_\delta$ and define equilibrium point $0 = f(x_0, u_0)$
- Use Taylor series approximation to obtain

$$\dot{x}_\delta = Ax_\delta + Bu_\delta \tag{62}$$

with $A = \frac{\delta f(x,u)}{\delta x}|_{x_0,u_0}$ and $B = \frac{\delta f(x,u)}{\delta u}|_{x_0,u_0}$

8 State-Space Control Design

8.1 Recap eigenvalues / eigenvectors / determinants

For the linear operation Ax, the vector x is called the eigenvector if the multiplication satisfies

$$Ax = \lambda x \tag{63}$$

For some scalar λ, which is the eigenvalue associated with eigenvector x.

- Computing eigenvalues: $(A - \lambda I)x = 0$ for some nontrivial x.
- Such a nontrivial vector exists if $A - \lambda I$ is singular $\rightarrow$ solve $\det(A - \lambda I) = 0$
- The eigenvector x of eigenvalue λ is in the nullspace of $A - \lambda I$.

8.2 State-Space Control Design

- Plant ($D = 0$ in most cases)
  - $\dot{x} = Ax + Bu$
  - $y = Cx$
- State Feedback: $u = -K\hat{x} + Nr$
- State observer
  - $\dot{\hat{x}}(A - BK)\hat{x} + L(y - \hat{y} + Mr$
  - $\hat{y} = C\hat{x}$
- Integral action ($N = 0$)
  - $\dot{x}_I = y - r$
  - $u = -K_I x_I - K\hat{x}$

8.3 Controllability / Observability

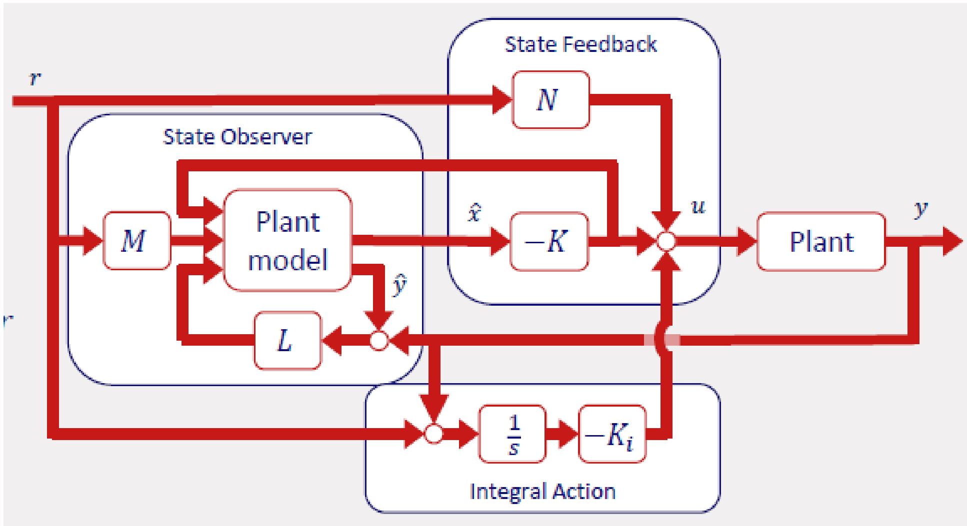
Controllability/observability:

- Ability to independently control and observe each state: actuator/sensor location
- Controllability/observability matrices (where n is the nr. of states / order of the system:

$$\Gamma = \begin{bmatrix} B & AB & ...A^{n-1}B \end{bmatrix} \tag{64}$$

$$\Omega = \begin{bmatrix} C \\ CA \\ ... \\ CA^{n-1} \end{bmatrix} \tag{65}$$

- Controllable if rank $\Gamma = n$ and observable if rank $\Omega = n$



If the plant/system is uncontrollable/unobservable:

- Closed-loop poles cannot be placed freely
- Some "dynamics" (poles) are missing in the transfer function

8.4 State Feedback

- Design of state feedback controllers $u = -Kx + Nr$ to achieve $\dot{x} = (A - BK)x + BNr$
- Matrix K to achieve desired closed-loop poles
  - Convert plant to control canonical form with transformation matrix T (previous section)
  - Specify desired characteristic polynomial $(s - \gamma_1)...(s - \gamma_n) = s^n + \alpha_1 s^{n-1} + ... + \alpha_n = 0$
  - Compute $\hat{K} = \begin{bmatrix} \alpha_1 - \hat{\alpha_1} & ... & \alpha_n - \hat{\alpha_n} \end{bmatrix}$
  - Let $K = \hat{K}T^{-1}$
  - MATLAB "place" command
- Matrix N to achieve steady-state reference tracking:

$$N = N_u + KN_x \quad \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{66}$$

8.5 Observer Design

- Controller equations
  - $\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + Mr$
  - $u = -K\hat{x} + Nr$
- Observation error ($\tilde{x} = x - \hat{x}$):
  - $\dot{\tilde{x}}(A - LC)\tilde{x} + (BN - M)r$
- Matrix L to achieve desired transient response of observation error
  - Duality of control and estimation: placing closed-loop poles of $\dot{w} = (A^T - C^T L^T)w$
  - Separation property: stabilizing state feedback + stable estimation error = stable closed-loop system

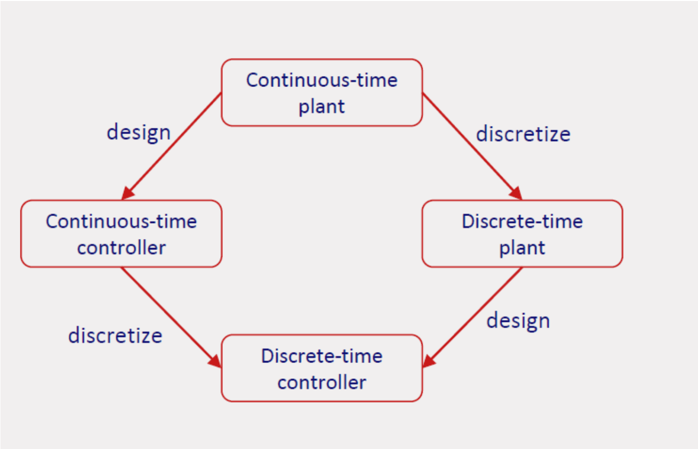
- Matrix M to achieve desired reference tracking properties:
  - Choose $M = BN$ to make estimation error independent of reference
  - Choose $N = 0$ and $M = -L$ to get "classical control scheme"

8.6 Reference Tracking

- With state feedback $u = -Kx + Nr$, reference not in feedback path and no integral action
- Tracking for inexact plant model (robustness):
  - Augment the plant model with integrator
  - $\dot{x} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix} x + \begin{bmatrix} 0 \\ B \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$
  - First element of K matrix is integral action
  - Base observer on plant without integrator
- Space-space controllers are more versatile (certain choices lead to "classical" controller)
- We discussed 3 different controller structures for reference tracking
  - $N = N_u + KN_x$ and $M = BN$ leads to controller:
    * $\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + BNr$
    * $u = -K\hat{x} + Nr$
  - $N = 0$ and $M = -L$ leads to controller:
    * $\dot{\hat{x}} = (A - BK - LC)\hat{x} + L(r - y)$
    * $u = -K\hat{x}$
  - $N = 0$, $M = 0$ and add integral action leads to controller:
    * $\dot{x}_I = y - r$
    * $\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly$
    * $u = -K_I x_I - K\hat{x}$

9 Digital Control

Control algorithms are often implemented on digital computers or microcontrollers.



9.1 Artifacts introduced in Digital Control

- Sensor quantization: perceived as noise
- Zero-Order Hold creates a delay of $T/2$
  - Transfer function of this delay: $H(s) = e^{-sT/2}$
    - Linear phase shift $\angle H(j\omega) = -\frac{180}{\pi}\frac{T}{2}\omega$
  - Approximation:

$$H(s) = \frac{1}{e^{sT/2}} \approx \frac{2/T}{s + 2/T} \tag{67}$$

$$H(s) = \frac{e^{-sT/4}}{e^{sT/4}} \approx \frac{4/T - s}{4/T + s} \tag{68}$$

  - Rule of thumb: choose sample frequency $\omega_s = \frac{2\pi}{T} > 40\omega_c > 20\omega_{BW}$

9.2 The Z-domain

Two things to remember:

- Stability: Unit circle instead of Left-Hand Plane
- z is a forward time shift, s is a derivative in Laplace

9.3 Three Different Mappings

- **Matched pole-zero:** just remember that $z = e^{st}$, because this one is not used often
- **ZOH:** used for discretizing plant to do discrete-time controller design
  - Solve $G(z) = \frac{z-1}{z}\mathrm{Z}(\frac{G(s)}{s})$
  - Compute $\frac{G(s)}{s}$, then do partial fraction expansion and consult Table 8.1 in the book
- **Tustin:** used for discretizing controller
  - Replace $s \to \frac{2}{T}(\frac{z-1}{z+1})$
  - Lead/lag filter: $D(s) = \frac{\tau_1 s+1}{\tau_a s+1}$ becomes $D(z) = \frac{(\tau_1+T/2)z+T/2-\tau_1}{(\tau_a+T/2)z+T/2-\tau_a}$

9.4 Two Different Design Approaches

- Design CT controller, then discretize the controller
  - Take sampling delay into account (increase PM)
  - Take $\omega_s \approx 20\omega_{BW}$
  - Discretize controller using Tustin/Bilinear transform
- Discretize CT plant to DT plant, then design DT controller
  - Example: $G(s) = \frac{1}{s+a'}$, let $\frac{G(s)}{s} = \frac{1}{s(s+a)} = \frac{1}{a}(\frac{1}{s} - \frac{1}{s+a})$ and $G(z) = \frac{z-1}{z}\frac{1}{a}(\frac{z}{z-1} - \frac{z}{z-e^{aT}}) = \frac{(1-e^{-aT})/a}{z-e^{-aT}}$
  - Sampling delay is considered, DT is not stable for all K, because of sampling (In CT: RHP zero)
- Design controllers using standard techniques (Root locus, Nyquist, loop shaping)

9.5 Compensation in Discrete-Time

- Proportional compensation
  - CT: $D_c(s) = K_p$
  - DT: $D_c(z) = K_p$
  - Apply ZOH rules
- Derivative compensation
  - CT: $D_c(s) = K_D s$
  - DT: $D_c(z) = K_D \frac{z-1}{z}$
  - Apply ZOH rules
- Integral compensation
  - CT: $D_c(s) = \frac{K_I}{s}$
  - DT: $D_c(z) = K_I \frac{z}{z-1}$
  - Apply ZOH rules
- Lead/lag compensation
  - CT: $D_c(s) = \frac{\tau_1 s+1}{\tau_a s+1}$
  - DT: $D_c(z) = \kappa \frac{z-\alpha}{z-\beta}$
  - The relation between $(\tau_1, \tau_b) \to (\alpha, \beta, \kappa)$ is not straightforward, but also not important