Communication Theory (5ETB0)
Module 5.2

Alex Alvarado
a.alvarado@tue.nl

Information and Communication Theory Lab
Signal Processing Systems Group
Department of Electrical Engineering
Eindhoven University of Technology, The Netherlands

www.tue.nl/ictlab/

Module 5.2

Presentation Outline

Gram-Schmidt Orthogonalization

Gram-Schmidt Theorem

For an arbitrary signal set, i.e., a set of waveforms $\{s_1(t), s_2(t), \ldots, s_{|\mathcal{M}|}(t)\}$ we can construct a set of $N \leq |\mathcal{M}|$ **building-block waveforms** $\{\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)\}$ and find coefficients s_{mi} such that for $m = 1, 2, \ldots, |\mathcal{M}|$ the signals can be synthesized as $s_m(t) = \sum_{i=1}^{N} s_{mi}\varphi_i(t)$.

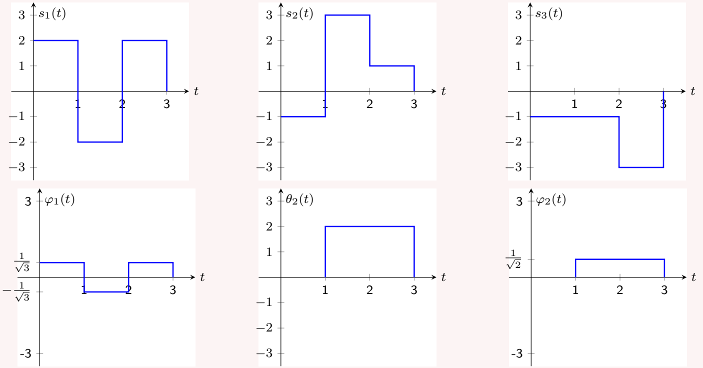
Gram-Schmidt Procedure

1. Compute $\varphi_1(t)$ via $\varphi_1(t) = s_1(t)/\sqrt{E_1}$, then $s_{11} = \sqrt{E_1}$
2. Compute $\theta_2(t) = s_2(t) - s_{21}\varphi_1(t)$, where $s_{21} = \int_{-\infty}^{\infty} s_2(t)\varphi_1(t)dt$
3. Compute $\varphi_2(t) = \theta_2(t)/\sqrt{E_{\theta_2}}$ and $s_{22} = \int_{-\infty}^{\infty} s_2(t)\varphi_2(t)dt$
4. Compute the function $\theta_m(t) \stackrel{\Delta}{=} s_m(t) - \sum_{i=1}^{m-1} s_{mi}\varphi_i(t)$, where the coefficients s_{mi} with $i = 1, \ldots, m-1$ are $s_{mi} = \int_{-\infty}^{\infty} s_m(t)\varphi_i(t)dt$
5. Two cases:
   - If $\theta_m(t) \equiv 0$ then we stop, or
   - If $\theta_m(t) \not\equiv 0$ then $\varphi_m(t) = \theta_m(t)/\sqrt{E_{\theta_m}}$ and go back to step 4

Gram-Schmidt Procedure Example

Example 5.3

Three waveforms $s_m(t)$ with $m = 1, 2, 3$



$s_{11} = 2\sqrt{3}$

$(s_{21}, s_{22}) = (-\sqrt{3}, 2\sqrt{2})$

$\theta_3(t) = 0$

$(s_{31}, s_{32}) = (-\sqrt{3}, -2\sqrt{2})$

Module 5.2

Presentation Outline

Geometric Interpretation of Signals (1/2)

Main Points

- We have seen that $s_m(t) = \sum_{i=1}^{N} s_{mi}\varphi_i(t)$.
- Thus, to each waveform $s_m(t)$, there corresponds a *vector* with N coefficients $\underline{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$.
- *Signal space* is the N-dimensional space where the set of waveforms $\{s_1(t), s_2(t), \ldots, s_{|\mathcal{M}|}(t)\}$ is represented as a set of vectors $\{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{|\mathcal{M}|}\}$.
- *Signal structure/signal constellation* is the set of vectors $\{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{|\mathcal{M}|}\}$

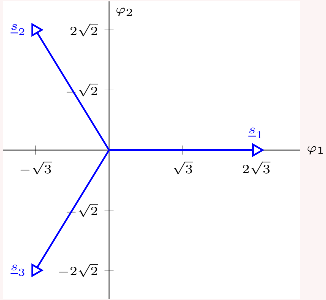
Geomnetric Interpretation of Signals (2/2)

Example 5.3

$$\underline{s}_1 = (s_{11}, s_{12}) = (2\sqrt{3}, 0), \quad (1)$$

$$\underline{s}_2 = (s_{21}, s_{22}) = (-\sqrt{3}, 2\sqrt{2}), \text{ and} \quad (2)$$

$$\underline{s}_3 = (s_{31}, s_{32}) = (-\sqrt{3}, -2\sqrt{2}). \quad (3)$$



Food for Thought

Example 5.3

- "One should also note that a given set of signals can be expanded in many different orthogonal sets of building-block waveforms, possibly with a larger dimension. What remains constant is the geometrical configuration of the vector representations of the signals."
- "It should be noted that, when using the Gram-Schmidt procedure, any ordering of the signals other than $s_1(t), s_2(t), \ldots, s_{|\mathcal{M}|}(t)$ will yield a basis, i.e., a set of building-block waveforms, of the same dimensionality, however in general with different building-block waveforms."

Module 5.2

Presentation Outline

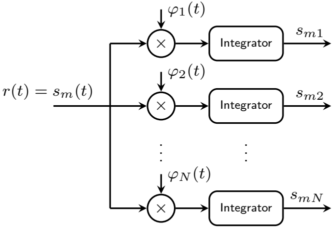
From Signals to Vectors: Without Noise

Determine signal vector $\underline{s}_m$ from $r(t) = s_m(t)$

$$\int_{-\infty}^{\infty} s_m(t)\varphi_i(t)dt = \int_{-\infty}^{\infty}\left(\sum_{j=1}^{N} s_{mj}\varphi_j(t)\right)\varphi_i(t)dt$$
$$= \sum_{j=1}^{N} s_{mj}\int_{-\infty}^{\infty}\varphi_j(t)\varphi_i(t)dt = \sum_{j=1}^{N} s_{mj}\delta_{ji} = s_{mi}$$

If we carry this out for all $i = 1, \ldots, N$, we find all coefficients s_{mi} of the vector $\underline{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$.

"Note that it is definitely easier to check an N-dimensional vector than a waveform from $-\infty < t < \infty$..."

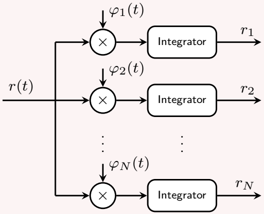


Recovery of Signal Vectors: With Noise

Receiving noisy version $\underline{s}_m$

Receiver gets $r(t) = s_m(t) + n_w(t)$ and computes r_i as

$$r_i \triangleq \int_{-\infty}^{\infty} r(t)\varphi_i(t)dt \text{ for } i = 1, 2, \ldots, N.$$



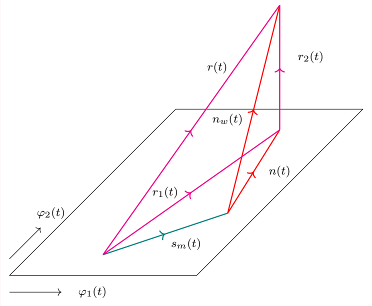
N-dimensional r-vector is:
$\underline{r}_1 = (r_1, r_2, \ldots, r_N)$

Irrelevant Data

Receiving noisy version $\underline{s}_m$

$r(t) = s_m(t) + n_w(t)$, and thus,

$$
\begin{aligned}
r_i &= \int_{-\infty}^{\infty} r(t)\varphi_i(t)dt \\
&= \int_{-\infty}^{\infty} s_m(t)\varphi_i(t)dt \\
&\quad + \int_{-\infty}^{\infty} n_w(t)\varphi_i(t)dt \\
&= s_{mi} + n_i
\end{aligned}
$$



Hence also

$$\underline{r}_1 = \underline{s}_m + \underline{n}$$

with $\underline{r}_1 = (r_1, r_2, \ldots, r_N)$, $\underline{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$, and $\underline{n} = (n_1, n_2, \ldots, n_N)$.

Joint Density of Relevant Noise

Irrelevant Noise

Only noise on the signal space is relevant for detection. Noise in all other dimensions can be safely discarded. Proof follows from (i) theorem of reversibility and (ii) theorem of irrelevance.

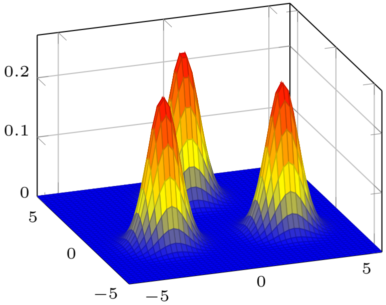
Joint PDF of Relevant Noise

Therefore the joint PDF of the relevant noise vector $\underline{n}$ is

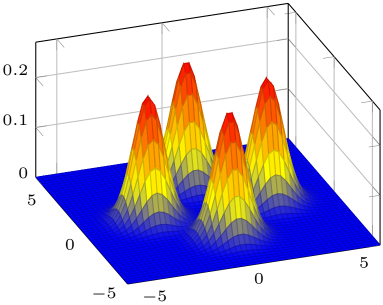
$$p_{\underline{N}}(\underline{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\underline{n}\|^2}{N_0}\right),$$

hence the noise is **spherically symmetric** and depends on the magnitude but not on the direction of $\underline{n}$. Noise projected on each direction has variance $\frac{N_0}{2}$.

PDF of 2D ($N=2$) Signals with $|\mathcal{M}|=3,4$



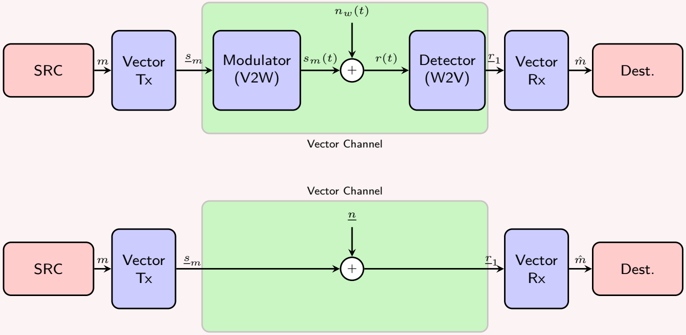
Conditional PDFs for Example 5.3.



Conditional PDFs for Example 5.4.

Relation Between Waveform and Vector Channel

Waveform channel $\Longleftrightarrow$ Vector channel



Summary Module 5.2

Take Home Messages

- Waveform channels are difficult to deal with
- The waveform channel we considered (AWGN) can be converted into a vector DICO channel

Communication Theory (5ETB0)
Module 5.2

Alex Alvarado
a.alvarado@tue.nl

Information and Communication Theory Lab
Signal Processing Systems Group
Department of Electrical Engineering
Eindhoven University of Technology, The Netherlands
www.tue.nl/ictlab/