Communication Theory (5ETB0)
Module 2.1

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Module 2.1

Presentation Outline

Part I Signals and Systems Review
Part II Notation Convention
Part III Discrete Random Variables
Part IV Continuous Random Variables

Signals and Systems Review: Fourier Transform (Appendix A)

The Fourier Transform

The Fourier Transform pair is defined as

$$\mathcal{F}\{x(t)\} = X(f) \triangleq \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\jmath 2\pi f t}\, dt \Longleftrightarrow x(t) = \int_{-\infty}^{\infty} X(f)\mathrm{e}^{\jmath 2\pi f t}\, df,$$

or alternatively,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\jmath\omega t}\, dt \Longleftrightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\jmath\omega t}\, d\omega.$$

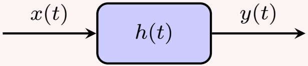
Fourier Transform Properties

- Linearity, i.e., $\mathcal{F}\{ax_1(t) + bx_2(t)\} = aX_1(f) + bX_2(f)$
- Transform of a convolution: $\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(f) \cdot X_2(f)$
- If $x(t) \in \mathbb{R}$, its Fourier transform satisfies $X(f) = X^*(-f)$
- If $x(t) \in \mathbb{R}$ and even (i.e., symmetric respect to zero: $x(-t) = x(t)$) its Fourier transform is real ($X(f) \in \mathbb{R}$) and even ($X(f) = X(-f)$)

Signals and Systems Review: LTI System (Appendix C)

An LTI system

- The impulse response of the LTI system is given by $h(t)$
- In the time domain, $y(t) = x(t) * h(t)$
- In the frequency domain, $Y(f) = X(f)H(f)$



An LTI system. The output $y(t)$ is the convolution between the input $x(t)$ and the impulse response $h(t)$.

Signals and Systems Review: Parseval Relation (Appendix A)

Parseval's Theorem

Parseval's Relation states that for two real signals $g_1(t)$ and $g_2(t)$

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\,df,$$

which for the particular case of $g_1(t) = g_2(t) = g(t)$ translates into

$$\int_{-\infty}^{\infty} g^2(t)\,dt = \int_{-\infty}^{\infty} |G(f)|^2\,df,$$

where

$$E_g \triangleq \int_{-\infty}^{\infty} g^2(t)\,dt$$

is the energy of the signal.

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Mathematical Notation (1/3)

- *Sets* are denoted by calligraphic letters: $\mathcal{M}$

- The *cardinality* of a set is denoted by $|\mathcal{M}|$

- A *definition* is denoted by $\triangleq$

Mathematical Notation (2/3)

- A set definition is therefore: $\mathcal{M} \triangleq \{1, 2, \cdots, |\mathcal{M}|\}$
- Exceptions include the sets of real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$
- Cartesian product of sets: $\mathcal{M}^2 = \mathcal{M} \times \mathcal{M}$
- Notation $\mathcal{M} \subset \mathbb{R}$ means $\mathcal{M}$ is a subset of $\mathbb{R}$

Mathematical Notation (3/3)

- *Estimated* variables are denoted using a hat: $\hat{m}$
- *Scalars* are denoted by small letters: x
- *Vectors* are denoted using underlined letters: $\underline{x}$
- The function $\min_{m\in\mathcal{M}}\{\cdot\}$ is not the same as $\operatorname{argmin}_{m\in\mathcal{M}}\{\cdot\}$

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Discrete Random Variables (1/4)

- Random Variables (RVs) are denoted by capital letters and their realizations by small letters: X is not the same as x

- $\Pr\{X = x\} \geq 0$ denotes the probability that the r.v. X takes the value x

- The support of the r.v. X will be denoted by a set: $\mathcal{X} = \{x_1, x_2, \ldots, x_{|\mathcal{X}|}\}$

Discrete Random Variables (2/4)

- The probability mass function (PMF) of a random variable is denoted by $\Pr\{X = x\}$ for all $x \in \mathcal{X}$.

- PMFs satisfy

$$\sum_{x \in \mathcal{X}} \Pr\{X = x\} = 1$$

Discrete Random Variables (3/4)

- Joint PMFs are denoted by $\Pr\{X = x, Y = y\}$

- Conditional PMFs are denoted by $\Pr\{Y = y|X = x\}$.

- Conditional PMFs satisfy

$$\begin{aligned}\Pr\{Y = y, X = x\} &= \Pr\{Y = y|X = x\} \cdot \Pr\{X = x\}\\ &= \Pr\{X = x|Y = y\} \cdot \Pr\{Y = y\}\end{aligned}$$

Discrete Random Variables (4/4)

- Bayes' Rule:

$$\Pr\{Y=y|X=x\}=\frac{\Pr\{X=x|Y=y\}\cdot\Pr\{Y=y\}}{\Pr\{X=x\}}$$

- Law of total probability:

$$\begin{aligned}\Pr\{Y=y\}&=\sum_{x\in\mathcal{X}}\Pr\{X=x,Y=y\}\\&=\sum_{x\in\mathcal{X}}\Pr\{Y=y|X=x\}\cdot\Pr\{X=x\}\end{aligned}$$

- Independence

$$\Pr\{Y=y|X=x\}=\Pr\{Y=y\}$$

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Continuous Random Variables (1/2)

- Just like for discrete RVs, continuous RVs are denoted by capital letters and their realizations by small letters: R and r

- The support of the random variable is denoted by a calligraphic letter $\mathcal{R}$ ($\mathbb{R}$ for real numbers)

- The probability density function (PDF) of a random variable is $p_R(r)$ for all $r \in \mathcal{R}$

- $p_R(r)dr$ denotes the probability that the r.v. R takes a value between r and $r + dr$

Continuous Random Variables (2/2)

- PDFs satisfy

$$\int_{r \in \mathcal{R}} p_R(r) dr = 1$$

- Joint PDFs are denoted by $p_{R,S}(r,s)$ and also satisfy

$$\int_{r \in \mathcal{R}} \int_{s \in \mathcal{S}} p_{R,S}(r,s) ds dr = 1$$

- Conditional PDFs are $p_R(r|X=x)$. Law of total probability is

$$p_R(r) = \sum_{x \in \mathcal{X}} \Pr\{X=x\} \cdot p_R(r|X=x)$$

Summary Module 2.1

Take Home Messages

- Often used in the course:
  - Fourier Transforms
  - Convolutions
  - LTI Systems
- Notation Convention
- Discrete and Continuous Random Variables

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