Communication Theory (5ETB0)
Module 6.1

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Module 6.1

Presentation Outline

Vector Representation of Signals and Operations

Functions $f(t)$ and $g(t)$ and their Vector Representations

Waveforms $f(t)$ and $g(t)$ and orthonormal base $\{\varphi_i(t), i = 1, 2, \cdots, N\}$:

$$f(t) = \sum_{i=1}^{N} f_i \varphi_i(t), \qquad g(t) = \sum_{i=1}^{N} g_i \varphi_i(t)$$

sentations $f(t)$ and $g(t)$:

$$\underline{f} = (f_1, f_2, \cdots, f_N), \qquad \underline{g} = (g_1, g_2, \cdots, g_N)$$

Vector Operations

$$(\underline{f} \cdot \underline{g}) \triangleq \sum_{i=1}^{N} f_i g_i, \quad \|\underline{f}\|^2 \triangleq (\underline{f} \cdot \underline{f}) = \sum_{i=1}^{N} f_i^2$$

$$\|\underline{f} - \underline{g}\|^2 = \|\underline{f}\|^2 + \|\underline{g}\|^2 - 2(\underline{f} \cdot \underline{g})$$

Vector Representation of Signals: Correlation ⇔ Dot product

Correlation between $f(t)$ and $g(t)$ is the dot product

$$\int_{-\infty}^{\infty} f(t)g(t)dt = (\underline{f} \cdot \underline{g})$$

Proof

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)g(t)dt &= \int_{-\infty}^{\infty} \sum_{i=1}^{N} f_i \varphi_i(t) \sum_{j=1}^{N} g_j \varphi_j(t) dt \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} f_i g_j \int_{-\infty}^{\infty} \varphi_i(t) \varphi_j(t) dt \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} f_i g_j \delta_{ij} dt = \sum_{i=1}^{N} f_i g_i = (\underline{f} \cdot \underline{g})\end{aligned}$$

Vector Representation of Signals: Energy $\Leftrightarrow$ Square Norm

Signal Energy

If $g(t) = f(t)$, then the previous result gives

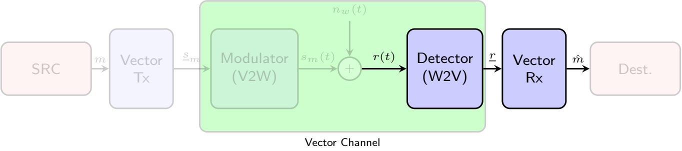
$$\int_{-\infty}^{\infty} f^2(t)dt = (\underline{f} \cdot \underline{f}) = \|\underline{f}\|^2$$

Energy of $f(t)$ is therefore $\|\underline{f}\|^2$.

Energy of $s_m(t)$ is the square norm

$$E_m \triangleq \int_{-\infty}^{\infty} s_m^2(t)dt = \|\underline{s}_m\|^2$$

Optimum Receiver Implementation



Optimum (MAP) Receiver

Computes $\underline{r} = (r_1, r_2, \ldots, r_N)$ where

$$r_i = \int_{-\infty}^{\infty} r(t)\varphi_i(t)dt \text{ for } i = 1, 2, \ldots, N$$

and solves

$$\min_{m \in \mathcal{M}} \{\|\underline{r} - \underline{s}_m\|^2 - 2\sigma^2 \ln \Pr\{M = m\}\} = \min_{m \in \mathcal{M}} \{\|\underline{r} - \underline{s}_m\|^2 - N_0 \ln \Pr\{M = m\}\}$$

Optimum Receiver requires... A-priori probabilities, Transmitted vectors, Noise variance, and N-dimensional r-values

Optimum Receiver Implementation

Optimum Receiver

The optimum receiver applies the rule

$$\hat{m}^{\mathsf{MAP}}(\underline{r}) = \underset{m\in\mathcal{M}}{\operatorname{argmax}}\{(\underline{r}\cdot\underline{s}_m) + c_m\}$$

where

$$c_m = \frac{N_0}{2}\ln\Pr\{M=m\} - \frac{E_m}{2}$$

and E_m is the energy of $s_m(t)$, for $m\in\mathcal{M}=\{1,2,\ldots,|\mathcal{M}|\}$.

Proof

$$\begin{aligned}
\hat{m}^{\mathsf{MAP}}(\underline{r}) &= \underset{m\in\mathcal{M}}{\operatorname{argmin}}\{\|\underline{r}-\underline{s}_m\|^2 - N_0\ln\Pr\{M=m\}\} \\
&= \underset{m\in\mathcal{M}}{\operatorname{argmin}}\{\|\underline{r}\|^2 + \|\underline{s}_m\|^2 - 2(\underline{r}\cdot\underline{s}_m) - N_0\ln\Pr\{M=m\}\} \\
&= \underset{m\in\mathcal{M}}{\operatorname{argmin}}\{\|\underline{s}_m\|^2/2 - (\underline{r}\cdot\underline{s}_m) - N_0/2\ln\Pr\{M=m\}\} \\
&= \underset{m\in\mathcal{M}}{\operatorname{argmax}}\{(\underline{r}\cdot\underline{s}_m) + N_0/2\ln\Pr\{M=m\} - \|\underline{s}_m\|^2/2\}
\end{aligned}$$

Module 6.1

Presentation Outline

How to Implement MAP with Filters?

Correlations are dot products

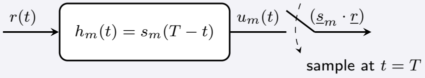
$$\int_{-\infty}^{\infty} r(t)s_m(t)dt = \int_{-\infty}^{\infty} r(t)\sum_{i=1}^{N} s_{mi}\varphi_i(t)dt$$

$$= \sum_{i=1}^{N} s_{mi}\int_{-\infty}^{\infty} r(t)\varphi_i(t)dt = \sum_{i=1}^{N} s_{mi}r_i = (\underline{s}_m \cdot \underline{r})$$

Correlation as Linear Filter and Sampling

$$u_m(t) = \int_{-\infty}^{\infty} r(\alpha)h_m(t-\alpha)d\alpha = \int_{-\infty}^{\infty} r(\alpha)s_m(T-t+\alpha)d\alpha$$

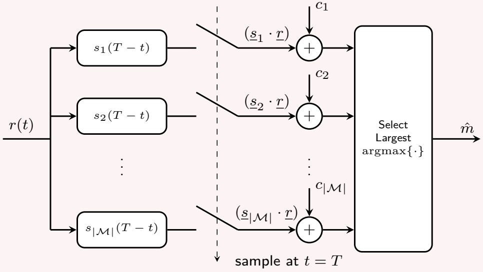
$$\stackrel{t=T}{=} \int_{-\infty}^{\infty} r(\alpha)s_m(\alpha)d\alpha = (\underline{s}_m \cdot \underline{r})$$



Optimum Receiver: Direct Receiver

Direct Receiver Structure

$$\hat{m}^{\mathrm{MAP}} = \underset{m\in\mathcal{M}}{\operatorname{argmax}}\{(\underline{r}\cdot\underline{s}_m) + c_m\}, \quad c_m = \frac{N_0}{2}\ln \Pr\{M=m\} - \frac{E_m}{2}$$



Two Questions

- Q1: Design a constellation for which the direct receiver is simple
- Q2: How many filters does this receiver need?

Summary Module 6.1

Take Home Messages

- Vectorial representation of signals:
  - Correlations are dot products
  - Energies are square norms
- Correlations as linear filters plus sampling
- Optimum receiver:
  - Finds maximum "shifted" correlations
  - Direct receiver ($|\mathcal{M}|$ Filters)

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