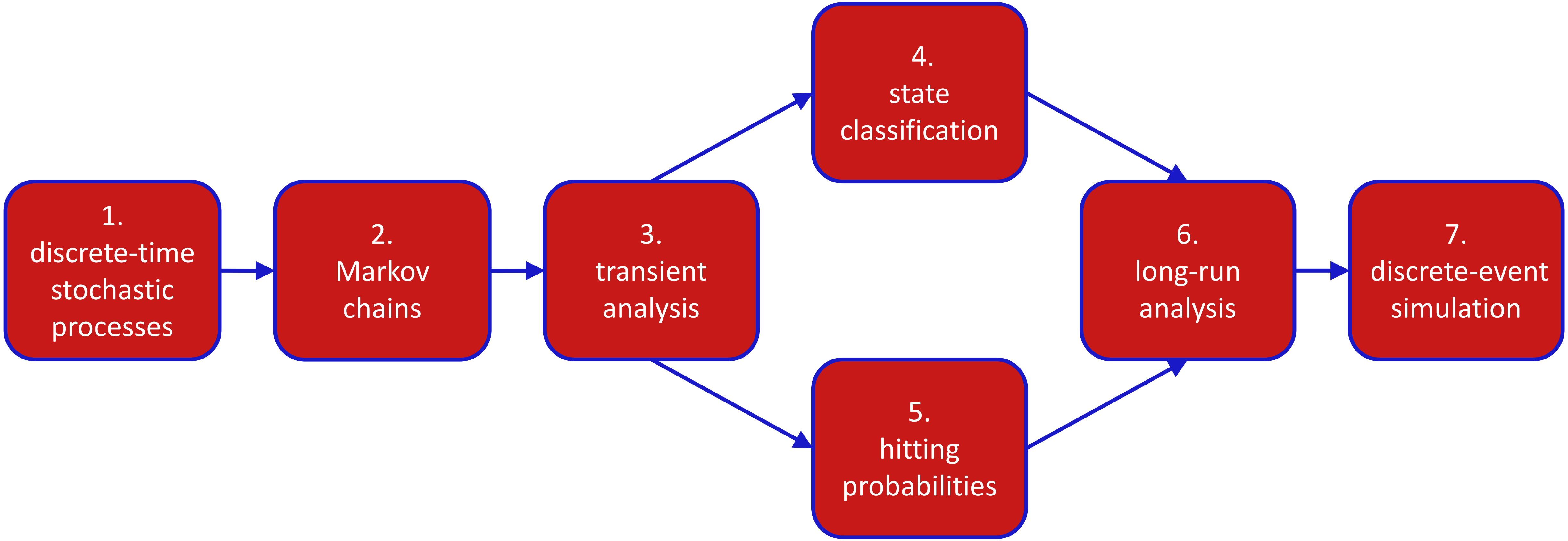
Markov modeling, discrete-event simulation – Exercises module B.6

5XIE0 Computational Modeling

Twan Basten, Marc Geilen, Jeroen Voeten
Electronic Systems Group, Department of Electrical Engineering

module B - submodules and dependencies



B.6 – long-run analysis

long-run analysis – exercises

- Section B.6 in the course notes
  - Exercise B.28 (Limiting matrix ergodic unichain)
    - use CMBW (DTMC) to compute / verify answer
      1. create the model
      2. select ‘Determine MC Type’ to determine the type of Markov chain (to find out what limit is actually computed)
      3. select ‘Limiting Matrix’ to compute the limiting matrix
  - Exercise B.29 (Video application – limiting distribution)
    - use CMBW (DTMC) verify answer for different values of p
      1. create the model (the same model as Exercise B.24) for different transition probabilities
      2. select ‘Determine MC Type’ to determine the type of Markov chain
      3. select ‘Limiting Distribution’ to compute the limiting distribution
- answers are provided in Section B.8 of the course notes

long-run analysis – exercises

- Section B.6 in the lecture notes on Markov modeling, discrete event simulation
  - Exercise B.30 (Packet generator – generated load)
    - use CMBW (DTMC) to compute / verify answer
      1. create the model for different values of p
      2. select 'Determine MC Type' to determine the type of Markov chain (to find out what limit is actually computed)
      3. press 'Limiting Distribution' to compute the limiting distribution
  - Exercise B.31 (Expected fraction of time spent in a state equals reciprocal of expected return time)
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      1. create Markov reward model
      2. select 'Determine MC Type' to determine the type of Markov chain
      3. select 'Limiting Distribution' to compute the limiting distribution
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long-run analysis– exercises

- Section B.6 in the course notes
  - Example B.18 (Computing limiting matrix)
    - use CMWB (DTMC) to verify answer
      - create the model
      - select ‘Limiting Matrix’ to compute the Cezàro limiting matrix
  - Exercise B.32 (Composition of two parallel packet generators – generated load)
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      1. create Markov reward model for different values of p
      2. select ‘Determine MC Type’ to determine the type of Markov chain
      3. select ‘Limiting Distribution’ to compute the limiting distribution
      4. select ‘Long-run Reward’ to compute the long-run expected (average) reward
      5. select ‘Limiting Matrix’ to compute the (normal or Cezàro) limiting matrix
  - Exercise B.33 (Computer system – throughput)
    - use CMWB (DTMC) to verify answer
- answers are provided in Section B.8 of the course notes

Exercise B.28 (Limiting matrix ergodic unichain)

Exercise B.28 (Limiting matrix ergodic unichain). Consider the Markov chain of Exercise B.21 again having transition probability matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$.

(a) Approximate P^5, P^{10} and P^{15}

(b) Compute P^∞

Use CMBW (DTMC) to compute / verify answer

1. create the model
2. select 'Determine MC Type' to determine the type of Markov chain (to find out what limit is actually computed)
3. select 'Limiting Matrix' to compute the limiting matrix

Exercise B.28 (Limiting matrix ergodic unichain).

(a) $P^5 \approx \begin{bmatrix} 0.125 & 0.250 & 0.625 \\ 0.312 & 0.125 & 0.562 \\ 0.281 & 0.312 & 0.406 \end{bmatrix}$, $P^{10} \approx \begin{bmatrix} 0.270 & 0.258 & 0.473 \\ 0.236 & 0.27 & 0.494 \\ 0.247 & 0.236 & 0.517 \end{bmatrix}$, $P^{15} \approx \begin{bmatrix} 0.247 & 0.247 & 0.505 \\ 0.253 & 0.247 & 0.500 \\ 0.250 & 0.253 & 0.497 \end{bmatrix}$

(b) First notice the matrix corresponds to an ergodic unichain and therefore the limit P^∞ exists. The rows of P^∞ are equal to the unique solution of the balance equations. These equations are: $x_1 = \frac{1}{2}x_3$, $x_2 = x_1$, $x_3 = x_2 + \frac{1}{2}x_3$, $x_1 + x_2 + x_3 = 1$. Solving yields: $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{4}$, $x_3 = \frac{1}{2}$. Hence $P^\infty = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$. Notice that P^{15} is already a good approximation of P^∞.

The limiting distribution $\lim_{n\to\infty} \pi^{(n)} = \pi^{(\infty)}$ exists for every ergodic unichain (B.45)

The limiting distribution $\pi^{(\infty)}$ of an ergodic unichain is a solution to the balance equations $\pi = \pi P, \sum_{i\in\mathcal{S}} \pi_i = 1$ (B.46)

The balance equations of an ergodic unichain have a unique solution (B.47)

The limiting matrix P^∞ exists for any ergodic unichain, where each row of P^∞ equals $\pi^{(\infty)}$ (B.48)

Exercise B.29 (Video application – limiting distribution)

Exercise B.29 (Video application - limiting distribution). Consider the Markov chain of the video application in Exercise B.24.

- Compute (e.g. using a symbolic solver) the probability that the buffer contains two frames in the long-run.
- For what value of p is this probability maximal? What is the maximal probability?

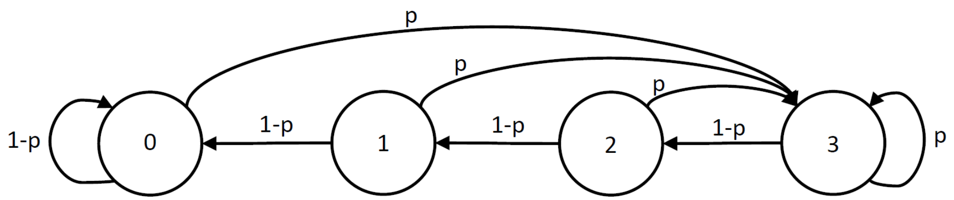


Figure B.16: Transition diagram of movie application

Use CMBW (DTMC) verify answer for different values of p

1. create the model (the same model as Exercise B.24) for different transition probabilities
2. select 'Determine MC Type' to determine the type of Markov chain
3. select 'Limiting Distribution' to compute the limiting distribution

Exercise B.29 (Video application - limiting distribution).

(a) Notice that the Markov chain is an ergodic unichain since $0 < p < 1$. Therefore the limiting distribution exists. We can find it by solving the balance equations: $\pi_0 = (1-p)\pi_0 + (1-p)\pi_1$, $\pi_1 = (1-p)\pi_2$, $\pi_2 = (1-p)\pi_3$, $\pi_3 = p\pi_0 + p\pi_1 + p\pi_2 + p\pi_3$, $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$. Solving yields $\pi_0 = 1 - 3p + 3p^2 - p^3$, $\pi_1 = p - 2p^2 + p^3$, $\pi_2 = p - p^2$ and $\pi_3 = p$. Hence the probability that the buffer contains two frames in the long run is $p - p^2$.

(b) $p - p^2$ corresponds to a downward opening parabola, the top of which lies at $p = \frac{1}{2}$. The corresponding maximal probability equals $\frac{1}{4}$.

The limiting distribution $\lim_{n\to\infty} \pi^{(n)} = \pi^{(\infty)}$ exists for every ergodic unichain (B.45)

The limiting distribution $\pi^{(\infty)}$ of an ergodic unichain is a solution to the balance equations $\pi = \pi P, \sum_{i\in\mathcal{S}} \pi_i = 1$ (B.46)

The balance equations of an ergodic unichain have a unique solution (B.47)

Exercise B.30 (Packet generator – generated load)

Exercise B.30 (Packet generator - generated load). A packet generator for a slotted communication medium behaves as a Markov chain with two states. The transition diagram is shown in Figure B.11. State 1 represents a wait state and state 2 represents a transmit state. A packet is produced only when the generator is in the transmit state. p $(0 \leq p \leq 1)$ is a parameter to control the generated load, i.e. the expected number of packets produced per time slot.

(a) Give an expression of the generated load in terms of p.

(b) What are the minimal and the maximal loads that can be generated?

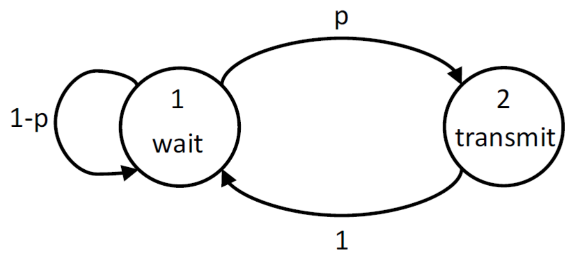


Figure B.11: Transition diagram of packet generator

Use CMBW (DTMC) to compute / verify answer

1. create the model for different values of p
2. select 'Determine MC Type' to determine the type of Markov chain (to find out what limit is actually computed)
3. press 'Limiting Distribution' to compute the limiting distribution

Exercise B.30 (Packet generator - generated load).

(a) We have to compute the long-run expected fraction of visits to state 2. For $p = 1$, the chain is a non-ergodic unichain. For $0 \leq p < 1$, the chain is an ergodic unichain. Hence for any value of p, the chain is a unichain and therefore the balance equations have a unique solution. The balance equations are given by $\pi_1 = (1-p)\pi_1 + \pi_2$, $\pi_2 = p\pi_1$, $\pi_1 + \pi_2 = 1$. Solving yields $\pi_1 = \frac{1}{1+p}$ and $\pi_2 = \frac{p}{1+p}$. Hence the generated load is $\frac{p}{1+p}$ packets per time slot.

(b) The minimal load is 0 and is obtained for $p = 0$. The maximal load is $\frac{1}{2}$ and is obtained for $p = 1$.

The balance equation have a unique stationary distribution if and only if the chain is a unichain (B.51)

The unique solution to the balance equations of a unichain equals the Cezàro limiting distribution $\pi^{(\propto)}$ (B.52)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \pi^{(k)}$$

$\pi_i^{(\propto)}$: long-run expected fraction of time the chain spends in state *i*

The Cezàro limiting matrix $P^{\propto}$ exists for any unichain, where each row of $P^{\propto}$ equals $\pi^{(\propto)}$ and $\pi^{(\propto)} = \pi^{(0)} P^{\propto}$ (B.53)

Exercise B.31 (Expected fraction of time spent in a state equals reciprocal of expected return time)

Exercise B.31 (Expected fraction of time spent in a state equals reciprocal of expected return time). One can show that for any state in a recurrent class, the long-run expected fraction of time the chain spends in this state equals the reciprocal of the expected return time to that state. Consider the Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(a) Show that this chain is irreducible (implying that it consists of single recurrent class), but not ergodic.

(b) Show that the above property holds for state 1 of this chain.

Exercise B.31 (Expected fraction of time spent in a state equals reciprocal of expected return time).

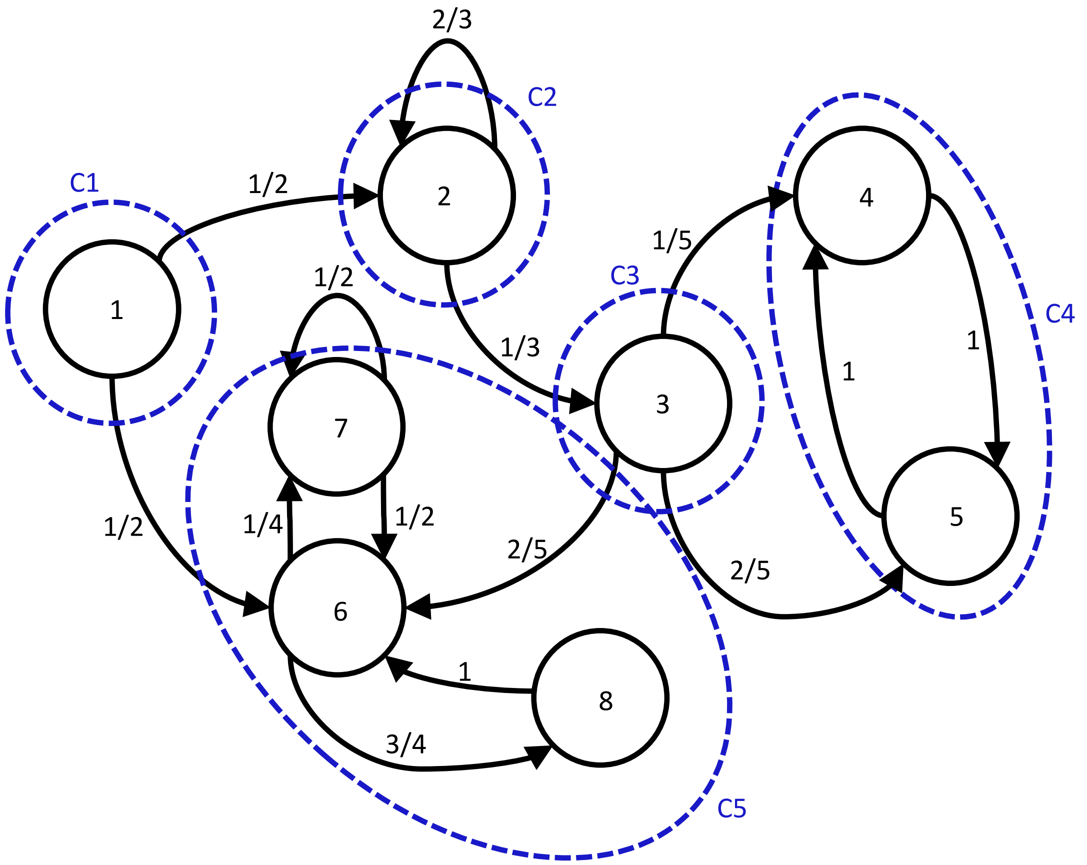
(a) From the transition diagram, it is immediately clear that all states in this chain communicate. Hence the chain has a single class of recurrent states and is thus irreducible. To understand why this chain is not ergodic, consider state 1. The length of the paths from state 1 to 1 are $4, 6, 8, \cdots$ and hence $d(1) = 2$. State 1 is thus periodic and so are all the other states.

(b) The long-run expected fraction of time the chain spends in state 1 equals $\pi_1^{(\infty)}$, where $\pi^{(\infty)}$ is the unique solution to the balance equations: $\pi_1 = \pi_4$, $\pi_2 = \pi_1 + \frac{5}{6}\pi_3$, $\pi_3 = \pi_2$, $\pi_4 = \frac{1}{6}\pi_3$ and $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$. Solving yields $\pi^{(\infty)} = [\frac{1}{14}, \frac{3}{7}, \frac{3}{7}, \frac{1}{14}]$. Hence $\pi_1^{(\infty)} = \frac{1}{14}$. The expected return time to state 1 is given by $\frac{f_{11}^r}{f_{11}}$, where reward r assigns the value 1 to each state. Since state 1 is recurrent, $f_{11} = 1$. To compute f_{11}^r we solve the system of equations in (B.39): $x_1 = 1 + x_2$, $x_2 = 1 + x_3$, $x_3 = 1 + \frac{5}{6}x_2 + \frac{1}{6}x_4$, $x_4 = 1$. Solving yields $x_1 = 14$, $x_2 = 13$, $x_3 = 12$ and $x_4 = 1$. Hence $f_{11}^r = 14 = \frac{1}{\pi_1^{(\infty)}}$.

Use CMBW (DTMC) to verify answer

1. create Markov reward model
2. select 'Determine MC Type' to determine the type of Markov chain
3. select 'Limiting Distribution' to compute the limiting distribution
4. select 'Reward until Hit' to compute the expected return times

Example B.18 (Computing limiting matrix)



States 1, 2 and 3 are transient

Stationary distribution class C4

Stationary distribution class C5

$P_{14} = h_{1C4} \cdot \pi_4^{C4} = 3/10 \cdot 1/2 = 3/20$

$P_{24} = h_{2C4} \cdot \pi_4^{C4} = 3/5 \cdot 1/2 = 3/10$

Etcetera

$$P^{\propto} = \begin{bmatrix} 0 & 0 & 0 & \frac{3}{20} & \frac{3}{20} & \frac{14}{45} & \frac{7}{45} & \frac{7}{30} \\ 0 & 0 & 0 & \frac{3}{10} & \frac{3}{10} & \frac{8}{45} & \frac{4}{45} & \frac{2}{15} \\ 0 & 0 & 0 & \frac{3}{10} & \frac{3}{10} & \frac{8}{45} & \frac{4}{45} & \frac{2}{15} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{9} & \frac{2}{9} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{9} & \frac{2}{9} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{9} & \frac{2}{9} & \frac{1}{3} \end{bmatrix}$$

$$P^{\propto}_{ij} = \begin{cases} h_{iC} \cdot \pi_j^C & \text{if } j \text{ is in recurrent class } C \text{ and } \pi_j^C \text{ is the solution} \\ & \text{for state } j \text{ of the balance equations for } C \\ 0 & \text{if } j \text{ is a transient state} \end{cases} \qquad \text{(B.56)}$$

Use CMWB (DTMC) to verify answer

- create the model
- select 'Limiting Matrix' to compute the Cezàro limiting matrix

Exercise B.32 (Composition of two parallel packet generators – generated load)

Exercise B.32 (Composition of two parallel packet generators - generated load). Two packet generators from Exercise B.30 are offering packets to a time-slotted communication medium, both using configuration parameter p. When both generators offer a packet in the same time slot, a collision occurs resulting in the loss of both packets. When precisely one generator offers a packet, this packet is successfully transmitted. In case no packet is offered, the time-slot is left unused.

(a) Model this system as a Markov chain. Explain the meaning of the chosen states and transitions.

(b) Compute the utilization of the medium.

(c) For what setting of the configuration parameter p do we obtain optimal utilization. What is the optimal utilization?

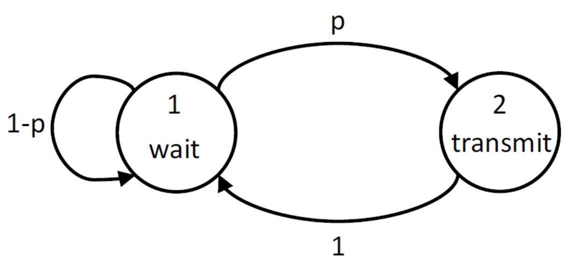


Figure B.11: Transition diagram of packet generator

Exercise B.32 (Composition of two parallel packet generators - generated load).

(a) The system can be modelled by a product of two traffic generators, see Figure B.18. State 1, for instance, indicates that both traffic generators are in their wait states and state 2 indicates that one traffic generator is transmitting while the other is waiting. Transitions refer to the logic AND of the corresponding transitions of the generators. E.g the probability of the transition from state 1 to state 2 is the product of probabilities that one generator transits from the wait to the transmit state (p) and the other remains in the wait state ($1 - p$).

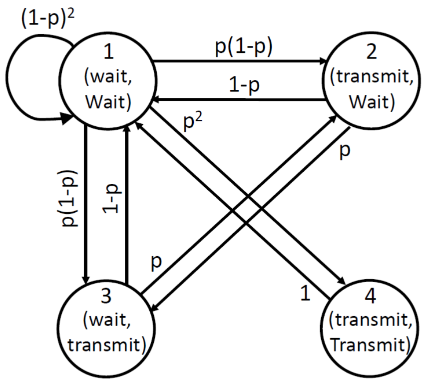


Figure B.18: Transition diagram of time-slotted medium with two packet generators

Exercise B.32 (Composition of two parallel packet generators – generated load)

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(a) Model this system as a Markov chain. Explain the meaning of the chosen states and transitions.

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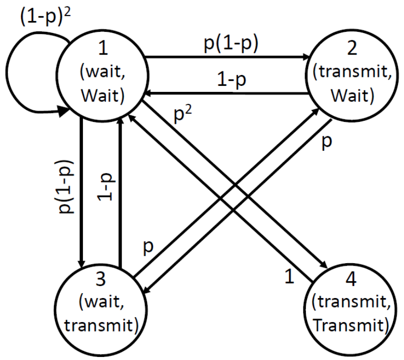


Figure B.18: Transition diagram of time-slotted medium with two packet generators

(b) We define a reward r as follows: $r(1) = 0$, $r(4) = 0$, $r(2) = 1$, $r(3) = 1$. The utilization is then given by long-run expected (average) reward is given by $\pi^{(\infty)} r^T = \pi_2^{(\infty)} + \pi_3^{(\infty)}$. We have to distinguish two cases:

(1) For $0 \leq p < 1$ we are dealing with an ergodic unichain. This means that the limit distribution $\pi^{(\infty)}$ exists, is independent of the initial distribution, and is given by the unique solution to the balance equations. These equations are: $\pi_1 = (1-p)^2\pi_1 + (1-p)\pi_2 + (1-p)\pi_3 + \pi_4$, $\pi_2 = p(1-p)\pi_1 + p\pi_3$, $\pi_3 = p(1-p)\pi_1 + p\pi_2$, $\pi_4 = p^2\pi_1$ and $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$. Solving yields $\pi_1 = \frac{1}{(1+p)^2}$, $\pi_2 = \frac{p}{(1+p)^2}$, $\pi_3 = \frac{p}{(1+p)^2}$ and $\pi_3 = \frac{p^2}{(1+p)^2}$. Thus $\pi^{(\infty)} r^T = \frac{2p}{(1+p)^2}$.

(2) For $p = 1$ the chain is a non-ergodic non-unichain; it consists of two recurrent periodic classes $\{1, 4\}$ and $\{2, 3\}$. Thus the normal limit $\pi^{(\infty)}$ does not exist and neither does the limit of the probability matrix. Further $\pi^{(\infty)}$ will depend on the initial distribution in this case. It is easy to find out that

$$P^{\infty} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

so that $\pi^{(\infty)} = \pi^{(0)} P^{\infty} = [\frac{1}{2}\pi_1^{(0)} + \frac{1}{2}\pi_4^{(0)}, \frac{1}{2}\pi_2^{(0)} + \frac{1}{2}\pi_3^{(0)}, \frac{1}{2}\pi_2^{(0)} + \frac{1}{2}\pi_3^{(0)}, \frac{1}{2}\pi_1^{(0)} + \frac{1}{2}\pi_4^{(0)}]$. Thus $\pi^{(\infty)} r^T = \pi_2^{(\infty)} + \pi_3^{(\infty)} = \pi_2^{(0)} + \pi_3^{(0)}$.

Exercise B.32 (Composition of two parallel packet generators – generated load)

Exercise B.32 (Composition of two parallel packet generators - generated load). Two packet generators from Exercise B.30 are offering packets to a time-slotted communication medium, both using configuration parameter p. When both generators offer a packet in the same time slot, a collision occurs resulting in the loss of both packets. When precisely one generator offers a packet, this packet is successfully transmitted. In case no packet is offered, the time-slot is left unused.

(a) Model this system as a Markov chain. Explain the meaning of the chosen states and transitions.

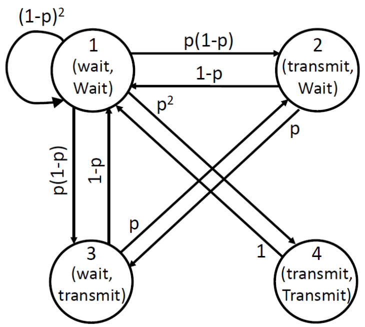
(b) Compute the utilization of the medium.

(c) For what setting of the configuration parameter p do we obtain optimal utilization. What is the optimal utilization?

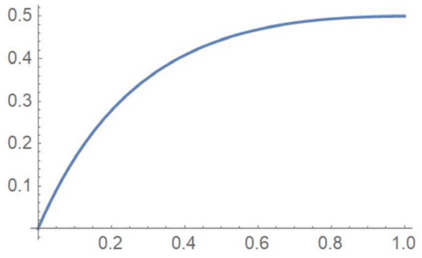
(c) We have to distinguish the same two cases again:

(1) For $0 \leq p < 1$ we have to find the optimum of $\frac{2p}{(1+p)^2}$. $\frac{2p}{(1+p)^2}$ is increasing in p, so the optimal value is obtained when p approaches 1. The optimal utilization is $\frac{1}{2}$. Apparently, even though both generators can produce a load of 50%, collisions prevent the system from reaching a utilization beyond 50%.

(2) For $p = 1$, we have to find the optimum of $\pi_2^{(0)} + \pi_3^{(0)}$. The optimum is 1 in case the sum of the initial probabilities of states 2 and 3 equals 1. This means that the packets generators are never in the same states and thus collisions are always avoided.



$$\frac{2p}{(1+p)^2}$$



Use CMWB (DTMC) to verify answer

1. create Markov reward model for different values of p
2. select 'Determine MC Type' to determine the type of Markov chain
3. select 'Limiting Distribution' to compute the limiting distribution
4. select 'Long-run Reward' to compute the long-run expected (average) reward
5. select 'Limiting Matrix' to compute the (normal or Cezàro) limiting matrix

Exercise B.33 (Computer system – throughput)

Exercise B.33 (Computer system - throughput). A computer system executes tasks of three different types, A, B and C. After the execution of a task of type A, such a task is executed again with probability $1-p$ and with probability p this is a task of type B. After the execution of a type B task, a type C task follows with probability p and a type A task with probability $1-p$. The execution of a task of type C is always followed by a task of type A.

(a) Construct a Markov chain that models the behaviour of the computer system.

(b) Let π_X denote the probability that the system is in state X in the long-run. Compute π_A, π_B and π_C.

(c) The execution of tasks of types A, B and C take respectively 1, $3+3p$ and $2p$ seconds. Give an expression of the throughput of this computer system, i.e. the the number of tasks executed per second in the long-run.

(d) What is the minimal throughput of the computer system?

Exercise B.33 (Computer system - throughput).

(a) The computer system can be modeled with three states A, B and C. Transition probabilities are $P_{AA} = 1-p$, $P_{AB} = p$, $P_{BA} = 1-p$, $P_{BC} = p$ and $P_{CA} = 1$.

(b) For $0 \leq p < 1$ the Markov chain is an ergodic unichain and for $p = 1$ the Markov chain is a non-ergodic unichain. In any case, we are dealing with a unichain for which the Cezàro limiting distribution exists. It can be found by solving the balance equations. Solving yields $\pi_A = \frac{1}{1+p+p^2}$, $\pi_B = \frac{p}{1+p+p^2}$ and $\pi_C = \frac{p^2}{1+p+p^2}$.

(c) We can model the execution times by assigning rewards to the states. In states A, B and C rewards 1, $3+3p$ and $2p$ are obtained respectively. The long-run expected average reward obtained is then $\pi_A + (3+3p)\pi_B + 2p\pi_C = \frac{1+3p+3p^2+2p^3}{1+p+p^2} = 2p+1$. This means that the expected average time per execution equals $2p+1$ seconds. Hence the throughput is $\frac{1}{2p+1}$ tasks per second.

(d) Expression $\frac{1}{2p+1}$ is decreasing in p and is thus minimal for $p = 1$. For $p = 1$ the throughput is $\frac{1}{3}$ tasks per second.

Use CMWB (DTMC) to verify answer