Note: The following MATLAB functions are useful for dealing with systems in state-space representation:

SS, TF2SS, SS2TF, ZP2SS, CTRB, CTRBF, OBSV, OBSVF, CANON, RESIDUE, SSDATA, RANK, STEP, IMPULSE, INITIAL, PLACE, ACKER

Type `help` *function_name* in MATLAB to see more details on the description of the functions. If a function does not exist, then you have to install the CONTROL SYSTEM TOOLBOX.

1. **State-Space Models, State Feedback:**
Consider a system described by the state-space equations

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 3 \end{bmatrix} x \end{aligned}$$

(a) Determine if the system is observable and/or controllable.

(b) Find the closed-loop characteristic equation if the feedback control law is

i. $u = -[K_1,\, K_2]x$,

ii. $u = -K_3 y$.

(c) Design a state-feedback control law ($u = -Kx = -[K_1,\, K_2]x$) such that the closed-loop poles have a damping coefficient of $\zeta = 1/\sqrt{2}$ and natural frequency $\omega_\mathrm{n} = \sqrt{2}$. Investigate whether this is possible via output feedback ($u = -K_3 y$).

Solution:

(a) The system is both controllable and observable, since

$$\begin{aligned} \operatorname{rank}(\mathcal{O}) &= \operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 3 \\ 21 & -11 \end{bmatrix} = 2, \\ \operatorname{rank}(\mathcal{C}) &= \operatorname{rank}\begin{bmatrix} B & AB \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = 2. \end{aligned}$$

(b) Two types of feedback control laws are considered:

(i.) State-feedback case:

Plugging in the control law gives

$$\dot{x} = Ax - BKx = (A - BK)\,x$$

Hence we can compute the characteristic equation by

$$\begin{aligned} \det(\lambda I - A + BK) &= \det\begin{bmatrix} \lambda + K_1 & -1 + K_2 \\ -7 + 2K_1 & \lambda + 4 + 2K_2 \end{bmatrix} \\ &= \lambda^2 + \lambda(4 + 2K_2 + K_1) + (6K_1 + 7K_2 - 7) = 0. \end{aligned}$$

(ii.) Output-feedback case:

First we write the control law as function of x using

$$u = -K_3 y = -K_3 C x = -K_3 \begin{bmatrix} 1 & 3 \end{bmatrix} x = -\begin{bmatrix} K_3 & 3K_3 \end{bmatrix} x,$$

which then leads to the characteristic equation

$$\lambda^2 + \lambda(7K_3 + 4) + (27K_3 - 7) = 0.$$

(c) By taking into account the closed-loop specifications, the roots of the closed-loop system will be $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, hence $s^2 + 2s + 2 = 0$.
By equating the coefficients of the closed-loop roots when employing a state-feedback control law, we have

$$\begin{aligned} 4 + 2K_2 + K1 &= 2 \\ 6K_1 + 7K_2 - 7 &= 2, \end{aligned}$$

hence $K_1 = 6.4$, $K_2 = -4.2$.

This is not possible for the output-feedback case, since there does not exist $K_3 \in \mathbb{R}$ that satisfies both the equations $7K_3 + 4 = 2$ and $27K_3 - 7 = 2$.

2. **State-Space Reference Tracking:**
The normalized equations of motion for an inverted pendulum at angle θ on a cart are

$$\ddot{\theta} = \theta + u, \quad \ddot{x} = -\beta\theta - u,$$

where x is the cart position and the control input is a force acting on the cart. The measured state variable is x.

(a) With the state defined as $\boldsymbol{x} = [\theta \ \dot{\theta} \ x \ \dot{x}]^\top$, find the feedback gain K for the state feedback control law ($u = -Kx$), that places the closed-loop poles at $s = -1, -1, -1 \pm j$.

(b) Assume that $\beta = 0.5$. Plot the responses of the closed- and open-loop system with initial condition $[\theta \ \dot{\theta} \ x \ \dot{x}]^\top = [10° \ 0 \ 0 \ 0]^\top$, by using MATLAB.

(c) Compute control law of the form $u(x, r) = -Kx + \bar{N}r$ such that the plant output y tracks a constant command input r, which denotes the desired cart position. Assume that $\beta = 0.5$.

Solution:

(a) The state space equations of motion are

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u.$$

We require the closed-loop characteristic equation to be

$$\alpha_c(s) = (s+1)^2(s^2+2s+2) = s^4 + 4s^3 + 7s^2 + 6s + 2.$$

From the above state equations, we obtain

$$\det(sI - A + BK) = s^4 + (k_2 - k_4)s^3 + (k_1 - k_3 - 1)s^2 + k_4(1-\beta)s + k_3(1-\beta).$$

Comparing coefficients we have

$$k_1 = \frac{10-8\beta}{1-\beta}, \quad k_2 = \frac{10-4\beta}{1-\beta}$$
$$k_3 = \frac{2}{1-\beta}, \quad k_4 = \frac{6}{1-\beta}.$$

(b) By setting $\beta = 0.5$, we have

$$K = [12 \quad 16 \quad 4 \quad 12].$$

The time response to the indicated initial condition is shown in Figure 1.

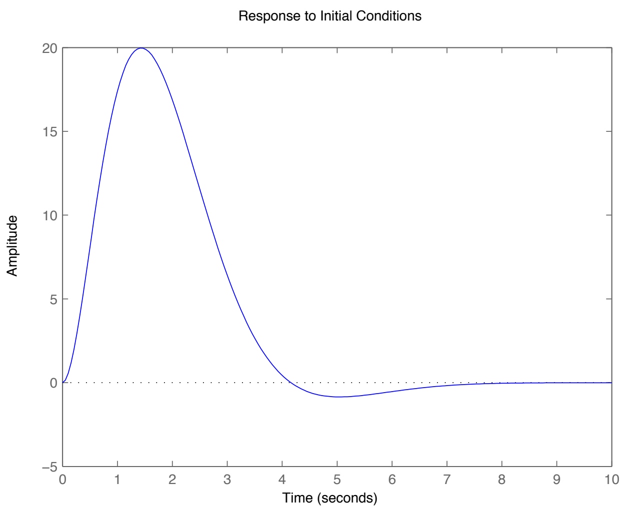

Figure 1: Problem 3: Time response of the closed-loop system

(c) We have that $\bar{N} = N_u + K\,N_x$ with

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Since the output y tracks the cart position x we have $C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ and hence $\bar{N} = 4$.

Different solution:
If we have $u = -Kx + \bar{N}r$ then we can write our state space equations as

$$\begin{cases} \dot{x} = (A - BK)x + B\bar{N}r \\ y = (C - DK)x + D\bar{N}r \end{cases}$$

We want to have zero steady state error. Steady state means that $\dot{x} = 0$ and hence

$$\begin{aligned} \dot{x} &= (A - BK)x + B\bar{N}r \\ 0 &= (A - BK)x + B\bar{N}r \\ (A - BK)x &= -B\bar{N}r \\ x &= -(A - BK)^{-1}B\bar{N}r \end{aligned}$$

Substituting x in the output equation then gives

$$\begin{aligned} y &= (C - DK)x + D\bar{N}r \\ &= -(C - DK)(A - BK)^{-1}B\bar{N}r + D\bar{N}r \\ &= \left[D - (C - DK)(A - BK)^{-1}B\right]\bar{N}r \end{aligned}$$

Note that we want zero offset, which means $r = y$ and hence

$$\begin{aligned} y &= \left[D - (C - DK)(A - BK)^{-1}B\right]\bar{N}r = r \\ 1 &= \left[D - (C - DK)(A - BK)^{-1}B\right]\bar{N} \\ \bar{N} &= \left[D - (C - DK)(A - BK)^{-1}B\right]^{-1} \end{aligned}$$

which for $D = 0$ also equals to 4.

3. **Observer Design:**
The equations of motion for a station-keeping satellite (such as a weather satellite), as shown in Figure 2, are

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - 3\omega^2 x &= 0 \\ \ddot{y} + 2\omega\dot{x} &= u, \end{aligned}$$

where x is the radial perturbation, y is the longitudinal position perturbation, and u is the engine thrust in the y-direction.

If the orbit is synchronous with the Earth's rotation, then $\omega = \frac{2\pi}{3600 \cdot 24}$ rad/s. Define $\boldsymbol{x} = [x \;\; \dot{x} \;\; y \;\; \dot{y}]^\top$ as the state vector and output y as the longitudinal position perturbation. Design a full-order observer with poles placed at $s = -2\omega, -3\omega$ and $-3\omega \pm 3\omega j$.

Solution:
The state-space equations are

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ z &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x \end{aligned}$$

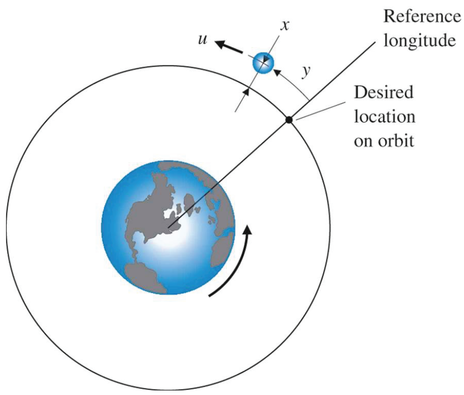

Figure 2: A station-keeping satellite in orbit.

First, we check the observability to verify that we can arbitrarily place the estimator poles. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix},$$

hence the system is observable since $\mathcal{O}$ is full rank. The desired estimator characteristic equation is

$$\alpha_{\text{des}}(s) = s^4 + 11\omega s^3 + 54\omega^2 s^2 + 126\omega^3 s + 108\omega^4,$$

while the estimator characteristic equation is

$$\alpha_{\text{est}}(s) = s^4 + l_3 s^3 + (l_4 + \omega^2)s^2 + (-2\omega l_2 + \omega l_3^2)s + (-3\omega^2 l_4 - 6\omega^3 l_1).$$

By equating the coefficients, we obtain

$$\begin{aligned} l_1 &= -44.5\omega \\ l_2 &= -57.5\omega^2 \\ l_3 &= 11\omega \\ l_4 &= 53\omega^2. \end{aligned}$$

4. **Dynamic Output Feedback:**
A certain process has the transfer function

$$G(s) = \frac{4}{s^2 - 4}$$

(a) Find the matrices A, B and C for this system in observer canonical form (see page 493, 7th Edition).

(b) If $u = -Kx$, compute K such that the closed-loop control poles are located at $s = -2 \pm 2j$.

(c) Compute L such that the estimator poles are located at $s = -10 \pm 10j$.

(d) Determine the transfer function of the resulting controller, i.e., the transfer function from $y(t)$ to $u(t)$, when

$$\begin{aligned} \dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly \\ u &= -K\hat{x}. \end{aligned}$$

Solution:

(a) We can construct the state space matrices directly from the transfer function in observability cannonical form:

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Note that there is duality between the controlability and observability canonical form meaning

$$A_o = A_c^\top \quad B_o = C_c^\top \quad C_o = B_c^\top \quad D_o = D_c^\top$$

(b) By setting $u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x$, we can write the characteristic equation as

$$\begin{aligned} \det(\lambda I - A + BK) &= \\ \lambda^2 + 4k_2\lambda - 4 + 4k_1 &= (s + 2 + 2j)(s + 2 - 2j) \\ &= s^2 + 4s + 8 \end{aligned}$$

Equating the coefficients then gives $k_1 = 3$ and $k_2 = 1$.

(c) The estimator roots are determined by the characteristic equation of the closed-loop system. By taking $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^\top$ we get

$$\begin{aligned} \det(\lambda I - A + LC) &= \\ \lambda^2 + l_1\lambda - 4 + l_2 &= (s + 10 + 10j)(s + 10 - 10j) \\ &= s^2 + 20s + 200 \end{aligned}$$

Again, equating the coefficients, we obtain $l_1 = 20$, $l_2 = 204$.

(d) We have

$$\begin{cases} \dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly \\ u = -K\hat{x} \end{cases}$$

Applying Laplace we can rewrite this as

$$\begin{aligned} s\hat{x} &= (A - BK - LC)\hat{x} + Ly \\ (sI - A + BK + LC)\hat{x} &= Ly \\ \hat{x} &= (sI - A + BK + LC)^{-}1Ly \end{aligned}$$

and thus

$$\begin{aligned} u &= -K\hat{x} \\ &= -K(sI - A + BK + LC)^{-}1Ly \end{aligned}$$

and hence the transfer function from y to u is given by

$$\begin{aligned}
D(s) &= \frac{U(s)}{Y(s)} = -K(sI - A + BK + LC)^{-1}L \\
&= -\left[\begin{array}{cc} 3 & 1 \end{array}\right]\left[\begin{array}{cc} s+20 & -1 \\ 212 & s+4 \end{array}\right]^{-1}\left[\begin{array}{c} 20 \\ 204 \end{array}\right] \\
&= \frac{-264s - 692}{s^2 + 24s + 292}.
\end{aligned}$$

This can be verified using the `ss2tf` command in MATLAB.

Suggestions by Sam:

5. **Lotka-Volterra model (Exam Level Question):**
Consider the nonlinear Lotka-Volterra model that describes the population of a predator and a prey. It is given by the following state-space equations:

$$\begin{cases} \dot{x}_1 = x_1 x_2 - x_1 + x_1 u \\ \dot{x}_2 = x_2 - x_1 x_2 \\ y = x_1 \end{cases}$$

where $x = \left[\begin{array}{cc} x_1 & x_2 \end{array}\right]^\top$ denotes the state, in which x_1 is he population of the prey. The control input u indicates the extra release or hunting of the predator species, and the population of predator species can be directly observed as output y. 4b Linearize this nonlinear differential equation around equilibrium position $x^0 = \left[\begin{array}{cc} 1 & 1 \end{array}\right]^\top, u^0 = 0$ and $y^0 = 1$ to arrive at the linearized model:

$$\begin{cases} \dot{x}^\delta(t) = Ax^\delta(t) + Bu^\delta(t) \\ y^\delta(t) = Cx^\delta(t) + Du^\delta(t). \end{cases}$$

a) Give the matrices A, B, C and D.

b) Compute the transfer function of the obtained linearized state-space model.

c) Since we can only measure the number of predators y, we would like to estimate the prey population by building an observer for the linearized model of the form $\dot{\hat{x}}^\delta = A\hat{x}^\delta + B\hat{u}^\delta + L\left(y^\delta - \hat{y}^\delta\right)$. Design L such that the observer has both closed-loop poles at -1.

Solution:

a)

$$A = \left[\begin{array}{cc} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} \end{array}\right] = \left[\begin{array}{cc} x_2 - 1 + u & x_1 \\ -x_2 & 1 - x_1 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right]$$

$$B = \left[\begin{array}{c} \frac{\delta f_1}{\delta u} \\ \frac{\delta f_2}{\delta u} \end{array}\right] = \left[\begin{array}{c} x_1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

$$C = \left[\begin{array}{cc} 1 & 0 \end{array}\right], \quad D = 0$$

b) To find the transfer function corresponding to the linearized state space we use,

$$G = C(sI - A)^{-1}B + D.$$

$$
\begin{aligned}
G &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{1}{s^2+1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2+1} \begin{bmatrix} s & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{s}{s^2+1}
\end{aligned}
$$

c) By taking $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^\top$ we get

$$
\begin{aligned}
\det\left(\lambda I - A + LC\right) &= \\
\det \begin{pmatrix} \lambda - l_1 & -1 \\ 1 - l_2 & \lambda \end{pmatrix} &= \\
\lambda^2 - l_1\lambda + 1 - l_2 &= (s+1)(s+1) \\
&= s^2 + 2s + 1
\end{aligned}
$$

Hence, equating the coefficients, we obtain $l_1 = -2$, $l_2 = 0$.