Digital Signal Processing Fundamentals (5ESC0)

Introduction Stochastic Signal Processing

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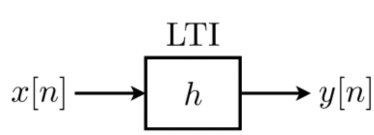
Same System, Different Signal

* We consider the same Linear Time Invariant (LTI) system:

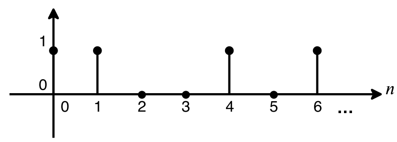


* Until now $x[n]$ was considered deterministic: a mathematical expression existed for each n of $x[n]$
* What will happen when we replace $x[n]$ by a stochastic signal?

Pre-knowledge: “Probability Theory” from course “Mathematics II” [5EMA0; Q4; year 2]

Discrete Time Stochastic Process Example

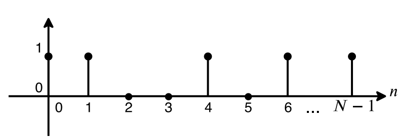
* Flipping a coin
* Probability for heads the same as tails: 50%
* Let us toss the coin many times and suppose that for time index n, our measurement is 1 for heads or 0 for tails



* We generate a probabilistic process with a mean value of 0.5: the average value between one and zero

Discrete Time Stochastic Process Example

* Now let us toss N times



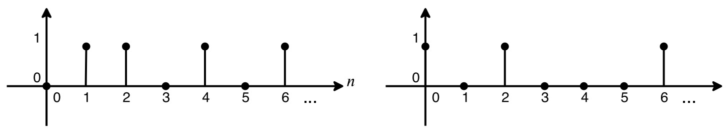
* To estimate the mean

$$\hat{\mu} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

* The chance that this estimate is equal to 0.5 is small
* The larger the N, the better the estimate will be

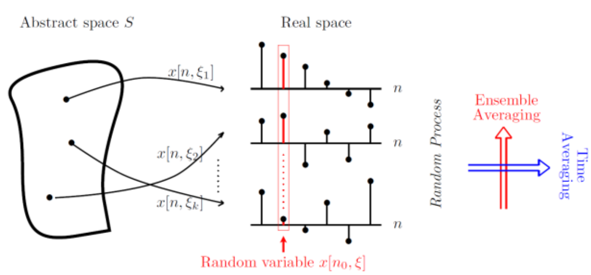
Discrete Time Stochastic Process Example

- If we repeat the same probabilistic process multiple times, different measurements are found, e.g.



- An infinite amount of these measurements can be made, each a different realization of the same process
- All these measurements together are called an "ensemble" of measurements

Discrete Time Stochastic Process

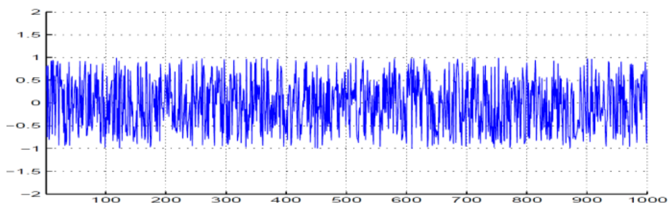


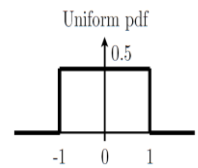
- Ensemble: Set of all possible sequences $\{x[n,\xi]\}$
- Realization: One sample sequence of the ensemble
- Random variable: Fixed $n = n_0$

Notation further on:
Skip ξ, thus $x[n]$ for random process or single realization

Discrete time stochastic processes: Examples

- Uniform
- White noise: uniformly distributed between -1 and 1

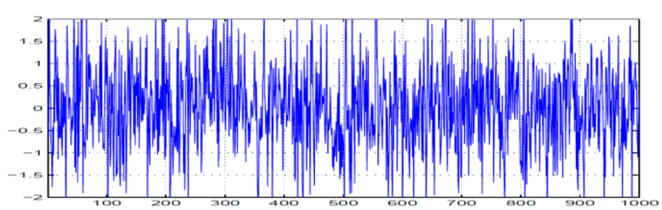


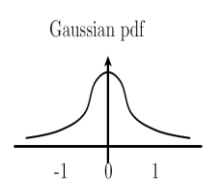


pdf – probability density function

Discrete time stochastic processes: Examples

- Gaussian: Allowing for any possible (confined) value





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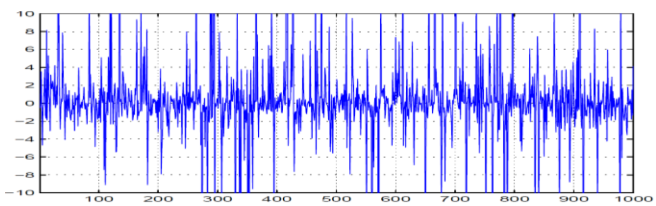
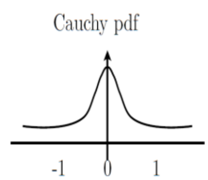
Discrete time stochastic processes: Examples

* Cauchy: Containing spikes and therefore has high variability





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Discrete time stochastic processes

* Features amplitude- vs time-plot random signal:
  - Probability distribution: frequency of occurrence of various amplitudes
  - Correlation: degree of dependency between 2 samples
  - Signal power spectrum
  - Indication of variability in mean, variance, etc.

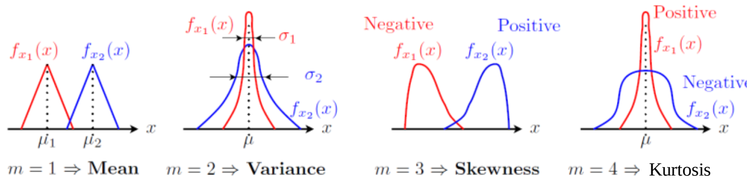
* Discrete mathematical expectation (ensemble average):

$E\{x(\xi)\} \doteq \mu_x = \sum_k x[k] \cdot p_k$ "center of gravity" of density function

p_k is probability of occurrence of the random variable $x[k]$

Discrete time stochastic processes

* Different moments:



skewness is a measure of the asymmetry of the probability distribution
kurtosis – quantifies the tail form of the probability distribution

2^{nd} – order statistics

* Used to describe stochastic signals
* Let us start with the formal definitions

Definitions 2^{nd} – order statistics: General

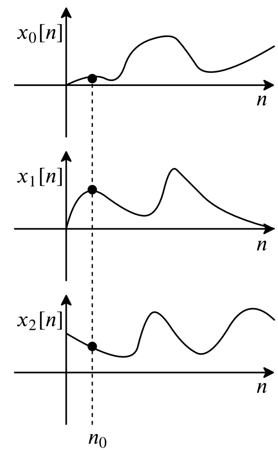
* Statistical properties stochastic process $x[n]$:

Mean: $\mu_x[n] = E\{x[n]\}$

* For the different realizations of the same stochastic process, we take the values corresponding to the same time index n_0 and estimate the mean using

$$\mu_x[n_0] = E\{x[n_0]\} = \frac{1}{K}\sum_{k=0}^{K-1} x_k[n_0]$$

where K is the number of realizations



Definitions 2^{nd} – order statistics: General

* Statistical properties stochastic process $x[n]$:

Mean: $\mu_x[n] = E\{x[n]\}$

Variance: $\sigma_x^2[n] = E\{|x[n] - \mu_x[n]|^2\} = E\{|x[n]|^2\} - E\{|\mu_x[n]|^2\}$

* The variance indicates the spread of the measurements; how far they are away from the mean

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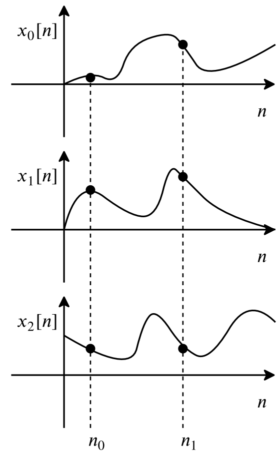
Variance: $\sigma_x^2[n] = E\{|x[n] - \mu_x[n]|^2\}$
$= E\{|x[n]|^2\} - E\{|\mu_x[n]|^2\}$

Autocorrelation: $r_x[n_1, n_2] = E\{x[n_1] \cdot x^*[n_2]\}$

* In a process with many different realizations, autocorrelation describes the relation between n_0 and n_1



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Autocorrelation: $r_x[n_1, n_2] = E\{x[n_1] \cdot x^*[n_2]\}$

Autocovariance: $\gamma_x[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1]) \cdot (x[n_2] - \mu_x[n_2])^*\}$
$= r_x[n_1, n_2] - \mu_x[n_1] \cdot \mu_x^*[n_2]$

* Autocovariance is sometimes easier to use
* It used to measure the same relation, but the average value is subtracted

Definitions 2^{nd} – order statistics: General

* Now consider two processes: $x[n]$ and $y[n]$
* The statistical properties now become statistical relations between the processes

* Statistical relation between stochastic processes $x[n]$ and $y[n]$:

Cross-correlation: $r_{xy}[n_1, n_2] = E\{x[n_1] \cdot y^*[n_2]\}$

Cross-covariance: $\gamma_{xy}[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1]) \cdot (y[n_2] - \mu_y[n_2])^*\}$
$= r_{xy}[n_1, n_2] - \mu_x[n_1] \cdot \mu_y^*[n_2]$

Normalized γ_{xy}: $\rho_{xy}[n_1, n_2] = \frac{\gamma_{xy}[n_1, n_2]}{\sigma_x[n_1] \cdot \sigma_y[n_2]}$

Definitions 2^{nd} – order statistics: General overview

* Statistical properties stochastic process $x[n]$:

Mean: $\mu_x[n] = E\{x[n]\}$

Variance: $\sigma_x^2[n] = E\{|x[n] - \mu_x[n]|^2\} = E\{|x[n]|^2\} - E\{|\mu_x[n]|^2\}$

Autocorrelation: $r_x[n_1, n_2] = E\{x[n_1] \cdot x^*[n_2]\}$

Autocovariance: $\gamma_x[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1]) \cdot (x[n_2] - \mu_x[n_2])^*\}$
$= r_x[n_1, n_2] - \mu_x[n_1] \cdot \mu_x^*[n_2]$

* Statistical relation between stochastic processes $x[n]$ and $y[n]$:

Cross-correlation: $r_{xy}[n_1, n_2] = E\{x[n_1] \cdot y^*[n_2]\}$

Cross-covariance: $\gamma_{xy}[n_1, n_2] = E\{(x[n_1] - \mu_x[n_1]) \cdot (y[n_2] - \mu_y[n_2])^*\}$
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Normalized γ_{xy}: $\rho_{xy}[n_1, n_2] = \frac{\gamma_{xy}[n_1, n_2]}{\sigma_x[n_1] \cdot \sigma_y[n_2]}$

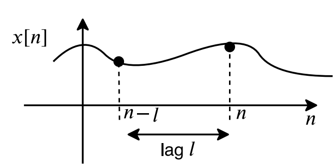
Note: * indicates complex conjugate

2^{nd} – Order statistics: Real and WSS

- We will further assume that the processes are real, so that we do not need the complex conjugation
- We also assume the signals to satisfy the Wide Sense Stationary (WSS) criteria
- WSS signals have time invariant statistics: e.g. the mean at n_0 equals the mean at n_1
- Although many signals are not stationary, in practice we still use this assumption by cutting the signal in parts and assuming stationarity within these parts

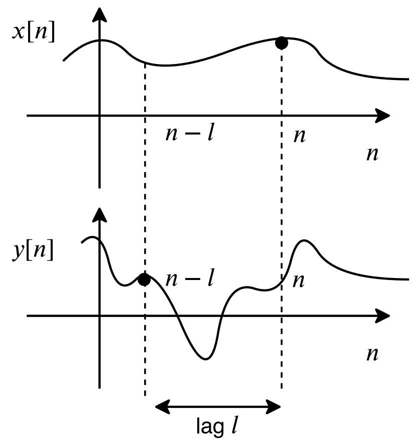
2^{nd} – Order statistics: lag

- For autocorrelation, we now have to define a lag l
- Autocorrelation: $r_x[l] = E\{x[n] \cdot x[n-l]\} = E\{x[n+l] \cdot x[n]\}$
- We take a couple of samples separated by the lag l
- The autocorrelation has the largest value for $l = 0$
- The autocovariance is computed likewise, but we subtract the mean



2^{nd} – Order statistics: lag

- The cross-correlation is defined between $x[n]$ and $y[n]$
- Cross-correlation:
  $r_{xy}[l] = E\{x[n] \cdot y[n-l]\}$
- The cross-covariance is computed likewise, but the product of the means of $x[n]$ and $y[n]$ is subtracted



Definitions 2^{nd} – order statistics: Real and WSS

- Mean: $\mu_x[n] = E\{x[n]\}$
- Variance: $\sigma_x^2 = E\{(x[n]-\mu_x)^2\} = E\{x^2[n]\} - \mu_x^2$
- Autocorrelation: $r_x[l] = E\{x[n] \cdot x[n-l]\} = E\{x[n+l] \cdot x[n]\}$

  Notes: Power $E\{x^2[n]\} = r_x[0] \geq 0$ and $r_x[0] \geq r_x[l] \quad \forall\, l$

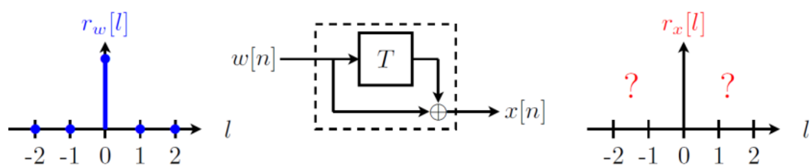
  Symmetry $r_x[l] = r_x[-l]$
- Autocovariance: $\gamma_x[l] = E\{(x[n]-\mu_x) \cdot (x[n-l]-\mu_x)\}$

  $= E\{(x[n+l]-\mu_x) \cdot (x[n]-\mu_x)\} = r_x[l] - \mu_x^2$
- Cross-correlation: $r_{xy}[l] = E\{x[n] \cdot y[n-l]\} = E\{x[n+l] \cdot y[n]\}$
- Cross-covariance: $\gamma_{xy}[l] = E\{(x[n]-\mu_x) \cdot (y[n-l]-\mu_y)\}$

  $= r_{xy}[l] - \mu_x \cdot \mu_y$
- Normalized γ_{xy}: $\rho_{xy}[l] = \frac{\gamma_{xy}[l]}{\sigma_x \cdot \sigma_y}$

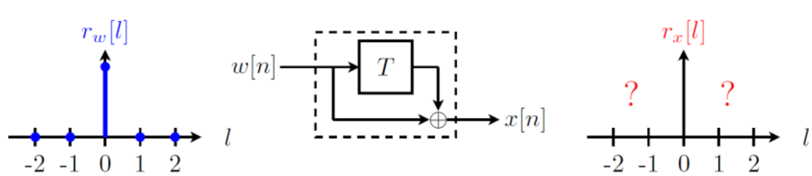
Examples: 1

- The signal $w[n]$ is the input of an LTI system
- $w[n]$ is WSS Gaussian white noise with zero mean $\mu_w = 0$, unit variance $\sigma_w^2 = 1$ and its autocorrelation is $r_w[l] = \sigma_w^2 \cdot \delta[l] = \delta[l]$
- Calculate the mean μ_x and autocorrelation $r_x[l]$ of output $x[n]$



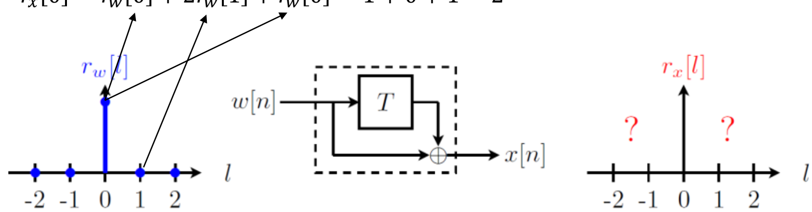
Examples: 1

- Start with the difference equation (DE) of the system: $x[n] = w[n] + w[n-1]$
- $w[n]$ is white noise and zero mean is assumed: $\mu_w = 0$
- Fill in: $\mu_x = E(x[n]) = E(w[n] + w[n-1]) = \underbrace{E(w[n])}_{=\mu_w = 0} + \underbrace{E(w[n-1])}_{=\mu_w = 0 \text{ (WSS} \rightarrow \text{ time invariant)}} = 0$
- We find that $\mu_x = 0$ if $\mu_w = 0$



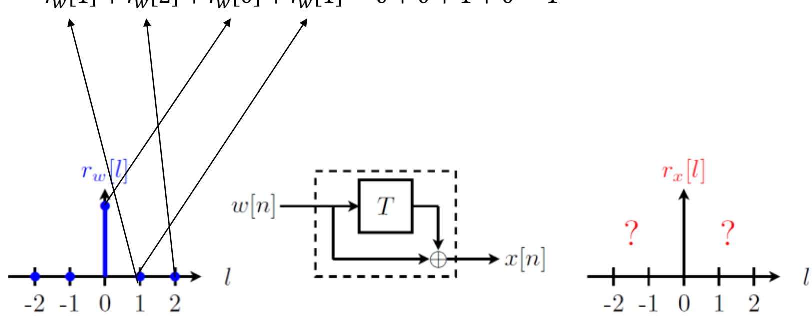
Examples: 1

* To find $r_x[l]$, start with the definition: $r_x[l] = E(x[n] \cdot x[n-l])$
* We have to find $r_x[l]$ for all l: $r_x[l]\ \ \forall\ \ l \in (-\infty;\ +\infty)$, let us start with $l = 0$
* $r_x[0] = E(x[n] \cdot x[n-0]) = E(x^2[n]) \overset{DE}{\triangleq} E\big((w[n] + w[n-1])^2\big)$ $\quad DE$ =Difference Equation
* $= E(w^2[n]) + 2E(w[n] \cdot w[n-1]) + E(w^2[n-1]) = r_w[0] + 2r_w[1] + r_w[0]$
* Note: for $E(w^2[n-1])$ we use the equation for $r_w[0]$, where we fill in $n-1$ for n: $r_w[0] = E(w[n-1] \cdot w[(n-1)-0]) = E(w^2[n-1])$
* $r_x[0] = r_w[0] + 2r_w[1] + r_w[0] = 1 + 0 + 1 = 2$



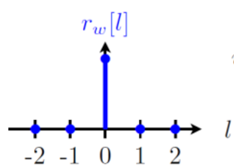
Examples: 1

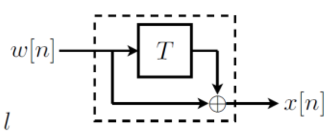
* $r_x[0] = 2$, now let us continue with $l = 1$
* $r_x[1] = E(x[n] \cdot x[n-1]) \overset{DE}{\triangleq} E((w[n] + w[n-1]) \cdot (w[n-1] + w[n-2]))$
* $= E(w[n] \cdot w[n-1]) + E(w[n] \cdot w[n-2]) + E(w^2[n-1]) + E(w[n-1] \cdot w[n-2])$
* $= r_w[1] + r_w[2] + r_w[0] + r_w[1] = 0 + 0 + 1 + 0 = 1$

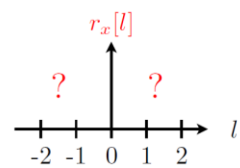


Examples: 1

* $r_x[0] = 2$, $r_x[1] = 1$, now let us continue with $l = 2$
* $r_x[2] = E(x[n] \cdot x[n-2]) \overset{DE}{\triangleq} E((w[n] + w[n-1]) \cdot (w[n-2] + w[n-3]))$
* $= E(w[n] \cdot w[n-2]) + E(w[n] \cdot w[n-3]) + E(w[n-1] \cdot w[n-2]) + E(w[n-1] \cdot w[n-3])$
* $= r_w[2] + r_w[3] + r_w[1] + r_w[2] = 0 + 0 + 0 + 0 = 0$
* It also becomes clear that the value for $l \geq 2$ will be 0 too

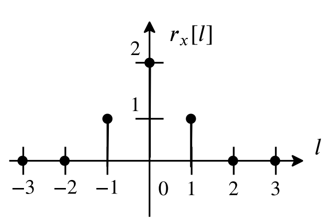






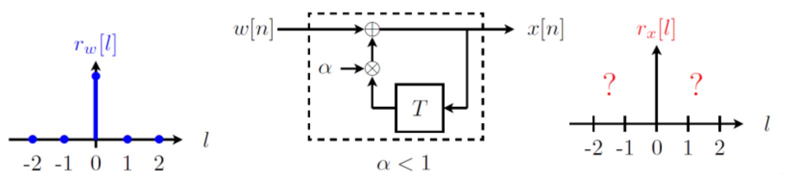
Examples: 1

* $r_x[0] = 2$, $r_x[1] = 1$, $r_x[\geq 2] = 0$, now what happens if $l = -1$?
* $w[n]$ is wide sense stationary → time invariant
* $r_x[-1] = E(x[n] \cdot x[n+1]) = E(x[n-1] \cdot x[n]) = r_x[1] = 1$
* Autocorrelation is symmetric
* We can plot $r_x[l]$ $\forall$ $l \in (-\infty; +\infty)$



Examples: 2

* The signal $w[n]$ is the input of a system shown below
* $w[n]$ is WSS Gaussian white noise with zero mean $\mu_w = 0$, variance $\sigma_w^2 = 1$, and its autocorrelation $r_w[l]$ is shown on the plot below
* Calculate the autocorrelation $r_x[l]$ of output $x[n]$

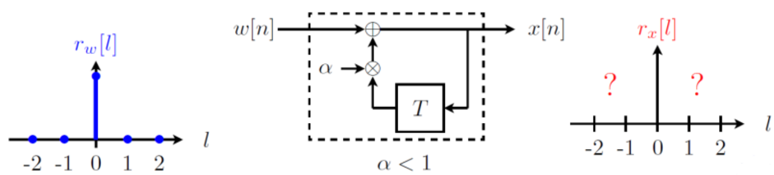


Examples: 2

* Let us start with $l = 0$
* By definition: $r_x[0] = E(x[n] \cdot x[n-0]) = E(x^2[n])$
* Find the difference equation: $x[n] = w[n] + \alpha x[n-1]$
* $E(x^2[n]) \overset{DE}{\triangleq} E\big((w[n] + \alpha x[n-1])^2\big)$
* $= E(w^2[n]) + 2\alpha E(w[n] \cdot x[n-1]) + \alpha^2 E(x^2[n-1])$
* Let us break this up in terms

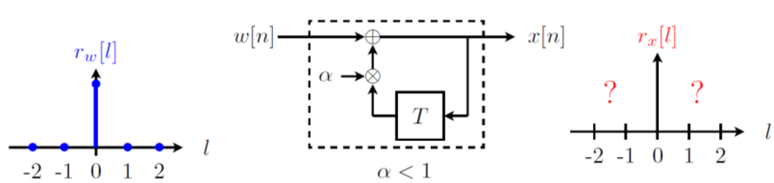
Examples: 2

- $E(w^2[n]) + 2\alpha E(w[n] \cdot x[n-1]) + \alpha^2 E(x^2[n-1])$
- $E(w^2[n]) = r_w[0] = 1$
- $2\alpha E(w[n] \cdot x[n-1])$ → α is a constant and the statistics of $x[n-1]$ are unknown
- Let us find an expression for $E(w[n] \cdot x[n-1])$ through the DE for $x[n-1]$
- $E(w[n] \cdot x[n-1]) \overset{DE}{\triangleq} E\big(w[n](w[n-1] + \alpha x[n-2])\big)$



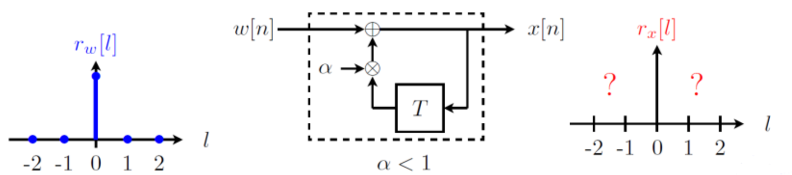
Examples: 2

- $E(w[n] \cdot x[n-1]) \overset{DE}{\triangleq} E\big(w[n](w[n-1] + \alpha x[n-2])\big)$
- $= \underbrace{E(w[n] \cdot w[n-1])}_{= r_w[1] = 0} + \alpha E(w[n] \cdot x[n-2]) = 0 + \alpha E(w[n] \cdot x[n-2])$
- We could repeat this step and involve samples $x[n-3]$, $x[n-4]$, etc., so from sample $x[n-1]$ we notice that $w[n-1]$, $w[n-2]$, etc. contribute to it
- Therefore it is a feedback loop: $x[n-1] = \sum_{p=1}^{\infty} w[n-p]$



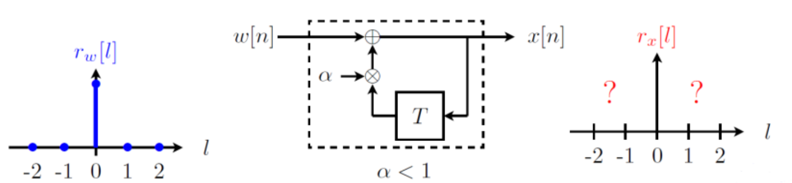
Examples: 2

- Since $x[n-1] = \sum_{p=1}^{\infty} w[n-p]$, we find $E(w[n]x[n-1]) = 0$
- Now let us express the final term: $\alpha^2 E(x^2[n-1])$
- WSS: $\alpha^2 E(x^2[n-1]) = \alpha^2 E(x^2[n]) = \alpha^2 r_x[0]$
- We find $r_x[0] = 1 + 0 + \alpha^2 r_x[0] \rightarrow r_x[0] - \alpha^2 r_x[0] = 1 \rightarrow r_x[0](1-\alpha^2) = 1$
- $r_x[0] = \frac{1}{1-\alpha^2}$



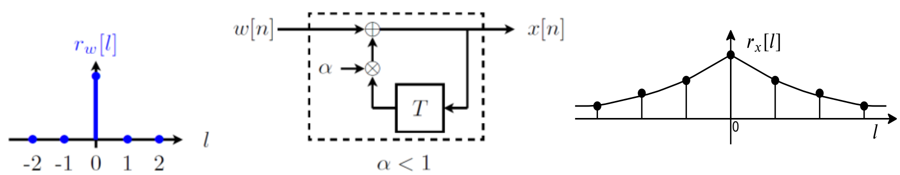
Examples: 2

- $r_x[0] = \frac{1}{1-\alpha^2}$
- Now let us look at $l = 1$
- $r_x[1] = E(x[n] \cdot x[n-1]) \stackrel{DE}{\triangleq} E\big((w[n] + \alpha x[n-1]) \cdot x[n-1]\big)$
  $= \underbrace{E(w[n] \cdot x[n-1])}_{=\,0\text{, as described before}} + \alpha E(x^2[n-1]) = 0 + \alpha r_x[0]$
- For $l = 2$, we obtain $r_x[2] = \alpha r_x[1] = \alpha^2 r_x[0]$



Examples: 2

- We find a general solution:
  $r_x[l] = \alpha^l \cdot \frac{1}{1-\alpha^2}$ for $l > 0$ and $r_x[l] = \alpha^{|l|} \cdot \frac{1}{1-\alpha^2}$ for $l < 0$
- We can see that with one recursive feedback loop the autocorrelation becomes infinitely long, as shown in the lower right corner
- The filtering process influences the statistics of the signal



Ergodicity

- An ensemble of measurements may not be available
- A process is ergodic if the statistics can be found from one single realization
- Only stationary signals can be ergodic
- Stationarity ensures time invariance of statistics of random signal
- Ergodicity implies that the statistics can be calculated by time-averaging over a single representative member of the ensemble.
  Practice: Number of measured samples is limited to, say, N
  → "Replace" ensemble-averaging by time-averaging:

$$E\{\cdot\} = \frac{1}{N}\sum_{n=0}^{N-1}(\cdot)$$

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Definitions 2^{nd}-order statistics: Practice

- Statistics in case of ergodic signals
- Mean: $\hat{\mu}_x[n] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$
- Variance: $\hat{\sigma}_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x[n] - \hat{\mu}_x)^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n] - \hat{\mu}_x{}^2$
- Autocorrelation: $\hat{r}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|}(x[n]x[n+|l|])$ for $|l| \leq L-1$
- Autocovariance: $\hat{\gamma}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|}(x[n] - \hat{\mu}_x)(x[n+|l|] - \hat{\mu}_x)$
  $= \hat{r}_x[l] - (\frac{N-|l|}{N})\hat{\mu}_x^2$
- Red indicates that the equations are biased

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Stochastic Signal Processing

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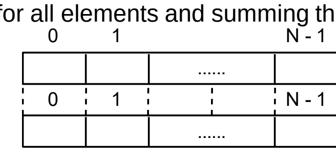
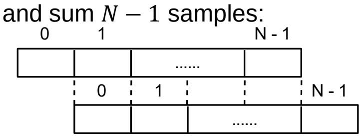
Definitions 2^{nd}-order statistics: Practice

- Autocorrelation: $\hat{r}_x[l] = \frac{1}{N}\sum_{n=0}^{N-1-|l|}(x[n] \cdot x[n+|l|])$ for $|l| \leq L-1$
- Autocorrelation for $l = 0$ will mean multiplying the signal with its copy element by element for all elements and summing the results:



- For $l = 1$ we multiply and sum $N-1$ samples:



- Lags are often taken lower than $N/4$

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Stochastic Signal Processing

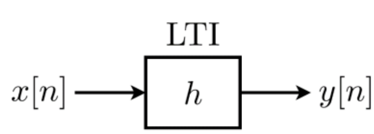
Definitions 2^{nd}-order statistics: Practice

- Statistics in case of ergodic signals
- Cross-correlation: $\hat{r}_{xy}[l] = \frac{1}{N}\sum_{n=0}^{N-1-l}(x[n]y[n+l]);\ 0 \le l \le L-1$
  $\hat{r}_{xy}[l] = \frac{1}{N}\sum_{n=|l|}^{N-1}(x[n]y[n+l]);\ -(L-1) \le l \le 0$
- Cross-correlation is not a symmetric function
- Cross-covariance: $\hat{\gamma}_{xy}[l] = \frac{1}{N}\sum_{n=0}^{N-1-l}(x[n]-\hat{\mu}_x)(y[n+l]-\hat{\mu}_y)$
  $= \hat{r}_{xy}[l] - \hat{\mu}_x \cdot \hat{\mu}_y;\ 0 \le l \le L-1$
- Normalized $\hat{\gamma}_{xy}$: $\hat{\rho}_{xy}[l] = \frac{\hat{\gamma}_{xy}[l]}{\hat{\sigma}_x \cdot \hat{\sigma}_y}$

- Red indicates that the equations are biased

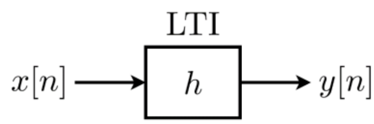
Linear systems with stationary inputs

- We have previously considered examples in which we calculated the autocorrelation of the output signal of an LTI system
- Input $x[n]$ is stationary and system h BIBO → $y[n]$ stationary
- Now we will look at a more general way: how can we find the statistics of $y[n]$ if the statistics of $x[n]$ are given?



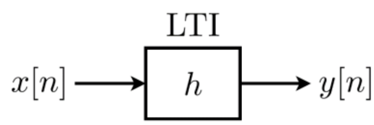
Linear systems with stationary inputs

- We can express $y[n]$ through a convolution: $y[n] = x[n] * h[n]$
- Apply convolution equation: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- Assume the autocorrelation function of $x[n]$ is $r_x[l]$ (no delta pulse)
- We want to express $r_y[l]$ as a function of $r_x[l]$ and $h[n]$
- Start with the definition: $r_y[l] = E(y[n] \cdot y[n-l])$
- Use the convolution equation for $y[n]$: $r_y[l] = E(\sum_k h[k]x[n-k] \cdot y[n-l])$
- Ensemble average is used for stochastic signals x and y, but h is deterministic
- We take out the sum and $h[k]$: $r_y[l] = \sum_k h[k] \cdot E(x[n-k] \cdot y[n-l])$



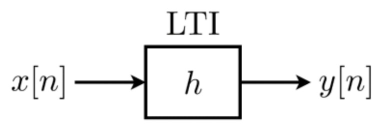
Linear systems with stationary inputs

- We take out the sum and $h[k]$: $r_y[l] = \sum_k h[k] \cdot E(x[n-k] \cdot y[n-l])$
- We notice that $x[n-k] \cdot y[n-l]$ is the cross-correlation between x and y with a lag of $n-k-(n-l) = l-k$, and fill this in: $r_y[l] = \sum_k h[k] \cdot r_{xy}[l-k]$
- In the product $h[k] \cdot r_{xy}[l-k]$, we can see that $h[n]$ is a sequence that runs through k and $r_{xy}[l-k]$ is a mirrored sequence for k → this is a convolution
- $r_y[l] = h[l] * r_{xy}[l]$, where we do not know the cross-correlation r_{xy}
- Let us use the definition to find $r_{xy}[l] = E(x[n] \cdot y[n-l])$
- As $y[n]$ is a result of convolution, we can substitute $y[n-l]$:
  $r_{xy}[l] = E(x[n] \cdot \sum_k h[k] \cdot x[n-l-k])$ (the k comes from convolution equation)



Linear systems with stationary inputs

- $r_{xy}[l] = E(x[n] \cdot \sum_k h[k] \cdot x[n-l-k])$ (the k comes from convolution equation)
- Again we take the sum and h out: $r_{xy}[l] = \sum_k h[k] \cdot E(x[n] \cdot x[n-l-k])$
- We notice the autocorrelation $r_x[l+k] = E(x[n] \cdot x[n-l-k])$
- Thus we find $r_{xy}[l] = \sum_k h[k] \cdot r_x[l+k] = h[-l] * r_x[l]$, because:
- $h[-l] * r_x[l] = h[l] * r_x[-l] = \sum_k (h[k] \cdot r_x[-l-k]) = \sum_k (h[k] \cdot r_x[l+k])$ (symmetry)
- So finally we can express $r_y[l]$:
- $r_y[l] = h[l] * r_{xy}[l] = h[l] * h[-l] * r_x[l] = r_h[l] * r_x[l]$



Power Spectral Density (PSD): Definition

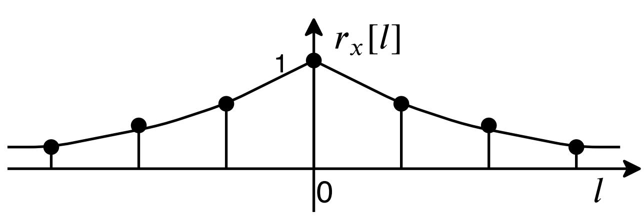
- The Power Spectral Density and autocorrelation are an FTD pair:
$$P_x(e^{j\theta}) = \sum_{l=-\infty}^{\infty} r_x[l] e^{-j\theta l} \multimap r_x[l] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{j\theta}) e^{j\theta l} d\theta$$
- This is called the Wiener-Kintchine relation
- The statistics can be described in the lag domain
- If the lag domain is transformed to the frequency domain, the PSD is found

Properties PSD

* $P_x(e^{j\theta})$ is a real-valued periodic function of frequency (period 2π)
* If the underlying process $x[n]$ is real, there is symmetry in the PSD: $P_x(e^{j\theta}) = P_x(e^{-j\theta})$
* PSD is nonnegative: $P_x(e^{j\theta}) \geq 0$
* The average power of the signal $x[n]$ is its autocorrelation function for a lag of 0: $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_x(e^{j\theta})\, d\theta = r_x[0] = E\{|x[n]|^2\} \geq 0$

Example

* Calculate $P_x(e^{j\theta})$ of the autocorrelation function $r_x[l] = a^{|l|}$ with $|a| < 1$
* The autocorrelation function looks like the graph below



Example

* We can use the Wiener-Kintchine relation to calculate the PSD function $P_x(e^{j\theta})$ using the autocorrelation function $r_x[l]$. Apply FTD:

$$P_x\left(e^{j\theta}\right) = \sum_{l=-\infty}^{\infty} a^{|l|} e^{-jl\theta}$$

* We can split this sum in two parts:

$$P_x\left(e^{j\theta}\right) = \sum_{l=-\infty}^{-1} a^{|l|} e^{-jl\theta} + \sum_{l=0}^{\infty} a^{|l|} e^{-jl\theta}$$

* We want to express $|l|$, so we look at the bounds of the sums:

$$P_x\left(e^{j\theta}\right) = \sum_{l=-\infty}^{-1} a^{-l} e^{-jl\theta} + \sum_{l=0}^{\infty} a^{l} e^{-jl\theta}$$

Example

* Now the exponent $a^{|l|}$ will be positive for all l

$$P_x\left(e^{j\theta}\right) = \sum_{l=-\infty}^{-1} a^{-l} e^{-jl\theta} + \sum_{l=0}^{\infty} a^{l} e^{-jl\theta}$$

* The terms in the equation above consist of two summation series that we know the solution to, if the bounds go from 0 to ∞. Therefore we add the element for $l = 0$ also to the left term by adding 1 to the upper bound. We then have to subtract the value (1) for this element because otherwise we would add it twice:

$$P_x\left(e^{j\theta}\right) = \sum_{l=-\infty}^{0} a^{-l} e^{-jl\theta} + \sum_{l=0}^{\infty} a^{l} e^{-jl\theta} - 1$$

Example

- We can then replace the negative values of l with another index, say $-p$:

$$P_x\left(e^{j\theta}\right) = \sum_{p=0}^{\infty} a^p e^{jp\theta} + \sum_{l=0}^{\infty} a^l e^{-jl\theta} - 1$$
$$= \sum_{p=0}^{\infty} (ae^{j\theta})^p + \sum_{l=0}^{\infty} (ae^{-j\theta})^l - 1$$

- We can use the following series: $\sum_{n=0}^{\infty} (z_0)^n = \frac{1}{1-z_0}$:

$$P_x\left(e^{j\theta}\right) = \frac{1}{1-ae^{j\theta}} + \frac{1}{1-ae^{-j\theta}} - 1$$

Example

$$P_x\left(e^{j\theta}\right) = \frac{1}{1-ae^{j\theta}} + \frac{1}{1-ae^{-j\theta}} - 1$$

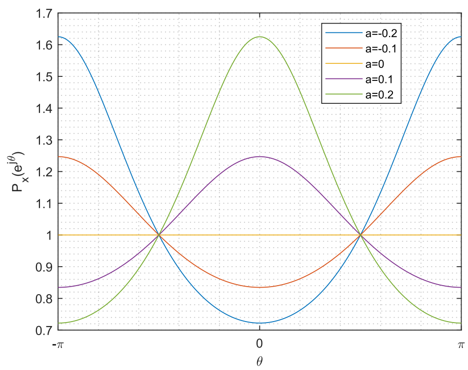
- We can rewrite this expression through finding the common denominator and Euler's expression to:

$$P_x\left(e^{j\theta}\right) = \frac{1+a^2}{1+a^2-2a\cos\theta}$$

Example

$$P_x(e^{j\theta}) = \frac{1+a^2}{1+a^2-2a\cos\theta}$$

* The plot below shows this function for some values of a, $|a| < 1$



Summary

* We have defined discrete time stochastic processes
* These processes can be described with 2nd-order statistics
  – The statistical properties of one process
  – The statistical relation between two processes
  – Wide Sense Stationarity (WSS) and assumed real
  – Lag
* We discussed ergodicity
* We derived a relation between the autocorrelation of the output and the autocorrelation of the input
* Power spectral density – signal characterization in frequency domain