Appendix A

Formulae Electromagnetics II

A.1 Vector algebra

Let **a**, **b**, **c** and **d** denote arbitrary three-dimensional vectors. They satisfy the following identies

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a},$$
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}),$$
$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}),$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},$$
$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

A.2 Differentiation of scalar and vector fields

Let $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ denote unit vectors in a cartesian system of coordinates. Φ is a scalar field, and **A** a vector field, satisfying the following relations

$$\nabla\Phi = \mathbf{a}_x \frac{\partial\Phi}{\partial x} + \mathbf{a}_y \frac{\partial\Phi}{\partial y} + \mathbf{a}_z \frac{\partial\Phi}{\partial z},$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z},$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right),$$

$$\Delta\Phi = \nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)\Phi.$$

A.3 Vector identities

Let us introduce a second scalar field Ψ and a second vector field **B**. We have

$$\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi,$$
$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$
$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\Phi\Psi) = \Phi\,\nabla\Psi + \Psi\nabla\Phi,$$
$$\nabla \cdot (\Phi\mathbf{A}) = \Phi\,\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\Phi,$$
$$\nabla \times (\Phi\mathbf{A}) = \Phi\,\nabla \times \mathbf{A} - \mathbf{A} \times \nabla\Phi,$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + \mathbf{A}\nabla \cdot \mathbf{B},$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A},$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\Phi = \mathbf{0},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \tfrac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

$$\nabla_t \cdot (\Phi\nabla_t\Phi) = \Phi\nabla_t^2\Phi + (\nabla_t\Phi) \cdot (\nabla_t\Phi).$$

A.4   Integral relations

For an arbitrary volume $\mathcal{V}$, bounded by a surface $\mathcal{S} = \partial\mathcal{V}$ with unit normal $\mathbf{n}$ pointing outward from $\mathcal{V}$, we have

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{A} \, dV = \int_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{S}, \qquad \text{(Gauss' theorem)}$$

$$\int_{\mathcal{V}} \nabla\Phi \, dV = \int_{\mathcal{S}} \Phi \, d\mathbf{S},$$

$$\int_{\mathcal{V}} \nabla \times \mathbf{A} \, dV = -\int_{\mathcal{S}} \mathbf{A} \times d\mathbf{S}.$$

By combining Gauss' law and a relation from Section A.3, one may deduce that

$$\int_{\mathcal{V}} (\Phi\Delta\Psi - \Psi\Delta\Phi) \, dV = \int_{\mathcal{S}} (\Phi\nabla\Psi - \Psi\nabla\Phi) \cdot d\mathbf{S},$$
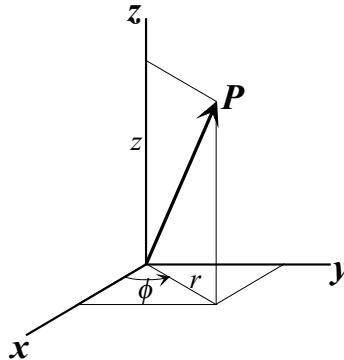
(Green's theorem).

For an orientable surface $\mathcal{S}$ with normal $\mathbf{n}$ bounded by a boundary curve $\mathcal{C} = \partial\mathcal{S}$ with tangent vector $\boldsymbol{\tau}$ and outward normal vector $\boldsymbol{\nu}$, we have

$$\int_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{A} \cdot d\boldsymbol{\ell}, \qquad \text{(Stokes' theorem)}$$

$$\int_{\mathcal{S}} \nabla_t \cdot \mathbf{A} \, dS = \oint_{\mathcal{C}} \mathbf{A} \cdot \boldsymbol{\nu} \, d\ell. \qquad \text{(Gauss' theorem in two dimensions)}$$

Here, $\boldsymbol{\nu}$ is the normal to the boundary $\mathcal{C}$.

A.5   Coordinate systems

Cylindrical polar coordinates (r, φ, z)

Coordinates: $x = r\cos(\varphi)$, $y = r\sin(\varphi)$, $z = z$.
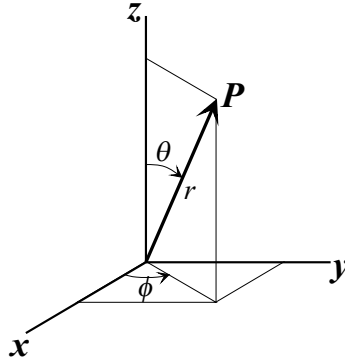Line elements: $\mathrm{d}r$, $r\,\mathrm{d}\varphi$, $\mathrm{d}z$.
Vector operators:

$$\nabla\Psi = \mathbf{a}_r \frac{\partial\Psi}{\partial r} + \mathbf{a}_\varphi \frac{1}{r}\frac{\partial\Psi}{\partial\varphi} + \mathbf{a}_z \frac{\partial\Psi}{\partial z},$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z},$$

$$\nabla\times\mathbf{A} = \mathbf{a}_r\left[\frac{1}{r}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right] + \mathbf{a}_\varphi\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] + \mathbf{a}_z\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial\varphi}\right],$$

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{\partial^2\Psi}{\partial z^2}.$$

Spherical polar coordinates (r, θ, φ)

Coordinates: $x = r\sin(\theta)\cos(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$, $z = r\cos(\theta)$.
Line elements: $\mathrm{d}r$, $r\,\mathrm{d}\theta$, $r\sin(\theta)\,\mathrm{d}\varphi$.
Vector operators:

$$\nabla\Psi = \mathbf{a}_r \frac{\partial\Psi}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \mathbf{a}_\varphi \frac{1}{r\sin(\theta)}\frac{\partial\Psi}{\partial\varphi},$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}[\sin(\theta)A_\theta] + \frac{1}{r\sin(\theta)}\frac{\partial A_\varphi}{\partial\varphi},$$

$$\nabla\times\mathbf{A} = \mathbf{a}_r\frac{1}{r\sin(\theta)}\left\{\frac{\partial}{\partial\theta}[\sin(\theta)A_\varphi] - \frac{\partial A_\theta}{\partial\varphi}\right\} +$$
$$\mathbf{a}_\theta\left[\frac{1}{r\sin(\theta)}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right] + \mathbf{a}_\varphi\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right],$$

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\Psi}{\partial\varphi^2}.$$

A.6   Miscellaneous formulae

Taylor approximations:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4),$$

$$\sin x = x - \frac{x^3}{6} + O(x^5),$$

$$\cos x = 1 - \frac{x^2}{2} + O(x^4).$$

Geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x| < 1.$$

A.7 Reflection and refraction

$$\theta_1 = \theta_1' \qquad\qquad \frac{\sin \theta_1}{\sin \theta_2} = \frac{k_2}{k_1}$$

$$\Gamma^s = \frac{Z_2 \cos \theta_1 - Z_1 \cos \theta_2}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2} \qquad \tau^s = \frac{2Z_2 \cos \theta_1}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2}$$

$$\Gamma^p = \frac{Z_2 \cos \theta_2 - Z_1 \cos \theta_1}{Z_2 \cos \theta_2 + Z_1 \cos \theta_1} \qquad \tau^p = \frac{2Z_2 \cos \theta_1}{Z_2 \cos \theta_2 + Z_1 \cos \theta_1}$$

$$\tan^2 \theta_B = \frac{\varepsilon_2}{\varepsilon_1} \qquad\qquad \sin \theta_c = \frac{n_2}{n_1}$$

A.8 Waveguides

$$E_x = \frac{-j\omega\mu}{k_t^2} \partial_y H_z + \frac{1}{k_t^2} \partial_z \partial_x E_z \quad H_y = \frac{1}{k_t^2} \partial_z \partial_y H_z + \frac{-j\omega\varepsilon}{k_t^2} \partial_x E_z$$

$$E_y = \frac{j\omega\mu}{k_t^2} \partial_x H_z + \frac{1}{k_t^2} \partial_z \partial_y E_z \quad H_x = \frac{1}{k_t^2} \partial_z \partial_x H_z + \frac{j\omega\varepsilon}{k_t^2} \partial_y E_z$$

$$= \sqrt{k_t^2 - k^2}$$

Where κ_t is either κ_m, κ_{mp} or κ_{mn} depending on the problem under consideration.

$$\frac{\omega_m}{c} = \frac{m\pi}{a}$$

$$\frac{\omega_{mn}}{c} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\frac{\omega_{mnp}}{c} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

A.9 Antennas

$$\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Psi = 0$$

$$G_d = \frac{\overline{S_h}(\theta,\phi)_{\max}}{\overline{P_h}/(4\pi r^2)}$$

$$F_{\text{array}} = \frac{\sin^2\left(\frac{N\Psi}{2}\right)}{\sin^2\left(\frac{\Psi}{2}\right)} \qquad\qquad \Psi = \beta d \cos\varphi + \alpha$$

$$R_{\text{rad}} = \frac{1}{|I_0|^2} \oiint_{S_{\text{gen}}} \text{Re}(\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{a}_r \, dA$$

A.10 Miscellaneous formulae part 1

A-Matrix

$$\mathbf{A} = \mathbf{D_1 T C_2} = \frac{1}{2} \begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ Y_2 & -Y_2 \end{pmatrix}$$

Effective impedance

$$Z_{in} = Z_{\text{eff}}(-\ell) = Z_1 \frac{Z_2 + jZ_1 \tan(k_1\ell)}{Z_1 + jZ_2 \tan(k_1\ell)}$$

T-matrix

$$\mathbf{T}(\mathbf{0}, \ell) = \begin{pmatrix} \cos(k\ell) & jZ\sin(k\ell) \\ jY\sin(k\ell) & \cos(k\ell) \end{pmatrix}$$

Wave number

$$k = \omega\sqrt{\epsilon\mu} = \beta - j\alpha$$