Markov modeling, discrete-event simulation – Exercises module B1

5XIE0 Computational Modeling

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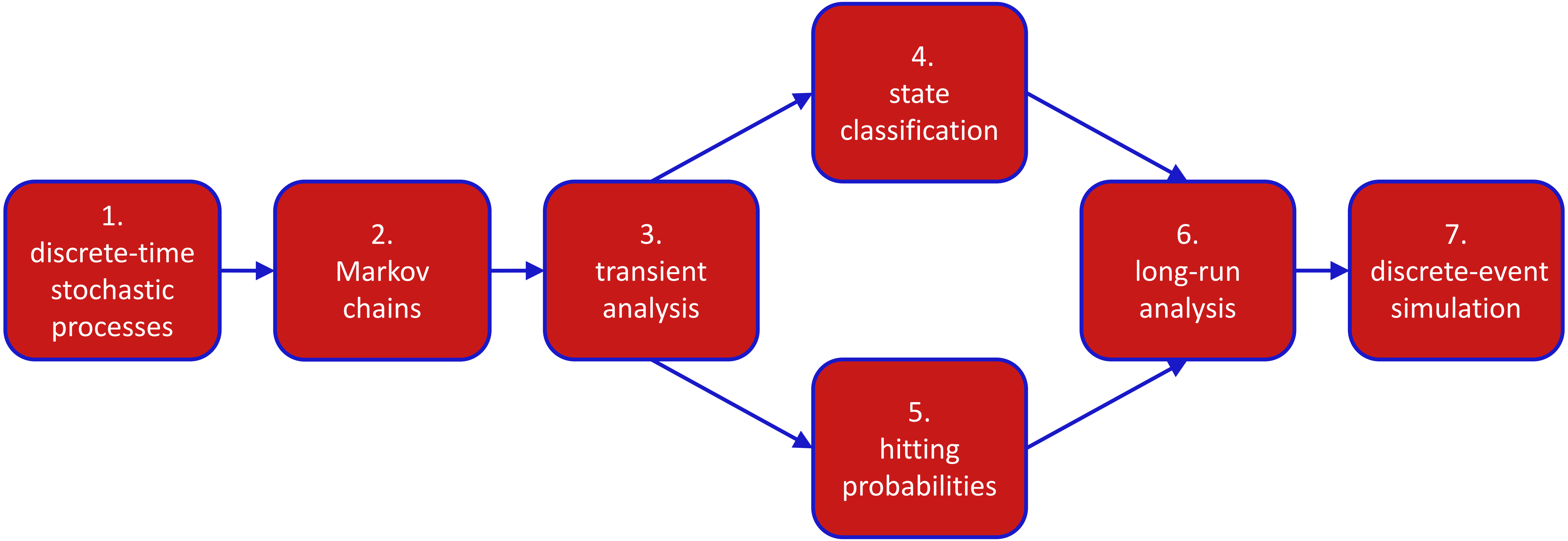
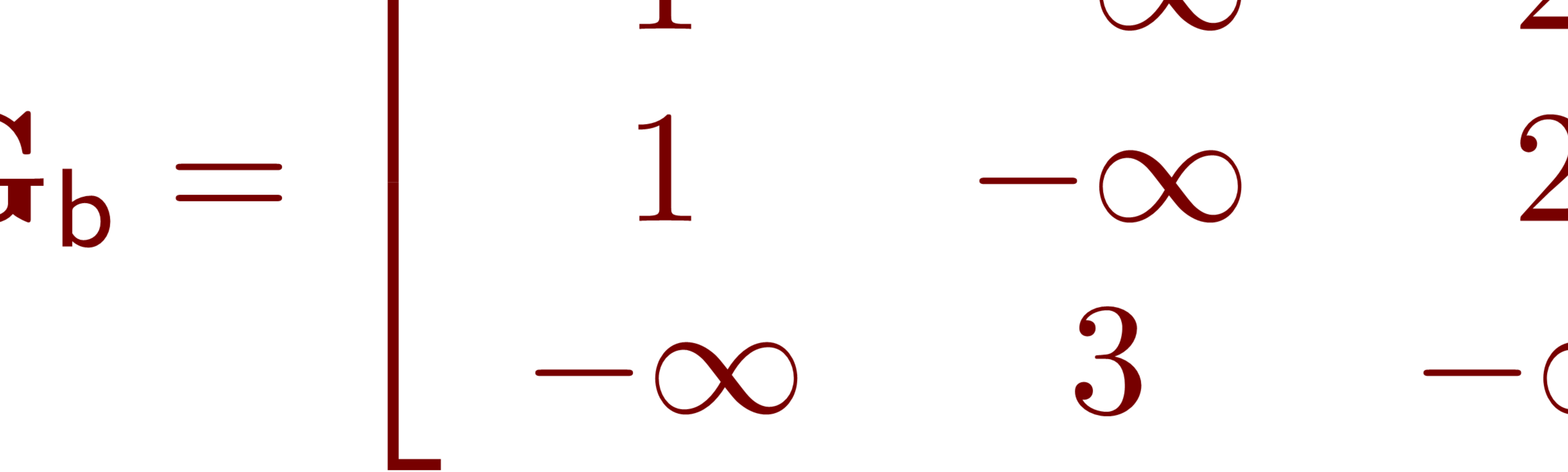
module B - submodules and dependencies





B.1 - discrete-time stochastic processes

discrete-time stochastic processes – exercises

- Section B.1 in the course notes
  - Exercise B.1 (Wealthy gambler – expected reward)
  - Exercise B.2 (Time-slotted Ethernetwork – throughput)
  - Exercise B.3 (Expected reward – computation)
- answers are provided in Section B.8 of the course notes

Exercise B.1 (Wealthy gambler – expected reward)

Exercise B.1 (Wealthy gambler - expected reward). A very wealthy gambler (having a practically unlimited amount of cash) is playing a game of roulette at the casino. The pockets of the roulette wheel are numbered from 0 to 36 where odd numbers are red, even numbers are black and number 0 is green. At each spin the gambler places €500 on red. If the outcome of the spin is red the player earns €500 (and receives the €500 on this game back) and otherwise the €500 are lost.

(a) Model this game as stochastic process, explain how the state-space of the process is defined and give the probability distributions.

(b) Define a reward function that models the amount of money earned or lost after each spin of the wheel.

(c) Compute the expected reward earned after the fifth spin of the wheel.

(d) Compute the probability that the player loses 20 times in a row.

Exercise B.1 (Wealthy gambler - expected reward).

(a) The game can be modelled as a stochastic process $X_0, X_1, \cdots$ with state-space $\mathcal{S} = \{0, 1, \cdots, 36\}$, where state $i \in \mathcal{S}$ corresponds to pocket number i. X_n represents the pocket that the roulette ball enters after spin $n+1$. The probability that the roulette ball ends in pocket i equals $\frac{1}{37}$. Hence $P(X_n = i) = \pi_i^{(n)} = \frac{1}{37}$.

(b) The players earns €500 when the ball enters one of the pockets $1, 3, \cdots 35$, and loses €500 otherwise. This can be modelled by a reward $r : \mathcal{S} \to \mathbb{R}$ with $r(i) = 500$ for $i = 1, 3, \cdots, 35$ and $r(i) = -500$ for $i = 0, 2, 4, \cdots, 36$.

(c) The expected reward at any time n (so also for $n = 4$) is given by $E(r(X_n)) = \pi^{(n)} r^T$ $= 18 \cdot \frac{1}{37} \cdot 500 + 19 \cdot \frac{1}{37} \cdot -500 = -\frac{500}{37} \approx -13.5$ €.

(d) The loss probability per spin equals $\frac{19}{37}$. Since random variables X_n are independent, the probability of losing 20 times in a row is given by $\frac{19}{37}^{20} \approx 1.6 \cdot 10^{-6}$.

$$\pi_i^{(n)} = P(X_n = i) \qquad \text{(B.2)}$$

The expected reward at time n is given by $E(r(X_n)) = \pi^{(n)} r^T$ (B.5)

Exercise B.2 (Time-slotted Ethernetwork - throughput)

Exercise B.2 (Time-slotted Ethernetwork - throughput). Two senders, A and B, offer frames to a time-slotted Ethernetwork. During each time slot sender A offers a frame with probability $\frac{1}{3}$ and sender B offers a frame with probability $\frac{1}{5}$. When two frames are offered during the same time slot, both frames are lost. When precisely one frame is offered, this frame is transmitted. When no frame is offered, the slot remains idle.

(a) Model this communication system as a stochastic process, explain the state-space of the process and give the probability distribution.

(b) Define a reward denoting the number of transmitted frame during a time slot.

(c) Compute the probability that the medium does not transmit a frame during time slot n.

(d) Compute the throughput of the medium, i.e. the expected number of transmitted frames per time slot.

Exercise B.2 (Time-slotted Ethernetwork - throughput).

(a) The communication system can be modelled as a stochastic process $X_0, X_1, \cdots$ with state-space $\mathcal{S} = \{0, 1, 2\}$, where state $i \in \mathcal{S}$ corresponds to the number of packets offered to the Ethernetwork. The probability distribution is as follows: $P(X_n = 0) = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$, $P(X_n = 1) = \frac{1}{3} \cdot \frac{4}{5} + \frac{2}{3} \cdot \frac{1}{5} = \frac{6}{15}$ and $P(X_n = 2) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$. Hence $\pi^{(n)} = [\frac{8}{15}, \frac{6}{15}, \frac{1}{15}]$.

(b) The number of transmitted frames during a time slot can be modelled by reward $r : \mathcal{S} \to \mathbb{R}$ defined by $r(0) = r(2) = 0$ and $r(1) = 1$.

(c) The probability that the medium does not transmit during time slot n is given by $P(r(X_n) = 0)$. This expression is computed by $\sum_{i \in S, r(i)=0} P(X_n = i) = P(X_n = 0) + P(X_n = 2) = \frac{8}{15} + \frac{1}{15} = \frac{3}{5}$.

(d) The expected number of transmitted frames per time slot is computed by $E(r(X_n)) = \pi^{(n)} r^T = [\frac{8}{15}, \frac{6}{15}, \frac{1}{15}][0, 1, 0]^T = \frac{6}{15} = \frac{2}{5}$ frames per time slot.

The probability that $r(X_n)$ takes value $v \in \mathcal{R}$, written $P(r(X_n) = v)$, is given by $P(r(X_n) = v) = \sum_{i \in S, r(i)=v} P(X_n = i)$ (B.4)

The expected reward at time n is given by $E(r(X_n)) = \pi^{(n)} r^T$ (B.5)

Exercise B.3 (Expected reward - computation)

Exercise B.3 (Expected reward - computation). Show that equation (B.5) holds.

$$\text{The expected reward at time } n \text{ is given by } E(r(X_n)) = \pi^{(n)} r^T \tag{B.5}$$

Exercise B.3 (Expected reward - computation). The expected reward at time n is given by $E(r(X_n)) = \sum_{v \in \mathcal{R}} v \cdot P(r(X_n) = v)$. By using (B.4) this expression is rewritten as $\sum_{v \in \mathcal{R}} v \cdot \sum_{i \in \mathcal{S}, r(i)=v} P(X_n = i)$. By using (B.2) we obtain $\sum_{v \in \mathcal{R}} v \cdot \sum_{i \in \mathcal{S}, r(i)=v} \pi_i^{(n)} = \sum_{v \in \mathcal{R}} \sum_{i \in \mathcal{S}, r(i)=v} v \cdot \pi_i^{(n)} = \sum_{v \in \mathcal{R}} \sum_{i \in \mathcal{S}, r(i)=v} r(i) \cdot \pi_i^{(n)} = \sum_{i \in \mathcal{S}} r(i) \cdot \pi_i^{(n)}$. The latter expression equals $\pi^{(n)} r^T$ when r is considered to be the row vector $[r(1), r(2), \cdots, r(N)]$.

The probability that $r(X_n)$ takes value $v \in \mathcal{R}$, written $P(r(X_n) = v)$, is given by $P(r(X_n) = v) = \sum_{i \in S, r(i)=v} P(X_n = i)$ (B.4)

$$\pi_i^{(n)} = P(X_n = i) \tag{B.2}$$