Communication Theory (5ETB0)
Module 10.2

Alex Alvarado
a.alvarado@tue.nl

Information and Communication Theory Lab
Signal Processing Systems Group
Department of Electrical Engineering
Eindhoven University of Technology, The Netherlands

www.tue.nl/ictlab/

Module 10.2

Presentation Outline

Part I The Nyquist Criterion

Part II Orthonormal Pulses and Bandwidth

Part III Receiver Implementation

Orthonormal Pulses: the Nyquist Criterion

Problem and Solution

Problem: Inter-symbol interference.
Solution: Pulses $p(t)$ such that all time shifts by T [s] of the $p(t)$ form an orthonormal basis.

The Nyquist Result in the Time domain

- Pulse $p(t)$ has to satisfy for integer k and k'

$$\int_{-\infty}^{\infty} p(t-kT)p(t-k'T)dt = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases}$$

- Equivalently:

$$\int_{-\infty}^{\infty} p(\alpha)p(\alpha-kT)d\alpha = p(t)*p(-t)\Big|_{t=kT} = h(kT) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

  where $h(t) \triangleq p(t)*p(-t)$.
- Time-domain restriction on the pulse $p(t)$ is called zero-forcing (ZF) criterion

Orthonormal Pulses: the Nyquist Criterion

The Nyquist Result in the Frequency Domain

Let $H(f)$ be the Fourier transform of $h(t) = p(t) * p(-t)$, where $H(f) = P(f)P^*(f) = |P(f)|^2$. The Nyquist criterion in the frequency domain is then

$$Z(f) = \frac{1}{T} \sum_{m=-\infty}^{\infty} H(f + m/T) = 1 \qquad \text{for all } f,$$

or equivalently

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} |P(f + m/T)|^2 = 1 \qquad \text{for all } f.$$

Two Comments

- Note that the Fourier transform of $p(t) * p(-t)$ is $|P(f)|^2$ because $p(t)$ is real
- When discussing orthogonality, T is important (T-orthogonality)

Module 10.2

Presentation Outline

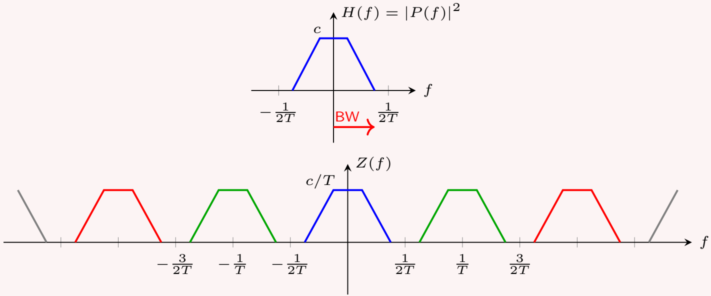
Part I The Nyquist Criterion

Part II Orthonormal Pulses and Bandwidth

Part III Receiver Implementation

Case 1: Negative

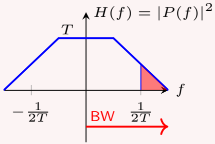
Negative Case: $W < 1/(2T)$

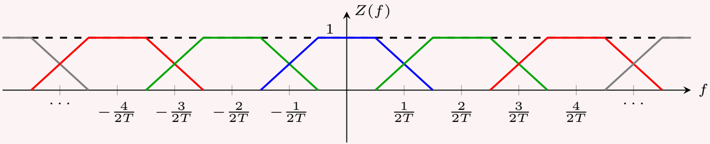


- Bandwidth W of the pulse $p(t)$ is strictly smaller than $1/(2T)$
- Nyquist criterion ($H(f) = |P(f)|^2$) can not be satisfied

Case 2: Excess

Excess Bandwidth Case $W > 1/(2T)$





- Bandwidth W of the pulse $p(t)$ is larger than $1/(2T)$
- *Excess bandwidth*: bandwidth minus $1/(2T)$
- Popular pulse with an excess bandwidth (yet orthonormal): square root-raised cosine

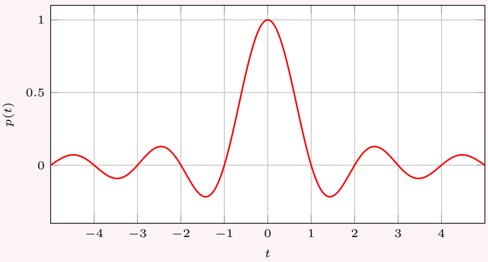
Case 3: Match (1/2)

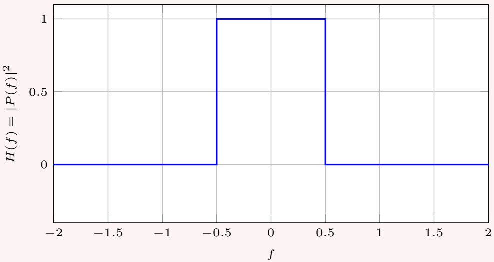
Matching Case $W = 1/(2T)$ (1/2)

- Basic pulse $p(t)$ that corresponds to the ideally bandlimited spectrum is the sinc-pulse:

$$p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t/T)}{\pi t/T}$$

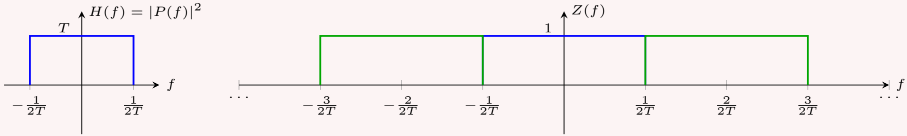
- For $T = 1$





Case 3: Match (2/2)

Matching Case $W = 1/(2T)$ (2/2)



- Smallest possible bandwidth W of a pulse that satisfies the Nyquist criterion is $1/(2T)$
- "Basic" pulse $P(f)$ with $W = 1/(2T)$ for which the Nyquist criterion holds has a brick-wall (ideally bandlimited) spectrum
- Basic pulse $p(t)$ that corresponds to the ideally bandlimited spectrum is the sinc-pulse

$$P(f) = \begin{cases} \sqrt{T} & \text{if } |f| < 1/(2T), \\ 0 & \text{if } |f| > 1/(2T) \end{cases}$$

Pulses with smallest Bandwidth

Sinc Pulses

The smallest possible bandwidth W of a pulse that satisfies the Nyquist criterion is $W = \frac{1}{2T}$. The sinc-pulse $p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t/T)}{\pi t/T}$ has this property. This way of serial pulse transmission leads to exactly $\frac{1}{T} = 2W$ dimensions per second.

Extra Considerations

Q1: What problems do sinc-pulses cause in practice?

- Infinitely long
- Difficult to generate
- Noncausal

Q2: What are the positive aspects of sinc pulses?

- Orthogonal pulses (no ISI)
- Best possible use of bandwidth
- Only one building-block waveform (one new dimension) every T [s]

Module 10.2

Presentation Outline

Part I The Nyquist Criterion

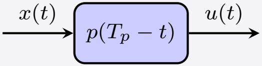
Part II Orthonormal Pulses and Bandwidth

Part III Receiver Implementation

Before Getting Started...

A Matched Filter

- Suppose we have a linear filter with impulse response $h(t) = p(T_p - t)$ and fed with a signal $x(t)$.
- Delay T_p is to make the filter causal



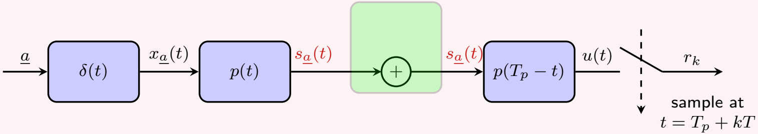
Q1: What is the output of the filter $u(t)$?

Answer:

$$u(t) = \int_{-\infty}^{\infty} x(\tau)p(\tau - t + T_p)d\tau$$

Optimum Receiver: Without Noise

Serial PAM Transceiver



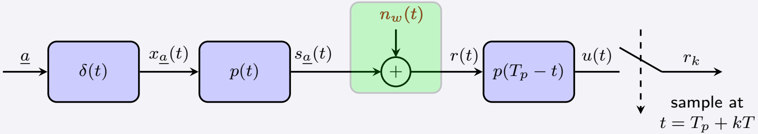
$$r_k = u(T_p + kT) = \int_{-\infty}^{\infty} r(\tau)p(T_p - t + \tau)d\tau\Big|_{t=T_p+kT} = \int_{-\infty}^{\infty} r(\tau)p(-kT + \tau)d\tau$$

$$= \int_{-\infty}^{\infty} \left(\sum_{k'=0}^{K-1} a_{k'} p(\tau - k'T) \right) p(-kT + \tau)d\tau$$

$$= \sum_{k'=0}^{K-1} a_{k'} \underbrace{\int_{-\infty}^{\infty} p(\tau - k'T)p(-kT + \tau)d\tau}_{0 \text{ unless } k' = k} = a_k$$

Optimum Receiver: With Noise

Serial PAM Transceiver



$$r_k = u(T_p + kT) = \int_{-\infty}^{\infty} r(\tau)p(T_p - t + \tau)d\tau \Big|_{t=T_p+kT} = \int_{-\infty}^{\infty} r(\tau)p(-kT + \tau)d\tau$$

$$= \int_{-\infty}^{\infty} \left(\sum_{k'=0}^{K-1} a_{k'} p(\tau - k'T) + n_w(\tau) \right) p(-kT + \tau)d\tau$$

$$= a_k + \int_{-\infty}^{\infty} n_w(\tau)p(-kT + \tau)d\tau = a_k + z_k \qquad z_k \sim \mathcal{N}(0, N_0/2)$$

Serial PAM Transceiver

Concluding Remarks

- (Serial) PAM transceiver uses a single filter (very simple!)
- No ISI is present at the receiver
- If $p(t)$ are sinc pulses: best use of bandwidth
- Abstract waveform transmission into DICO channel: optimum detection

Summary Module 10.2

Take Home Messages

- PAM transceiver
- Nyquist criterion (time and frequency): zero ISI!
- Bandwidth considerations: Negative, excess and matching

Communication Theory (5ETB0)
Module 10.2

Alex Alvarado
a.alvarado@tue.nl

Information and Communication Theory Lab
Signal Processing Systems Group
Department of Electrical Engineering
Eindhoven University of Technology, The Netherlands

www.tue.nl/ictlab/