Digital Signal Processing Fundamentals (5ESC0)

Filter Design

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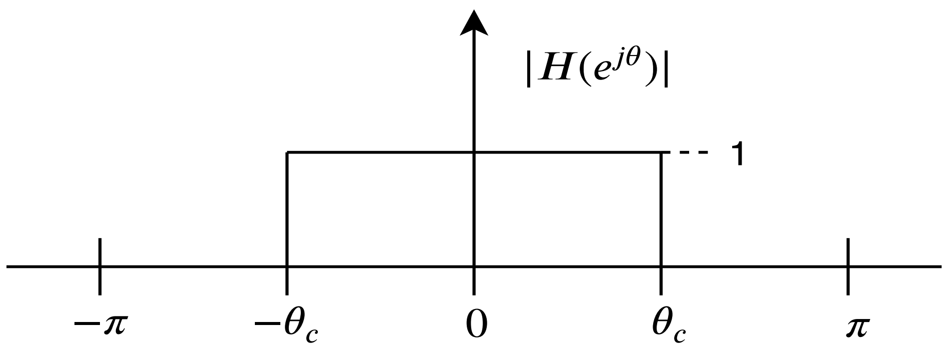
Filter Examples

* Let us start by looking at some examples of filters that may need to be designed

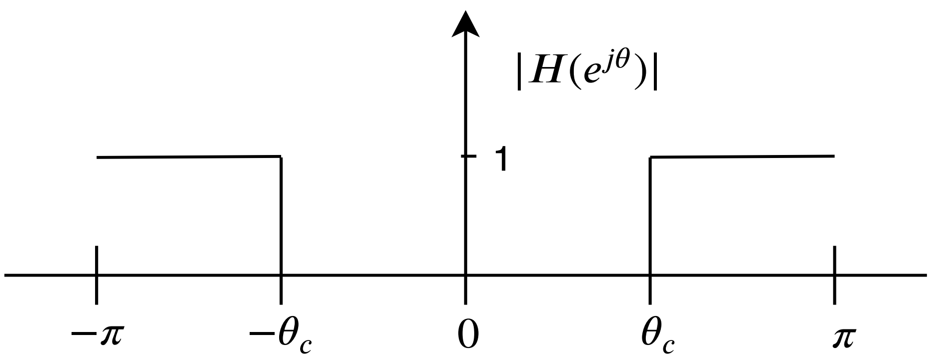
Filter Examples

- Ideal low-pass filter
- It allows all frequencies below the cutoff frequency θ_c to pass through: $\left|H\left(e^{j\theta}\right)\right| = 1 \;\; \forall \;\; |\theta| \leq \theta_c$
- It attenuates all frequencies above θ_c: $\left|H\left(e^{j\theta}\right)\right| = 0 \;\; \forall \;\; |\theta| > \theta_c$



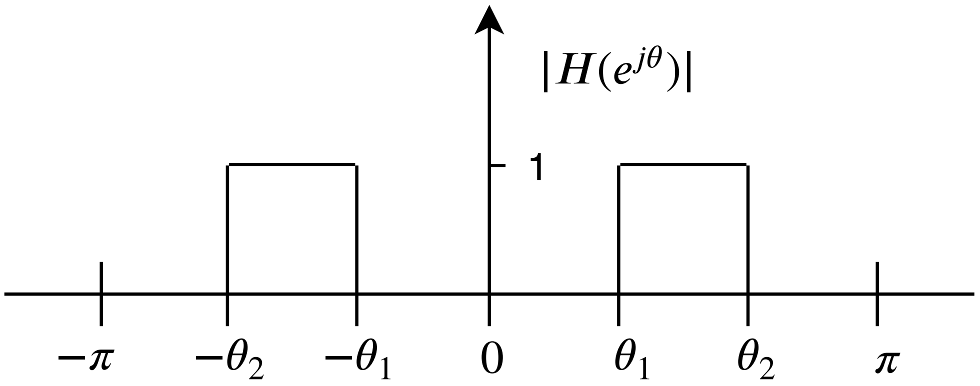
Filter Examples

* Ideal high-pass filter
* It allows all frequencies above the cutoff frequency θ_c to pass through: $\left|H\left(e^{j\theta}\right)\right| = 1 \;\; \forall \;\; |\theta| \geq \theta_c$
* It attenuates all frequencies below θ_c: $\left|H\left(e^{j\theta}\right)\right| = 0 \;\; \forall \;\; |\theta| < \theta_c$



Filter Examples

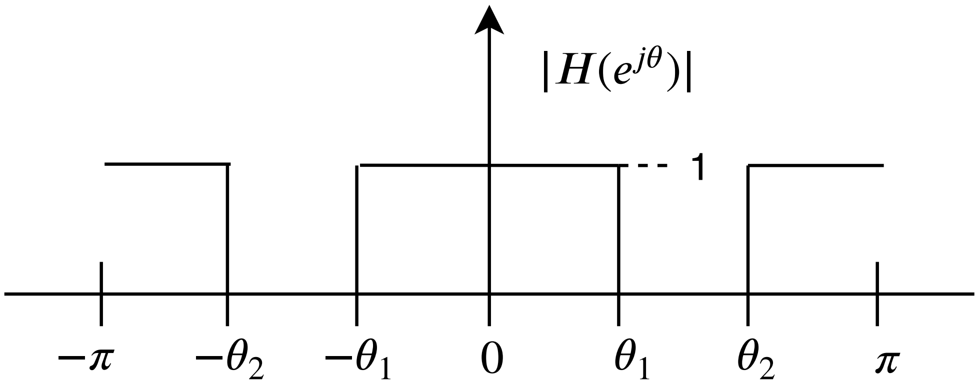
- Ideal band-pass filter
- It allows all frequencies within the specified bounds θ_1 and θ_2 of the band to pass through: $\left|H\left(e^{j\theta}\right)\right| = 1 \;\; \forall \;\; \theta_1 \leq |\theta| \leq \theta_2$
- It attenuates all frequencies outside the band: $\left|H\left(e^{j\theta}\right)\right| = 0 \;\; \forall \; |\theta| < \theta_1$ and $|\theta| > \theta_2$



Filter Examples

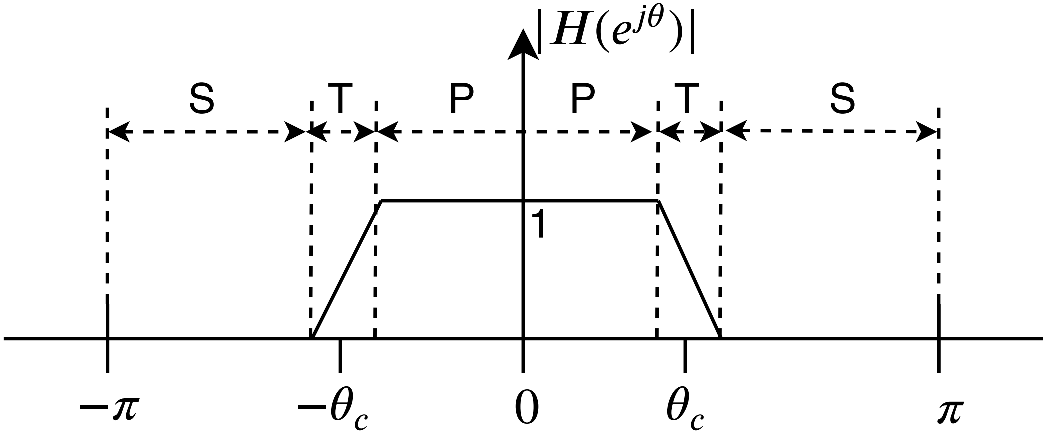
* Ideal band-stop filter
* It allows all frequencies outside of the specified bounds θ_1 and θ_2 of the band to pass through: $\left|H\left(e^{j\theta}\right)\right| = 1 \;\; \forall \, |\theta| \leq \theta_1$ and $|\theta| \geq \theta_2$
* It attenuates all frequencies inside the band:

$$\left|H\left(e^{j\theta}\right)\right| = 0 \;\; \forall \;\; \theta_1 < |\theta| < \theta_2$$



Filter Examples

- Ideal filters are symmetric in the frequency domain
- In practice, these filters change slightly
- Take the low-pass filter: its impulse response is a sinc pulse
- The sinc pulse is infinitely long, hence it cannot be realized
- A realistic, non ideal low-pass filter in frequency domain looks like this:



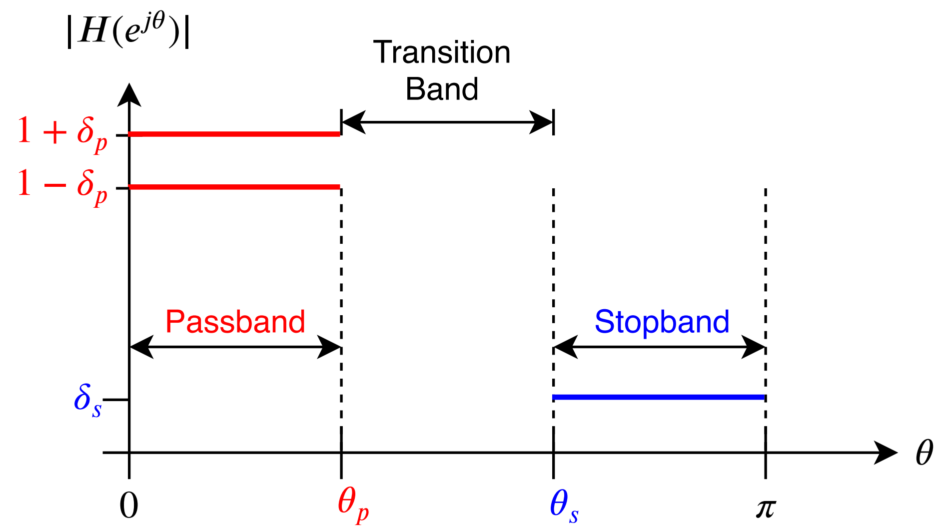
P=Pass band
T=Transition band
S=Stop band
θ_c=Cutoff frequency
θ_c is normally in the middle of T

Filter Design Process

* For the rest of the slides we will consider the low-pass filter
* The design process of other filters is similar

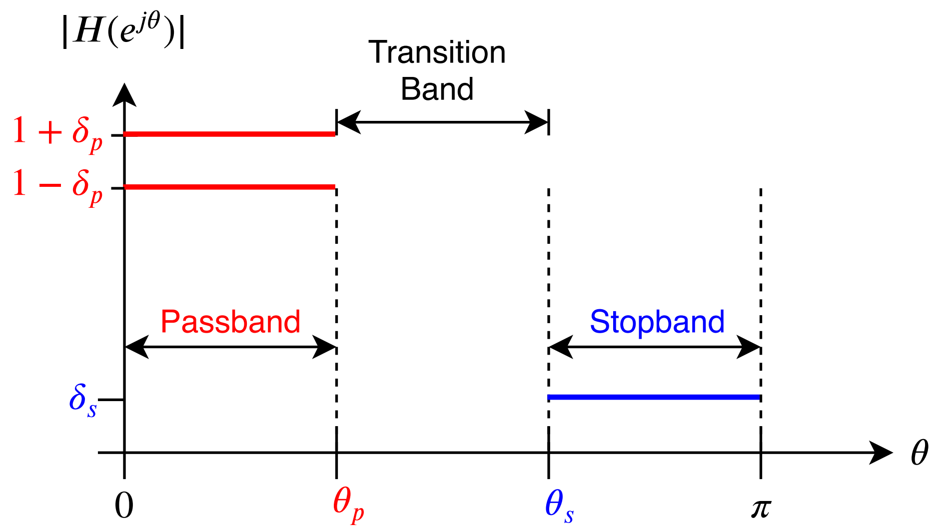
Filter Design Process: Specifications

- We want to design a low-pass filter
- We have a certain passband, transition band and stopband: the design specifications
- The figure below shows the design process in practice



Filter Design Process: Specifications

* In practice, it is impossible for the passband to be a straight line; it will be a ripple
* The ripple should stay between the bounds specified by δ_p
* For the stopband we define a lower bound δ_s, under which we accept the filter values for the stopband. The bounds are usually given in dB



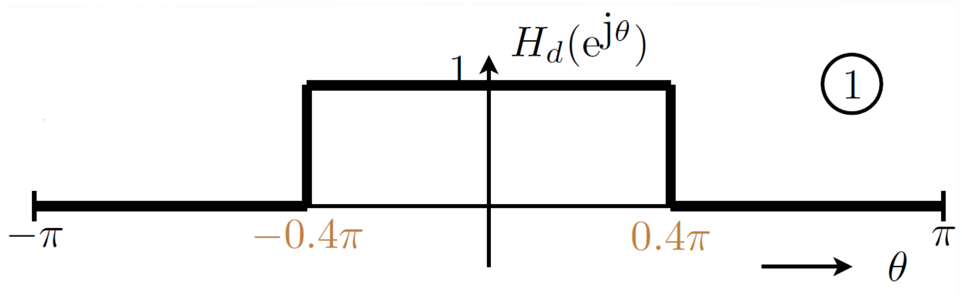
Filter Design Process

Below is the step by step approach of the filter design process.
The previous slides exemplified the specifications.

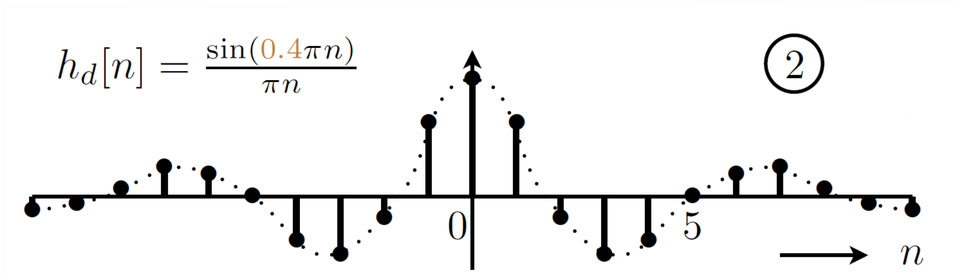
1. Specifications: Among others constraints on magnitude and/or phase
2. Type and order: Choose FIR/IIR and filter order (for this course only FIR)
3. Design (this chapter): Find coefficients producing an acceptable filter
4. Implement (hard- or software): Choose an appropriate structure and quantize the coefficients (if necessary)
5. Check resulting filter: Iterate steps 2-5 if necessary

FIR filter design based on FTD and windowing

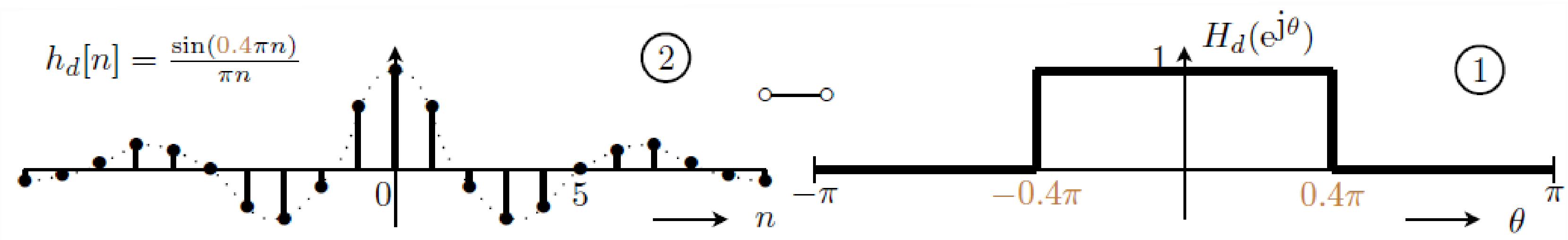
* We want to design the filter below, with a pass band of up to 0.4π and a stopband from 0.4π to π



* The time domain representation of the filter above is a sinc function:



FIR filter design based on FTD and windowing

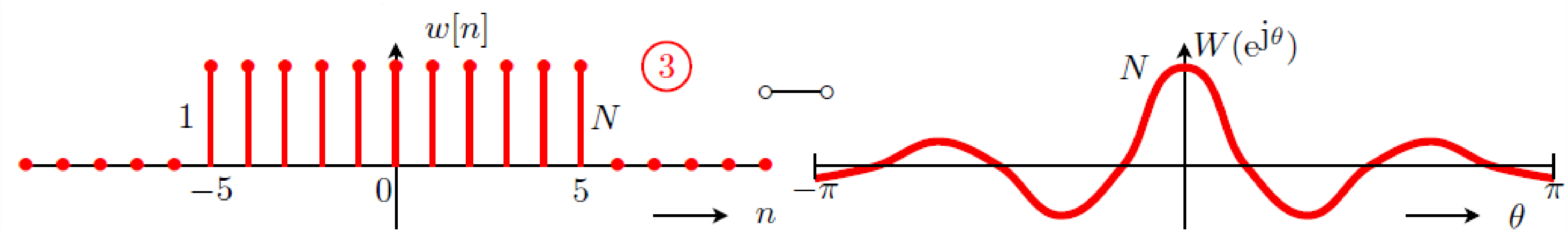


* To build this time domain function, an infinite number of coefficients is needed
* Therefore, the filter is not realizable

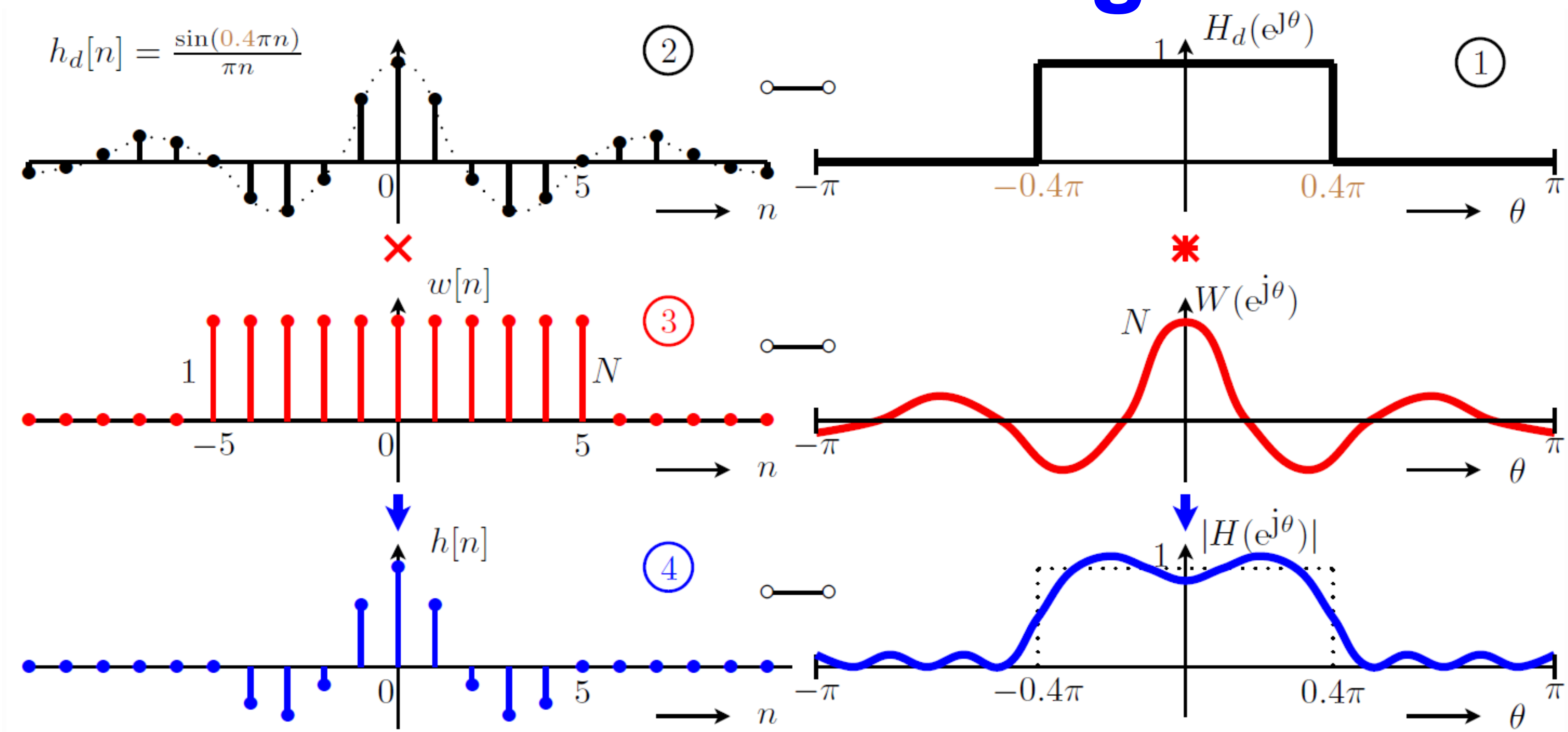
FIR filter design based on FTD and windowing

* The first step to a solution is limiting the number of coefficients by applying a filter of length N

* This filter may be a rectangular window of length N in time domain corresponding to a sinc pulse in frequency domain



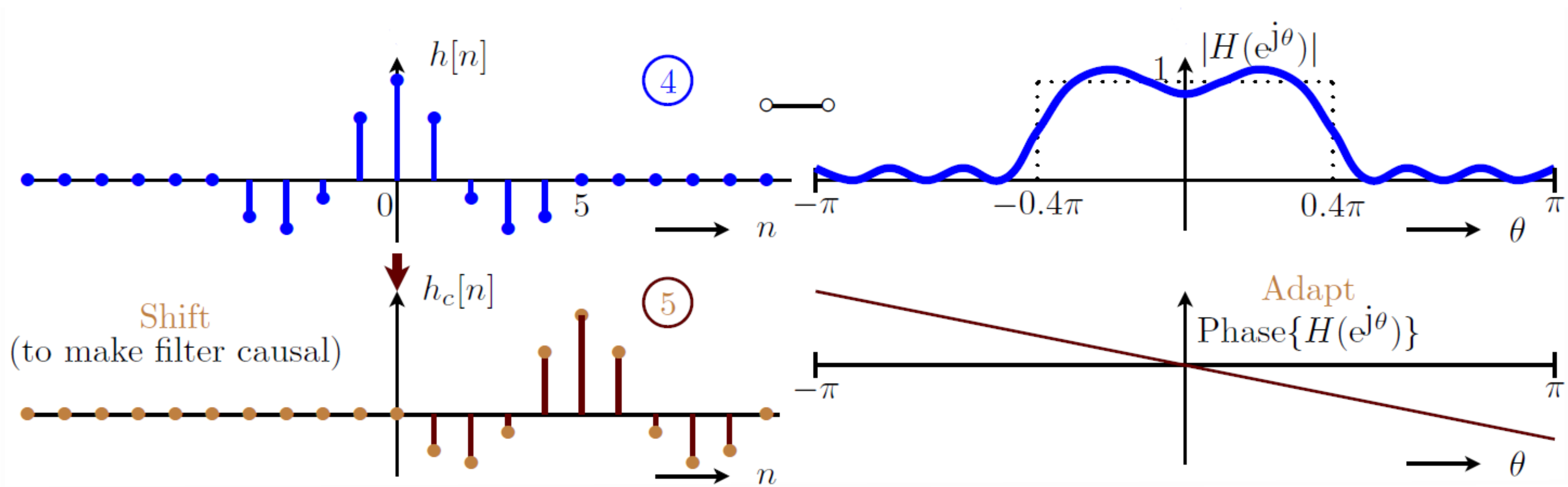
FIR filter design based on FTD and windowing

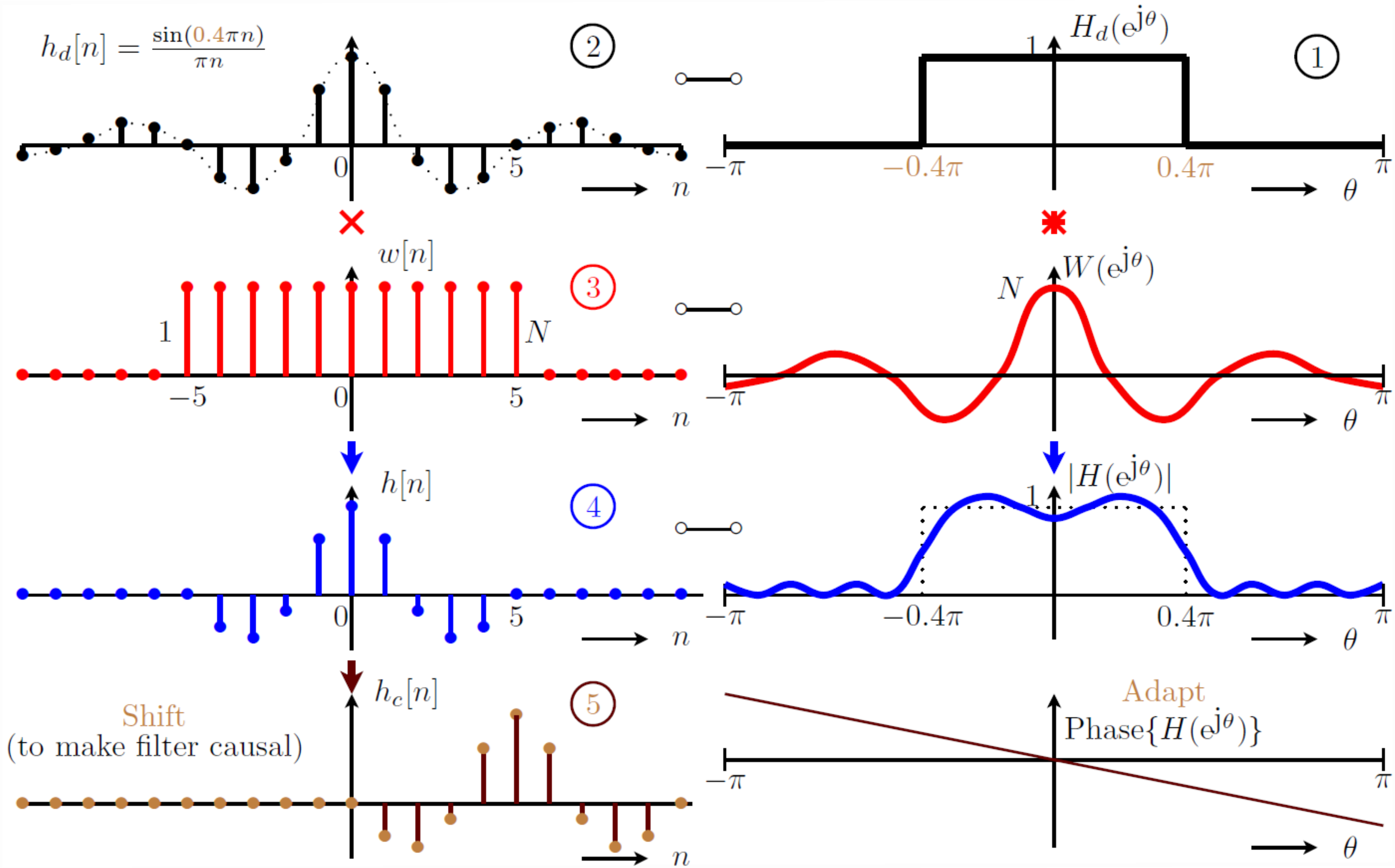


* We multiply the time domain sinc pulse with the rectangular window to limit the sinc pulse to N samples

FIR filter design based on FTD and windowing

* Now the filter is finite in length, but not causal yet
* We shift the sinc pulse in time domain, which adapts the phase in the frequency domain



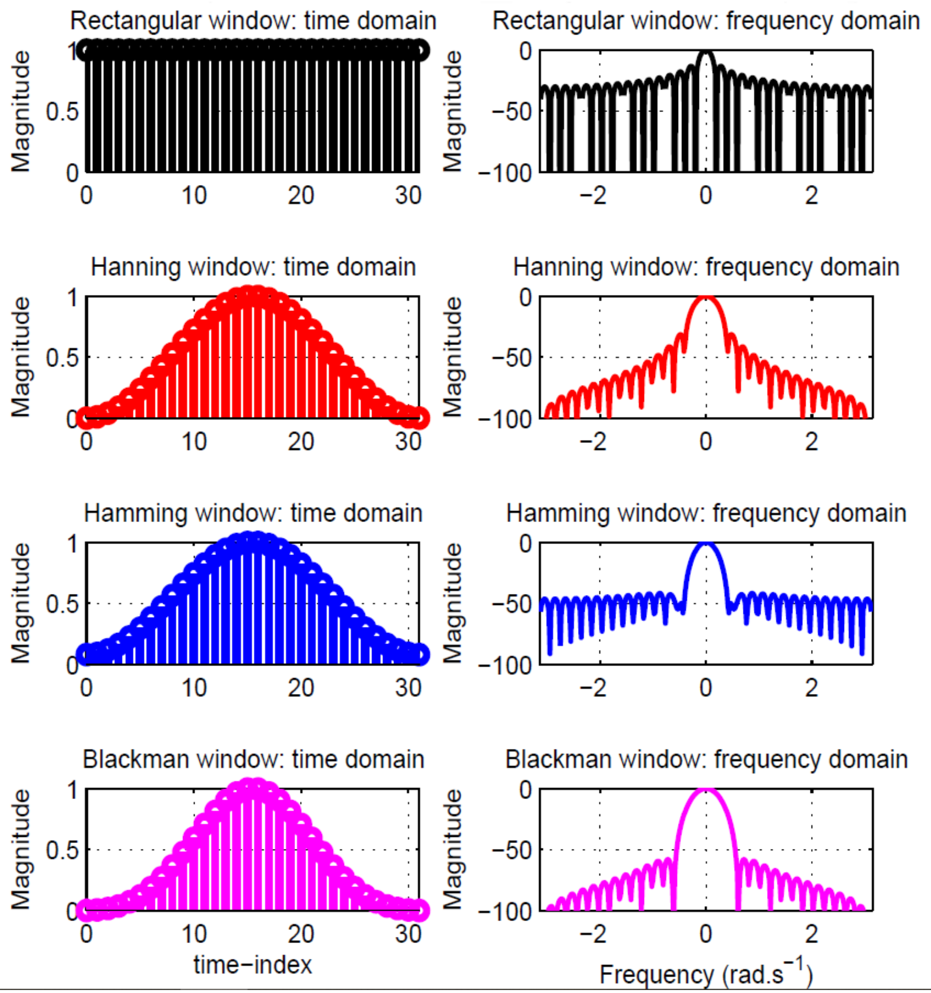


$h_d[n] = \frac{\sin(0.4\pi n)}{\pi n}$
2
0
5
n
$H_d(e^{j\theta})$
1
1
$-\pi$
-0.4π
0.4π
π
θ
$w[n]$
3
1
N
-5
0
5
n
$W(e^{j\theta})$
N
$-\pi$
π
θ
$h[n]$
4
0
5
n
$|H(e^{j\theta})|$
1
$-\pi$
-0.4π
0.4π
π
θ
$h_c[n]$
5
Shift
(to make filter causal)
n
Adapt
Phase$\{H(e^{j\theta})\}$
$-\pi$
π
θ

FIR filter design based on FTD and windowing

* In the previous example we used a rectangular window which is a sinc pulse in frequency domain
* If we choose a length $N = 32$, the rectangular window will consist of 32 ones in time domain
* Next we will look at other types of windows

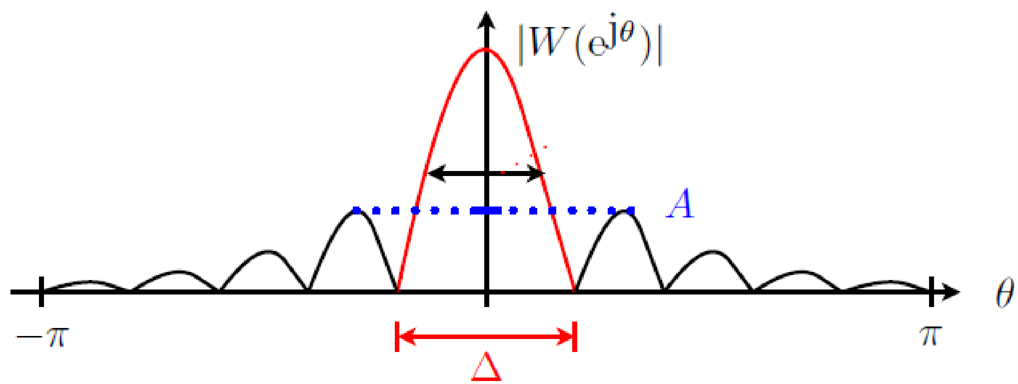
FIR filter design based on FTD and windowing



- $N = 32,\ w[n] \circ\!\!-\!\!\circ W(e^{j\theta})$
- The frequency domain representations are plotted in dB-scale $(20\log_{10}(\text{magnitude}))$
- The frequency domain representations contain a main lobe and side lobes

FIR filter design based on FTD and windowing

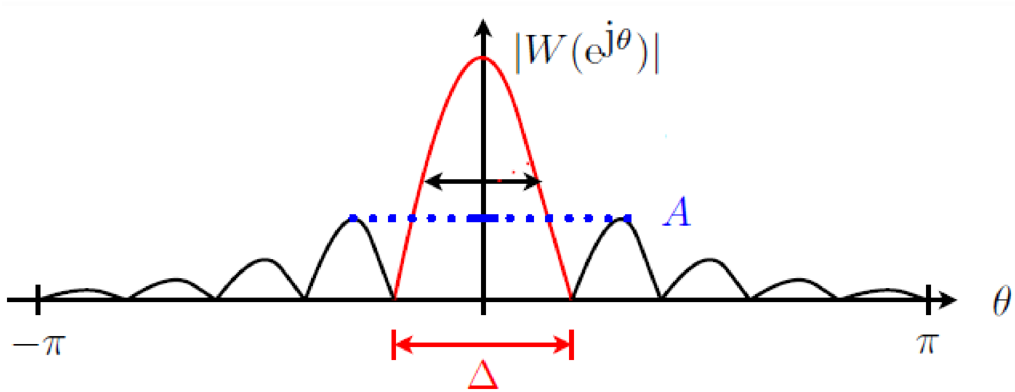
* The figure below shows the characteristics of the frequency response of a rectangular impulse response



* Main characteristics $W(e^{j\theta})$:
  - Main lobe width: Δ
  - Peak side-lobe: A
  - Ideally we want Δ and A to be small
  - Sharp transition in time domain (e.g. a rectangular window) will result in many frequencies in frequency domain

FIR filter design based on FTD and windowing

* General properties window $W(e^{j\theta})$, with window length N:
  - As N increases, the width of the main lobe decreases, which results in a decrease in the transition width $\Delta\theta$ between passband and stopband. $N \cdot \Delta f = c$, where c is a constant and Δ can be expressed in frequency or radial frequency: $\Delta f = \frac{\Delta\theta}{2\pi}$; Δf is the transition width
  - Peak A depends on window shape, not length N
  - If the window shape is changed to decrease A, generally Δ increases



FIR filter design based on FTD and windowing

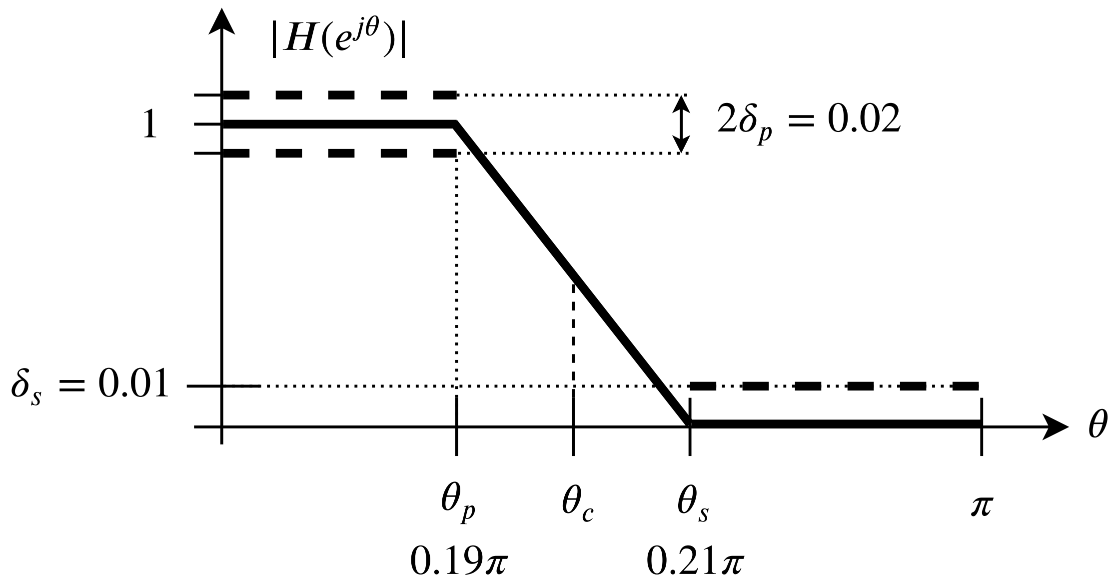
* The table below contains some general parameters for some window types

Window	A [dB]	Transition Δ	Stopband [dB]
Rectangular	-13	$1 \times (2\pi/N)$	-21
Hanning	-31	$3.1 \times (2\pi/N)$	-44
Hamming	-41	$3.3 \times (2\pi/N)$	-53
Blackman	-57	$5.5 \times (2\pi/N)$	-74

* **Note: The FTD and windowing strategy is more appropriate when I want to design filters with a smooth, continuous frequency response where control over the overall shape of the filter, including the transition band and ripple, is important**

Example

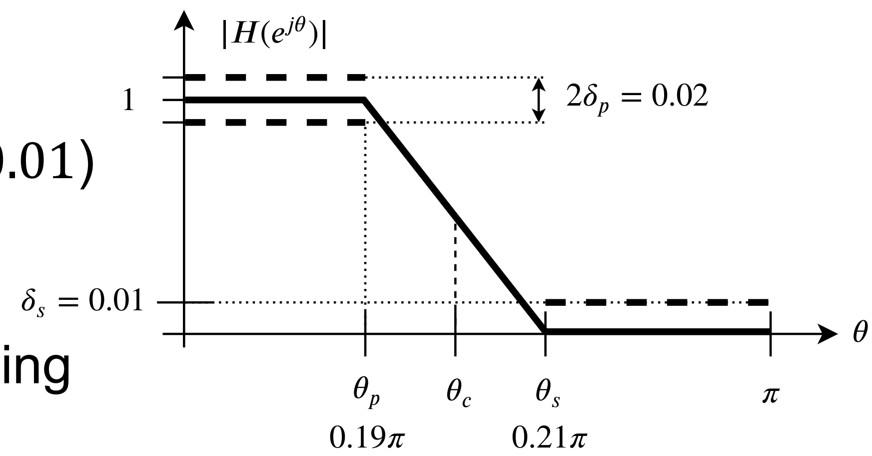
- We would like to design the low-pass filter with the parameters shown in the plot below



- The passband runs until the point 0.19π and the stopband starts at 0.21π

Example

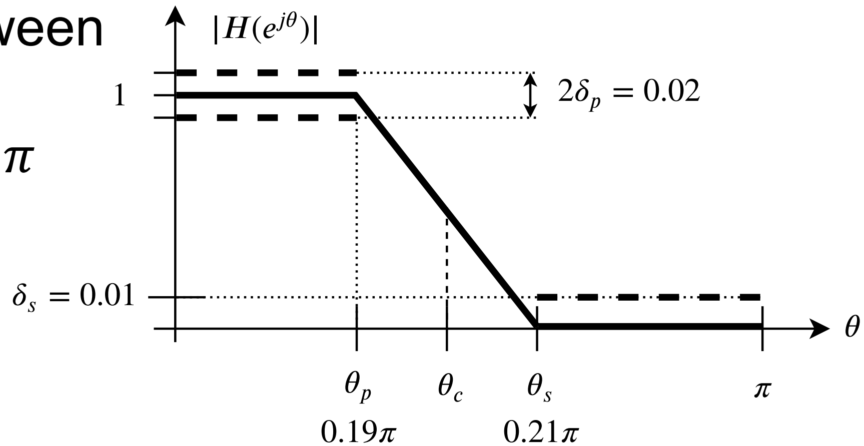
* We start by calculating the stopband attenuation in dB:
  $20\log_{10}(\delta_s) = -40$ dB $(\delta_s = 0.01)$
* Now we have a look at the table on slide 22. A stopband of at most -40 dB can be achieved with a Hanning window
* We know for a Hanning window:



Window	A [dB]	Transition Δ	Stopband [dB]
Hanning	-31	$3.1 \times (2\pi/N)$	-44

Example

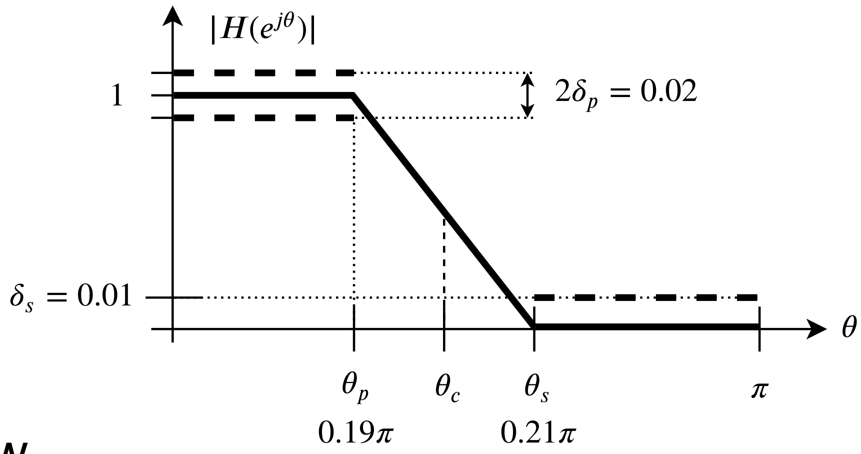
* Now we calculate the difference between the stopband and the passband, $\Delta\theta$:
  $\Delta\theta = \theta_s - \theta_p = 0.21\pi - 0.19\pi = 0.02\pi$
* $\Delta f = \frac{\Delta\theta}{2\pi} = \frac{0.02\pi}{2\pi} = 0.01$
* We know that $N\Delta f = c$, therefore
  $N = \frac{c}{\Delta f} = \frac{3.1}{0.01} = 310$ (c from table)
* This means that we need 310 samples in the Hanning window



Window	A [dB]	Transition Δ	Stopband [dB]
Hanning	-31	3.1 $\times (2\pi/N)$	-44

Example

* $N = 310$, delay: $\frac{N}{2} = 155$
* Cut off: $\theta_c = \frac{(\theta_s + \theta_p)}{2} = 0.2\pi$
* The impulse response of the filter window is the product of the sinc function and the Hanning window



* The definition of the Hanning window:

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right), & \text{if } 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$$

Example

* $N = 310$, delay: $\frac{N}{2} = 155$, cut off: $\theta_c = \frac{(\theta_s + \theta_p)}{2} = 0.2\pi$

  $\rightarrow h[n] = \frac{\sin(0.2\pi(n-155))}{(n-155)\pi} \times \{0.5 - 0.5\cos(\frac{2\pi}{310} \cdot n)\}$

* Note that we did not use the parameter δ_p in the design process

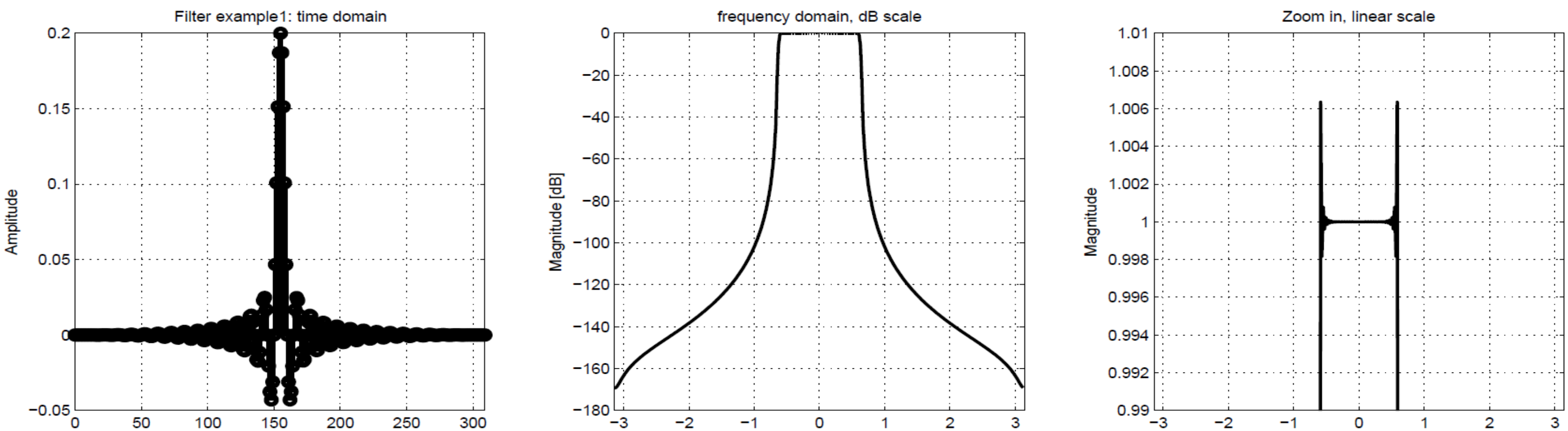
* Designing a LPF with the window design method generally produces filters that have ripples with the same amplitude in the stopband and passband

Example

* $N = 310$, delay: $\frac{N}{2} = 155$, cut off: $\theta_c = \frac{(\theta_s+\theta_p)}{2} = 0.2\pi$

$$\rightarrow h[n] = \frac{\sin(0.2\pi(n-155))}{(n-155)\pi} \times \{0.5 - 0.5\cos(\frac{2\pi}{310} \cdot n)\}$$

* The plots below show the impulse response, frequency response and a zoomed in version from left to right



FIR filter design based on frequency sampling

* Say we want to design a low-pass filter, we start in frequency domain
* We define the area in which we want to have the passband and the area in which we want to have the stopband
* The phase is also important as it is needed to make the filter causal

* **Note: this strategy is appropriate when designing filters where the response at specific, discrete frequencies is critical**

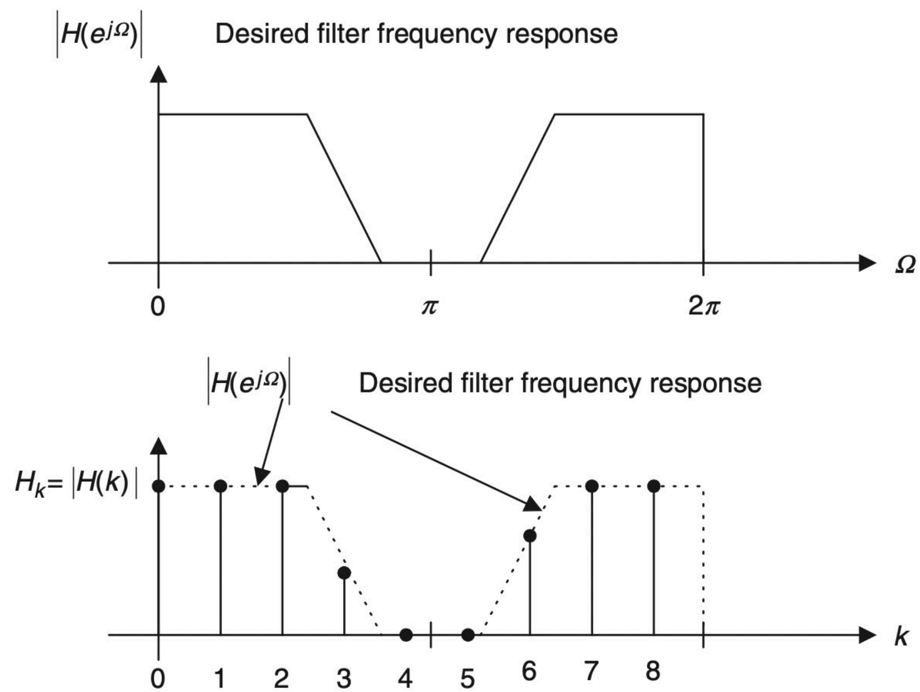
FIR filter design based on frequency sampling

The key feature of frequency sampling is that the filter coefficients can be calculated based on the specified magnitudes of the desired filter frequency response uniformly in Frequency domain. Hence, it has design flexibility.

To begin with development, we let $h(n)$, for $n = 0, \dots, N-1$, be the FIR filter coefficients that approximates the FIR filter, and we let $h(k)$,

for $k = 0, \dots, N-1$, represent the corresponding discrete Fourier transform (DFT) coefficients.

We obtain $h(k)$ by sampling the desired frequency filter response $h(k) = h(e^{j\omega})$ at equally spaced instants in frequency domain, as shown below.



FIR filter design based on frequency sampling

Then, according to the definition of the inverse DFT (IDFT), we can calculate the FIR coefficients:

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) W_N^{-kn}, \text{ for } n = 0, \ldots, N-1$$

$$\text{where } W_N = e^{-j\frac{2\pi}{N}}, \text{ is the twiddle factor.}$$

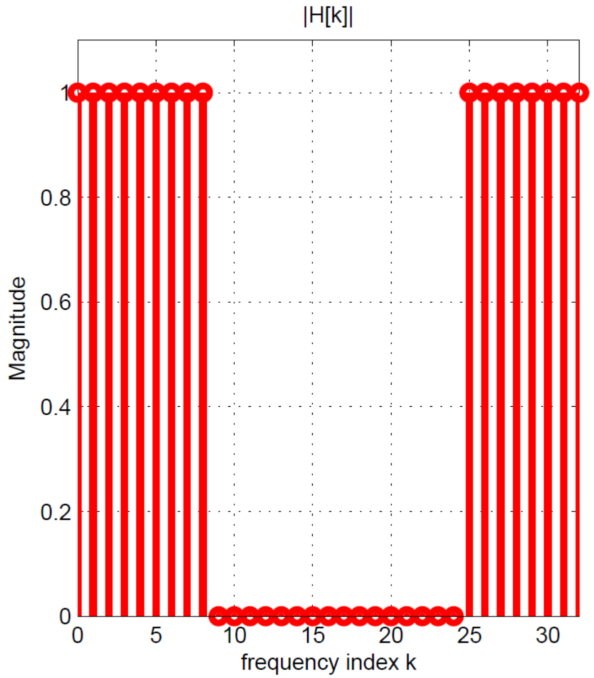
FIR filter design based on frequency sampling

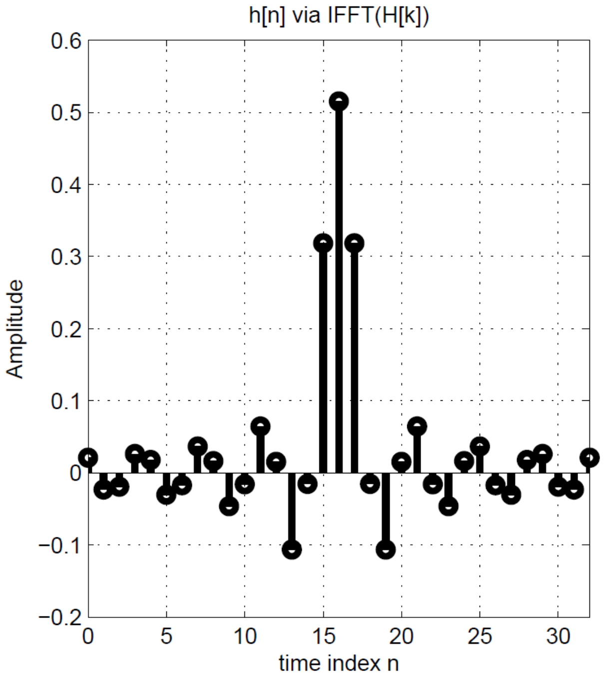
Furthermore, we assume that the FIR filter has linear phase with $N = 2M + 1$. The equation can be significantly simplified as

$$h(n) = \frac{1}{2M+1}\left\{H(0) + 2\sum_{k=1}^{M} H(k)\cos\left(\frac{2\pi(n-M)}{2M+1}\right)\right\}, \qquad \text{for } n = 0,1,\dots M$$

FIR filter design based on frequency sampling

- We find the impulse response $h[n]$ through inverse DFT
- If we do so, we obtain the following two plots: the filter as we designed it on the left, on the right a sinc-like function in time domain





Frequency sampling - Example

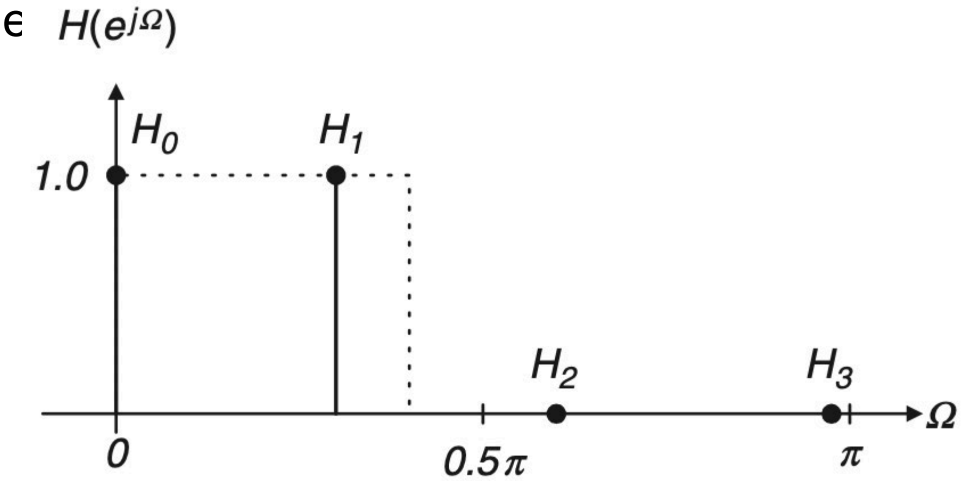
Design a linear phase lowpass FIR filter with 7 samples and a cutoff frequency of $f_c = 150$ Hz with a sampling frequency of $f_s = 1000$ Hz using the frequency sampling method.

Frequency sampling – Example solution

$$\theta_c = 2\pi \frac{150}{1000} = 0.3\pi \text{ [rad]}$$

Since $N = 2M + 1 = 7$ and $M = 3$, the sampled frequencies are given by $\theta_k = \frac{2\pi}{7}k$ radians, for $k = 0, 1, 2, 3$.

Next we specify the magnitude frequencies as follows:



$\theta_0 = 0$ radians, $\rightarrow H(0) = 1.$

$\theta_1 = \frac{2\pi}{7}$ radians, $\rightarrow H(1) = 1.$

$\theta_2 = \frac{4\pi}{7}$ radians, $\rightarrow H(2) = 0.$

$\theta_3 = \frac{6\pi}{7}$ radians, $\rightarrow H(3) = 0.$

Frequency sampling – Example solution

Thus, the filter coefficients can be calculated as follows:

$$
\begin{aligned}
h(n) &= \frac{1}{7}\left\{1 + 2\sum_{k=1}^{3} H_k \cos\left[2\pi k(n-3)/7\right]\right\}, \; n = 0,\ 1, \ldots, 3. \\
&= \frac{1}{7}\{1 + 2\cos\left[2\pi(n-3)/7\right]\}
\end{aligned}
$$

Frequency sampling – Example solution

Computing the FIR filter coefficients yields:

$$h(0) = \frac{1}{7}\{1 + 2\cos(-6\pi/7)\} = -0.11456$$
$$h(1) = \frac{1}{7}\{1 + 2\cos(-4\pi/7)\} = 0.07928$$
$$h(2) = \frac{1}{7}\{1 + 2\cos(-2\pi/7)\} = 0.32100$$
$$h(3) = \frac{1}{7}\{1 + 2\cos(-0 \times \pi/7)\} = 0.42857.$$

From symmetry, we obtain the rest of the coefficients:

$$h(4) = h(2) = 0.32100$$
$$h(5) = h(1) = 0.07928$$
$$h(6) = h(0) = -0.11456.$$

FIR filter design based on equiripple

* The last filter type that we will discuss is the equiripple linear phase filter
* For most windows, the ripple is not uniform in either the passband or the stopband
* The ripple generally decreases when moving away from the transition band
* Allowing the ripple to be uniformly distributed over the entire band would produce a smaller peak ripple

Summary

* We discussed the step by step approach of filter design
* We designed a filter through the FTD and windowing
* We looked at different types of windows