Digital Signal Processing Fundamentals (5ESC0)

System function

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Lecture content

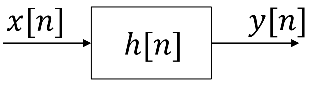
We have learnt Z-transform and the ways to calculate it

Now we will learn how to use the Z-transform tot analyse the system

What is the context of this lecture?

Lecture context

In this course we deal with the following system:



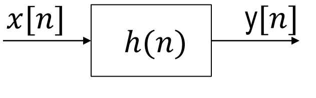
The input signal $x[n]$ is in the discrete time domain. The signal is input to the system.

The system can be described by an impuls response $h[n]$.

The system produces the output signal $y[n]$.

Lecture context

We assume that the system is Linear Time-Invariant (LTI)



The input signal $x[n]$ is in the discrete time domain. The signal is input to the system.

What is an LTI system?

LTI system definition

LTI system is a system which is *linear* and *shift-invariant*

Linear system is a system which is both additive and homogeneous:

- Additive system is a system for which the response to a sum of inputs is equal to sum of responses for each of these inputs
- Homogeneous system is a system for which scaling the input results in scaling of the output by the same amount

Shift-invariant system:

- Shift (delay) in the input by n_0 results in a shift in the output by n_0

System function

System function is obtained by Z-transform of the impulse response:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

We remember from Z-transform:

FTD $\equiv$ ZT **evaluated on unit circle:** $X(z)|_{z=e^{j\theta}} = X(e^{j\theta})$

System function

Frequency response:

$$H(e^{j\theta}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\theta}$$

Generalization by substituting $e^{j\theta}$ by complex variable z

$$\boxed{\text{System function:} \quad H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}}$$

LTI property: convolution

One of the important properties of an LTI system is that its output can be described as a convolution of the input signal with the impulse response:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n]*x[n]$$

So we can evaluate the output of the system once we know the input and the impulse response by applying the convolution

LTI property: convolution

Calculating convolution in the time domain is equivalent to multiplication in the frequency domain:

$$Y(\mathrm{e}^{\mathrm{j}\theta}) = X(\mathrm{e}^{\mathrm{j}\theta}) \cdot H(\mathrm{e}^{\mathrm{j}\theta})$$

In the Z-domain we have an equivalent property

$$Y(z) = H(z) \cdot X(z)$$

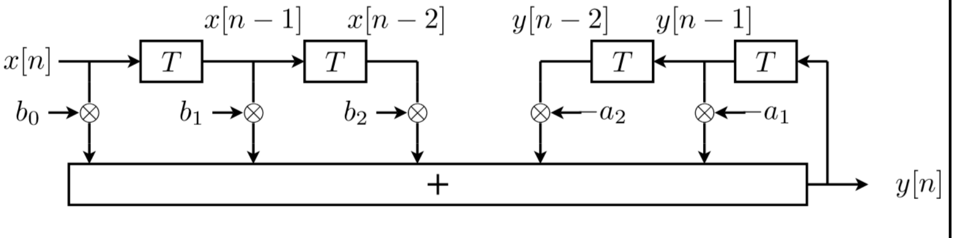
The system function can therefore be expressed as

$$H(z) = \frac{Y(z)}{X(z)}$$

Example 1

In this example the frequency response is calculated from the system function

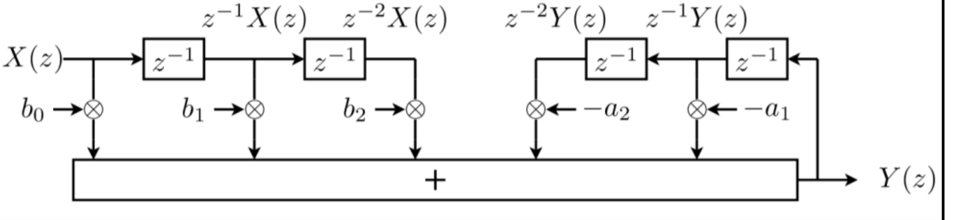
Let's consider the system with input, output and delay lines: input is delayed several times and there also a recursion – we reuse the output in the delay line



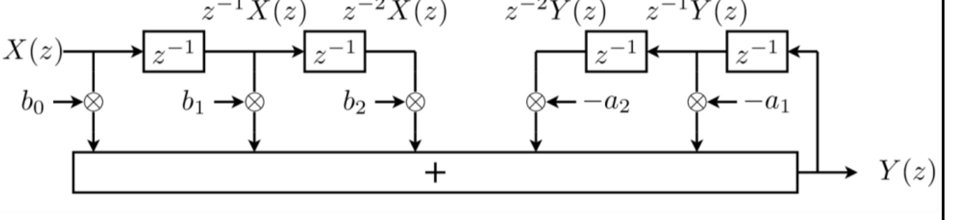
Example 1

The system on the previous slide can be described by a difference equation

The same system can be depicted in the Z-domain



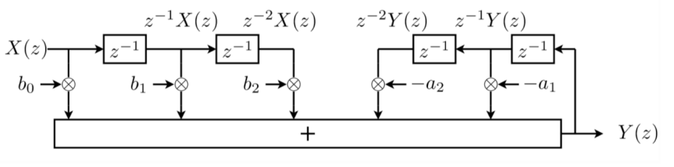
Example 1



In Z-domain the delay is represented by the symbol z^{-1}
Input and outputs are replaced by their Z-transforms
Multiplications and additions stay the same

We will now find $H(z)$, knowing that $H(z) = \dfrac{Y(z)}{X(z)}$

Example 1



Let's define the system output in Z-domain:

$$Y(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) - a_1z^{-1}Y(z) - a_2z^{-2}Y(z)$$

We can group all sum elements with $Y(z)$ on one side of the equality and with $X(z)$ – on the other side:

$$Y(z) + a_1z^{-1}Y(z) + a_2z^{-2}Y(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z)$$

Example 1

This can be rewritten:

$$Y(z)(1 + a_1z^{-1} + a_2z^{-2}) = \mathrm{X(z)}(b_0 + b_1z^{-1} + b_2z^{-2})$$

Now the expression for the output is the following (we divide both sides by the multiplier of $Y(z)$):

$$Y(z) = X(z)\frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$$

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Example 1

We can therefore substitute $Y(z)$ by $X(z)\frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$ and calculate the system function:

$$H(Z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + b_2z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}$$

Now $H(\mathbf{e}^{\mathbf{j}\theta}) = H(z)\big|_{z=\mathbf{e}^{\mathbf{j}\theta}}$

Notice: we have polynomial descriptions in both numerator and denominator

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Poles and zeros

The Difference Equation (DE) can be generalised and described by polynomials in the numerator and denominator:

$$y[n] = \sum_{k=0}^{q} b_k x[n-k] - \sum_{k=1}^{p} a_k y[n-k]$$

$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})} = b_0 \cdot z^{p-q} \frac{\prod_{k=1}^{q}(z-\beta_k)}{\prod_{k=1}^{p}(z-\alpha_k)}$$

β_k values are the zeros of $H(z)$

α_k are the poles of $H(z)$

Poles and zeros

Notes:

- Calculation system function $H(z)$: via Z-transform of $h[n]$ or DE
- In schemes often $T \leftrightarrow z^{-1}$
- Besides scale factor b_0, $H(z)$ defined by poles and zeros

$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}$$

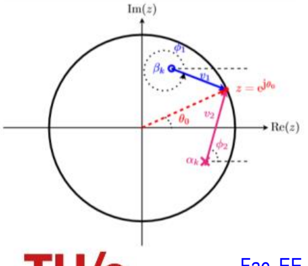
Frequency response can be calculated from the pole-zero plot

Frequency response from pole-zero plot

$$H(z)\big|_{z=\mathsf{e}^{\mathsf{j}\theta}} = |H(\mathsf{e}^{\mathsf{j}\theta})| \cdot \mathsf{e}^{\mathsf{j}\Phi(\mathsf{e}^{\mathsf{j}\theta})} \quad \text{with} \quad H(z) = A \cdot z^{p-q} \frac{\prod_{k=1}^{q}(z-\beta_k)}{\prod_{k=1}^{p}(z-\alpha_k)}$$

$$|H(\mathsf{e}^{\mathsf{j}\theta})| = |A| \times \left(\prod_{k=1}^{q} \text{length}(\mathsf{e}^{\mathsf{j}\theta} - \beta_k)\right) \Big/ \left(\prod_{k=1}^{p} \text{length}(\mathsf{e}^{\mathsf{j}\theta} - \alpha_k)\right)$$

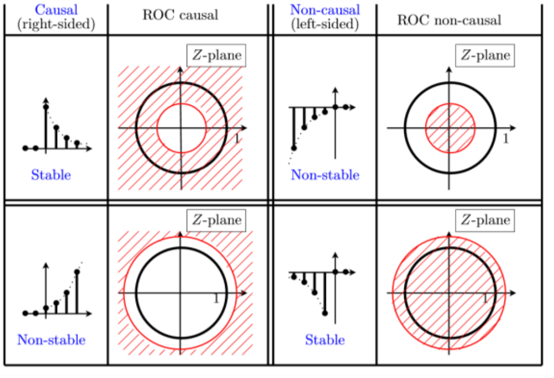
$$\Phi(\mathsf{e}^{\mathsf{j}\theta}) = (p-q)\cdot\theta + \sum_{k=1}^{q} \arg(\mathsf{e}^{\mathsf{j}\theta} - \beta_k) - \sum_{k=1}^{p} \arg(\mathsf{e}^{\mathsf{j}\theta} - \alpha_k)$$



$$|H(\mathsf{e}^{\mathsf{j}\theta_0})| = v_1/v_2$$
$$\Phi(\mathsf{e}^{\mathsf{j}\theta_0}) = \phi_1 - \phi_2$$

Stability and causality

Region Of Convergence (ROC) for a causal system is shown in the left part of the table below:



For a causal system ROC is exterior to the circle, we can see it on the plots

In practice, we are interested in a stable and causal system (upper left part of the table)

Stability

BIBO (Bounded Input Bounded Output) stability:

Sum of the values of impulse response has to be smaller than infinity

$$\sum_{-\infty}^{\infty} |h[n]| < \infty$$

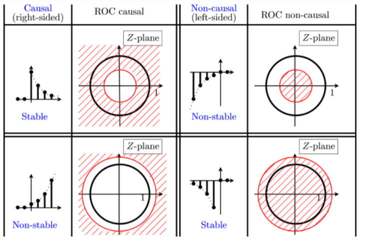
ROC of impulse response in Z-domain

$$(\sum_{-\infty}^{\infty} |h[n]||z^{-n} < \infty)|_{|z|=1}$$

ROC $H(z)$ must include $|z| = 1$

Causality, Realizability

Causal systems have right-sided impulse responses and ROC exterior to the circle $|z| > a$

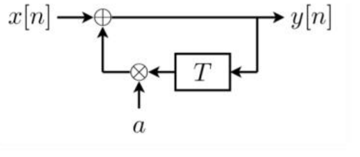


Realizability:

Both stable and causal $\Rightarrow$ Form of ROC: $|z| > a$ with $0 \leq a < 1$

All poles must lie inside unit circle

Example 2



$$y[n] = x[n] + ay[n-1] \quad \circ\!\!-\!\!\circ \quad Y(z) = X(z) + az^{-1}Y(z)$$

$$\Rightarrow H_3(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}} = \sum_{k=0}^{\infty} (az^{-1})^k \quad \text{iff} \quad |az^{-1}| < 1 \leftrightarrow |z| > |a|$$

Thus for $|a| < 1$ realizable system and

$$h_3[n] = \sum_{k=0}^{\infty} a^k \delta[n-k] = a^n u[n]$$

Poles for FIR and IIR filters

General form rational $H(z)$ of LTD system (p poles, q zeros):

$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}$$

- Case $p = 0$: (Finite Impulse Response (FIR) filters)

FIR filter can have poles at $z = 0$

$$H(z) = \sum_{k=0}^{q} b_k z^{-k} \Rightarrow h[n] = \sum_{k=0}^{q} b_k \delta[n-k]$$

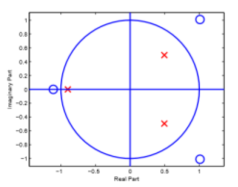
- Case $p \neq 0$: (Infinite Impulse Response (IIR) filters)

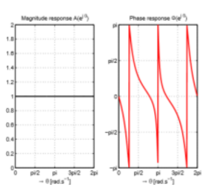
$$H(z) = \sum_{k=0}^{q-p} B_k z^{-k} + \sum_{k=1}^{p} \frac{A_k}{1-\alpha_k z^{-1}} \Rightarrow h[n] = \sum_{k=0}^{q-p} B_k \delta[n-k] + \sum_{k=1}^{p} A_k \alpha_k^n u[n]$$

All-pass system

- All-pass:

Poles, zeros mirrored pairs

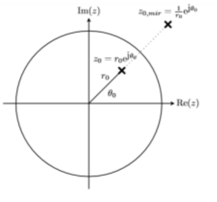




Take a point $z_0 = r_0 e^{j\theta_0}$

Complex conjugation: $z_0^* = r_0 e^{-j\theta_0}$

Mirroring: $z_{0,mirr} = (\frac{1}{z_0})^* = \frac{1}{r_0} e^{j\theta_0}$

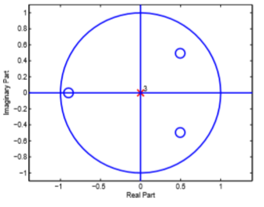


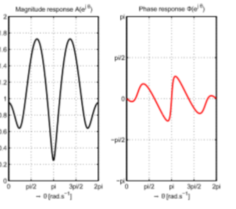
Zeros are circles, poles are crosses

Minimum, maximum phase

- **Minimum phase:**

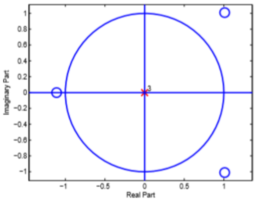
  Zeros inside unit circle

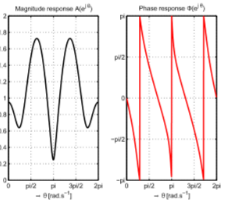




- **Maximum phase:**

  Mirror zeros of minimum phase





Summary

We considered the definition of system function, its properties

poles and zeros,

system stability and realizability

and examples