7 Course Reader

7.1 Chapter 2, Equation 2.7

In Chapter 2, the following derivation should be added for equation (2.7):

We first note that the from equation (B.6) in the course reader, the PSD of the output of the signal out of a linear filter is the multiplication of the FT of the impulse response squared and the PSD of the input

$$S_Y(f) = |H(f)|^2 \cdot S_X(f) \tag{1}$$

where $x(t)$ is the input of the linear filter, $h(t)$ is the impulse response of the filter and $y(t)$ is the output. Therefore

$$S_U(f) = |W_b(f)|^2 \cdot S_{U_W}(f) \tag{2}$$

$$= \begin{cases} \frac{U_0}{2}, & \text{if } -W < f < W \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Hence,

$$E[U^2(t)] = \int_{-\infty}^{\infty} S_U(f)\, df = \frac{U_0}{2} \cdot 2W \tag{4}$$

7.2 Chapter 2, Equation 2.21

The following has to be also be added in the course reader as a derivation for equation (2.21):

What is the power of $d(t)$ in (2.21)?

$$d(t) = \sqrt{\frac{U_0 W}{P}} \cdot u(t) * W_0(t) \tag{5}$$

$w_b(t)$

where $u(t)$ is defined as (not the same as the $u(t)$ at the Tx in Fig. 2.8):

$$u(t) \triangleq n_w(t) \cdot \sqrt{2}\cos(2\pi f_0 t) \tag{6}$$

Therefore

$$E[D^2(t)] = \int_{-\infty}^{\infty} |W_b(f)|^2 S_U(f) \cdot \frac{U_0 W}{P}\, df = \frac{U_0 W}{P} \int_{-W}^{W} S_U(f)\, df \tag{7}$$

The question is then: What is the PSD of $u(t)$? For this, we use an alternative definition of the PSD:

$$S_U(f) \triangleq E[|U(f)|^2] \tag{8}$$

where $U(f)$ is the FT of $u(t)$ and the expectation is over the realizations of the random process $u(t)$.

Now what is $U(f)$?

$$U(f) = \mathcal{F}\left\{\sqrt{2}\cos(2\pi f_0 t) \cdot n_w(t)\right\} \tag{9}$$
$$= \frac{\sqrt{2}}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right) * N_w(f) \tag{10}$$
$$= \frac{\sqrt{2}}{2}\left[N_w(f - f_0) + N_w(f + f_0)\right] \tag{11}$$

where $N_w(f)$ is the FT of $n_w(t)$. We can now compute the PSD of $u(t)$:

$$E\left[|U(f)|^2\right] = E\left[\left|\frac{\sqrt{2}}{2}\right|^2 |N_w(f - f_0) + N_w(f + f_0)|^2\right] \tag{12}$$
$$= \frac{1}{2}\left[E\left[|N_w(f - f_0)|^2\right] + E\left[|N_w(f + f_0)|^2\right] + 2E\left[N_w(f - f_0)N_w(f + f_0)\right]\right] \tag{13}$$
$$= \frac{1}{2}\left[E\left[|N_w(f - f_0)|^2\right] + E\left[|N_w(f + f_0)|^2\right]\right] \tag{14}$$
$$= \frac{1}{2}\left(S_{N_w}(f - f_0) + S_{N_w}(f + f_0)\right) \tag{15}$$
$$= \frac{1}{2}\left(\frac{N_0}{2} + \frac{N_0}{2}\right) \tag{16}$$
$$= \frac{N_0}{2}, \tag{17}$$

where the term with the multiplication of expectations is zero because $n_w(t)$ is a wide-sense stationary process.

Finally, equation 7 can be solved

$$E[D^2(t)] = \frac{U_0 W}{P}\int_{-W}^{W} \frac{N_0}{2}\, df \tag{18}$$
$$= \frac{U_0 W}{P}\frac{N_0}{2} 2W \tag{19}$$