Electromagnetics II

Question 4.1

$$\begin{pmatrix} V_{so}^- \\ V_{se}^+ \end{pmatrix} = S \begin{pmatrix} V_{so}^+ \\ V_{se}^- \end{pmatrix}$$

$$A = \underbrace{\frac{1}{2}\begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix}}_{D_1} T \underbrace{\begin{pmatrix} 1 & 1 \\ Y_2 & -Y_2 \end{pmatrix}}_{C_2} \quad \text{AND} \quad \begin{pmatrix} V_{so} \\ I_{so} \end{pmatrix} = T \begin{pmatrix} V_{se} \\ I_{se} \end{pmatrix}$$

Decomposition in medium 1 Composition in medium 2 $\Rightarrow C_2 D_2 = I$

$$\Rightarrow \underbrace{\frac{1}{2}\begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix}}_{D_1}\begin{pmatrix} V_{so} \\ I_{so} \end{pmatrix} = \underbrace{A}_{D_1 T C_2} \cdot \underbrace{\frac{1}{2}\begin{pmatrix} 1 & Z_2 \\ 1 & -Z_2 \end{pmatrix}}_{D_2}\begin{pmatrix} V_{se} \\ I_{se} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} V_{so}^+ \\ V_{so}^- \end{pmatrix} = A \begin{pmatrix} V_{se}^+ \\ V_{se}^- \end{pmatrix} = \begin{pmatrix} A_{11} V_{se}^+ + A_{12} V_{se}^- \\ A_{21} V_{se}^+ + A_{22} V_{se}^- \end{pmatrix}$$

$$\Rightarrow \begin{aligned} V_{so}^- - A_{21} V_{se}^+ &= A_{22} V_{se}^- \\ - A_{11} V_{se}^+ &= - V_{so}^+ + A_{12} V_{se}^- \end{aligned}$$

$$\Rightarrow \begin{pmatrix} V_{so}^- \\ V_{se}^+ \end{pmatrix} = \begin{pmatrix} 1 & -A_{21} \\ 0 & -A_{11} \end{pmatrix}^{-1} \begin{pmatrix} 0 & A_{22} \\ -1 & A_{12} \end{pmatrix} \begin{pmatrix} V_{so}^+ \\ V_{se}^- \end{pmatrix} = S \begin{pmatrix} V_{so}^+ \\ V_{se}^- \end{pmatrix}$$

$$\Rightarrow S = \begin{pmatrix} 1 & -\frac{A_{21}}{A_{11}} \\ 0 & \frac{-1}{A_{11}} \end{pmatrix} \begin{pmatrix} 0 & A_{22} \\ -1 & A_{12} \end{pmatrix} = \frac{1}{A_{11}} \begin{pmatrix} +A_{21} & (A_{11}A_{22} - A_{12}A_{21}) \\ 1 & -A_{12} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} T_{11} + Z_1 Y_2 T_{22} + Z_1 T_{21} + Y_2 T_{12} & T_{11} - Z_1 Y_2 T_{22} + Z_1 T_{21} - Y_2 T_{12} \\ T_{11} - Z_1 Y_2 T_{22} - Z_1 T_{21} + Y_2 T_{12} & T_{11} + Z_1 Y_2 T_{22} - Z_1 T_{21} - Y_2 T_{12} \end{pmatrix}$$

Note that $A_{11} A_{22} - A_{12} A_{21} = \det A = Z_1 Y_2$

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Question 4.2

$$\Gamma = \frac{r-1+jx}{r+1+jx} = 1 - \frac{2}{r+1+jx} \quad \begin{cases} r \geq 0 & \text{constant} \\ x \in \mathbb{R} \end{cases}$$

$$\lim_{x \to \pm\infty} \Gamma = 1 , \quad \Gamma\Big|_{x=0} = \frac{r-1}{r+1} \Rightarrow$$ The centre of the circle should be at $\frac{1}{2}\left(1+\frac{r-1}{r+1}\right) = \frac{r}{r+1}$

$$\Rightarrow \Gamma - \frac{r}{r+1} = \frac{1}{r+1} - \frac{2}{r+1+jx} = \frac{1}{r+1}\left[\frac{-(r+1)+jx}{(r+1)+jx}\right]$$

$$= \frac{-1}{r+1}\left[\frac{1-j\frac{x}{r+1}}{1+j\frac{x}{r+1}}\right] \quad \frac{\frac{x}{r+1} = \tan\varphi}{\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})} \quad \frac{-1}{r+1}\,\frac{1-j\tan\varphi}{1+j\tan\varphi} = \frac{-e^{-2j\varphi}}{r+1}$$

$$\Rightarrow \Gamma = \frac{r}{r+1} - \frac{e^{-2j\varphi}}{r+1}$$ $\Rightarrow$ circle with centre at $\frac{r}{r+1}$ and radius $\frac{1}{r+1}$

Question 4.3

Horizontal line segment $r+jx$, $\begin{cases} x \in \mathbb{R} & \text{constant} \\ r \geq 0 \end{cases}$

$\lim_{r \to \infty} \Gamma = 1$ where it touches the real axis

$r=0 \Rightarrow \Gamma = \frac{-1+jx}{1+jx} \Rightarrow |\Gamma| = 1 \Rightarrow$ point on unit circle

For $r=-1$ (outside the range of passive loads) we would have $\Gamma = 1 - \frac{2}{jx} \Rightarrow$ centre of circle must be at $1 - \frac{1}{jx}$

$$\Rightarrow \Gamma - 1 + \frac{1}{jx} = \frac{1}{jx}\left[1 - \frac{2jx}{r+1+jx}\right] = \frac{1}{jx}\left[\frac{r+1-jx}{r+1+jx}\right]$$

Again, set $\tan\varphi = \frac{x}{r+1} \Rightarrow |\tan\varphi| \in (0, |x|)$

$$\Rightarrow \Gamma = 1 - \frac{1}{jx} + \frac{1}{jx}e^{-2j\varphi}$$ $\Rightarrow$ circle segment

Question 4.4.

a) $k_1 = \omega\sqrt{\varepsilon_1\mu_1} = \omega\sqrt{\varepsilon_0\mu_0}\, n_1 = k_0 n_1$; $k_2 = k_0 n_2$; $Z_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{n_1} = \frac{Z_0}{n_1}$

$Z_2 = \frac{Z_0}{n_2}$ $\quad Y_1 = \frac{n_1}{Z_0}$ $\quad Y_2 = \frac{n_2}{Z_0}$

b) $\left.\begin{aligned} -\frac{\partial V}{\partial z} &= j\omega\mu I \\ -\frac{\partial I}{\partial z} &= j\omega\varepsilon V \end{aligned}\right\} \Rightarrow \underbrace{\begin{aligned} -\frac{\partial V}{\partial z} &= jk_0 Z_0 I \\ -\frac{\partial I}{\partial z} &= jk_0 \frac{n_1}{Z_0} V \end{aligned}}_{0<z<d} \qquad \underbrace{\begin{aligned} -\frac{\partial V}{\partial z} &= jk_0 Z_0 I \\ -\frac{\partial I}{\partial z} &= jk_0 \frac{n_2}{Z_0} V \end{aligned}}_{z>d}$

c) $I = \frac{-1}{jk_0 Z_0}\frac{\partial}{\partial z}\left[-jZ_1 I_s(0)\sin k_1 z\right] = \frac{k_1 Z_1}{k_0 Z_0} I_s(0)\cos k_1 z = I_s(0)\cos k_1 z$ for $0<z<d$.

$I = \frac{+jk_2}{jk_0 Z_0} V_s(d)\, e^{-jk_2(z-d)} = \frac{n_2}{Z_0} V_s(d)\, e^{-jk_2(z-d)}$ for $z>d$.

d) $\vec{a}_z \times \left[\vec{E}_s(d+0) - \vec{E}_s(d-0)\right] = 0 \Rightarrow V_s|_{z\downarrow d} = V_s|_{z\uparrow d}$

$\vec{a}_z \times \left[\vec{H}_s(d+0) - \vec{H}_s(d-0)\right] = \vec{J}_{sS} \Rightarrow -\vec{a}_x\left[I_s(d+0) - I_s(d-0)\right] = J_{sS}\,\vec{a}_x$

$\underbrace{-jZ_1 I_s(0)\sin(k_1 d)}_{V_s(z\uparrow d)} = \underbrace{V_s(d)}_{V_s(z\downarrow d)} \Rightarrow V_s(d) = -j\frac{Z_0}{n_1} I_s(0)\sin(k_1 d)$

$I_s(z\downarrow d) - I_s(z\uparrow d) = -J_{sS} \Rightarrow \frac{n_2}{Z_0}V_s(d) - I_s(0)\cos(k_1 d) = -J_{sS}$.

$\Rightarrow \left[j\frac{n_2}{n_1}\sin(\underbrace{k_1 d}_{k_0 n_1 d}) + \cos(\underbrace{k_1 d}_{k_0 n_1 d})\right] I_s(0) = J_{sS}$

e) $\vec{S} = \vec{E}\times\vec{H}^* = \frac{n_2}{Z_0}|V_s(d)|^2 = \frac{n_2}{Z_0}\frac{Z_0^2}{n_1^2}|I_s(0)|^2\sin^2(k_1 d)$

$\mu_0|\vec{H}_s|^2 + \varepsilon_1|\vec{E}_s|^2 \underset{0<z<d}{=} \mu_0|I_s(0)|^2\cos^2(k_1 z) + \underbrace{\varepsilon_0 n_1^2 Z_1^2}_{\mu_0}|I_s(0)|^2\sin^2(k_1 z)$

$= \mu_0|I_s(0)|^2 \Rightarrow \langle W_{em}\rangle_T = \frac{d}{4}\mu_0|I_s(0)|^2$

$Q = \dfrac{\frac{d}{4}\omega\mu_0|I_s(0)|^2}{\frac{1}{2}Z_0\frac{n_2}{n_1^2}|I_s(0)|^2\sin^2(k_1 d)} = \dfrac{k_0 d n_1^2}{2n_2\sin(k_0 n_1 d)}$