Communication Theory (5ETB0)
Module 4.1

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Module 4.1

Presentation Outline

Definitions

Source $\xrightarrow{m}$ Transmitter $\xrightarrow{s_m \in \mathcal{S}}$ DICO Channel $\xrightarrow{r \in \mathbb{R}}$ Receiver $\xrightarrow{\hat{m}}$ Dest.

Definitions

- Source: Produces a *message* $m \in \mathcal{M} \triangleq \{1, 2, \ldots, |\mathcal{M}|\}$ with probability $\Pr\{M = m\}$ for $m \in \mathcal{M}$. The r.v. is M
- Transmitter: Sends a *signal* $s_m \in \mathcal{S} \subset \mathbb{R}$ if message m is to be transmitted. The r.v. is S
- DICO Channel: Produces output $r \in (-\infty, \infty) = \mathbb{R}$ (r.v. is R) with probability density function $p_R(r|S = s_m) = p_R(r|M = m)$
- Receiver: Forms an *estimate* $\hat{m}$ by observing the received channel output $r \in \mathbb{R}$ using a mapping $\hat{m} = f(r) \in \mathcal{M}$. The r.v. is $\hat{M}$

Decision Variables, MAP and ML

Decision Variables for DICO Channels

The **decision variables** are

$\Pr\{M = m, R = r\} = \Pr\{M = m\} p_R(r|S = s_m) = \Pr\{M = m\} p_R(r|M = m)$

MAP decision rule

The maximum a-posteriori probability (MAP) decision rule is

$$\hat{m}^{\mathsf{MAP}}(r) \triangleq \operatorname*{argmax}_{m \in \mathcal{M}} \Pr\{M = m | R = r\} \tag{1}$$

$$= \operatorname*{argmax}_{m \in \mathcal{M}} \Pr\{M = m\} p_R(r|M = m) \tag{2}$$

ML decision rule

The maximum likelihood (ML) decision rule is

$$\hat{m}^{\mathsf{ML}}(r) \triangleq \operatorname*{argmax}_{m \in \mathcal{M}} p_R(r|M = m) \tag{3}$$

Module 4.1

Presentation Outline

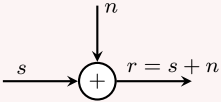
The AGN Channel

Scalar AGN channel

The **AGN channel** adds Gaussian noise N to the input signal S.

This Gaussian noise N has variance σ^2 and mean 0.
Its PDF is

$$p_N(n) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n^2}{2\sigma^2}\right)$$



The noise variable N is assumed to be independent of the signal S.

Two Questions

$$\text{Q1: } \int_{-\infty}^{\infty} p_N(n)\, dn = ? \quad \text{and} \quad \text{Q2: } p_N(n|S = s_m) \stackrel{?}{=} p_N(n) \tag{4}$$

The AGN Channel: A Matlab Example

Conditional AGN PDF

The conditional PDF of $R = r$ when the signal is $S = s_m$ is a Gaussian PDF, i.e.,

$$p_R(r|S = s_m) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r - s_m)^2}{2\sigma^2}\right) \tag{5}$$

where σ^2 is the variance of the AGN.

Error Probability

For the Matlab example, if a threshold ad $r^* = 0$ is used, the error probability can be obtained by solving the following integral

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r - 1)^2}{2\sigma^2}\right) dr \tag{6}$$

This type of integral is very popular.

Module 4.1

Presentation Outline

MAP Rule for the bi-AGN Channel (1/2)

Two Messages: Binary-input AGN (bi-AGN) Channel

- Assume that $|\mathcal{M}| = 2$: two messages. M can be either 1 or 2.
- The conditional PDF of $R = r$ when the signal is $S = s_m$ is

$$p_R(r|S = s_m) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r - s_m)^2}{2\sigma^2}\right), \text{ for } m = 1, 2$$

MAP Rule bi-AGN Channel

MAP receiver for $\hat{m} = f(r) = 1$ if

$$\Pr\{M = 1\} p_R(r|S = 1) \geq \Pr\{M = 2\} p_R(r|S = 2) \tag{7}$$

and $\hat{m} = 2$ otherwise. This means $\hat{m} = f(r) = 1$ if

$$\Pr\{M = 1\} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r - s_1)^2}{2\sigma^2}\right) \geq \Pr\{M = 2\} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r - s_2)^2}{2\sigma^2}\right)$$

and $\hat{m} = 2$ otherwise.

MAP Rule for the bi-AGN Channel (2/2)

MAP Threshold

One can show (complete derivation in course reader) that $\geq$ becomes $=$ in the MAP ir $r = r^*$, where

$$r^* \triangleq \frac{\sigma^2}{s_1 - s_2} \ln \frac{\Pr\{M=2\}}{\Pr\{M=1\}} + \frac{s_1 + s_2}{2}$$

Optimum receiver for the bi-AGN channel

A receiver that decides $\hat{m} = 1$ if

$$r \geq r^*$$

and $\hat{m} = 2$ otherwise, is optimum.

What is needed by MAP?

MAP threshold splits the real line (two intervals) and it depends on noise variance, a-priori probabilities, and transmitted symbols.

ML Rule for the bi-AGN Channel

Optimum Threshold for Equiprobable Messages: ML Rule

When the a-priori probabilities $\Pr\{M=1\}$ and $\Pr\{M=2\}$ are equal, the optimum threshold is

$$r^* = \frac{s_1+s_2}{2}.$$

This corresponds to the ML receiver, which chooses $\hat{m}=f(r)=1$ if

$$p_R(r|S=1) \geq p_R(r|S=2)$$

and $\hat{m}=2$ otherwise. This means $\hat{m}=f(r)=1$ if

$$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(r-s_1)^2}{2\sigma^2}\right) \geq \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(r-s_2)^2}{2\sigma^2}\right)$$

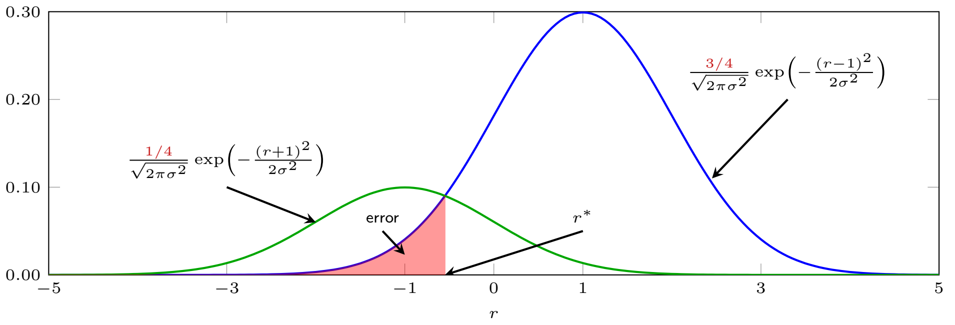
and $\hat{m}=2$ otherwise.

Example: MAP vs. ML for bi-AGN (1/3)

MAP Thresholds

$$\Pr\{M=1\}=3/4,\ s_1=+1, \qquad \Pr\{M=2\}=1/4,\ s_2=-1,$$

$$\implies r^* = -\frac{\ln(3)}{2} \approx -0.549$$

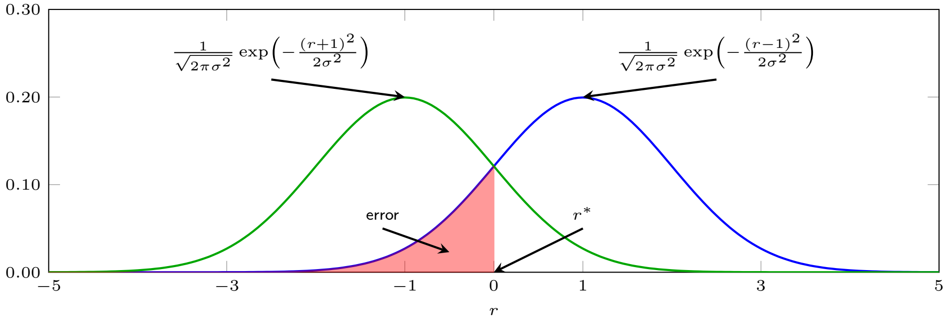


Example: MAP vs. ML for bi-AGN (2/3)

ML Thresholds

$$\Pr\{M=1\}=1/2,\ s_1=+1, \qquad \Pr\{M=2\}=1/2,\ s_2=-1,$$

$$\implies r^* = \frac{1-1}{2} = 0$$



Example: MAP vs. ML for bi-AGN (3/3)

MAP vs. ML

- Threshold in Matlab example was in fact ML detection
- MAP is optimum but more complex than ML
- ML is simple to implement (fixed threshold, one-bit decisions) but suboptimal in general

Summary Module 4.1

Take Home Messages

- In DICO channels the output is continuous (PMFs $\rightarrow$ PDFs)
- AGN model studied in detail
- MAP and ML detectors
  - For bi-AGN, MAP and ML create two intervals via a threshold
  - MAP is optimum but more complex
  - Error probabilities of MAP and ML are not the same

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