Communication Theory (5ETB0)
Module 9.2

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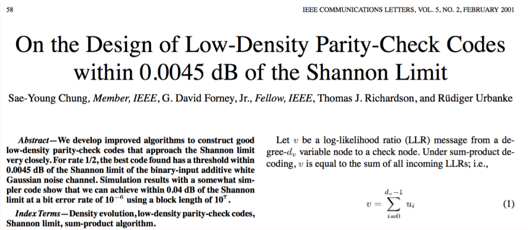
Module 9.2

Presentation Outline

How do we achieve capacity?

Modern Codes

- Turbo Codes, invented in 1990-1991, “to good to be true”. Used in 3G and 4G standards. [1]
- Low-density parity check codes, invented in 1960 and re-discovered in 1996. Used in WiFi and DVB.



58 IEEE COMMUNICATIONS LETTERS, VOL. 5, NO. 2, FEBRUARY 2001

On the Design of Low-Density Parity-Check Codes within 0.0045 dB of the Shannon Limit

Sae-Young Chung, *Member, IEEE*, G. David Forney, Jr., *Fellow, IEEE*, Thomas J. Richardson, and Rüdiger Urbanke

***Abstract*—We develop improved algorithms to construct good low-density parity-check codes that approach the Shannon limit very closely. For rate 1/2, the best code found has a threshold within 0.0045 dB of the Shannon limit of the binary-input additive white Gaussian noise channel. Simulation results with a somewhat simpler code show that we can achieve within 0.04 dB of the Shannon limit at a bit error rate of 10^{-6} using a block length of 10^7.**

***Index Terms*—Density evolution, low-density parity-check codes, Shannon limit, sum-product algorithm.**

Let v be a log-likelihood ratio (LLR) message from a degree-d_v variable node to a check node. Under sum-product decoding, v is equal to the sum of all incoming LLRs; i.e.,

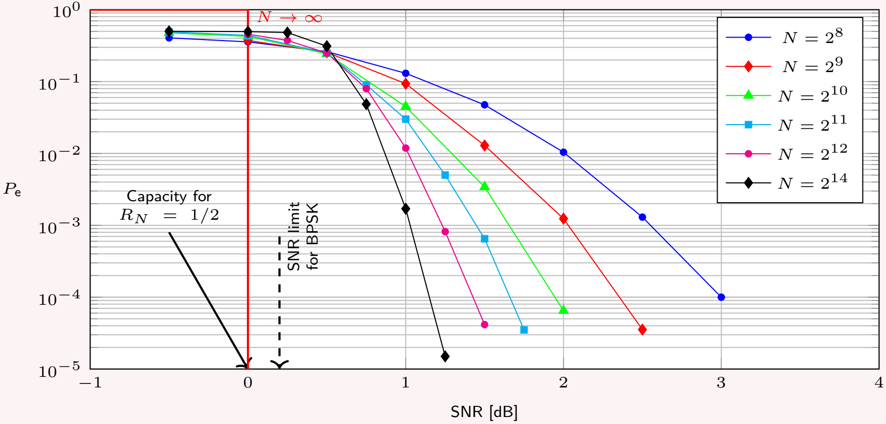
$$v = \sum_{i=0}^{d_v-1} u_i \qquad (1)$$

- Polar Codes, first codes with explicit construction that can be proven to achieve the channel capacity. Used in 5G NR. [2]

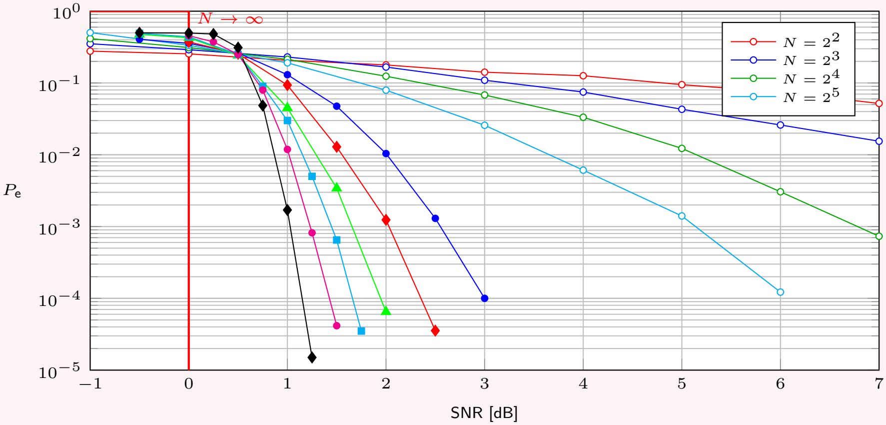
[1] C. Berrou et al., “Near Shannon limit error-correcting coding and decoding: Turbo-codes,” in Proc. IEEE ICC 1993.

[2] E. Arikan, “Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels,” in IEEE Trans. Inf. Theory, July 2009.

Example: Polar Codes, $R_N = 1/2$

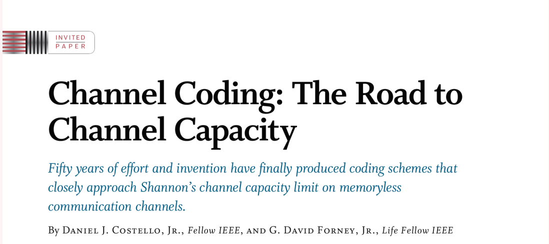


Example: Random Codes vs. Polar Codes



Channel Coding: The Road to Channel Capacity

Turbo Codes vs. NASA Codes

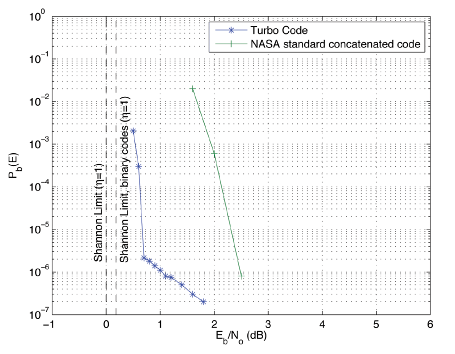


INVITED PAPER

Channel Coding: The Road to Channel Capacity

Fifty years of effort and invention have finally produced coding schemes that closely approach Shannon's channel capacity limit on memoryless communication channels.

By Daniel J. Costello, Jr., *Fellow IEEE*, and G. David Forney, Jr., *Life Fellow IEEE*



Excellent review paper [3] where coding history is explained, including NASA codes and the so-called turbo revolution

All this and more in 5LSF0, "Applications of Information Theory" (Q4) and "Information Theory" 5XSE0 (Q3)

[3] Daniel J. Costello, Jr., and G. D. Forney, Jr., "Channel Coding: The Road to Channel Capacity," Proc. of the IEEE, vol. 95, no. 6, June 2007.

Module 9.2

Presentation Outline

Part I Motivation: Modern Codes

Part II Capacity Proof

Capacity Proof

Sketch of the Proof

Four key ingredients:

Part 1: Sphere Hardening of Gaussian Vectors

Part 2: Random Coding Generation

Part 3: Everything is Gaussian

Part 4: Error Probability

Part 1: Sphere Hardening of Gaussian Vectors (1/2)

Gaussian Vectors

Consider a random Gaussian vector $\underline{G}$ with N components, each with mean 0 and variance σ_g^2 and a normalized version of this vector: $\underline{G}' = \underline{G}/\sqrt{N}$. The square norms (lengths) of these vectors ($\|\underline{G}\|^2$ and $\|\underline{G}'\|^2$) are random variables.

Sphere Hardening of Gaussian Vectors

The normalized vector $\underline{G}'$ can be shown to have the following mean and variance:

$$E\left[\|\underline{G}'\|^2\right] = \sigma_g^2, \qquad \text{var}\left[\|\underline{G}'\|^2\right] = \frac{2\sigma_g^4}{N}.$$

Thus, vectors $\underline{G}'$ are **on the surface of a hypersphere** with radius σ_g. Fluctuations are possible, however for $N \to \infty$ these fluctuations disappear.

Part 1: Sphere Hardening of Gaussian Vectors (2/2)

Part 2: Random Code Generation

Generating a Random Code

- Fix the number of signal vectors $|\mathcal{M}|$ and their number of components (dimensions) N
- Select $|\mathcal{M}|$ signal vectors $\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{|\mathcal{M}|}$ *at random*, independently of each other
- Make each vector component is a random sample from a Gaussian density with mean 0 and variance E_N:
  - By the sphere-hardening argument, *energies of the vectors* $E[\|\underline{S}\|^2]$ are actually roughly NE_N.
  - The *expected energy per dimension* is E_N.
- Consider the *ensemble* of all signal sets that can be chosen in this way

Part 3: Everything is Gaussian

Gaussian Inputs, Gaussian Outputs

- The channel output vector is $\underline{r} = \underline{s}_m + \underline{n}$
- The components of the noise vector $\underline{n}$ are Gaussian with mean 0 and variance $N_0/2$
- The sum of two independent Gaussian vectors is also Gaussian, with all components having mean 0 and variance $E_N + N_0/2$

Consider Normalized Quantities with $N \to \infty$

- Output is $\underline{r}' = \underline{s}'_m + \underline{n}'$, where $\underline{s}'_m = \underline{s}_m/\sqrt{N}$ and $\underline{n}' = \underline{n}/\sqrt{N}$
- Normalized vector $\underline{s}'_m$ is on the surface of a hypersphere with radius $\sqrt{E_N}$, i.e., $\|\underline{s}'_m\|^2 = E_N$
- Normalized noise vector $\underline{n}'$ is on the surface of a hypersphere with radius $\sqrt{N_0/2}$, i.e., $\|\underline{n}'\|^2 = N_0/2$
- Normalized received vector $\underline{r}'$ is on the surface of a hypersphere with radius $\sqrt{E_N + N_0/2}$, i.e., $\|\underline{r}'\|^2 = E_N + N_0/2$

Part 4: Error Probability

Average Error Probability

- We are interested in $P_{\mathsf{e}}^{\mathsf{av}}$, i.e., the error probability P_{e} *averaged over the ensemble of signal sets*, i.e.,

$$P_{\mathsf{e}}^{\mathsf{av}} = \int_{\mathbb{R}^{N\cdot|\mathcal{M}|}} p(\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{|\mathcal{M}|}) P_{\mathsf{e}}(\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{|\mathcal{M}|}) d\underline{s}_1 d\underline{s}_2 \ldots d\underline{s}_{|\mathcal{M}|}$$

- Once we know $P_{\mathsf{e}}^{\mathsf{av}}$ we claim that there exists *at least one signal set* $\{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{|\mathcal{M}|}\}$ with error probability $P_{\mathsf{e}}(\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{|\mathcal{M}|}) \leq P_{\mathsf{e}}^{\mathsf{av}}$
- It can then be shown that if $R_N = C_N - \delta$

$$\lim_{N\to\infty} P_{\mathsf{e}}^{\mathsf{av}} \leq \lim_{N\to\infty} 2^{-\delta N} \sqrt{\frac{E_N + N_0/2}{N_0/2}} = 0$$

Summary Module 9.2

Take Home Messages

- Four key ingredients in the capacity proof
- Codes with Gaussian inputs achieve capacity
- Modern codes exist that approach capacity
- Is channel capacity mportant? Yes because...
  - It is a fundamental limit with beautiful and simple derivations
  - We want to approach such limits
  - Signal shaping

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