Question 2. Transfer & scatter matrices

2A. Although the $\frac{1}{4}\lambda$ transformer matches the load to the transmission line, it is only for <u>1 frequency</u> only. So, this limits the usage, e.g. <u>very limited bandwidth</u>

2B.

$$T(0,\ell) = \begin{pmatrix} \cos(k\ell) & jZ_1 \sin(k\ell) \\ jY \sin(k\ell) & \cos(k\ell) \end{pmatrix}$$

now: $\ell = \frac{1}{4}\lambda \qquad k = \frac{2\pi}{\lambda} \Rightarrow k\ell = \frac{1}{4}\lambda \frac{2\pi}{\lambda} = \frac{\pi}{2}$

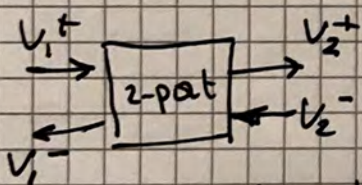
$$\Rightarrow T(0, \lambda/4) = \begin{pmatrix} 0 & jZ_1 \\ jY_1 & 0 \end{pmatrix}$$

2C. So, we have the transfer matrix and we need to find the scatter matrix.

From theory:

* <u>scatter matrix</u> $\quad \begin{pmatrix} V_1^- \\ V_2^+ \end{pmatrix} = \bar{\bar{S}} \begin{pmatrix} V_1^+ \\ V_2^- \end{pmatrix}$

$$\underbrace{\begin{pmatrix} V_1^- \\ V_2^+ \end{pmatrix}}_{\text{outgoing}} \qquad \underbrace{\begin{pmatrix} V_1^+ \\ V_2^- \end{pmatrix}}_{\text{incoming}}$$



* <u>transfer matrix</u> $\quad \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \bar{\bar{T}} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$

In this case it is handy to use the $\bar{\bar{A}}$-matrix

$$\bar{\bar{A}} = D_1 T C_2$$

$$\Rightarrow \begin{pmatrix} V_1^+ \\ V_1^- \end{pmatrix} = \bar{\bar{A}} \begin{pmatrix} V_2^+ \\ V_2^- \end{pmatrix}$$

with $D = \frac{1}{2}\begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ Y_0 & -Y_0 \end{pmatrix}$

So, first calculate the $\bar{\bar{A}}$-matrix and then rearrange to get the S-matrix

$$\bar{\bar{A}} = \bar{\bar{D}}_1 . T . \bar{\bar{C}}_2 = \underbrace{\underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2}Z_1 \\ \frac{1}{2} & -\frac{1}{2}Z_1 \end{pmatrix}\begin{pmatrix} 0 & jZ_1 \\ jY_1 & 0 \end{pmatrix}}_{\begin{pmatrix} \frac{1}{2}j & \frac{1}{2}jZ_1 \\ -\frac{1}{2}j & \frac{1}{2}jZ_1 \end{pmatrix}}\begin{pmatrix} 1 & 1 \\ Y_0 & -Y_0 \end{pmatrix}}_{\begin{pmatrix} \frac{1}{2}j + \frac{1}{2}j & \frac{1}{2}j - \frac{1}{2}j \\ -\frac{1}{2}j + \frac{1}{2}j & -\frac{1}{2}j - \frac{1}{2}j \end{pmatrix}} = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$$

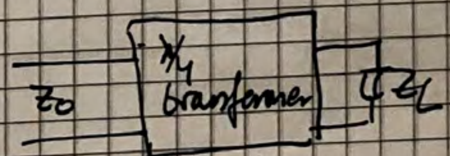
$$\Rightarrow \begin{pmatrix} V_1^+ \\ V_1^- \end{pmatrix} = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}\begin{pmatrix} V_2^+ \\ V_2^- \end{pmatrix}$$

$$\Rightarrow \begin{cases} V_1^+ = jV_2^+ \Rightarrow V_2^+ = -jV_1^+ \\ V_1^- = -jV_2^- \end{cases}$$

$$\Rightarrow \begin{pmatrix} V_1^- \\ V_2^+ \end{pmatrix} = \bar{\bar{S}}\begin{pmatrix} V_1^+ \\ V_2^- \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}}}\begin{pmatrix} V_1^+ \\ V_2^- \end{pmatrix}$$

<u>2d.</u> $Z_1 = \sqrt{Z_0 Z_L}$ → although some students might know this answer, the question is to deduce the answer.

situation:



$$\uparrow$$
$$\text{T-matrix} = \begin{pmatrix} 0 & jZ_1 \\ jY_1 & 0 \end{pmatrix}$$

<u>idea</u> → use T-matrix to get an expression of V_1^+ and V_1^- (input 2-port = $\lambda/4$ transformer)
V_1^- should be <u>zero</u> (= matched)

Decomposition matrix:

$$\begin{pmatrix} V_1^+ \\ V_1^- \end{pmatrix} = \bar{\bar{D}} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

$$\bar{\bar{D}} \longrightarrow = \frac{1}{2}\begin{pmatrix} 1 & Z \\ 1 & -Z \end{pmatrix}$$

now
$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = T \cdot \begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & jZ_1 \\ jY_1 & 0 \end{pmatrix}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} V_1^+ \\ V_1^- \end{pmatrix} = \begin{pmatrix} V_1^+ \\ \Gamma \cdot V_1^+ \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{pmatrix}\begin{pmatrix} 0 & jZ_1 \\ jY_1 & 0 \end{pmatrix}\begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

$$\hookrightarrow = \begin{pmatrix} T \cdot V_1^+ \\ T \cdot V_1^+ \cdot Y_L \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ \Gamma \end{pmatrix} = \frac{T}{2}\begin{pmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{pmatrix}\begin{pmatrix} 0 & jZ_1 \\ jY_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ Y_L \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ \Gamma \end{pmatrix} = \frac{T}{2}\begin{pmatrix} jY_1 Z_0 & jZ_1 \\ -jY_1 Z_0 & jZ_1 \end{pmatrix}\begin{pmatrix} 1 \\ Y_L \end{pmatrix} = \frac{jT}{2}\begin{pmatrix} Y_1 Z_0 + Z_1 Y_L \\ Y_L Z_1 - Y_1 Z_0 \end{pmatrix}$$

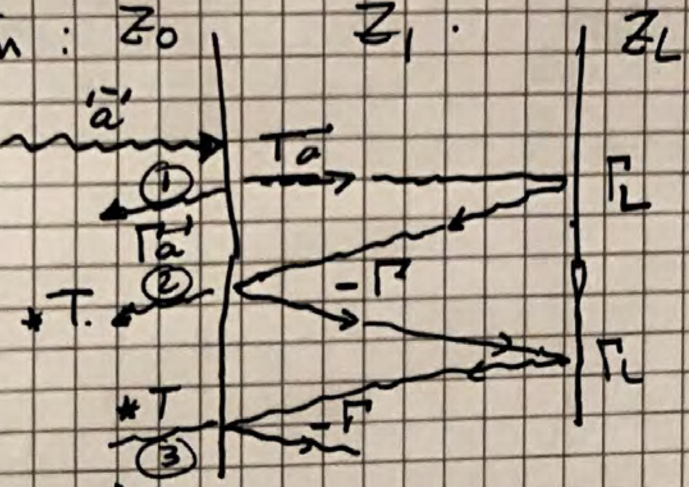
now, we know Γ should be ϕ (Z_L is matched to the transmission line, so no reflection)

$$\Rightarrow Y_L Z_1 - Y_1 Z_0 = \phi$$

$$\Rightarrow Y_L \cdot Z_1 = Y_1 Z_0 \Rightarrow \frac{Z_1}{Z_L} = \frac{Z_0}{Z_1}$$

$$\rightarrow \underline{\underline{Z_1 = \sqrt{Z_0 Z_L}}}$$

2e. Situation :



$V_1^+ = a$

$V_1^- = \Gamma a + (2) + (3) + (4) + \ldots$

and should be <u>zero</u> <u>of course</u>

So, task here is to identify all the waves and make this equal to zero — this should give the same answer as in 2d:

$$Z_1 = \sqrt{Z_0 Z_L}$$

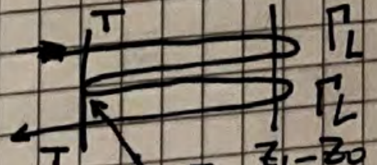
(A) First one is $\Gamma \cdot \bar{a}$ with $\Gamma = \dfrac{Z_1 - Z_0}{Z_1 + Z_0}$

(B) second one

$\underbrace{}_{\frac{1}{4}\lambda}$, $\rightarrow$ total $= \frac{1}{2}\lambda$ = $-$sign

$\rightarrow -T^2 \cdot \Gamma_L \cdot \bar{a}$

(C) third one



$\Gamma_* = \dfrac{Z_1 - Z_0}{Z_1 + Z_0} = -\Gamma$

$\underbrace{}_{\frac{1}{4}\lambda}$, $\rightarrow$ total $= 4 \cdot \frac{1}{4}\lambda = \lambda$ = +-sign.

$\Rightarrow -T^2 \Gamma_L^2 \Gamma \cdot \bar{a}$

(D) fourth one $= -T^2 \Gamma_L^3 \Gamma^2 \cdot \bar{a}$

etc. etc.

So, the complete V_1^- wave is:

$$\Gamma - T^2\Gamma_L - T^2\Gamma_L^2\Gamma - T^2\Gamma_L^3\Gamma^2 - \ldots\ldots$$

$$\Rightarrow \Gamma - T^2\Gamma_L(\Gamma_L\Gamma)^0 - T^2\Gamma_L(\Gamma_L\Gamma)^1 - T^2\Gamma_L(\Gamma_L\Gamma)^2 - \ldots$$

$$\Rightarrow \Gamma - T^2\Gamma_L\left(\sum_{n=0}^{\infty}(\Gamma_L\Gamma)^n\right)$$

$$\downarrow \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } r<1$$

$$\Rightarrow \Gamma - T^2\Gamma_L\left(\frac{1}{1-\Gamma_L\Gamma}\right)$$

$$\rightarrow \frac{\Gamma(1-\Gamma_L\Gamma)}{1-\Gamma_L\Gamma} - \frac{T^2\Gamma_L}{1-\Gamma_L\Gamma} = \frac{\Gamma - \Gamma_L\Gamma^2 - T^2\Gamma_L}{1-\Gamma_L\Gamma}$$

So, what we have:

$$V_1^- = \left(\frac{\Gamma - \Gamma_L\Gamma^2 - T^2\Gamma_L}{1-\Gamma_L\Gamma}\right) V_1^+$$

and — as mentioned before — should be ϕ

that means:

$$\Gamma - \Gamma_L\Gamma^2 - T^2\Gamma_L = \phi$$

we know:

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

$$\Gamma_L = \frac{Z_L - Z_1}{Z_L + Z_1}$$

$$T = 1 + \Gamma = 1 + \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{2Z_1}{Z_1 + Z_0}$$

now we fill this in the equation:

$$\Gamma - \Gamma_L \Gamma^2 - T^2 \Gamma_L$$

$$\Rightarrow \Gamma - \Gamma_L \Gamma^2 - (1 - \Gamma^2)\Gamma_L \qquad \boxed{T^2 = 1 - \Gamma^2}$$

$$\Rightarrow \Gamma - \Gamma_L$$

$$= \frac{Z_1 - Z_0}{Z_1 + Z_0} - \frac{Z_L - Z_1}{Z_L + Z_1}$$

$$\Rightarrow \frac{(Z_1 - Z_0)(Z_L + Z_1) - (Z_L - Z_1)(Z_1 + Z_0)}{(Z_1 + Z_0)(Z_L + Z_1)}$$

$$\Rightarrow \frac{Z_1 Z_L - Z_0 Z_L + Z_1^2 - Z_0 Z_1 - (-Z_1^2 + Z_L Z_1 + Z_L Z_0 - Z_1 Z_0)}{(Z_1 + Z_0)(Z_L + Z_1)}$$

$$\rightarrow \frac{2Z_1^2 - 2Z_0 Z_L}{(Z_1 + Z_0)(Z_L + Z_1)} = \frac{2(Z_1^2 - Z_0 Z_L)}{(Z_1 + Z_0)(Z_L + Z_1)}$$

this means $$Z_1^2 = Z_0 Z_L$$

$$\rightarrow Z_1 = \sqrt{Z_0 Z_L}$$

indeed the same as before.