Proposed Solutions EM II (5EPB0) - Rectangular Waveguides

1) Field visualization in rectangular waveguides

- Sketch the induced surface currents $\underline{J}_S$ at the walls of a rectangular waveguide propagating the TE_{10} mode. Hints:

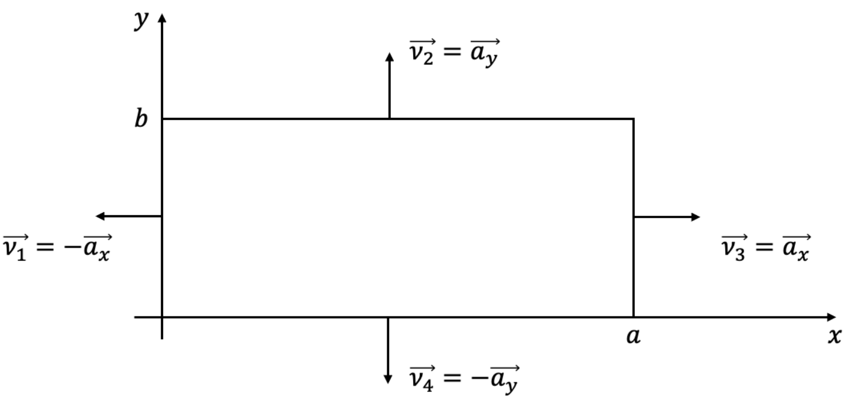
a. Find the explicit nonzero components for the TE_{10} mode.

$$E_y = \frac{-j\omega\mu A}{k_{mp}^2} k_m \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z}$$

$$H_x = \frac{\gamma A}{k_{mp}^2} k_m \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z}$$

$$H_z = A \cos\left(\frac{\pi x}{a}\right) e^{-\gamma z}$$

b. Identify a normal vector for every wall.



c. Apply the proper boundary condition in order to get an expression for $\underline{J}_S$.

General expression for the B.C. $\quad \rightarrow \quad -\vec{\nu} \times \vec{H} = -\vec{J}_s$

d. Find a mathematical expression for $\underline{J}_S$ at every wall $x = 0, a$ and $y = 0, b$.

Wall #1 $\quad x = 0 \quad \vec{a_x} \times \vec{H}_z \vec{a_z} = \vec{J}_s \quad \Rightarrow \quad \vec{J}_{s_{(x=0)}} = -Ae^{-\gamma z}\vec{a_y}$

Wall #2 $\quad y = b \quad -\vec{a_y} \times (H_z\vec{a_z} + H_x\vec{a_x}) = -H_z\vec{a_x} + H_x\vec{a_z} = \vec{J_s}$

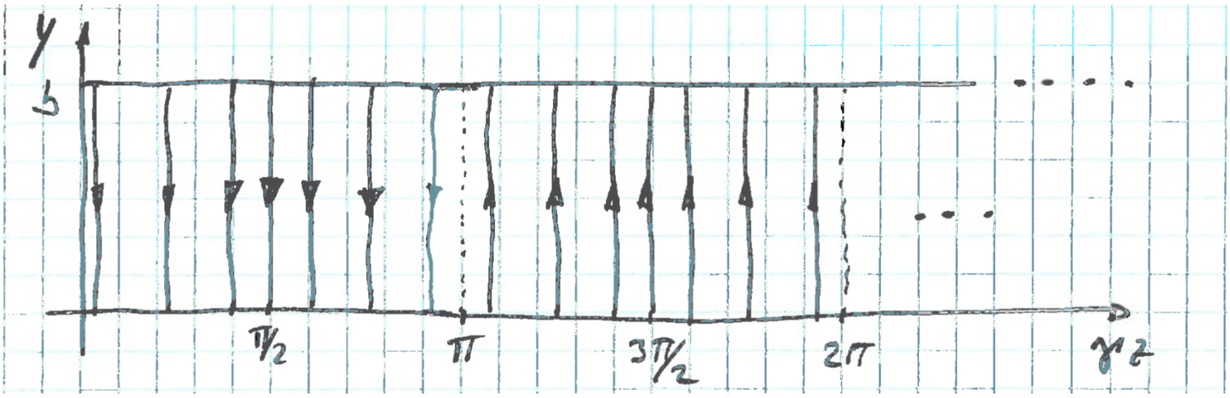
$$\Rightarrow \vec{J}_{s_{(y=b)}} = A\left(\cos\left(\frac{\pi x}{a}\right)\vec{a_x} - \frac{\gamma}{k_{10}}\sin\left(\frac{\pi x}{a}\right)\vec{a_z}\right)e^{-\gamma z}$$

Wall #3 $\quad x = a \quad -\vec{a_x} \times \vec{H}_z\vec{a_z} = \vec{J_s} \quad \Rightarrow \quad \vec{J}_{s_{(x=a)}} = -Ae^{-\gamma z}\vec{a_y}$

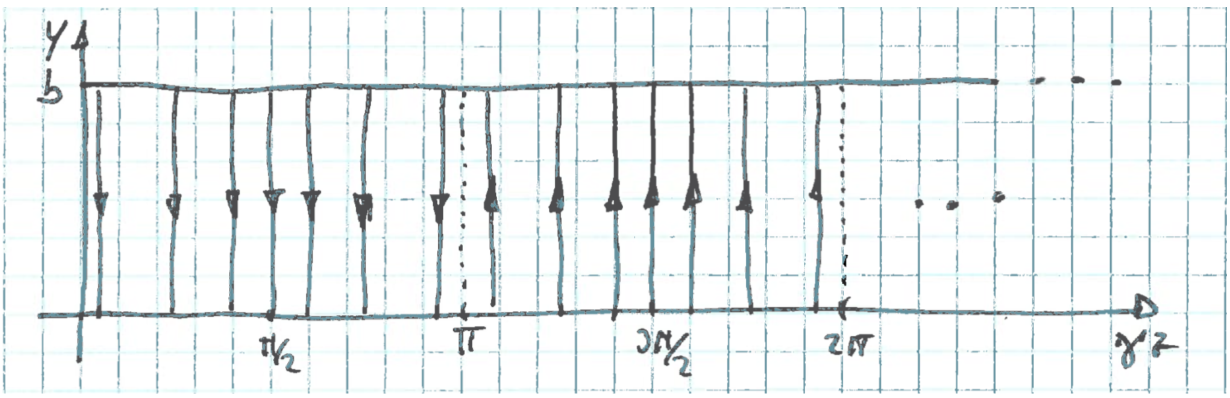
Wall #4 $\quad y = 0 \quad \vec{a_y} \times (H_z\vec{a_z} + H_x\vec{a_x}) = H_z\vec{a_x} - H_x\vec{a_z} = \vec{J_s}$

$$\Rightarrow \vec{J}_{s_{(y=0)}} = A\left(\cos\left(\frac{\pi x}{a}\right)\vec{a_x} - \frac{\gamma}{k_{10}}\sin\left(\frac{\pi x}{a}\right)\vec{a_z}\right)e^{-\gamma z}$$

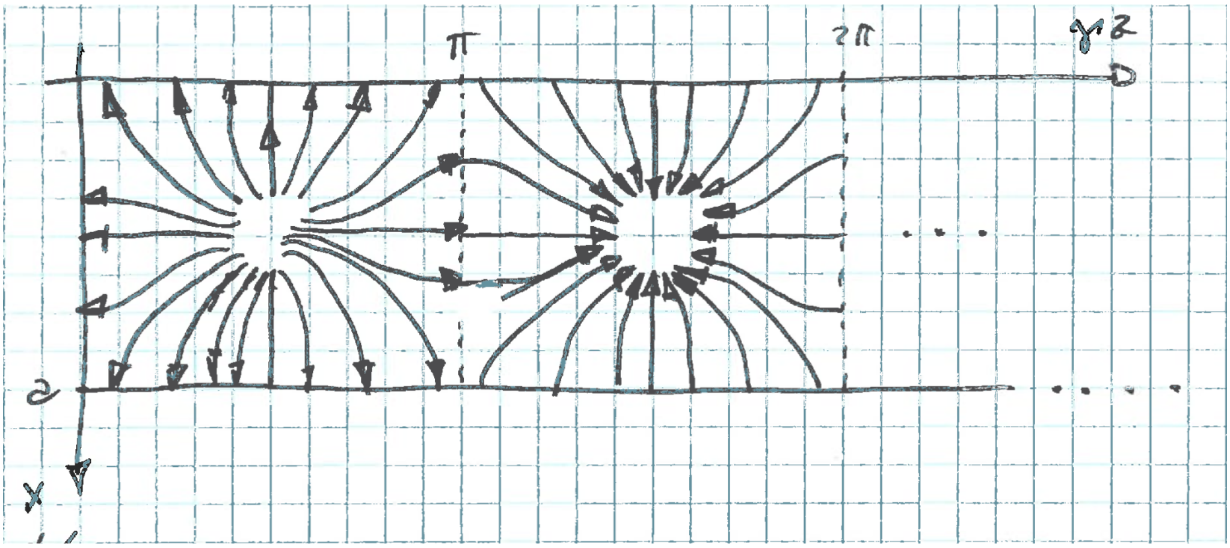
e. With the above solutions, sketch $\underline{J}_S$ at the walls.



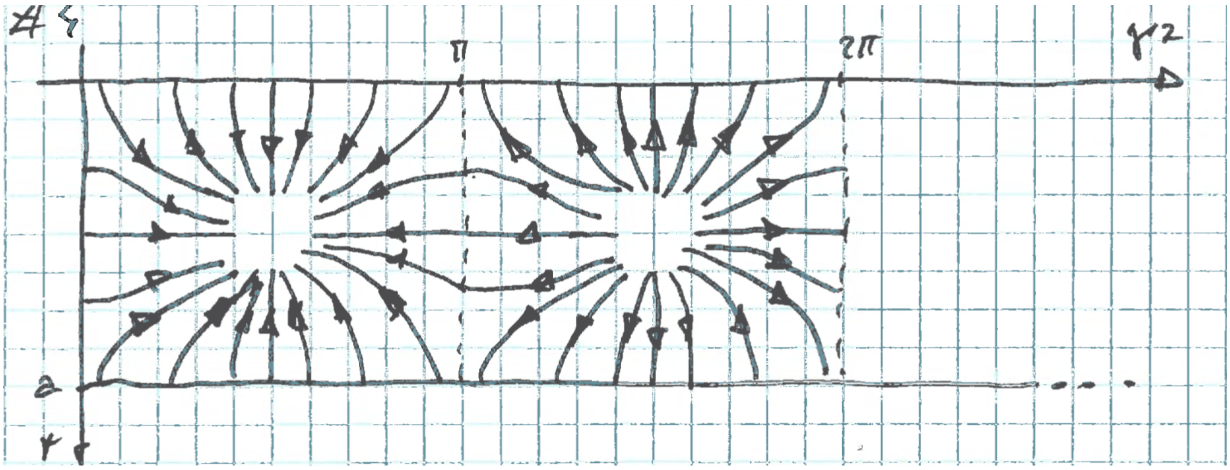
Wall 1



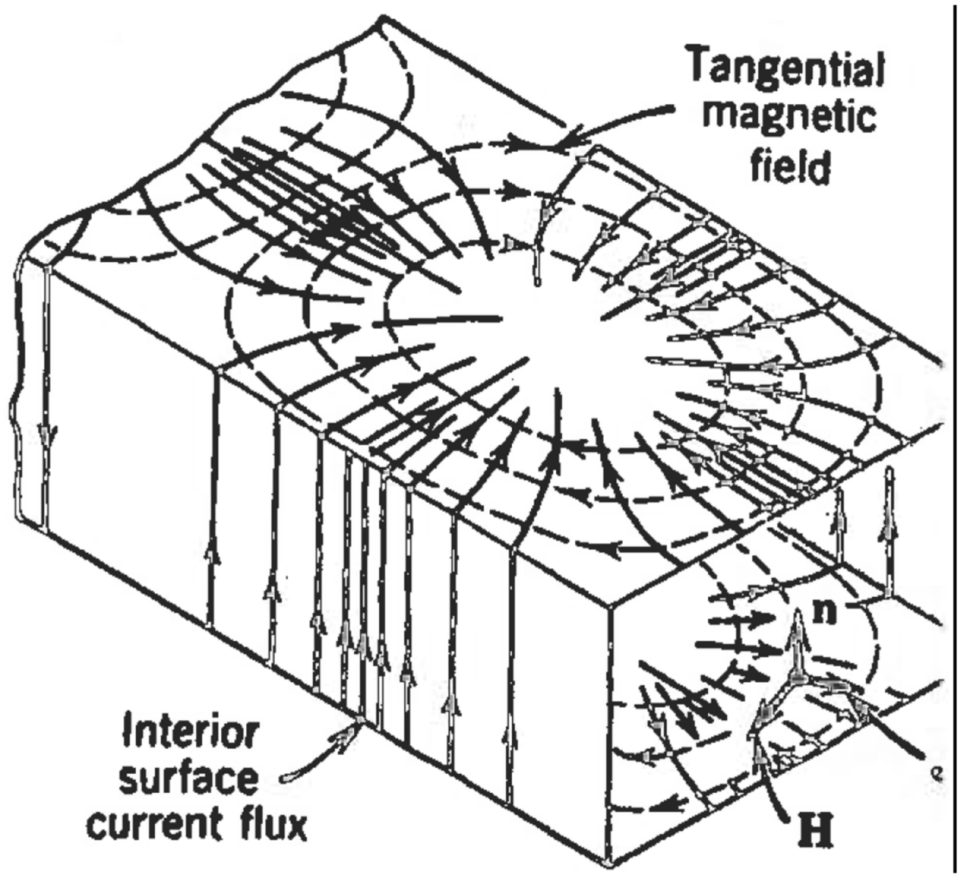
Wall 3



Wall 2



Wall 4



A combined 3D plot of $\vec{J}_s$

f. Sketch the surface charge density ρ_S. Hint: follow an analogous procedure as for $\underline{J}_S$ but applying the proper boundary conditions.

B.C. $-\vec{\nu} \cdot \vec{D} = \rho_s$

In walls #1 and #3 $\rho_s = 0$

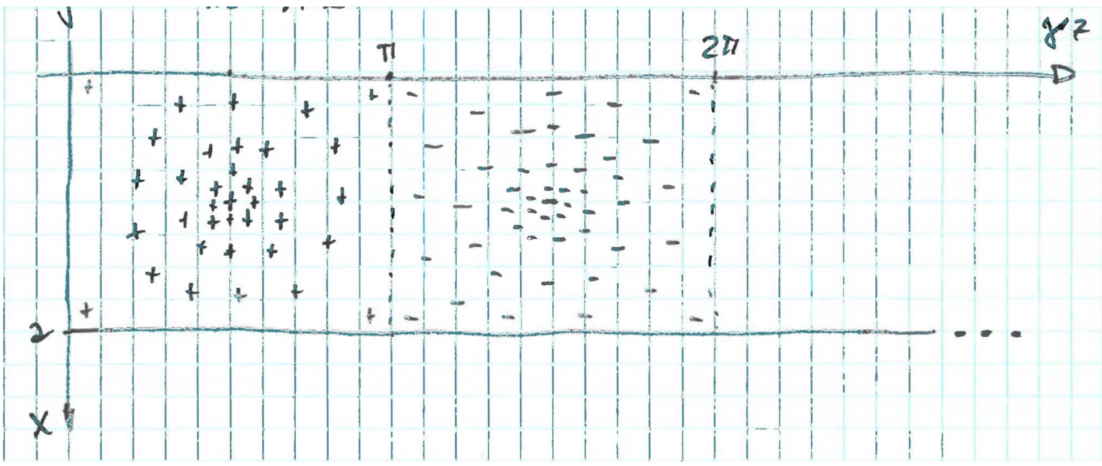
Wall #2

$$-\vec{a_y} \cdot \frac{(-j\omega\varepsilon\mu A)}{k_{10}} \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z} \vec{a_y} \quad \Rightarrow \quad \rho_{s_{(y=b)}} = \frac{j\omega\varepsilon\mu A}{k_{10}} \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z}$$

Wall #4

$$\vec{a_y} \cdot \varepsilon E_y \vec{a_y} = \rho_{s_{(y=0)}} = \frac{-j\omega\varepsilon\mu A}{k_{10}} \sin\left(\frac{\pi x}{a}\right) e^{-\gamma z}$$

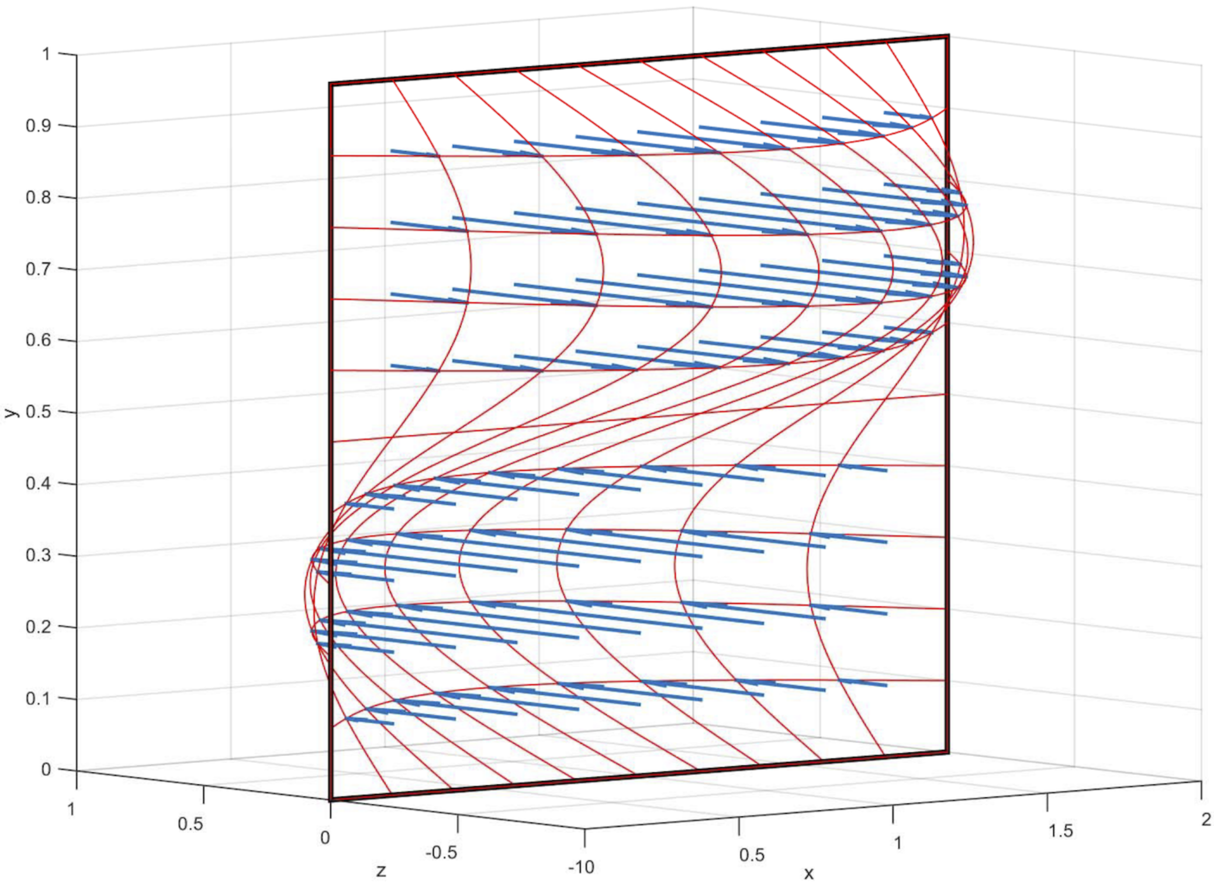
e.g. in wall #2



2) Field visualization in rectangular waveguides

Sketch a diagram showing the z-directed electric field component of the TM_{12} mode

$$E_z^{TM_{12}} = B \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right) e^{-\gamma z}$$



3) AM vs. FM radios in tunnels

Which radio signal, AM or FM, do you loose more easily in a tunnel[1] (and why)?

It's easier to loose AM, since its frequencies are much lower than cutoff, for fixed dimensions of the tunnel. Assumption: the walls of a tunnel act as a WG.

[1] Assuming the tunnel does not feature any amplification, retransmitter or internal transmitters

4) Waveguide design

a. Based on its cutoff frequency, determine the inside dimension of the smallest air-filled square metallic waveguide that will just propagate the lowest TM mode at the operating frequency: (a) 5 GHz, (b) 5 MHz, (c) 5 kHz.

Lowest TM is TM_{11} square WG $\Rightarrow a = b$

$$f_{11} = 5 \text{ GHz} = \frac{c}{2}\sqrt{\frac{2}{a^2}} \quad \Rightarrow \quad a = 4.24 \text{ cm}$$
$$f_{11} = 5 \text{ MHz} \quad \Rightarrow \quad a = 42.4 \text{ m}$$
$$f_{11} = 5 \text{ kHz} \quad \Rightarrow \quad a = 42.4 \text{ km}$$

b. Repeat a. for the TE_{10} mode.

$$f_{10} = \frac{c}{2}\sqrt{\frac{1}{a^2}} \quad \Rightarrow \quad a = \frac{c}{2f_{10}}$$

$$\begin{aligned} a &= 3 \text{ cm} \quad @\ 5 \text{ GHz} \\ a &= 30 \text{ cm} \quad @\ 5 \text{ MHz} \\ a &= 30 \text{ km} \quad @\ 5 \text{ kHz} \end{aligned}$$

c. A rectangular air-filled waveguide with dimensions 2.286×1.016 cm carries the dominant TE_{10} mode at the source frequency $f = 9.375$ GHz. Determine, for this mode: (a) the cutoff frequency f_{10}, (b) the propagation constant γ, (c) the wavelength λ_{10} in the waveguide, (d) the speed of the wave, (e) the wave impedance $\mathrm{Z}^{TE_{10}}$. (f) What is the cutoff frequency for the TE_{20} mode in this size waveguide? What do the propagation constant and the wave impedance become for the TE_{10} mode at $f = 4.5$ GHz? (g) Compare answers (b) through (e) with the values expected for a uniform plane wave in free space at the same frequency.

c-a.

$$f_{10} = \frac{c}{2a} = 6.56 \text{ GHz}$$

c-b.

$$\gamma_{10} = \sqrt{k_{10}^2 - \omega^2\varepsilon\mu} = \sqrt{\frac{\pi^2}{a^2} - \omega^2\varepsilon\mu} = j140.36 \ \ {}^{1}/_{m}$$

c-c.

$$\lambda_{10} = \frac{2\pi}{\gamma_{10}} = 4.57 \text{ cm}$$

c-d.

$$c_{10} = \lambda_{10} \cdot f \approx 1.43 c_0$$

c-e.

$$Z^{TE_{10}} = \frac{-E_y}{H_x} = \frac{j\omega\mu}{\gamma_{10}} = 527.37\ \Omega$$

c-f.

$$f_{20} = \frac{c}{a} = 13.12 \text{ GHz}$$

@ 4.5 GHz, the propagation coefficient becomes real, and $Z^{TE_{10}}$ imaginary (below cutoff)

$$\frac{\gamma_{10}}{j\beta_0} = 0.7145 \qquad \frac{c_{10}}{c_0} = 1.43$$

$$\frac{\lambda_{10}}{\lambda} = 1.4 \qquad \frac{Z^{TE_{10}}}{Z_0} = 1.4$$

5) Waveguide design

A rectangular waveguide of dimensions a and b ($a > b$), as shown in Fig. 13.7 from the book, is to be operated in a single mode regime. This means that only the dominant TE_{10} mode will be excited.

1. Determine the smallest ratio of the a/b dimensions that will allow the largest bandwidth of the single-mode operation.

If $a > b$, the dominant mode is the TE_{10} mode, with $f_{10} = \frac{c}{2a}$

The mode with the next higher cutoff frequency would be either the TE_{20} of the TE_{01}, whose cutoff frequencies are respectively:

$$f_{20} = \frac{c}{a} = 2f_{10}$$
$$f_{01} = \frac{c}{2b}$$

The largest BW for single-mode operation is then

$$f_{10} \leq f \leq 2f_{10} = f_{20} \leq f_{01}$$

$$\frac{c}{2a} \leq f \leq \frac{c}{2b} \;\Rightarrow\; a \geq b$$

$$\frac{c}{a} \leq \frac{c}{2b} \;\Rightarrow\; a \geq 2b \;\Rightarrow\; 2 \leq \frac{a}{b}$$

2. Design an air-filled rectangular waveguide with dimensions a and b that will operate in the dominant TE_{10} mode at $f = 10$ GHz. The dimensions a and b should be chosen according to the design criteria of the preceding question (single-mode operation), together with the requirement that $f = 10$ GHz is *simultaneously* 25% above the cutoff frequency of the dominant TE_{10} mode and 25% below the next higher-order TE_{01} mode.

$$f \geq 1.25\frac{c}{2a} \;\Rightarrow\; a \geq 1.875 \text{ cm}$$

The next higher order mode is the TE_{01} $\Rightarrow$

$$f \leq 0.75\frac{c}{2b} \quad\Rightarrow\quad b \leq 1.125 \text{ cm}$$

Many solutions are possible, e.g. $a = 2$ cm and $b = 1$ cm

3. With the inner dimensions estimated before, determine the cutoff frequencies, in ascending order, of the first 10 TE and/or TM modes.

1) $TE_{10} = 7.5$ GHz
2) and 3) TE_{01} & $\mathrm{TE}_{20} = 15$ GHz
4) and 5) TE_{11} & $\mathrm{TM}_{11} = 16.77$ GHz
6) and 7) TE_{21} & $\mathrm{TM}_{21} = 21.21$ GHz

8) TE_{30} = 22.5 GHz
9) and 10) TE_{31} & TM_{31} = 27.04 GHz

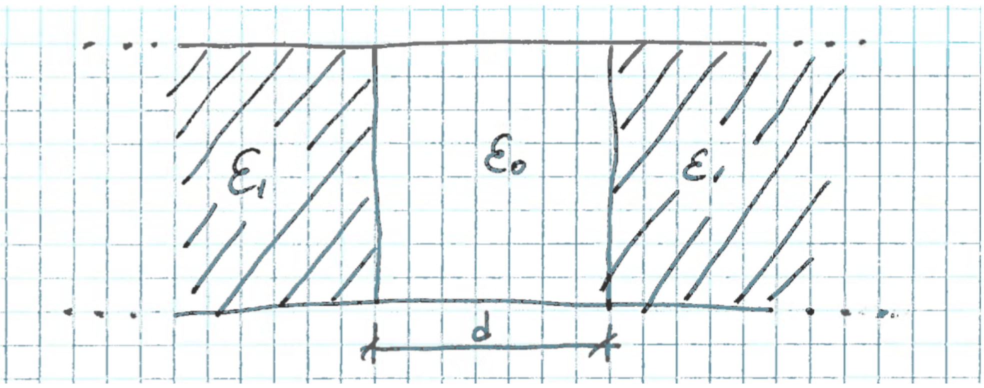
4. We would like to reduce the dimensions of the waveguide by half on each side, but without loosing the propagation properties (same cutoff frequencies, same dominant mode, same bandwidth, etc). We can do that by filling the waveguide with a dielectric different from air with $\varepsilon_r > 1$. Find the value of the new dielectric permittivity.

$$\varepsilon_r = 4$$

6) Waveguide design

In order to increase the bandwidth of a waveguide, designers often use the so-called "air gap". The idea behind this is that we would only want to propagate the dominant TE_{10} mode, even for a frequency of operation above the cut-off frequency of the next higher-order mode. The waveguide is filled by dielectric material with $\varepsilon_r > 1$ but interrupted with a 'piece' of air (i.e. the air-gap).

1. Provide a qualitative explanation of the basic fundament behind this idea.



Cutoff frequencies in dielectrics < cutoff frequencies in air

Higher order modes decay in air and even though they are propagating in the dielectric, if d is sufficiently long, their amplitudes can be neglected after evanescence.

2. In which percentage the frequency band of the waveguide is increased with the use of

such an air-gap?

$$BW^{\text{air}} = f_{20}^{\text{air}} - f_{10}^{\text{air}} = \frac{c}{a}\left(1 - \tfrac{1}{2}\right) \qquad \text{(assuming } a = 2b\text{)}$$

$$BW^{\text{air-gap}} = f_{20}^{\text{air}} - f_{10}^{\text{dielectric}} = \frac{c}{a}\left(1 - \frac{1}{2\sqrt{\varepsilon_r}}\right)$$

$$\Delta BW_{\%} = \frac{BW^{\text{air-gap}} - BW^{\text{air}}}{BW^{\text{air}}} \cdot 100 = 100\left(1 - \frac{1}{\sqrt{\varepsilon_r}}\right)$$

3. Consider a standard waveguide WR-975 ($a = 24.765$ cm and $b = 12.383$ cm) filled with polystyrene ($\varepsilon_r = 2.56$). Calculate the length of the air-gap that would reduce all the higher-order modes up to at least 1% of their amplitudes.

$$E|_{z=0} = A_0 \qquad E|_{z=d} = A_0 e^{-\gamma d}$$

$$\frac{E|_{z=d}}{E|_{z=0}} < 0.01 \quad \Rightarrow \quad \gamma d > 4.6$$

$$\gamma_{20} = \sqrt{\left(\frac{2\pi}{a}\right)^2 - \omega^2 \varepsilon\mu} \qquad \text{when } \omega = \omega_{20}^{\text{dielectric}} \quad \Rightarrow \quad d = 23.2 \text{ cm}$$

Book. Solve problems 13.12, 13.17, 13.19 and 13.21.

13.12)

$$f < f_{c_1} \quad \Rightarrow \quad f < \frac{c}{2nd} \quad \Rightarrow \quad d = 3.45 \text{ cm}$$