Communication Theory (5ETB0)

Exercise Bundle with Solutions

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Version 1.1.1

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2 Analog versus Digital Communication

Exercise 2.1: Consider an analog communication system in which the signal $u(t)$ is not white, but has power spectrum

$$S_U(f) = \frac{U_0/2}{1+(f/B)^2} \left[\tfrac{W}{Hz}\right], \text{ for all } f,$$

where B is the 3 dB bandwidth. The signal is transmitted over a channel and $r(t)$ is received. This received signal $r(t) = u(t) + n_w(t)$ is now passed through an ideal low-pass filter with bandwidth W and unity gain. The channel noise $n_w(t)$ is white with power spectral density $S_W(f) = N_0/2$. The output $v(t)$ of the low-pass filter is used as an estimate for the signal $u(t)$.

(a) First determine what the total power of the signal $u(t)$ is and the part of this power that is filtered out by the low-pass filter. Note that this last part contributes to the distortion. The other part of the distortion is the low-pass filtered white noise.

Hint: Use $\int \frac{1}{1+x^2}dx = arctan(x)$ to evaluate the integral.

(b) What is the total power of the distortion and the SDR as a function of W/B where U_0/N_0 is a parameter? Make a plot of this function for $U_0/N_0 = 50$.

(c) Give an expression for W/B that maximizes the SDR, again with U_0/N_0 as a parameter.

Solution:

(a) Determine the total signal power.

$$\begin{aligned} E[U^2(t)] = \int_{-\infty}^{\infty} S_U(f)df &= \int_{-\infty}^{\infty} \frac{U_0/2}{1+(f/B)^2}df \\ &= B\int_{-\infty}^{\infty} \frac{U_0/2}{1+(f/B)^2}d(f/B) \\ &= [B(U_0/2)\arctan(f/B)]_{-\infty}^{\infty} \\ &= B(U_0/2)\pi \end{aligned}$$

Note that $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$.

Part that passes through the low-pass filter:

$$\begin{aligned} E[U_w^2(t)] = \int_{-W}^{W} S_U(f)df &= [B(U_0/2)\arctan(f/B)]_{-W}^{W} \\ &= BU_0 \arctan(W/B) \end{aligned}$$

Note that $\arctan(-x) = -\arctan(x)$.

Part that is filtered out by the low-pass filter:

$$E[U^2(t)] - E[U_w^2(t)] = B(U_0/2)\pi - BU_0\arctan(W/B)$$

Low-pass filtered noise is

$$\begin{aligned} E[N_w^2(t)] = \int_{-W}^{W} S_N(f)df &= [(N_0/2)f]_{-W}^{W} \\ &= (N_0/2)2W \\ &= N_0W. \end{aligned}$$

(b) Thus, the power of the distortion waveform is

$$\begin{aligned} E[D^2(t)] &= |E[R_W^2(t)] - E[U^2(t)]| \\ &= |(E[N_W^2(t)] + E[U_W^2(t)]) - E[U^2(t)]| \\ &= N_0 W + BU_0\pi/2 - BU_0 \arctan(W/B)) \end{aligned}$$

Finally, the SDR can be found by noting that $SDR = \frac{E[U^2]}{E[D^2]}$, so

$$\begin{aligned} \text{SDR} &= \frac{BU_0\pi/2}{N_0 W + BU_0(\pi/2 - \arctan(W/B))} \\ &= \frac{U_0/N_0\pi/2}{U_0/N_0(\pi/2 - \arctan(W/B)) + W/B}. \end{aligned}$$

Remarks: If $W = 0 \rightarrow$, there is no noise but also no signal, and SDR = 1. If $W \rightarrow \infty$, the complete signal passes, but also the noise has infinite power, and SDR = 0. See Fig. 1 for a plot of the SDR for $U_0/N_0 = 50$.

(c) For maximizing the SDR, we should minimize the denominator: take the derivative of the denominator and solve it for 0.

Define: $x = W/B \geq 0$,

$$\begin{aligned} \frac{d(\text{SDR})}{dx} &= \frac{d}{dx}\left(U_0/N_0(\pi/2 - \arctan(x)) + x\right) \\ &= 1 - \frac{U_0/N_0}{1+x^2} \\ &\rightarrow 0 \end{aligned}$$

Therefore,

$$\begin{aligned} 1 + x^2 &= U_0/N_0 \\ x &= \sqrt{U_0/N_0 - 1}. \end{aligned}$$

You can verify this solution in Fig 1 for $U_0/N_0 = 50$, where we calculate that the maximum is achieved for $W/B = 7$.

Note that if $U_0/N_0 < 1$, then the above does not have a valid solution. However, it means that the noise has more power than the signal. In this case the choice for $W = 0$ is optimum.

Exercise 2.2: Consider a source that emits a single Gaussian sample u, with mean 0 and variance σ_u^2. This sample u is amplified by a factor $\sqrt{P/\sigma_u^2}$ which results in an input $s = u\sqrt{P/\sigma_u^2}$. This input is subsequently transmitted over an additive Gaussian noise channel. For the resulting channel output we can write $r = s + n$, where the mean and variance of the noise variable n is 0 and σ_n^2 respectively.

The receiver observing r makes an estimate $v = \alpha r$ for an α such that the mean squared error (MSE) distortion $E[D^2] = E[(V - U)^2]$ is minimized.

Hint: Note that when two RV are independent, then $E[AB] = E[A]E[B]$.

(a) The variance of the source-signal U is σ_u^2. What is the variance of the channel-input signal S?

(b) What value of α minimizes the MSE-distortion?

(c) How large is the resulting minimal MSE-distortion?

(d) Now also express the source signal-to-distortion ratio SDR $= \sigma_u^2/E[D^2]$ in terms of the channel signal-to-noise ratio SNR $= P/\sigma_n^2$. (We call the resulting procedure minimal MSE (MMSE) estimation.)

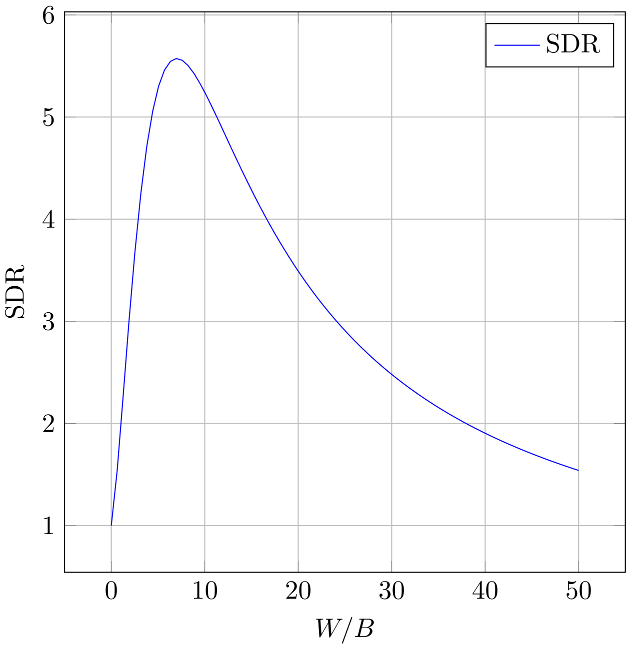

Figure 1: Plot of the SDR as a function of W/B with $U_0/N_0 = 50$ (Sol. 2.1)

Solution:

(a) Use $S = \sqrt{P/\sigma_u^2}U$, then the variance: $E[S^2] = \frac{P}{\sigma_u^2}E[U^2] = \frac{P}{\sigma_u^2}\sigma_u^2 = P$

(b) Minimize Mean Square Error (MSE) distortion $E[D^2] = E[(V-U)^2]$:

$$\begin{aligned} E[D^2] &= E[(V-U)^2] \\ &= E[(\alpha(\sqrt{P/\sigma_u^2}U + N) - U)^2] \\ &= E[(\alpha\sqrt{P/\sigma_u^2} - 1)U + \alpha N)^2] \\ &= (\alpha\sqrt{P/\sigma_u^2} - 1)^2 E[U^2] + \alpha^2 E[N^2] \\ &= (\alpha\sqrt{P/\sigma_u^2} - 1)^2\sigma_u^2 + \alpha^2\sigma_n^2 \end{aligned}$$

$$\begin{aligned} \frac{dE[D^2]}{d\alpha} &= 2(\alpha\sqrt{P/\sigma_u^2} - 1)\sqrt{P/\sigma_u^2}\sigma_u^2 + 2\alpha\sigma_n^2 \\ &\to 0 \end{aligned}$$

Therefore,

$$\begin{aligned} 2\alpha P + 2\alpha\sigma_n^2 &= 2\sqrt{P\sigma_u^2} \\ \alpha_{\text{opt}} = \frac{\sqrt{P\sigma_u^2}}{P + \sigma_u^2} &= \sqrt{\frac{\sigma_u^2}{P}}\frac{P}{P + \sigma_n^2}. \end{aligned}$$

Which we can interpret as the standard solution: $\sqrt{\frac{\sigma_u^2}{P}}$ with a correction $\frac{P}{P+\sigma_n^2}$.

(c) Now, the minimum MSE is

$$\begin{aligned}
\text{MSE} &= \left(\frac{P}{P+\sigma_n^2}-1\right)^2\sigma_u^2+\frac{\sigma_u^2}{P}\left(\frac{P}{P+\sigma_n^2}\right)^2\sigma_n^2 \\
&= \frac{\sigma_n^4\sigma_u^2}{(P+\sigma_n^2)^2}+\frac{\sigma_u^2 P\sigma_n^2}{(P+\sigma_n^2)^2} \\
&= \frac{\sigma_u^2\sigma_n^2(P+\sigma_u^2)}{(P+\sigma_n^2)^2} = \sigma_u^2\frac{\sigma_n^2}{(P+\sigma_n^2)}
\end{aligned}$$

(d) The signal-to-distortion ratio is

$$\text{SDR} = \frac{\sigma_u^2}{E[D^2]} = \sigma_u^2/\left(\sigma_u^2\frac{\sigma_n^2}{P+\sigma_n^2}\right) = \frac{P+\sigma_n^2}{\sigma_n^2} = \text{SNR}+1$$

This is an improvement w.r.t. $\alpha_{ML}=\sqrt{\sigma_u^2/P}$ (maximum likelihood estimate).

Exercise 2.3: We consider quantization and discrete encoding of Gaussian samples u. Assume that $E[U]=0$ and $E[U^2]=1$.

interval	point	representation	alternative representation
$[-\infty,-1.5]$	$q_1=-1.75$	000	0111
$[-1.5,-1.0]$	$q_2=-1.25$	001	0110
$[-1.0,-0.5]$	$q_3=-0.75$	010	010
$[-0.5,0.0]$	$q_4=-0.25$	011	00
$[0.0,+0.5]$	$q_5=+0.25$	100	10
$[+0.5,+1.0]$	$q_6=+0.75$	101	110
$[+1.0,+1.5]$	$q_7=+1.25$	110	1110
$[+1.5,+\infty]$	$q_8=+1.75$	111	1111

We quantize these variables by choosing eight quantization points $q_1, q_2, \cdots, q_8$, see table. The quantizer always chooses the quantization point closest to the observed source output u.

(a) Determine (numerically) the mean-square error (MSE) distortion defined as $E[D^2]=E[(Q(U)-U)^2]$, where $Q(u)$ is the value of the quantized version of u. e.g., for $u=1.23$ quantization point $q_7=1.25$ is chosen and therefore $Q(1.23)=1.25$.

(b) To represent one out of eight quantization points three binary digits can be used, see table. It will be clear that the expected representation-length is $E[L]=3$ binary digits in that case. An alternative representation is given in the table. Compute the expected representation length $E[L_a]$ also for this alternative representation.

Solution:

(a) The mean-square error (MSE) is defined as $E[D^2]=E[(Q(U)-U)^2]$.

$$\begin{aligned}
E[(Q(U)-U)^2] &= E[U^2-2UQ(U)+Q^2(U)] \\
&= E[U^2]+E[Q^2(U)]-2E[UQ(U)]
\end{aligned}$$

Given that u are Gaussian samples, i.e. $U\sim\mathcal{N}(0,1)$, the probability of having $u\in[-1.5,-1.0]$, i.e. q_2, can be compute as

$$Pr(Q=q_2)=\int_{-1.5}^{-1.0}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,du = 0.0918. \tag{1}$$

interval	probability	codeword
$[-\infty, -1.5]$	0.0668	0111
$[-1.5, -1.0]$	0.0918	0110
$[-1.0, -0.5]$	0.1499	010
$[-0.5, 0]$	0.1915	00
$[0, 0.5]$	0.1915	10
$[0.5, 1.0]$	0.1499	110
$[1.0, 1.5]$	0.0918	1110
$[1.5, \infty]$	0.0668	1111

Table 1: Probability of occurrence for each interval (Sol. 2.3)

The rest of the probabilities $Pr(Q = q_i)$ are shown in Table 1.

Next, $E[Q^2(U)]$ and $E[UQ(U)]$ can be computed as follows:

$$E[Q^2(U)] = \sum_{Q(u)\in S_{Q(u)}} Q^2(u) \cdot Pr(Q(u)) = \sum_{i=1}^{8} q_i^2 \cdot Pr(Q = q_i) = 0.8888. \tag{2}$$

$$E[UQ(U)] = \int_{-\infty}^{-1.5} uq_1 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + \int_{-1.5}^{-1.0} uq_2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + ... + \int_{1.5}^{+\infty} uq_8 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$
$$E[UQ(U)] = 0.9230.$$

Therefore,

$$\begin{aligned} E[(U - Q(U))^2] &= E[U^2] + E[Q^2(U)] - 2E[UQ(U)] \\ &= 1 + 0.8888 - 2 \cdot 0.9230 \\ &= 0.0427 \end{aligned}$$

(b) In the table 1 we show for each interval and alternative representation the corresponding probability of occurrence.

$E[L_a] = 2(0.1915 \cdot 2 + 0.1499 \cdot 3 + 0.0918 \cdot 4 + 0.0668 \cdot 4) = 2.9344$. This is somewhat better than $E[L] = 3$ binary digits (per symbol).

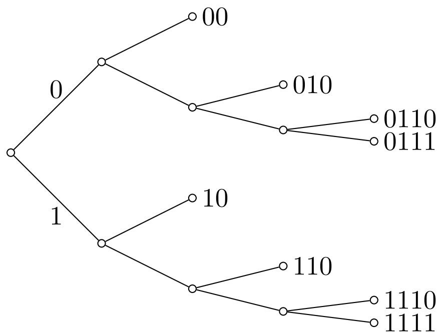

Figure 2: Prefix tree of codebook (Sol. 2.3)

Since the codewords are only leaves in the tree, we can conclude that no codeword is a prefix of any other codeword.

Exercise 2.4: The channel in Figure 3 has a single input s and two outputs r_1 and r_2. The additive noise variables n_1 and n_2 are Gaussian, have mean $E[N_1] = E[N_2] = 0$ and variances $E[N_1^2] = \sigma_1^2$ and $E[N_2^2] = \sigma_2^2$ respectively. The variance of the input is constrained, hence $E[S^2] = P$. The signal-to-noise ratios for both branches can now be defined as

$$\text{SNR}_1 \triangleq \frac{P}{\sigma_1^2} \text{ and } \text{SNR}_2 \triangleq \frac{P}{\sigma_2^2}.$$

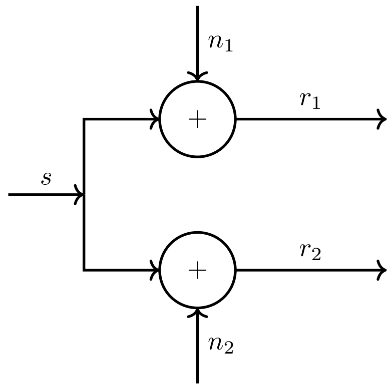

Figure 3: Channel with two branches (Ex. 2.4).

We now would like to replace the pair (r_1, r_2) by a single output

$$r = \alpha r_1 + (1 - \alpha) r_2,$$

where α is a constant, chosen in such a way that the signal-to-noise ratio SNR for the channel from s to r is maximized.

(a) Find an expression for the α that maximizes the signal-to-noise ratio SNR for the channel from s to r.

(b) Express the resulting maximized $\text{SNR}_{\max}$ in terms of SNR_1 and SNR_2. Show that $\text{SNR}_{\max} \geq \max(\text{SNR}_1, \text{SNR}_2)$.

Solution: Known: $E[N_1] = E[N_2] = 0$, $E[N_1^2] = \sigma_1^2$, $E[N_2^2] = \sigma_2^2$, $E[S^2] = P$.

(a)

$$\begin{aligned} r &= \alpha r_1 + (1-\alpha) r_2 \\ &= \alpha(s + n_1) + (1-\alpha)(s + n_2) \\ &= \alpha s + (1-\alpha)s + \alpha n_1 + (1-\alpha) n_2 \\ &= s + \alpha n_1 + (1-\alpha) n_2 \end{aligned}$$

Now, $E[N_1^2] = \sigma_1^2$, $E[N_2^2] = \sigma_2^2$ and $E[S^2] = P$.

$$\begin{aligned} \text{SNR} &= \frac{E[S^2]}{\alpha^2 E[N_1^2] + (1-\alpha)^2 E[N_2^2]} \\ &= \frac{P}{\alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2} \end{aligned}$$

In order to maximize the SNR, the denominator need to be minimized with respect to α. To calculate the optimal α (α_{opt}), solve the derivative of the denominator for $d/d\alpha = 0$:

$$\begin{aligned} &\Rightarrow 2\alpha\sigma_1^2 - 2(1-\alpha)\sigma_2^2 = 0 \\ &\Rightarrow 2\alpha\sigma_1^2 + 2\alpha\sigma_2^2 = 2\sigma_2^2 \\ &\Rightarrow \alpha_{\text{opt}} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \end{aligned}$$

(b)

$$
\begin{aligned}
\mathrm{SNR} &= \frac{E[S^2]}{\alpha^2 E[N_1^2] + (1-\alpha)^2 E[N_2^2]} \\
&= \frac{P}{\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2} \\
\mathrm{SNR}_{\max} &= \frac{P}{\left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 \sigma_1^2 + \left(1-\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 \sigma_2^2} \\
&= \frac{P(\sigma_1^2+\sigma_2^2)^2}{\sigma_2^4\sigma_1^2 + \sigma_1^4\sigma_2^2} = \frac{P(\sigma_1^2+\sigma_2^2)^2}{\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)} = \frac{P(\sigma_1^2+\sigma_2^2)}{\sigma_1^2\sigma_2^2} \\
&= \frac{P}{\sigma_1^2} + \frac{P}{\sigma_2^2} = \mathrm{SNR}_1 + \mathrm{SNR}_2
\end{aligned}
$$

$\mathrm{SNR}_{\max} = \mathrm{SNR}_1 + \mathrm{SNR}_2$ implies that $\mathrm{SNR}_{\max} \geq \max(\mathrm{SNR}_1, \mathrm{SNR}_2)$.

Exercise 2.5: In the previous exercise we considered communication over "parallel" branches (channels). Here we will investigate channels in an "in series" setting, see Fig. 4.

The additive noise variables n_1 and n_2 are Gaussian, have mean $E[N_1] = E[N_2] = 0$ and variances $E[N_1^2] = \sigma_1^2$ and $E[N_2^2] = \sigma_2^2$ respectively. The variances of the channel inputs s_1 and s_2 are constrained, hence $E[S_1^2] = P_1$ and $E[S_2^2] = P_2$. The signal-to-noise ratios for both channels can now be defined as

$$\mathrm{SNR}_1 \triangleq \frac{P_1}{\sigma_1^2} \text{ and } \mathrm{SNR}_2 \triangleq \frac{P_2}{\sigma_2^2}.$$

There is a repeater between the channels, whose only task it is to scale the output r_1 of first channel such that the result s_2 meets the power constraint $E[S_2^2] = P_2$.

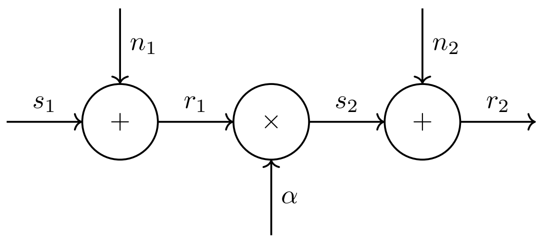

Figure 4: Two channel in series, with a repeater in between (Ex. 2.5).

(a) Express α in terms of the channel parameters P_1, P_2, σ_1^2, and σ_2^2.

(b) Express the signal-to-noise ratio SNR of the channel with input s_1 and output r_2 in terms of the channel parameters P_1, P_2, σ_1^2, and σ_2^2.

(c) Now express the signal-to-noise ratio SNR of the channel with input s_1 and output r_2 in terms of SNR_1 and SNR_2. Show that $\mathrm{SNR} \leq \min(\mathrm{SNR}_1, \mathrm{SNR}_2)$.

Solution: We are given that $E[S_1^2] = P_1$, $E[S_2^2] = P_2$, $E[N_1^2] = \sigma_1^2$, and $E[N_2^2] = \sigma_2^2$ where N_1 and N_2 are stated to be zero-mean random variables.

(a) Observing that $r_1 = s_1 + n_1$, we can write the average power of r_1 as

$$
\begin{aligned}
E[R_1^2] =& E[S_1^2] + E[N_1^2], && (3)\\
=& P_1 + \sigma_1^2, && (4)
\end{aligned}
$$

where we used the independence of s_1 and n_1. Observing that $E[S_2^2] = \alpha^2 E[R_1^2]$, the scaling parameter should be

$$\alpha = \sqrt{\frac{P_2}{P_1 + \sigma_1^2}}, \tag{5}$$

to satisfy the requirement which is $E[S_2^2] = P_2$.

(b) It is straightforward to write that

$$s_2 = \sqrt{\frac{P_2}{P_1 + \sigma_1^2}}(s_1 + n_1), \tag{6}$$

$$r_2 = \sqrt{\frac{P_2}{P_1 + \sigma_1^2}}(s_1 + n_1) + n_2, \tag{7}$$

$$= \sqrt{\frac{P_2}{P_1 + \sigma_1^2}} s_1 + \sqrt{\frac{P_2}{P_1 + \sigma_1^2}} n_1 + n_2. \tag{8}$$

Since we are required to write SNR of the channel with input s_1 and output r_2, the first term in (8) is our signal and the rest is the equivalent noise. It is known that the channel scales input signal s_1 by a parameter α, so at the receiver it is expected to observe $\alpha \cdot s_1$. Therefore the SNR is the ratio between the power of the signal $\alpha \cdot s_1$ and the power of the noise of the channel:

$$\mathrm{SNR} = \frac{\frac{P_2}{P_1+\sigma_1^2} E[S_1^2]}{\frac{P_2}{P_1+\sigma_1^2} E[N_1^2] + E[N_2^2]}, \tag{9}$$

$$= \frac{\frac{P_1 P_2}{P_1+\sigma_1^2}}{\frac{P_2 \sigma_1^2}{P_1+\sigma_1^2} + \sigma_2^2}, \tag{10}$$

$$= \frac{P_1 P_2}{P_2 \sigma_1^2 + P_1 \sigma_2^2 + \sigma_1^2 \sigma_2^2}, \tag{11}$$

$$= \frac{\frac{P_1}{\sigma_1^2} \frac{P_2}{\sigma_2^2}}{\frac{P_2}{\sigma_2^2} + \frac{P_1}{\sigma_1^2} + 1}. \tag{12}$$

(c) Eq. (12), using the definitions of SNR_1 and SNR_2 given in the question, reduces to

$$\mathrm{SNR} = \frac{\mathrm{SNR}_1 \mathrm{SNR}_2}{\mathrm{SNR}_2 + \mathrm{SNR}_1 + 1}, \tag{13}$$

$$= \mathrm{SNR}_1 \frac{\mathrm{SNR}_2}{\mathrm{SNR}_2 + \mathrm{SNR}_1 + 1}, \tag{14}$$

$$= \mathrm{SNR}_2 \frac{\mathrm{SNR}_1}{\mathrm{SNR}_2 + \mathrm{SNR}_1 + 1}. \tag{15}$$

Eq. (14) implies $\mathrm{SNR} \leq \mathrm{SNR}_1$ and Eq. (15) implies $\mathrm{SNR} \leq \mathrm{SNR}_2$, which together imply

$$\mathrm{SNR} \leq \min(\mathrm{SNR}_1, \mathrm{SNR}_2). \tag{16}$$

3 Decision Rules for DIDO Channels

Exercise 3.1: Consider a communication system based on a DIDO channel. The number of messages is five i.e., $\mathcal{M} = \{1,2,3,4,5\}$. The a-priori probabilities are $\Pr\{M = 1\} = \Pr\{M = 5\} = 0.1$, $\Pr\{M = 2\} = \Pr\{M = 4\} = 0.2$, and $\Pr\{M = 3\} = 0.4$. The signal corresponding to message m is equal to m, thus $s_m = m$, for $m \in \mathcal{M}$. This signal is the input for the discrete channel. The channel adds the noise n to the signal, hence the channel output

$$r = s_m + n. \tag{17}$$

The noise variable N is independent of the channel input and can have the values $-1, 0$, and $+1$ only. It is given that $\Pr\{N = 0\} = 0.4$ and $\Pr\{N = -1\} = \Pr\{N = +1\} = 0.3$.

(a) Note that the channel output takes values from $\{0,1,2,3,4,5,6\}$. Determine the probability $\Pr\{R = 2\}$. For each message $m \in \mathcal{M}$ compute the a-posteriori probability $\Pr\{M = m|R = 2\}$.

(b) Note that an optimum receiver minimizes the probability of error P_e. Give for each possible channel output the estimate $\hat{m}$ that an optimum receiver will make. Determine the resulting error probability.

(c) Consider a maximum-likelihood receiver. Give for each possible channel output the estimate $\hat{m}$ that a maximum-likelihood receiver will make. Again determine the resulting probability of error.

Solution:
From the assignment, the message a-priori probabilities are $\Pr\{M = 1\} = \Pr\{M = 5\} = 0.1$, $\Pr\{M = 2\} = \Pr\{M = 4\} = 0.2, \Pr\{M = 3\} = 0.4$.
The noise probabilities are $\Pr\{N = 0\} = 0.4, \Pr\{N = -1\} = \Pr\{N = +1\} = 0.3$.
Now the probability of observing a channel output 2 is

$$\Pr\{R = 2\} = \sum_m \Pr\{R = 2, M = m\} = \sum_m \Pr\{R = 2|M = m\} \Pr\{M = m\}. \tag{18}$$

(a) The event $R = m + n = 2$ occurs only when the following pairs are observed: $(M = 1, N = 1)$, $(M = 2, N = 0)$ and $(M = 3, N = -1)$. The probability that the received message is $R = 2$ given that the message $M = 1$ was sent is equivalent to the probability of observing noise $N = 1$, i.e.

$$\Pr\{R = 2|M = 1\} = \Pr\{N = R - M\} = \Pr\{N = 1\} = 0.3. \tag{19}$$

Similarly, $\Pr\{R = 2|M = 3\} = \Pr\{N = -1\}$ and $\Pr\{R = 2|M = 2\} = \Pr\{N = 0\}$ can be found.

Next, the joint probabilities are found, for example,

$$\begin{aligned}\Pr\{R = 2, M = 1\} &= \Pr\{R = 2|M = 1\} \cdot \Pr(M = 1) \\ &= \Pr\{N = 1\} \cdot \Pr(M = 1) = 0.3 \cdot 0.1 = 0.03.\end{aligned}$$

All the joint probabilities $\Pr\{R = r, M = m\} = \Pr\{R = 2|M = m\} \Pr\{M = m\}$ are tabulated in the table below:

m	Pr{M=m}	R=0	1	2	3	4	5	6
1	0.1	**0.03**	0.04	0.03				
2	0.2		**0.06**	0.08	0.06			
3	0.4			**0.12**	**0.16**	**0.12**		
4	0.2				0.06	0.08	**0.06**	
5	0.1					0.03	0.04	**0.03**

This results in

$$\Pr\{R = 2\} = \sum_m \Pr\{R = 2, M = m\} = 0.03 + 0.08 + 0.12 = 0.23.$$

Now, use Bayes rule to compute the a-posteriori probabilities:

$$\Pr\{M = m|R = 2\} = \frac{\Pr\{R = 2|M = M\}\Pr\{M = m\}}{\Pr\{R = 2\}} = \frac{\Pr\{R = 2, M = m\}}{\Pr\{R = 2\}}$$

This results in:

$$\begin{aligned}
\Pr\{M = 1|R = 2\} &= 0.03/0.23 = 3/23 \\
\Pr\{M = 2|R = 2\} &= 8/23 \\
\Pr\{M = 3|R = 2\} &= 12/23 \\
\Pr\{M = 4|R = 2\} &= 0 \\
\Pr\{M = 5|R = 2\} &= 0
\end{aligned}$$

(b) In order to minimize the probability of error, we should use the MAP decisions rule

$$\hat{m}(r) = \text{argmax}_m \Pr\{M = m\}\Pr\{R = r|M = m\}.$$

So we should decode for each received signal r the corresponding $\hat{m}$ that has maximum a-posteriori probability, which corresponds to the maximum value of each column in the table above. We have marked these maxima in bold in the table. The resulting optimal decision rule is given as

r	0	1	2	3	4	5	6
$\hat{m}(r)$	1	2	3	3	3	4	5

And the resulting error probability can be written as a sum of non-bold probabilities in the table above, so P_e is given by

$$P_e = \Pr\{\hat{m} \neq m\} = \sum_r \sum_{m \neq \hat{m}(r)} \Pr\{m, r\} = 2 * (0.04 + 0.03 + 0.08 + 0.06) = 0.42$$

(c) The ML-rule says

$$\hat{m}_{\text{ML}}(r) = \text{argmax}_m \Pr\{R = r|M = m\}.$$

In other words, ML-rule performs the decision based on the likelihood, $\Pr\{R = r|M = m\}$. The following table shows the likelihoods for each pair (R, M). The maxima for each column (maximum likelihood) is again marked by bold.

m	Pr{M=m}	R=0	1	2	3	4	5	6
1	0.1	**0.3**	**0.4**	0.3				
2	0.2		0.3	**0.4**	0.3			
3	0.4			0.3	**0.4**	0.3		
4	0.2				0.3	**0.4**	0.3	
5	0.1					0.3	**0.4**	**0.3**

By looking at the decision variables in the above table, the following decision rule is obtained

r	0	1	2	3	4	5	6
$\hat{m}_{ML}(r)$	1	1	2	3	4	5	5

$$P_e^{\text{ML}} = \Pr\{\hat{m}_{\text{ML}} \neq m\} = \sum_r \sum_{m \neq \hat{m}_{\text{ML}}(r)} \Pr\{m, r\} = 2 * (0.03 + 0.06 + 0.06 + 0.12) = 0.54.$$

The error probability in (b) is smaller than in (c), so MAP-rule is more optimal decision rule in this case. This is because not all messages are equally likely (messages do not have the same a-priori probability $\Pr\{M = m\}$).

Exercise 3.2: Consider an optical communication system. A binary message is transmitted by switching the intensity λ of a laser "on" or "off" within the time-interval $[0, T]$. For message $m = 1$ ("off") the intensity $\lambda = \lambda_{\text{off}} \geq 0$. For message $m = 2$ ("on") the intensity $\lambda = \lambda_{\text{on}} > \lambda_{\text{off}}$.

The laser produces photons that are counted by a detector. The number K of photons is a non-negative random variable, and the corresponding probabilities are

$$\Pr\{K = k\} = \frac{(\lambda T)^k}{k!} \exp(-\lambda T), \text{ for } k = 0, 1, 2, \cdots .$$

This probability distribution is known as Poisson distribution.

The a-priori message probabilities are $\Pr\{M = 1\} = p_{\text{off}}$ and $\Pr\{M = 2\} = p_{\text{on}}$. It is assumed that $0 < p_{\text{on}} = 1 - p_{\text{off}} < 1$.

The detector forms the message estimate $\hat{m}$ based on the number k of photons that are counted.

(a) Suppose that the detector chooses $\hat{m}$ such that the probability of error $P_e = \Pr\{\hat{M} \neq M\}$ is minimized. For what values of k does the detector choose $\hat{m} = 1$ and when does it choose $\hat{m} = 2$?

(b) Consider a maximum-likelihood detector. For what values of k does this detector choose $\hat{m} = 1$ and when does it choose $\hat{m} = 2$ now?

(c) Assume that $\lambda_{\text{off}} = 0$. Consider a maximum-a-posteriori detector. For what values of p_{on} does the detector choose $\hat{m} = 2$ no matter what k is? What is in that case the error probability P_e?

Solution: We are given that $m = 1$ and $m = 2$ corresponds to off and on states, respectively.

(a) To minimize word error probability $P_e = \Pr\{\hat{M} \neq M\}$, we can use the Maximum A-Posteriori (MAP) decision in general. For MAP decision, the decision variables are defined as the joint probability density of the transmitted and received signals (e.g, the message and the received signal)

$$\begin{aligned} \Pr\{M = m, R = r\} &= \Pr\{M = m\} \Pr\{R = r | M = m\}, && (20) \\ &= \Pr\{M = m\} \Pr\{R = r | S = s_m\}. && (21) \end{aligned}$$

For this question, decision variables corresponding to $M = 1$ and $M = 2$ cases are

$$\begin{aligned} \Pr\{M = 1\} \Pr\{R = k | M = 1\} &= p_{\text{off}} \frac{(\lambda_{\text{off}} T)^k}{k!} \exp(-\lambda_{\text{off}} T), && (22) \\ \Pr\{M = 2\} \Pr\{R = k | M = 2\} &= p_{\text{on}} \frac{(\lambda_{\text{on}} T)^k}{k!} \exp(-\lambda_{\text{on}} T). && (23) \end{aligned}$$

Therefore we decide $\hat{m} = 1$ when

$$\begin{aligned} p_{\text{off}} \frac{(\lambda_{\text{off}} T)^k}{k!} \exp(-\lambda_{\text{off}} T) &> p_{\text{on}} \frac{(\lambda_{\text{on}} T)^k}{k!} \exp(-\lambda_{\text{on}} T), && (24) \\ \left(\frac{\lambda_{\text{on}} T}{\lambda_{\text{off}} T} \right)^k &< \frac{p_{\text{off}}}{p_{\text{on}}} \exp(\lambda_{\text{on}} T - \lambda_{\text{off}} T), && (25) \\ k \ln \left(\frac{\lambda_{\text{on}}}{\lambda_{\text{off}}} \right) &< \ln \left(\frac{p_{\text{off}}}{p_{\text{on}}} \right) + (\lambda_{\text{on}} - \lambda_{\text{off}}) T, && (26) \\ k &< \frac{\ln \left(\frac{p_{\text{off}}}{p_{\text{on}}} \right) + (\lambda_{\text{on}} - \lambda_{\text{off}}) T}{\ln \left(\frac{\lambda_{\text{on}}}{\lambda_{\text{off}}} \right)}, && (27) \end{aligned}$$

and $\hat{m} = 2$ otherwise.

(b) A maximum-likelihood detector is a class of MAP detector where the messages are equally likely to be sent, e.g., $p_{\text{on}} = p_{\text{off}}$ for this problem. Then our decision rule reduces to deciding $\hat{m} = 1$ when

$$k < \frac{(\lambda_{\text{on}} - \lambda_{\text{off}})T}{\ln\left(\frac{\lambda_{\text{on}}}{\lambda_{\text{off}}}\right)}, \tag{28}$$

and $\hat{m} = 2$ otherwise.

(c) Let's examine the decision variables (e.g., (22) and (23)) when $\lambda_{\text{off}} = 0$ and we are using a MAP detector:

- $K = 0$ case: For detector to choose $\hat{m} = 2$

$$p_{\text{off}} \times \frac{(\lambda_{\text{off}}T)^k}{k!} e^{-\lambda_{\text{off}}T} < p_{\text{on}} \exp(-\lambda_{\text{on}}T), \tag{29}$$

$$p_{\text{off}} \times 1 < p_{\text{on}} \exp(-\lambda_{\text{on}}T), \tag{30}$$

should be satisfied.

- $K > 0$ case: For detector to select $\hat{m} = 2$ all the time

$$p_{\text{off}} \times 0 < p_{\text{on}} \frac{(\lambda_{\text{on}}T)^k}{k!} \exp(-\lambda_{\text{on}}T), \tag{31}$$

should be satisfied which is always satisfied for $k > 0$.

Therefore necessary condition is

$$p_{\text{off}} < p_{\text{on}} \exp(-\lambda_{\text{on}}T), \tag{32}$$

$$1 - p_{\text{on}} < p_{\text{on}} \exp(-\lambda_{\text{on}}T), \tag{33}$$

$$1 < p_{\text{on}}(1 + \exp(-\lambda_{\text{on}}T)), \tag{34}$$

$$p_{\text{on}} > \frac{1}{1 + \exp(-\lambda_{\text{on}}T)} \tag{35}$$

to choose $\hat{m} = 2$ all the time.

Since the detector always chooses $\hat{m} = 2$, detection is correct if $m = 2$ and wrong if $m = 1$. Therefore $P_e = p_{\text{off}}$.

Exercise 3.3: Consider transmission over a binary symmetric channel with cross-over probability

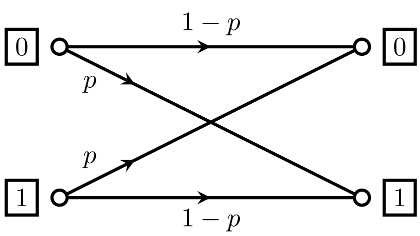

Figure 5: A binary symmetric channel with cross-over probability p (Ex. 3.3).

$0 < p < 1/2$ (see Fig. 5). This means that if message 0 is sent to the channel, then the probability of observing unchanged message 0 at the receiver side is $1-p$ and the probability to observe an altered message 1 is p. All messages $m \in \mathcal{M}$ are equally likely. Each message m now corresponds to a signal vector (codeword) $\underline{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$ consisting of N binary digits, hence $s_{mi} \in \{0, 1\}$ for $i = 1, \ldots, N$. Such a codeword (sequence) is transmitted over the binary symmetric channel, the resulting channel output sequence is denoted by $\underline{r}$.

The Hamming-distance $D_H(\underline{x}, \underline{y})$ between two sequences $\underline{x}$ and $\underline{y}$, both consisting of N binary digits, is the number of positions at which they differ.

(a) Show that the optimum receiver should choose the codeword that has minimum Hamming distance to the received channel output sequence.

(b) What is the optimum receiver for $1/2 < p < 1$?

(c) Assume that the messages are not equally likely and that the cross-over probability $p = 1/2$. What is the minimum error probability now? What does the corresponding receiver do?

Solution: Given that equally likely messages are send over a Binary Symmetric Channel (BSC). Therefore, optimum decision rule is Maximum Likelihood decision rule.

(a)

$$\begin{aligned}\hat{m}_{\text{ML}}(\underline{r}) &= \underset{m}{\text{argmax}}\ \Pr\{\underline{R} = \underline{r}|M = m\} \\ &= \underset{m}{\text{argmax}}\ p^{D_H(\underline{s}_m,\underline{r})}(1-p)^{N-D_H(\underline{s}_m,\underline{r})} \\ &= \underset{m}{\text{argmax}}\ (1-p)^N \left(\frac{p}{1-p}\right)^{D_H(\underline{s}_m,\underline{r})}.\end{aligned}$$

Since $0 < p < 1/2$, this means that $0 < \frac{p}{1-p} < 1$ and the above equation is maximized for a minimized $D_H(\underline{s}_m, \underline{r})$.

(b) In the case $1/2 < p < 1$ this means that $1 < \frac{p}{1-p}$ and the above equation is maximized for a maximized $D_H(\underline{s}_m, \underline{r})$. Therefore, the optimum receiver should select the codeword with maximum Hamming distance to the received sequence.

(c) If the messages are not equally likely, we should use MAP decision rule. Furthermore, since the cross-over probability $p = 1/2$ the likelihood $\Pr\{\underline{R} = \underline{r}|M = m\} = 1/2^N$ for all m. Therefore, the receiver should always select the message that has highest a-priori probability $\Pr\{M = m\}$. The probability of error P_e is the probability that another message was send $1 - \max_m \Pr\{M = m\}$.

Exercise 3.4: Consider a DIDO channel, over which one out of four possible messages is transmitted, hence $m \in \{1, 2, 3, 4\}$. The channel-output can assume four values, or $r \in \{a, b, c, d\}$. For the transition probabilities we have that:

m	$\Pr\{R = a\|M = m\}$	$\Pr\{R = b\|M = m\}$	$\Pr\{R = c\|M = m\}$	$\Pr\{R = d\|M = m\}$
1	1	0	0	0
2	1/2	1/2	0	0
3	1/3	1/3	1/3	0
4	1/4	1/4	1/4	1/4

The probability that channel-output b is produced when message 3 is transmitted is therefore 1/3.

(a) First assume that all messages are equally likely, hence for the a-priori message probabilities we have that $\Pr\{M = 1\} = \Pr\{M = 2\} = \Pr\{M = 3\} = \Pr\{M = 4\} = 1/4$. How does an optimal receiver decide in this case? Specify for each channel output the estimate $\hat{m}$ for the transmitted message. What is the resulting error probability? Is the "maximum-likelihood decision rule" optimal here?

(b) Now the a-priori message probabilities differ from each other. We assume that $\Pr\{M = 1\} = 1/16$, $\Pr\{M = 2\} = 3/16$, $\Pr\{M = 3\} = 5/16$, and $\Pr\{M = 4\} = 7/16$. How does an optimum receiver decide now for each of the four possible channel outputs? What is the resulting error probability? Is the "maximum-likelihood decision rule" optimal here?

(c) Again we assume that $\Pr\{M = 1\} = \Pr\{M = 2\} = \Pr\{M = 3\} = \Pr\{M = 4\} = 1/4$, hence all messages are equally likely. Assume that the channel has an extra output $r_{\text{o/e}}$ that indicates whether the transmitted message was odd or even. The receiver uses this extra channel-output to make decisions. Instead of four channel-outputs, there are now eight channel-output-combinations possible, and $(r, r_{\text{o/e}}) \in \{(a,\text{odd}),(a,\text{even}), (b,\text{odd}),(b,\text{even}),\cdots,(d,\text{even})\}$, however combination (d,odd) does not occur. Construct a table, consisting of four rows, one for every message, with in each row the transition probabilities $\Pr\{(R, R_{\text{o/e}}) = (r, r_{\text{o/e}})|M = m\}$ for each of the seven possible channel-output-combinations. How should an optimal receiver decide for each of the seven output-combinations? What is the resulting error-probability?

Solution: The following conditional probabilities were given in the assignment:

m	$\Pr\{R=a\|M=m\}$	$\Pr\{R=b\|M=m\}$	$\Pr\{R=c\|M=m\}$	$\Pr\{R=d\|M=m\}$
1	1	0	0	0
2	1/2	1/2	0	0
3	1/3	1/3	1/3	0
4	1/4	1/4	1/4	1/4

(a) When we assume that each message is equally likely, the MAP decision rule simplifies to the ML decision rule. Therefore, the decoder selects the message sucht that

$$\hat{m}_{\text{ML}}(r) = \text{argmax}_m \Pr\{R = r|M = m\}$$

The resulting decision rule for the decoder is given by

r	a	b	c	d
$\hat{m}_{\text{ML}}(r)$	1	2	3	4

The probability of error can be calculated in two ways

$$P_e = 1 - P_c = 1 - \sum_r \Pr(R = r, M = \hat{m}(r)) = 1 - 1/4(1 + 1/2 + 1/3 + 1/4) = 0.4792 \tag{36}$$

$$P_e = \sum_r \Pr(R = r, M \neq \hat{m}(r)) = \sum_r \sum_{m\neq\hat{m}(r)} \Pr(R = r|M = m) \Pr(M = m) \tag{37}$$

$$= 1/4 * (0 + 1/2 + 2/3 + 3/4) = 23/48 = 0.4792 \tag{38}$$

(b) In this case we use the MAP decision rule

$$\hat{m}_{\text{MAP}}(r) = \text{argmax}_m \Pr\{R = r|M = m\} \Pr\{M = m\}$$

In the table below we calculated the joint probabilities $\Pr\{R = r, M = m\} = \Pr\{R = r|M = m\} \Pr\{M = m\}$.

m	$\Pr\{R=a,M=m\}$	$\Pr\{R=b,M=m\}$	$\Pr\{R=c,M=m\}$	$\Pr\{R=d,M=m\}$
1	1/16	0	0	0
2	3/32	3/32	0	0
3	5/48	5/48	5/48	0
4	7/64	7/64	7/64	7/64

The resulting decision rule for the decoder is given by

r	a	b	c	d
$\hat{m}_{\text{MAP}}(r)$	4	4	4	4

Apparently, we should always decode $\hat{m}_{\text{MAP}} = 4$. The resulting error probability

$$P_e^{\text{MAP}} = \Pr(M \neq 4) = 1 - 7/16 = 9/16 = 0.5625$$

If we would use the ML decision rule the error probability would be

$$P_e^{\mathrm{ML}} = 1 - Pc = 1 - \sum_r \Pr(R = r, M = \hat{m}(r)) \tag{39}$$

$$= 1 - (1/16 + 1/2 * 3/16 + 1/3 * 5/16 + 1/4 * 7/16) = 0.6302 \tag{40}$$

$$P_e^{\mathrm{ML}} = \Pr(R = r, M \neq \hat{m}(r)) = \sum_r \sum_{m \neq \hat{m}(r)} \Pr(R = r|M = m) \Pr(M = m) \tag{41}$$

$$= (0 * 1/16 + 1/2 * 3/16 + 2/3 * 5/16 + 3/4 * 7/16) = 121/192 = 0.6302 \tag{42}$$

Since $P_e^{\mathrm{MAP}} < P_e^{\mathrm{ML}}$ this shows that the maximum-likelihood decision rule is not optimal in this case, as was expected.

(c) Note that in the table we kept the notation concise s.t. $\Pr\{a, o|m\} = \Pr\{R = a, R_{o.e} = o|M = m\}$.

m	$\Pr\{a,o\|m\}$	$\Pr\{a,e\|m\}$	$\Pr\{b,o\|m\}$	$\Pr\{b,e\|m\}$	$\Pr\{c,o\|m\}$	$\Pr\{c,e\|m\}$	$\Pr\{d,e\|m\}$
1	1	0	0	0	0	0	0
2	0	1/2	0	1/2	0	0	0
3	1/3	0	1/3	0	1/3	0	0
4	0	1/4	0	1/4	0	1/4	1/4

Since all messages are equally likely we can use the ML decision rule. The resulting optimum decision rule for the decoder is given by

$r, r_{o/e}$	a,o	a,e	b,o	b,e	c,o	c,e	d,e
$\hat{m}_{\mathrm{ML}}(r)$	1	2	3	2	3	4	4

The error probability

$$P_e^{\mathrm{ML}} = 1 - P_c = 1 - \sum_{r, r_{o/e}} \Pr(R = (r, r_{o/e}), M = \hat{m}(r, r_{o/e})) \tag{43}$$

$$= 1 - 1/4 * (1 + 1/2 + 1/3 + 1/2 + 1/3 + 1/4 + 1/4) = 0.208 \tag{44}$$

$$P_e^{\mathrm{ML}} = \Pr(R = r, r_{o/e}, M \neq \hat{m}(r, r_{o/e})) = \sum_{r, r_{o/e}} \sum_{m \neq \hat{m}(r, r_{o/e})} \Pr(R = r|M = m) \Pr(M = m) \tag{45}$$

$$= 1/4 * (0 + 0 + 1/3 + 1/2) = 0.208 \tag{46}$$

Exercise 3.5: The weather type in Paris is a random variable that is denoted by P. Similarly the weather type in Marseille is a random variable denoted by M. Both random variables can assume the values $\{s, c, r\}$. Here s stands for sunny, c for cloudy, and r for rainy. We assume that the weather type in Marseille M depends on the weather type in Paris P. The joint probability of a certain weather type in Paris together with a certain weather type in Marseille can be found in the table below.

p	$\Pr\{P = p, M = s\}$	$\Pr\{P = p, M = c\}$	$\Pr\{P = p, M = r\}$
s	0.25	0.05	0.00
c	0.20	0.10	0.10
r	0.05	0.15	0.10

The table shows, e.g., that the joint probability of cloudy weather in Paris and sunny weather in Marseille is 0.20.

(a) Suppose that you have to predict a combination of weather types for Paris and Marseille such that your prediction is correct with a probability as large as possible. What combination do you choose and what is the probability that your prediction will be wrong?

(b) Suppose that you have to guess the weather type in Marseille knowing the weather type in Paris. How do you decide for each of the three possible types of weather in Paris if you want to be correct with the highest possible probability? How large is the probability that your guess is wrong?

(c) Suppose that you know the weather type in Marseille and that you must suggest two types of weather in Paris such that the probability that at least one of your suggestions is correct is maximal. What is your pair of suggestions for each of the three possible types of weather in Marseille? Compute the probability that both your suggestions are wrong also for this case.

Solution:

(a) Optimum prediction would be the combination with maximum joint probability, which is $\Pr\{P = s, M = s\} = 0.25$. The probability that this prediction is wrong is $1 - 0.25 = 0.75$.

(b) In case we know the weather type in Paris, we can use the MAP decision rule to have smallest error probability. This results in

p	$\hat{m}$
s	s
c	s
r	c

Now the probability of error is $P_e = 1 - 0.25 - 0.2 - 0.15 = 0.4$.

(c) In this case we will use the MAP decision rule to select the first option and then use the MAP decision rule on the remaining possibilities for the second option. Resulting in

m	$\hat{p}$
s	s c
c	r c
r	c r

Now the probability of error is $P_e = 0.05 + 0.05 + 0.00 = 0.1$.

Exercise 3.6: Consider a set of two messages, hence $\mathcal{M} = \{1, 2\}$. One of these messages is chosen as input of a channel. The output R of this channel takes values fro the set $\mathcal{R} = \{0, 1, 2, 3, 4, 5\}$. When message $M = 1$ is sent, the probability that channel output r occurs is equal to $\Pr\{R = r|M = 1\} = \binom{5}{r}(1/2)^5$. When message $M = 2$ is sent, output r occurs with probability $\Pr\{R = r|M = 2\} = \binom{5}{r}(1/3)^{5-r}(2/3)^r$. Note that in both cases the output has a binomial distribution.

(a) First assume that the a priori message probabilities are equal hence that $\Pr\{M = 1\} = \Pr\{M = 2\} = 1/2$. What are the values of r for which an optimal receiver decides $\hat{M} = 1$? Determine the error probability that is realized by such an optimal receiver.

(b) Next assume that $\Pr\{M = 1\} = 2/3$ and $\Pr\{M = 2\} = 1/3$. Determine the values r for which an optimal receiver receiver should decide $\hat{M} = 1$ now. What is the resulting error probability?

Solution: It is given that

$$\Pr\{R = r|M = 1\} = \binom{5}{r}(1/2)^5$$
$$\Pr\{R = r|M = 2\} = \binom{5}{r}(1/3)^{5-r}(2/3)^r$$

(a) Since each message is equally probable we may use the ML decision rule

$$\hat{m}_{\text{ML}}(r) = \text{argmax}_m \Pr\{R = r|M = m\}$$

Therefore, we choose $\widehat{m} = 1$ if

$$\begin{aligned}
\binom{5}{r}(1/2)^5 &> \binom{5}{r}(1/3)^{5-r}(2/3)^r \\
(1/2)^5 &> (1/3)^{5-r}(2/3)^r \\
\left(\frac{1/2}{1/3}\right)^5 &> \left(\frac{2/3}{1/3}\right)^r \\
\left(\frac{3}{2}\right)^5 &> 2^r \\
5\log_2\left(\frac{3}{2}\right) &> r \\
r < 2.95 &\to r \in \{0,1,2\}
\end{aligned}$$

Now the probability of error

$$\begin{aligned}
P_e^{\mathrm{ML}} &= 1/2 \cdot (\Pr\{R=0|M=2\} + \Pr\{R=1|M=2\} + \Pr\{R=2|M=2\} + \\
&\quad \Pr\{R=3|M=1\} + \Pr\{R=4|M=1\} + \Pr\{R=5|M=1\}) \\
&= 1/2 \cdot (1/243 + 10/243 + 40/243 + 10/32 + 5/32 + 1/32) \\
&= 345/972 = 0.3549
\end{aligned}$$

(b) In this case we must use the MAP decision rule

$$\hat{m}_{\mathrm{MAP}}(r) = \mathrm{argmax}_m \Pr\{R=r|M=m\}\Pr\{M=m\}$$

Therefore, we choose $\hat{m} = 1$ if

$$\begin{aligned}
\binom{5}{r}(1/2)^5 \cdot 2/3 &> \binom{5}{r}(1/3)^{5-r}(2/3)^r \cdot 1/3 \\
(1/2)^5 \cdot 2 &> (1/3)^{5-r}(2/3)^r \\
\left(\frac{1/2}{1/3}\right)^5 \cdot 2 &> \left(\frac{2/3}{1/3}\right)^r \\
\left(\frac{3}{2}\right)^5 \cdot 2 &> 2^r \\
5\log_2\left(\frac{3}{2}\right) + 1 &> r \\
r < 3.95 &\to r \in \{0,1,2,3\}
\end{aligned}$$

Now the probability of error

$$\begin{aligned}
P_e^{\mathrm{MAP}} &= 1/3 \cdot (\Pr\{R=0|M=2\} + \Pr\{R=1|M=2\} + \Pr\{R=2|M=2\} + \Pr\{R=3|M=2\}) \\
&\quad + 2/3 \cdot (\Pr\{R=4|M=1\} + \Pr\{R=5|M=1\}) \\
&= 1/3 \cdot (1/243 + 10/243 + 40/243 + 80/243) \\
&\quad + 2/3 \cdot (5/32 + 1/32) \\
&= 0.3047
\end{aligned}$$

4 Decision Rules for DICO Channels

Exercise 4.1: A communication system is used to transmit one of two equally likely messages, 1 and 2. The channel output is a real-valued random variable R, the conditional density functions of which are shown in Fig. 6. Determine the optimum receiver decision rule and compute the resulting probability of error.

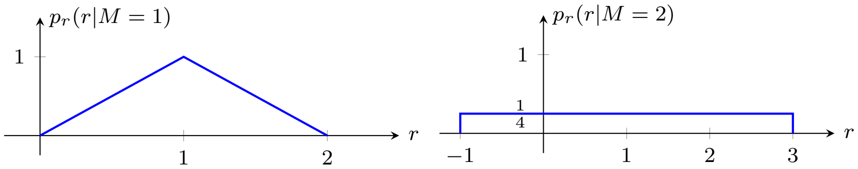

Figure 6: Conditional PDFs (Ex. 4.1).

Solution: We have two messages equaly likely, therefore we use ML detection.

Decide

$$\begin{cases} \hat{m} = 2 & \text{if } r < 1/4 \\ \hat{m} = 1 & \text{if } 1/4 < r < 7/4 \\ \hat{m} = 2 & \text{if } r > 7/4 \end{cases}$$

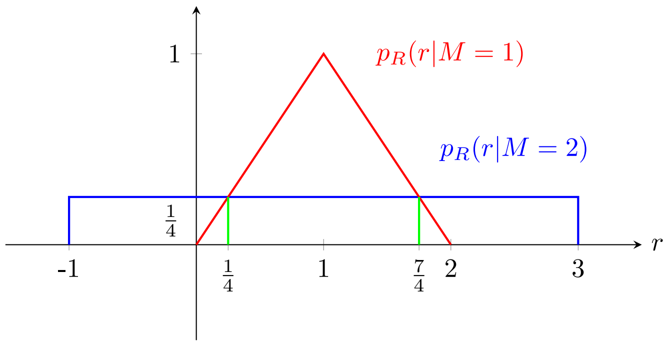

Figure 7: CBA (Sol. 4.1)

To compute the probability of error, we have:

$$\Pr\{E|M=1\} = 2\int_0^{1/4} x dx = x^2|_0^{1/4} = \frac{1}{16} \tag{47}$$

$$\Pr\{E|M=2\} = \frac{1}{4}\int_{1/4}^{7/4} dx = \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{8} \tag{48}$$

Therefore,

$$P_e = \Pr\{M=1\}\Pr\{E|M=1\} + \Pr\{M=2\}\Pr\{E|M=2\} \tag{49}$$

$$= \frac{1}{2}\cdot\frac{1}{16} + \frac{1}{2}\cdot\frac{3}{8} = \frac{7}{32} \tag{50}$$

Obs.: MAP consider $\Pr\{M=m\}\Pr\{r|M=m\}$. ML consider $\Pr\{r|M=m\}$.

Exercise 4.2: The noise n in Fig. 8(a) is Gaussian with zero mean, i.e., $E[N] = 0$. If one of two equally likely messages is transmitted, using the signals of 8(b), an optimum receiver yields $P_e = 0.01$.

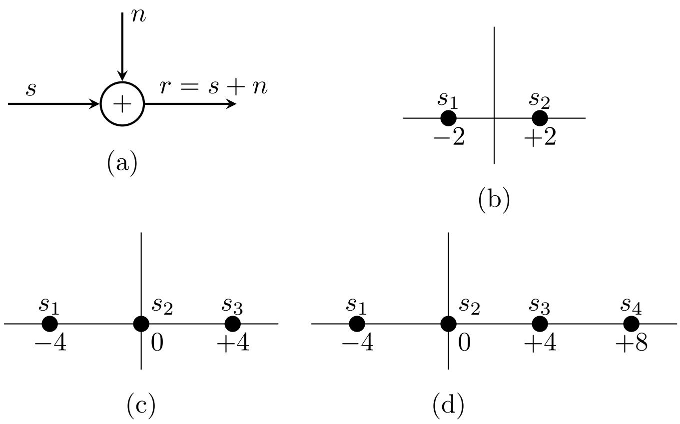

Figure 8: Channel and three sets of signals (Ex. 4.2).

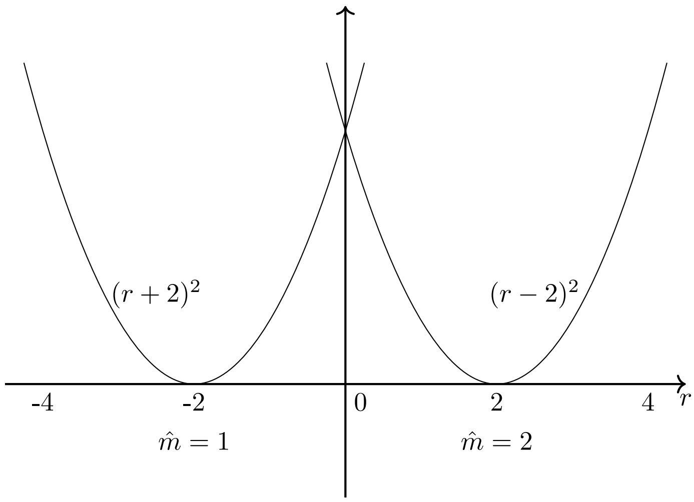

Figure 9: Visualisation of the decision variables (Sol. 4.2)

(a) What is the minimum attainable probability of error $P_e^{\min}$ when the channel of 8(a) is used with three equally likely messages and the signals of 8(c)? And with four equally likely messages and the signals of 8(d)?

(b) How do the answers to the previous questions change if it is known that $E[N] = 1$ instead of 0?

Solution: Noise variable N, with Gaussian distribution and mean $\mu = 0$. Thus, the decision variable for message m is

$$\Pr\{M = m\}\Pr\{r|M = m\} = \Pr\{M = m\}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(r-s_m)^2}{2\sigma^2}}. \tag{51}$$

The optimum receiver will select the message with the biggest decision variable. Because $\Pr\{M = 1\} = \Pr\{M = 2\}$, this is equivalent to choosing the message for which $(r - s_m)^2$ is minimized. That is, $(r + 2)^2$ for $m = 1$ and $(r - 2)^2$ for $m = 2$. The threshold results in $r^* = 0$. Fig. 9 shows the visual representation of the decision variables.

For the case shown in Fig. 8 b), the error occurs when message $m = 1$ is sent, but the receiver estimates message $\hat{m} = 2$ according to the decision rule shown in Fig. 9. The opposite, when $m = 2$ and $\hat{m} = 1$, also results in the error. Thus, for the probability of error we have

$$P_e = \Pr\{M = 1\}\Pr\{r > 0|M = 1\} + \Pr\{M = 1\}\Pr\{r < 0|M = 2\} \tag{52}$$

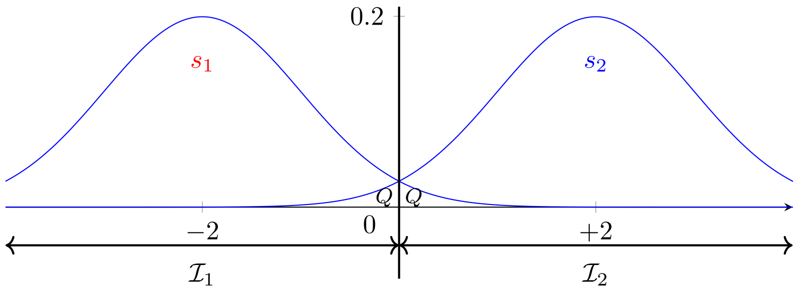

Figure 10: 'tails' have area Q (Sol. 4.2)

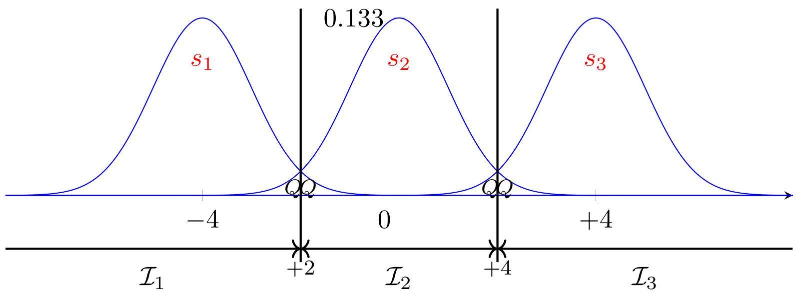

Figure 11: three equally like messages (Sol. 4.2)

If we look at $\Pr\{r > 0|M = 1\}$ we find that

$$\Pr\{r > 0|M = 1\} = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r+2)^2}{2\sigma^2}} dr \tag{53}$$

Making the change of variables $\lambda = \frac{r+2}{\sigma}$ results in

$$\Pr\{r > 0|M = 1\} = \int_{2/\sigma}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda = Q\left(\frac{2}{\sigma}\right) \tag{54}$$

Following the same steps as for $\Pr\{r > 0|M = 1\}$, we can conclude that $\Pr\{r < 0|M = 2\} = Q\left(\frac{2}{\sigma}\right)$. Then

$$\begin{aligned} P_e &= \Pr\{M = 1\} \Pr\{r > 0|M = 1\} + \Pr\{M = 2\} \Pr\{r < 0|M = 2\} \\ &= \frac{1}{2} Q\left(\frac{2}{\sigma}\right) + \frac{1}{2} Q\left(\frac{2}{\sigma}\right) = Q\left(\frac{2}{\sigma}\right). \end{aligned}$$

It's given that $P_e = 0.01$, so $Q\left(\frac{2}{\sigma}\right) = 0.01$ and then, as $Q(2.326) = 0.01$, we obtain $\frac{2}{\sigma} = 2.326 \implies \sigma = 0.86$.

(a) For the three equally likely messages (Fig. 11):

$$\begin{aligned} P_e &= \Pr\{M = 1\} \Pr\{r > 2|M = 1\} + (\Pr\{M = 2\} \Pr\{r < 2|M = 2\} \\ &+ \Pr\{M = 2\} \Pr\{r > 4|M = 2\}) + \Pr\{M = 3\} \Pr\{r < 4|M = 3\} \\ &= \Pr\{M = 1\}Q + \Pr\{M = 2\}2Q + \Pr\{M = 3\}Q = \frac{1}{3} \cdot 4Q = 0.0133 \end{aligned}$$

For the four equally like messages (Fig. 12):

$$P_e = \Pr\{M = 1\}Q + \Pr\{M = 2\}2Q + \Pr\{M = 3\}2Q + \Pr\{M = 4\}Q \tag{55}$$

$$= \frac{1}{4} \cdot 6Q = 0.015 \tag{56}$$

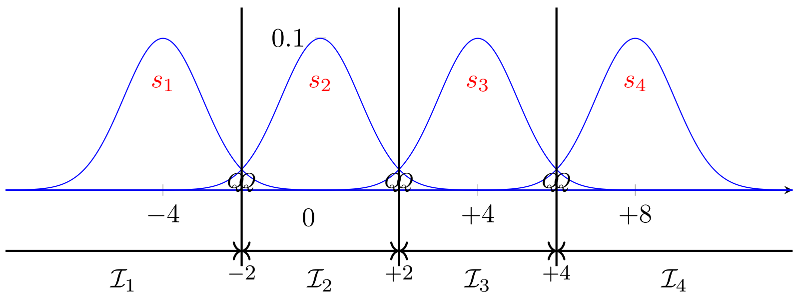

Figure 12: four equally like messages (Sol. 4.2)

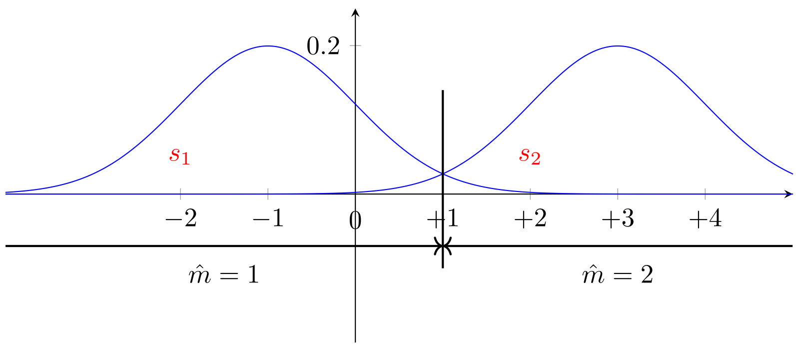

Figure 13: four equally like messages (Sol. 4.2)

(b) $E[N] = 1$, thus $\mu = 1$.

Therefore, the Gaussian distributions shift by $+1$ and also the threshold shifts to $+1$ (instead of 0), see Fig. 13. This does not result in a change of the error probability. Precisely, since

$$p_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(n-1)^2}{2\sigma^2}}, \tag{57}$$

it can be shown that

$$p_R(r|s_m) = p_N(r - s_m) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-s_m-1)^2}{2\sigma^2}}. \tag{58}$$

Thus, concerning to the decision variables:

$$\Pr\{M = m\} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-s_m-1)^2}{2\sigma^2}} \tag{59}$$

etc.

Exercise 4.3: One of four equally likely messages is to be communicated over a vector channel which adds a (different) statistically independent zero-mean Gaussian random variable with variance $N_0/2$ to each transmitted vector component. Assume that the transmitter uses the signal vectors shown in Fig. 14. Express the P_e produced by an optimum receiver in terms of the function $Q(x)$. Compute an upper bound based on the union bound and compare in a plot the exact expression with the bound. In what region is the bound good? What is the intuition behind this?

Solution: It is given that each message is equally likely and the noise $\underline{N} \sim \mathcal{N}(\mu = 0, N_0/2)$. The vector signal $\underline{s}_m = (s_{m1}, s_{m2})$ corresponds to a message m. when this signal is transmitted through the

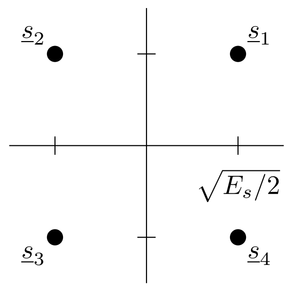

Figure 14: Signal structure (Ex. 4.3).

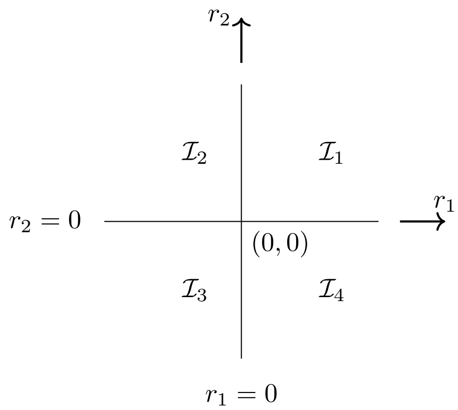

Figure 15: Visualization of the decision regions (Sol. 4.3). Each decision region corresponds to a particular message $\underline{s}_m$ in Fig. 14.

vector channel, the channel adds Gaussian noise to signal components. The channel produces an output vector

$$\underline{r} = (r_1, r_2) = \underline{s} + \underline{n} = (s_{m1} + n_1, s_{m2} + n_2), \tag{60}$$

The conditional PDF of the channel output is

$$p_{\underline{R}}(\underline{r}|M = m) = p_{\underline{R}}(\underline{r}|\underline{S} = \underline{s}_m) = p_{\underline{N}}(\underline{r} - \underline{s}_m|\underline{S} = \underline{s}_m) = p_{\underline{N}}(\underline{r} - \underline{s}_m). \tag{61}$$

Thus, the decision variables for receiving (r_1, r_2):

$$\Pr\{M = m\}\Pr\{\underline{r}|M = m\} \tag{62}$$

$$= \Pr\{M = m\}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(r_1-s_{m1})^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(r_2-s_{m2})^2}{2\sigma^2}} \tag{63}$$

$$= \Pr\{M = m\}\frac{1}{2\pi\sigma^2}e^{-\frac{(r_1-s_{m1})^2+(r_2-s_{m2})^2}{2\sigma^2}} \tag{64}$$

for $m = 1, 2, 3$ and 4. Because all messages are equally likely we should minimize $(r_1-s_{m1})^2+(r_2-s_{m2})^2 = ||\underline{r} - \underline{s}_m||^2$. This gives the decision regions as shown in Fig. 15.

To estimate the probability of error, it is sufficient (because of symmetry) to look at the probability of correct decoding for s_1 only. According to the decision regions in Fig. 15, when message $m = 1$ is sent, the receiver estimates the message correctly, $\hat{m} = 1$, only when $\underline{r}$ satisfies $(0 \leq r_1 \leq \infty, 0 \leq r_2 \leq \infty)$.

Thus, with $\underline{s}_1 = (s_{11}, s_{12}) = \left(\sqrt{E_s/2}, \sqrt{E_s/2}\right)$:

$$P_c = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_1-s_{11})^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_2-s_{12})^2}{2\sigma^2}} dr_1 dr_2 \tag{65}$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_1-s_{11})^2}{2\sigma^2}} dr_1 \cdot \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_2-s_{12})^2}{2\sigma^2}} dr_2 \tag{66}$$

$$= \left(1 - Q\left(\frac{\sqrt{E_s/2}}{\sigma}\right)\right)\left(1 - Q\left(\frac{\sqrt{E_s/2}}{\sigma}\right)\right) \tag{67}$$

$$= \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2. \tag{68}$$

Due to the symmetry,

$$P_c^1 = P_c^2 = P_c^3 = P_c^4. \tag{69}$$

Therefore total probability of the correct decision is

$$\begin{aligned} P_c &= \sum_{m\in\mathcal{M}} \Pr\{M = m\} P_c^m \\ &= \sum_{m=1}^{4} \frac{1}{4}\left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \\ &= \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2. \end{aligned}$$

Finally:

$$P_e = 1 - P_c = 1 - (1-Q)^2 = 2Q - Q^2 \tag{70}$$

$$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right). \tag{71}$$

Now, the horizontal distance between the constellation points is $\|\underline{s}_1 - \underline{s}_2\| = \|\underline{s}_4 - \underline{s}_3\| = \sqrt{2E_s}$, while the vertical distance between the constellation points is $\|\underline{s}_1 - \underline{s}_4\| = \|\underline{s}_2 - \underline{s}_3\| = \sqrt{2E_s}$.

Considering that $\underline{s}_1$ is transmitted. The conditioned error probability $P_e^1 = \Pr\{\hat{\underline{s}} \neq \underline{s}_1 | \underline{s}_1\}$ will be upper-bounded by the error probability of detecting $\underline{s}_2$ or $\underline{s}_4$:

$$\begin{aligned} P_e^1 &\leq \Pr\{\hat{\underline{s}} = \underline{s}_2 | \underline{s}_1\} + \Pr\{\hat{\underline{s}} = \underline{s}_4 | \underline{s}_1\} \\ &= Q\left(\frac{\|\underline{s}_2 - \underline{s}_1\|}{2\sqrt{N_0/2}}\right) + Q\left(\frac{\|\underline{s}_4 - \underline{s}_1\|}{2\sqrt{N_0/2}}\right) \\ &= Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{E_s}{N_0}}\right) = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) \end{aligned}$$

Due to the symmetry of the problem, the conditioned error probabilities will be the same for every transmitted message $\underline{s}_m$. Thus, the total error probability P_e is bounded by

$$P_e = \frac{1}{4}\sum_{m=1}^{4} P_e^m = P_e^1 \leq 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Exercise 4.4: In the communication system diagrammed in Fig. 16 the transmitted signal s and the noises n_1 and n_2 are all random voltages and all statistically independent. Assume that $|\mathcal{M}| = 2$ i.e., $\mathcal{M} = \{1, 2\}$ and that

$$\begin{aligned} \Pr\{M = 1\} = \Pr\{M = 2\} &= 1/2 \\ s_1 = -s_2 &= \sqrt{E_s} \\ p_{N_1}(n) = p_{N_2}(n) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n^2}{2\sigma^2}\right). \end{aligned} \tag{72}$$

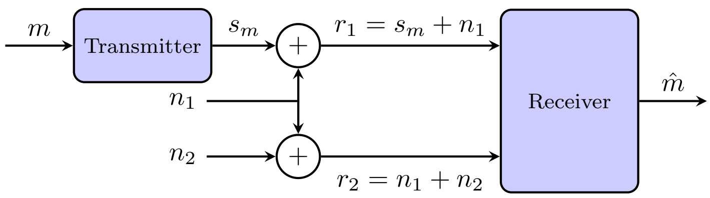

Figure 16: Communication system (Ex. 4.4).

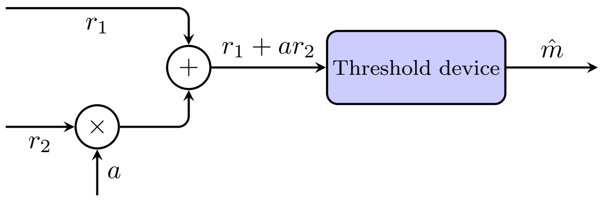

Figure 17: Receiver realization (Ex. 4.4).

(a) Show that the optimum receiver can be realized as diagrammed in Fig. 17 where a is an appropriately chosen constant.

(b) What is the optimum value for a?

(c) What is the optimum threshold setting?

(d) Express the resulting P_e in terms of $Q(x)$.

(e) By what factor would E_s have to be increased to yield this same probability of error if the receiver were restricted to observing *only* r_1?

Solution:

(a) We can write that

$$\begin{aligned} n_1 &= r_1 - s_m, \\ n_2 &= r_2 - n_1 = r_2 - r_1 + s_m, \end{aligned}$$

The decision variable corresponding to m is defined as

$$\Pr\{M = m\} p_{\underline{R}}(\underline{r}|S = s_m)$$

where

$$\begin{aligned} p_{\underline{R}}(\underline{r}|S = s_m) &= p_{R_1}(r_1|S = s_m) p_{R_2}(r_2|S = s_m, R_1 = r_1), \\ &= p_{N_1}(r_1 - s_m) p_{N_2}(r_2 - r_1 + s_m). \end{aligned}$$

Therefore, we can write the decision rule as

$$\begin{aligned} \hat{m} &= \arg\max_m \frac{1}{2} \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^2 \exp\left(-\frac{(r_1 - s_m)^2}{2\sigma^2} \right) \exp\left(-\frac{(r_2 - r_1 + s_m)^2}{2\sigma^2} \right), \\ &= \arg\min_m (r_1 - s_m)^2 + (r_2 - r_1 + s_m)^2, \\ &= \arg\min_m r_1^2 - 2r_1 s_m + s_m^2 + (r_2 - r_1)^2 + 2(r_2 - r_1)s_m + s_m^2. \end{aligned}$$

The minimization is performed with respect to m, so only terms with s_m are taken into account. However, note that $s_m^2 = E_s$ is a constant, so it can be left out. Overall, the equation is simplified to

$$(2r_2 - 4r_1)s_m.$$

The resulting expression can be divided by 4 (minimization problem is not affected) in order to obtain the form $r_1 + \alpha r_2$, and the following decision rule can be obtained

$$\hat{m} = \arg\min_m \left(\frac{1}{2}r_2 - r_1\right) s_m = \arg\max_m \left(r_1 - \frac{1}{2}r_2\right) s_m \tag{73}$$

Note that $\arg\max f(x) = \arg\min -f(x)$.

If $\left(r_1 - \frac{1}{2}r_2\right) > 0$, then we decide that $s_m = \sqrt{E_s}$ is sent, i.e., $\hat{m} = 1$, and vice versa.

(b) As Eq. (73) implies, the optimum value for a is -1/2.

(c) As Eq. (73) implies, the optimum threshold setting is to compare with 0.

(d) Consider

$$\begin{aligned}
r =& r_1 - \frac{1}{2}r_2, \\
=& s_m + n_1 - \frac{1}{2}(n_1 + n_2), \\
=& s_m + \frac{1}{2}n_1 - \frac{1}{2}n_2, \\
=& s_m + n,
\end{aligned}$$

where $n = \frac{1}{2}(n_1 - n_2)$. We know $S_m = \pm\sqrt{E_s}$. What is n?

- Random variable N is Gaussian and the expected value of N is

$$E[N] = \frac{1}{2}E[N_1] - \frac{1}{2}E[N_2] = 0.$$

- The variance of N is

$$\begin{aligned}
E[N^2] =& \frac{1}{4}E[(N_1 - N_2)^2], \\
=& \frac{1}{4}E[N_1^2] - \frac{1}{2}E[N_1 N_2] + \frac{1}{4}E[N_2^2], \\
=& \frac{1}{4}\sigma^2 - \frac{1}{2}E[N_1]E[N_2] + \frac{1}{4}\sigma^2, \\
\sigma_n^2 =& \frac{1}{2}\sigma^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
P_e =& Q\left(\frac{d/2}{\sigma_n}\right), \\
=& Q\left(\frac{\sqrt{E_s}}{\sigma/\sqrt{2}}\right), \\
P_e =& Q\left(\frac{\sqrt{2E_s}}{\sigma}\right).
\end{aligned}$$

(e) Decision based only on r_1 leads to a probability of error

$$\tilde{P}_e = Q\left(\frac{\sqrt{E_s}}{\sigma}\right).$$

Therefore, we should double E_s to have the same error probability.

Exercise 4.5: Consider a communication channel with an input s that can only assume positive values. The PDF of the output R when the input is s is given by

$$p_R(r|S=s) = \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{r^2}{2s^2}\right).$$

Note that conditionally on the input s, this R is Gaussian, with mean zero and variance s^2. In Fig. 18 this density is plotted for $s = 1$.

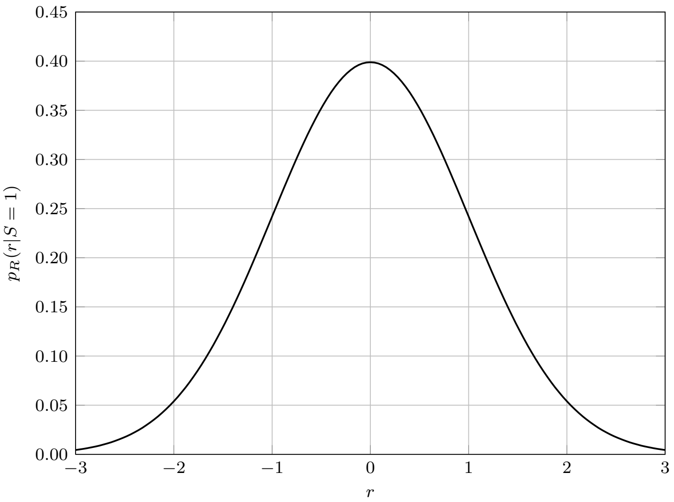

Figure 18: The PDF $p_R(r|S=1)$ (Ex. 4.5).

There are two messages i.e., $\mathcal{M} = \{1, 2\}$ and the corresponding signals are $s_1 = 1$ and $s_2 = \sqrt{e}$. Let the a-priori probabilities $\Pr\{M = 1\} = \Pr\{M = 2\} = 1/2$.

(a) Sketch both PDFs $p_R(r|S = s_1)$ and $p_R(r|S = s_2)$ in a single figure. For what values r does an optimum receiver decide $\hat{M} = 1$?

(b) Determine the corresponding error probability P_e. Express it in terms of the $Q(\cdot)$ function.

(c) If the a-priori probability $\Pr\{M = 1\}$ is small enough, an optimum receiver will choose $\hat{M} = 2$ for all r. What is the largest value of $\Pr\{M = 1\}$ for which this happens? What is the error probability for this value of $\Pr\{M = 1\}$?

Solution: It is given that $\mathcal{M} = \{1, 2\}$, $s_1 = 1$, $s_2 = \sqrt{e}$ and both messages are equally likely: $\Pr\{M = 1\} = \Pr\{M = 2\} = \frac{1}{2}$.

(a) Fig. 20.

ML-rule can be used to estimate the message at the receiver side. The decision is made based on the likelihood probabilities $\Pr\{r|M = 1\}$ and $\Pr\{r|M = 2\}$. Making the equations equal in order to find the threshold value of r:

$$\Pr\{r|M=1\} = \Pr\{r|M=2\} \tag{74}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} = \frac{1}{\sqrt{2\pi e}} e^{-\frac{r^2}{2e}} \tag{75}$$

$$e^{\frac{r^2}{2} - \frac{r^2}{2e}} = \sqrt{e} = e^{\frac{1}{2}} \tag{76}$$

Thus

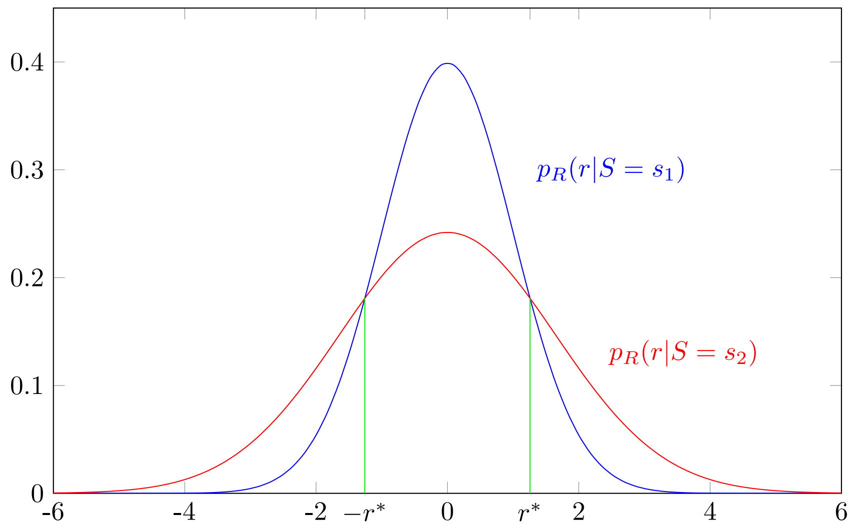

Figure 19: Sketch of the PDFs (Sol. 4.5).

$$\frac{r^2}{2} - \frac{r^2}{2e} = \frac{1}{2} \tag{77}$$
$$er^2 - r^2 = e \tag{78}$$
$$r^2(e-1) = e \tag{79}$$
$$r^2 = \frac{e}{e-1} \tag{80}$$
$$r^* = \sqrt{\frac{e}{e-1}} \approx 1.2578 \tag{81}$$

Thus, the receiver estimates $\hat{M} = 1$ if it receives $|r| < \sqrt{\frac{e}{e-1}}$.

(b) The error occurs when the receiver estimates the wrong message. In other words when message $m = 1$ is sent, but the receiver estimates $\hat{m} = 2$, and the other way around when $m = 2$, but the receiver estimates $\hat{m} = 1$. The probabilities of the two events are $\Pr\{M = 1\} \Pr\{|r| > r^*\}$ and $\Pr\{M = 2\} \Pr\{-r^* < r < r^*\}$ respectively. Thus, the overall error probability is

$$P_e = \Pr\{M = 1\} \Pr\{|r| > r^*|S = s_1\} + \Pr\{M = 2\} \Pr\{-r^* < r < r^*|S = s_2\} \tag{82}$$

While $\Pr\{M = 1\} = \Pr\{M = 2\} = \frac{1}{2}$, the likelihood probabilities can be seen in Fig. 19.
$\Pr\{r|S = s_1\} = \mathcal{N}(r|0, 1)$ is a standard normal distribution, so finding $\Pr\{|r| > r^*\}$ is equivalent to finding two tails of $\Pr\{r|S = s_1\}$ that can be shown using Q-function:

$$\Pr\{|r| > r^*|S = s_1\} = (1 - Q(-r^*)) + Q(r^*) = Q(r^*) + Q(r^*) = 2Q(r^*). \tag{83}$$

Next, $\Pr\{r|S = s_2\} = \mathcal{N}(r|0, e)$ is a normal distribution, so to simplify the integral it is easier to bring the distribution into the standard form by performing the change of variables $r = \frac{r-0}{\sqrt{e}}$. The result is

$$\Pr\{-r^* < r < r^*|S = s_2\} = \Pr\left\{-\frac{r^*}{\sqrt{e}} < \frac{r}{\sqrt{e}} < \frac{r^*}{\sqrt{e}}|S = s_2\right\} \tag{84}$$
$$= 1 - \left(1 - Q\left(-\frac{r^*}{\sqrt{e}}\right)\right) - Q\left(\frac{r^*}{\sqrt{e}}\right) \tag{85}$$
$$= 1 - 2Q\left(\frac{r^*}{\sqrt{e}}\right). \tag{86}$$

Substituting the above results into (82),

$$P_e = \frac{1}{2} \cdot 2Q(r^*) + \frac{1}{2}\left(1 - 2Q\left(\frac{r^*}{\sqrt{e}}\right)\right) \tag{87}$$

$$= Q(r^*) - Q\left(\frac{r^*}{\sqrt{e}}\right) + \frac{1}{2} \tag{88}$$

$$= Q\left(\sqrt{\frac{e}{e-1}}\right) - Q\left(\sqrt{\frac{1}{e-1}}\right) + \frac{1}{2} \approx 0.3815 \tag{89}$$

(c) The receiver always estimates $\hat{m} = 2$ if

$$\Pr\{M = 1\}\Pr\{r|M = 1\} \le \Pr\{M = 2\}\Pr\{r|M = 2\} \tag{90}$$

From Fig. 20, note that the receiver always estimates $\hat{m} = 2$ when $r^* = 0$.

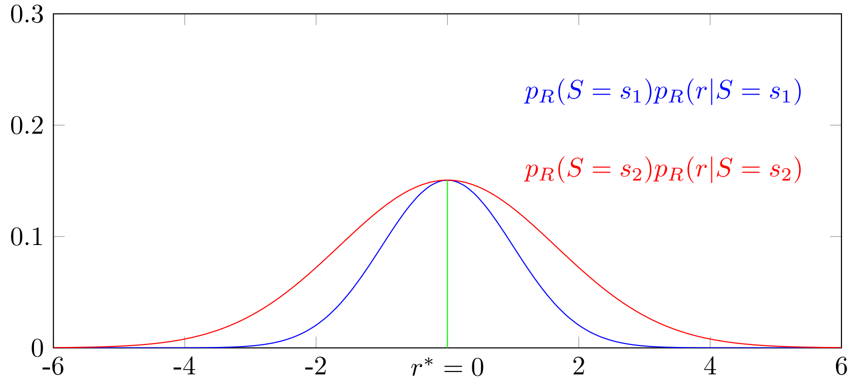

Figure 20: Sketch of the decision variables that satisfy (c) (Sol. 4.5).

It will always be decided to select message 2 when

$$p_1 \frac{1}{\sqrt{2\pi}} \le p_2 \frac{1}{\sqrt{2\pi e}} \tag{91}$$

$$p_1\sqrt{e} = p_2 = 1 - p_1 \tag{92}$$

$$p_1(\sqrt{e} + 1) = 1 \tag{93}$$

$$p_1 = \frac{1}{\sqrt{e} + 1} \approx 0.3775 \tag{94}$$

For the error probability we have

$$P_e = p_1, \tag{95}$$

since whenever message $m = 1$ is sent, the receiver makes wrong estimation $\hat{m} = 2$.

Exercise 4.6: Consider a communication channel with an input s that can only assume positive values. The PDF of the output R when the input is s is given by

$$p_R(r|S = s) = \begin{cases} \frac{s-|r|}{s^2} & \text{for } 0 \le |r| \le s \\ 0 & \text{for } |r| > s \end{cases},$$

see Fig. 21.

There are two messages i.e., $\mathcal{M} = \{1, 2\}$ and the corresponding signals are $s_1 = 1/2$ and $s_2 = 2$. Let the a-priori probabilities $\Pr\{M = 1\} = \Pr\{M = 2\} = 1/2$.

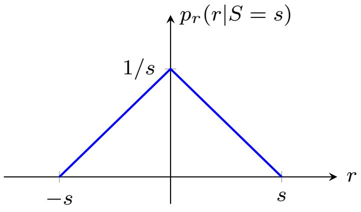

Figure 21: The PDF $p_R(r|S = s)$ (Ex. 4.6).

(a) Sketch the decision variables as a function of the output r in a single figure. For what values r does an optimum receiver decide $\hat{M} = 1$?

(b) Determine the corresponding error probability P_e.

(c) If the a-priori probability $\Pr\{M = 1\}$ is small enough, an optimum receiver will choose $\hat{M} = 2$ for all r. What is the largest value of $\Pr\{M = 1\}$ for which this happens? What is the error probability for this value of $\Pr\{M = 1\}$?

Solution:

(a) The two messages are equally likely, so the ML-rule can be used. Therefore, the likelihood probabilities can be used as the decision variables. These decision variables are shown in Figure 22.

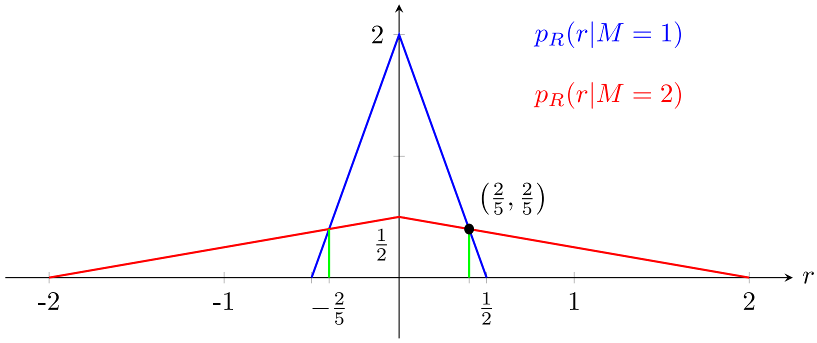

Figure 22: Sketch of the decision variables (Sol. 4.6).

The equations of the two lines are

$$y = 2 - 4x \tag{96}$$

$$y = \frac{1}{2} - \frac{1}{4}x \tag{97}$$

Equating them to obtain the interception:

$$2 - 4x = \frac{1}{2} - \frac{1}{4}x \tag{98}$$

$$\frac{3}{2} = \frac{15}{4}x \tag{99}$$

$$x = \frac{2}{5} \tag{100}$$

$$y = 2 - 4x = \frac{2}{5} \tag{101}$$

So the thresholds are $|r^*| = x = \frac{2}{5}$

(b) The probability of mistakenly choosing $M = 2$ when $M = 1$ is

$$P_e^1 = \left(\frac{1}{2} - \frac{2}{5}\right)\frac{2}{5} = \frac{1}{25}$$

The probability of mistakenly choosing $M = 1$ when $M = 2$ is

$$P_e^2 = 1 - \left(2 - \frac{2}{5}\right)\frac{2}{5} = \frac{9}{25}$$

Then, the total probability of error is

$$P_e = \frac{1}{2}P_e^1 + \frac{1}{2}P_e^2 = \frac{1}{2}\left(\frac{1}{25} - \frac{9}{25}\right) = \frac{1}{5}$$

(c) For that to happen, because of the PDF shapes, the triangle with respect to $M = 1$ times the probability p_1 should be inside the triangle with respect to $M = 2$ times the probability p_2, which is equivalent to make

$$\frac{1}{2}p_2 = 2p_1 \tag{102}$$

$$\frac{1}{2}(1 - p_1) = 2p_1 \tag{103}$$

$$\frac{1}{2} = \left(2 + \frac{1}{2}\right)p_1 \tag{104}$$

$$p_1 = \frac{1}{5} \tag{105}$$

$$p_2 = 1 - p_1 = \frac{4}{5} \tag{106}$$

$$P_e = p_1 = \frac{1}{5} \tag{107}$$

Exercise 4.7: Consider a communication system which is used to transmit one out of three equally likely messages. The corresponding signals are $s_1 = 0, s_2 = 2$, and $s_3 = 6$. The output of the channel $r = s_m + n$ where the noise N is Gaussian with $E[N] = 0$ and $E[N^2] = 1$.

(a) What is the smallest possible error probability that can be achieved? Describe how the best possible (optimum) receiver should decide.

(b) What is the largest possible error probability that can be achieved? Describe how the worst possible receiver should decide.

Note that the receiver has to decide $\hat{m} \in \{1, 2, 3\}$ for all real-valued r. Use the $Q(\cdot)$-function in your answers.

Solution:
We are given that $m = 1$, $m = 2$, and $m = 3$ are equally likely messages corresponding to $s_1 = 0$, $s_2 = 2$, and $s_3 = 6$, respectively. Received signal is defined as $r = s_m + n$ where N is Gaussian with zero mean and unit variance.

(a) In the course reader on page 40, it is shown that for equally likely messages and an additive Gaussian noise distribution, Maximum-Likelihood detector uses

$$\frac{s_1 + s_2}{2}, \tag{108}$$

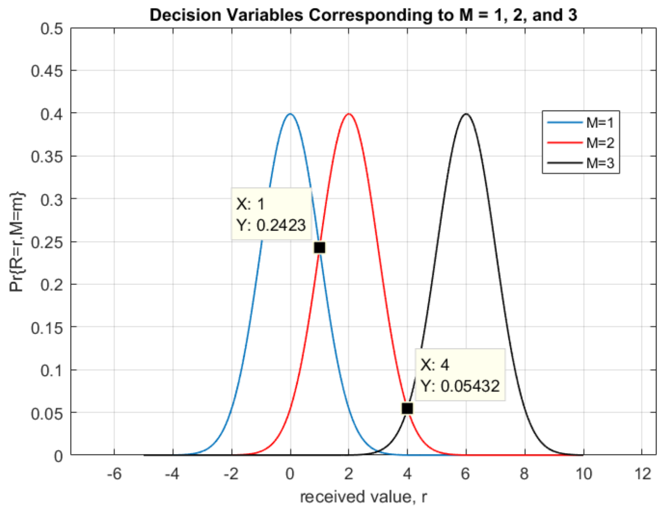

Figure 23: Decision variables corresponding to different M values (Sol. 4.7).

as the decision threshold which corresponds to the minimum Euclidean distance decision. Therefore our decision thresholds are

$$r_{12}^* = \frac{s_1 + s_2}{2} = 1, \tag{109}$$

$$r_{23}^* = \frac{s_2 + s_3}{2} = 4, \tag{110}$$

which are illustrated in Fig. 23, and corresponds to a decision procedure according to

$$\hat{m} = \begin{cases} 1: & 1 > r \\ 2: & 4 > r > 1 \\ 3: & r > 4 \end{cases} \tag{111}$$

Finally the minimum probability of error is defined to be

$$P_e = \sum_{i=1}^{3} \Pr\{M = i\} \Pr\{f(r) \neq i | M = i\} = \sum_{i=1}^{3} \Pr\{M = i\} P_e^i \tag{112}$$

$$= \frac{1}{3} Q(1) + \frac{1}{3}(Q(1) + Q(2)) + \frac{1}{3} Q(2) = 0.1209, \tag{113}$$

where $f(r)$ is the decision function which outputs 1, 2, or 3 as $\hat{m}$.

(b) Maximizing P_e corresponds to minimizing P_c by definition. Consider an interval of $r < R < r + dr$. The contribution of this interval to P_c is

$$\Pr\{M = f(r), r < R < r + dr\} = \Pr\{M = f(r)\} p_R(r | M = f(r)) \tag{114}$$

$$= \Pr\{M = f(r)\} \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{(r - f(r))^2}{2} \right) dr, \tag{115}$$

where the only part whose value depends on M is $(r - f(r))^2$. Therefore our new task is to maximize it over $f(r)$ which is Maximum Euclidean-Distance Decoding.

Intuitively, if we select $(s_1 + s_3)/2 = 3$ as our decision variable and write our decision rule as

$$\hat{m} = \begin{cases} 1: & r > 3 \\ 3: & 3 > r \end{cases} \tag{116}$$

as illustrated in Fig. 24, P_c and P_e become

$$P_c = \frac{1}{3}(Q(3) + 0 + Q(3)), \tag{117}$$

$$P_e = 1 - P_c = 1 - \frac{2}{3} Q(3) = 0.9991, \tag{118}$$

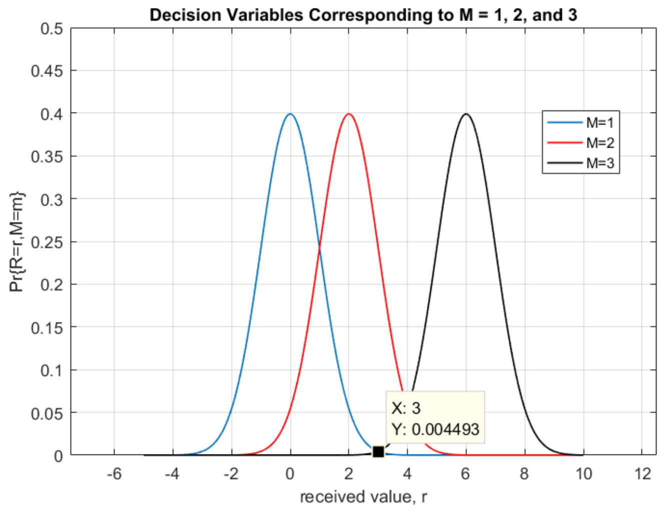

Figure 24: Decision variables corresponding to different M values (Sol. 4.7).

which leads to the maximum P_e.

Exercise 4.8: Two binary symbols b_1 and b_2 are transmitted over a Gaussian channel. We want to find out what the resulting bit-error probabilities are.

b_1	b_2	s
0	0	-3
0	1	-1
1	1	+1
1	0	+3

The binary symbols are uniform and independent, i.e., $\Pr\{B_1 = 0, B_2 = 0\} = \Pr\{B_1 = 0, B_2 = 1\} = \Pr\{B_1 = 1, B_2 = 0\} = \Pr\{B_1 = 1, B_2 = 1\} = 1/4$. A Gray-mapping, see table, is used to map the binary symbols onto the signals $s \in \{-3, -1, +1, +3\}$. The output of the channel $r = s_{b_1 b_2} + n$ where the noise N is Gaussian with $E[N] = 0$ and $E[N^2] = 1$.

(a) A receiver produces bit-pair estimates $\hat{B}_1\hat{B}_2 \in \{00, 01, 11, 10\}$. It minimizes the bit-pair error probability $P_e^{12} = \Pr\{\hat{B}_1\hat{B}_2 \neq B_1B_2\}$. What is the resulting bit-pair probability P_e^{12}. How does this receiver decide?

(b) A second receiver produces two bit estimates $\hat{B}_1 \in \{0, 1\}$, and $\hat{B}_2 \in \{0, 1\}$. It minimizes both the bit-error probability $P_e^1 = \Pr\{\hat{B}_1 \neq B_1\}$ and the bit-error probability $P_e^2 = \Pr\{\hat{B}_2 \neq B_2\}$. What are the resulting bit-error probabilities P_e^1 and P_e^2? How does the receiver decide?

Use the $Q(\cdot)$-function in your answers. The solutions of $\frac{\exp(3x)+\exp(-3x)}{\exp(x)+\exp(-x)} = \exp(4)$ are $x = \pm 2.0089$.

Solution:

(a) Receiver decision (estimate) for b_1b_2 is given by $\widehat{b_1b_2} = \begin{cases} 00, & \text{if } r < -2 \\ 01, & \text{if } -2 \leq r < 0 \\ 11, & \text{if } 0 \leq r < 2 \\ 10, & \text{if } 2 \leq r. \end{cases}$

The decision variables for every message can be seen in Fig. 25. The error probability for every region can be shown as

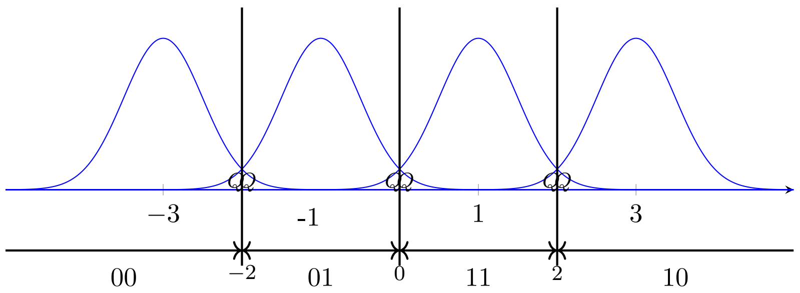

Figure 25: visualisation of the distribution of the channel output for each signal (Sol. 4.8).

$$P_e^{12} = \frac{1}{4}Q(1) + \frac{1}{4}\cdot 2Q(1) + \frac{1}{4}\cdot 2Q(1) + \frac{1}{4}Q(1) \tag{119}$$
$$= \frac{3}{2}Q(1) = 0.2380. \tag{120}$$

(b) The receiver estimates $\hat{B}_1 = 1$ if

$$\Pr\{B_1B_2 = 00\}\frac{1}{\sqrt{2\pi}}e^{-\frac{(r+3)^2}{2}} + \Pr\{B_1B_2 = 01\}\frac{1}{\sqrt{2\pi}}e^{-\frac{(r+1)^2}{2}} \tag{121}$$
$$< \Pr\{B_1B_2 = 11\}\frac{1}{\sqrt{2\pi}}e^{-\frac{(r-1)^2}{2}} + \Pr\{B_1B_2 = 10\}\frac{1}{\sqrt{2\pi}}e^{-\frac{(r-3)^2}{2}} \tag{122}$$

Then

$$e^{-\frac{(r+3)^2}{2}} + e^{-\frac{(r+1)^2}{2}} < e^{-\frac{(r-1)^2}{2}} + e^{-\frac{(r-3)^2}{2}} \tag{123}$$

which implies that the receiver always estimates $\hat{B}_1 = 1$ when

$$r > 0. \tag{124}$$

Therefore

$$P_e^1 = \frac{1}{4}Q(3) + \frac{1}{4}Q(1) + \frac{1}{4}Q(1) + \frac{1}{4}Q(3) \tag{125}$$
$$= \frac{1}{2}Q(3) + \frac{1}{2}Q(1) \approx 0.0800 \tag{126}$$

Decision $\hat{B}_2 = 1$

$$\Pr\{B_1B_2 = 00\}\frac{1}{\sqrt{2\pi}}e^{-\frac{(r+3)^2}{2}} + \Pr\{B_1B_2 = 10\}\frac{1}{\sqrt{2\pi}}e^{-\frac{(r-3)^2}{2}} < \tag{127}$$
$$< \Pr\{B_1B_2 = 01\}\frac{1}{\sqrt{2\pi}}e^{-\frac{(r+1)^2}{2}} + \Pr\{B_1B_2 = 11\}\frac{1}{\sqrt{2\pi}}e^{-\frac{(r-1)^2}{2}} \tag{128}$$

Then

$$e^{-\frac{(r+3)^2}{2}} + e^{-\frac{(r-3)^2}{2}} < e^{-\frac{(r+1)^2}{2}} + e^{-\frac{(r-1)^2}{2}} \tag{129}$$
$$e^{-3r-\frac{9}{2}} + e^{+3r-\frac{9}{2}} < e^{-r-\frac{1}{2}} + e^{+r-\frac{1}{2}} \tag{130}$$
$$\left[e^{-3r} + e^{+3r}\right]e^{-\frac{9}{2}} < \left[e^{-r} + e^{+r}\right]e^{-\frac{1}{2}} \tag{131}$$
$$\frac{e^{-3r} + e^{+3r}}{e^{-r} + e^{+r}} < e^4 \tag{132}$$

which implies that the receiver always estimates $\hat{B}_2 = 1$ when

$$-2.01 < r < +2.01 \tag{133}$$

Thus, the error probability can be found,

$$\begin{aligned}
P_e^2 &= \Pr\{(B_1B_2 = 00)\} \Pr\{r > -2.01, r < 2.01 | s = -3\} && (134)\\
&+ \Pr\{(B_1B_2 = 01)\} \Pr\{r < -2.01, r > 2.01 | s = -1\} && (135)\\
&+ \Pr\{(B_1B_2 = 11)\} \Pr\{r < -2.01, r > 2.01 | s = 1\} && (136)\\
&+ \Pr\{(B_1B_2 = 10)\} \Pr\{r < 2.01, r > -2.01 | s = 3\} && (137)\\
&= \frac{1}{4}\left[Q(0.99) - Q(5.01)\right] && (138)\\
&+ \frac{1}{4}\left[Q(1.01) + Q(3.01)\right] && (139)\\
&+ \frac{1}{4}\left[Q(3.01) + Q(1.01)\right] && (140)\\
&+ \frac{1}{4}\left[Q(0.99) - Q(5.01)\right] && (141)\\
&= \frac{1}{2}Q(0.99) + \frac{1}{2}Q(1.01) - \frac{1}{2}Q(5.01) + \frac{1}{2}Q(3.01) && (142)\\
&\approx 0.1593 && (143)
\end{aligned}$$

5 Waveform Channels

Exercise 5.1: Calculate $P_e^{\min}$ when the signal sets (a), (b), and (c) specified in Fig. 26 are used to communicate one of two equally likely messages over a channel disturbed by additive Gaussian noise with $S_{N_w}(f) = 0.125$.

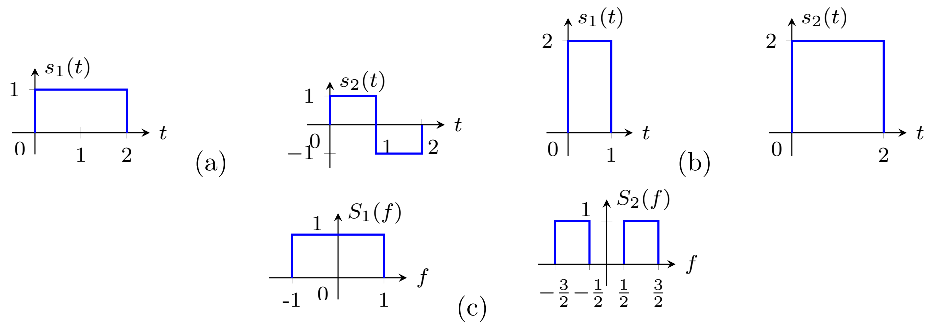

Figure 26: Signals $s_m(t)$ or spectra $S_m(f)$ for $m = 1, 2$ (Ex. 5.1).

Solution: It is given that $\Pr\{M = 1\} = \Pr\{M = 2\} = 1/2$ and $S_{N_w} = N_0/2 = 1/8$.

(a) First find building block waveforms and constellation, using Gramm-schmidt procedure:

$$E_1 = \int_0^2 s_1^2(t)dt = 2 \tag{144}$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{2}} \tag{145}$$

$$s_{11} = \sqrt{2} \tag{146}$$

$$s_{21} = \int_0^2 s_2(t)\phi_1(t)dt = 0 \tag{147}$$

To find the second building block waveform $\phi_2(t)$ we have

$$\theta_2(t) = s_2(t) - s_{21}\phi_1(t) = s_2(t) \tag{148}$$

$$E_{\theta_2} = E_2 = \int_0^2 s_2^2(t)dt = 2 \tag{149}$$

$$\phi_2(t) = \frac{\theta_2(t)}{\sqrt{E_{\theta_2}}} = \frac{s_2(t)}{\sqrt{2}} \tag{150}$$

$$s_{22} = \sqrt{2} \tag{151}$$

We conclude that the signal constellation is

$$\underline{s}_1 = (\sqrt{2}, 0) \tag{152}$$

$$\underline{s}_2 = (0, \sqrt{2}) \tag{153}$$

Fig. 27 shows the corresponding signal constellation and the decision boundary used by the receiver. Hence, the probability of error is given as the probability that the noise is more than $d/2$, with d the distance between the two constellation points:

$$P_e = Q\left(\frac{d}{2\sigma}\right) = Q\left(\frac{\Delta}{\sigma}\right). \tag{154}$$

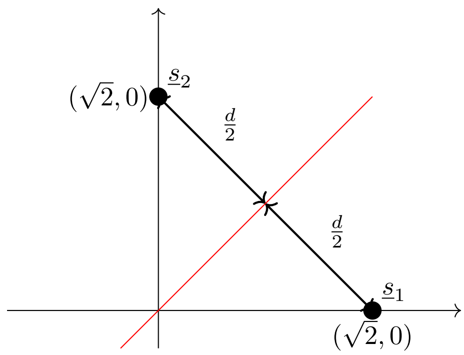

Figure 27: Visualisation of the two messages, the decision boundary (red line) and the distance between them.

Next, calculate $d/2$:

$$\begin{aligned}\frac{d}{2} &= \frac{\sqrt{(s_{11}-s_{21})^2+(s_{12}-s_{22})^2}}{2} \\ &= \frac{\sqrt{(\sqrt{2}-0)^2+(0-\sqrt{2})^2}}{2} = \frac{2}{2}1.\end{aligned}$$

Next, calculate σ:

$$\sigma = \sqrt{S_{N_w}} = \sqrt{\frac{1}{8}} = \frac{1}{4}\sqrt{2}. \tag{155}$$

Finally, P_e^{min} can be found:

$$P_e^{min} = \frac{1}{2}Q\left(\frac{1}{\frac{\sqrt{2}}{4}}\right) + \frac{1}{2}Q\left(\frac{1}{\frac{\sqrt{2}}{4}}\right) = Q(2\sqrt{2}).$$

(b) First find building block waveforms and constellation, using Gramm-schmidt procedure:

$$E_1 = \int s_1^2(t)dt = 4 \tag{156}$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{2} \tag{157}$$

$$s_{11} = 2 \tag{158}$$

$$s_{21} = \int s_2(t)\phi_1(t)dt = 2 \tag{159}$$

To find the building block waveform $\phi_2(t)$ we have

$$\theta_2(t) = s_2(t) - s_{21}\phi_1(t) \tag{160}$$

$$E_{\theta_2} = \int \theta_2^2(t)dt = 4 \tag{161}$$

$$\phi_2(t) = \frac{\theta_2(t)}{\sqrt{E_{\theta_2}}} = \frac{1}{2}\theta_2(t) \tag{162}$$

$$s_2(t) = s_{21}\phi_1(t) + \theta_2(t) \tag{163}$$

$$= s_{21}\phi_1(t) + 2\phi_2(t) \tag{164}$$

$$s_{22} = 2 \tag{165}$$

We conclude that the signal constellation is

$$\underline{s}_1 = (2, 0) \tag{166}$$
$$\underline{s}_2 = (2, 2) \tag{167}$$

Hence, the probability of error is given as the probability that the noise is more than $d/2$, with d the distance between the two constellation points:

$$P_e = Q\left(\frac{d/2}{\sigma}\right), \tag{168}$$

where σ and d are given by

$$\sigma = \frac{1}{4}\sqrt{2} \tag{169}$$
$$d = 2 \tag{170}$$

Thus:

$$P_e^{\min} = Q\left(\frac{1}{\frac{\sqrt{2}}{4}}\right) = Q(2\sqrt{2}) \tag{171}$$

(c) First find building block waveforms and constellation, using Gramm-schmidt procedure (and Parseval Thm.):

$$E_1 = \int s_1^2(t)dt = \int S_1(f)S_1^*(f)df = 2 \tag{172}$$
$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{2}} \tag{173}$$
$$s_{11} = \sqrt{2} \tag{174}$$

To calculate s_{21} we can compute the integral:

$$s_{21} = \int s_2(t)\phi_1(t)dt \tag{175}$$
$$= \int s_2(t)\frac{1}{\sqrt{2}}s_1(t)dt \tag{176}$$
$$= \frac{1}{\sqrt{2}}\int S_2(f)S_1^*(f)df \tag{177}$$
$$= \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{2}\sqrt{2} \tag{178}$$

Now, in order to find ϕ_2, we first compute

$$\theta_2(t) = s_2(t) - s_{21}\phi_1(t) = s_2(t) - \frac{1}{2}s_1(t) \tag{179}$$

and the respective value of E_{θ_2}:

$$E_{\theta_2} = \int \theta_2^2(t)dt \tag{180}$$
$$= \int \left[s_2(t) - \frac{1}{2}s_1(t)\right]^2 dt \tag{181}$$
$$= \int s_2^2(t)\mathrm{d}t - \int s_2(t)s_1(t)dt + \frac{1}{4}\int s_1^2(t)dt \tag{182}$$
$$= \int S_2(f)S_2^*(f)\mathrm{d}f - \int S_2(f)S_1^*(f)\mathrm{d}f + \frac{1}{4}\int S_1(f)S_1^*(f)df \tag{183}$$
$$= 2 - 1 + \frac{1}{4} \cdot 2 = \frac{3}{2} \tag{184}$$

Then, the normalized function ϕ_2 is given by

$$\phi_2(t) = \frac{\theta_2(t)}{\sqrt{E_{\theta_2}}} = \frac{s_2(t) - \frac{1}{2}s_1(t)}{\sqrt{\frac{3}{2}}}. \tag{185}$$

We then derive

$$s_2(t) - \frac{1}{2}s_1(t) = \sqrt{\frac{3}{2}}\phi_2(t) \tag{186}$$
$$s_2(t) = \frac{1}{2}s_1(t) + \sqrt{\frac{3}{2}}\phi_2(t) \tag{187}$$
$$= \frac{1}{2}\sqrt{2}\phi_1(t) + \sqrt{\frac{3}{2}}\phi_2(t). \tag{188}$$

Hence

$$s_{22} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2} \tag{189}$$
$$\underline{s}_1 = (\sqrt{2}, 0) \tag{190}$$
$$\underline{s}_2 = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{6}\right). \tag{191}$$

Finally, we can find σ and d to be

$$\sigma = \frac{1}{4}\sqrt{2} \tag{192}$$
$$d = \sqrt{2} \tag{193}$$

So the probability of error is given by:

$$P_e^{\min} = Q\left(\frac{\frac{\sqrt{2}}{2}}{\frac{1}{4}\sqrt{2}}\right) = Q(2) \tag{194}$$

Alternative Solution:
We only need to find the distance between the two signal vectors:

$$||\underline{s}_1 - \underline{s}_2||^2 = (\underline{s}_1 \cdot \underline{s}_1) - 2(\underline{s}_1 \cdot \underline{s}_2) + (\underline{s}_2 \cdot \underline{s}_2) \tag{195}$$
$$= ||\underline{s}_1||^2 - 2(\underline{s}_1 \cdot \underline{s}_2) + ||\underline{s}_2||^2 \tag{196}$$
$$||\underline{s}_1||^2 = \int s_1^2(t)dt \tag{197}$$
$$= \int S_1(f)S_1^*(f)df = 2 \tag{198}$$
$$||\underline{s}_2||^2 = \int s_2^2(t)dt \tag{199}$$
$$= \int S_2(f)S_2^*(f)\mathrm{d}f = 2 \tag{200}$$
$$(\underline{s}_1 \cdot \underline{s}_2) = \int s_1(t)s_2(t)dt \tag{201}$$
$$= \int S_1(f)S_2^*(f)df = 1 \tag{202}$$

Thus

$$||\underline{s}_1 - \underline{s}_2||^2 = 2 - 2 \cdot 1 + 2 = 2 \tag{203}$$

$$d = \sqrt{2} \tag{204}$$

$$P_e = Q\left(\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{4}}\right) = Q(2) \tag{205}$$

Exercise 5.2: One of three equally likely messages is to be transmitted over an AWGN channel with $S_{N_w}(f) = N_0/2 = 1$ by means of pulse amplitude modulation. Specifically

$$\begin{aligned} s_1(t) &= +2\sqrt{2}\sin(2\pi t), \\ s_2(t) &= 0, \\ s_3(t) &= -2\sqrt{2}\sin(2\pi t), \end{aligned} \tag{206}$$

for $0 \le t \le 1$ and zero elsewhere.

(a) What mathematical operations are performed by the optimum receiver?

(b) Express the resulting average error probability in terms of $Q(\cdot)$.

(c) Calculate the minimum attainable average error probability if, instead of being equal, the message probabilities satisfy

$$\Pr\{M=1\} = \Pr\{M=3\} = \frac{1}{e+2} \text{ and } \Pr\{M=2\} = \frac{e}{e+2}, \tag{207}$$

where e is the base of the natural logarithm.

Solution:

(a) The energy of the waveform $s_1(t)$ is

$$E_{s_1} = \int_0^1 8\sin^2(2\pi t)dt, \tag{208}$$

$$= \int_0^1 8\frac{1-\cos(4\pi t)}{2}dt, \tag{209}$$

$$= 4. \tag{210}$$

Therefore, the building block waveform is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \sqrt{2}\sin(2\pi t), \tag{211}$$

and $s_1 = 2$, $s_2 = 0$, and $s_3 = -2$.

Thus, there's only one building block waveform here. The receiver correlates $r(t)$ with this waveform $\phi_1(t)$. Then, compares the result with two thresholds.

(b) Fig. 28 shows the decision variables. We can write the average error probability as follows:

$$P_e = \frac{1}{3}Q\left(\frac{d/2}{\sigma}\right) + \frac{1}{3}\left[Q\left(\frac{d/2}{\sigma}\right) + Q\left(\frac{d/2}{\sigma}\right)\right] + \frac{1}{3}Q\left(\frac{d/2}{\sigma}\right). \tag{212}$$

Given that $\sigma = S_{N_w}(f) = N_0/2 = 1$, and $d = 2$ the average error probability is

$$P_e = \frac{4}{3}Q(1). \tag{213}$$

(c) Consider first the decision variables corresponding to $m = 1$ and $m = 2$. The receiver estimates

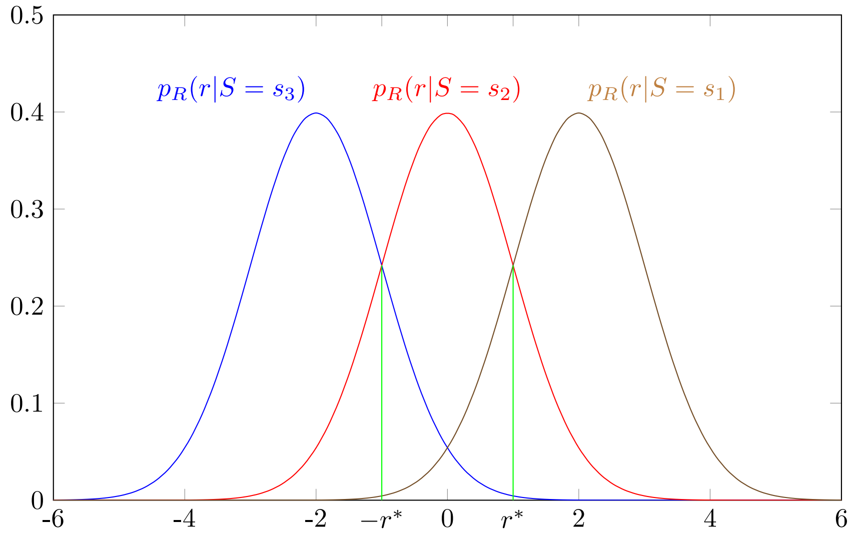

Figure 28: Decision variables that correspond to 3 messages.

$\hat{M}=1$ if

$$\Pr\{M=1\}\cdot\Pr\{r|S=s_1\} > \Pr\{M=2\}\cdot\Pr\{r|S=s_2\}, \tag{214}$$

$$p_1\cdot\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{(r-s_1)^2}{\sigma^2}\right) > p_2\cdot\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{(r-s_2)^2}{2\sigma^2}\right), \tag{215}$$

$$\ln\left(p_1\exp\left(-\frac{(r-s_1)^2}{2}\right)\right) > \ln\left(p_2\exp\left(-\frac{(r-s_2)^2}{2}\right)\right), \tag{216}$$

$$\ln(p_1)-\frac{(r-s_1)^2}{2} > \ln(p_2)-\frac{(r-s_2)^2}{2}, \tag{217}$$

$$r > \frac{1}{s_1-s_2}\ln\left(\frac{p_2}{p_1}\right)+\frac{s_1+s_2}{2}, \tag{218}$$

$$r > \frac{3}{2}. \tag{219}$$

Due to symmetry, the other threshold is -3/2. Overall, the threshold between $m=1$ and $m=2$ is $r^*=\frac{3}{2}$ and the threshold between $m=2$ and $m=3$ is $-r^*=-\frac{3}{2}$.
Finally, the minimum attainable average error probability is

$$\begin{aligned}P_e &= \Pr\{M=1\}\Pr\{r<r^*|S=s_1\}+\Pr\{M=2\}\Pr\{|r|>r^*|S=s_2\}\\ &\quad+\Pr\{M=3\}\Pr\{r>-r^*|S=s_3\}\\ &= \frac{1}{e+2}Q(1/2)+2\frac{e}{e+2}Q(3/2)+\frac{1}{e+2}Q(1/2).\end{aligned}$$

Exercise 5.3:

Consider a waveform channel with AWGN for which the channel-output $r(t)=s_m(t)+n_w(t)$. For the power spectral density of the noise we can say that it equals $N_0/2=2$ W/Hz for all frequencies. There are four messages hence $m\in\{1,2,3,4\}$ and the corresponding signals $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_4(t)$ are shown in Fig. 29. All four messages are equally likely hence $\Pr\{M=1\}=\Pr\{M=2\}=\Pr\{M=3\}=\Pr\{M=4\}=1/4$.

(a) The first building-block waveform $\varphi_1(t)$ for our four signals is given, see Fig. 30. Do we need more building-block waveforms to represent the signals $s_1(t)$, $s_2(t), s_3(t)$, and $s_4(t)$? If yes, sketch the extra building-block waveform(s).

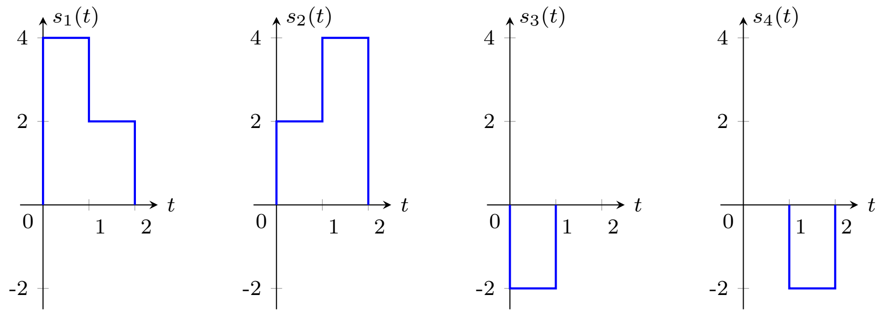

Figure 29: Four signals (waveforms). Horizontally we have t, vertically $s_m(t)$ is shown, for $m = 1, 2, 3, 4$ (Ex. 5.3).

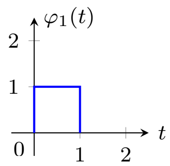

Figure 30: The first building-block waveform $\varphi_1(t)$ (Ex. 5.3).

(b) What are now the signal-vectors corresponding to $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$? Sketch the signal-structure. Observe that all signal-vectors can be regarded as corner-points of a rectangle.

(c) Describe for which received vectors $\underline{r}$ an optimum receiver decides $\hat{M} = 1$, $\hat{M} = 2$, $\hat{M} = 3$, and $\hat{M} = 4$. Determine the resulting error probability realized by an optimum receiver. Use the Q-function.

Solution:

(a) From Fig. 29 it is clear that the second building-block waveform is necessary to represent signals $s_1(t)$, $s_2(t)$ and $s_4(t)$. First, let us find the component of $s_1(t)$ projected on the the first building-block waveform,

$$s_{11} = \int_{-\infty}^{\infty} s_1(t)\varphi_1(t)dt = \int_0^2 s_1(t)\varphi_1(t)dt = \int_0^1 4(t)dt = 4. \tag{220}$$

$$s_{21} = \int_{-\infty}^{\infty} s_2(t)\varphi_1(t)dt = \int_0^2 s_2(t)\varphi_1(t)dt = \int_0^1 2(t)dt = 2. \tag{221}$$

Then the auxiliary signal $\theta_2(t)$ can be found

$$\theta_2(t) = s_2(t) - s_{21}(t)\varphi_1(t) \tag{222}$$

The signal is shown in Fig. 31.

Next, $\phi_2(t)$ can be found,

$$E_{\theta_2} = \int_{-\infty}^{\infty} \theta_2^2(t)dt = 16, \text{ so} \tag{223}$$

$$\varphi_2(t) = \frac{\theta_2(t)}{\sqrt{E_{\theta_2}}}. \tag{224}$$

The sketch of the second building block $\varphi_2(t)$ is shown in Fig. 31

(b) The signal vectors are:

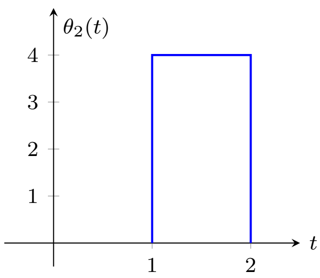

Figure 31: Sketch of the auxiliary signal $\theta_2(t)$ (Sol. 5.3)

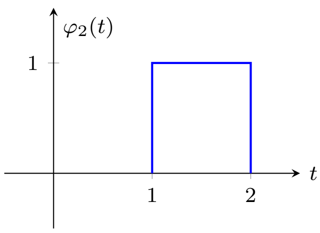

Figure 32: Sketch of building-block waveform (Sol. 5.3)

$$\underline{s}_1 = (4, 2) \tag{225}$$
$$\underline{s}_2 = (2, 4) \tag{226}$$
$$\underline{s}_3 = (-2, 0) \tag{227}$$
$$\underline{s}_4 = (0, -2) \tag{228}$$

The signal-structure sketch is on Fig. 33

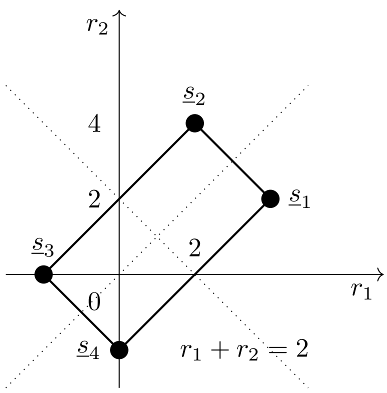

Figure 33: Sketch of signal structure (Sol. 5.3).

(c) The receiver estimates $\hat{M}$ following the rules:

$$\hat{M} = 1 \text{ if } r_1 > r_2 \text{ and } r_1 + r_2 > 2 \tag{229}$$
$$\hat{M} = 2 \text{ if } r_1 < r_2 \text{ and } r_1 + r_2 > 2 \tag{230}$$
$$\hat{M} = 3 \text{ if } r_1 < r_2 \text{ and } r_1 + r_2 < 2 \tag{231}$$
$$\hat{M} = 4 \text{ if } r_1 > r_2 \text{ and } r_1 + r_2 < 2 \tag{232}$$

The distance between signal points $\underline{s}_1$ and $\underline{s}_2$ is

$$d_{12} = \sqrt{(4-2)^2 + (2-4)^2} = 2\sqrt{2}. \tag{233}$$

Similarly,

$$d_{34} = 2\sqrt{2},$$
$$d_{23} = \sqrt{(2-(-2))^2 + (4-0)^2} = 4\sqrt{2},$$
$$d_{14} = 4\sqrt{2}.$$

First lets calculate the probability of correct decision. The probability of error in one direction is probability that noise is bigger than $\frac{2\sqrt{2}}{2}$ and in perpendicular direction that the noise is bigger than $\frac{4\sqrt{2}}{2}$. Given that $d_{12}/2 = \sqrt{2}$, $d_{14}/2 = 2\sqrt{2}$ and $\sigma = \sqrt{N_0/2} = \sqrt{2}$, the probability of correct decision for message $M = 1$ is

$$P_{c_1} = \left[1 - Q\left(\frac{\Delta_{12}}{\sigma}\right)\right]\left[1 - Q\left(\frac{\Delta_{14}}{\sigma}\right)\right] \tag{234}$$
$$= \left[1 - Q\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\right]\left[1 - Q\left(\frac{2\sqrt{2}}{\sqrt{2}}\right)\right] \tag{235}$$
$$= [1 - Q(1)][1 - Q(2)] \tag{236}$$
$$= 1 - Q(1) - Q(2) + Q(1)Q(2). \tag{237}$$

Same result can be obtained for the other messages, $P_{c_1} = P_{c_2} = P_{c_3} = P_{c_4}$. Then the total probability of the correct decision is

$$P_c = \sum_{i=1}^{4} \Pr\{M = i\} P_{c_i} = 1 - Q(1) - Q(2) + Q(1)Q(2). \tag{238}$$

Finally, the exact error probability realized by an optimum receiver can be found,

$$P_e = 1 - P_c = Q(1) + Q(2) - Q(1)Q(2). \tag{239}$$

Exercise 5.4: One of four equally likely messages is to be transmitted over an AWGN channel with $S_{N_w}(f) = N_0/2 = 1$ using the signals given in Fig. 34.

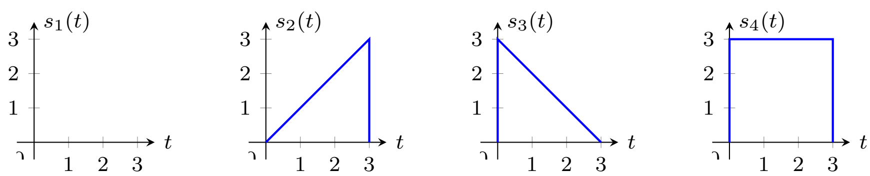

Figure 34: Signals $s_m(t)$ for $m = 1, 2, 3, 4$ (Ex. 5.4).

(a) First determine the building blocks for these signals. The first building block should be a scaled version of $s_2(t)$. What is this first building block waveform $\varphi_1(t)$? Next use $s_3(t)$ to find the second building block waveform. What is the second building block waveform $\varphi_2(t)$?

(b) What are the vector representations $\underline{s}_1, \underline{s}_2, \underline{s}_3$, and $\underline{s}_4$ of the four signals? Make a figure in which you draw these four vectors, i.e., the signal constellation. Also draw the decision regions $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$, and $\mathcal{I}_4$ that are used by an optimal receiver.

(c) What are the distances between the signal vectors? Use the union bound to find an upper bound for the error probability realized by an optimal receiver. Use the $Q(\cdot)$-function.

Solution: We are given that $S_{N_w}(f) = \frac{N_0}{2} = 1$, which means $\sigma = 1$.

(a) It is stated that the first building block should be a scaled version of $s_2(t)$. To scale it to a unit-energy signal, let's find its energy:

$$E_{s_2} = \int_0^3 t^2 dt = \frac{1}{3}t^3\Big|_0^3 = 9. \tag{240}$$

Therefore the first building block is

$$\phi_1(t) = \frac{s_2(t)}{3} = \frac{t}{3}, \tag{241}$$

which tells us that

$$s_{21} = 3, \tag{242}$$

which means there are 3 $\phi_1(t)$ in $s_2(t)$.
Let's calculate how many $\phi_1(t)$ in $s_3(t)$

$$s_{31} = \int_0^3 s_3(t)\phi_1(t)\mathrm{d}t, \tag{243}$$

$$= \int_0^3 (3-t)\frac{1}{3}t\mathrm{d}t, \tag{244}$$

$$= \left(\frac{t^2}{2} - \frac{t^3}{9}\right)\Big|_0^3, \tag{245}$$

$$s_{31} = \frac{3}{2}. \tag{246}$$

Then the remaining waveform form s_3 when the contribution of $\phi_1(t)$ is removed

$$\theta_3(t) = s_3(t) - \frac{3}{2}\phi_1(t) = 3 - \frac{3t}{2}. \tag{247}$$

To scale $\theta_3(t)$ to a unit energy signal to make it the second building block, let's calculate its energy

$$E_{\theta_3} = \int_0^3 \left(3 - \frac{3t}{2}\right)^2 dt, \tag{248}$$

$$= \frac{9}{4}\int_0^3 (2-t)^2\, dt, \tag{249}$$

$$= \frac{9}{4}\int_{-2}^1 \alpha^2 d\alpha, \tag{250}$$

$$E_{\theta_3} = \frac{27}{4}. \tag{251}$$

Therefore the second building block is

$$\phi_2(t) = \frac{\theta_3(t)}{\sqrt{27/4}} = \frac{2}{3\sqrt{3}}\left(3 - \frac{3t}{2}\right) = \frac{2-t}{\sqrt{3}}, \tag{252}$$

which tells us that

$$s_3(t) = \theta_3(t) + \frac{3}{2}\phi_1(t) \tag{253}$$

which means there are $3\sqrt{3}/2$ $\phi_2(t)$ in $s_3(t)$. Basically

$$s_{32} = \frac{3\sqrt{3}}{2}. \tag{254}$$

Therefore, the building block waveforms are

$$\phi_1(t) = \frac{t}{3}, \tag{255}$$

$$\phi_2(t) = \frac{2-t}{\sqrt{3}}, \tag{256}$$

(b) Considering the fact that $s_4(t) = s_2(t) + s_3(t)$, we can write the vector representations of the signals in terms of building blocks as

$$\underline{s}_1 = (0, 0), \tag{257}$$
$$\underline{s}_2 = (3, 0), \tag{258}$$
$$\underline{s}_3 = (3/2, 3\sqrt{3}/2), \tag{259}$$
$$\underline{s}_4 = (9/2, 3\sqrt{3}/2). \tag{260}$$

The signal constellation and decision regions are shown in Fig. 35.

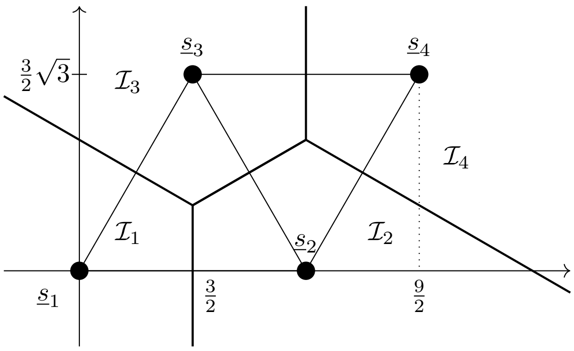

Figure 35: Signal vectors, and decision regions (Ans. 5.4).

(c) The distances between all possible signal vector pairs is 3, except the distance between $\underline{s}_1$ and $\underline{s}_4$, which is $3\sqrt{3}$. The upper bound is calculated using nearest neighbors, so the error between $\underline{s}_1$ and $\underline{s}_4$ can be omitted. In addition, we know that $\sigma = 1$. The upper bound of the total error probability can be found using the Union bound:

$$P_e \leq \sum_{m \in \mathcal{M}} \Pr\{M = m\} P_e^m \tag{261}$$
$$= \frac{1}{4}\left(Q(3/2) + Q(3/2)\right) + \frac{1}{4}\left(Q(3/2) + Q(3/2) + Q(3/2)\right) \tag{262}$$
$$+ \frac{1}{4}\left(Q(3/2) + Q(3/2) + Q(3/2)\right) + \frac{1}{4}\left(Q(3/2) + Q(3/2)\right), \tag{263}$$
$$P_e \leq \frac{5}{2} Q(3/2). \tag{264}$$

Exercise 5.5: A transmitter uses the signals $s_1(t), s_2(t), \ldots, s_{|\mathcal{M}|}(t)$ to communicate one of $|\mathcal{M}|$ equally likely messages over an AWGN channel with power density $\frac{N_0}{2}$, where for $m = 1, 2, \ldots, |\mathcal{M}|$

$$s_m(t) = \begin{cases} \sqrt{\frac{2E_s}{T}} \cos\left(2\pi f_0 t + 2\pi \frac{m-1}{|\mathcal{M}|}\right) & \text{for } 0 \leq t < T \text{ and} \\ 0 & \text{elsewhere,} \end{cases} \tag{265}$$

where f_0 is a multiple of $1/T$.

(a) Sketch the signal vectors and optimum decision regions for $|\mathcal{M}| = 5$.

(b) Use geometric arguments to show that the minimum attainable P_e is bounded by

$$p \leq P_e \leq 2p,$$

where $p = Q\left(\sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{|\mathcal{M}|}\right)$.

Solution:

(a) First find the building block waveforms. Use:

$$\cos\left(2\pi f_0 t + 2\pi\frac{m-1}{|\mathcal{M}|}\right) = \cos(2\pi f_0 t)\cos\left(2\pi\frac{m-1}{|\mathcal{M}|}\right) - \sin\left(2\pi f_0 t\right)\sin\left(2\pi\frac{m-1}{|\mathcal{M}|}\right) \tag{266}$$

Take as building-block waveforms

$$\varphi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_0 t) \tag{267}$$

$$\varphi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_0 t) \tag{268}$$

Then

$$\int_0^T \left[\sqrt{\frac{2}{T}}\cos(2\pi f_0 t)\right]^2 dt = \frac{1}{T}\int_0^T 2\cos^2(2\pi f_0 t)dt \tag{269}$$

$$= \frac{1}{T}\int_0^T [\cos(4\pi f_0 t) + 1]dt \tag{270}$$

$$= 1 \tag{271}$$

and

$$\int_0^T \sqrt{\frac{2}{T}}\cos(2\pi f_0 t)\sqrt{\frac{2}{T}}\sin(2\pi f_0 t)dt = \frac{1}{T}\int_0^T \sin(4\pi f_0 t)dt \tag{272}$$

$$= 0 \tag{273}$$

Now, we can find the signal constellation
$|\mathcal{M}| = 5$ such that $\underline{s}_m = \sqrt{E_s}\left(\cos\left(2\pi\frac{m-1}{5}\right), -\sin\left(2\pi\frac{m-1}{5}\right)\right)$

$$\underline{s}_1 = \sqrt{E_s}(1, 0) \tag{274}$$

$$\underline{s}_2 = \sqrt{E_s}(\cos(\alpha), -\sin(\alpha)) \tag{275}$$

$$\underline{s}_3 = \sqrt{E_s}(\cos(2\alpha), -\sin(2\alpha)) \tag{276}$$

$$\underline{s}_4 = \sqrt{E_s}(\cos(3\alpha), -\sin(3\alpha)) \tag{277}$$

$$\underline{s}_5 = \sqrt{E_s}(\cos(4\alpha), -\sin(4\alpha)) \tag{278}$$

where $\alpha = \dfrac{2\pi}{5}$.

(b) Fig. 36 shows the signal space and the decision regions. Let us consider the distance between the signal vectors $\underline{s}_1$ and $\underline{s}_5$,

$$\begin{aligned}
d_{15} &= \sqrt{(s_{11} - s_{51})^2 + (s_{12} - s_{52})^2} \\
&= \sqrt{(\sqrt{E_s}\cdot 1 - \sqrt{E_s}\cos(\alpha))^2 + ((\sqrt{E_s}\cdot 0 - \sqrt{E_s}(\sin(-\alpha))^2} \\
&= \sqrt{E_s}\sqrt{1 - 2cos(\alpha) + \cos^2(\alpha) + \sin^2(\alpha)} \\
&= \sqrt{E_s}\sqrt{2 - 2\cos(\alpha)} \\
&= 2\sqrt{E_s}\sqrt{\frac{1}{2} - \frac{1}{2}\cos(\alpha)} \\
&= 2\sqrt{E_s}\sin\left(\frac{\alpha}{2}\right)
\end{aligned}$$

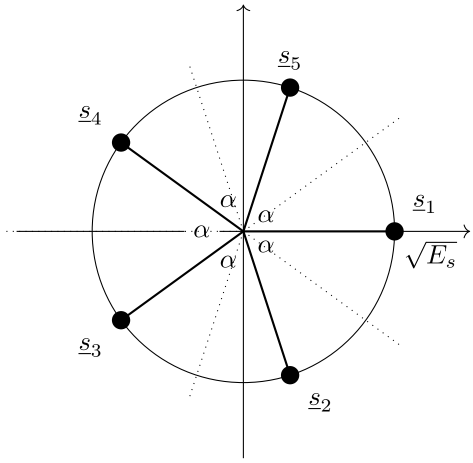

Figure 36: Signal vectors and decision regions (Sol. 5.5).

Exactly the same distance be observed between all the neighboring points, so for all points

$$\frac{d}{2} = \sqrt{E_s} \sin\left(\frac{\alpha}{2}\right). \tag{279}$$

The error probability for s_1 is

$$\begin{aligned} P_e^1 &= Q\left(\frac{d/2}{\sigma}\right) + Q\left(\frac{d/2}{\sigma}\right) \\ &= 2Q\left(\frac{\sqrt{E_s}\sin\left(\frac{\alpha}{2}\right)}{\sqrt{\frac{N_0}{2}}}\right). \end{aligned}$$

Due to the symmetry, $P_e^1 = P_e^2 = P_e^3 = P_e^4 = P_e^5$. Thus, using the union bound:

$$P_e \leq \sum_{m \in \mathcal{M}} \Pr\{M = m\} P_e^m \tag{280}$$

$$= |\mathcal{M}| \frac{1}{\mathcal{M}} 2Q\left(\frac{\sqrt{E_s}\sin\left(\frac{\alpha}{2}\right)}{\sqrt{\frac{N_0}{2}}}\right) \tag{281}$$

$$= 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{|\mathcal{M}|}\right)\right) \tag{282}$$

The lower bound of the error probability can be found by considering only one of the error events as shown in Fig. 37.

$$P_e \geq Q\left(\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{|\mathcal{M}|}\right)\right). \tag{283}$$

Exercise 5.6: Consider two messages with corresponding signals $s_1(t)$ and $s_2(t)$ and probabilities $\Pr\{M = 1\}$ and $\Pr\{M = 2\}$.

(a) Describe an optimum receiver for these signals in terms of the vector representations of the signals.

(b) What is the resulting error probability in terms of: the vectors $\underline{s}_1$ and $\underline{s}_2$, the corresponding probabilities, and the power spectral density $\frac{N_0}{2}$ of the noise?

Solution:

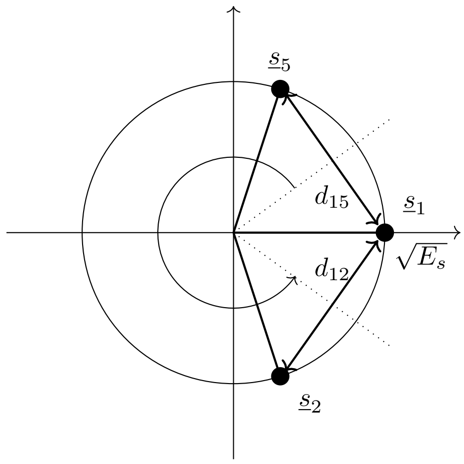

Figure 37: Signal vectors and decision regions (Sol. 5.5).

(a) Decide $\hat{m} = 1$ with the MAP detector when (see also Chapter 4 page 39 from the reader)

$$\Pr\{M=1\}\exp\left(-\frac{(\underline{r}-\underline{s}_1)^2}{2}\right) > \Pr\{M=2\}\exp\left(-\frac{(\underline{r}-\underline{s}_2)^2}{2}\right), \tag{284}$$

$$\ln\Pr(M=1) - \frac{||\underline{s}_1||^2}{2\sigma^2} + \frac{2(\underline{r}\cdot\underline{s}_1)}{2\sigma^2} \geq \ln\Pr(M=2) - \frac{||\underline{s}_2||^2}{2\sigma^2} + \frac{2(\underline{r}\cdot\underline{s}_2)}{2\sigma^2} \tag{285}$$

$$\frac{N_0}{2}\ln\Pr(M=1) - \frac{||\underline{s}_1||^2}{2} + (\underline{r}\cdot\underline{s}_1) \geq \frac{N_0}{2}\ln\Pr(M=2) - \frac{||\underline{s}_2||^2}{2} + (\underline{r}\cdot\underline{s}_2) \tag{286}$$

$$\frac{||\underline{s}_2||^2 - ||\underline{s}_1||^2}{2} + (\underline{r}\cdot(\underline{s}_1-\underline{s}_2)) \geq \frac{N_0}{2}\ln\frac{\Pr(M=2)}{\Pr(M=1)} \tag{287}$$

$$\frac{(\underline{s}_2+\underline{s}_1)(\underline{s}_2-\underline{s}_1)}{2} + (\underline{r}\cdot(\underline{s}_1-\underline{s}_2)) \geq \frac{N_0}{2}\ln\frac{\Pr(M=2)}{\Pr(M=1)} \tag{288}$$

$$(\underline{r} - \frac{\underline{s}_1+\underline{s}_2}{2})\cdot(\underline{s}_1-\underline{s}_2) \geq \frac{N_0}{2}\ln\frac{\Pr(M=2)}{\Pr(M=1)} \tag{289}$$

(This is what the receiver verifies, i.e. MAP decision rule).

(b) We want to find the minimum noise vectors length that will result in an error for message 1 or message 2.

First we $\underline{r}*$ as the threshold point on the line between the signal vectors $\underline{s}_1$ and $\underline{s}_2$, where the decision changes from decoded message $M = 1$ to $M = 2$ (and viseversa).

$$\underline{r} = \alpha(\underline{s}_1 - \underline{s}_2) + \frac{\underline{s}_1+\underline{s}_2}{2} \tag{290}$$

where α is found to be

$$\alpha(\underline{s}_1-\underline{s}_2)\cdot(\underline{s}_1-\underline{s}_2) = \frac{N_0}{2}\ln\frac{\Pr\{M=2\}}{\Pr\{M=1\}} \tag{291}$$

$$\alpha^* = \frac{N_0}{2||\underline{s}_1-\underline{s}_2||^2}\ln\frac{\Pr\{M=2\}}{\Pr\{M=1\}} \tag{292}$$

Now, the probability that message $\hat{m} = 2$ is detected in case that $\underline{s}_1$ was send (error probability) is

$$\Pr(\hat{M}=2|M=1) = Q\left(\frac{\|\underline{s}_1-\underline{r}^*\|}{\sqrt{N_0/2}}\right), \tag{293}$$

where

$$||\underline{s}_1 - \underline{r}^*|| = \left|\left|\underline{s}_1 - \alpha^*(\underline{s}_1 - \underline{s}_2) - \frac{\underline{s}_1 + \underline{s}_2}{2}\right|\right| \tag{294}$$

$$= \left|\left|(\underline{s}_1 - \underline{s}_2)\left(\frac{1}{2} - \alpha^*\right)\right|\right| \tag{295}$$

$$= ||(\underline{s}_1 - \underline{s}_2)|| \left|\frac{1}{2} - \alpha^*\right|. \tag{296}$$

Now, for probability that message $\hat{m} = 1$ is detected in case that $\underline{s}_2$ was send (error probability) is

$$\Pr(\hat{M} = 1|M = 2) = Q\left(\frac{\|\underline{r}^* - \underline{s}_2\|}{\sqrt{N_0/2}}\right), \tag{297}$$

where

$$||\underline{r}^* - \underline{s}_2|| = \left|\left|\alpha^*(\underline{s}_1 - \underline{s}_2) + \frac{\underline{s}_1 + \underline{s}_2}{2} - \underline{s}_2\right|\right| \tag{298}$$

$$= ||(\underline{s}_1 - \underline{s}_2)|| \left|\frac{1}{2} + \alpha^*\right|. \tag{299}$$

Finally, the total error probability

$$P_e = \Pr\{M = 1\} \Pr(\hat{M} = 2|M = 1) + \Pr\{M = 2\} \Pr(\hat{M} = 1|M = 2)$$
$$= \Pr\{M = 1\} Q\left(\frac{||\underline{s}_1 - \underline{s}_2||}{\sqrt{N_0/2}} \left|\frac{1}{2} - \alpha^*\right|\right) + \Pr\{M = 2\} Q\left(\frac{||\underline{s}_1 - \underline{s}_2||}{\sqrt{N_0/2}} \left|\frac{1}{2} + \alpha^*\right|\right).$$

Exercise 5.7: One out of four equally likely messages is transmitted over an AWGN channel with noise power spectral density $S_{N_w}(f) = \frac{N_0}{2} = 1/9$. The signals corresponding to the four messages are depicted in Fig. 38.

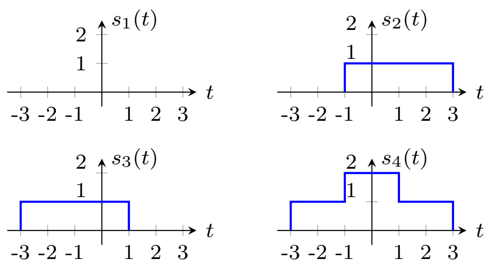

Figure 38: Signals $s_m(t)$ for $m = 1, 2, 3, 4$ (Ex. 5.7).

(a) Find first the building-block waveforms for these four signals. The first building-block waveform should be a scaled version of the second signal $s_2(t)$. What is this first building-block waveform $\varphi_1(t)$? Next use $s_3(t)$ to find the second building-block waveform with the Gram-Schmidt method. What is this second building-block waveform $\varphi_2(t)$? Is there a third building-block waveform needed? What are the resulting vector representations $\underline{s}_1, \underline{s}_2, \underline{s}_3$, and $\underline{s}_4$ for the four signals?

(b) Draw the four signal-vectors (the signal-structure) in a figure. Also depict in this figure the decision regions $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$, and $\mathcal{I}_4$, that are used by an optimum receiver.

(c) Determine the distances between the signal-vectors. Now use the union bound to find an upper bound on the error probability of an optimum receiver. Use in your answer the $Q(\cdot)$ function.

Solution:

$$E_1 = 0 \tag{300}$$
$$E_2 = E_3 = 4 \tag{301}$$
$$E_4 = 12 \tag{302}$$

(a) The first building-block waveform is

$$\phi_1(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{1}{2}s_2(t)$$
$$s_{21} = 2$$

We can then find now s_{31}:

$$s_{31} = \int s_3(t)\phi_1(t)dt = 1 \tag{303}$$

hence

$$\theta_3(t) = s_3(t) - \phi_1(t) \tag{304}$$
$$E_{\theta_3} = 2 + 4 \cdot \frac{1}{4} = 3 \tag{305}$$
$$\phi_2(t) = \theta_3(t)/\sqrt{E_{\theta_3}} = \frac{s_3(t) - \phi_1(t)}{\sqrt{3}} \tag{306}$$

and θ_3 and ϕ_2 are represented in Fig. 39.

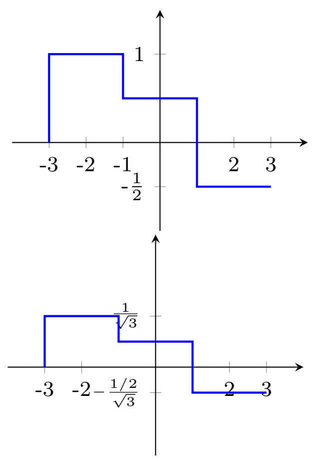

Figure 39: Representation of θ_3 and ϕ_2 (Sol. 5.7).

We don't need a third building-block waveform, and the vector representations are

$$\underline{s}_1 = (0, 0) \tag{307}$$
$$\underline{s}_2 = (2, 0) \tag{308}$$
$$\underline{s}_3 = (1, \sqrt{3}) \tag{309}$$
$$\underline{s}_4 = (3, \sqrt{3}) \tag{310}$$

(b) The sketch can be seen on figure 40.

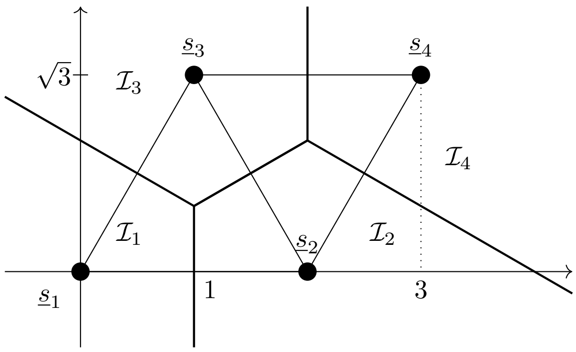

Figure 40: Sketch of signal vectors and decision regions (Sol. 5.7).

(c) The distances between the nearest points:

$$||\underline{s}_1 - \underline{s}_3|| = ||\underline{s}_1 - \underline{s}_2|| = ||\underline{s}_2 - \underline{s}_4|| = ||\underline{s}_3 - \underline{s}_4|| = ||\underline{s}_2 - \underline{s}_3|| = 2 \tag{311}$$

$$||\underline{s}_1 - \underline{s}_4|| = 2\sqrt{3} \tag{312}$$

The union bound is calculated between the nearest neighbors, since such error probability covers the errors between further points.

The same error probability can be obtained between $\mathcal{I}_1$ and $\mathcal{I}_4$. Same between $\mathcal{I}_2$ and $\mathcal{I}_3$. Thus, the upper bound error probability is

$$P_e = \frac{10}{4}Q(3) = \frac{5}{2}Q(3).$$

6 Receiver Implementation, Matched Filters

Exercise 6.1: One of two equally likely messages is to be transmitted over an AWGN channel with $S_{N_w}(f) = \frac{N_0}{2} = 1$ by means of binary pulse-position modulation. Specifically

$$\begin{aligned} s_1(t) &= p(t), \\ s_2(t) &= p(t-2), \end{aligned}$$

for which the pulse $p(t)$ is shown in Fig. 41.

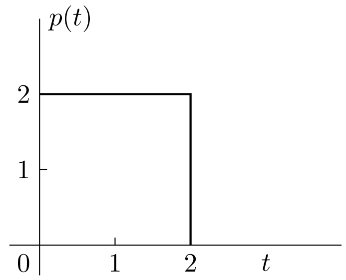

Figure 41: A rectangular pulse (Ex. 6.1).

(a) There are only two messages, so a maximum of two building block waveforms are needed to represent the signal space. This means that any kind of receiver, whether it is direct, matched or correlation receiver, is optimal. Describe and sketch the direct receiver. Express the resulting error probability in terms of $Q(\cdot)$.

(b) Give the implementation of an optimum receiver which uses a single linear filter followed by a sampler and comparison device. Assume that two samples from the filter output are fed into the comparison device. Sketch the receiver structure and the impulse response of the filter.

(c) What is the output of the filter at both sample moments when the filter input is $s_1(t)$? What are these outputs for filter input $s_2(t)$?

(d) Calculate the minimum attainable average error probability if

$$s_1(t) = p(t) \text{ and } s_2(t) = p(t-1).$$

Solution: It is given that $s_1(t) = p(t)$, $s_2(t) = p(t-2)$, they are equally likely and they are transmitted over an AWGN channel with $S_{N_w}(t) = N_0/2 = 1$.

(a) The direct receiver is shown in Fig. 42. See also Fig. 6.5 in the reader where constants c_1 and c_2 are equal.

For the resulting error probability, consider the building blocks in Fig. 43.

Then $\underline{s}_1 = (2\sqrt{2}, 0)$ and $\underline{s}_2 = (0, 2\sqrt{2})$. Thus, $d = ||\underline{s}_2 - \underline{s}_1||$, and

$$P_e = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) = Q(2) \tag{313}$$

(b) The direct receiver extracts the components of the transmitted signal $s_i(t)$ from the received signal $r(t)$. This is done by passing the received signal $r(t)$ through a filter whose impulse response is

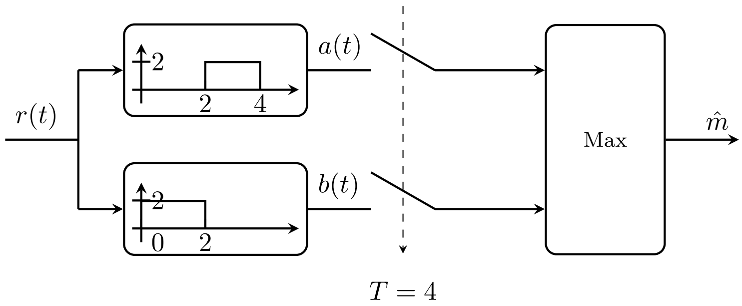

Figure 42: Optimum receiver for this case (Sol. 6.1).

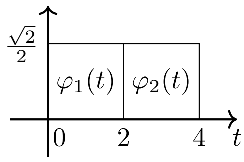

Figure 43: Building block waveforms (Sol. 6.1).

a delayed time-reversed version of a signal $s_i(t)$, i.e. $s_i(T-t)$. Let us denote the outputs of the filters:

$$a(t) = \int r(\tau)s_1(T-t+\tau)d\tau \tag{314}$$

$$b(t) = \int r(\tau)s_2(T-t+\tau)d\tau, \tag{315}$$

where T is a pulse duration of the signal $s_i(t)$ and τ is a dummy variable to perform the convolution.

Now, let us derive the optimum receiver which uses a single linear filter followed by a sampler and comparison device. Using the fact that $s_1(t) = s_2(t+2)$, we have

$$a(t) = \int r(\tau)s_1(T-t+\tau)d\tau = \int r(\tau)s_2(T-t+2+\tau)d\tau. \tag{316}$$

$$= b(t-2). \tag{317}$$

Thus, $a(t)$ is simply a time delayed version on $b(t)$.

When $b(t)$ is sampled at $t = T$,

$$b(t=T) = \int r(\tau)s_2(\tau)d\tau = \underline{r} \cdot \underline{s}_2. \tag{318}$$

$$\begin{aligned} b(t = T-2) &= \int r(\tau)s_2(2+\tau)d\tau = \int r(\tau)s_1(\tau)d\tau \\ &= \underline{r} \cdot \underline{s}_1 \\ &= a(t=T). \end{aligned}$$

Given that the signal pulse duration is 4 seconds, the receiver that uses the single linear filter is shown in Fig. 44. Here the output of the filter, $b(t)$, is sampled at $T = 2$ that results in $\underline{r} \cdot \underline{s}_1$ at one branch and at another branch, $b(t)$ is sampled at $T = 4$ (2 seconds difference) to obtain $\underline{r} \cdot \underline{s}_2$.

(c) The response on $s_1(t)$ can be calculated as $y(t) = \int s_1(\tau)p(t-\tau)d\tau$, which can be evaluated by the following equation, graphically obtained in Figure 45:

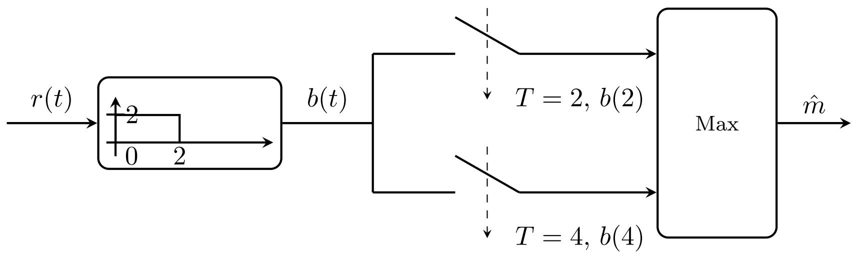

Figure 44: Optimum receiver that uses a single linear filter (Sol. 6.1).

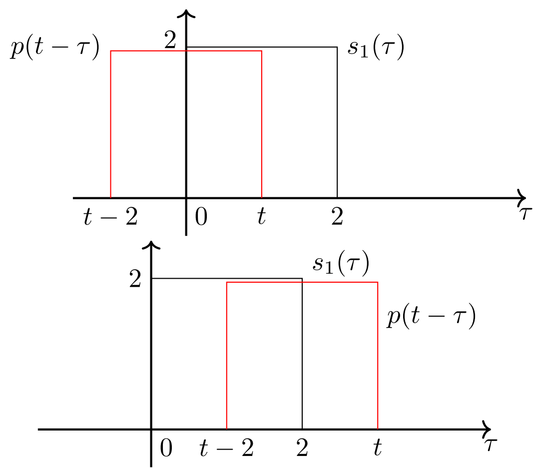

Figure 45: Graphical representation of the convolution between p and s_1 (Sol. 6.1).

$$y(t) = \begin{cases} 0, & \text{if } t < 0 \\ 4t, & \text{if } 0 \leq t < 2 \\ 4(4-t), & \text{if } 2 \leq t < 4 \\ 0, & \text{if } t > 4 \end{cases} \tag{319}$$

The result is shown in Figure 46. So at the first sample moment the filter output is $y(2) = 8$ and at the second sample moment the filter output is $y(4) = 0$.

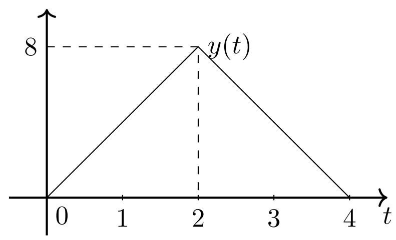

Figure 46: Filter response on s_1 as function of time (Sol. 6.1).

Now, since $s_2(t) = s_1(t-2)$, the response for input $s_2(t)$ is the same, but shifted 2 time instances later, as shown in Figure 47

So at the first sample moment the filter output is $y(2-2) = 0$ and at the second sample moment the filter output is $y(4-2) = 8$.

(d) Now we have that $s_1(t) = p(t)$ and $s_2(t) = p(t-1)$, as in Figure 48. We first want to determine $||\underline{s}_1 - \underline{s}_2||^2$:

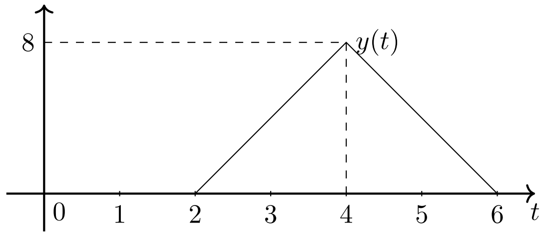

Figure 47: Filter response on s_2 as function of time (Sol. 6.1).

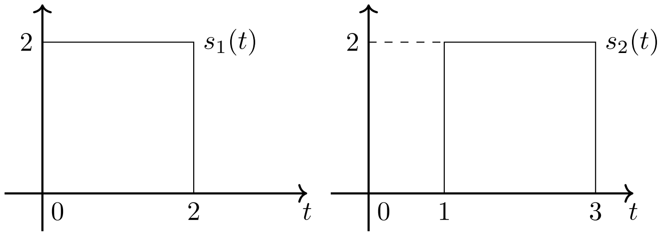

Figure 48: Graphical representation of s_1 and s_2 (Sol. 6.1).

$$||\underline{s}_1 - \underline{s}_2||^2 = ||\underline{s}_1||^2 - 2(\underline{s}_1 \cdot \underline{s}_2) + ||\underline{s}_2||^2 \tag{320}$$

we have $||\underline{s}_1||^2 = ||\underline{s}_2||^2 = 8$ and $(\underline{s}_1 \cdot \underline{s}_2) = \int s_1(t)s_2(t)dt = 4$. Then

$$||\underline{s}_1 - \underline{s}_2||^2 = 8 + 8 - 2 \cdot 4 = 8 \tag{321}$$

Hence

$$P_e = Q\left(\frac{\frac{1}{2}\sqrt{8}}{\sigma}\right) = Q(\sqrt{2}) \tag{322}$$

Exercise 6.2: In a communication system based on an AWGN waveform channel six signals (waveforms) are used. All signals are zero for $t < 0$ and $t \geq 8$. For $0 \leq t < 8$ the signals are

$$\begin{aligned}
s_1(t) &= 0 \\
s_2(t) &= +2\cos(\pi t/2) \\
s_3(t) &= +2\cos(\pi t/2) + 2\sin(\pi t/2) \\
s_4(t) &= +2\cos(\pi t/2) + 4\sin(\pi t/2) \\
s_5(t) &= +4\sin(\pi t/2) \\
s_6(t) &= +2\sin(\pi t/2)
\end{aligned}$$

The messages corresponding to the signals all have probability $1/6$. The power spectral density of the noise process $N_w(t)$ is $\frac{N_0}{2} = 4/9$ for all f. The receiver observes the received waveform $r(t) = s_m(t)+n_w(t)$ in the time interval $0 \leq t < 8$.

(a) Determine a set of building-block waveforms for these six signals. Sketch these building-block waveforms. Show that they are orthonormal over $[0, 8)$. Give the vector representations of all six signals and sketch the resulting signal structure.

(b) Describe for what received vectors $\underline{r}$ an optimum receiver chooses $\hat{M} = 1, \hat{M} = 2, \ldots,$ and $\hat{M} = 6$. Sketch the corresponding decision regions $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_6$. Give an expression for the error probability P_e obtained by an optimum receiver. Use the $Q(\cdot)$-function.

(c) Sketch and specify a matched-filter implementation of an optimum receiver for the six signals (all occurring with equal a-priori probability). Use only two matched filters.

Solution: Note $\sigma = \sqrt{4/9} = 2/3$.

(a) We can directly see that the building blocks are in the form of $\phi_1(t) = K_1 \cos(\pi t/2)$ and $\phi_2(t) = K_2 \sin(\pi t/2)$, where K_1 and K_2 are the constants chosen such that $\phi_1(t)$ and $\phi_2(t)$ are unit-energy signals.

The energy of signal $s_2(t)$ is

$$E_2 = \int_0^8 2^2 \cos^2\left(\frac{\pi t}{2}\right) dt = 4 \int_0^8 \frac{\cos(\pi t) + 1}{2} dt = 16. \tag{323}$$

Therefore the first building block waveform is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{1}{2} \cos\left(\frac{\pi t}{2}\right). \tag{324}$$

The second building block waveform is

$$\phi_2(t) = \frac{1}{2} \sin\left(\frac{\pi t}{2}\right). \tag{325}$$

Note that

$$\int_0^8 \cos\left(\frac{\pi t}{2}\right) \sin\left(\frac{\pi t}{2}\right) dt = \int_0^8 \frac{1}{2} \sin(\pi t)\, dt = 0, \tag{326}$$

Therefore, building block waveforms are orthonormal over $[0, 8)$.

The vector representation of the signals are

$$\underline{s}_1 = (0, 0), \tag{327}$$
$$\underline{s}_2 = (4, 0), \tag{328}$$
$$\underline{s}_3 = (4, 4), \tag{329}$$
$$\underline{s}_4 = (4, 8), \tag{330}$$
$$\underline{s}_5 = (0, 8), \tag{331}$$
$$\underline{s}_6 = (0, 4), \tag{332}$$

and can be seen in Fig. 49.

(b) The decision rule can be written as

$$\hat{m} = \begin{cases} 1, & \text{if } r_1 < 2 \text{ and } r_2 < 2 \\ 2, & \text{if } r_1 > 2 \text{ and } r_2 < 2 \\ 3, & \text{if } r_1 > 2 \text{ and } 2 < r_2 < 6 \\ 4, & \text{if } r_1 > 2 \text{ and } r_2 > 6 \\ 5, & \text{if } r_1 < 2 \text{ and } r_2 > 6 \\ 6, & \text{if } r_1 < 2 \text{ and } 2 < r_2 < 6 \end{cases}. \tag{333}$$

The probability of correct decision for the corner points of the constellation (e.g., $\underline{s}_1$, $\underline{s}_2$, $\underline{s}_4$, and $\underline{s}_5$) is

$$P_c = (1 - Q)(1 - Q).$$

The probability of correct decision for the points $\underline{s}_3$ and $\underline{s}_6$ is

$$P_c = (1 - Q)(1 - 2Q).$$

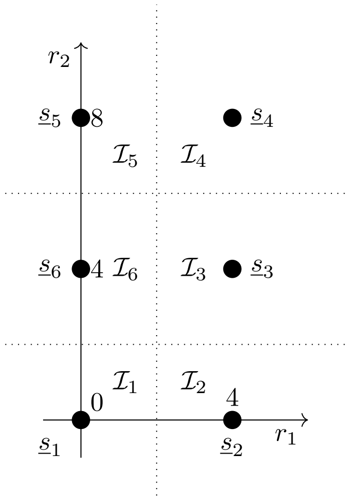

Figure 49: Resulting signal structure (Sol. 6.2).

The distance between the nearest neighbors is $d = 4$, so the distance from the constellation point to the decision region is $d/2 = 2$. The noise variance is $\sigma = \sqrt{N_0/2} = 2/3$. Thus,

$$Q\left(\frac{d/2}{\sigma}\right) = Q(3). \tag{334}$$

The total correct error probability is

$$\begin{aligned}
P_c &= \sum_{m=1}^{6} \Pr\{M = m\} P_c^m \\
&= \frac{4(1-Q)^2 + 2(1-Q)(1-2Q)}{6} \\
&= \frac{(1-Q)(4-4Q+2-4Q)}{6} \\
&= \frac{(1-Q)(6-8Q)}{6} \\
&= \frac{6-14Q+8Q^2}{6}.
\end{aligned}$$

Thus the error probability obtained by an optimum receiver is

$$P_e = 1 - P_c = \frac{7}{3}Q(3) - \frac{4}{3}Q^2(3). \tag{335}$$

(c) The matched filters for $T = 8$ are

$$h_1(t) = \phi_1(T-t) = \frac{1}{2}\cos\left(\pi\frac{T-t}{2}\right) = \frac{1}{2}\cos\left(\frac{\pi t}{2}\right), \tag{336}$$

$$h_2(t) = \phi_2(T-t) = \frac{1}{2}\sin\left(\pi\frac{T-t}{2}\right) = -\frac{1}{2}\sin\left(\frac{\pi t}{2}\right). \tag{337}$$

The matched filter implementation can be sketched as in Fig. 50. Sampler samples with period $T = 8$, and the decision block $(*)$ implements the rule given in (333).

Exercise 6.3: Consider two messages with corresponding signals $s_1(t)$ and $s_2(t)$ and probabilities $\Pr\{M = 1\}$ and $\Pr\{M = 2\}$. Describe an optimum matched-filter receiver that uses a single matched filter, sampling device, and a threshold. What is the impulse response of the matched filter and the

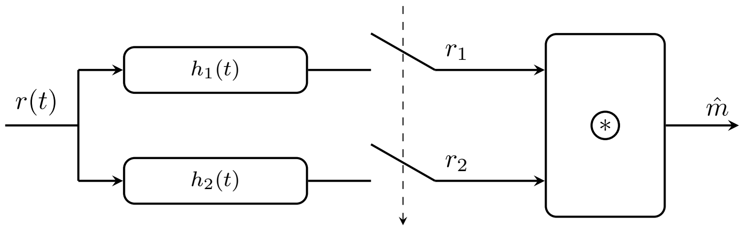

Figure 50: Matched filter implementation of an optimum receiver (Sol. 6.2).

value of the threshold in terms of the waveforms $s_1(t)$, and $s_2(t)$, the corresponding probabilities, and the power spectral density $\frac{N_0}{2}$ of the noise?

Solution:
See Exercise 5.6 (vector extension to result 4.3 of the reader), then the receiver decides $\hat{m} = 1$ when

$$\underline{r} \cdot (\underline{s}_1 - \underline{s}_2) \geq \frac{N_0}{2} \ln \frac{\Pr\{M=2\}}{\Pr\{M=1\}} + \frac{||\underline{s}_1||^2 - ||\underline{s}_2||^2}{2} \tag{338}$$

Or in other words

$$\int r(t) \cdot (s_1(t) - s_2(t)) dt \geq \left[\frac{N_0}{2} \ln \frac{\Pr\{M=2\}}{\Pr\{M=1\}} + \frac{||\underline{s}_1||^2 - ||\underline{s}_2||^2}{2} \right] \tag{339}$$

This can also be done with a matched-filter, where the filter is matched on $s_1(t) - s_2(t)$.
Error probability, see again Exercise 5.6, with

$$P_e = \Pr\{M=1\} Q\left(\frac{||\underline{s}_1 - \underline{s}_2||}{\sqrt{N_0/2}} \left(\frac{1}{2} - \alpha^* \right) \right) + Pr\{M=2\} Q\left(\frac{||\underline{s}_1 - \underline{s}_2||}{\sqrt{N_0/2}} \left(\frac{1}{2} + \alpha^* \right) \right) \tag{340}$$

$$\alpha^* = \frac{N_0}{2||\underline{s}_1 - \underline{s}_2||^2} \ln \frac{\Pr\{M=2\}}{\Pr\{M=1\}} \tag{341}$$

$$||\underline{s}_1 - \underline{s}_2|| = \sqrt{\int (s_1(t) - s_2(t))^2 \, dt}. \tag{342}$$

Exercise 6.4: Consider a matched-filter receiver for a signal $s(t)$ that is observed in additive noise, but assume that the additive noise $N(t)$ process is non-white now. However there exists a $g(t)$ that is *reversible*, and that whitens the noise i.e., $S_N(f)|G(f)|^2 = \frac{N_0}{2}$ for all f. This results in a new system as is depicted in Fig. 51.

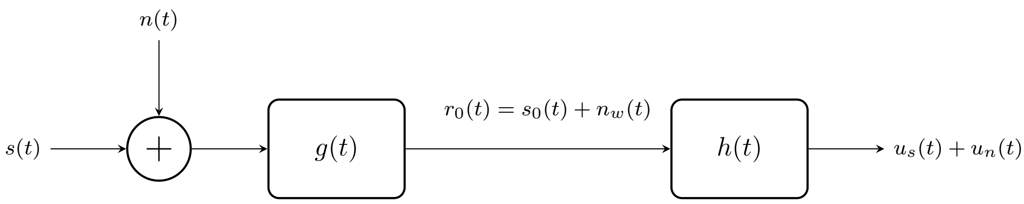

Figure 51: Scheme with additional filter with impulse response $g(t)$ before the matched filter $h(t)$, in order to whiten the noise (Ex. 6.4).

(a) Derive an expression for the maximum achievable signal-to noise-ratio in terms of the signal $s(t)$, the impulse response $g(t)$ of the whitening filter, and $\frac{N_0}{2}$.

(b) What is the impulse response of the *total* filter ($g(t) * h(t)$) that achieves this maximal SNR, expressed as a function of $s(t)$, and $g(t)$? This filter is called *whitened matched filter*.

(c) Express the achievable SNR as a function of the signal energy and the noise in the frequency domain.

(d) Finally, interpret your result from (c). For example, assume that two frequency bands are available for your signal, and it is given that the energy of the noise is larger in the first band than in the second band. How would you divide the energy of your signal over the two bands? (see waterfilling)

Solution:

(a) It is given that $g(t)$ is reversible, whitened filter and $S_N(f)|G(f)|^2 = N_0/2$ for all f. We have that

$$s_0(t) = s(t) * g(t) \tag{343}$$
$$r_0(t) = r(t) * g(t) \tag{344}$$
$$= s(t) * g(t) + n(t) * g(t) \tag{345}$$
$$= s_0(t) + n_w(t) \tag{346}$$

By the matched filter expression we have $h(t) = s_0(T - t)$. Also

$$\text{SNR}_{\max} = \frac{\int s_0^2(t)dt}{\frac{N_0}{2}} = \frac{\int (s(t) * g(t))^2 dt}{\frac{N_0}{2}} \tag{347}$$

(b) The total filter is given by $g(t) * s_0(T - t)$. We have that

$$s_0(t) = s(t) * g(t) \tag{348}$$
$$= \int s(t - \alpha)g(\alpha)d\alpha \tag{349}$$

Hence

$$h(t) = s_0(T - t) \tag{350}$$
$$= \int s(T - t - \alpha)g(\alpha)d\alpha \tag{351}$$
$$\overset{\alpha = T_1 - \beta}{=} \int s(T - t - T_1 + \beta)g(T_1 - \beta)d\beta \tag{352}$$
$$\overset{T_1 = T - T_2}{=} \int s(T_2 - (t - \beta))g(T_1 - \beta)d\beta \tag{353}$$
$$= s(T_2 - t) * g(T_1 - t) \tag{354}$$

The total is given by $g(t) * g(T_1 - t) * s(T_2 - t)$, where $g(T_1 - t)$ is matched to the whitened filter and $s(T_2 - t)$ is matched to the signal.

(c) To calculate the SNR, we can go to the frequency domain as:

$$\int s_0^2(t)dt = \int S_0(f)S_0^*(f)df \tag{355}$$
$$= \int S(f)G(f)S^*(f)G^*(f)df \tag{356}$$
$$= \int |S(f)|^2|G(f)|^2 df \tag{357}$$
$$= \int |S(f)|^2 \frac{N_0/2}{S_N(f)} df \tag{358}$$

Hence, the SNR is given by

$$\text{SNR} = \int \frac{|S(f)|^2}{S_N(f)} df \tag{359}$$

with $S_N(f)$ the weighted signal energy.

(d) This shows that you should divide the energy of your signal in such a way that you focus on the frequency bands that have small noise energy. However, note that this does not mean that you shouldn't use the bands that have noise. Use a water filling technique to divide the total energy (signal + noise) evenly over the (available) bands.

7 Signal Energy Considerations, Orthogonal Signals

Exercise 7.1: In a communication system based on an AWGN waveform channel four signals (waveforms) are used. All signals are zero for $t < 0$ and $t \geq 8$. For $0 \leq t < 8$ the signals are

$$\begin{aligned} s_1(t) &= 0, \\ s_2(t) &= +2\cos(\pi t/4), \\ s_3(t) &= +2\cos(\pi t/2), \\ s_4(t) &= +2\cos(\pi t/4) + 2\cos(\pi t/2). \end{aligned} \tag{360}$$

The messages corresponding to the signals all have probability $1/4$. The power spectral density of the noise process $N_w(t)$ is $\frac{N_0}{2} = 2$ for all f. The receiver observes the received waveform $r(t) = s_m(t) + n_w(t)$ in the time interval $0 \leq t < 8$.

(a) Determine a set of building-block waveforms for the four signals. Sketch these building-block waveforms. Show that they are orthonormal over $[0, 8)$. Give the vector representations of all four signals and sketch the resulting signal structure.

(b) Describe for what received vectors $\underline{r}$ an optimum receiver chooses $\hat{M} = 1, \hat{M} = 2, \hat{M} = 3$, and $\hat{M} = 4$. Sketch the corresponding decision regions $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$. Give an expression for the error probability P_e obtained by an optimum receiver. Use the $Q(\cdot)$-function.

(c) Sketch and specify a matched-filter implementation of an optimum receiver for the four signals (all occurring with equal a-priori probability). Use only two matched filters.

(d) Compute the average energy of the signals. We can translate the signal structure over a vector $\underline{a}$ such that the average signal energy is minimal. Determine this vector $\underline{a}$. What is the minimal average signal energy? Specify the modified signals $s_1'(t), s_2'(t), s_3'(t)$, and $s_4'(t)$ that correspond to this translated signal structure.

(e) Consider the signals as specified in (360) again. There we assumed that all messages were equally likely. Assume next that

$$\begin{aligned} \Pr\{M = 1\} &= \frac{1}{1 + 2e^2 + e^4}, \\ \Pr\{M = 2\} = \Pr\{M = 3\} &= \frac{e^2}{1 + 2e^2 + e^4}, \\ \Pr\{M = 4\} &= \frac{e^4}{1 + 2e^2 + e^4}. \end{aligned}$$

Sketch the corresponding decision regions $\mathcal{I}_1', \mathcal{I}_2', \mathcal{I}_3', \mathcal{I}_4'$. Give an expression for the resulting error probability P_e' obtained by an optimum receiver. Use the $Q(\cdot)$-function again.

Solution:

Note that

$$\sin(Ax)\sin(Bx) = \frac{1}{2}\left[\cos(A - B)x - \cos(A + B)x\right] \tag{361}$$

(a)

$$\begin{aligned} E_{s_2} &= \int_0^T s_2^2(t)dt = \int_0^8 4\cos^2\left(\frac{\pi t}{4}\right)dt \\ &= 16 \end{aligned}$$

Then:

$$\varphi_1(t) = \frac{s_2(t)}{\sqrt{E_{s_2}}} = \frac{1}{2}\cos\left(\frac{\pi t}{4}\right) \tag{362}$$

and also

$$\varphi_2(t) = \frac{1}{2}\cos\left(\frac{\pi t}{2}\right) \tag{363}$$

To show that they are orthogonal over $[0, 8)$, we can compute the integral:

$$\int_0^8 \varphi_1(t)\varphi_2(t)\mathrm{d}t = \int_0^8 \frac{1}{4}\cos\left(\frac{\pi t}{4}\right)\cos\left(\frac{\pi t}{2}\right) dt \tag{364}$$

$$= \int_0^8 \left[\frac{1}{8}\cos\left(\frac{3\pi t}{4}\right) + \frac{1}{8}\cos\left(\frac{\pi t}{4}\right)\right] dt \tag{365}$$

$$= 0 \tag{366}$$

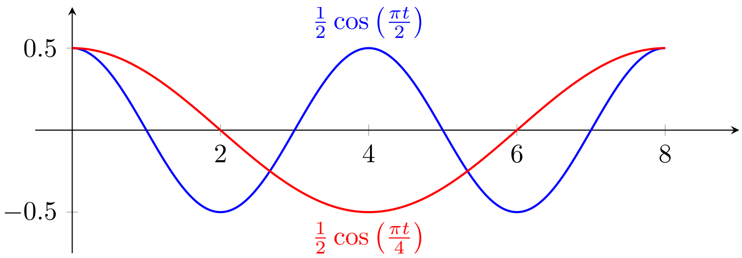

Figure 52: Signals φ_1 and φ_2 (Sol. 7.1).

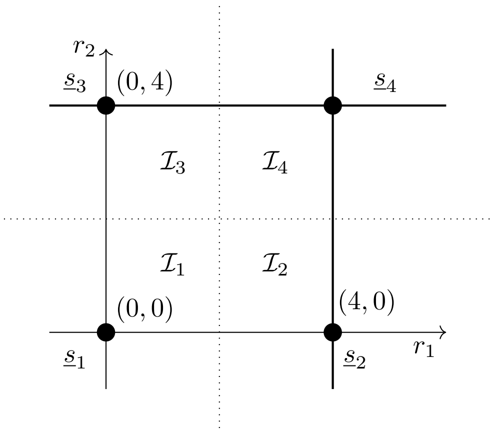

Figure 53: Signal constellation (Sol. 7.1).

(b) Decision rule:

$$\hat{M} = 1 \text{ if } r_1 < 2 \text{ and } r_2 < 2 \tag{367}$$

$$\hat{M} = 2 \text{ if } r_1 > 2 \text{ and } r_2 < 2 \tag{368}$$

$$\hat{M} = 3 \text{ if } r_1 < 2 \text{ and } r_2 > 2 \tag{369}$$

$$\hat{M} = 4 \text{ if } r_1 > 2 \text{ and } r_2 > 2 \tag{370}$$

Here the exact error probability can be calculated.

The probability for correctly estimating $\hat{m} = 1$ when signal $\underline{s}_1$ was sent is

$$\begin{aligned} P_c^1 &= \Pr\{\hat{m} = 1 | \underline{s} = \underline{s}_1\} \\ &= \Pr\{r_1 < 2 | \underline{s} = \underline{s}_1\} \cdot \Pr\{r_2 < 2 | \underline{s} = \underline{s}_1\} \\ &= (1 - Q)(1 - Q) \\ &= 1 - 2Q + Q^2, \end{aligned}$$

where

$$Q = Q\left(\frac{d/2}{\sigma}\right) = Q\left(\frac{2}{\sqrt{2}}\right) = Q(\sqrt{2}). \tag{371}$$

Due to the symmetry in the constellation,

$$P_c^1 = P_c^2 = P_c^3 = P_c^4 \tag{372}$$

and the total correct probability is

$$P_c = \sum_{m=1}^{4} \Pr\{M = m\} P_c^m = 1 - 2Q + Q^2 \tag{373}$$

Then the exact error probability is

$$P_e = 1 - P_c = 2Q - Q^2 \tag{374}$$

(c) See figure 54.

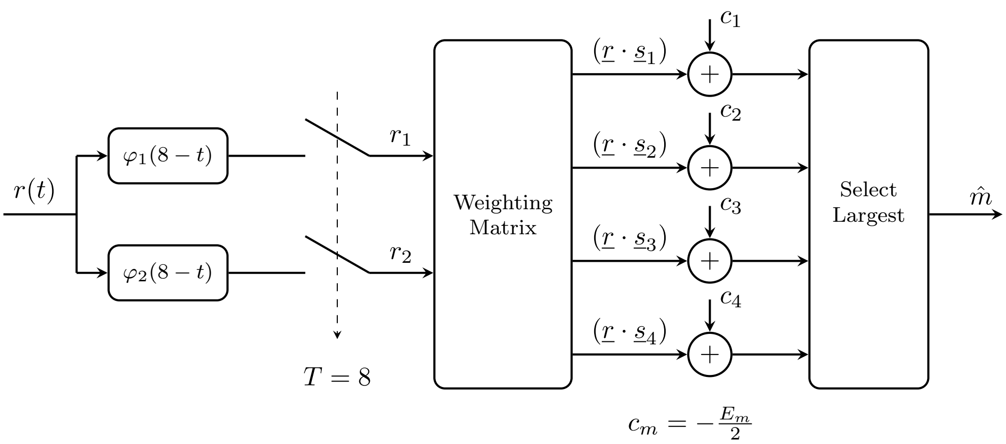

Figure 54: Matched filter implementation (Sol. 7.1).

(d) The average signal energy is

$$E_{\text{av}} = \frac{0 + 16 + 16 + 32}{4} = 16 \tag{375}$$

The vector $\underline{a}$ is the translation vector from zero origin to the average of the signal structure. It can be found as follows:

$$\underline{a} = \frac{\underline{s}_1 + \underline{s}_2 + \underline{s}_3 + \underline{s}_4}{4} = (2, 2). \tag{376}$$

The average signal energy can be minimized by translating the signal structure over $-\underline{a}$, or equivalently when the origin of the coordinate system is moved to $\underline{a}$.

The translated signal structure is represented now by

$$\underline{s}_1 = (-2, -2) \tag{377}$$
$$\underline{s}_2 = (2, -2) \tag{378}$$
$$\underline{s}_3 = (-2, 2) \tag{379}$$
$$\underline{s}_4 = (2, 2) \tag{380}$$

The new average energy is

$$E_{\text{av}} = \frac{8 + 8 + 8 + 8}{4} = 8 \tag{381}$$

The new signal waveforms are

$$s_1(t) = -\cos\left(\frac{\pi t}{4}\right) - \cos\left(\frac{\pi t}{2}\right) \tag{382}$$

$$s_2(t) = +\cos\left(\frac{\pi t}{4}\right) - \cos\left(\frac{\pi t}{2}\right) \tag{383}$$

$$s_3(t) = -\cos\left(\frac{\pi t}{4}\right) + \cos\left(\frac{\pi t}{2}\right) \tag{384}$$

$$s_4(t) = +\cos\left(\frac{\pi t}{4}\right) + \cos\left(\frac{\pi t}{2}\right) \tag{385}$$

(e) There are four decision variables. For message m, the decision variable is

$$\begin{aligned}\Pr\{M=m\}\Pr\{R=\underline{r}|S=\underline{s}_m\} &= \Pr\{M=m\}p_N(\underline{r}-\underline{s}_m)\\ &= \Pr\{M=m\}\prod_{i=1}^{2}\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{(r_i-s_{mi})^2}{2\sigma^2}}.\end{aligned}$$

It can be noticed that the parts $\frac{1}{1+2e^2+e^4}$ and $\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^2$ are common for all the decision variables (for every message m). Thus, these parts can be removed (cancelled out) from the decision rules. The resulting simplified decision variables are shown in Fig. 55.

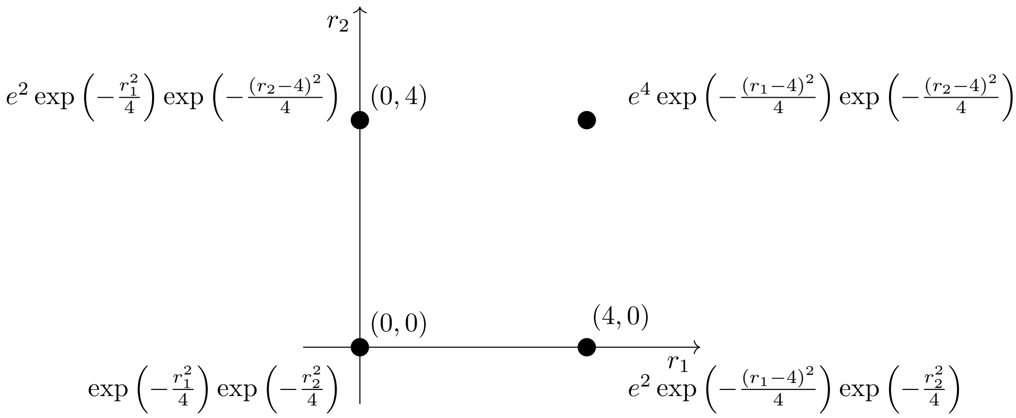

Figure 55: Decision variables (Sol. 7.1).

Now apply the natural logarithm "ln" to all the decision variables and multiply by 4 to simplify. The result is shown in Fig. 56. The expressions can be further simplified by removing commonly present $r_1^2 + r_2^2$ that results in Fig. 57.

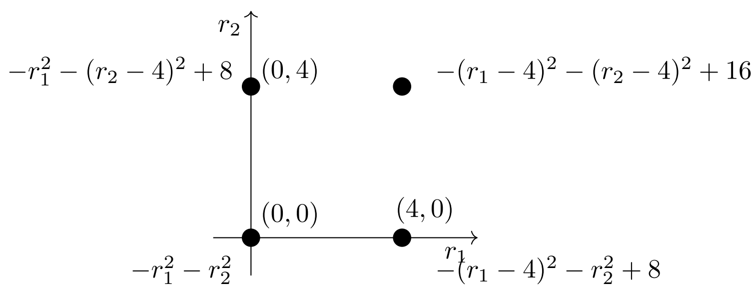

Figure 56: After applying ln and multiply 4 (Sol. 7.1).

We can finally create the decision rules, for example, the receiver will estimate message $m=2$ rather than message $m=1$ if

$$\begin{aligned}\Pr\{M=2\}\Pr\{R=\underline{r}|S=\underline{s}_2\} &> \Pr\{M=1\}\Pr\{R=\underline{r}|S=\underline{s}_1\}\\ -8(1-r_1) &> 0,\end{aligned}$$

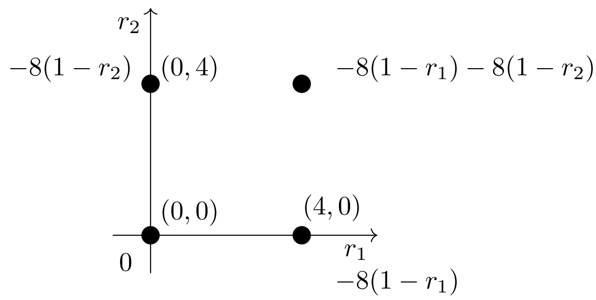

Figure 57: Equations to be minimized by the decoder (Sol. 7.1).

that is equivalent to condition

$$r_1 > 1. \tag{386}$$

Similarly, the receiver decides for $m = 2$ over $m = 4$ if $r_2 < 1$. Thus, the receiver estimates $\hat{m} = 2$ if $r_1 > 1$ and $r_2 < 1$.

Overall, after manipulation of the decision variables, we find that a decoder should select the message that gives for the received values r_1, r_2 the minimum that minimizes the corresponding equations given in Fig. 57, we find the optimum decision regions as shown in figure 58.

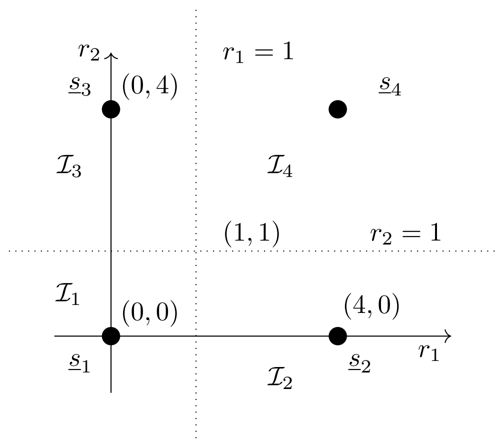

Figure 58: Optimum decision regions (Sol. 7.1).

Then the correct error probability is

$$\begin{aligned} P_c = \frac{1}{1+2e^2+e^4} \Bigg\{ & e^4 \left[1 - Q\left(\frac{3}{\sqrt{2}}\right)\right]^2 + \\ & + 2e^2 \left[1 - Q\left(\frac{1}{\sqrt{2}}\right)\right] \left[1 - Q\left(\frac{3}{\sqrt{2}}\right)\right] + \left[1 - Q\left(\frac{1}{\sqrt{2}}\right)\right]^2 \Bigg\} \end{aligned} \tag{387}$$

and the exact probability of error is given by

$$P_e = 1 - P_c \tag{388}$$

It is also true that

$$P_c = \left\{ \frac{e^2 \left[1 - Q\left(\frac{3}{\sqrt{2}}\right)\right] + \left[1 - Q\left(\frac{1}{\sqrt{2}}\right)\right]}{e^2 + 1} \right\}^2 \tag{389}$$

because of symmetry in r_1 and r_2 direction.

Exercise 7.2: A Hadamard matrix is a matrix whose elements are ± 1. When n is a power of 2, an $n \times n$ Hadamard matrix is constructed by means of the recursion:

$$\begin{aligned} \mathbf{H}_2 &\triangleq \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}, \\ \mathbf{H}_{2n} &\triangleq \begin{pmatrix} +\mathbf{H}_n & +\mathbf{H}_n \\ +\mathbf{H}_n & -\mathbf{H}_n \end{pmatrix}. \end{aligned} \tag{390}$$

Let n be a power of 2 and $\mathcal{M} = \{1, 2, \ldots, n\}$. Consider for $m \in \mathcal{M}$ the signal vectors $\underline{s}_m \triangleq \sqrt{\frac{E_s}{n}} \underline{h}_m$ where $\underline{h}_m$ is the m-th row of the Hadamard matrix $\mathbf{H}_n$.

(a) Show that the signal set $\{\underline{s}_m, m \in \mathcal{M}\}$ consists of orthogonal vectors all having energy E_s.

(b) What is the error probability P_e if the signal vectors correspond to waveforms that are transmitted over a waveform channel with AWGN having spectral density $\frac{N_0}{2}$?

(c) What is the advantage of using the Hadamard signal set over the orthogonal set from definition 7.1 in the Course Reader if we assume that the building-block waveforms are in both cases time-shifts of a pulse? And the disadvantage?

(d) The matrix $n \times 2n$ matrix

$$\mathbf{H}_n^* \triangleq \begin{vmatrix} +\mathbf{H}_n \\ -\mathbf{H}_n \end{vmatrix}. \tag{391}$$

defines a set of $2n$ signals which is called bi-orthogonal. Determine the error probability of this signal set if the energy of each signal is E_s.

A bi-orthogonal "code" with $n = 32$ was used for an early deep-space mission (Mariner, 1969). A fast Hadamard transform was used as decoding method.

Solution:

It is advised to first consult the solution of Exercise 7.4.

(a) For the energy:

$$E_m = ||\underline{s}_m||^2 = \sum_{i=1,n} s_{mi}^2 = \sum_{i=1,n} \frac{E_S}{n} = E_S. \tag{392}$$

For orthogonality: Since H_2 has orthogonal rows $H_2 H_2^T = 2I_2$. Furthermore, $H_n H_n^T = nI_n$, and thus

$$H_{2n} H_{2n}^T = \begin{bmatrix} 2nI_n & 0 \\ 0 & 2nI_n \end{bmatrix}. \tag{393}$$

Therefore, also the rows of H_{2n} are orthogonal.

(b) The signals are n-dimensional, all have length $\sqrt{E_s}$ and are orthogonal. Now, choose a new basis with normalised signal vectors, such that $\underline{s}_m = \sqrt{E_s}\underline{\phi}'_m$. From Section 7.7 of the reader, we find that for orthogonal signals the error probability is given by

$$P_e = 1 - \int_{-\infty}^{\infty} p_N\left(\alpha - \sqrt{E_S}\right) \left(\int_{-\infty}^{\alpha} p_N(\beta) d\beta\right)^{n-1} d\alpha, \tag{394}$$

where

$$p_N(n) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{n^2}{N_0}\right). \tag{395}$$

(c) Advantage: Hadamard has regular amplitudes, orthogonal set from Def. 7.1 has all zeros but one. Disadvantage: Hadamard has complex receiver $\hat{m} = \text{argmax} \sum_{i=1}^{n} r_i h_{mi}$, orthogonal set from Def. 7.1 has $\hat{m} = \text{argmax}\, r_m$.

(d) First of all, we notice that the center of gravity $\frac{1}{2n}\sum \underline{s}_m = 0$ since $\underline{s}_m = -\underline{s}_{m+n}$. Again define a new basis with normalised signal vectors, such that

$$\underline{s}_m = \sqrt{E_s}\underline{\phi}'_m \qquad 1 \le m \le n \tag{396}$$
$$\underline{s}_m = \sqrt{E_s}\underline{\phi}'_m \qquad n+1 \le m \le 2n \tag{397}$$

For this case the error probability is given by

$$P_e = 1 - \int_0^{\infty} p_N\left(\alpha - \sqrt{E_S}\right)\left(\int_{-\alpha}^{\alpha} p_N(\beta)d\beta\right)^{n-1} d\alpha. \tag{398}$$

Exercise 7.3:

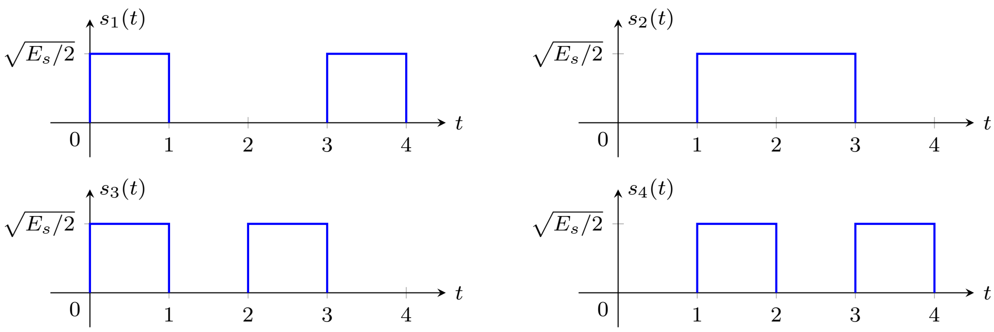

Figure 59: Signal set (a) (Ex. 7.3).

Either of the two waveform sets illustrated in Figs. 59 and 60 may be used to communicate one of four equally likely messages over an AWGN channel.

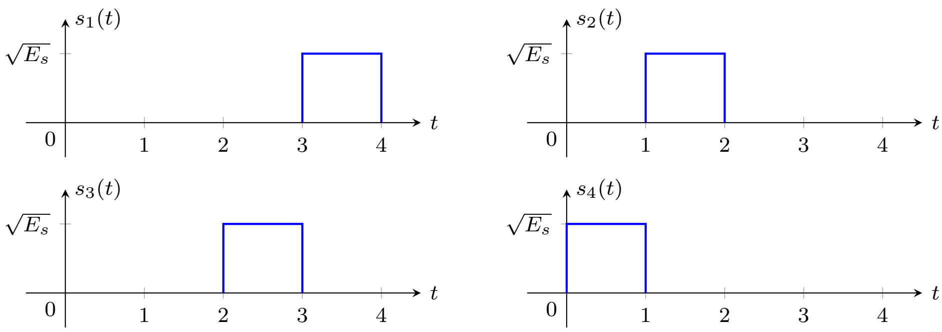

Figure 60: Signal set (b) (Ex. 7.3).

(a) Show that both sets use the same energy.

(b) Exploit the union bound to show that the set of Fig. 60 uses energy almost 3 dB more effectively than the set of Fig. 59 when a small P_e is required.

Solution:

(a) The energy of set "a" is:

$$E_1 = 2\int_0^1 \frac{E_s}{2} dt = E_s \tag{399}$$

which applies to all signals of that set.

The energy of set "b" is:

$$E_1 = \int_3^4 E_s dt = E_s \tag{400}$$

which applies to all signals of that set.

(b) Choose for the sets a simple basis given by:

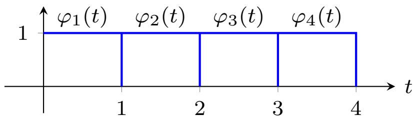

Figure 61: Simple basis chosen from the sets (Sol. 7.3).

Then, for the first set ("a"):

$$\underline{s}_1 = \sqrt{\frac{E_s}{2}}(1, 0, 0, 1) \tag{401}$$

$$\underline{s}_2 = \sqrt{\frac{E_s}{2}}(0, 1, 1, 0) \tag{402}$$

$$\underline{s}_3 = \sqrt{\frac{E_s}{2}}(1, 0, 1, 0) \tag{403}$$

$$\underline{s}_4 = \sqrt{\frac{E_s}{2}}(0, 1, 0, 1) \tag{404}$$

For the second set ("b"):

$$\underline{s}_1 = \sqrt{E_s}(0, 0, 0, 1) \tag{405}$$

$$\underline{s}_2 = \sqrt{E_s}(0, 1, 0, 0) \tag{406}$$

$$\underline{s}_3 = \sqrt{E_s}(0, 0, 1, 0) \tag{407}$$

$$\underline{s}_4 = \sqrt{E_s}(1, 0, 0, 0) \tag{408}$$

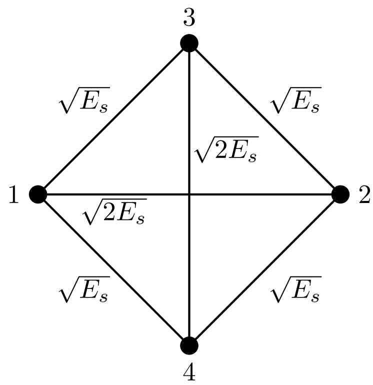

Figure 62: Sketch of distances between the constellation points - set "a" (Sol. 7.3).

Distances for set ("a") (symmetric) are shown in Fig. 62. Let $d_{1,2}$ refer to the distance between the messages s_1 and s_2 and let $d_{2,3}$ refer to the distance between the messages s_2 and s_3, then

$$d_{1,2} = \sqrt{\left(0-\sqrt{\frac{E_s}{2}}\right)^2 + \left(\sqrt{\frac{E_s}{2}}-0\right)^2 + \left(\sqrt{\frac{E_s}{2}}-0\right)^2 + \left(0-\sqrt{\frac{E_s}{2}}\right)^2} = \sqrt{2E_s},$$
$$d_{2,3} = \sqrt{\left(\sqrt{\frac{E_s}{2}}-0\right)^2 + \left(0-\sqrt{\frac{E_s}{2}}\right)^2 + \left(\sqrt{\frac{E_s}{2}}-\sqrt{\frac{E_s}{2}}\right)^2 + (0-0)^2} = \sqrt{E_s}.$$

Now: $P_e = \Pr(R$ closer to S_2 than S_1 OR R closer to S_3 than S_1 OR R closer to S_4 than $S_1)$.

$$\begin{aligned} P_e &\leq \sum_{m=2,3,4} \Pr\{\underline{r} \text{ closer to } \underline{s}_m \text{ than to } \underline{s}_1 | M=1\} \\ &= \sum_{m=2,3,4} Q\left(\frac{d(\underline{s}_m,\underline{s}_1)}{2\sigma}\right) \\ &= Q\left(\frac{\sqrt{2E_s}}{2\sqrt{N_0/2}}\right) + Q\left(\frac{\sqrt{E_s}}{2\sqrt{N_0/2}}\right) + Q\left(\frac{\sqrt{E_s}}{2\sqrt{N_0/2}}\right) \\ &\leq 3Q\left(\sqrt{\frac{E_s}{2N_0}}\right) \end{aligned} \tag{409}$$

Distances for set "b" (also symmetric) are shown in Fig. 63. Now:

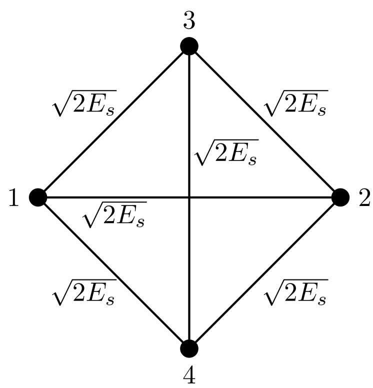

Figure 63: Sketch of distances between the constellation points - set "b" (Sol. 7.3).

$$P_e \leq 3Q\left(\frac{\sqrt{2E_s}}{2\sqrt{N_0/2}}\right) \tag{410}$$
$$= 3Q\left(\sqrt{\frac{E_s}{N_0}}\right) \tag{411}$$

To get the same P_e the E_s should be two times as big in set a as in set b. (assuming that the estimates are correct)

Remark: The union bound is good in case of large SNR. Therefore, in the current case the estimate of P_e is not that good. We can find the exact error probability instead to get a better estimation.

It is not really convenient to calculate the exact error probability for the current signal structure. The calculations can be simplified with the use of the translation vector $\underline{a}$. By moving the origin of the signal structure from zero to the average of the signal structure, the signal energy is minimized.

For signal set "a", the translation vector is

$$\underline{a} = \sum_{m\in\mathcal{M}} \Pr\{M=m\}\underline{s}_m = \sqrt{\frac{E_s}{2}}\frac{(2,2,2,2)}{4} = \sqrt{\frac{E_s}{2}}\frac{(1,1,1,1)}{2}. \tag{412}$$

The signal structure and translation vector are shown in Fig. 64. The signals can be shown as

$$\underline{s}_1 = \underline{a} + \sqrt{\frac{E_s}{2}}\frac{(1,-1,-1,1)}{2} \tag{413}$$

$$\underline{s}_2 = \underline{a} - \sqrt{\frac{E_s}{2}}\frac{(1,-1,-1,1)}{2} \tag{414}$$

$$\underline{s}_3 = \underline{a} + \sqrt{\frac{E_s}{2}}\frac{(1,-1,1,-1)}{2} \tag{415}$$

$$\underline{s}_4 = \underline{a} - \sqrt{\frac{E_s}{2}}\frac{(1,-1,1,-1)}{2}. \tag{416}$$

Notice that the signal representation in (404) is defined in 4 dimensions, but the new representation in (416) is effectively defined in 2 dimensions. In other words, the vectors $\underline{s}_1$, $\underline{s}_2$,$\underline{s}_3$ and $\underline{s}_4$ lie in one plane as shown in Fig. 64.

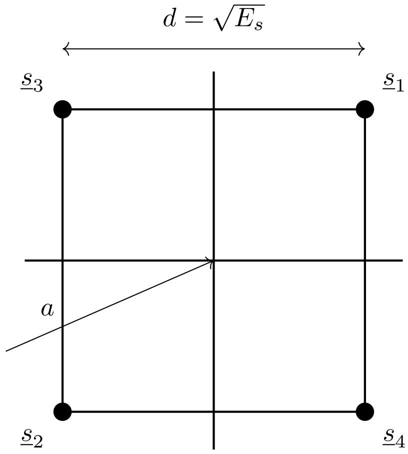

Figure 64: Representation of the vectors in the plane - set "a" (Sol. 7.3).

The probability for correctly estimating $\hat{m} = 1$ when signal $\underline{s}_1$ was sent is

$$P_c^1 = (1-Q)(1-Q)$$

with

$$Q = Q\left(\sqrt{\frac{E_s}{2N_0}}\right). \tag{417}$$

Due to the symmetry, $P_{c_1} = P_{c_2} = P_{c_3} = P_{c_4}$, so total correct probability is

$$P_c = 1 - 2Q + Q^2. \tag{418}$$

Then the corresponding error probability is

$$P_e = 2Q - Q^2. \tag{419}$$

Set "b": Orthogonal signals

The procedure is exactly the same as for "a":

$$P_c = (1-Q)^2 \tag{420}$$

$$P_e = 1-(1-Q)^2 = 2Q - Q^2 \tag{421}$$

Exercise 7.4: Consider a communication system based on frequency-shift keying (FSK). There are 8 equiprobable messages, hence $\Pr\{M = m\} = 1/8$ for $m \in \{1, 2, \ldots, 8\}$. The signal waveform corresponding to message m is

$$s_m(t) = A\sqrt{2}\cos(2\pi mt), \text{ for } 0 \leq t < 1.$$

For $t < 0$ and $t \geq 1$ all signals are zero. The signals are transmitted over an AWGN waveform channel. The power spectral density of the noise is $S_n(f) = \frac{N_0}{2}$ for all frequencies f.

(a) First show that the signals $s_m(t), m \in \{1, 2, \ldots, 8\}$ are orthogonal[1]. Give the energies of the signals. What are the building-block waveforms $\varphi_1(t), \varphi_2(t), \ldots, \varphi_8(t)$ that result in the signal vectors

$$\begin{aligned} \underline{s}_1 &= (A, 0, 0, 0, 0, 0, 0, 0), \\ \underline{s}_2 &= (0, A, 0, 0, 0, 0, 0, 0), \\ &\ldots \\ \underline{s}_8 &= (0, 0, 0, 0, 0, 0, 0, A)? \end{aligned}$$

(b) The optimum receiver first determines the correlations $r_i = \int_0^1 r(t)\varphi_i(t)dt$ for $i = 1, 2, \ldots, 8$. Here $r(t)$ is the received waveform, so $r(t) = s_m(t) + n_w(t)$. For what values of the vector $\underline{r} = (r_1, r_2, \ldots, r_8)$ does the receiver decide that message m was transmitted?

(c) Give an expression for the error probability P_e obtained by the optimum receiver.

(d) Next consider a system with 16 messages all having the same a-priori probability. The signal waveform corresponding to message m is now given by

$$\begin{aligned} s_m(t) &= A\sqrt{2}\cos(2\pi mt), \text{ for } 0 \leq t < 1 \text{ for } m = 1, 2, \ldots, 8, \\ s_m(t) &= -A\sqrt{2}\cos(2\pi(m-8)t), \text{ for } 0 \leq t < 1 \text{ for } m = 9, 10, \ldots, 16. \end{aligned}$$

For $t < 0$ and $t \geq 1$ these signals are again zero. What are the signal vectors $\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{16}$ if we use the building-block waveforms mentioned in part (a) again?

(e) The optimum receiver again determines the correlations $r_i = \int_0^1 r(t)\varphi_i(t)dt$ for $i = 1, 2, \ldots, 8$. For what values of the vector $\underline{r} = (r_1, r_2, \ldots, r_8)$ does the receiver now decide that message m was transmitted? Give an expression for the error probability P_e obtained by the optimum receiver now.

Solution:

(a) We can check for the orthogonality of the signals and find the signals energies as follows:

$$\int_0^1 s_m(t)s_k(t)dt = \int_0^1 A^2 2\cos(2\pi mt)\cos(2\pi kt)dt \tag{422}$$

$$= A^2 \int_0^1 \left[\cos(2\pi(m-k)t) + \cos(2\pi(m+k)t)\right] dt \tag{423}$$

$$= \begin{cases} 0 & \text{if } m \neq k \text{ (orthogonal)} \\ A^2 & \text{if } m = k \text{ (energy } A^2\text{)} \end{cases} \tag{424}$$

Building blocks (for $0 \leq t < 1$)

[1] Hint: $2\cos(a)\cos(b) = \cos(a-b) + \cos(a+b)$.

$$\varphi_1(t) = \sqrt{2}\cos(2\pi t) \tag{425}$$
$$\varphi_2(t) = \sqrt{2}\cos(4\pi t) \tag{426}$$
$$\varphi_3(t) = \sqrt{2}\cos(6\pi t) \tag{427}$$
$$\vdots$$
$$\varphi_8(t) = \sqrt{2}\cos(16\pi t) \tag{428}$$

(b) Signal vectors, $\underline{s}_i = (s_1, s_2, ..., s_8)$ are

$$\underline{s}_1 = (A, 0, \cdots, 0) \tag{429}$$
$$\vdots$$
$$\underline{s}_8 = (0, 0, \cdots, A) \tag{430}$$

and the received vector is $\underline{r} = (r_1, r_2, ..., r_8)$.
The decision variable for the vector channel is

$$\begin{aligned}\Pr\{M = m\}\Pr\{\underline{R} = \underline{r}|M = m\} &= \Pr\{M = m\}p_{\underline{N}}\{\underline{r} - \underline{s}_m\} \\ &= \Pr\{M = m\}\frac{1}{(2\pi\sigma^2)^{N/2}}e^{-\frac{||\underline{r}-\underline{s}_m||^2}{2\sigma^2}}.\end{aligned}$$

The receiver estimates message m that minimizes

$$||\underline{r} - \underline{s}_m||^2 = ||\underline{r}||^2 + ||\underline{s}_m||^2 - 2(\underline{r} \cdot \underline{s}_m) \tag{431}$$
$$= ||\underline{r}||^2 + A^2 - 2r_m A. \tag{432}$$

Overall, the decision rule is

$$\begin{gathered}\Pr\{\underline{R} = \underline{r}, M = \hat{m}\} \geq \Pr\{\underline{R} = \underline{r}, M = m\} \\ ||\underline{r} - \underline{s}_{\hat{m}}||^2 \leq ||\underline{r} - \underline{s}_m||^2 \\ r_{\hat{m}}A \geq r_m A,\end{gathered}$$

so the optimum receiver for orthogonal signaling chooses $\hat{m}$ such that

$$r_{\hat{m}} \geq r_m \text{ for all } m \in \mathcal{M}. \tag{433}$$

(c) Let us assume that the message $m = 1$ was sent. Then $\underline{s}_1 = (s_1, ..., s_8) = (A, ..., 0)$, so the receiver receives

$$\begin{aligned}\underline{r} = \underline{s}_1 + \underline{n} &= (r_1, r_2, ..., r_8) \\ &= (s_1 + n_1, s_2 + n_2, ..., s_8 + n_8) \\ &= (A + n_1, n_2, ..., n_8).\end{aligned}$$

Therefore,

$$r_1 = A + n_1 \tag{434}$$

and

$$r_m = n_m \text{ for } m = 2, 3, \cdots, 8. \tag{435}$$

The probability that $m = 1$ is sent, i.e. $s_1 = A$, and the receiver observes some value $r_1 = \alpha$ is

$$\Pr\{R_1 = \alpha|M = 1\} = p_N(R_1 - S_1) = p_N(\alpha - A) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{||\alpha-A||^2}{2\sigma^2}}. \tag{436}$$

Following (433), the receiver estimates $\hat{m} = 1$ only if $r_m = n_m < \alpha$ for $m = 2, 3, \cdots, 8$. Thus, given that $m = 1$ is sent, i.e. $s_1 = A$, and the receiver observes $r_1 = \alpha$, then the probability that the receiver correctly estimates $\hat{m} = 1$ is

$$\begin{aligned}
\Pr\{\hat{M} = 1 | M = 1, R_1 = \alpha\} &= \Pr\{N_2 \leq \alpha, N_3 \leq \alpha, ..., N_8 \leq \alpha\} && (437)\\
&= \Pr\{N_2 \leq \alpha\} \cdot \Pr\{N_3 \leq \alpha\} \cdot ... \cdot \Pr\{N_8 \leq \alpha\} && (438)\\
&= \prod_{m=2}^{8} \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{||r_m - s_m||^2}{2\sigma^2}} d(r_m - s_m) && (439)\\
&= \prod_{m=2}^{8} \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{||n_m||^2}{2\sigma^2}} dn_m && (440)\\
&= \left[1 - Q\left(\frac{\alpha}{\sqrt{N_0/2}}\right)\right]^7 && (441)
\end{aligned}$$

Finally, given that $m = 1$ was sent, i.e. $s_1 = A$, the probability that the receiver correctly estimates $\hat{m} = 1$ for any value of $r_1 = \alpha$ is

$$P_c^1 = \int_{-\infty}^{\infty} p_N(\alpha - A) \left[1 - Q\left(\frac{\alpha}{\sqrt{N_0/2}}\right)\right]^7 d\alpha. \tag{442}$$

The correct probability for other messages can be derived in the same way, such that

$$P_c^1 = ... = P_c^8. \tag{443}$$

The exact error probability is

$$P_e = 1 - \int_{-\infty}^{\infty} p_N(\alpha - A) \left[1 - Q\left(\frac{\alpha}{\sqrt{N_0/2}}\right)\right]^7 d\alpha.$$

(d)

$$\begin{aligned}
\underline{s}_1 &= (A, 0, \cdots, 0) && (444)\\
\underline{s}_2 &= (0, A, \cdots, 0) && (445)\\
&\vdots\\
\underline{s}_8 &= (0, 0, \cdots, A) && (446)\\
\underline{s}_9 &= (-A, 0, \cdots, 0) && (447)\\
\underline{s}_{10} &= (0, -A, \cdots, 0) && (448)\\
&\vdots\\
\underline{s}_{16} &= (0, 0, \cdots, -A) && (449)
\end{aligned}$$

(e) The receiver receives a vector

$$\underline{r} = \underline{s} + \underline{n} = (r_1, r_2, \cdots, r_8) = (\pm A + n_1, \pm A + n_2, \cdots, \pm A + n_8), \tag{450}$$

The decision rule is again equivalent to maximization of (**??**), i.e. the receiver selects such m that maximizes the dot product $(\underline{r} \cdot \underline{s}_m)$.

Thus, the receiver estimates $\hat{m}$ based on the largest component in vector $\underline{r}$ and its sign. If the receiver checks for $\hat{m} = 1, \cdots, 8$, then the decision rule is

$$r_{\hat{m}} \geq |r_i|, \text{ for } i = 1, \cdots, 8 \tag{451}$$

and if the receiver checks for $\hat{m} = 9, \cdots, 16$, then the decision rule is

$$-r_{\hat{m}} \geq |r_i|, \text{ for } i = 1, \cdots, 8. \tag{452}$$

Similarly to c): When S_1 was send and $r_1 = \alpha$, then $\hat{m}$ is only decoded as $\hat{m} = 1$ when $r_2 \leq \alpha$ and $r_3 \leq \alpha$ and ... and $r_8 \leq \alpha$ and $-r_2 \leq \alpha$ and $-r_3 \leq \alpha$ and ... and $-r_8 \leq \alpha$ or:

$$r_2, r_3, \cdots, r_8, -r_1, -r_2, \cdots, -r_8 \leq \alpha, \tag{453}$$

where $-r_1 \leq \alpha = r_1$ implies that $r_1 = \alpha \geq 0$.
Now:

$$\Pr\{\hat{M} = 1 | M = 1, R_1 = \alpha \geq 0\} = \Pr\{-\alpha \leq N_2 \leq \alpha, \cdots, -\alpha \leq N_8 \leq \alpha\} \tag{454}$$

$$= \left[1 - 2Q\left(\frac{\alpha}{\sqrt{N_0/2}}\right)\right]^7 \tag{455}$$

Therefore

$$P_c = \int_0^\infty p_N(\alpha - A)\left[1 - 2Q\left(\frac{\alpha}{\sqrt{N_0/2}}\right)\right]^7 d\alpha \tag{456}$$

with

$$p_N(n) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n^2}{N_0}}. \tag{457}$$

Furthermore

$$P_e = 1 - P_c \tag{458}$$

8 Message Sequences, Bandwidth

Exercise 8.1:

Consider $N = 2K + 1$ orthogonal waveforms that are always zero except for $-\frac{T}{2} \le t < \frac{T}{2}$. The waveforms are defined as

$$\begin{aligned}
\phi_0(t) &= \sqrt{1/T} \\
\phi_1^c(t) &= \sqrt{2/T}\cos(2\pi t/T) \quad \text{and} \quad \phi_1^s(t) = \sqrt{2/T}\sin(2\pi t/T) \\
\phi_2^c(t) &= \sqrt{2/T}\cos(4\pi t/T) \quad \text{and} \quad \phi_2^s(t) = \sqrt{2/T}\sin(4\pi t/T) \\
&\dots \\
\phi_K^c(t) &= \sqrt{2/T}\cos(2K\pi t/T) \quad \text{and} \quad \phi_K^s(t) = \sqrt{2/T}\sin(2K\pi t/T).
\end{aligned} \tag{459}$$

(a) Determine the spectra of all these waveforms.

(b) Sketch the spectra of $\phi_K^c(t)$ and $\phi_K^s(t)$ assuming that $T = 1$ and $K = 5$. What is the number of dimensions per second?

(c) Find out (numerically) how much spectral energy of these two waveforms is outside $[-6, +6]$ Hz. What about the other nine waveforms?

(d) Now sketch the spectra of $\phi_K^c(t)$ and $\phi_K^s(t)$ assuming that $T = 5$ and $K = 25$. What is now the number of dimensions per second?

(e) Find out how much spectral energy of these two waveforms is outside $[-6, +6]$ Hz.

(f) Interpret your results. Explain the effect of T and K on the spectrum of the waveforms. Furthermore, explain whether your results correspond with the dimensionality theorem (Result 8.2 reader).

Solution: Note that $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$ and $\text{sinc}(x) = \frac{\sin x}{x}$.

(a) The spectrum of $\phi_0(t) = \sqrt{\frac{1}{T}}$ is

$$\Phi_0(f) = \int_{-T/2}^{T/2} \sqrt{\frac{1}{T}} e^{-j2\pi f t} dt \tag{460}$$

$$= \sqrt{\frac{1}{T}} \frac{-1}{j2\pi f}\left(e^{-j2\pi f\frac{T}{2}} - e^{j2\pi f\frac{T}{2}}\right) \tag{461}$$

$$= \sqrt{\frac{1}{T}} \frac{\sin(\pi f T)}{\pi f} \tag{462}$$

$$= \sqrt{T}\frac{\sin(\pi f T)}{\pi f T} \tag{463}$$

$$= \sqrt{T}\text{sinc}(fT) \tag{464}$$

The spectrum of

$$\phi_k^c(t) = \sqrt{\frac{2}{T}}\cos\left(2\pi k\frac{t}{T}\right) \tag{465}$$

$$= \phi_0(t)\sqrt{2}\cos\left(2\pi k\frac{t}{T}\right) \tag{466}$$

$$= \phi_0(t)\sqrt{2}\left[\frac{e^{j2\pi k\frac{t}{T}} + e^{-j2\pi k\frac{t}{T}}}{2}\right] \tag{467}$$

is given by

$$\Phi_k^c(f) = \frac{1}{\sqrt{2}}\left[\Phi_0\left(f - \frac{k}{T}\right) + \Phi_0\left(f + \frac{k}{T}\right)\right] \tag{468}$$
$$= \sqrt{\frac{T}{2}}\,[\mathrm{sinc}(fT - k) + \mathrm{sinc}(fT + k)] \tag{469}$$

The spectrum of

$$\phi_k^s(t) = \sqrt{\frac{2}{T}}\sin\left(2\pi k\frac{t}{T}\right) \tag{470}$$
$$= \phi_0(t)\sqrt{2}\sin\left(2\pi k\frac{t}{T}\right) \tag{471}$$
$$= \phi_0(t)\sqrt{2}\left[\frac{e^{j2\pi k\frac{t}{T}} - e^{-j2\pi k\frac{t}{T}}}{2j}\right] \tag{472}$$

is given by

$$\Phi_k^s(f) = \frac{1}{j\sqrt{2}}\left[\Phi_0\left(f - \frac{k}{T}\right) - \Phi_0\left(f + \frac{k}{T}\right)\right] \tag{473}$$
$$= \frac{1}{j}\sqrt{\frac{T}{2}}\,[\mathrm{sinc}(fT - k) - \mathrm{sinc}(fT + k)] \tag{474}$$

(b) See the spectrum for $K = 5$ and $T = 1$ in Fig. 65. Note that K acts like a frequency shifting parameter.

The number of dimensions per second is $\frac{2K+1}{T} = 11$. This means that in one second, we can send 11 dimensions (orthogonal waveforms), so that these waveforms do not interfere with each other.

For the pulse duration $T = 1$, the minimum possible frequency that the building block waveform can carry is 1 Hz. $\phi_0(f)$ has its first zeros at $+-1$ Hz.

(c)

$$E_{\text{inside}} = \int_{-w}^{w} |\Phi(f)|^2 df \tag{475}$$
$$E_{\text{outside}} = 1 - E_{\text{inside}} \tag{476}$$
$$w = 6 \tag{477}$$

Here the case of only $T = 1$ is considered. Then if the energy is calculated for $\phi_2^s(t)$ (for $k = 2$),

$$E_{\text{inside}} = \int_{-6}^{6}\left|\sqrt{\frac{T}{2}}\,[\mathrm{sinc}(fT - 2) - \mathrm{sinc}(fT + 2)]\right|^2 df = 0.9986 \tag{478}$$
$$E_{\text{outside}} = 1 - E_{\text{inside}} = 0.0014 \tag{479}$$

The values of E_{outside} for different waveforms are listed in table below. These values can be calculated either manually or numerically using the Matlab code provided below. Also, see Fig. 66 for visual representations of the power spectrum for $T = 1$ and $K = 0, 5$.

E_{outs}		$K = 1$	$K = 2$	$K = 3$	$K = 4$	$K = 5$
ϕ_0	0.0169					
ϕ_K^c		0.0344	0.0365	0.0409	0.0504	0.0773
ϕ_K^s		0.0003	0.0014	0.0039	0.0098	0.0291

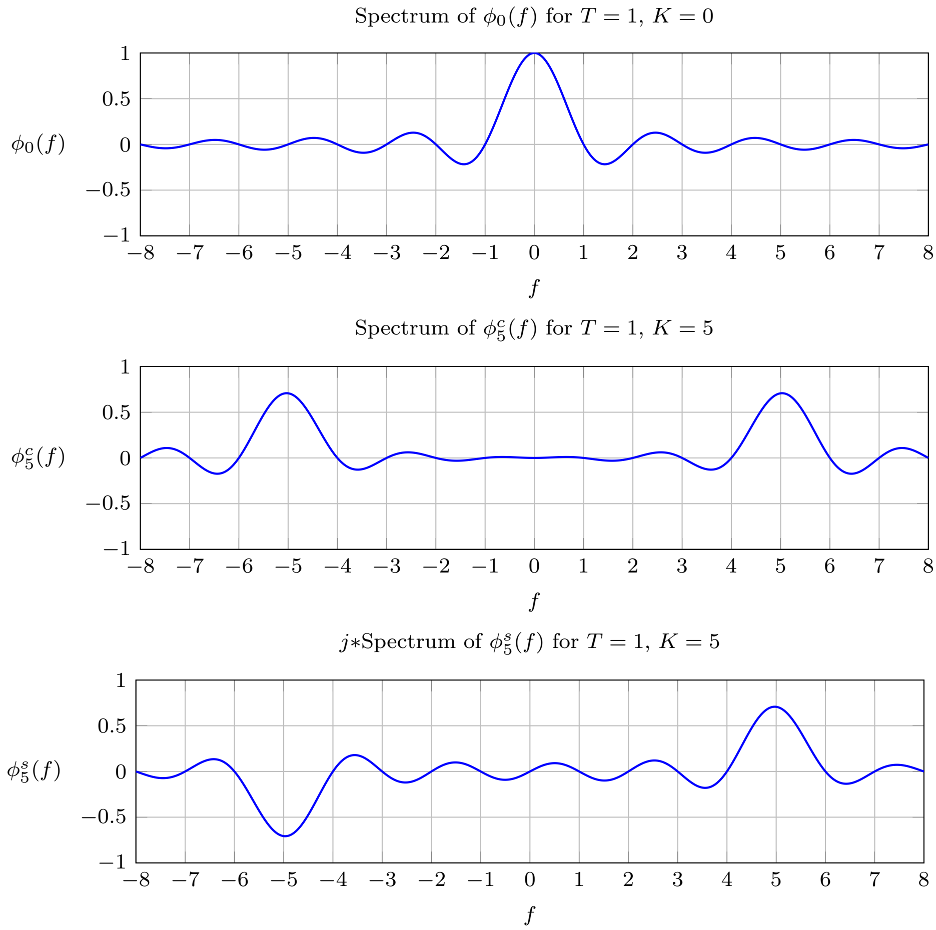

Figure 65: Fourrier spectra for $\phi_K^c(t)$ and $\phi_K^s(t)$, for $K = 5$ and $T = 1$ (Sol. 8.1).

```
clc; clear; close all
% Set the time period T
T = 1;
% Calculate Energy related to phi_0(f)
phi_0 = @(f) abs(sqrt(T)*sinc(f*T)).^2;   % ||phi_0(f)||^2
E_inside_0 = abs(integral(phi_0, -6, 6));
E_outside_0 = 1 - E_inside_0
% Choose the value of k in order to find phi_c_k(f) and phi_s_k(f)
k = 5;
% Find Energy related to phi_c_k(f)
fun_c = @(f) abs(sqrt(T/2)*(sinc(f*T - k)+sinc(f*T + k))).^2; % ||
    phi_c_k(f)||^2
E_inside_c = integral(fun_c, -6, 6);     % Find E_inside
E_outside_c = 1 - E_inside_c             % Caclulate E_outside
% Find Energy related to phi_s_k(f)
fun_s = @(f) abs(1/(1i)*sqrt(T/2)*(sinc(f*T - k)-sinc(f*T + k))).^2;  %
     ||phi_s_k(f)||^2
E_inside_s = integral(fun_s, -6, 6);
E_outside_s = 1 - E_inside_s
```

(d)

$$\frac{2K+1}{T} = \frac{51}{5} = 10.2 \text{ dim/sec} \tag{480}$$

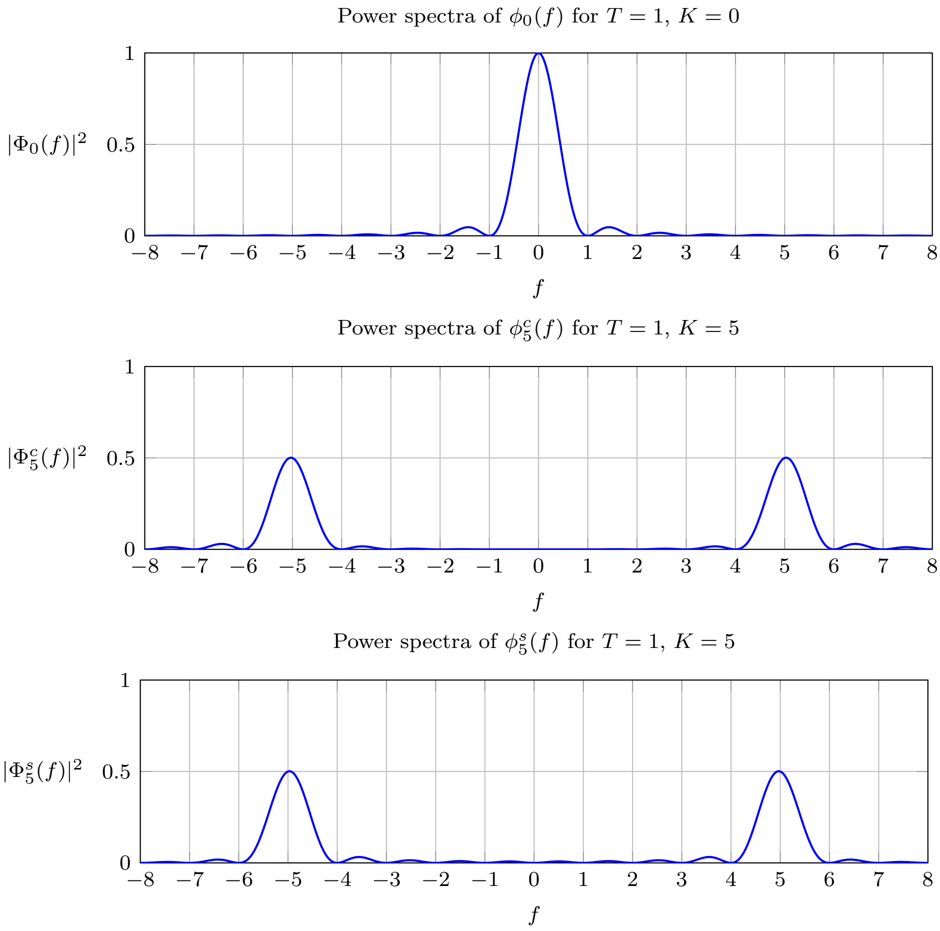

Figure 66: Power spectra for $\phi_K^c(t)$ and $\phi_K^s(t)$, for $K = 5$ and $T = 1$ (Sol. 8.1).

For the increased pulse duration $T = 5$, the minimum possible frequency that the building block waveform can carry is $1/T = 0.2$ Hz. Thus, the signal can be carried by the building block waveforms that have the smallest of 0.2 Hz difference between each other. For example, in this case $\phi_2^c(t)$ and $\phi_3^c(t)$ carry information at 0.4 Hz and 0.6 Hz respectively. This increase in dimensionality can be seen in Fig. 67, where it is clearly seen that the *sinc* of $\phi_0(f)$ has its first zeros at $+-0.2$ Hz.

(e) For $K = 25$ and $T = 5$:

	E_{outside}
ϕ_0	0.0034
ϕ_{25}^c	0.0159
ϕ_{25}^s	0.0062

See Fig. 68.

(f) We can see that the width of the sinc puls decreases for increased T. Therefore, the overlap between the waveforms in frequency domain will decrease for increased T. Furthermore, when K is increased the used frequency band is increased (when T is kept constant).

From the dimensionality theorem we would expect that $2WT$ dimensions are maximum possible per T seconds. For the bandwidth of $W = 6$ HZ this corresponds to maximum $12T$ dimensions. We can clearly see from our equations that indeed as soon as we cross this bound when for example we increase K to 6 and choose $T = 1$, the spectra of ϕ_K^c and ϕ_K^s go outside of the bandwidth.

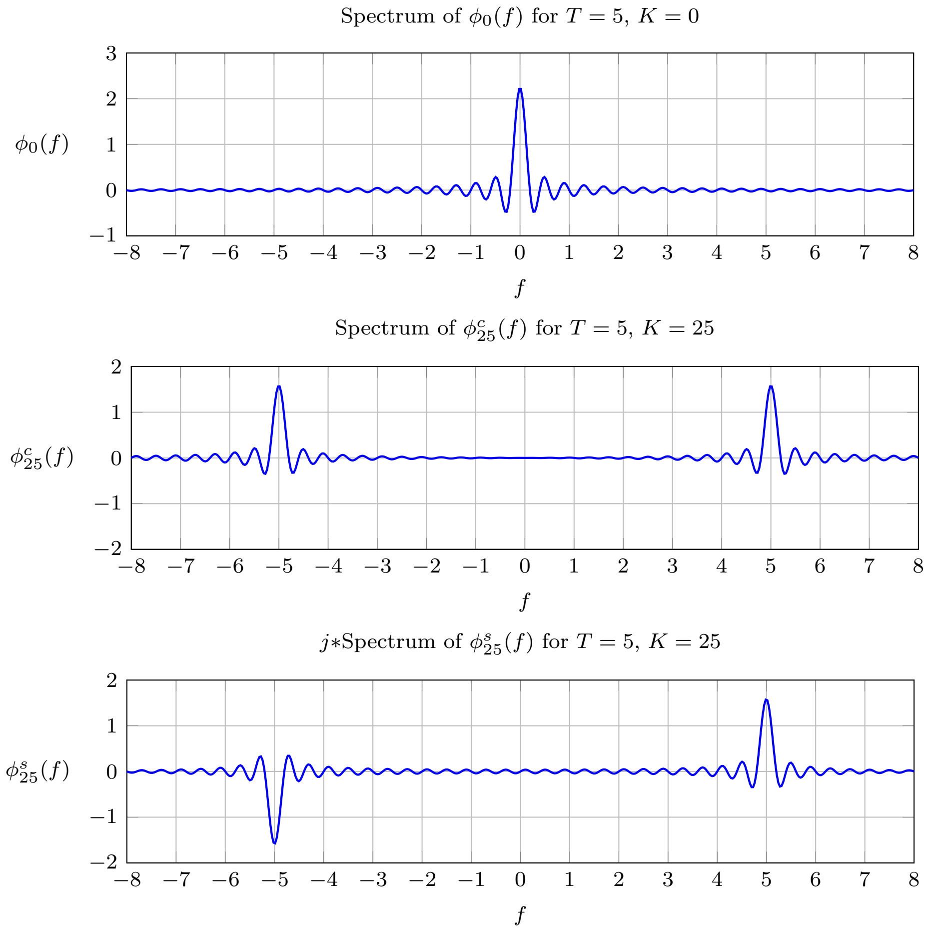

Figure 67: Fourrier spectra for $\phi^c_K(t)$ and $\phi^s_K(t)$, for $K = 25$ and $T = 5$ (Sol. 8.1).

Exercise 8.2: Consider the waveforms described in the previous exercise (Exercise 8.1). Show that all these waveforms form an orthonormal set of waveforms, and therefore can be used as building-block waveforms. You can assume here that $0 \le t < T$.

Solution: Consider $\phi_0(t) = \sqrt{\frac{1}{T}}$ and for integers $K \ge 1$ the functions $\phi^c_K(t) = \sqrt{\dfrac{2}{T}} \cos\left(2\pi K \frac{t}{T}\right)$ and $\phi^s_K(t) = \sqrt{\dfrac{2}{T}} \sin\left(2\pi K \frac{t}{T}\right)$ over $0 \le t < T$ beyound 0.
Note that

$$
\begin{aligned}
cos A \sin B &= \frac{\sin(A+B) - \sin(A+B)}{2}, \\
\cos A \cos B &= \frac{\cos(A+B) + \cos(A-B)}{2}, \\
\sin A \sin B &= \frac{\cos(A-B) - \cos(A+B)}{2}
\end{aligned}
$$

These are building blocks because

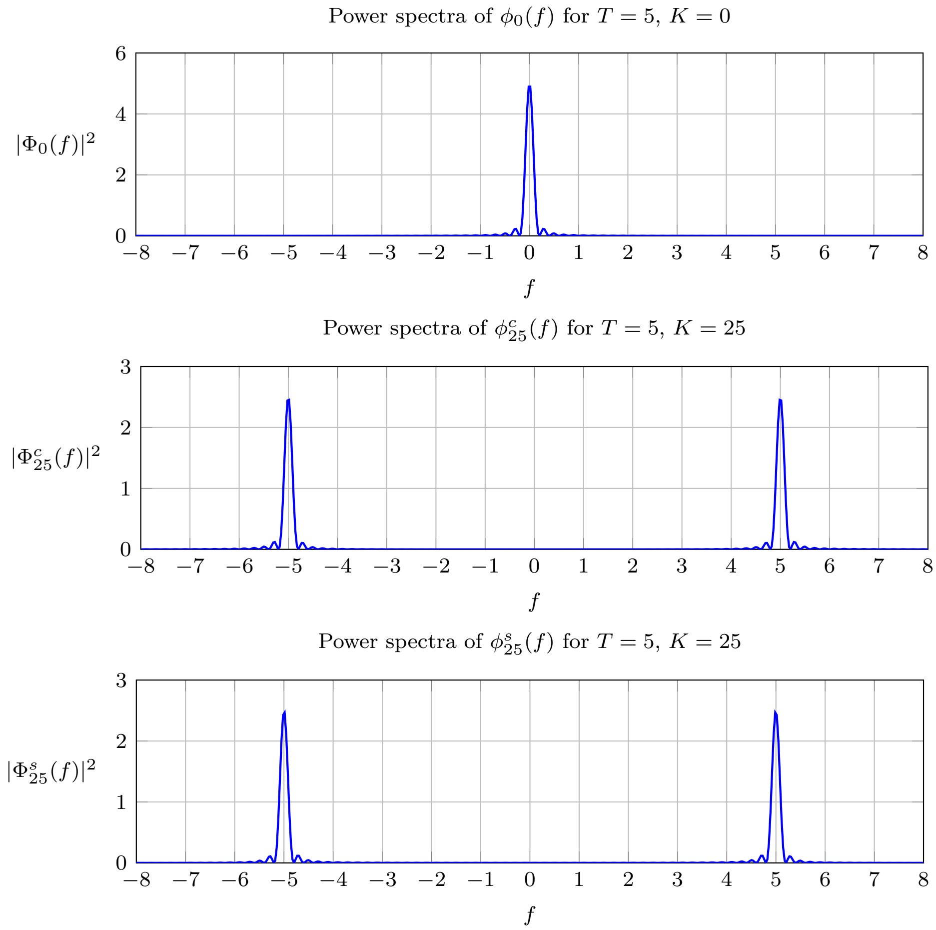

Figure 68: Power spectra for $\phi^c_K(t)$ and $\phi^s_K(t)$, for $K = 25$ and $T = 5$ (Sol. 8.1).

$$\int \phi_0^2(t)dt = \int_0^T \frac{1}{T}dt = 1 \tag{481}$$

$$\int \phi_0(t)\phi^c_K(t)dt = \int_0^T \frac{\sqrt{2}}{T}\cos\left(2\pi K\frac{t}{T}\right)dt = 0 \tag{482}$$

$$\int \phi_0(t)\phi^s_K(t)dt = \int_0^T \frac{\sqrt{2}}{T}\sin\left(2\pi K\frac{t}{T}\right)dt = 0 \tag{483}$$

Also

$$\int \phi^c_K(t)\phi^c_m(t)dt = \frac{1}{T}\int_0^T 2\cos\left(2\pi K\frac{t}{T}\right)\cos\left(2\pi m\frac{t}{T}\right)dt \tag{484}$$

$$= \frac{1}{T}\int_0^T \cos\left(2\pi(K-m)\frac{t}{T}\right)dt + \frac{1}{T}\int_0^T \cos\left(2\pi(K+m)\frac{t}{T}\right)dt \tag{485}$$

$$= \begin{cases} 1 & \text{if } K = m \\ 0 & \text{otherwise} \end{cases} \tag{486}$$

$$\int \phi_K^s(t)\phi_m^s(t)dt = \frac{1}{T}\int_0^T 2\sin\left(2\pi K\frac{t}{T}\right)\sin\left(2\pi m\frac{t}{T}\right)dt \tag{487}$$

$$= \frac{1}{T}\int_0^T \cos\left(2\pi(K-m)\frac{t}{T}\right)dt - \frac{1}{T}\int_0^T \cos\left(2\pi(K+m)\frac{t}{T}\right)dt \tag{488}$$

$$= \begin{cases} 1 & \text{if } K = m \\ 0 & \text{otherwise} \end{cases} \tag{489}$$

$$\int \phi_K^c(t)\phi_m^s(t)dt = \frac{1}{T}\int_0^T 2\cos\left(2\pi K\frac{t}{T}\right)\sin\left(2\pi m\frac{t}{T}\right)dt \tag{490}$$

$$= \frac{1}{T}\int_0^T \sin\left(2\pi(K-m)\frac{t}{T}\right)dt + \frac{1}{T}\int_0^T \sin\left(2\pi(K+m)\frac{t}{T}\right)dt \tag{491}$$

$$= 0 \tag{492}$$

Exercise 8.3:

"A": To transmit a message consisting of three binary digits we can use one of the eight vectors $\sqrt{E_s/3}(-1,-1,-1)$, $\sqrt{E_s/3}(-1,-1,+1)$, …, $\sqrt{E_s/3}(+1,+1,+1)$.

"B": To transmit a message containing two binary digits we can use one of the four vectors $\sqrt{E_s/3}(-1,-1,-1)$, $\sqrt{E_s/3}(-1,+1,+1)$, $\sqrt{E_s/3}(+1,-1,+1)$, $\sqrt{E_s/3}(+1,+1,-1)$.

(a) What are in both cases the distances between the vectors?

(b) Investigate the message error probabilities for both cases. We assume that $S_{N_w}(f) = \frac{N_0}{2}$ is the power spectral density of the additive noise. Use in both cases the union bound to find approximations for these error probabilities as a function of E_s/N_0. Plot these approximations.

(c) Demonstrate that the second method needs less energy per bit E_b to achieve the same error probability. Express the difference in E_b/N_0 in dB.

Solution:

(a) Case "A": 8 messages 3 bits, see Fig. 69. Also note that the factor $\sqrt{E_s/3}$ is not shown in the figure. Due to the cubic structure, there are only three different types of distances, the examples are

$$||\underline{s}_5 - \underline{s}_6|| = 2\sqrt{E_S/3}$$
$$||\underline{s}_5 - \underline{s}_2|| = \sqrt{8E_S/3}$$
$$||\underline{s}_5 - \underline{s}_4|| = \sqrt{12E_S/3}$$

Case "B": 4 messages 2 bits. The same Fig. 69 can be considered, but now only signal vectors $\underline{s}_1$, $\underline{s}_7$, $\underline{s}_6$ and $\underline{s}_4$ are taken into account. Thus, there is only one kind of distance between all these vectors, such as

$$||\underline{s}_1 - \underline{s}_7|| = \sqrt{8E_s/3} \tag{493}$$

(b) Error Probability: Because only nearest neighbors are considered for the error probability, in case "A" only distances $2\sqrt{E_S/3}$ are counted, while in case "B" the only distance is $\sqrt{8E_S/3}$. Thus, the argument of the Q-functions are

$$\text{"A"}:\ Q\left(\frac{d_A/2}{\sigma}\right) = Q\left(\frac{2\sqrt{E_S/3}}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_S}{3N_0}}\right)$$

$$\text{"B"}:\ Q\left(\frac{d_B/2}{\sigma}\right) = Q\left(\frac{\sqrt{8E_S/3}}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{4E_S}{3N_0}}\right).$$

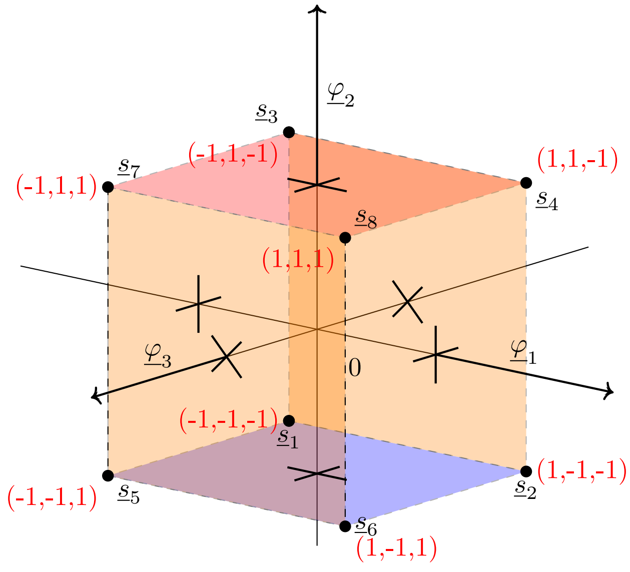

Figure 69: Bit-by-bit signal structure, solution 8.3.

Use the Union bound: $P_e = \sum_{m\in\mathcal{M}} \Pr\{M=m\}P_e^m = 3Q$. (Note that the exact error probability is $P_e = 1 - P_c = 1-(1-Q)^3 = 3Q - 3Q^2 + Q^3$, but the term $3Q$ is the largest).

$$\text{Case "A": } P_e \leq 3Q\left(\sqrt{\frac{2E_S}{3N_0}}\right), \tag{494}$$

$$\text{Case "B": } P_e \leq 3Q\left(\sqrt{\frac{4E_S}{3N_0}}\right), \tag{495}$$

From the above is clear that for the same value of E_s/N_0, the error probability in case "A" is larger than in case "B". In fact, case "A" requires 3dB (factor 2) more SNR for same P_e as case "B".

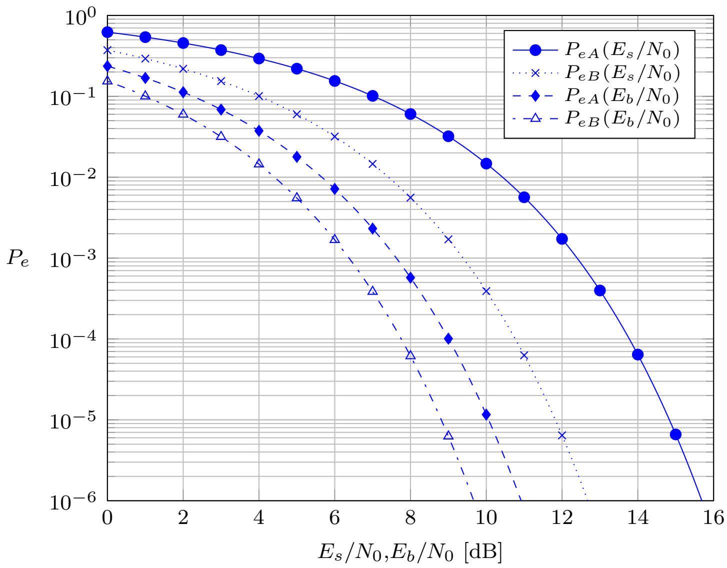

Figure 70: Approximated error probabilities (Sol. 8.3)

(c) The energy per bit E_b is calculated as follows:

$$E_b = \frac{E_s}{log_2|\mathcal{M}|}, \tag{496}$$

where $\mathcal{M}$ is the number of messages, so $\mathcal{M}_A = 8$ and $\mathcal{M}_B = 4$. Then the substitution of E_s from (496) into (495) results in

$$\text{Case "A": } P_e \leq 3Q\left(\sqrt{\frac{2E_b}{N_0}}\right), \tag{497}$$

$$\text{Case "B": } P_e \leq 3Q\left(\sqrt{\frac{8E_b}{3N_0}}\right). \tag{498}$$

Case "A" needs 4/3 times more SNR in E_b/N_0 to get same P_e as case "B", $10\log_{10}(4/3) = 1.25dB$.

9 Capacity of the Baseband and Wideband Channels

Exercise 9.1: Consider a normalized signal vector $\underline{s}'_m$ of dimension N with length $\sqrt{E_N}$. Each vector component is a random sample from a Gaussian density with mean 0 and variance E_N. We consider an AGN channel with variance $N_0/2$ for each dimension.

(a) Show that the expected squared length of the normalized received vector is not larger than $E_N + N_0/2$.

(b) Now use this fact to show that the number of vectors that can be reliably transmitted can not exceed $\left(\frac{E_N+N_0/2}{N_0/2}\right)^{N/2}$, i.e., the volume of the hypersphere containing the received vectors divided by the volume of a noise sphere.

Use that the volume of a hypersphere with radius R (in N dimensions) is given by

$$V_n(R) = \frac{\pi^{N/2}}{\Gamma(N/2+1)} R^N$$

from https://en.wikipedia.org/wiki/Volume_of_an_n-ball.

(c) The above argument gives an upper bound on the transmission rate. Therefore, what does this tell us about the capacity result (Result 9.1)?

Solution:

(a) The normalized received vector $\underline{s}'_m + \underline{n}'$ and thus the expected squared length of this vector is

$$\begin{aligned} E[||\underline{S}'_m + \underline{N}'||^2] &= E[||\underline{S}'_m||^2] + E[||\underline{N}'||^2] + 2E[||\underline{S}'_m \underline{N}'||] \\ &= \frac{1}{N}\sum_{i=1}^{N} E[S_i^2] + \frac{1}{N}\sum_{i=1}^{N} E[U_i^2] + \frac{2}{N}\sum_{i=1}^{N} E[S_i G_i] \\ &= E_N + N_0/2 + 0 \end{aligned}$$

Therefore, the expected length of the normalized received vector is equal to $\sqrt{E_N + N_0/2}$.

(b) Intuitively, for large enough N all the received vectors lie within a sphere of radius $\sqrt{E_N + N_0/2}$, with high probability. Now, a receiver can correctly decode the received vectors as long as the expected noise does not result in pushing the received vector outside of the correct decision region. Therefore, as long as the diameter of the decision regions are large enough (have radius at least $\sqrt{N_0/2}$), we expect a reliable transmission. Then, the maximum number of decision regions that can be created is given by the total available volume $V_n(\sqrt{E_N + N_0/2})$ divided by the volume for the (non-overlapping) decision regions $V_n(\sqrt{N_0/2})$.

(c) The capacity result (Result 9.1) tells us, that for N large enough, codes exists that can achieve a rate

$$R_N \triangleq \frac{\log_2 |\mathcal{M}|}{N} < C_N \triangleq \frac{1}{2}\log_2\left(\frac{E_N + \frac{N_0}{2}}{\frac{N_0}{2}}\right) \qquad \left[\frac{\text{bit}}{\text{dimension}}\right], \tag{499}$$

In the above derivation we have shown that the maximum achievable bit rate per dimension is equal to

$$\frac{1}{N}\frac{N}{2}\log_2 \frac{E_N + N_0/2}{N_0/2}. \tag{500}$$

Therefore,the achievable rate C_N of Result 9.1, actually achieves the upper bound (it is not possible to achieve a higher rate than this).

Exercise 9.2: A transmitter having power $P_s = 10$ W is connected to a receiver by two waveform channels (see Fig. 71). Each of these channels adds white Gaussian noise to its input waveform. The power densities of the two noise processes are $N_0^a/2 = 1$ and $N_0^b/2 = 2$ for $-\infty < f < \infty$, while the frequency bandwidth $W = 1$ Hz.

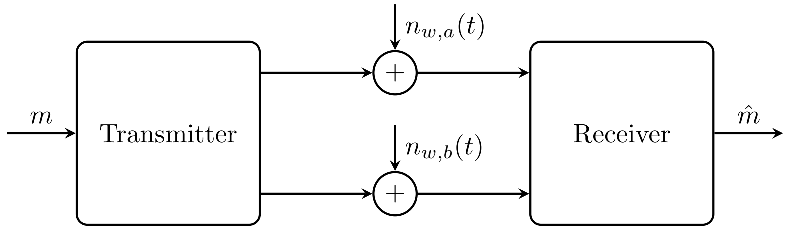

Figure 71: Water-filling situation (Ex. 9.2)

(a) What is the total capacity of this parallel channel in bit per second?

(b) What happens if $P_s = 1$ W?

The power allocation procedure that should be used here is called "water-filling".

Solution:

(a) The bandwidth $W = 1$ and spectral noise density $N_0/2$ are given, so it is a baseband (AWGN) channel. For such channel the capacity is given by

$$C = W log_2 \left(1 + \frac{P_s}{WN_0}\right) \frac{bit}{second}. \tag{501}$$

Then for the each channel the capacitance is

$$C_a = \log_2 \left(1 + \frac{P_a}{N_0^a}\right) \tag{502}$$

and

$$C_b = \log_2 \left(1 + \frac{P_b}{N_0^b}\right) \tag{503}$$

The total capacity of the two parallel channels can be found as

$$C = C_a + C_b = \log_2 \left[\left(1 + \frac{P_a}{N_0^a}\right)\left(1 + \frac{P_b}{N_0^b}\right)\right]. \tag{504}$$

Next, it is given that the transmitter in total transmits power P_s through the two channels, so $P_s = P_a + P_b$. In addition, it is assumed that $0 \leq P_a \leq P_s$. Therefore, by setting $P_b = P_s - Pa$, the capacity in terms of P_a is

$$C = \log_2 \left[\left(1 + \frac{P_a}{N_0^a}\right)\left(1 + \frac{P_s - P_a}{N_0^b}\right)\right]. \tag{505}$$

In general, we aim to allocate the available power P_s in such a way that it maximizes the total capacity of the parallel channel. Thus, in order to find the optimum P_a and P_b, we can solve $dC/dP_a = 0$ for optimum P_a^*.

Next, note that $\log_2$ in (505) is a non-decreasing function, so by maximizing the argument of $\log_2$, the capacitance C is also maximized. We can write the argument of the logarithm in (505) as

$$\left(\frac{N_0^a + P_a}{N_0^a}\right)\left(\frac{N_0^b + P_s - P_a}{N_0^b}\right) = \frac{N_0^a N_0^b + N_0^a P_s - N_0^a P_a + P_a N_0^b + P_a P_s - P_a^2}{N_0^a N_0^b} \tag{506}$$

Now, the expression in (506) can be maximized over $0 \leq P_a \leq P_s$, also see Fig. 72, by setting the derivative of (506) to 0,

$$\frac{d(506)}{dP_a} = -\frac{1}{N_0^b} + \frac{1}{N_0^a} + \frac{P_s}{N_0^a N_0^b} - \frac{2P_a^*}{N_0^a N_0^b} = 0. \tag{507}$$

$$-N_0^a + N_0^b + P_s - 2P_a^* = 0 \tag{508}$$

that results in

$$P_a^* = \frac{P_s + N_0^b - N_0^a}{2} = 6. \tag{509}$$

Using the same approach, by solving $dC/dP_b = 0$ for optimum P_b^*, the optimum power for channel b can be found,

$$P_b^* = \frac{P_s + N_0^a - N_0^b}{2} = 4. \tag{510}$$

Notice that if the noise in both channels would be equal, $N_0^a = N_0^b$, then the available power P_s would be distributed equally between the channels, $P_s/2$. However, the difference in the noise levels, $|N_0^b - N_0^a|$ results in uneven power distribution, with more power allocated to the channel with the least noise, i.e. channel a.

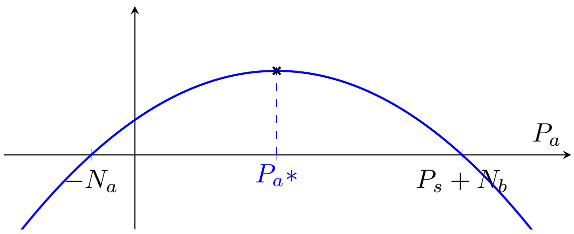

Figure 72: Expression in (506) as a function of P_a that is maximized at $P_a = P_a^*$ (Sol. 9.2).

We conclude that

$$C_a = \log_2 4 = 2 \tag{511}$$

$$C_b = \log_2 2 = 1 \tag{512}$$

$$C = C_a + C_b = 3 \tag{513}$$

Final remark:

From (509), since $N_0^b > N_0^a$, it follows that $P_a \geq 0$. Then we have $P_a^* \leq P_s$, i.e.

$$\frac{P_s + N_0^b - N_0^a}{2} \leq P_s \longleftrightarrow P_s \geq N_0^b - N_0^a. \tag{514}$$

Notice that now we have (see Fig. 73)

$$P_a^* + N_0^a = \underbrace{\frac{P_s}{2} + \underbrace{\frac{N_0^a + N_0^b}{2}}_{\text{average noise}}}_{\text{average power per channel}} = P_b^* + N_0^b \tag{515}$$

(b) Recall that $P_a^* \leq P_s$. However, now P_s is $0 \leq P_s < N_0^b - N_0^a$, so (514) becomes

$$P_s < \frac{P_s + N_0^b - N_0^a}{2} \tag{516}$$

Maximum is now achieved for $P_a = P_s$ and $P_b = 0$. See Figure 74 and 75.

In this case,

$$C = C_a = W \log_2\left(1 + \frac{P_a}{W N_0^a}\right) = \log_2(3) - 1.$$

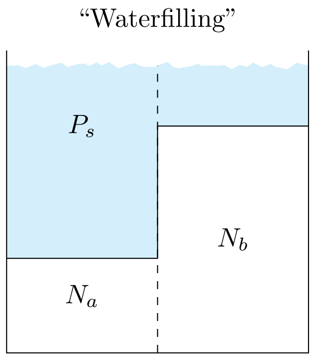

Figure 73: Waterfilling (Sol. 9.2)

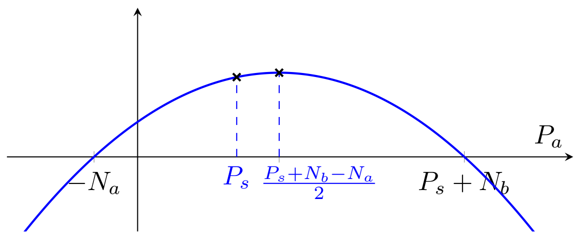

Figure 74: (Sol. 9.2)

Exercise 9.3: Find out whether a telephone line channel ($W = 3400$Hz, $C = 34000$ bit/sec) is power- or bandwidth-limited by determining its SNR. Also determine E_b/N_0 and R_N.

Solution: It is given that, for the telephone channel, $C = 34000$ bit/s and $W = 3400$ Hz.

$$C = W\log_2\left(1 + \frac{P_s}{N_0 W}\right) = W\log\left(1 + \text{SNR}\right) \tag{517}$$

then

$$\log_2(1 + \text{SNR}) = \frac{C}{W} = 10 \tag{518}$$

$$\implies \text{SNR} = 2^{10} - 1 = 1023 \tag{519}$$

Because of SNR >> 1, we are speaking of a bandwidth limited channel. Now, if we consider $R = C$, then

$$R_N = \frac{C}{2W} = 5 \text{ bit/dim} \tag{520}$$

Furthermore:

$$\frac{E_b}{N_0} = \frac{E_N/R_N}{N_0} \tag{521}$$

$$= \frac{1}{2R_N}\frac{E_N}{N_0/2} \tag{522}$$

$$= \frac{1}{2R_N}\text{SNR} \tag{523}$$

hence

“Waterfilling”

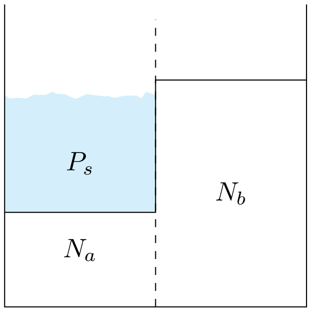

Figure 75: (Sol. 9.2)

$$\frac{E_b}{N_0} = \frac{1023}{2 \cdot 5} = 102.3 \tag{524}$$

Notice that $\left.\frac{E_b}{N_0}\right|_{\text{minimal}} = \ln 2$, which is much smaller than 102.3.

Exercise 9.4: For a fixed ratio E_b/N_0, for some values of R/W reliable transmission is possible, for other values of R/W this is not possible. Therefore the $E_b/N_0 \times R/W$ - plane can be divided in a region in which reliable transmission is possible and a region where this is not possible. Find out what function of R/W describes the boundary between these two regions. (The ratio R/W is sometimes called the bandwidth efficiency.)

Solution:

$$R < W \log_2 \left(1 + \frac{P_s}{N_0 W}\right) = C \text{ bit/s}. \tag{525}$$

Because

$$\frac{P_s}{N_0 W} = \frac{R}{W}\frac{E_b}{N_0}, \tag{526}$$

we get

$$\frac{R}{W} < \log_2 \left(1 + \frac{R}{W}\frac{E_b}{N_0}\right) \tag{527}$$

that can be raised to the power of 2 to remove $\log_2$,

$$\frac{2^{R/W} - 1}{R/W} < \frac{E_b}{N_0} \tag{528}$$

See Fig. 76 for a plot of this function.

Note that for

$$\lim_{R/W \to 0} \frac{2^{(R/W)} - 1}{R/W} = \lim_{R/W \to 0} \frac{2^{(R/W)} \ln(2)}{1} = \ln 2, \tag{529}$$

where we utilized L’Hospital’s Rule,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \tag{530}$$

Exercise 9.5: A receiver is connected to a transmitter by two channels, a bandlimited channel and a wideband channel, see Fig. 77. The transmitter’s signalling power $P_s = 90$ W. The transmitter can use all its power to communicate only via the bandlimited channel, only via the wideband channel, or it may distribute its power over the bandlimited channel and the wideband channel.

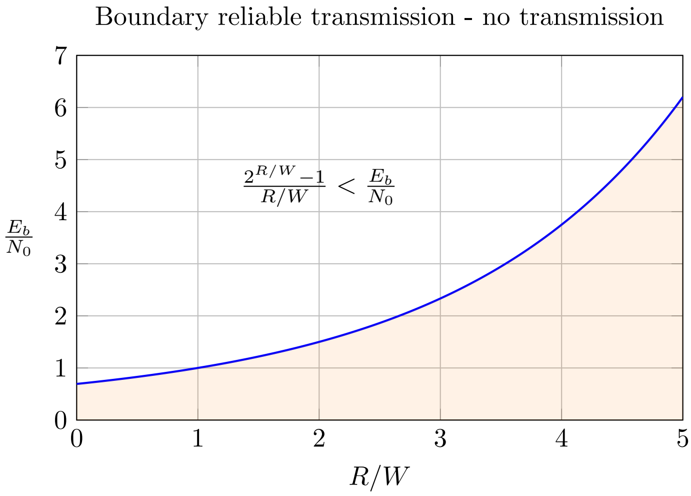

Figure 76: (Sol. 9.4)

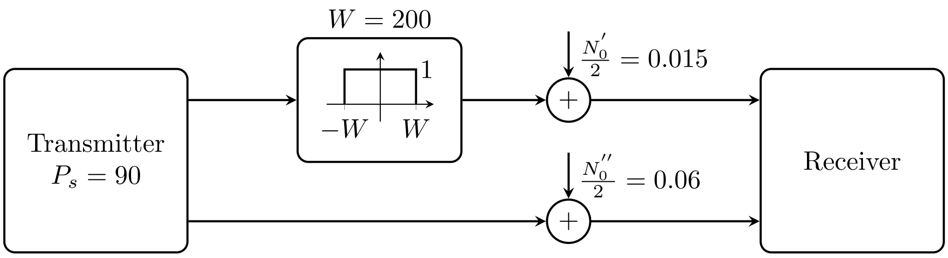

Figure 77: Transmitter, two parallel channels (bandlimited and wideband channel), and receiver (Ex. 9.5).

(a) The bandlimited channel has bandwidth $W = 200$Hz. The power spectral density of the noise in this channel $N_0'/2 = 0.015$ (W/Hz). Assume that the transmitter uses all of its power to communicate over the bandlimited channel. First determine the capacity C_N' of this channel in bits per dimension. Then determine the capacity C' of the bandlimited channel in bits per second.

(b) The spectral density of the noise in the wideband channel $N_0''/2 = 0.06$ (W/Hz). Assume now that the transmitter only communicates via the wideband channel to the receiver. What is the capacity C'' of this wideband channel in bits per second?

(c) How should the transmitter split up its power over the bandlimited channel and the wideband channel to achieve the maximum total capacity in bits per second? How large is this maximum total capacity C_{tot}?

Solution:

(a) (See page 100 of the reader)

The capacitance of the bandlimited channel is

$$\begin{aligned} C_N' =& \frac{1}{2}\log_2\left(1 + \frac{P_S}{N_0'W}\right), \\ =& \frac{1}{2}\log_2\left(1 + \frac{90}{0.03 \times 200}\right), \\ =& 2 \text{ bits/dim.} \end{aligned}$$

Therefore,

$$C' = 2WC_N' = 400 \times 2 = 800 \text{ bits/sec.} \tag{531}$$

(b) (See page 101 of the reader)

For the wideband channel, the capacitance is

$$C'' = \frac{P_S}{N_0'' \ln 2} = \frac{90}{0.12 \times \ln 2} = 1082 \text{ bits/sec.} \tag{532}$$

Note that the capacity per dimension is 0!

(c) αP_S is the power in the band-limited (baseband) channel. Then

$$C(\alpha) = W \log_2 \left(\frac{\alpha P_S + N_0' W}{N_0' W} \right) + \frac{(1-\alpha) P_S}{N_0'' \ln 2} \tag{533}$$

$$\frac{dC}{d\alpha} = \frac{W}{\ln 2} \frac{P_S}{\alpha P_S + N_0' W} - \frac{P_S}{N_0'' \ln 2} = 0. \tag{534}$$

Then

$$\begin{aligned} \frac{W}{\ln 2} \frac{P_S}{\alpha P_S + N_0' W} =& \frac{P_S}{N_0'' \ln 2}, & (535) \\ \alpha P_S + N_0'' W =& N_0'' W, & (536) \\ \alpha P_S =& W \left(N_0'' - N_0' \right), & (537) \\ =& 18. & (538) \end{aligned}$$

Therefore

$$C_{\text{tot}} = 200 \log_2 \left(1 + \frac{18}{0.03 \times 200} \right) + \frac{72}{0.12 \ln 2} = 1266 \text{ bits/sec.} \tag{539}$$

Exercise 9.6: Consider two AWGN channels, a baseband channel (Channel A) and a wideband channel (Channel B), see Fig. 78. The noise power spectral densities of $n_w^A(t)$ and $n_w^B(t)$ are $N_0^A/2 = 0.2$ W/Hz and $N_0^B/2 = 0.8$ W/Hz respectively, for all frequencies f. For the bandwidth W of the baseband channel we have that $W = 100$ Hz, see Fig. 78.

For all sub-questions listed below we have that the (total) transmit power $P = 280$ W.

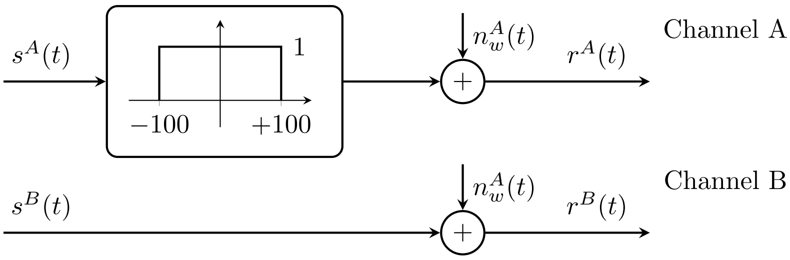

Figure 78: A baseband channel (bandwidth $W = 100$ Hz, noise power spectral density $N_0^A/2 = 0.2$ W/Hz) and a wideband channel (noise power spectral density $N_0^B/2 = 0.8$ W/Hz) (Ex. 9.6).

(a) Determine the capacity of the baseband channel (channel A) for transmit power $P = 280$ W in bit per second. What is the capacity in bit per dimension?

(b) Determine the capacity of the wideband channel (channel B) for transmit power $P = 280$ W in bit per second. What is the capacity in bit per dimension?

(c) Find the "in-parallel" capacity of the link when channel A and B are used in parallel, see Fig. 79. Note that the total transmit power $P = 280$ W has to be divided over both the channels A and B.

(d) Now the two channels are used in series, see Fig. 80. The total transmit power $P = 280$ W again has to be divided over both channels. Assume that $P^A = 40$ W is the transmit power chosen for baseband channel A and $P^B = 240$ W is then the remaining transmit power for wideband channel B. What is the capacity of this "in-series" link now?

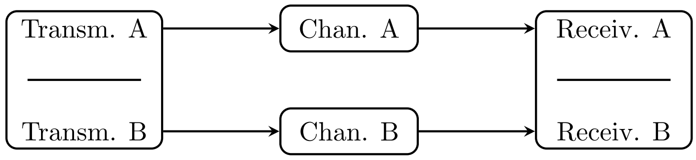

Figure 79: A communication link consisting of our two channels used in parallel (Ex. 9.6).

Figure 80: A communication link consisting of our two channels used in series (Ex. 9.6).

Solution:

(a)

$$\begin{aligned} C^A &= W\log_2\left(1+\frac{P}{N_0^A W}\right) \tag{540}\\ &= 100\log_2\left(1+\frac{280}{0.4\cdot 100}\right) \tag{541}\\ &= 100\log_2(8) \tag{542}\\ &= 300 \text{ bits/sec} \tag{543}\end{aligned}$$

$$C_N^A = \frac{C^A}{2W} = \frac{3}{2} \text{ bits/dim} \tag{544}$$

(b)

$$\begin{aligned} C^B &= \frac{P}{\ln 2 N_0^B} \tag{545}\\ &= \frac{280}{1.6\ln 2} \tag{546}\\ &= 252.47 \text{ bits/sec} \tag{547}\end{aligned}$$

$$\#\ \text{dim/sec} \ = \infty \text{ then } C_N^B = 0 \text{ bits/dim}$$

(c)

$$C_{\text{par}} = W\log_2\left(1+\frac{\alpha P}{N_0^A W}\right) + \frac{(1-\alpha)P}{N_0^B \ln 2} \tag{548}$$

but we have that

$$\begin{aligned} \frac{W}{N_0^A W+\alpha P}\frac{P}{\ln 2} + \frac{-P}{N_0^B \ln 2} &= 0 \tag{549}\\ \frac{W}{N_0^A W+\alpha P} &= \frac{1}{N_0^B} \tag{550}\\ N_0^B W &= N_0^A W + \alpha P \tag{551}\\ \alpha P &= (N_0^B - N_0^A)W \tag{552}\\ &= (1.6-0.4)100 \tag{553}\\ &= 120 \tag{554}\end{aligned}$$

Therefore

$$C_{\text{par}} = 100 \log_2 \left(1 + \frac{120}{40}\right) + \frac{160}{1.6 \ln 2} \quad (555)$$
$$= 200 + 144.27 \quad (556)$$
$$= 344.27 \text{ bits/sec} \quad (557)$$

(d) $P^A = 40$ W, then $P^B = 240$ W. Now we have

$$C^A = 100 \log_2 \left(1 + \frac{40}{0.4 \cdot 100}\right) = 100 \text{ bits/sec} \quad (558)$$

and

$$C^B = \frac{240}{1.6 \ln 2} = 216.4 \text{ bits/sec} \quad (559)$$

$$C_{\text{ser}} = \min(C^A, C^B) = 100 \text{ bits/sec} \quad (560)$$

Observe that $\alpha = 0.3237$ gives

$$W \log_2 \left(1 + \frac{\alpha P}{N_0^A W}\right) = \frac{(1-\alpha)P}{N_0^B \ln 2} = 170.75 \text{ bits/sec} \quad (561)$$

10 Pulse Transmission

Exercise 10.1: Consider the pulse $p(t)$ with spectrum $P(f)$ as in Fig. 81. We want to apply this pulse in a pulse-amplitude-modulation system. We are now interested in finding out whether there

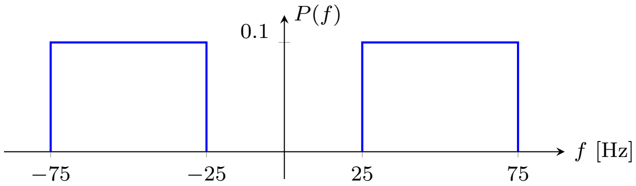

Figure 81: The frequency-spectrum $P(f)$ of the pulse $p(t)$ (Ex. 10.1).

exists a T such that this pulse and all its shifted versions form an orthonormal base (set of building-block waveforms), i.e., for integers k

$$\int_{-\infty}^{\infty} p(t)p(t-kT)dt = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

(a) Check first that pulse $p(t)$ has unit energy. Use Parseval.

(b) Find out (in a graphical manner) whether the pulse-spectrum satisfies the Nyquist criterion for zero intersymbol interference for $T = 0.02$ sec.

(c) Assume that the pulse-spectrum satisfies the Nyquist criterion for zero intersymbol interference for $T = \alpha$. Is this also true then for $T = 2\alpha$? Why?

Solution:

(a)

$$\begin{aligned} E_p &= \int p^2(t)dt && (562) \\ &= \int ||P(f)||^2 df && (563) \\ &= \int_{-75}^{-25} 0.1^2 df + \int_{25}^{75} 0.1^2 df && (564) \\ &= 50 \cdot 0.1^2 + 50 \cdot 0.1^2 && (565) \\ &= 1 && (566) \end{aligned}$$

(b) Consider $H(f) = |P(f)|^2$, which is shown in Fig. 82.

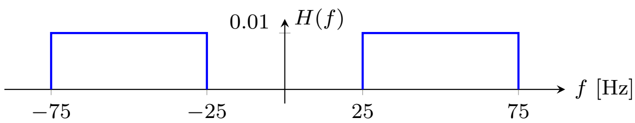

Figure 82: The frequency-spectrum $H(f)$ (Ex. 10.1).

We want to know if the following expression is true

$$Z(f) = \frac{1}{T} \sum_{m=-\infty}^{\infty} H\left(f + \frac{m}{T}\right) = 1. \qquad (567)$$

As $T = 0.02$ s, we have $1/T = 50$ Hz, resulting in

$$Z(f) = 50 \sum_{m=-\infty}^{\infty} H\left(f + 50m\right), \tag{568}$$

which we can graphically represent in Fig. 83.

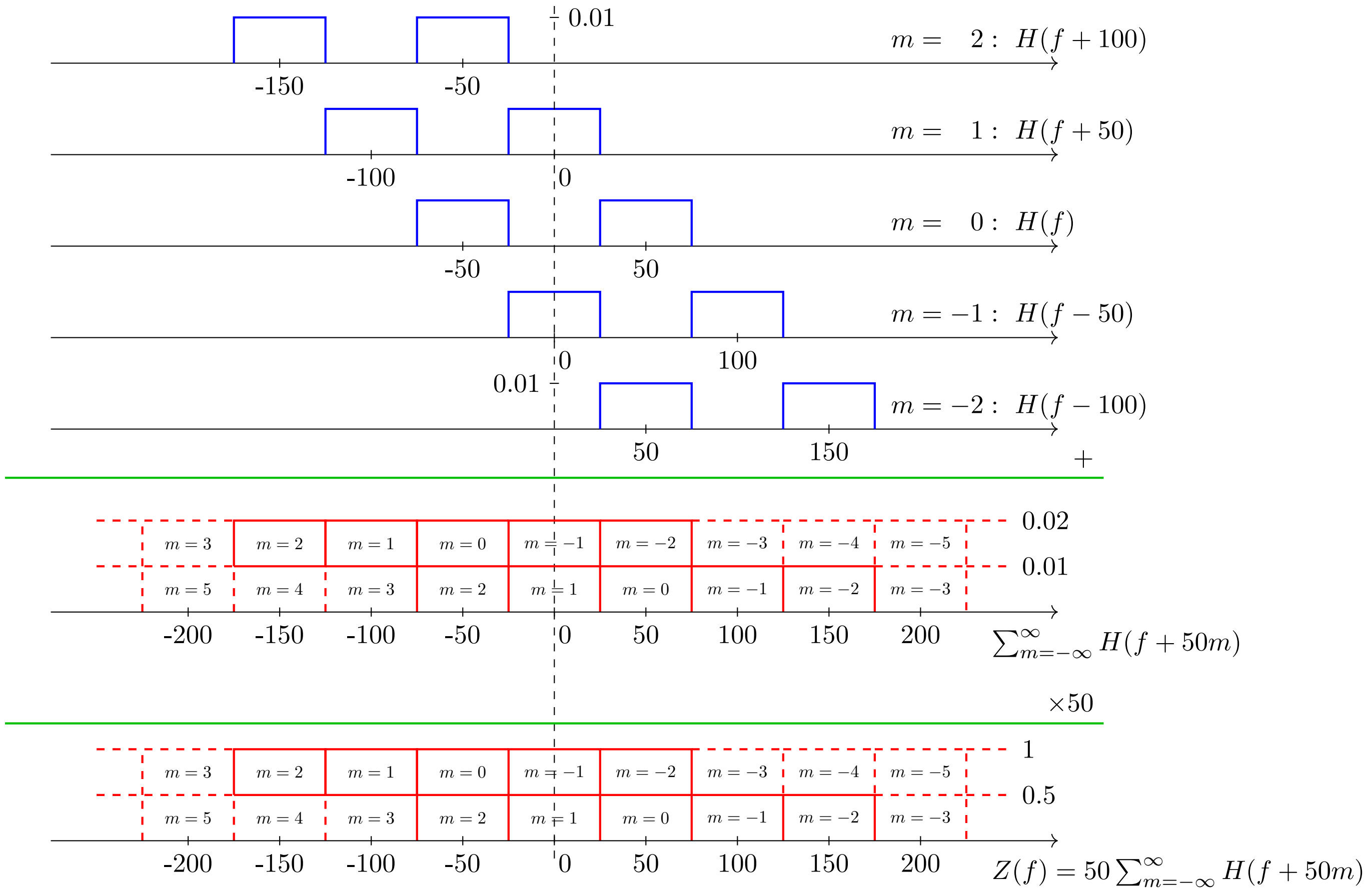

Figure 83: Graphical representation of the Nyquist criterion for zero intersymbol interference (Sol. 10.1).

From Fig. 83, it is clear that the spectrum of $Z(f)$ is equal to 1 for all frequencies f. This means that the pulse-spectrum $P(f)$ satisfies the Nyquist criterion for zero intersymbol interference (ISI) for $T = 0.02$ sec.

(c) For $T = \alpha$ we have

$$\int p(t)p(t - k\alpha)dt = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0, \end{cases} \tag{569}$$

where $p(t)$ is a *sinc* pulse.

Now, if we make $k = 2m$, we have that

$$\int p(t)p(t - 2m\alpha)dt = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases} \tag{570}$$

In other words:

$$\int p(t)p(t - mT)dt = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases} \tag{571}$$

for $T = 2\alpha$. If $T = \alpha$ satisfies the Nyquist criterion, then $T = 2\alpha$ also satisfies the Nyquist criterion.

Exercise 10.2: Consider the two pulses

$$
\begin{aligned}
p_1(t) &= \frac{\sin(\pi t)}{\pi t} \\
p_2(t) &= \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t}.
\end{aligned} \tag{572}
$$

(a) Determine the Fourier spectra $P_1(f)$ and $P_2(f)$ of these pulses.

(b) Show that the set of all shifted versions of these two pulses $\{\ldots, p_1(t+1), p_2(t+1), p_1(t), p_2(t), p_1(t-1), p_2(t-1), p_1(t-2), p_2(t-2), \ldots\}$ is a set of building blocks (energy 1, orthogonal). Shifts are over multiples of 1 second.

We conclude that with these two pulses, **multi-pulse transmission** is possible, hence

$$
s(t) = \sum_{k=0}^{K-1} a_{k1} p_1(t-k) + a_{k2} p_2(t-k)
$$

is a linear combination of shifted versions of **two** pulses.

Solution:

(a) Let us consider a pulse

$$
p(t) = \frac{1}{\sqrt{T}} \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}}, \tag{573}
$$

and its transform

$$
P(f) = \begin{cases} \sqrt{T}, & \text{if } |f| \leq \frac{1}{2T} \\ 0, & \text{if } |f| > \frac{1}{2T}. \end{cases} \tag{574}
$$

If $T = 1$, then $p(t)$ is equivalent to pulse $p_1(t)$, and has the transform as illustrated in Fig. 84.

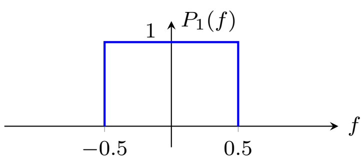

Figure 84: Transform of $p(t)$ when $T = 1$ (Sol. 10.2).

If $T = 0.5$, then

$$
p_{T=0.5}(t) = \frac{1}{\sqrt{2}} \frac{\sin(2\pi t)}{\pi t} \tag{575}
$$

that has the transform as illustrated in Fig. 85.

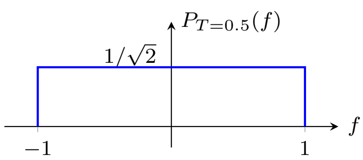

Figure 85: Transform of $p(t)$ when $T = 0.5$ (Sol. 10.2).

Pulse $p_2(t)$ combines the pulses $p(t)$ for $T = 1$ and for $T = 0.5$,

$$p_2(t) = \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t} = \sqrt{2} \cdot p_{T=0.5}(t) - p_1(t), \tag{576}$$

and has the corresponding transform

$$P_2(f) = \sqrt{2} \cdot P_{T=0.5}(f) - P_1(f) \tag{577}$$

as shown in Fig. 86.

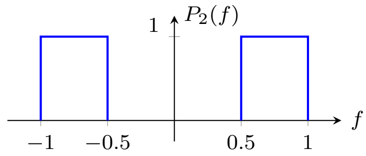

Figure 86: Transform of $p_2(t)$ (Sol. 10.2).

Therefore, $p_1(t) + p_2(t)$ has the transform as illustrated in Fig. 87.

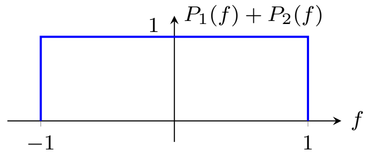

Figure 87: Transform of $p_1(t) + p_2(t)$ (Sol. 10.2).

(b) Note that

$$H_1(f) = P_1(f)P_1^*(f) = 1 \text{ for } |f| \le 1/2, \tag{578}$$
$$H_2(f) = P_2(f)P_2^*(f) = 1 \text{ for } 1/2 < |f| \le 1. \tag{579}$$

We determine the first graph (see Fig. 88),

$$Z_1(f) = \sum_{m=-\infty}^{\infty} H_1\left(f + \frac{m}{T}\right) \text{ where } T = 1. \tag{580}$$

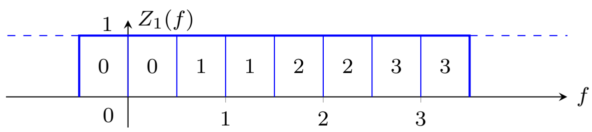

Figure 88: Graph of $Z_1(f)$ (Sol. 10.2).

Therefore $Z_1(f) = 1$ and pulse $p_1(t)$ satisfies the Nyquist criteria, i.e. for $T = 1$ the pulses do not overlap (we can use Parseval's theorem):

$$\begin{aligned}
\int p_1(t)p_1(t-kT)dt = &\int P_1(f)P_1^*(f)e^{j2\pi fk}df, &(581)\\
=&\int_{-1}^{-1/2} e^{j2\pi fk}df + \int_{1/2}^{1} e^{j2\pi fk}df, &(582)\\
=&\int_{0}^{1/2} e^{j2\pi fk}df + \int_{-1/2}^{0} e^{j2\pi fk}df, &(583)\\
=&\int_{-1/2}^{1/2} e^{j2\pi fk}df, &(584)\\
=&\left[\frac{e^{j2\pi fk}}{j2\pi k}\right]_{f=-1/2}^{f=1/2} &(585)\\
=&\frac{sin(\pi k)}{\pi k} &(586)\\
=&0, \text{when } k \neq 0 &(587)\\
=&1, \text{when } k = 0, \text{ using L'Hôpital's rule.} &(588)
\end{aligned}$$

Overall, the Nyquist criteria is satisfied for $T = 1$,

$$\int p_1(t)p_1(t-kT)dt = \begin{cases} 0, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases}. \tag{589}$$

For $p_2(t)$, we also determine (see Fig 89),

$$Z_2(f) = \sum_{m=-\infty}^{\infty} H_2\left(f + \frac{m}{T}\right) \text{ where } T = 1. \tag{590}$$

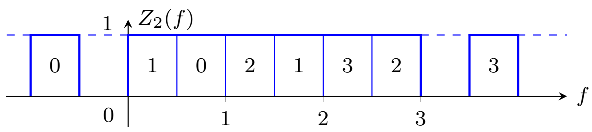

Figure 89: Graph of $Z_2(f)$ (Sol. 10.2).

Therefore $Z_2(f) = 1$ and pulse $p_2(t)$ satisfies the Nyquist criteria, i.e. for $T = 1$ the pulses do not overlap:

$$\int p_2(t)p_2(t-kT)dt = \begin{cases} 0, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases}. \tag{591}$$

We also need to check for orthogonality of the two pulses:

$$\int p_1(t)p_2(t-k)dt = \int P_1(f)P_2^*(f)e^{j2\pi fk}df = 0, \tag{592}$$

which can easily be proven by observing that $P_1(f)$ and $P_2(f)$ are not overlapping.

Exercise 10.3: The "square-root raised cosine pulse" $p(t)$ has spectrum

$$P(f) = \begin{cases} \sqrt{T}\cos\left(\frac{\pi f T}{2}\right) & \text{for } -\frac{1}{T} \leq f \leq \frac{1}{T}, \\ 0 & \text{everywhere else.} \end{cases}.$$

(a) Check whether $p(t)$ has unit energy.

(b) Show that $p(t)$ satisfies he Nyquist criterion for modulation-interval size T.

(c) Demonstrate that $p(t)$ "has a shorter tail" than the sinc-pulse corresponding to T.

Solution:

(a)

$$\begin{aligned}\int p^2(t)dt &= \int ||P(f)||^2 df && (593)\\ &= \int_{-\frac{1}{T}}^{+\frac{1}{T}} T\cos^2\left(\frac{\pi f T}{2}\right) df && (594)\\ &= T\int_{-\frac{1}{T}}^{+\frac{1}{T}} \frac{1+\cos(\pi f T)}{2} df && (595)\\ &= T\cdot\frac{2}{T}\cdot\frac{1}{2} && (596)\\ &= 1 && (597)\end{aligned}$$

then $p(t)$ has unit energy.

(b)

$$H(f) = ||P(f)||^2 = T\cos^2\left(\frac{\pi f T}{2}\right) \quad (598)$$

for $-\frac{1}{T} \leq f \leq \frac{1}{T}$. Notice that

$$\begin{aligned}H\left(f-\frac{1}{T}\right) &= T\cos^2\left(\frac{\pi\left(f-\frac{1}{T}\right)T}{2}\right) && (599)\\ &= T\sin^2\left(\frac{\pi f T}{2}\right) && (600)\end{aligned}$$

Now

$$Z(f) = \frac{1}{T}\sum_{m=-\infty}^{\infty} H\left(f-\frac{m}{T}\right) \quad (601)$$

Due to the symmetry, let us consider only f such that $0 \leq f \leq \frac{1}{T}$, then the sum has two overlapping terms: $H(f)$ and $H\left(f-\frac{1}{T}\right)$.

Now

$$\begin{aligned}Z(f) &= \frac{1}{T}\left[H(f) + H\left(f-\frac{1}{T}\right)\right] && (602)\\ &= \frac{1}{T}\left[T\cos^2\left(\frac{\pi f T}{2}\right) + T\sin^2\left(\frac{\pi f T}{2}\right)\right] && (603)\\ &= 1 && (604)\end{aligned}$$

Thus, the Nyquist criterion is satisfied.

(c) Now we determine $p(t)$.

$$p(t) = \int_{-\frac{1}{T}}^{\frac{1}{T}} \sqrt{T}\cos\left(\frac{\pi f T}{2}\right) e^{j2\pi f t} df \tag{605}$$

$$= \sqrt{T}\int_{-\frac{1}{T}}^{\frac{1}{T}} \cos\left(\frac{\pi f T}{2}\right)\cos(2\pi f t)df + j\sqrt{T}\underbrace{\int_{-\frac{1}{T}}^{\frac{1}{T}} \cos\left(\frac{\pi f T}{2}\right)\sin(2\pi f t)df}_{=0} \tag{606}$$

$$= \frac{\sqrt{T}}{2}\int_{-\frac{1}{T}}^{\frac{1}{T}} \cos\left(2\pi f\left(t-\frac{T}{4}\right)\right) + \cos\left(2\pi f\left(t+\frac{T}{4}\right)\right) df \tag{607}$$

$$= \frac{\sqrt{T}}{2}\left[\frac{\sin\left(2\pi f\left(t-\frac{T}{4}\right)\right)}{2\pi\left(t-\frac{T}{4}\right)} + \frac{\sin\left(2\pi f\left(t+\frac{T}{4}\right)\right)}{2\pi\left(t+\frac{T}{4}\right)}\right]\Bigg|_{-\frac{1}{T}}^{+\frac{1}{T}} \tag{608}$$

$$= \frac{\sqrt{T}}{2}\left[\frac{\sin\left(2\pi\frac{\left(t-\frac{T}{4}\right)}{T}\right)}{2\pi\left(t-\frac{T}{4}\right)} + \frac{\sin\left(2\pi\frac{\left(t+\frac{T}{4}\right)}{T}\right)}{2\pi\left(t+\frac{T}{4}\right)}\right] \tag{609}$$

$$= \sqrt{T}\left[-\frac{\cos\left(2\pi\frac{t}{T}\right)}{2\pi\left(t-\frac{T}{4}\right)} + \frac{\cos\left(2\pi\frac{t}{T}\right)}{2\pi\left(t+\frac{T}{4}\right)}\right], \text{ we used } \cos(t) = \sin(t+\pi/2) \tag{610}$$

$$= \frac{\sqrt{T}\cos\left(2\pi\frac{t}{T}\right)}{2\pi}\left(\frac{1}{t+\frac{T}{4}} - \frac{1}{t-\frac{T}{4}}\right) \tag{611}$$

$$= \frac{\sqrt{T}\cos\left(2\pi\frac{t}{T}\right)}{2\pi}\cdot\frac{-\frac{T}{2}}{t^2-\left(\frac{T}{4}\right)^2} \tag{612}$$

$$= \frac{T\sqrt{T}}{4\pi}\cdot\cos\left(2\pi\frac{t}{T}\right)\cdot\frac{-1}{t^2-\left(\frac{T}{4}\right)^2} \tag{613}$$

For $t = 0$ we have

$$p(t) = \frac{4}{\pi\sqrt{T}} \tag{614}$$

For $t \to \infty$

$$|p(t)| = \frac{c}{t^2} \tag{615}$$

For the sinc pulse

$$p'(t) = \frac{1}{\sqrt{T}}\frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} \tag{616}$$

it applies that $p'(0) = \frac{1}{\sqrt{T}}$ and for $t \to \infty$ we have $|p'(t)| = \frac{c'}{t}$. Therefore the sinc pulse decreases slower with time than the ”squared-root raised cosine pulse”.

Exercise 10.4: Consider a sinc-pulse $p_1(t)$ with spectrum $P_1(f)$ and a second sinc-pulse $p_2(t)$ with spectrum $P_2(f)$. The rectangular spectra $P_1(f)$ and $P_2(f)$ are shown in Fig. 91.

A third pulse is defined as

$$p_3(t) = \left(\sqrt{2/3} - \sqrt{1/3}\right)p_1(t) + \left(\sqrt{2/3}\right)p_2(t).$$

We want to use these pulses in pulse-amplitude-modulation systems. Therefore we would like to know, for which values of T, shifted versions over multiples of T, of a pulse form an orthonormal basis, i.e., a set of building-block waveforms. Then, for integer k we have that

$$\int_{-\infty}^{\infty} p(t)p(t-kT)dt = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for all } k \neq 0, \end{cases}$$

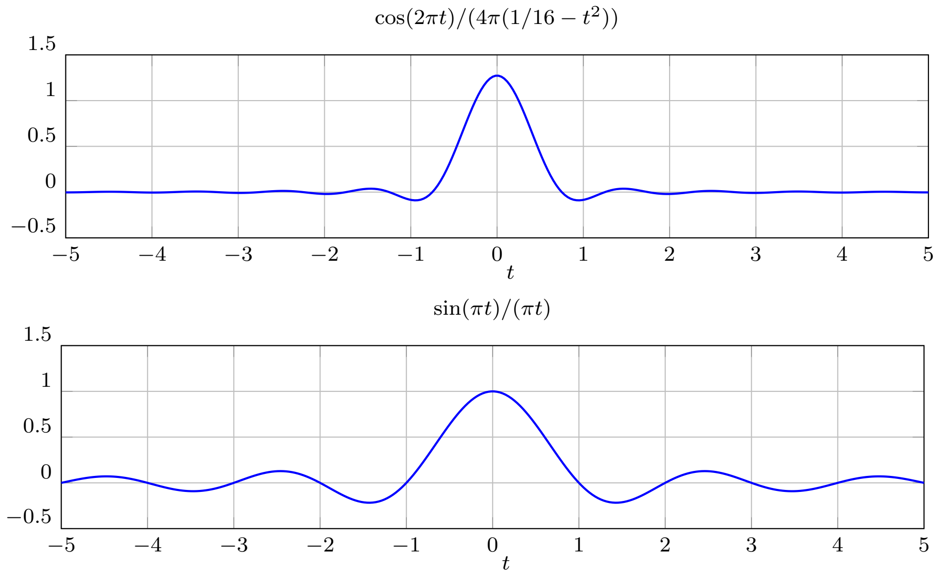

Figure 90: Plot of sinc pulse vs "squared-root raised cosine pulse" (Sol. 10.3).

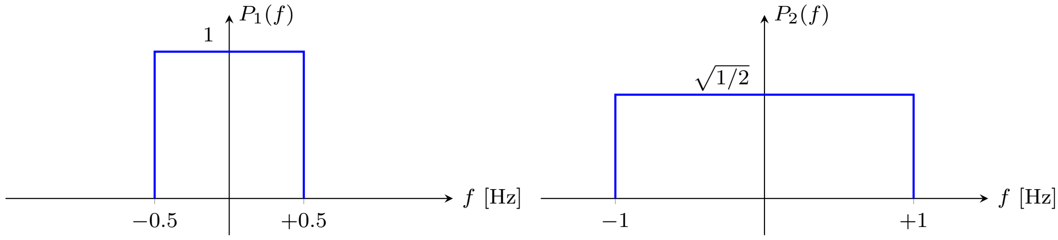

Figure 91: The frequency-spectra $P_1(f)$ and $P_2(f)$ of the pulses $p_1(t)$ and $p_2(t)$ respectively (Ex. 10.4).

where $p(t)$ is either $p_1(t)$, $p_2(t)$, or $p_3(t)$. To check this orthogonality in the frequency domain we can use the Nyquist criterium.

(a) Check first that pulse $p_1(t)$ has unit energy. Use Parseval. Then show graphically that $p_1(t)$ satisfies the Nyquist criterion for $T = 1$ second.

(b) Check also that $p_2(t)$ has unit energy. Use Parseval. Pulse $p_2(t)$ satisfies the Nyquist criterion for $T = 1/2$ second. Show graphically that $p_2(t)$ also satisfies the Nyquist criterion for $T = 1$ second.

(c) Determine the spectrum $P_3(f)$ of $p_3(t)$. Check that $p_3(t)$ has unit energy. Use Parseval.

(d) Show graphically that $p_3(t)$ satisfies the Nyquist criterion for $T = 1$ second.

(e) Show graphically that $p_3(t)$ also satisfies the Nyquist criterion for $T = 2/3$ second.

Solution:

First, let us recall what kind of pulse in time domain corresponds to the spectrum

$$P(f) = \begin{cases} \sqrt{T} & \text{for } |f| \leq \frac{1}{2T} \\ 0 & \text{for } |f| > \frac{1}{2T} \end{cases} \tag{617}$$

The inverse Fourier transform of such $P(f)$ is

$$p(t) = \sqrt{T}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{j2\pi ft}df \tag{618}$$

$$= \sqrt{T}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} \cos(2\pi ft)df \tag{619}$$

$$= \sqrt{T}\left.\frac{\sin(2\pi ft)}{2\pi t}\right|_{-\frac{1}{2T}}^{\frac{1}{2T}} \tag{620}$$

$$= \sqrt{T}\frac{\sin\left(2\pi\frac{t}{2T}\right) - \sin\left(2\pi\frac{-t}{T}\right)}{2\pi t} \tag{621}$$

$$= \frac{1}{\sqrt{T}} \cdot \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} \tag{622}$$

$$= \frac{1}{\sqrt{T}} \cdot \text{sinc}\left(\frac{t}{T}\right). \tag{623}$$

(a) Pulse $p_1(t)$ has unit energy that can be shown using Parseval's relation as follows:

$$\int p_1^2(t)dt = \int ||P_1(f)||^2 df = 1 \cdot 1 = 1. \tag{624}$$

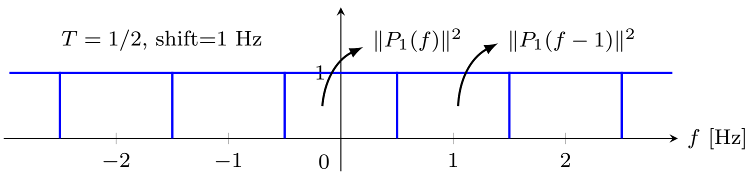

Figure 92: Graphical representation for $P_1(f)$ Nyquist criterion (Sol. 10.4).

Figure 92 shows

$$\sum_{m=-\infty}^{\infty} H_1\left(f + \frac{m}{T}\right), \tag{625}$$

where $H(f) = ||P_1(f)||^2$ and $T = 1$. Multiplying (625) by $\frac{1}{T} = 1$ yields $Z(f) = 1$ for all f. Hence, Nyquist is satisfied.

(b) Pulse $p_2(t)$ has unit energy that can be shown using Parseval's relation as follows:

$$\int p_2^2(t)dt = \int ||P_2(f)||^2 df = 2 \cdot \frac{1}{2} = 1. \tag{626}$$

For pulse duration $T = 0.5$ seconds, the frequency spacing (shift) is $f = 1/T = 2$ Hz. Figure 93 shows

$$\sum_{m=-\infty}^{\infty} H_2\left(f + \frac{m}{T}\right) = \sum_{m=-\infty}^{\infty} H_2\left(f + 2m\right). \tag{627}$$

Multiplying (627) by $\frac{1}{T} = 2$ yields $Z(f) = 1$ for all f. Hence, Nyquist is satisfied.

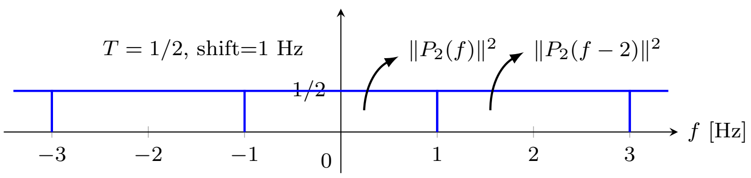

Figure 93: Graphical representation for $P_2(f)$ Nyquist criterion for $T = 1/2$ (Sol. 10.4).

If the pulse duration is increased to $T = 1$ second, then the frequency spacing is $\frac{1}{T} = 1$ Hz. Figure 94 shows

$$\sum_{m=-\infty}^{\infty} H_2 (f + m), \tag{628}$$

and in the figure the red blocks denote $H(f+m)$ for odd $m = ..., -3, -1, 1, 3, ...$ and the blue blocks denote $H(f + m)$ for even $m = ..., -2, 0, 2, ...$. Both red and blue blocks are of amplitude 1/2, but they are visualized on top of each other to demonstrate that the sum in (628) results in 1 for all f. Finally, multiplying (628) by $\frac{1}{T} = 1$ yields $Z(f) = 1$ for all f. Hence, Nyquist is satisfied.

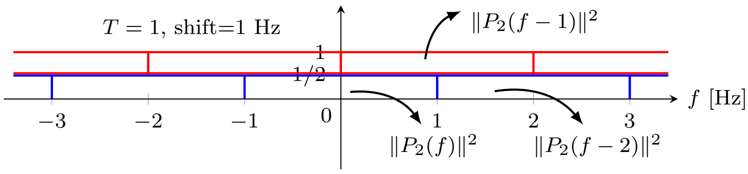

Figure 94: Graphical representation for $P_2(f)$ Nyquist criterion for $T = 1$ (Sol. 10.4).

(c) Pulse $p_3(t)$ has unit energy that can be shown using Parseval's relation (see spectrum in Fig 95 and power spectrum in Fig 96) as follows:

$$\int p_3^2(t)dt = \int ||P_3(f)||^2 df = 1 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = 1 \tag{629}$$

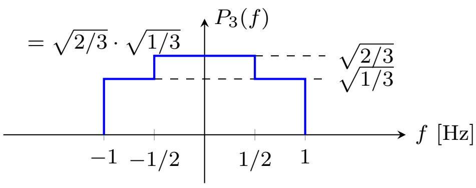

Figure 95: $P_3(f)$ (Sol. 10.4).

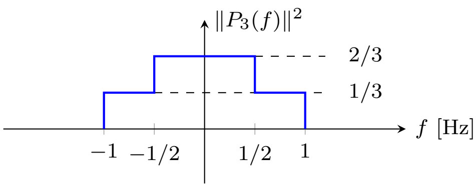

Figure 96: Calculating the energy for $p_3(t)$ (Sol. 10.4).

(d) For $T = 1$ second, the frequency spacing is $f = 1/T = 1$ Hz, that results in

$$\sum_{m=-\infty}^{\infty} H_2 (f + m), \tag{630}$$

as shown in Fig. 97. Similarly to Fig.94, here the red blocks refer to odd values of m and blue blocks refer to even values of m. Finally, Multiplying (630) by $\frac{1}{T} = 1$ gives $Z(f) = 1$. Hence, Nyquist is satisfied.

(e) For $T = 2/3$ seconds, the frequency spacing is $f = 1/T = 3/2$ Hz, so using $H_3(f) = ||P_3(f)||^2$,

$$\sum_{m=-\infty}^{\infty} H_3 \left(f + \frac{m}{T}\right) = \sum_{m=-\infty}^{\infty} H_3 \left(f + \frac{3m}{2}\right) \tag{631}$$

that is shown in Fig. 98. Multiplying (631) by $\frac{1}{T} = \frac{3}{2}$ gives $Z(f) = 1$. Hence, Nyquist is satisfied.

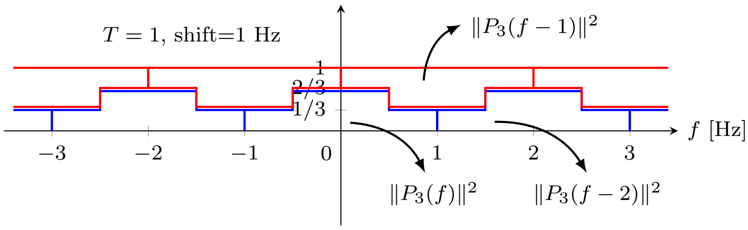

Figure 97: Graphical representation for $P_3(f)$ Nyquist criterion for $T = 1$ (Sol. 10.4).

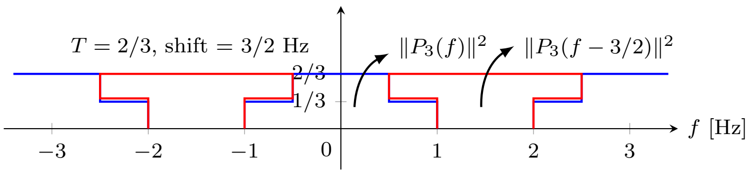

Figure 98: Graphical representation for $P_3(f)$ Nyquist criterion for $T = 2/3$ (Sol. 10.4).

11 Pass-Band Channels

Exercise 11.1: Consider a baseband pulse $p(t)$ with energy $\int_{-\infty}^{\infty} p^2(t)dt = 1$. There are now two equally likely signals

$$\begin{aligned} s_1(t) &= 8p(t)\sqrt{2}\cos(2\pi f_0 t), \text{ and} \\ s_2(t) &= 6p(t)\sqrt{2}\sin(2\pi f_0 t), \end{aligned} \tag{632}$$

where f_0 is the carrier frequency which is large compared to the baseband bandwidth. Give the vector-representations of both signals. Then compute the error probability for $N_0/2 = 5$.

Solution: The Building Blocks are

$$\phi_1(t) = p(t)\sqrt{2}\cos(2\pi f_0 t) \tag{633}$$

$$\phi_2(t) = p(t)\sqrt{2}\sin(2\pi f_0 t) \tag{634}$$

Then

$$\underline{s}_1 = (8, 0) \tag{635}$$

$$\underline{s}_2 = (0, 6) \tag{636}$$

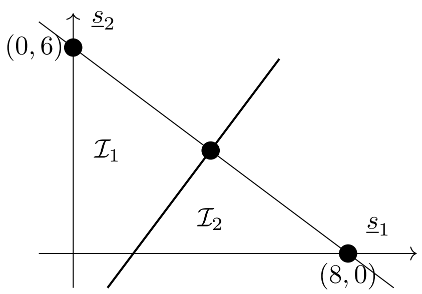

Figure 99: Vector representation of the signals (Sol. 11.1).

$$P_e = Q\left(\frac{\frac{d}{2}}{\sqrt{\frac{N_0}{2}}}\right) \tag{637}$$

$$d^2 = (0-8)^2 + (6-0)^2 = 100 \tag{638}$$

$$\implies \frac{d}{2} = 5 \tag{639}$$

Then, as we have $\sqrt{N_0/2} = \sqrt{5}$:

$$P_e = Q(\sqrt{5}). \tag{640}$$

Exercise 11.2: A signal structure must contain $|\mathcal{M}|$ signal points. These point are to be chosen on an integer grid. Now consider a hypercube and a hypersphere in N dimensions. Both have their center in the origin of the coordinate system.

We can either choose our signal structure as all the grid points inside the sphere or all the grid points inside the cube. Assume that the dimensions of the sphere and cube are such that they contain equally many signal points.

Let $|\mathcal{M}| \gg 1$ so that the signal points can be assumed to be uniformly distributed over the sphere and the cube.

(a) Let $N = 2$. Find the ratio between the average signal energies of the sphere-structure and that of the cube-structure. What is the difference in dB?

(b) Let N be even. What is then the ratio for $N \to \infty$? In dB?
Hint: The volume of an N-dimensional sphere is $\pi^{N/2} R^N/(N/2)!$ for even N.

The computed ratios are called *shaping gains.*

Solution:

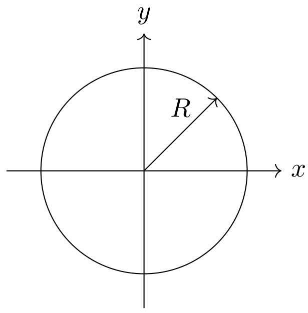

Figure 100: Sphere (Sol. 11.2).

(a) For the circle (see Fig. 100) with radius R we have an area $A_1 = \pi R^2$ and circumference $2\pi R$. The signal values can have amplitude of radius $\rho = \sqrt{x^2 + y^2}$. The probability of observing a point at certain radius ρ is $\Pr(\rho)$ that can be found as

$$\Pr(\rho) = \frac{2\pi\rho}{\pi R^2}. \tag{641}$$

The average signal energy can be found by finding the expected value of the squared amplitude of the signal, so $E(x^2 + y^2) = E(\rho^2)$. This can be found as follows:

$$\begin{aligned} E(\rho^2) &= \int_0^R \Pr(\rho)\rho^2 d\rho & (642)\\ &= \frac{1}{\pi R^2}\int_0^R \rho^2 2\pi\rho d\rho & (643)\\ &= \frac{1}{\pi R^2} 2\pi \frac{\rho^4}{4}\bigg|_0^R & (644)\\ &= \frac{1}{2}R^2. & (645)\end{aligned}$$

For the square, shown in Fig. 101, we have an area $A_2 = 4z^2$. Thus, the average signal energy is

$$E(x^2 + y^2) = 2E(x^2), \tag{646}$$

where

$$\begin{aligned} E(x^2) &= \frac{1}{2z}\int_{-z}^{z} \alpha^2 d\alpha & (647)\\ &= \frac{1}{2z}\cdot\frac{1}{3}\alpha^3\bigg|_{-z}^{z} & (648)\\ &= \frac{1}{3}z^2. & (649)\end{aligned}$$

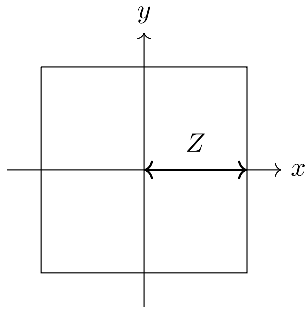

Figure 101: Square (Sol. 11.2).

Thus,

$$E(x^2+y^2) = \frac{2}{3}z^2. \tag{650}$$

If we consider that the two structures contain equally many signal points: i.e. the same area, then

$$\begin{aligned} \text{Area}_{\triangle} &= \text{Area}_{\square} \\ \pi R^2 &= 4z^2 \\ \implies \frac{R^2}{z^2} &= \frac{4}{\pi}. \end{aligned}$$

Using the result above, we can show that

$$\frac{E(x^2+y^2)_{\text{circle}}}{E(x^2+y^2)_{\text{square}}} = \frac{\frac{1}{2}R^2}{\frac{2}{3}z^2} \tag{651}$$

$$= \frac{3}{4}\cdot\frac{4}{\pi} \tag{652}$$

$$= \frac{3}{\pi} \tag{653}$$

which is approximately -0.20 dB.

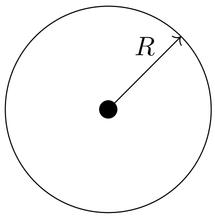

Figure 102: N-dimensional sphere (Sol. 11.2).

(b) For the sphere we have

$$\text{Vol.} = B_N R^N \tag{654}$$

with

$$B_N = \frac{\pi^{\frac{N}{2}}}{(N/2)!} \tag{655}$$

for even N.

The peel volume is given by

$$B_N\rho^N - B_N(\rho-\Delta\rho)^N \approx B_N N\rho^{N-1}\Delta\rho \tag{656}$$

and, for the vectors $v \in \mathbb{R}^N$ in the N-sphere

$$E(||v||^2) = \frac{1}{\text{Vol}} \int_0^R B_N N \rho^{N-1} \rho^2 d\rho \tag{657}$$
$$= \frac{1}{\text{Vol}} \int_0^R B_N N \rho^{N+1} d\rho \tag{658}$$
$$= \frac{1}{\text{Vol}} B_N \frac{N}{N+2} R^{N+2} \tag{659}$$
$$= \frac{N}{N+2} R^2 \tag{660}$$

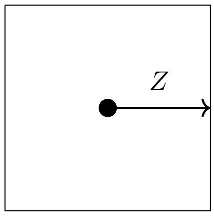

Figure 103: N-dimensional cube (Sol. 11.2).

For the cube-structure we have

$$\text{Vol} = 2^N z^N \tag{661}$$
$$E(||v||^2) = \frac{N}{3} z^2 \tag{662}$$

Equating the volumes

$$B_N R^N = 2^N z^N \tag{663}$$
$$\frac{R^2}{z^2} = \frac{4}{(B_N)^{\frac{2}{N}}} \tag{664}$$

For a large N, we can consider the Stirling's approximation:

$$B_N = \frac{\pi^{\frac{N}{2}}}{(N/2)!} \approx \frac{\pi^{\frac{N}{2}}}{\sqrt{\pi N} e^{-\frac{N}{2}} \left(\frac{N}{2}\right)^{\frac{N}{2}}} \tag{665}$$

Thus

$$(B_N)^{\frac{2}{N}} = \frac{\pi}{e^{-1}\frac{N}{2}} = \frac{2\pi e}{N} \tag{666}$$

Then we have

$$\frac{3}{N+2} \cdot \frac{4}{(B_N)^{\frac{2}{N}}} = \frac{3}{N+2} \cdot \frac{4N}{2\pi e} = \frac{N}{N+2} \cdot \frac{12}{2\pi e} \tag{667}$$

Making $N \to \infty$ we have

$$\lim_{N\to\infty} \frac{N}{N+2} \cdot \frac{12}{2\pi e} = \frac{6}{\pi e} \tag{668}$$

which is approximately -1.53 dB.

Exercise 11.3: In a wireless system the transmitter uses FSK (frequency shift keying) to transmit one of two messages. The message probabilities are equal, hence $\Pr\{M = 1\} = \Pr\{M = 2\} = 1/2$. The corresponding signals are

$$\begin{aligned} s_1(t) &= A\cos(2\pi f_1 t), \\ s_2(t) &= A\cos(2\pi f_2 t), \text{ for } -1\mu\text{s} \le t < +200\mu\text{s}, \end{aligned}$$

with $f_1 = 100$MHz, $f_2 = 105$MHz, and $A = 300$.

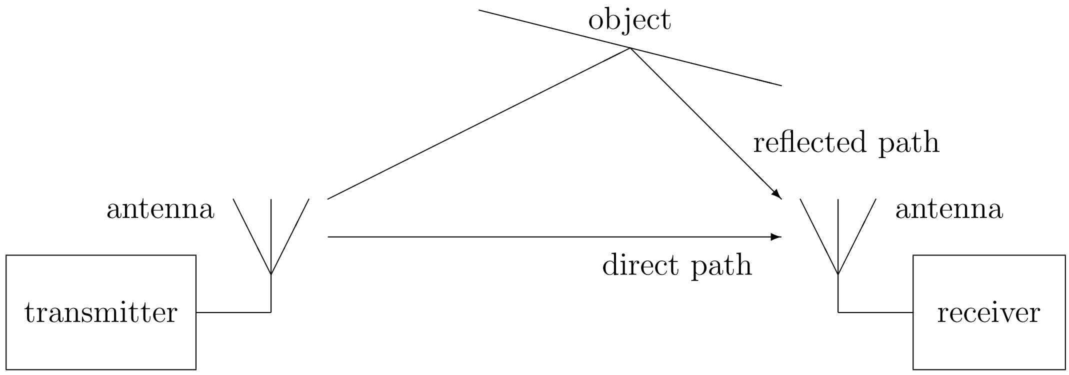

Figure 104: Transmitter and receiver, direct path, reflected path, and reflecting object (Ex. 11.3).

The receiver first correlates the received signal $r(t)$ with the building-block waveforms $\varphi_1(t) = B\cos(2\pi f_1 t)$ and $\varphi_2(t) = B\cos(2\pi f_2 t)$ in the time-interval $0 \le t < 200\mu$s. Then it makes a decision based on these two correlations. Denote

$$\begin{aligned} r_1 &= \int_0^{200\mu\text{s}} r(t)\varphi_1(t)dt, \\ r_2 &= \int_0^{200\mu\text{s}} r(t)\varphi_2(t)dt. \end{aligned}$$

(a) How should B be chosen? Note that building blocks $\varphi_1(t)$ and $\varphi_2(t)$ should be orthonormal.

(b) See Figure 104. Assume that the receiver receives **only the direct path** hence $r(t) = s(t - \Delta) + n_w(t)$. Here $\Delta = 0.2\mu$s is the time that the signal needs to travel from the transmit antenna to the receive antenna. The power spectral density of the white noise is $\frac{N_0}{2} = 1/2$. Assume that the receiver knows Δ. How large is the error probability if the receiver makes an optimum decision based on r_1 and r_2?

(c) See again Figure 104. Suppose that the receiver now receives, in addition to the direct path, a reflected path that has a total delay of $\Delta' = 0.3\mu$s which is attenuated by a factor of 3, hence $r(t) = s(t - \Delta) + \frac{1}{3}s(t - \Delta') + n_w(t)$. The power spectral density of the noise is again $\frac{N_0}{2} = 1/2$. What is the error probability now? Assume that the receiver knows both Δ and Δ' and makes an optimum decision based on r_1 and r_2.

Solution:

(a) The building blocks waveforms are orthonormal, so each waveform $\phi_1(t)$ and $\phi_2(t)$ must have unit energy. The energy of the waveform is

$$E_{\phi_i(t)} = \int_0^{200\cdot 10^{-6}} B^2\cos^2(2\pi f_i t)dt = 200 \cdot 10^{-6} B^2 \cdot \frac{1}{2} = 1. \tag{669}$$

Thus, in order to satisfy $E_{\phi_i(t)} = 1$, the required value is $B = 100$.

Furthermore, we note that both building block cosines fit a whole number of periods inside the time-interval and therefore they are orthogonal.

(b) Determine first

$$2\pi\Delta f_1 = 2\pi \cdot 0.2 \cdot 10^{-6} \cdot 100 \cdot 10^6 = 40\pi \tag{670}$$
$$2\pi\Delta f_2 = 2\pi \cdot 0.2 \cdot 10^{-6} \cdot 105 \cdot 10^6 = 42\pi \tag{671}$$

Now

$$\begin{aligned} r(t) &= A\cos(2\pi f_i(t-\Delta)) + n_w(t) &(672)\\ &= A\cos(2\pi f_i t - 2\pi f_i \Delta) + n_w(t) &(673)\\ &= A\cos(2\pi f_i\Delta)\cos(2\pi f_i t) + \sin(2\pi f_i\Delta)\sin(2\pi f_i t) + n_w(t) &(674)\\ &= A\cos(2\pi f_i t) + n_w(t) &(675)\end{aligned}$$

because

$$\sin(2\pi f_i\Delta) = 0 \tag{676}$$
$$\cos(2\pi f_i\Delta) = 1 \tag{677}$$

for $i = 1, 2$.

The vector notation of the received signals is

$$\underline{s}_1 = \left(\frac{A}{B}, 0\right) = (3, 0) \tag{678}$$
$$\underline{s}_2 = \left(0, \frac{A}{B}\right) = (0, 3) \tag{679}$$

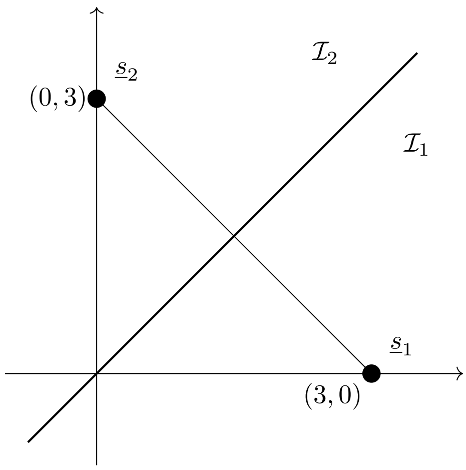

Figure 105: Figure Sol. 11.3

Now, because $\sigma = \sqrt{N_0/2} = \sqrt{1/2}$, we have

$$P_e = Q\left(\frac{3\sqrt{2}}{2\sqrt{1/2}}\right) = Q(3) \tag{680}$$

(c) Now we have an extra path with $\Delta' = 0.3$ μs. Thus $2\pi f_1 \Delta' = 60\pi$ and $2\pi f_2 \Delta' = 63\pi$, and which is attenuated by a factor of 3. Then

$$r(t) = A\cos(2\pi f_i t - 2\pi f_i \Delta) + \frac{A}{3}\cos(2\pi f_i t - 2\pi f_i \Delta') + n_w(t) \tag{681}$$
$$= A\cos(2\pi f_i \Delta)\cos(2\pi f_i t) + \frac{A}{3}\cos(2\pi f_i \Delta')\cos(2\pi f_i t) + n_w(t) \tag{682}$$
$$= \left(A\cos(2\pi f_i \Delta) + \frac{A}{3}\cos(2\pi f_i \Delta')\right)\cos(2\pi f_i t) + n_w(t) \tag{683}$$

(notice that $\sin(2\pi f_i \Delta)$ and $\sin(2\pi f_i \Delta')$ are both equal to 0).
For $m = 1$ we have

$$r(t) = \left(A + \frac{A}{3}\right)\cos(2\pi f_1 t) + n_w(t) \tag{684}$$

For $m = 2$ we have

$$r(t) = \left(A - \frac{A}{3}\right)\cos(2\pi f_2 t) + n_w(t) \tag{685}$$

The vector notation of the received signals is

$$\underline{s}_1 = \left(\frac{4A}{3B}, 0\right) = (4, 0) \tag{686}$$
$$\underline{s}_2 = \left(0, \frac{2A}{3B}\right) = (0, 2) \tag{687}$$

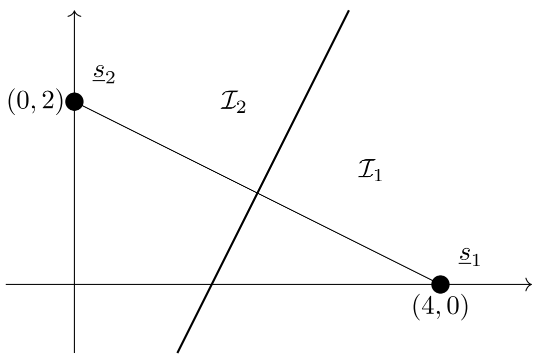

Figure 106: Figure Sol. 11.3

Now, because $d = \sqrt{16 + 4} = \sqrt{20}$, we have

$$P_e = Q\left(\frac{\sqrt{20}}{2\sqrt{1/2}}\right) = Q(\sqrt{10}) \tag{688}$$

12 Random Carrier-Phase

Exercise 12.1: Let X and Y be two independent Gaussian random variables with common variance σ^2. The mean of X is m and Y is a zero-mean random variable. We define the random variable V as $V = \sqrt{X^2 + Y^2}$. Show that

$$p_V(v) = \frac{v}{\sigma^2} I_0\left(\frac{mv}{\sigma^2}\right) \exp\left(-\frac{v^2 + m^2}{2\sigma^2}\right), \text{ for } v > 0, \tag{689}$$

and 0 for $v \leq 0$. Here $I_0(\cdot)$ is the modified Bessel function of the first kind and zero order.

The distribution of V is know as the *Rician distribution*. In the special case where $m = 0$, the Rician distribution simplifies to the *Rayleigh distribution*.

Solution:

$$p_{XY}(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \tag{690}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{(x-m)^2+y^2}{2\sigma^2}} \tag{691}$$

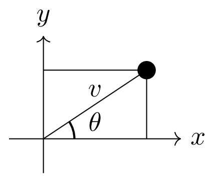

Figure 107: Vector v inclined by an angle θ (Sol. 12.1).

We have

$$v = \sqrt{x^2 + y^2} \tag{692}$$

$$x = v\cos(\theta) \tag{693}$$

$$y = v\sin(\theta) \tag{694}$$

Now

$$p_{V\Theta}(v, \theta) = p_{XY}(v\cos(\theta), v\sin(\theta))|\mathcal{J}| \tag{695}$$

where

$$\mathcal{J} = \begin{pmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -v\sin(\theta) \\ \sin(\theta) & v\cos(\theta) \end{pmatrix} \tag{696}$$

Therefore

$$|\mathcal{J}| = v\cos^2(\theta) + v\sin^2(\theta) = v \tag{697}$$

Now we have

$$p_{V\Theta}(v, \theta) = v p_{XY}(v\cos(\theta), v\sin(\theta)) \tag{698}$$

$$= \frac{v}{2\pi\sigma^2} e^{-\frac{[v\cos(\theta)-m]^2+[v\sin(\theta)]^2}{2\sigma^2}} \tag{699}$$

$$= \frac{v}{2\pi\sigma^2} e^{-\frac{v^2+m^2-2mv\cos(\theta)}{2\sigma^2}} \tag{700}$$

Finally, for $v > 0$, we can find $p_V(v)$ by marginalizing $p_{V\Theta}(v, \theta)$ over θ:

$$p_V(v) = \int_0^{2\pi} p_{V\Theta}(v, \theta) d\theta \tag{701}$$

$$= \frac{v}{\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} \int_0^{2\pi} \frac{1}{2\pi} e^{\frac{mv}{\sigma^2}\cos(\theta)} d\theta \tag{702}$$

$$= \frac{v}{\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} I_0\left(\frac{mv}{\sigma^2}\right) \tag{703}$$

For $m = 0$ notice that we have the Rayleigh distribution

$$p_V(v) = \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} \tag{704}$$

Exercise 12.2: In on-off keying of a carrier-modulated signal, the two possible signals are

$$\begin{aligned} s_1^b(t) &= \sqrt{E}\varphi(t), \text{ hence } s_1 = \sqrt{E}, \\ s_2^b(t) &= 0, \text{ hence } s_2 = 0. \end{aligned} \tag{705}$$

Note that s_1 and s_2 are signals in one-dimensional "vector"-notation. The signals are equiprobable, i.e., $\Pr\{M = 1\} = \Pr\{M = 2\} = 1/2$.

These signals are transmitted over a bandpass channel with a carrier with random phase θ with $p_\Theta(\theta) = 1/2\pi$ for $0 \leq \theta < 2\pi$. Power density of the noise is $\frac{N_0}{2}$ for all frequencies.

An optimum receiver first determines r^c and r^s for which we can write

$$\begin{aligned} r^c &= s_m \cos(\theta) + n^c, \\ r^s &= s_m \sin(\theta) + n^s, \end{aligned} \tag{706}$$

where both n^c and n^s are independent zero-mean Gaussian vectors with variance $\frac{N_0}{2}$.

Now let $E/N_0 = 10$.

(a) Determine the optimum decision regions $\mathcal{I}_m$ for $m = 1, 2$.
Hint: Note that $I_0(12.1571) = 22026$.

(b) Determine the expected error probability that is achieved by the optimum receiver.
Hint: Note that $\int_0^{2.7184} x \exp\left(-\frac{x^2}{2}\right) I_0\left(x\sqrt{20}\right) dx = 635.72$.

Solution:

$$m = 1 \rightarrow s_1 = \sqrt{E} \tag{707}$$

$$m = 2 \rightarrow s_2 = 0 \tag{708}$$

$$s^c = s_m \cos(\theta) \tag{709}$$

$$s^s = s_m \sin(\theta) \tag{710}$$

$$r^c = s_m \cos(\theta) + n^c \tag{711}$$

$$r^s = s_m \sin(\theta) + n^s \tag{712}$$

We are now going to determine the decision regions

$$x_m = \sqrt{(r^c s_m)^2 + (r^s s_m)^2} \tag{713}$$

$$= |s_m|\sqrt{(r^c)^2 + (r^s)^2} \tag{714}$$

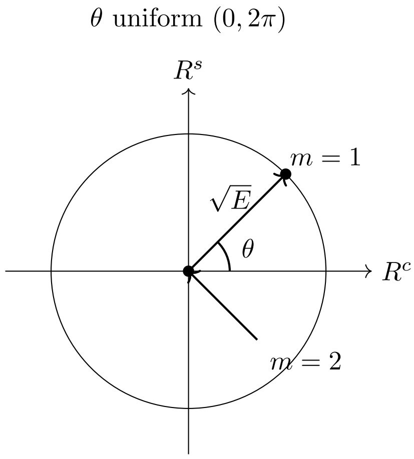

Figure 108: Vector representation of the received signals (Sol. 12.2).

$$m = 1 \rightarrow x_1 = \sqrt{E}\sqrt{(r^c)^2 + (r^s)^2} \tag{715}$$
$$m = 2 \rightarrow x_2 = 0 \tag{716}$$

Thus

$$m = 1 \rightarrow I_0\left(\frac{2\sqrt{E}\sqrt{(r^c)^2 + (r^s)^2}}{N_0}\right) e^{-\frac{E}{N_0}} \tag{717}$$
$$m = 2 \rightarrow I_0(0)e^{-0} = 1 \tag{718}$$

because

$$I_0(0) = \frac{1}{2\pi}\int_0^{2\pi} e^{0\cos(\theta)} d\theta = 1 \tag{719}$$

(a) Equate and make $\rho = \sqrt{(r^c)^2 + (r^s)^2}$:

$$I_0\left(\frac{2\rho\sqrt{E}}{N_0}\right) e^{-\frac{E}{N_0}} = 1 \tag{720}$$
$$I_0\left(\frac{\rho}{\sqrt{N_0/2}}\sqrt{\frac{E}{N_0/2}}\right) = e^{\frac{E}{N_0}} = e^{10} = 22026 \tag{721}$$

Therefore

$$\frac{\rho}{\sqrt{N_0/2}}\sqrt{\frac{E}{N_0/2}} = 12.1571 \tag{722}$$

Now notice that

$$E = 10N_0 = 20N_0/2 \tag{723}$$
$$\sqrt{\frac{E}{N_0/2}} = \sqrt{20} = 4.4721 \tag{724}$$

Thus

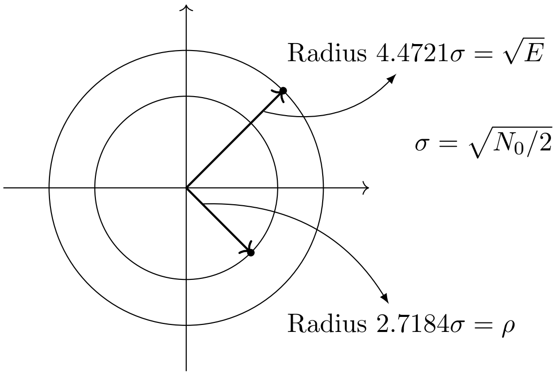

Figure 109: Decision regions (Sol. 12.2).

$$\frac{\rho}{\sqrt{N_0/2}} = \frac{12.1571}{\sqrt{20}} = 2.7184 \tag{725}$$

Within the circle of radius ρ choose $m = 2$, and outside choose $m = 1$. Obs.: outside, the Bessel function is greater than 22026, and inside smaller than 22026.

(b) First, we have

$$\begin{aligned}
\Pr\{\hat{M} = 1|M = 2\} &= \int_{x^2+y^2 \geq \rho^2} \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy && (726)\\
&= \int_0^{2\pi} \int_\rho^\infty \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r dr d\theta && (727)\\
&= \int_\rho^\infty \frac{1}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r dr && (728)\\
&= \int_\rho^\infty e^{-\frac{r^2}{2\sigma^2}} d\left(\frac{r^2}{2\sigma^2}\right) && (729)\\
&= -e^{-\frac{r^2}{2\sigma^2}} \Big|_\rho^\infty && (730)\\
&= e^{-\frac{\rho^2}{2\sigma^2}} && (731)\\
&= e^{-\frac{1}{2}\frac{\rho^2}{N_0/2}} && (732)\\
&= e^{-\frac{1}{2}(2.7184)^2} = 0.0249 && (733)
\end{aligned}$$

Now, for $\Pr\{\hat{M} = 2|M = 1\}$, we can use the equation of Exercise 12.1 by making $m^2 = E$:

$$\begin{aligned}
\Pr\{\hat{M} = 2|M = 1\} &= \int_0^\rho \frac{v}{\sigma^2} e^{-\frac{v^2+E}{2\sigma^2}} I_0\left(\frac{\sqrt{E}v}{\sigma^2}\right) dv && (734)\\
&= \int_0^\rho \frac{v}{\sigma}\frac{1}{\sigma} e^{-\frac{v^2}{2\sigma^2}} e^{-\frac{E}{2\sigma^2}} I_0\left(\frac{\sqrt{E}}{\sigma}\frac{v}{\sigma}\right) dv && (735)\\
&= e^{-\frac{E}{2\sigma^2}} \int_0^\rho \frac{v}{\sigma} e^{-\frac{v^2}{2\sigma^2}} I_0\left(\frac{\sqrt{E}}{\sigma}\frac{v}{\sigma}\right) d\left(\frac{v}{\sigma}\right) && (736)
\end{aligned}$$

We know $E/2\sigma^2 = E/N_0 = 10$, $\sqrt{E}/\sigma = \sqrt{2E/2\sigma^2} = \sqrt{20}$.

$$= e^{-10} \int_0^\rho \frac{v}{\sigma} e^{-\frac{v^2}{2\sigma^2}} I_0\left(\frac{v}{\sigma}\sqrt{20}\right) d\left(\frac{v}{\sigma}\right) \tag{737}$$

Apply substitution: Let $x = v/\sigma$, divide integration limits by σ to get updated limits. It is given that $\rho = 2.7184\sigma$, resulting in $\rho/\sigma = 2.7184$.

$$= e^{-10} \int_0^{\rho/\sigma} x e^{-\frac{x^2}{2}} I_0(x\sqrt{20}) dx \tag{738}$$

$$= e^{-10} \int_0^{2.7184} x e^{-\frac{x^2}{2}} I_0(x\sqrt{20}) dx \tag{739}$$

$$= \frac{635.72}{22026} = 0.0289 \tag{740}$$

Finally:

$$P_e = \frac{0.0249 + 0.0289}{2} = 0.0269 \tag{741}$$

Exercise 12.3: In on-off keying of a carrier modulated signal the two baseband signals are

$$\begin{aligned} s_1^b(t) &= A\varphi(t), \text{ hence } s_1 = A \\ s_2^b(t) &= 0, \text{ hence } s_2 = 0. \end{aligned}$$

Here $\varphi(t)$ is a baseband building-block waveform and s_1 and s_2 are the baseband signals in vector-representation (one dimensional). Both signals are equiprobable, hence $\Pr\{M = 1\} = \Pr\{M = 2\} = 1/2$.

The baseband signals are modulated with a carrier having frequency f_0 and random phase Θ, hence

$$s_m(t) = s_m^b(t)\sqrt{2}\cos(2\pi f_0 t - \theta).$$

The resulting signal $s_1(t)$ or $s_2(t)$ is transmitted over a bandpass channel, power spectral density of the AWGN is $\frac{N_0}{2}$ for all frequencies f. Assume that $A^2/N_0 = \ln(13/5)$.

An optimum receiver correlates the received signal $r(t) = s_m(t) + n_w(t)$ with the bandpass building-block waveforms $\varphi(t)\sqrt{2}\cos(2\pi f_0 t)$ and $\varphi(t)\sqrt{2}\sin(2\pi f_0 t)$ to obtain the received vector (r^c, r^s).

(a) Fix some θ. Express the received vector (r^c, r^s) as a signal vector (s^c, s^s) plus a noise vector (n^c, n^s). What is the signal vector (s^c, s^s) as a function of θ? What are the statistical properties of the noise vector (n^c, n^s)?

(b) The random variable Θ can have the values $+\pi/2$ and $-\pi/2$ only and $\Pr\{\Theta = +\pi/2\} = \Pr\{\Theta = -\pi/2\} = 1/2$. What are now the possible signal vectors (s^c, s^s)? Give the decision variables as function of (r^c, r^s) and specify[2] for what (r^c, r^s) an optimum receiver decides $\hat{M} = 2$.

(c) Give an expression for the error probability P_e that is achieved by an optimum receiver.

Solution:

(a)

$$s_m(t) = s_m^b(t)\sqrt{2}\cos(2\pi f_0 t - \theta) \tag{742}$$

$$= s_m^b\sqrt{2}\left(\cos(2\pi f_0 t)\cos(\theta) + \sin(2\pi f_0 t)\sin(\theta)\right) \tag{743}$$

$$= s_m\phi(t)\sqrt{2}\cos(2\pi f_0 t)\cos(\theta) + s_m\phi(t)\sqrt{2}\sin(2\pi f_0 t)\sin(\theta) \tag{744}$$

Building blocks:

$$\phi(t)\sqrt{2}\cos(2\pi f_0 t) \tag{745}$$

$$\phi(t)\sqrt{2}\sin(2\pi f_0 t) \tag{746}$$

[2]Hint: note that $x = \ln 5$ satisfies $\frac{26}{5} = \exp(x) + \exp(-x)$.

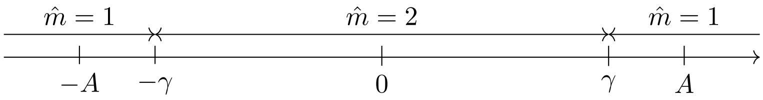

Figure 110: Decision region (Sol. 12.3).

Thus:

$$s_m^c = s_m \cos(\theta) \tag{747}$$
$$s_m^s = s_m \sin(\theta) \tag{748}$$

and

$$r^c = s_m \cos(\theta) + n^c \tag{749}$$
$$r^s = s_m \sin(\theta) + n^s \tag{750}$$

(n^c, n^s) is Gaussian, zero mean and with variance $N_0/2$.

(b) Two possibilities

$$(s^c, s^s) = (0, A) \qquad \to 1/4 \tag{751}$$
$$(s^c, s^s) = (0, -A) \qquad \to 1/4 \tag{752}$$
$$(s^c, s^s) = (0, 0) \qquad \to 1/2 \tag{753}$$

r^c is irrelevant! $r = r^s$.
Decision variables:

$$m = 2 \to \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \tag{754}$$
$$m = 1 \to \frac{1/2}{\sqrt{\pi N_0}} \left(e^{-\frac{(r-A)^2}{N_0}} + e^{-\frac{(r+A)^2}{N_0}} \right) \tag{755}$$

Now solve

$$2e^{-\frac{\gamma^2}{N_0}} = e^{-\frac{(\gamma-A)^2}{N_0}} + e^{-\frac{(\gamma+A)^2}{N_0}} \tag{756}$$
$$2 = e^{\frac{2\gamma A - A^2}{N_0}} + e^{-\frac{2\gamma A + A^2}{N_0}} \tag{757}$$
$$2e^{\frac{A^2}{N_0}} = e^{\frac{2\gamma A}{N_0}} + e^{-\frac{2\gamma A}{N_0}} \tag{758}$$
$$\frac{A^2}{N_0} = \ln 13/5 \Rightarrow 2e^{\frac{A^2}{N_0}} = 26/5 \tag{759}$$

Now we use the hint, and $A^2/N_0 = \ln(13/5)$ to find

$$\frac{2\gamma A}{N_0} = \ln 5 \quad (\text{or} - \ln 5) \tag{760}$$
$$\frac{\gamma}{\sqrt{N_0/2}} = \frac{\ln 5}{\sqrt{2\ln(13/5)}} \left(\text{or} - \frac{\ln 5}{\sqrt{2\ln(13/5)}} \right) \tag{761}$$

We decode $\hat{m} = 2$ if $-\gamma < r < \gamma$ and else $\hat{m} = 1$.

(c)

$$P_e = \frac{1}{2} 2Q(\frac{\gamma}{\sqrt{N_0/2}}) + \frac{2}{4} \left(Q(\frac{A-\gamma}{\sqrt{N_0/2}}) - Q(\frac{A+\gamma}{\sqrt{N_0/2}} \right) \tag{762}$$
$$\approx Q(1.16) + 1/2(Q(0.22) - Q(2.54)) \tag{763}$$

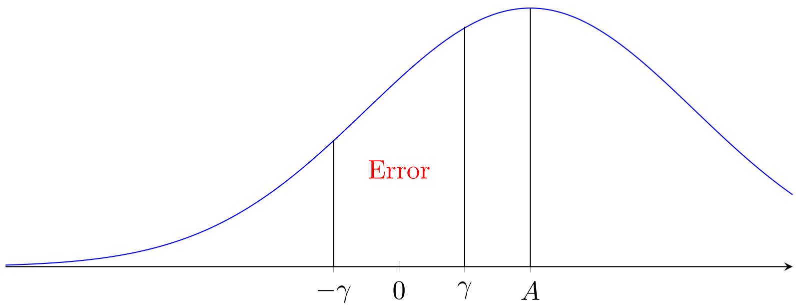

Figure 111: Visualize error probability $1/2\,\Pr(\hat{m}=2|m=1)$ (Sol. 12.3).

Exercise 12.4:

Consider two equally likely messages, so $\mathcal{M}=\{1,2\}$ and $\Pr\{M=1\}=\Pr\{M=2\}=1/2$. The following base-band signals belong to these messages

$$\begin{aligned} s_1^b(t) &= p(t)+s_0(t),\\ s_2^b(t) &= p(t)-s_0(t), \end{aligned} \tag{764}$$

where $p(t)=a\varphi_1(t)$ and $s_0(t)=b\varphi_2(t)$, and $\varphi_1(t)$ and $\varphi_2(t)$ are two base-band building block waveforms. Assume $a>0$ and $b>0$.

(a) What are the corresponding signal-vectors $\underline{s}_1$ and $\underline{s}_2$? Make a sketch of the signal structure. Express the error probability in a and/or b that the optimal receiver realizes if

$$r(t)=s^b(t)+n_w(t)$$

for which $S_{N_w}(f)=1/4$ for all f. Use the Q-function in the answer.

(b) The base-band signal is now being used to modulate a carrier with sufficiently large frequency f_0 with random phase θ, so

$$s(t)=s^b(t)\sqrt{2}\cos(2\pi f_0 t-\theta).$$

The phase θ is uniform over $[0,2\pi)$. The band-pass signal $s(t)$ is received as

$$r(t)=s(t)+n_w(t)$$

with $S_{N_w}(f)=N_0/2$ for all f. An optimal receiver first forms the vectors $\underline{r}^c=(r_1^c,r_2^c)$ and $\underline{r}^s=(r_1^s,r_2^s)$ with

$$\begin{aligned} r_1^c &= \int r(t)\varphi_1(t)\sqrt{2}\cos(2\pi f_0 t)dt,\\ r_1^s &= \int r(t)\varphi_1(t)\sqrt{2}\sin(2\pi f_0 t)dt,\\ r_2^c &= \int r(t)\varphi_2(t)\sqrt{2}\cos(2\pi f_0 t)dt,\\ r_2^s &= \int r(t)\varphi_2(t)\sqrt{2}\sin(2\pi f_0 t)dt. \end{aligned} \tag{765}$$

How large is the energy E_1 of signal 1 and the energy E_2 of signal 2?

For what combination of r_1^c, r_1^s, r_2^c, and r_2^s the optimal receiver chooses $\hat{M}=1$?

(c) Assume $r_1^c=r_1\cos(\alpha)$ and $r_1^s=r_1\sin(\alpha)$ is a well chosen $\alpha\in[0,2\pi)$ and $r_1=\sqrt{(r_1^c)^2+(r_1^s)^2}$. Also assume $r_2^c=r_2\cos(\beta)$ and $r_2^s=r_2\sin(\beta)$ is a well chosen $\beta\in[0,2\pi)$ and $r_2=\sqrt{(r_2^c)^2+(r_2^s)^2}$. For what values of $\alpha-\beta$ the receiver choses $\hat{M}=1$? Use $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$. Interpret this result as if you know α is an estimate of θ. This phase is being estimated with the 'pilot' $p(t)$.

Solution:

(a)

$$\underline{s}_1 = (a, b) \tag{766}$$
$$\underline{s}_2 = (a, -b) \tag{767}$$

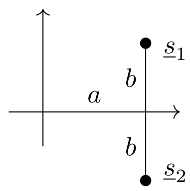

Figure 112: Signal structure (Sol. 12.4).

$$a > 0 \tag{768}$$
$$b > 0 \tag{769}$$

$$\sigma = \sqrt{\frac{1}{4}} = \frac{1}{2} \tag{770}$$
$$P_e = Q\left(\frac{b}{1/2}\right) = Q(2b) \tag{771}$$

(b) Same energy.

$$(\underline{r}^c \cdot \underline{s}_1) = ar_1^c + br_2^c \tag{772}$$
$$(\underline{r}^s \cdot \underline{s}_1) = ar_1^s + br_2^s \tag{773}$$
$$(\underline{r}^c \cdot \underline{s}_2) = ar_1^c - br_2^c \tag{774}$$
$$(\underline{r}^s \cdot \underline{s}_2) = ar_1^s - br_2^s \tag{775}$$

Choose $\hat{m} = 1$ for

$$(\underline{r}^c \cdot \underline{s}_1)^2 + (\underline{r}^s \cdot \underline{s}_1)^2 > (\underline{r}^c \cdot \underline{s}_2)^2 + (\underline{r}^s \cdot \underline{s}_2)^2 \tag{776}$$
$$(ar_1^c + br_2^c)^2 + (ar_1^s + br_2^s)^2 > (ar_1^c - br_2^c)^2 + (ar_1^s - br_2^s)^2 \tag{777}$$
$$2abr_1^c r_2^c + 2abr_1^s r_2^s > -2abr_1^c r_2^c - 2abr_1^s r_2^s \tag{778}$$
$$r_1^c r_2^c + r_1^s r_2^s > 0 \tag{779}$$

(c)

$$r_1 \cos(\alpha) r_2 \cos(\beta) + r_1 \sin(\alpha) r_2 \sin(\beta) = r_1 r_2 \cos(\alpha - \beta) > 0 \tag{780}$$
$$\implies |\alpha - \beta| < \frac{\pi}{2} \tag{781}$$

In the first dimension, the signal is always $(r_1^c, r_1^s) = (a\cos(\theta) + n_1^c, a\sin(\theta) + n_1^s)$, where θ is the unknown angle. In this first dimension, the unknown angle can be estimated using (r_1^c, r_1^s), leading to α.

In the second dimension, the signal is $(r_2^c, r_2^s) = (b\cos(\theta) + n_2^c, b\sin(\theta) + n_2^s)$, depending on the message. The angle β corresponding to (r_2^c, r_2^s) can be close to α, then $\hat{m} = 1$; or not, then $\hat{m} = 2$.

The pulse $p(t)$ is a pilot, used to estimate θ.

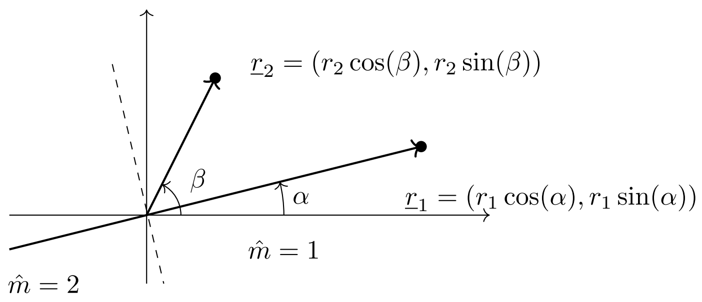

Figure 113: Vector representation of the received signal (Sol. 12.4).