2 - Root Locus design method

Root Locus method

Analyse changes in the system parameters that modify
$\underbrace{\text{the roots of the characteristic equation}}_{\text{Closed loop poles}}$

analyse closed loop behaviour using the open loop plant

understanding of system

Implication on changes

But first, a side step

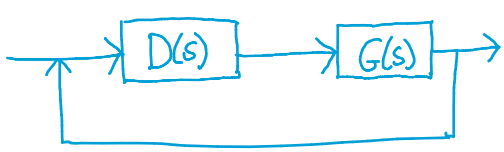
$n = \#$ Poles , $m = \#$ zeros

Product operator

$$G(s) = \frac{s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots a_n} = \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)} = \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

Monic: highest power has a coefficient of 1

Strictly proper	$n > m$
Bi Proper	$n = m$
non Proper	$n < m$

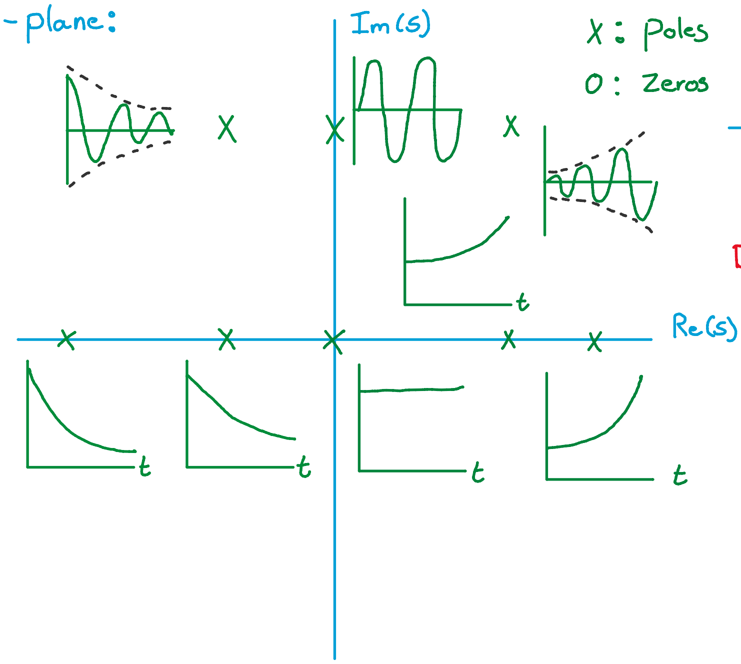


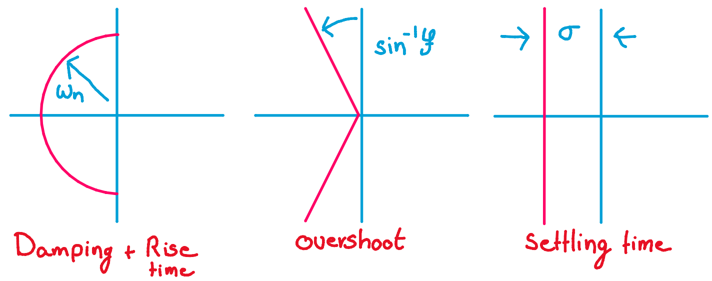
$$\frac{D(s)G(s)}{\underbrace{1 + D(s)G(s)}_{\substack{\text{Closed loop system} \\ \text{characteristic equation}}}}$$

$$\left.\begin{aligned} 1 + kL(s) &= 0 \\ 1 + k\frac{b(s)}{a(s)} &= 0 \\ a(s) + kb(s) &= 0 \\ L(s) &= -1/k \end{aligned}\right\} \text{ equivalent forms}$$

$$\frac{b(s)}{a(s)} = \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

S-plane:





Waveform : e^{st}

Root Locus

Plural: Loci

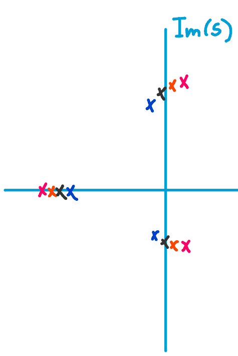
Locus: position or place where something occurs or is situated.

→ Curve or Figure formed by points satisfying an equation

Root locus: $1 + KL(s) = 0 \iff 1 + D(s)G(s) = 0$

$K \in [0, \infty)$

Open loop plant → dynamics closed loop plant



Tedious ✗

Drawing rules ✓ → "But my computer can do this"

→ - Understand change in dynamic response

- Understand consequences on response for controller design

Example:

$$G(s) = \frac{s+2c}{s(s+c)} \qquad D(s) = 1$$

$$1 + D(s)G(s) = 1 + \frac{s+2c}{s(s+c)}$$

characteristic equation: $s^2 + s + c(s+2) = 0$

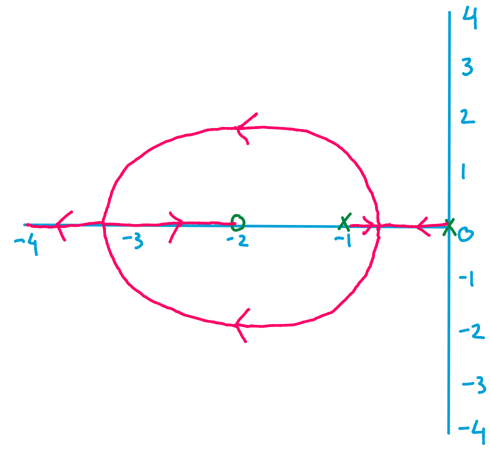
$$L(s) = \frac{s+2}{s(s+1)} \qquad b(s) = s+2 \qquad m = 1 \qquad z_i = -2$$

$$a(s) = s(s+1) \qquad n = 2 \qquad p_i = 0, -1$$

$$K = c$$

$$r_1, r_2 = -\frac{k+1}{2} \pm \frac{\sqrt{k^2 - 6k + 1}}{2}$$

Complex when <0
$0.172 < K < 5.828$



Summary

- Closed loop system analysis by varying K and using open loop system
- Solve characteristic equation $1 + KL(s) = 0$
- Changing pole location changes dynamic response
- Demonstration of simple example

2 - Root Locus method - Drawing rules

Root Locus drawing rules

$$\frac{D(s)\,G(s)}{1+D(s)G(s)} = \frac{K L(s)}{\underbrace{1+KL(s)}_{\text{Characteristic equation}}}$$

roots of $1+KL(s)=0$ → closed loop poles

Varying K changes dynamic response

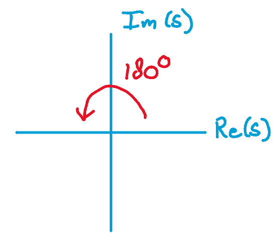
$$b(s) = \prod_{i=1}^{m} (s-z_i)$$

$$a(s) = \prod_{i=1}^{n} (s-p_i)$$

$$\left.\begin{aligned} &1+KL(s)=0 \\ &1+K\frac{b(s)}{a(s)}=0 \\ &a(s)+Kb(s)=0 \\ &L(s)=-1/K \end{aligned}\right\} \text{equivalent}$$

Key definitions

Definition I: $1+KL(s)=0$ must be satisfied under varying K, $K \in \underbrace{[0,\infty)}_{\mathbb{R}^+}$



$L(s) = -1/K$ $\quad Re^{st} = Re^{j\omega t}$ ($s=j\omega$ ← phase; ← amplitude)

Definition II: Root locus of $L(s)$ is the set of points in the s-plane where the phase of $L(s)$ is 180°

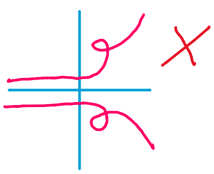
ℓ : integer value

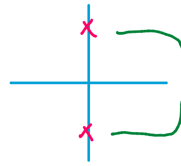
$$\sum \psi_i - \sum \phi_i = 180^\circ + 360^\circ(\ell-1)$$ ← easy to check!

$a(s) = s^n + a_1 s^{n-1} + \dots + a_n = 0$

hard to solve...

LTI system:





Complex conjugate behaviour

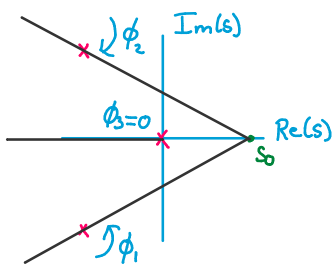
Drawing rules

Rule 1: n branches start at the poles of $L(s)$, m branches end at the zeros of $L(s)$

$$a(s) + K\,b(s) = 0 \quad \Rightarrow \quad \begin{cases} a(s) = 0 & K = 0 \quad \text{poles } (n) \\ b(s) = 0 & K \to \infty \quad \text{zeros } (m) \end{cases}$$

Strictly proper TF: n−m branches go to ∞

Rule 2: The loci are on the real axis to the left of an odd number of poles and zeros



S_0 does not satisfy Definition II

Rule 3: For large s and K, the loci that go to infinity are asymptotic lines radiating at a fixed angle from a central point, the centre of gravity of the poles.

$$\phi_\ell = \frac{180° + 360°(\ell - 1)}{n - m} \qquad \ell = 1, 2, \ldots, n-m$$

loci going to infinity: n−m

angles at which the asymptotes branch out

$$\alpha = \frac{\sum p_i - \sum z_i}{n - m}$$ only real parts needed!

Complex conjugate pair: $\sum c \pm di = c + di + c - di = 2c$

Centre of gravity of the asymptotes

Intermediate example

$$G(s) = \frac{s+1}{(s-1)(s^2+4s-8)} \qquad m=1 \qquad z_i=-1$$
$$n=3 \qquad p_i = 1,\ -2 \pm 2j$$

$D(s) = 1$

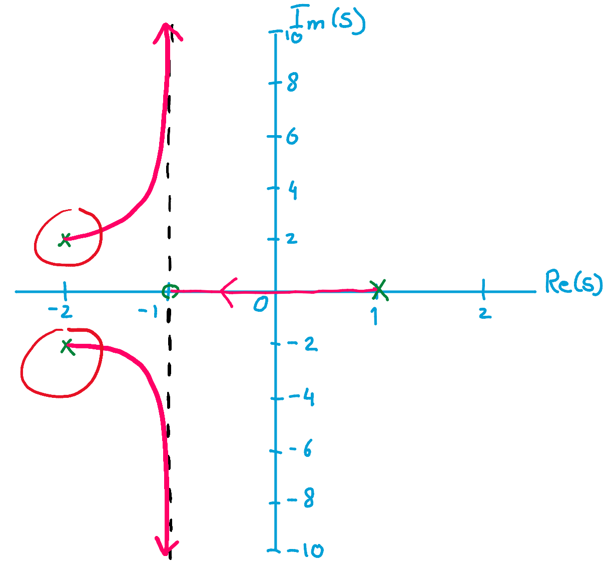
$1 + D(s)\,G(s) = 1 + kL(s) = 0 \qquad n-m=2$

$$\phi_\ell = \frac{180^\circ + 360^\circ(\ell-1)}{n-m} \qquad \ell = 1,\ \overset{=2}{n-m}$$

$$\phi_1 = \frac{180^\circ + 0^\circ}{2} = 90^\circ \qquad \phi_2 = \frac{180^\circ + 360}{2} = 270^\circ$$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{1-2-2-(-1)}{3-1} = -1$$

?



Rule 4: The departure angle of a branch of the locus from a single pole is given by

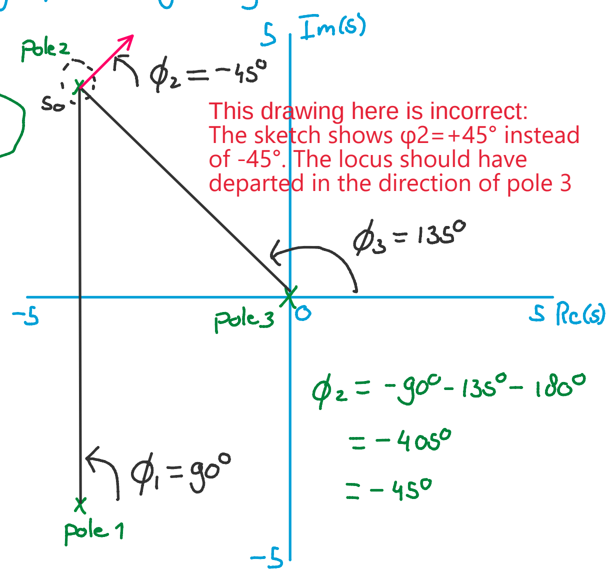
$$\phi_{dep} = \underbrace{\sum \psi_i}_{\text{sum of zeros}} - \underbrace{\sum_{i \neq dep} \phi_i}_{\text{sum of poles not considered}} - 180^\circ$$

Definition II

$$q\,\phi_{\ell,dep} = \sum \psi_i - \sum_{i \neq \ell,dep} \phi_i - 180^\circ - 360^\circ(\ell-1) \qquad \ell = 1, 2, \ldots, \overset{=1}{q}$$

q: repeated poles

$$q\,\psi_{\ell,arr} = \sum \phi_i - \sum_{i \neq \ell,arr} \psi_i + 180^\circ + 360^\circ(\ell-1) \qquad \ell = 1, 2, \ldots, q$$



$$\phi_2 = -90^\circ - 135^\circ - 180^\circ$$
$$= -405^\circ$$
$$= -45^\circ$$

Rule 5: The locus can have multiple roots at points on the locus and the branches approach a point of q roots with angles seperated by $\frac{180^\circ + 360^\circ(\ell-1)}{q}$

angles of departure and arrival: Continuation Locus

$$\left.\begin{aligned} G(s) &= \frac{s+2c}{s(s+c)} \\ D(s) &= 1 \end{aligned}\right\} \quad 1 + kL(s) = 1 + k\,\overset{=c}{\frac{s+2}{s(s+1)}} = 0$$

$$r_1, r_2 = -\frac{k+1}{2} \pm \frac{\sqrt{k^2-6k+1}}{2}$$

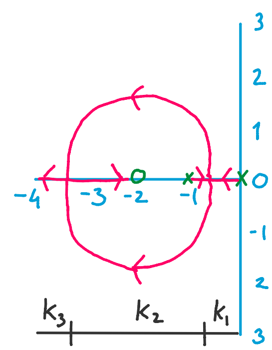
Complex when $0{,}172 < K < 5.828$

$K = K_1$

$K = K_1 + K_2$

$K = K_1 + K_2 + K_3$

apply Rule 4 ?!



Summary

* Drawing rules for Root Locus ▷ understanding closed-loop behaviour through its open-loop plant
* Two definitions 1: $1 + kL(s) = 0$ $k \in \mathbb{R}^+$ 2: $\sum \psi_i + \sum \phi_i = 180^\circ + 360^\circ(\ell - 1)$
* Rule 1: Branches from pole to zero or pole to infinity

  Rule 2: Phase requirement, Loci on the real axis to the left of odd numbers of poles and zeros

  Rule 3: Asymptotic behaviour (angles + central point) $\phi_\ell = \dfrac{180^\circ + 360^\circ(\ell-1)}{n-m}$ $\ell = 1, 2, \dots, n-m$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

Rule 4: Angles of departure and arrival

$q\,\phi_{\ell,dep} = \sum \psi_i + \sum_{i \neq \ell,dep} \phi_i - 180^\circ - 360^\circ(\ell - 1)$ $q\,\psi_{\ell,arr} = \sum_{i \neq \ell,arr} \psi_i + \sum \phi_i + 180^\circ + 360^\circ(\ell-1)$ $\ell = 1, 2, \dots, q$

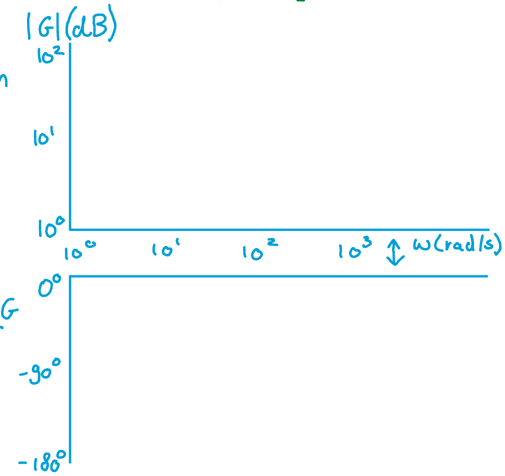
Rule 5: Multiple roots at point on the Locus $\dfrac{180^\circ + 360^\circ(\ell-1)}{q}$

↳▷ Continuation locus

Frequency response analysis

- Popular method for controller design
- Provides good designs even when plant model is not precisely known
- Experimental data can be used for design purposes

$[\text{Rad/s}] \propto [\text{Hz}]$



Frequency response: response of system to sinusoidal inputs

Knowledge on location poles and zeros

(Transfer function)

Mathematical preliminaries required

Steady state behaviour

Two Facts:

1. If all poles are stable (LHP), $\underbrace{\text{natural response} \to 0}$

   → Steady state response

   Purely sinusoidal

2. Frequency of steady state output remains equal to the applied sinusoidal input.

   Phase and magnitude change → bode plots

   - $G(t) \cdot a$
   - $G(t) \cdot 1/a$
   - $\frac{d\,G(t)}{dt}$
   - $\int G(t)\,dt$
   - $G_1(t) + G_2(t)$
   - $G_1(t) + (-G_2(t))$

Frequency dependent TF

$$G(s) = \frac{s^m + b_1 s^{m-1} + \dots b_m}{s^n + a_1 s^{n-1} + \dots + a_n} = \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

$$\frac{\prod_{i=1}^{m} (j\omega - z_i)}{\prod_{i=1}^{n} (j\omega - p_i)}$$

$$S = \underbrace{\cancel{\sigma}^{\,0}}_{\text{Real}} + \underbrace{j\omega}_{\text{Imaginary}} \leftarrow \text{Frequency}$$

Real: Transient response

Imaginary: steady-state response

only consider $s = j\omega$!

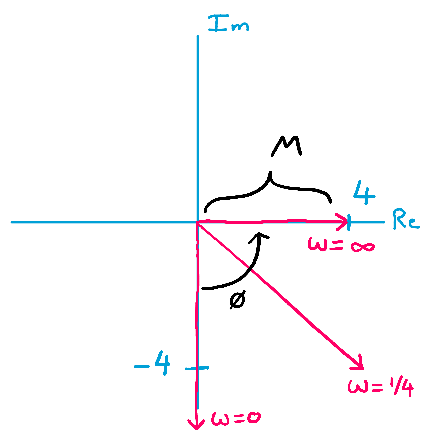
Pole / Zero representation

Polar form: $G(j\omega_0) = \underbrace{M}_{\text{Magnitude}} e^{\overbrace{j\phi}^{\text{Phase}}}$

$$M = |G(j\omega_0)| = |G(s)|_{s=j\omega_0} =$$

$$\sqrt{\{Re[G(j\omega_0)]\}^2 + \{Im[G(j\omega_0)]\}^2}$$

$$\phi = \angle G(j\omega_0) = \arctan\left(\frac{Im[G(j\omega_0)]}{Re[G(j\omega_0)]}\right)$$



$$y(t) = 4u(t) + \int u(t)\,dt$$

$$y(s) = (4 + 1/s)\,U(s)$$

$$G(s) = \frac{4s+1}{s} = \frac{4j\omega + 1}{j\omega} =$$

$$4 - \frac{1}{\omega} j$$

Decibels & composite system

- Measure power gain in dB

$$|G|_{dB} = 10\log_{10}\left(\frac{P_2}{P_1}\right) = 10\log_{10}\left(\frac{V_2}{V_1}\right)^2 = \overbrace{20}^{2\cdot 10}\log_{10}\left(\frac{V_2}{V_1}\right)$$

$$G(j\omega) = \frac{\vec{S_1}\,\vec{S_2}}{\underbrace{\vec{S_3}\,\vec{S_4}\,\vec{S_5}}_{\text{phasors}}} = \frac{M_1 e^{j\phi_1} M_2 e^{j\phi_2}}{M_3 e^{j\phi_3} M_4 e^{j\phi_4} M_5 e^{j\phi_5}} = \left(\frac{M_1 M_2}{M_3 M_4 M_5}\right) e^{j(\phi_1+\phi_2-\phi_3-\phi_4-\phi_5)}$$

$$\log_{10}|G(j\omega)| = \log_{10} M_1 + \log_{10} M_2 - \log_{10} M_3 - \log_{10} M_4 - \log_{10} M_5$$

Why bode plots?

- Quickly draw bode plots and identify properties of complex composite systems

- But Matlab can do this for me ...
  - ↳ understand effects of poles and zeros
  - ↳ Deduce form of compensation and at what frequencies
  - ↳ Sanity check

Bode form of transfer function

$s = j\omega$

$$K\,G(s) = K\frac{(s-z_1)(s-z_2)\dots}{(s-p_1)(s-p_2)\dots}$$

Not the same!

$$K\,G(j\omega) = K_0(j\omega)^n\frac{(j\omega\tau_1+1)(j\omega\tau_2+1)\dots}{(j\omega\underbrace{\tau_a}_{\text{Break point Frequencies}}+1)(j\omega\tau_b+1)\dots}$$

Three classes:

1. $K_0(j\omega)^n$
2. $(j\omega\tau+1)^{\pm 1}$
3. $\left[\left(\underbrace{j\omega/\omega_n}_{\text{natural Frequency}}\right)^2 + 2\underbrace{\zeta}_{\text{Damping}}\frac{j\omega}{\omega_n}+1\right]^{\pm 1}$

Break point: $\omega = 1/\tau$

$$j\omega\tau+1\begin{cases}\omega\tau \ll 1, & j\omega\tau+1 \cong 1\\ \omega\tau \gg 1, & j\omega\tau+1 \cong j\omega\tau\end{cases}$$

Contributions of individual terms add up in bode plot!

Summary

- Bode plots $\begin{cases} \text{magnitude plot} & \text{magnitude [dB] against frequency [rad/s]} \\ \text{Phase plot} & \text{phase [degrees] against frequency [rad/s]} \end{cases}$
- Frequency response covers sinusoidal inputs
  ↳ changes in magnitude and phase, no change in frequency
- Steady-state response $s = j\omega$ $\quad \dfrac{\prod_{i=1}^{m} (j\omega - z_i)}{\prod_{i=1}^{n} (j\omega - p_i)}$

  Polar form: $G(j\omega_0) = M e^{j\phi}$

- TF is composition of individual terms $\begin{cases} \text{Phase: addition of terms} \\ \text{Magnitude: product zeros / product poles} \xrightarrow{\text{Log}} \text{Summation of terms} \end{cases}$
- $|G|dB = 10 \log_{10} \left(P_2/P_1 \right) = 20 \log_{10} \left(V_2/V_1 \right)$
- Bode form

$$K G(j\omega) = K_0 (j\omega)^n \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\ldots}{(j\omega\underbrace{\tau_a}_{\text{Break point frequencies}} + 1)(j\omega\tau_b + 1)\ldots}$$

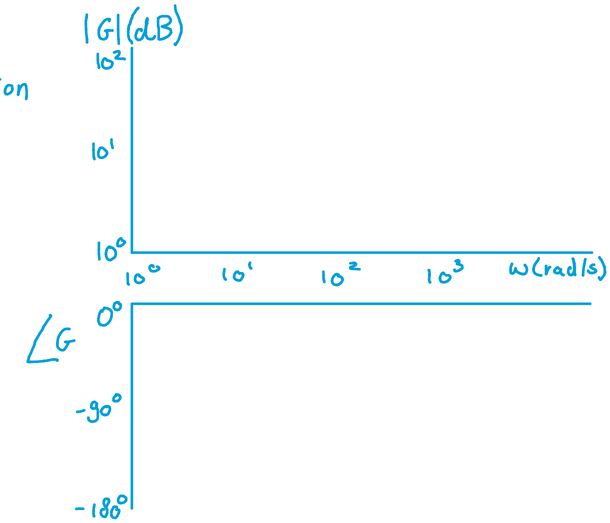
Three classes: ← up Next!

1. $K_0 (j\omega)^n$ 2. $(j\omega\tau + 1)^{\pm 1}$ 3. $\left[\left(j\omega/\omega_n \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^{\pm 1}$

3 - Frequency response - Type 1 and type 2 building blocks

Frequency response analysis

- Frequency response analysis — System identification / Controller design
  - ▷ visualisation with Bode Plots



- Steady state behaviour
  - ▷ only sinusoidal inputs
  - ▷ Math operations
    - addition/subtraction
    - multiplication/division
    - Integrating / derivating

  } change in magnitude & phase, not frequency

- Polar Form $G(j\omega_0) = \underbrace{M}_{\text{phase}} e^{j\phi}$ } magnitude

- Composite system

$$G(j\omega) = \frac{M_1 e^{j\phi_1} M_2 e^{j\phi_2}}{M_3 e^{j\phi_3} M_4 e^{j\phi_4} M_5 e^{j\phi_5}} = \frac{M_1 M_2}{\underbrace{M_3 M_4 M_5}} e^{j(\phi_1 + \phi_2 - \phi_3 - \phi_4 - \phi_5)}$$

$$|G(j\omega)| = \log(M_1 + M_2 - M_3 - M_4 - M_5)$$

- $s = j\omega$ Frequency dependent $K\,G(j\omega) = K_0 (j\omega)^n \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\ldots}{(j\omega\underbrace{\tau_a}_{}+1)(j\omega\tau_b + 1)\ldots}$

  Break point Frequencies

- Three classes:
  1. $K_0 (j\omega)^n$
  2. $(j\omega\tau + 1)^{\pm 1}$
  3. $\left[(j\omega/\omega_n)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^{\pm 1}$

Class 1: singularities at the origin

- $K\,G(s) = K\dfrac{(s-z_1)(s-z_2)\dots}{(s-p_1)(s-p_2)\dots} \xLeftrightarrow{s=j\omega} \underbrace{K_0(j\omega)^n}_{\text{Class 1}} \dfrac{(j\omega\tau_1+1)(j\omega\tau_2+1)\dots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\dots}$

  degree of poles/zeros

  $K \neq K_0$!

- at the origin:
  - Pole $1/s$ or s^{-1}, integration
  - Zero s or s^{1}, derivation

  } Laplace

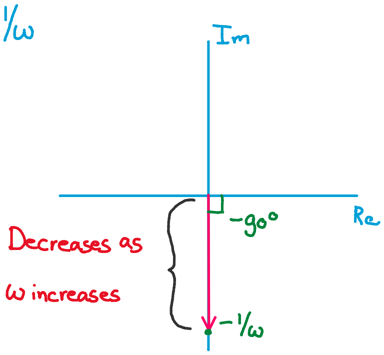
Illustration

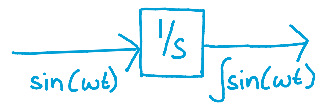
$\underline{\underline{\dfrac{1}{s}}}$ $\qquad \dfrac{1}{j\omega} = 0 + \dfrac{1}{\omega}\underbrace{\dfrac{1}{j}\dfrac{j}{j}} = 0 - \dfrac{1}{\omega}j$

$\dfrac{1}{\sqrt{-1}}\cdot\dfrac{\sqrt{-1}}{\sqrt{-1}} = \dfrac{\sqrt{-1}}{-1} = -j$

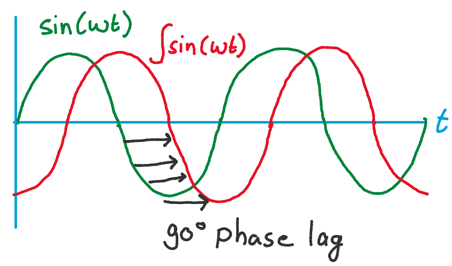
Real : 0

Imag : $-1/\omega$





$= -\dfrac{1}{\omega}\cos(\omega t) = -\dfrac{1}{\omega}\sin(\omega t + \underbrace{\phi}_{-90°})$



Class 1: in bode form $1/s$

$\omega = 1$: gain $= 1$ $20\log(1) = 0$

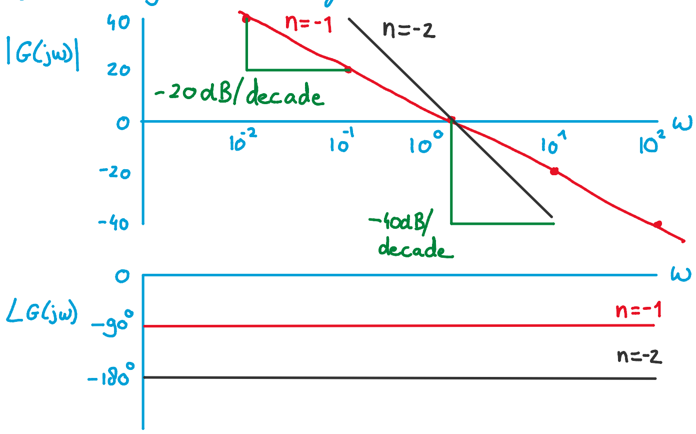
$\omega = 10$ gain $= 1/10$ $20\log(\frac{1}{10}) = -20$

$\omega = 100$ gain $= 1/100$ $20\log(1/100) -40$

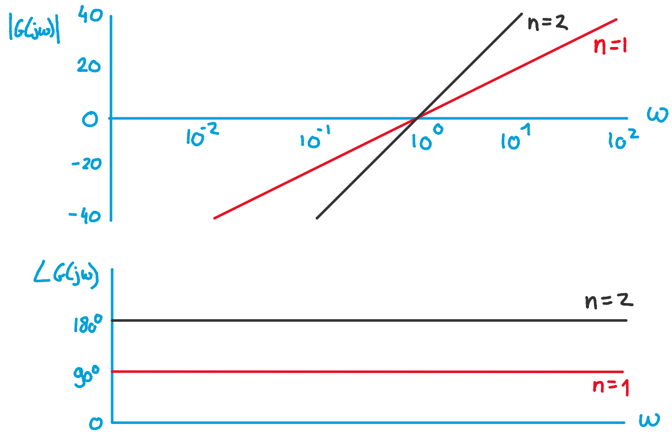
Bode gain-phase relationship

Gain : $n \times 20dB/decade$

Phase : $n \times 90^\circ$

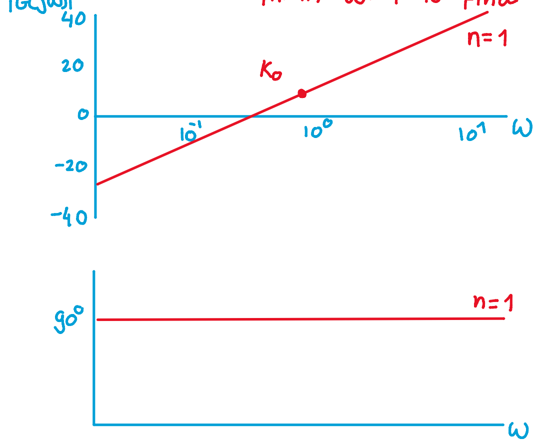


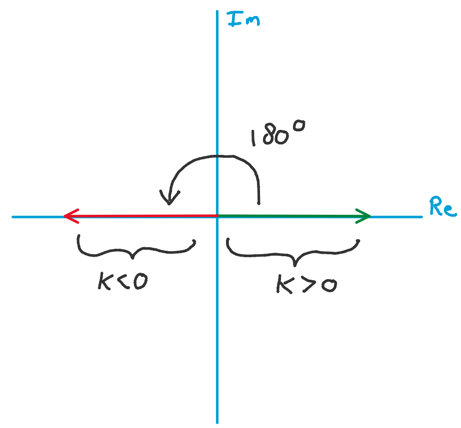
S^n



$\underline{K_0}(j\omega)^n$ $\log K_0|(j\omega)^n| = \log K_0 + n\log(j\omega)$

Fill in $\omega = 1$ to Find K_0





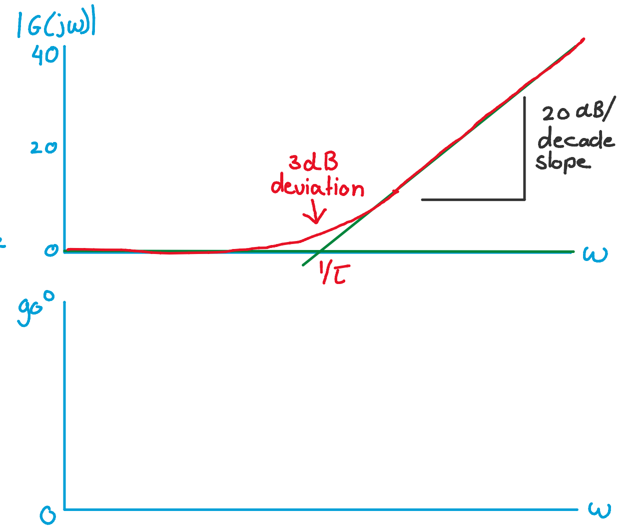
Class 2: First order terms

$$(j\omega\tau+1)^{\pm 1} \quad \begin{cases} +1 & \text{Zero} \\ -1 & \text{Pole} \end{cases}$$

$$\begin{cases} \omega\tau \ll 1, & j\omega\tau+1 \cong 1 \longrightarrow 0\,\text{dB} \\ \omega\tau \gg 1, & j\omega\tau+1 \cong j\omega\tau \longrightarrow \pm 20\,\text{dB/decade} \end{cases}$$

Break point frequency τ

$\omega = 1/\tau$ Break point



Pole

$$\frac{1}{j\omega\tau+1} \cdot \frac{1-j\omega\tau}{1-j\omega\tau} = \frac{1-j\omega\tau}{1+\omega^2\tau^2}$$

Re: $1/1+\omega^2\tau^2$

Im: $-\omega\tau/1+\omega^2\tau^2$

Gain: $\left[\left(\frac{1}{1+\omega^2\tau^2}\right)^2 + \left(\frac{-\omega\tau}{1+\omega^2\tau^2}\right)^2\right]^{1/2} = \left(\frac{\cancel{1+\omega^2\tau^2}}{(1+\omega^2\tau^2)^{\cancel{2}}}\right)^{1/2}$

$= -20\log\left(\sqrt{1+\omega^2\tau^2}\right)$ in decibels

Phase: $\arctan(Im/Re) = \arctan(-\omega\tau)$

Zero

$1+j\omega\tau$

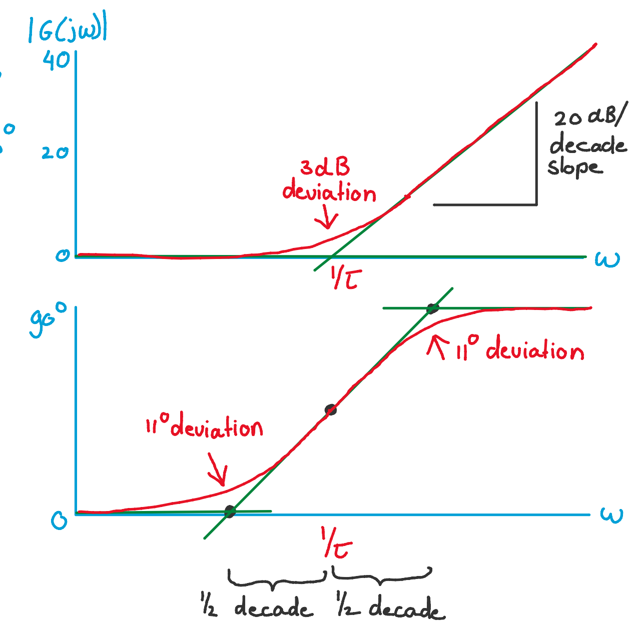
Re: 1

Im: $\omega\tau$

Gain: $\sqrt{1+\omega^2\tau^2} = 20\log\left(\sqrt{1+\omega^2\tau^2}\right)$

phase: $\arctan(\omega\tau)$

$$\begin{cases} \omega\tau \ll 1: & j\omega\tau+1 \cong 1, & \angle 1 = 0^\circ \\ \omega\tau \gg 1: & j\omega\tau+1 \cong j\omega\tau, & \angle j\omega\tau = 90^\circ \\ \omega\tau \cong 1: & & \angle(j\omega\tau+1) \cong 45^\circ \end{cases}$$



Summary:

- Class 1: $K_o(j\omega_o)^n$

  Class 2: $(j\omega\tau+1)^{\pm 1}$

evaluate at $\omega = 1$ rad/s

- Bode gain-phase relationship

  gain : $n \times 20$ dB/decade $\longrightarrow$ magnitude through K_o $\log K_o|(j\omega)^n| = \log K_o + n\log(j\omega)$

  phase : $n \times 90°$

- First order systems at break point frequency τ

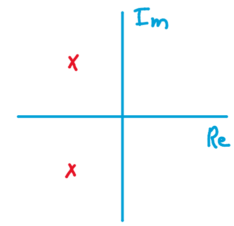
  $\hookrightarrow$ Assymptotic behaviour used to draw frequency behaviour

- Up next: Class 3 terms & Frequency response of poles and zeros in RHP

3 - Frequency response - Type 3 building blocks, nonminimum-phase systems & sketching procedure

Frequency response analysis

- 3rd class / Class 3: Second order complex conjugate pair of poles and zeros
- Frequency response of RHP zero
- Drawing rules bode plot



Class 3

$$\left[\underbrace{\left(\frac{j\omega}{\omega_n}\right)^2}_{\text{natural Frequency}} + 2\overbrace{\zeta}^{\text{Damping coefficient}}\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$$

Break point $\frac{1}{\omega_n}$

Behaviour:

- $\omega/\omega_n \gg 1$, ± 40dB/decade
- $\omega/\omega_n \cong 1$, resonant peak M_r
- Phase lead / lag 180°, 90° at ω_n

$$\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1}$$ (Complex conjugate pole pair)

Re: $$\frac{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}$$

Im: $$\frac{-2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}$$

Three ranges:

1) $\omega \ll \omega_n$
2) $\omega \gg \omega_n$
3) $\omega = \omega_n$

$\omega \ll \omega_n$

$\cancel{\frac{\omega}{\omega_n}} \to 0$ is very small

Re: $$\frac{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2} \cong \frac{1}{1+0}$$

Im: $$\frac{-2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2} \cong \frac{0}{1+0}$$

Gain: $\sqrt{1^2+0^2} = 1 = 0$dB

phase: $\arctan(0/1) = 0^\circ$

$\omega \gg \omega_n$

$$1-\left(\frac{\omega}{\omega_n}\right)^2 \cong -\left(\frac{\omega}{\omega_n}\right)^2$$

Re: $$\frac{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2} \cong \frac{-\left(\frac{\omega}{\omega_n}\right)^2}{\left(-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\underbrace{\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}_{\text{small compared to}}} \cong \frac{-\left(\frac{\omega}{\omega_n}\right)^2}{\left(\left(\frac{\omega}{\omega_n}\right)^2\right)^2} = -\left(\frac{\omega}{\omega_n}\right)^{-2}$$

Im: $$\frac{-2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2} \cong \frac{-2\zeta}{\left(\frac{\omega}{\omega_n}\right)^3}$$

Gain: $$\left[\left(1/(\omega/\omega_n)\right)^2+\underbrace{\left(-2\zeta/(\omega/\omega_n)^3\right)^2}_{\text{small compared to}}\right]^{1/2} \approx \left(\left(\frac{\omega}{\omega_n}\right)^{-4}\right)^{1/2} = \left(\frac{\omega}{\omega_n}\right)^{-2}$$

In decibels: $20\log_{10}\left(\frac{\omega}{\omega_n}\right)^{\pm2} = \pm 2\cdot 20\log_{10}(\omega/\omega_n)$

Phase: $$\arctan\left(\frac{-2\zeta\left(\frac{\omega}{\omega_n}\right)^{-3}}{\left(\frac{\omega}{\omega_n}\right)^{-2}}\right) = \arctan\left(-2\zeta\left(\frac{\omega}{\omega_n}\right)^{-1}\right)$$

$= 0^\circ$ sign information ignored, $\underline{\pm 180^\circ}$

$\omega = \omega_n$

$$\frac{\omega_0}{\omega_n} = 1$$

Re: $$\frac{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2} = 0$$

Im: $$\frac{-2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2} = \frac{-1}{2\zeta}$$

Gain: $\sqrt{\left(\frac{-1}{2\zeta}\right)^2} = \frac{1}{2\zeta}$, In decibels $\pm 20\log_{10}(2\zeta)$

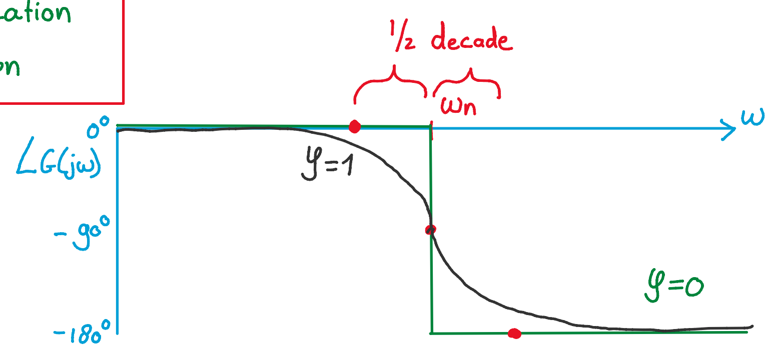
$$\begin{cases} \zeta = 1/2 & \log_{10}(2\zeta) = 0 \\ \zeta < 1/2 & \log_{10}(2\zeta) \quad \text{negative} \\ \zeta > 1/2 & \log_{10}(2\zeta) \quad \text{positive} \end{cases}$$

	Zeros	Poles
$\zeta < 1/2$	−	+
$\zeta > 1/2$	+	−

+ : amplification

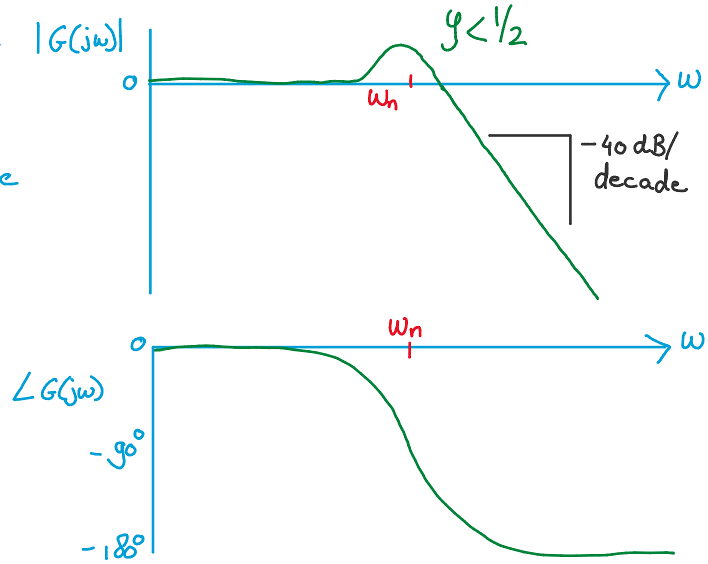
− : attenuation

Phase: $\arctan\left(\frac{Im}{\underbrace{Re}_{\text{Very small}}}\right) = \infty$



Class 3 bode plot

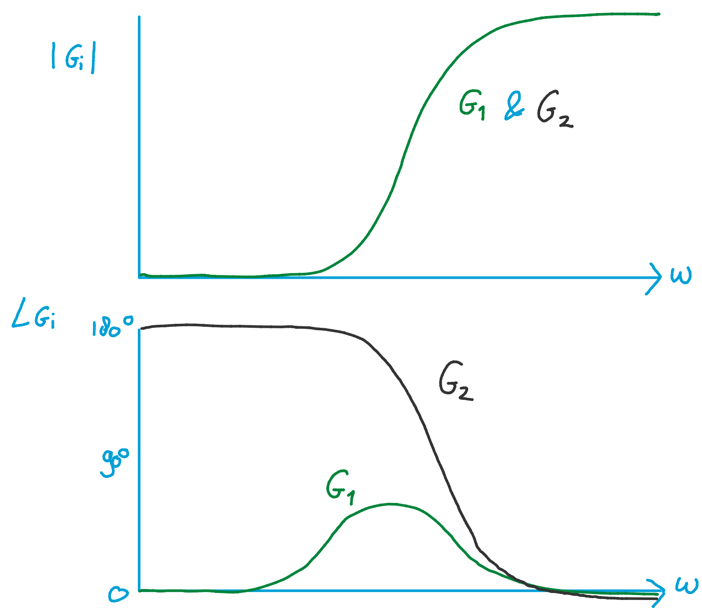
Complex conjugate pole pair



RHP zeros

$G_1 = 10\,\frac{s+1}{s+10}$

$G_2 = 10\,\frac{s-1}{s+10}$



RHP zero

Gain: 20 dB/decade

Phase: −90°

$|G_1(j\omega)| = |G_2(j\omega)|$

Design procedure bode plots

1.) Write TF in bode form, identify building blocks and their break point frequencies
2.) Determine n in $K_0 (j\omega_0)^n$, plot low frequency asymptote through K_0
    Bode gain-phase: gain $n \times 20$ dB/decade phase $n \times 90^\circ$
3.) Extend asymptote untill first break point. Change slope based on pole/zero at break points
    Continue in ascending order
4.) At the break points, magnitude deviates 3dB (1st order) or ζ-dependent (2nd order)
5.) Plot low frequency asymptote $n \times 90^\circ$
6.) Phase change of $\pm 90^\circ$ or $\pm 180^\circ$ at break points
7.) Draw asymptotes for phase and intersect them
8.) Account for 11° phase difference at asymptote intersection

Summary

- Frequency response of 2nd order system as function of ζ

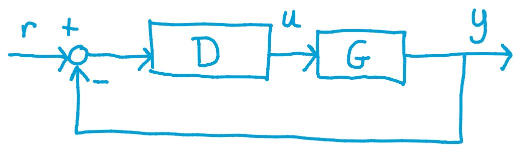
$$\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1\right]^{\pm 1}$$

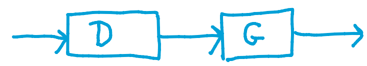
- Zero in LHP vs RHP
- Overall design procedure

  Transfer function → Bode plot → System properties

5 - Performance specifications

Frequency-Domain design Method



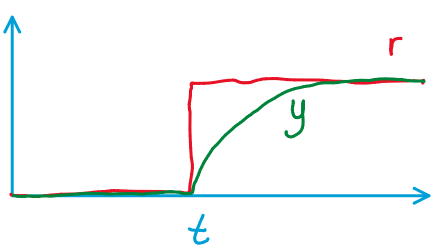


Design feedback controllers using open-Loop Frequency response Function

Loop shaping: "Shape" the loop gain $L(s) = D(s)\,G(s)$ by choosing $D(s)$

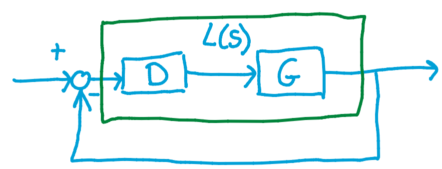
Closed-loop requirements	Open-loop properties
• Stability, Robustness	Nyquist, PM & GM
• Steady-state reference tracking	System type
• Overshoot	PM
• Bandwidth	Crossover Frequency ω_c

Steady state tracking error



Feedback system responds

Steady state tracking at best



Assume now $L(s) \approx \frac{1}{s^n}\hat{L}(0)$ For $s \to 0$

$$E(s) = \frac{1}{1+L(s)}R(s) \approx \frac{s^n}{s^n + \hat{L}(0)} \quad \text{For } s \to 0$$

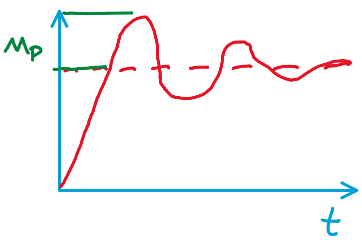
$$r(t) = \begin{cases} \text{Step} \\ \text{Ramp} \\ \text{Parabola} \end{cases} \Rightarrow R(s) = \begin{cases} 1/s \\ 1/s^2 \\ 1/s^3 \end{cases} \Rightarrow R(s) = \frac{1}{s^{k+1}} \quad k \geq 0$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s E(s) \approx \frac{s^{\cancel{n+1}}}{s^n + \hat{L}(0)} \frac{1}{s^{\cancel{k+1}}} = \begin{cases} \infty & n<k \\ \frac{1}{1+\hat{L}} & n=k=0 \\ \frac{1}{\hat{L}} & n=k>0 \\ 0 & n>k \end{cases}$$

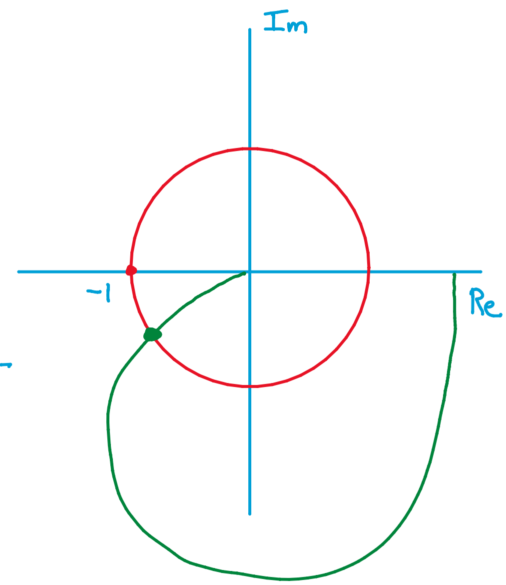
FVT

Steady-state error $\rightarrow 0$ if slope of Bode diagram of $L(s)$ is "steep" enough

Limited overshoot $\Leftrightarrow$ Damping $\Leftrightarrow$ Phase Margin



$$M_p = e^{\frac{-\pi \xi}{\sqrt{1-\xi^2}}}$$



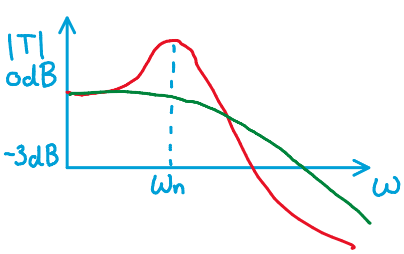
$$T(s) = \frac{L(s)}{1+L(s)} \approx \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \Rightarrow L(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

PM is the angle at which $|L(j\omega)| = 1$

$$\omega = \sqrt{\sqrt{4\xi^4+1} - 2\xi^2}\, \omega_n \qquad PM = \arctan\left(\frac{2\xi}{\sqrt{\sqrt{4\xi^4+1}-2\xi^2}}\right) \approx 100\xi$$

Limited overshoot $\Rightarrow$ enough PM

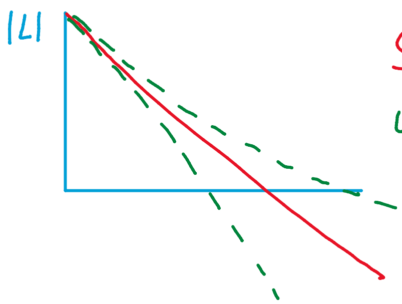
Bandwidth



$$T(s) = \frac{L(s)}{1+L(s)} \qquad |L(j\omega)| = 1$$

$$T(j\omega) = \begin{cases} 0\text{ dB} & \text{for } \omega \ll \omega_c \\ -3\text{ dB} & \text{for } \omega \approx \omega_c \\ L(j\omega) & \text{for } \omega \gg \omega_c \end{cases}$$

$$\omega_c \leq \omega_{BW} \leq 2\omega_c$$



System model uncertainty bound

Have enough bandwidth but not too much!

uncertainty larger at larger ω

So, many closed loop properties can be inferred for loop gain $L(s) = D(s)\,G(s)$

Stability & stability margins : Nyquist

Steady-state tracking error : Slope of Bode

Overshoot : Sufficient PM

Closed-loop bandwidth : 0dB of $L(j\omega)$

Given performance requirements, we can specify what $L = DG$ should look like

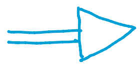
Q: How to tune $D(s)$ such that $L(s)$ has the desired shape

$\Longrightarrow$ Compensator design

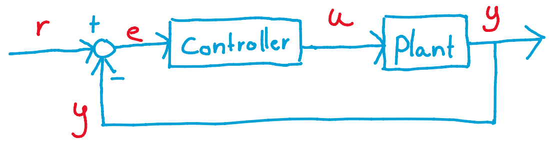
5 - Compensator design

Compensator design

open loop



Closed loop

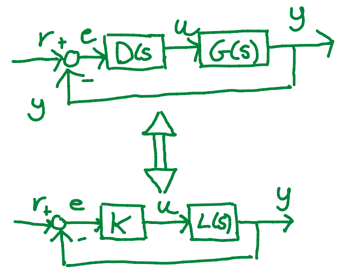


not able to track reference

May be able to track reference

Controller design:

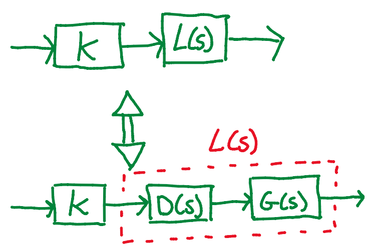
- Meet set of requirements (closed loop)
- closed loop properties ⟺ properties of open loop gain $L(s)$



Transfer Functions

- $K\,G(s) = K_o\,(j\omega)^n \dfrac{(j\omega\tau_1+1)(j\omega\tau_2+1)\ldots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\ldots}$
- Break point $\omega = 1/\tau$

$KL(s) \Longleftrightarrow K\,D(s)\,G(s)$



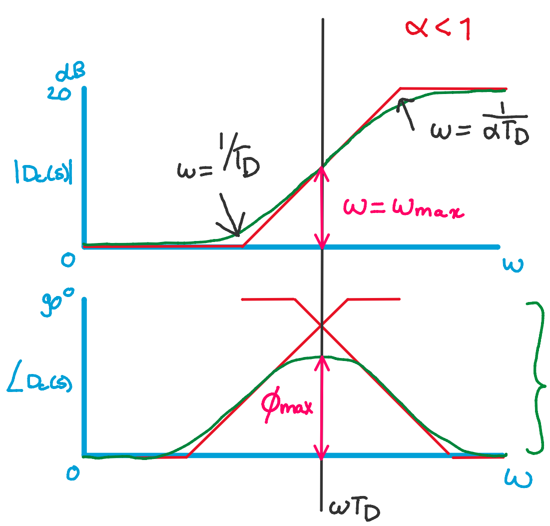
D(s) is the controller to design!

Lead compensation

Lead compensator: $D_c(s) = \dfrac{T_D s+1}{\alpha T_D s+1}$

Pole/zero $\underbrace{\text{break point}}_{1/T_D}$ frequency ratio $1/\alpha$

$\alpha < 1$



90° phase addition in theory
up to 55° practical

Mathematical properties:

- Phase contribution $\phi = \tan^{-1}(T_D\omega) - \tan^{-1}(\alpha T_D \omega)$
- Frequency at which phase is maximum $\omega = 1/T_D\sqrt{\alpha}$
  - ⊳ Take derrivative and set equal to 0 $\quad \frac{d}{dx}\tan^{-1}(ax) = \frac{a}{a^2x^2+1}$
- Maximum phase contribution $\sin(\phi_{max}) = \frac{1-\alpha}{1+\alpha} \Leftrightarrow \alpha = \frac{1-\sin(\phi_{max})}{1+\sin(\phi_{max})}$

1) $\frac{T_D s+1}{\alpha T_D s+1} = \frac{j/\sqrt{\alpha}+1}{\sqrt{\alpha}j+1}$

2) $\tan(\phi_{max}) = \frac{\frac{1}{\sqrt{\alpha}} + \sqrt{\alpha}}{2}$

3) $\sin^2(\phi_{max}) = \frac{\tan^2(\phi_{max})}{1+\tan^2(\phi_{max})} = \frac{(1-\alpha)^2}{(1+\alpha)^2}$

Phase lead only depends on α !

Design procedure

- Serves as a starting point
- More than one iteration might be necessary

Look at system type and do not add unneeded poles at origin

1.) Set value of gain K to
  - Satisfy error requirement. May not be possible with only Lead!
  - Meet bandwidth requirement.

2.) Evaluate PM with found value K

3.) Add $10°$ of phase ⊳ determine how much phase is needed (ϕ_{max})

4.) Determine α with $\alpha = \frac{1-\sin(\phi_{max})}{1+\sin(\phi_{max})}$

5.) Pick ω_{max} at ω_c. ⊳ Zero at $1/T_D = \omega_{max}\sqrt{\alpha}$ & pole at $\frac{1}{\alpha T_D} = \frac{\omega_{max}}{\sqrt{\alpha}}$

6.+7.) Draw compensated open-loop frequency response, evaluate PM ⊳ Iterate if needed

Guidelines!

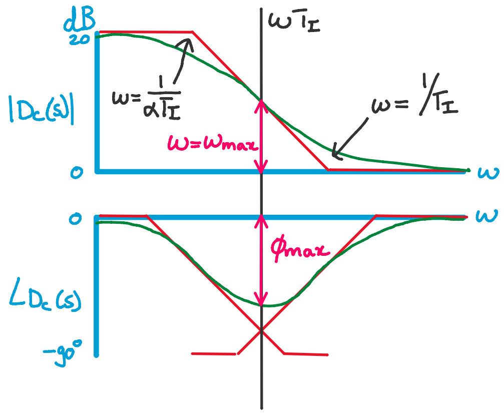
No guarantee!

Lag compensation

- Used to attenuate gain where PM is sufficient (decrease in phase!)

$$D_c(s) = \alpha \frac{T_I s+1}{\alpha T_I s+1} \qquad \alpha > 1$$

place pole and zero well below ω_c

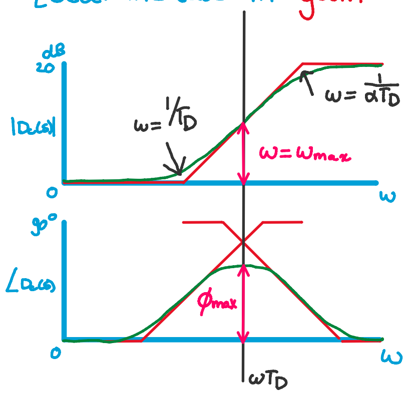
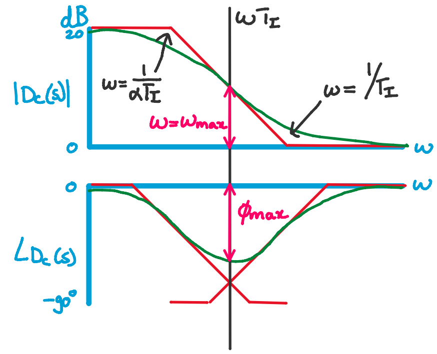


Minimizes this affecting PM

Design procedure

1.) Determine the open loop gain K that meets PM requirement
2.) Draw the bode plot
3.) Evaluate the low frequency gain
4.) Determine α to meet low frequency gain
5.) Choose $1/T_I$ to be one $\underbrace{\text{octave}}_{\times 2}$ to one $\underbrace{\text{decade}}_{\times 10}$ below the new ω_c
6.) Frequency of pole is $1/\alpha T_I$
7.) Iterate when needed

Guidelines

	Lead compensator	Lag compensator
• TF	$D_c(s) = \frac{T_D s+1}{\alpha T_D s+1}$ $\alpha<1$	$D_c(s) = \alpha\frac{T_I s+1}{\alpha T_I s+1}$ $\alpha>1$
• Goal	Increase PM	attenuate gain
• Side effect	Local increase in gain	adds phase lag
• Bode		

What about PID controllers?

	PD	Lead
gain (High rad/s)	infinite	Finite
Phase	90°	≤ 55°

	PI	Lag
gain (low rad/s)	Infinite	Finite
phase	- 90°	< 90°

Nyquist rotation, problems at -1 point

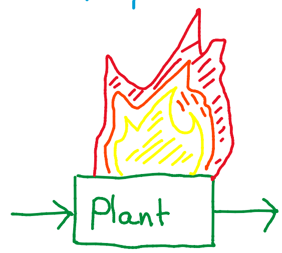
PID ⟶ Combine, advanced exercise

In reality...

- Can we compensate any LTI system? ✗
- actuators, sensors and system have speed response limitations

  → $\omega_c \gg \omega_n$ → Saturation

  Fail to meet objectives



Summary

- Controllers can be created out of building blocks

  Integrator, lead compensator, lag compensator

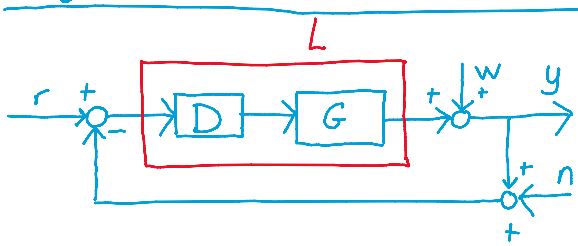
- Lead compensator $\dfrac{T_D s+1}{\alpha T_D s+1}$ $\quad \alpha<1 \qquad \omega = 1/(T_D\sqrt{\alpha})$

- Lag compensator $\alpha \dfrac{T_I s+1}{\alpha T_I s+1}$ $\qquad \alpha = \dfrac{1-\sin(\phi_{max})}{1+\sin(\phi_{max})}$

- Real life limitations such as saturation

6- Limitations - Closed-loop specifications

Design tradeoffs & Performance limitations



Performance: Time domain $\rightarrow$ PM, ω_c
Tracking $\rightarrow$ Low frequency slope, gain

} Loop gain L

$$y = w + DG(r-y-n)$$
$$(1+DG)\,y = w + DG(r-n)$$

$$y = \underbrace{\frac{1}{1+L}}_{\text{Low}} w + \underbrace{\frac{L}{1+L}}_{\text{Low}}(r-n)$$

Q: can we make both $S=\frac{1}{1+L}$ and $T=\frac{L}{1+L}$ Low?

$$\underbrace{r-y}_{\neq e \text{ (tracking error)}} = r - \frac{1}{1+L}w - \frac{L}{1+L}(r-n) = \underbrace{\frac{1}{1+L}}_{\text{Low}}(r-w) + \underbrace{\frac{L}{1+L}}_{\text{Low}}n$$

Sensitivity

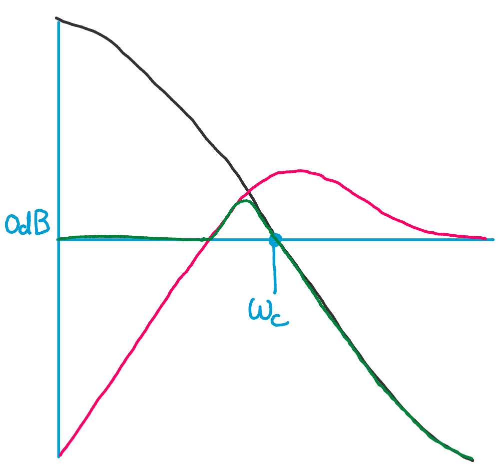
Sensitivity: $S=\frac{1}{1+L}$

Complementary sensitivity: $T=\frac{L}{1+L}$

$$S+T = \frac{1}{1+L} + \frac{L}{1+L} = \frac{1+L}{1+L} = 1 \quad \text{For all } \omega\,!$$

$\Rightarrow$ Reducing effects of disturbances and measurement noise not possible

$S+T=1$ is a fundamental limitation!



$|L| \gg 1$ For $\omega \to 0$

$|L| \to 0$ For $\omega \to \infty$

$$S = \frac{1}{1+L} \approx \begin{cases} 1/L \to 0 & \text{For } \omega \to 0 \\ 1 & \text{For } \omega \to \infty \end{cases}$$

$$T = \frac{L}{1+L} \approx \begin{cases} 1 & \text{For } \omega \to 0 \\ L & \text{For } \omega \to \infty \end{cases}$$

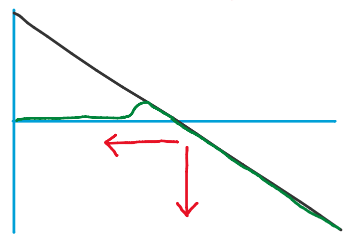
Examples

$y = Sw + T(r-n)$

$e \neq r-y = S(r-w) + Tn$

- $n(j\omega)$ high for some ω
  $\Rightarrow$ $T(j\omega)$ low at that ω
- $T(j\omega) = 1 \quad \omega < \omega_c$
  $\Rightarrow$ $T(j\omega) = L(j\omega) \quad \omega > \omega_c$

} Make L Low !



D = 0 $\Rightarrow$ no feedback

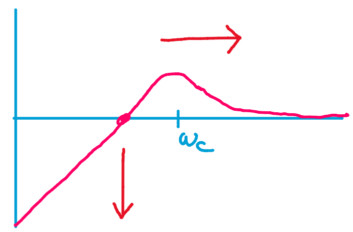
no measurement

noise amplification

- Given $r(j\omega)$
  $\Rightarrow$ good $r-y = Sr$, $S(j\omega)$ low at that ω

$$\begin{cases} S(j\omega) = 1 & \omega > \omega_c \\ S(j\omega) = 1/L & \omega < \omega_c \end{cases}$$

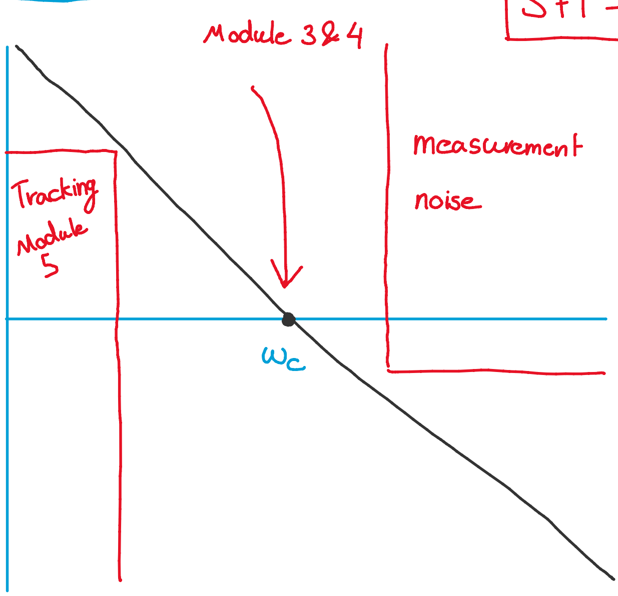
also for disturbances !



high ω_c

Conclusion

$\boxed{S+T=1}$



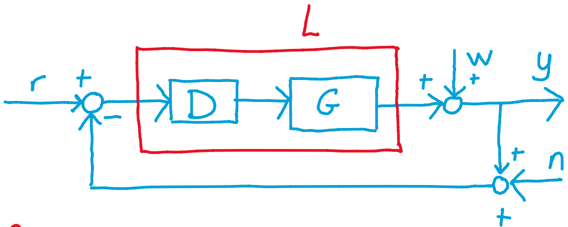
at ω_c: Slope of -1 ($\sim 90°$)

for good PM

(Bode phase gain relationship)

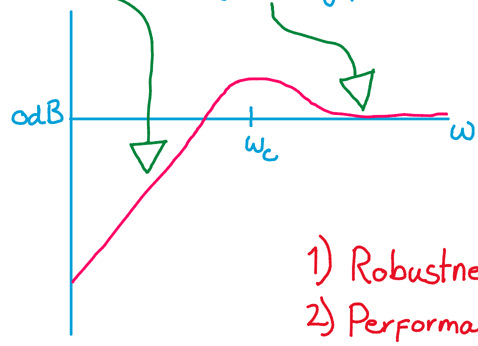
6 - limitations - Bode sensitivity integral

Bode sensitivity integral

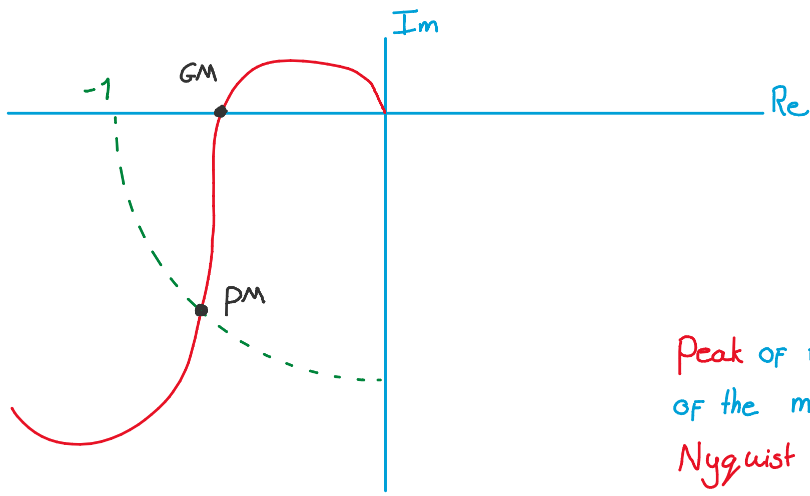


Sensitivity Function is important for controller design

$$S = \frac{1}{1+L} = \begin{cases} 1/L & \text{if } |L| \text{ is large} \quad \omega \ll \omega_c \\ 1 & \text{if } |L| \text{ is small} \quad \omega \gg \omega_c \end{cases}$$



1) Robustness
2) Performance limitations



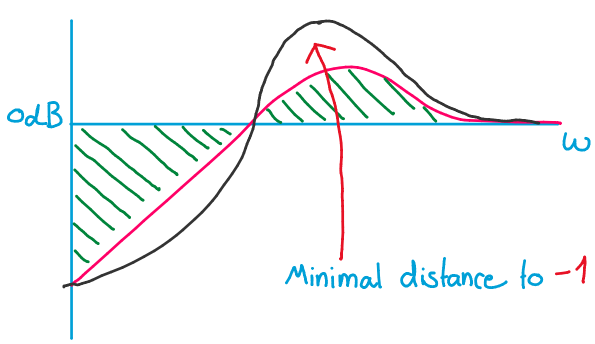
- PM & GM are important
- Minimal distance between −1 & $L(s)$?

$$\min_{\omega} |L(j\omega)+1| = \max_{\omega} \left| \frac{1}{L(j\omega)+1} \right|$$

Peak of the sensitivity function is the reciprocal of the minimum distance to −1 in the Nyquist diagram

Bode sensitivity integral

Waterbed effect



Green areas add up to zero!
- Natural log of $|S(j\omega)|$
- Linear scale ω

↳ under certain conditions

Cauchy's integral Formula

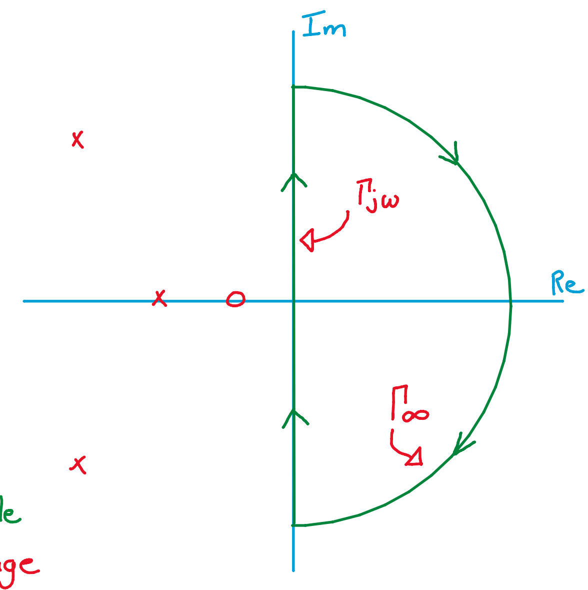
Cauchy's integral Formula for holomorphic functions that are analytic along a contour Γ

$$\oint_{\Gamma} \ln(|S(s)|)\,ds = 0$$

$$= \oint_{\underbrace{\Gamma_{j\omega}}_{\text{along imaginary axis}}} \ln(|S(s)|)\,ds + \oint_{\underbrace{\Gamma_{\infty}}_{\text{Circle with infinite radius}}} \ln(|S(s)|)\,ds$$

Laplace variable

s is huge



Along the semicircle Γ_∞

$$\ln(|S(s)|) = -\ln(|1+L(s)|) \approx -|L(s)| \approx g\underbrace{s}_{\text{Laplace variable}}^{-n_r}$$

If s is huge $\Rightarrow L(s)$ is small

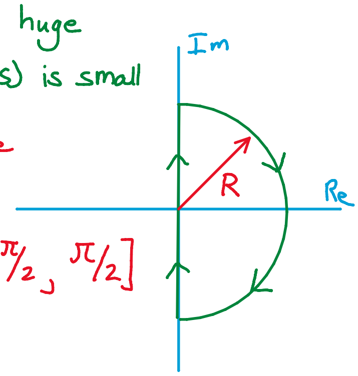
$$\oint_{\Gamma_\infty} \ln(|S(s)|)\,ds = \int_{-\pi/2}^{\pi/2} \ln(|S(Re^{j\theta})|)\,dRe^{j\theta}$$

change of variables: $\frac{dRe^{j\theta}}{d\theta}d\theta$

n_r is the relative degree

#poles − #zeros

$Re^{j\theta}$ $\quad \theta \in [-\pi/2, \pi/2]$



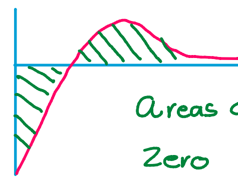
$$\approx \int_{-\pi/2}^{\pi/2} g(Re^{j\theta})^{-n_r}\,\overbrace{Rje^{j\theta}d\theta} = \int_{-\pi/2}^{\pi/2} gj\underbrace{R^{1-n_r}}_{=0 \text{ when } R\to\infty} e^{j\theta(1-n_r)}\,d\theta = 0 \quad \text{if } n_r > 1$$

$$\Rightarrow \oint_{\Gamma} \ln(|S(s)|)\,ds = \oint_{\Gamma_{j\omega}} \ln(|S(s)|)\,ds = 2j\underbrace{\int_0^\infty \ln(|S(j\omega)|)\,d\omega}_{=0} = 0$$

- For open-loop stable and minimum phase systems, with a relative degree of 2 or more

$\underbrace{\text{open-loop stable}}_{\text{No RHP poles}}$ and $\underbrace{\text{minimum phase}}_{\text{No RHP zeros}}$

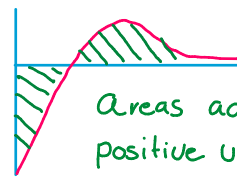
$$\int_0^{\infty} \ln(|S(j\omega)|)\,d\omega = 0$$



areas add upp to zero

- For open-loop unstable systems with unstable poles p_i

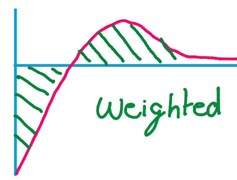
$$\int_0^{\infty} \ln(|S(j\omega)|)\,d\omega = \pi \sum_i \mathrm{Re}\{p_i\}$$



areas add up to positive value

- For open-loop nonminimum phase systems with RHP zero $z_0 = \sigma_0 + j\omega_0$

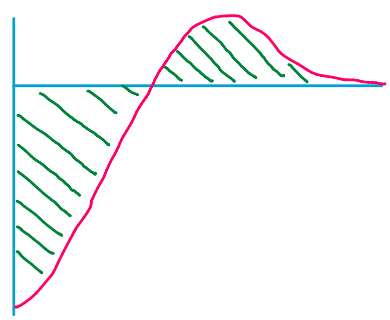
$$\int_{-\infty}^{\infty} \ln(|S(j\omega)|)\underbrace{\frac{\sigma_0}{\sigma_0^2 + (\omega-\omega_0)^2}}_{\text{weight}}\,d\omega = \pi \sum_i \ln\left|\frac{p_i + z_0}{\overline{p_i} + z_0}\right|$$



weighted sum

Summary

- Peak of Sensitivity Function can be used a measure of Robustness
  PM & GM are important
- Bode sensitivity integral $\longrightarrow$ waterbed effect
- $\oint_{\Gamma_{j\omega}} \ln(|S(s)|)\,ds + \oint_{\Gamma_{\infty}} \ln(|S(s)|)\,ds$



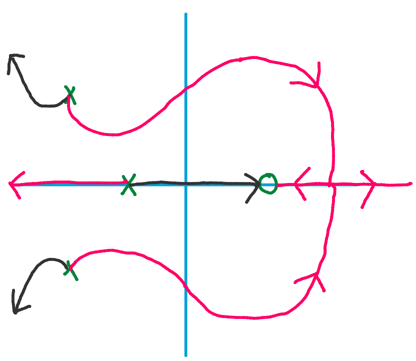
6 - Limitations - RHP poles & zeros

Performance limitations, RHP poles & zeros

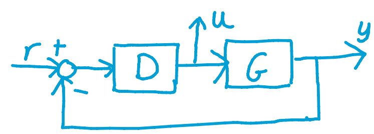
- Root Locus method
  - ⟶ shows how feedback changes closed loop behaviour
- $S + T = 1$
  - ⟶ asymptotic behaviour of sensitivity and complementary sensitivity
- Bode sensitivity integral & waterbed effect
  - ⟶ $\oint_{\Gamma} \ln(|S(s)|)\, ds = 0$ Cauchy's integral formula

⟹ Fundamental limitation in performance

Performance Limitations of RHP zeros



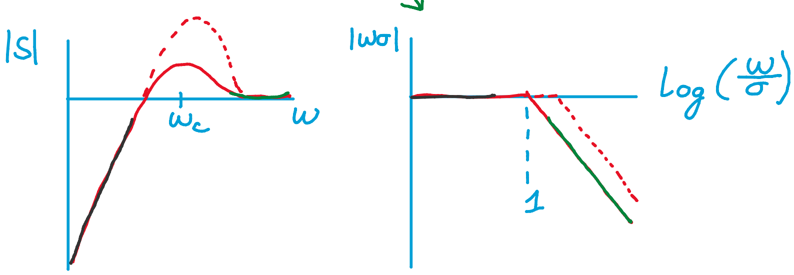
Eventually all closed-loop poles move to open-loop zeros



$$u = \underbrace{\frac{D}{1+DG}}_{\text{Control sensitivity}} r \approx \begin{cases} \frac{1}{G} r & \text{for } \omega_c > \omega \quad ✗ \\ D r & \text{for } \omega_c < \omega \quad ✓ \end{cases}$$

We don't want this approximation to hold for $\omega \approx$ RHP zero because $\frac{1}{G}$ is unstable

$$\int_{-\infty}^{\infty} \ln(|S(j\omega)|)\,\frac{\sigma}{\sigma^2+\omega^2}\,d\omega \qquad \text{RHP zero at } s=\sigma$$

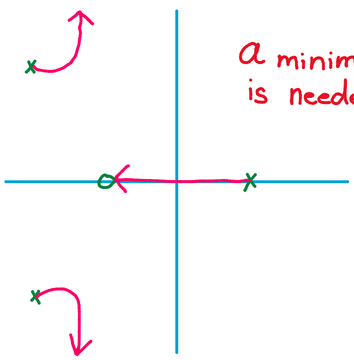


To minimize impact of the unavoidable bump

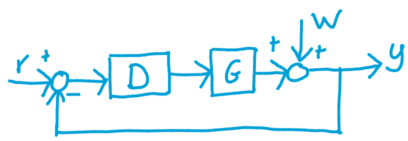
$$\frac{\omega_c}{\sigma} > 1 \iff \omega_c > \sigma$$

$\Longrightarrow$ The bandwidth of a closed-loop system is constrained from above by the open-loop RHP zero

Performance limitations of RHP poles



a minimal gain is needed

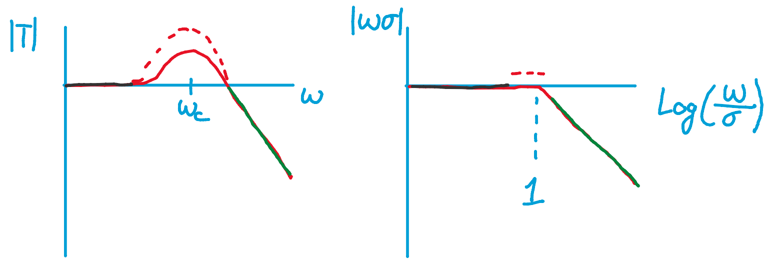


$$y = \underbrace{\frac{G}{1+DG}}_{\text{Process sensitivity}}\, w \approx \begin{cases} Gw & \text{for } \omega > \omega_c \quad X \\ \frac{1}{D} w & \text{for } \omega < \omega_c \quad V \end{cases}$$

unstable, we don't want this approximation to hold for $\omega =$ RHP pole so high bandwidth is needed

$T = \frac{L}{1+L} = \frac{1}{1+1/L}$ Sensitivity of the inverse loop gain

$$\int_{-\infty}^{\infty} \ln\left(|T(j\omega)|\right)\frac{\sigma}{\sigma^2+\omega^2}\,d\omega = \text{constant}$$ For RHP pole at $s=\sigma$



$\Rightarrow$ The bandwidth of a closed-loop system is constrained by an open-loop RHP pole

To minimize the effect of the bump $\omega_c > \sigma$

Summary

- Non minimum phase systems have a maximum bandwidth
- Unstable systems have a minimum bandwidth

7 - State-Space derivation

Deriving state space models

$$y(s)=G(s)\,u(s) \iff \begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t)+Du(t)\end{aligned}$$

- Laplace transform of impuls response (of differential equation)
- Set of first order differential equations

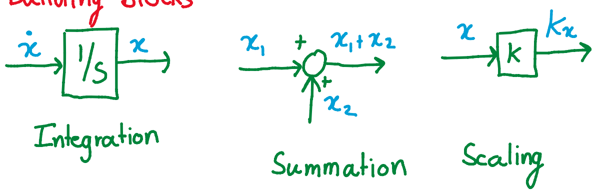
How to represent a differential equation as a state space model?

1) Simulation diagram
2) Modal decomposition
3) control canonical form

$\left.\begin{array}{l}2)\\3)\end{array}\right\}$ of transfer function

Simulation diagram approach

Building blocks



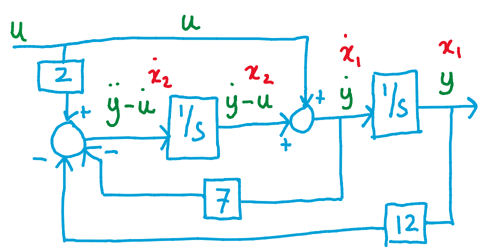
Approach:

1) Group highest derivatives (initially) ✓
2) Integrate (not differentiate) ✓
3) Complete block diagram ✓
4) Output of every integrator is a state ✓

Example

$$\ddot{y}+7\dot{y}+12y=\dot{u}+2u$$

$$\ddot{y}-\dot{u}=2u-7\dot{y}-12y$$



$$\dot{x}_1 = x_2+u$$

$$\dot{x}_2 = 2u-7\dot{x}_1-12x_1$$

$$= -5u-7x_2-12x_1$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -12 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ -5\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Make modal decomposition of TF

$$G(s) = \frac{n(s)}{d(s)} = \frac{n(s)}{(s+p_1)\cdots(s+p_n)}$$

Not always possible
(Complex poles, repeated poles)

$$= C_0 + \frac{C_1}{s+p_1} + \ldots + \frac{C_n}{s+p_n}$$

$$= G_0 + G_1 + \ldots + G_n$$

$$G_i(s) = \frac{C_i}{s+p_i} \iff (s+p_i)x_i(s) = c_i u(s)$$

$$\dot{x}_i(t) = -p_i x_i(t) + c_i u(t)$$

$$\dot{x} = \begin{bmatrix} -p_1 & & \\ & \ddots & \\ & & -p_n \end{bmatrix} x + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} x + C_0 u$$

$$(s^2 + 7s + 12)\, y(s) = (s+2)\, u(s)$$

$$G(s) = \frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$A(s+4) + B(s+3) = s+2$$
$$A + B = 1$$
$$4A + 3B = 2$$
$$A = -1$$
$$B = 2$$

$$\dot{x} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0\, u$$

Use control canonical Form

Strictly proper

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots a_n} = \frac{s+2}{s^2+7s+12}$$

Monic denominator

$n=2$, $b_1 = 1$, $b_2 = 2$, $a_1 = 7$, $a_2 = 12$

$$s x_1 = -a_1 x_1 - a_2 x_2 - \ldots - a_n x_n + u \Rightarrow (s^n + a_1 s^{n-1} + \ldots + a_n) x_n = u$$

$$s x_2 = x_1 \quad x_1 = s x_2 \Rightarrow x_1 = s^2 x_3 \Rightarrow x_i = s^{n-i} x_n$$

$$s x_3 = x_2 \quad x_2 = s x_3$$

$$\vdots$$

$$y = b_1 x_1 + b_2 x_2 + \ldots b_n x^n = (b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n) x_n$$

$$\Rightarrow \quad y(s) = \frac{b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 \quad 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} x + 0\, u$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x + 0\, u$$

3 different methods:

1) Simulation diagram

2) Modal decomposition

3) Control canonical form

1) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} x + \begin{bmatrix} 1 \\ -5 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

2) $\dot{x} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$

3) $\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 2 \end{bmatrix} u$

Similarity?

Analysis of State-Space models

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \Longrightarrow$$

3 different methods to derive State-Space models of

$\ddot{y} + 7\dot{y} + 12y = \dot{u} + 2u$

Simulation diagram

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} x + \begin{bmatrix} 1 \\ -5 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

modal decomposition

$$\dot{x} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

control canonical form

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

1 differential equation : 3 State Space models

1 transfer function

Q_1: Do these State-Space models describe the same system? If so, how do we go from one to the other?

$\hookrightarrow$ equivalent

Q_2: How do we compute transfer functions from these?

Q_3: Do State-Space models have poles and zeros? If so, how to compute/determine them?

TF from SS

$$s\overset{\times}{x} = Ax + Bu \Longrightarrow sIx - Ax = Bu \Longleftrightarrow (sI - A)x = Bu$$
$$y = Cx + Du$$
$$y = (C(sI-A)^{-1}B + D)u$$

$$G(s) = \boxed{C(sI-A)^{-1}B + D} = \frac{s+2}{s^2 + 7s + 12} \checkmark$$

Uniqueness

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

equivalence

Now define $x = Tz$ ← Full rank matrix, invertible matrix ← Similarity transformation

$$\dot{x} = T\dot{z}$$

$$\dot{z} = T^{-1}\dot{x} = T^{-1}Ax + T^{-1}Bu = T^{-1}ATz + T^{-1}Bu$$
$$y = CTz + Du$$

$$\boxed{\begin{aligned} \dot{z} &= \hat{A}z + \hat{B}u \\ y &= \hat{C}z + Du \end{aligned} \quad \text{with } \hat{A} = T^{-1}AT \quad \hat{B} = T^{-1}B \quad \hat{C} = CT}$$

Same transfer function?

$$G(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D = CT(sT^{-1}T - T^{-1}AT)^{-1}T^{-1}B + D \qquad (AB)^{-1} = B^{-1}A^{-1}$$
$$= CT(T^{-1}(sI-A)T)^{-1}T^{-1}B + D$$

Same transfer function!

Specific choices for T?

- Modal form: diagonal form $\hat{A} = T^{-1}AT$ $\quad Av = \lambda v$ (eigenvector, eigenvalue) $\quad A[v_1 \dots v_n] = [v_1 \dots v_n]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$
- Control Canonical form $\quad AT = T\hat{A}$

Fact:

n is the dimension of the x vector

1) Controllability matrix $\quad [\hat{B}\ \hat{A}\hat{B} \dots \hat{A}^{n-1}\hat{B}] = [T^{-1}B\ \ T^{-1}A\cancel{T}\cancel{T^{-1}}B \dots (T^{-1}A\cancel{T})^{n-1}\cancel{T^{-1}}B] = T^{-1}[B\ AB \dots A^{n-1}B]$

2) $\hat{A} = T^{-1}AT \Leftrightarrow \hat{A}T^{-1} = T^{-1}A$

$[\hat{B}\ \hat{A}\hat{B} \dots \hat{A}^{n-1}\hat{B}] = \begin{bmatrix}1 & x & \cdots & x\\ 0 & 1 & & x\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix} = \begin{bmatrix}t_1\\ \vdots\\ t_n\end{bmatrix}[B\ AB \dots A^{n-1}B]$ $\leftarrow t_i$ rows of T^{-1}

$t_n = [0\,0 \dots 0\,1][B\ AB \dots A^{n-1}B]^{-1}$ $\leftarrow$ If this inverse exists

$$\begin{bmatrix}-a_1 & -a_2 & \cdots & -a_n\\ 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 1 & 0\end{bmatrix}\begin{bmatrix}t_1\\ t_2\\ \vdots\\ t_n\end{bmatrix} = \begin{bmatrix}t_1\\ t_2\\ \vdots\\ t_n\end{bmatrix}A$$

$t_{n-1} = t_n A, \quad t_{n-2} = t_{n-1}A = t_n A^2$

$$T^{-1} = \begin{bmatrix}t_n A^{n-1}\\ \vdots\\ t_n A\\ t_n\end{bmatrix}$$ with $t_n = [0\,0 \dots 0\,1][B\ AB \dots A^{n-1}B]^{-1}$ will take any system to control canonical form

Poles and zeros of state space systems

Poles:

$G(s) = \frac{n(s)}{d(s)}$ $\quad d(s) = 0$ gives the poles

$= C(sI - A)^{-1}B = \underbrace{\frac{1}{\det(sI-A)}}_{1/d(s)}\underbrace{C\,\mathrm{adj}(sI-A)\,B}_{n(s)}$

Cramer's rule

$x^{-1} = \frac{1}{\det(x)}\mathrm{adj}(x)$ (determinant, adjugate)

$d(s) = \det(sI - A) = 0$, Recall how to compute eigenvalues? $Ax = \lambda x \Rightarrow (\lambda I - A)x = 0$

$\Rightarrow$ Poles of a state-space model are the eigenvalues of A

For non trivial λ? $\det(\lambda I - A) = 0$

Zeros: $y(t) = 0$ for $u(t) = u_0 e^{z_i t}$ $\quad z_i \in \mathbb{C}$

$\Downarrow \quad \dot{x} = Ax + Bu$

$x(t) = x_0 e^{z_i t}$

$\dot{x} = Ax + Bu \Rightarrow x_0 z_i e^{z_i t} = Ax_0 e^{z_i t} + Bu_0 e^{z_i t}$

$y = Cx + Du \Rightarrow 0 = Cx_0 e^{z_i t} + Du e^{z_i t}$

$\Rightarrow 0 = \underbrace{\begin{bmatrix}-A + z_i I & -B\\ C & D\end{bmatrix}}_{\det = 0}\begin{bmatrix}x_0\\ u_0\end{bmatrix}e^{z_i t}$ $\quad$ non trivial $x_0\ u_0$

Summary

- Transfer function from state-space model

  $G(s) = C(sI - A)^{-1}B + D$

- Equivalent state-space models $(A, B, C, D) = (T^{-1}AT, T^{-1}B, CT, D)$

  For non singular/invertible matrix T

  Specific choices for T to go to modal form

  controllable canonical form

- Poles of SS models, eigenvalues of A
- Zeros of SS models z for which $\det \begin{bmatrix} -A + z_i I & -B \\ C & D \end{bmatrix} = 0$

7 - Linearisation

Linearisation of state space models

Now states change = f(current state, current Input)

$\dot{x} = f(x, u)$

Input (points to u)

State (points to x)

Linear state space models

$\dot{x} = Ax + Bu$

n x 1 matrix

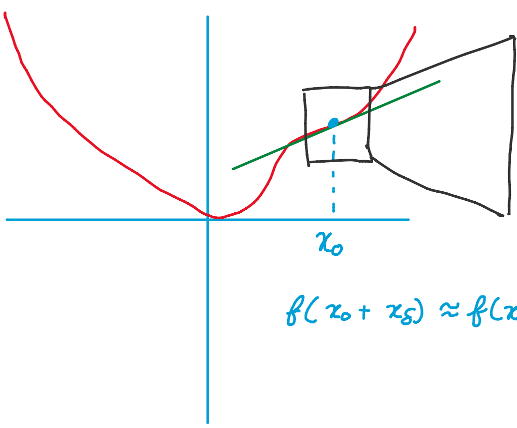
n x n matrix

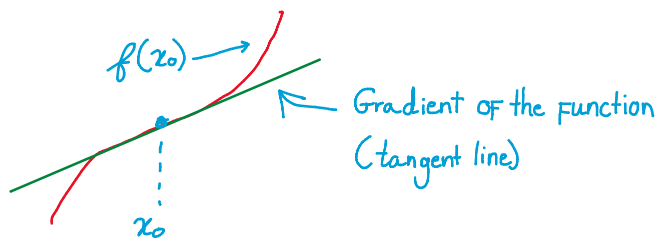
(Single input, Single output (SISO) models for now)

Q: How to apply linear design techniques to nonlinear dynamic systems?

A: Make a "Linear approximation (and hope for the best)

Main idea : Taylor series approximation





$$f(x_0 + x_\delta) \approx f(x_0) + \frac{\partial f}{\partial x}(x_0)\, x_\delta \qquad \frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Two Properties:

1) For $x_\delta = 0$, approximation is exact
2) For $x_\delta = 0$, the first derivative is exact

Now how to apply this "trick" to nonlinear state space models?

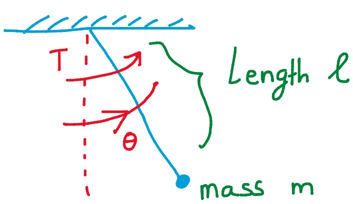
$x = x_0 + x_\delta \qquad u = u_0 + u_\delta$

equilibrium   Pertubations around this equilibrium

$$\dot{x} = \underbrace{\dot{x}_0}_{=0} + \dot{x}_\delta = f(x,u) \approx \underbrace{f(x_0, u_0)}_{=0} + \underbrace{\frac{\partial f}{\partial x}(x_0,u_0)}_{=A} x_\delta + \underbrace{\frac{\partial f}{\partial u}(x_0,u_0)}_{=B} u_\delta$$

$$\dot{x}_\delta = A x_\delta + B u_\delta \quad \text{with} \quad A = \frac{\partial f}{\partial x}(x_0,u_0) \qquad B = \frac{\partial f}{\partial u}(x_0,u_0)$$

Example



$$\underbrace{m\ell^2\ddot{\theta}}_{\text{Inertia}} + \underbrace{m\ell g \sin(\theta)}_{\text{Gravitational Force}} = T \Big\} \text{External Force}$$

$x = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \qquad u = T/m\ell^2 \qquad \omega_0^2 = g/\ell$

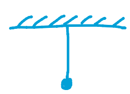
$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} x_2 \\ -\omega_0^2 \sin(x_1) + u \end{pmatrix} = f(x,u)$$

Equilibria where $\dot{x} = 0$

$\hookrightarrow x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad u_0 = 0$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}$$

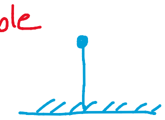
Stable



$\hookrightarrow x_0 = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \quad u_0 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{pmatrix}$$

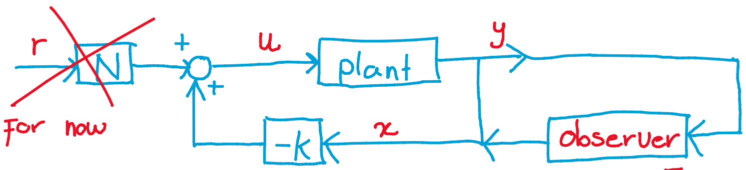
unstable



Linearised models can be used to design linear controllers
For nonlinear systems
(that work close to the equilibrium)

8 - Output feedback design

Output Feedback design method using state-space method



assume we can measure x

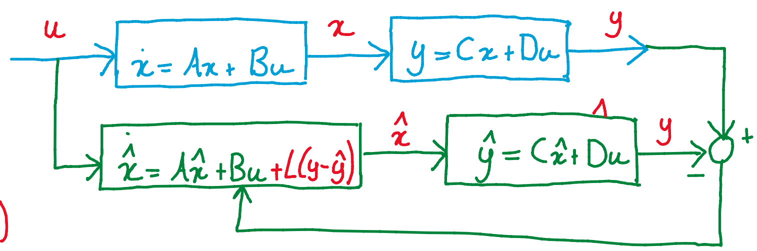
↳ Not realistic (too expensive, not possible)

State observer!

Can we estimate x from y?

$\dot{x} = Ax + Bu$

$y = Cx + Du$



$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$

$\hat{y} = C\hat{x} + Du$

$$\tilde{x} = x - \hat{x} \Rightarrow \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} - A\hat{x} - \cancel{Bu} - L(Cx + \cancel{Du} - C\hat{x} - \cancel{Du})$$

$$= A\tilde{x} - \underbrace{L}_{\text{State observer gain}} C\tilde{x}$$

$$\dot{x} = Ax - B\underbrace{K}_{\text{State Feedback gain}}x$$

State observer design ⟺ **State Feedback design**

Find matrix L such that $\dot{\tilde{x}} = (A - LC)\tilde{x}$ has desired closed-loop poles

$\det(sI - A + LC) = \det(sI - A^T + C^T L^T) = 0$

$L = \text{place}(A^T, C^T, \text{cl-poles})^T$

Find matrix K such that $\dot{x} = (A - BK)x$ has desired closed-Loop poles

$\det(sI - A + BK) = 0 \qquad K = \text{place}(A, B, \text{cl-poles})$

Design observer gain L ⟺ Design state feedback $\dot{\omega}^T = A^T\omega + C^T y$

Duality

Observer	Feedback
A^T	A
C^T	B
B^T	C

Observability

$$\dot{x} = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}x + \begin{bmatrix}1\\1\end{bmatrix}u$$

$$y = [\,0 \;\; 1\,]x$$

We cannot observe this state from the output

Not possible to design an observer for unobservable systems

(~~similar~~ DUAL to controllability)

Controllability

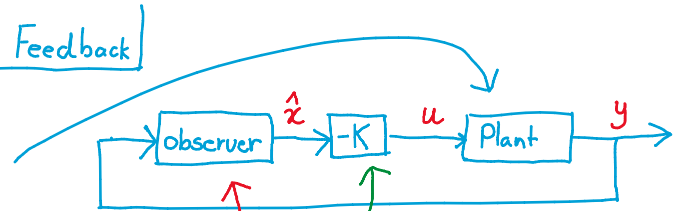
$[B \;\; AB \cdots A^{n-1}B]$ should have rank n

Observability

$[C^T \;\; A^TC^T \cdots (A^{n-1})^TC^T] \Longleftrightarrow \begin{bmatrix}C\\CA\\\vdots\\CA^{n-1}\end{bmatrix}$

should have rank n

Full output Feedback



$$\left.\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}\right\}$$

$$u = -k\hat{x}$$

$$\left.\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y-\hat{y})\\ \hat{y} &= C\hat{x} + Du\end{aligned}\right\}$$

$$\left.\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + LC\tilde{x}\\ \dot{x} &= Ax - Bk\hat{x} = (A-BK)x + Bk\tilde{x}\end{aligned}\right\}$$

$\tilde{x} = x - \hat{x}$

$\hat{x} = x - \tilde{x}$

$$\begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix}A-BK & BK\\ 0 & A-LC\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}$$

Fact: $\det\left(sI - \begin{bmatrix}x & y\\ 0 & z\end{bmatrix}\right) = \det\left(\begin{bmatrix}sI-x & -y\\ 0 & sI-z\end{bmatrix}\right)$

$= \underbrace{\det(sI-x)}_{=0} \cdot \underbrace{\det(sI-Z)}_{\text{or } =0} = 0 \}$ closed loop system Poles = poles of $A-BK$ and $A-LC$

One last remark

$$\left.\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y-\hat{y}) \\ \hat{y} &= C\hat{x} + Du \\ u &= -k\hat{x} \end{aligned}\right\} \begin{aligned} \dot{\hat{x}} &= (A - BK - LC + LDK)\hat{x} + Ly \\ u &= -K\hat{x} \end{aligned}$$

$$u(s) = -k(sI - A - Bk - LC + LDk)^{-1} L\, y(s)$$

Transfer function of state feedback controller and state observer

Summary

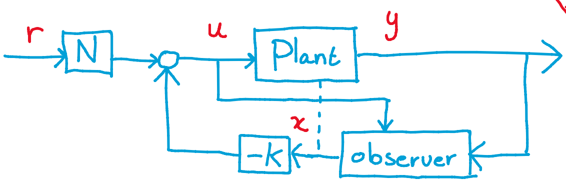
- Observer $\dot{\hat{x}} = A\hat{x} + Bu + L(y-\hat{y})$
  $\hat{y} = C\hat{x} + Du$
- Design of $L \Longrightarrow$ pole placement using A^T and C^T
  Duality between control and estimation
- Observer design requires observability

  $$\text{Rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

  where n is the order of the system
- Output feedback $\Longrightarrow$ design seperate K and L
  Seperation principle / property

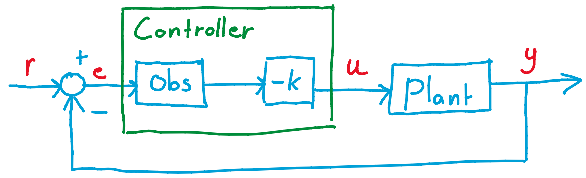
8 - Reference tracking - Integral action

Reference tracking & Integral Action

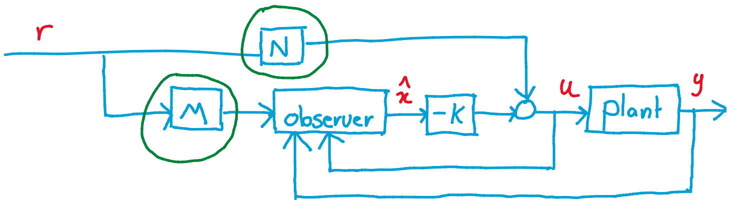


State feedback controller + state observer

?



Feeding back the tracking error into the state observer



More freedom in the state space design method!

$$\begin{cases}\dot{\hat{x}} = Ax + Bu + L(y-\hat{y}) + Mr\\ \hat{y} = C\hat{x} + Du\\ u = -K\hat{x} + Nr\end{cases}$$

How to choose N and M?

Several choices possible (and we'll discuss the two simplest ones)

1) Take $N = N_u + KN_x$ with $\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to ensure steady-state reference tracking

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} + BNr - (A\hat{x} + \cancel{Bu} + L(Cx + \cancel{Du} - C\hat{x} - \cancel{Du}) + Mr)$$
$$= (A-LC)\tilde{x} + BNr - Mr$$

Take $M = BN$ to have that observer error does not depend on the reference

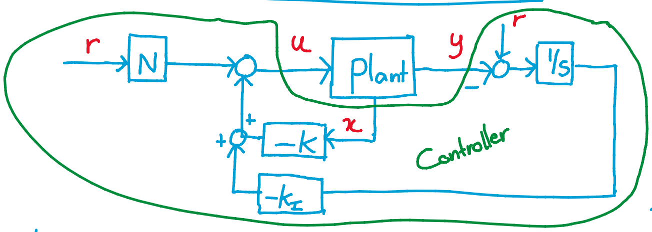
2) Take $N=0$ and $M=-L$ and the controller becomes

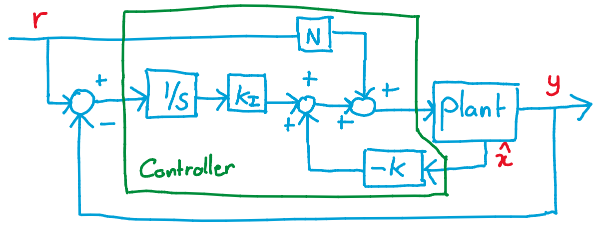
$$\dot{\hat{x}} = A\hat{x} - BK\hat{x} + B\cancel{N}r + L(y - C\hat{x} - D(-K\hat{x} + \cancel{N}r)) + \cancel{Mr}^{\,-L} = (A - BK - LC + LDK)\hat{x} + L(y-r)$$

$u = -K\hat{x}$ ← Negative feedback

Reference tracking like in the frequency design method

Integral Action (and Robust Tracking)





Choose N to achieve reference tracking

- N is not the feedback !
- If N is based on the 'wrong' A and B ⇒ no perfect tracking !

↳ Solution: Integral action !

$$\dot{x}_p = Ax_p + Bu$$

$$x_I = \int_0^t e(\tau)\,d\tau \iff \dot{x}_I = e = Cx + Du - r$$

$$\begin{bmatrix}\dot{x}_I\\ \dot{x}_p\end{bmatrix} = \underbrace{\begin{bmatrix}0 & C\\ 0 & A\end{bmatrix}}_{\hat{A}}\begin{bmatrix}x_I\\ x_p\end{bmatrix} + \underbrace{\begin{bmatrix}D\\ B\end{bmatrix}}_{\hat{B}}u + \begin{bmatrix}1\\ 0\end{bmatrix}r$$

Integral action

$$u = \underbrace{\begin{bmatrix}k_I & k_p\end{bmatrix}}_{\hat{K}}\begin{bmatrix}x_I\\ x_p\end{bmatrix} + Nr$$

Remark: observer can still be designed for plant without integrator (A,B) instead $(\hat{A},\hat{B})$

Integration is not a physical state

Summary

- Reference tracking for complete output feedback controller using state-space method

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y-\hat{y}) + Mr$$
$$\hat{y} = C\hat{x} + Du$$
$$u = -k\hat{x} + Nr$$

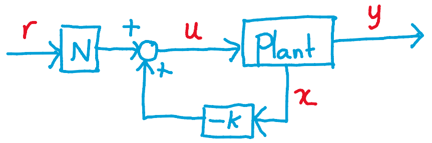
1) N using state feedback methods, $M = BN$

2) $N = 0$, $M = -L$ to obtain 'classical' structure

- State-Space design method does not automatically give integral action, but

$$\begin{bmatrix}\dot{x}_I\\ \dot{x}_p\end{bmatrix} = \begin{bmatrix}0 & C\\ 0 & A\end{bmatrix}\begin{bmatrix}x_I\\ x_p\end{bmatrix} + \begin{bmatrix}D\\ B\end{bmatrix}u - \begin{bmatrix}1\\ 0\end{bmatrix}r$$

State Feedback Control Design



Select K to ensure that $\underbrace{\text{closed-loop eigenvalues}}$ are at desired location

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$\dot{x} = Ax + Bu = Ax + B(-Kx + Nr) = (A-BK)x + BNr$

Select N to ensure proper reference tracking

Q: How to select K? (still simple for second order systems)

Can we find such a matrix K?

How to choose N?

Example

$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ Modal form

$y = [1 \ 1]x$

$u = -[k_1 \ k_2]x$ No reference for now

$\Downarrow$

$\dot{x} = \begin{bmatrix} -1-k_1 & -k_2 \\ 0 & -2 \end{bmatrix} x$

Poles are at $-1-k_1$ and -2 no matter what state we choose

We cannot 'access' the second state!

System is not fully controllable

$G(s) = C(sI-A)^{-1}B + D = \frac{1}{s+1}$ ← Pole at -2 does not appear in the transfer function

Controllability is a property of a state-space model

How to check controllability?

Controllability is the ability to control all states

$x(t) = \int_0^t e^{At} Bu(t-\tau)\, d\tau$ Convolution of the impulse response and input

$e^{At} = 1 + At + \frac{1}{2}A^2t^2 + \dots$

$= \int_0^t \underbrace{[B \ AB \ A^2B \dots]}_{\text{Full rank?}} \underbrace{\begin{bmatrix} 1 \\ t \\ \frac{1}{2}t^2 \\ \vdots \end{bmatrix}}_{\text{Full rank!}} u(t-\tau)\, d\tau$

Cayley-Hamilton

$\boxed{[B \ AB \dots A^{n-1}B] \text{ has rank } n}$

Controllability matrix

Controllability matrix used to transform state-space model to control canonical form

Computing the matrix K

$\dot{x} = Ax + Bu$ & $u = -kx \Rightarrow \dot{x} = (A - Bk)x$, such that the eigenvalues (poles) are at the desired place

Assume that the system is in control canonical form $\leftarrow$ Can always be done if system is controllable (see video state space analysis)

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ \emptyset & & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u, \quad u = -[k_1\ k_2 \cdots k_n]\, x$$

$$y = [\,b_1 \ \cdots\ \cdots\ b_n\,]\, x$$

$$\dot{x} = \begin{bmatrix} -a_1-k_1 & -a_2-k_2 & \cdots & -a_n-k_n \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \vdots \\ \emptyset & & 1 & 0 \end{bmatrix} x$$

Closed-loop eigenvalues where

$$s^n + (a_1 + k_1)s^{n-1} + (a_2 + k_2)s^{n-2} + \dots + (a_n - k_n) = 0$$

Suppose we want the eigenvalues (poles) to be at $\gamma_1 \dots \gamma_n$

$$(s-\gamma_1)(s-\gamma_2)\dots(s-\gamma_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \dots + \alpha_n s^{n-n} = 0$$

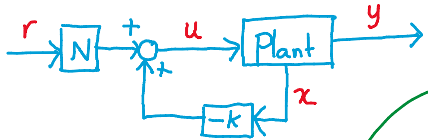
$s^{n-n} = s^0 = 1$

Solve for these coefficients

Procedure:

1) Convert system $\dot{x} = Ax + Bu$ to controllable canonical form
   $\dot{z} = \hat{A}z + \hat{B}u$ with $x = Tz$
2) Design state feedback $u = -\hat{k}z$ by solving for the coefficients as described above
3) Transform state feedback $u = -\hat{k}T^{-1}x$ because
   $$\dot{z} = (\hat{A} - \hat{B}\hat{k})z = (T^{-1}AT - T^{-1}B\hat{k})z$$
   $$\Rightarrow T\dot{z} = (A - B\hat{k}T^{-1})Tz$$
   $$\dot{x} = (A - Bk)x$$

Computing the matrix N



Matrix K for closed-loop stability and transient response

Matrix N for steady-state reference tracking

Consider $\dot{x} = Ax + Bu$ with $u = u_{ss} - K(x - x_{ss})$

How to select these such that y tracks r?

$$0 = Ax_{ss} + Bu_{ss}$$
$$y_{ss} = Cx_{ss} + Du_{ss}$$

take $\begin{cases} x_{ss} = N_x r \\ u_{ss} = N_u r \\ y_{ss} = r \end{cases}$ $\quad 0 = AN_x r + BN_u r$, $\quad r = N_x r + DN_u r$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} N_x \\ N_u \end{bmatrix} r = \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \implies \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Holds for all r

$$u = N_u r - K(x - N_x r) = \overbrace{(N_u + KN_x)}^{N} r - Kx$$

To summarize

- Controllability is the ability to control each state

  Rank $[B \ AB \ldots A^{n-1}B] = n$ with n the order of the system
- Controllability canonical form can be used to compute matrix K
- Steady-state reference tracking by taking

  $N = N_u + KN_x$ with $\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$