Signals 2:
Sampling & Interpolation

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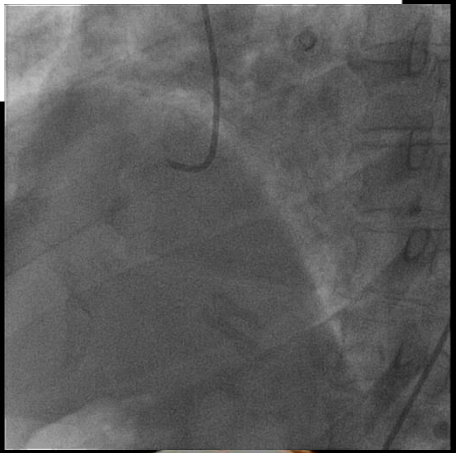
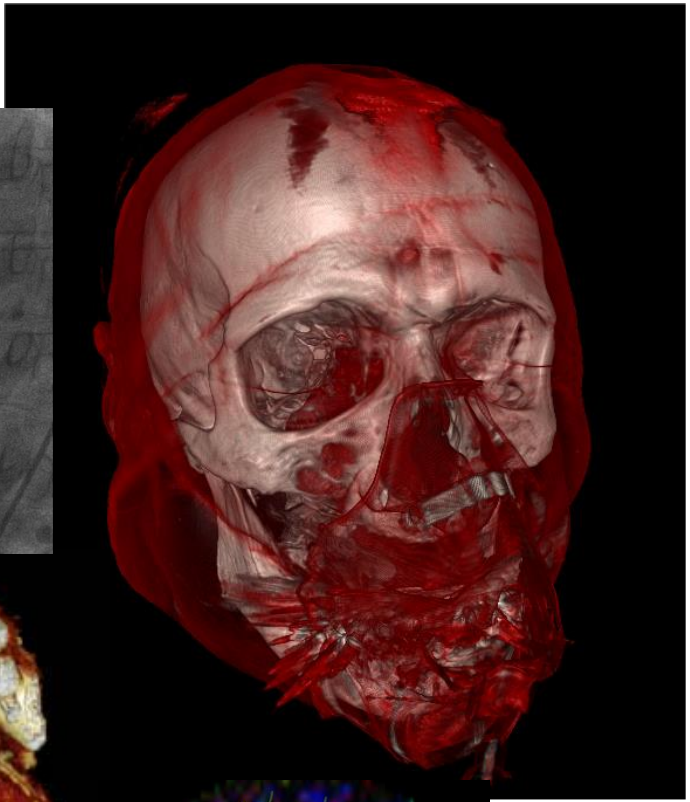
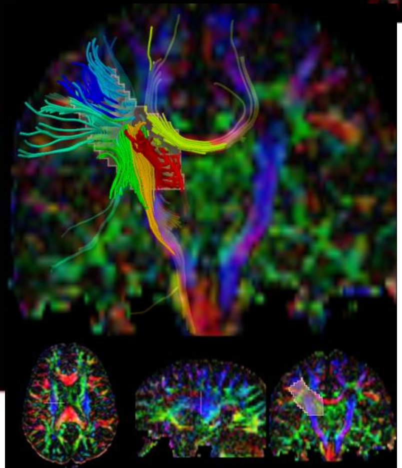
Produces images







Basics

(stuff you already should know)

link

Dirac impulse δ

- Also known as: Dirac delta, delta function, Dirac delta function, Dirac operator

Dirac delta function $\delta(t)$

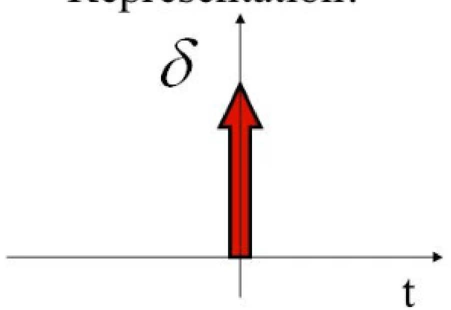
- This "unit impulse" function is defined by the conditions:

$$\delta(t) = 0, \ \ for \ \ t \neq 0.$$

$$\delta(0) = +\infty$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$

Representation:



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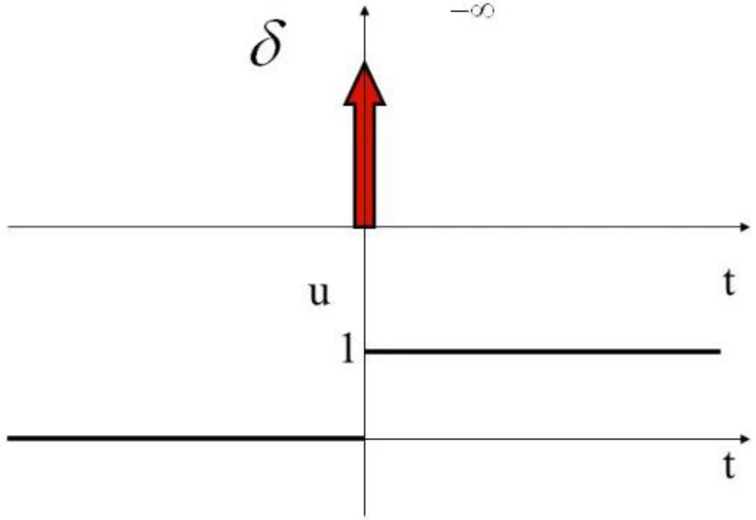
Dirac impulse δ

- Integral of the Dirac function:

Calculate: $u(t) = \int_{-\infty}^{t} \delta(\sigma) d\sigma$



$$u(t) = \begin{cases} 0 & if \quad t < 0 \\ 1 & if \quad t > 0 \end{cases}$$

Remark: value at t = 0 is not well defined, we adopt one by convention.

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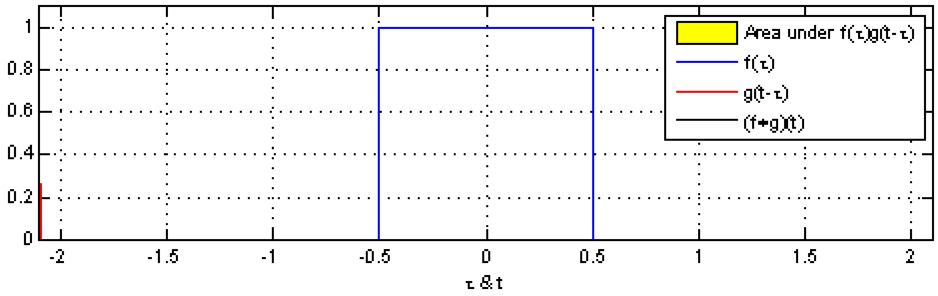
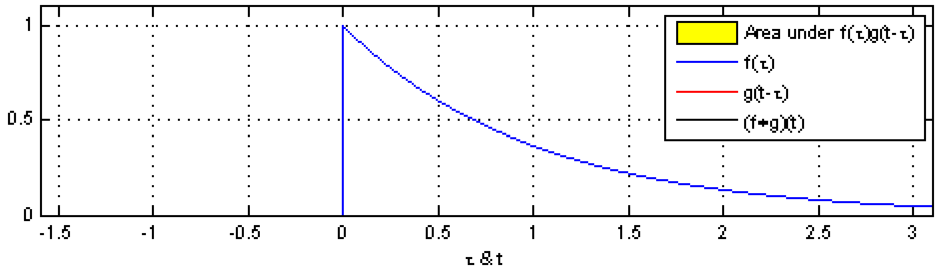
Convolution integral

$$y(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$





Convolution integral

$$y(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

- Time and frequency domain:
  - Convolution in time domain $f(t) * h(t)$ -> multiplication in frequency domain $\mathrm{F}(\omega) \cdot H(\omega)$
  - Multiplication in time domain $f(t) \cdot h(t)$ -> convolution in frequency domain $\mathrm{F}(\omega) * H(\omega)$

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Convolution with the Dirac function

$$y(t) = \delta(t) * h(t) = \int_{-\infty}^{\infty} \delta(\tau)\, h(t-\tau)\, d\tau = h(t)$$

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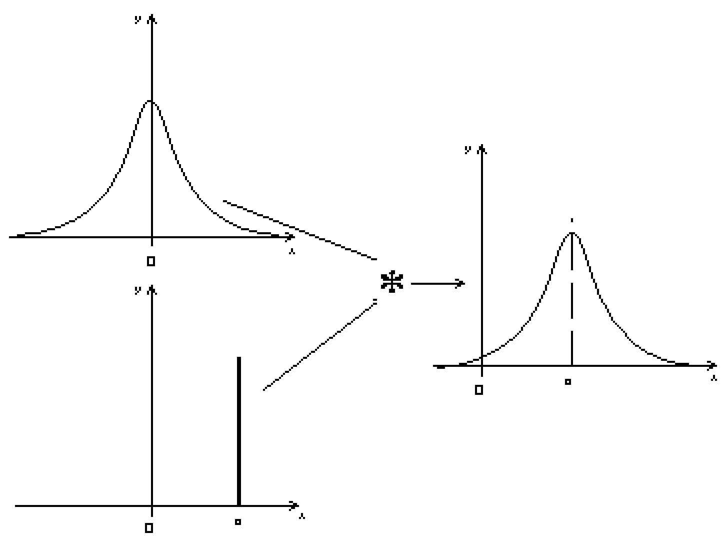
Convolution with a shifted Dirac function

$$y(t) = \delta(t-s) * h(t) = \int_{-\infty}^{\infty} \delta(\tau - s)\, h(t-\tau)\, d\tau = h(t-s)$$



Fourier Transform

- Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$$

- Inverse Fourier Transform

$$\omega = 2\pi f$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega$$

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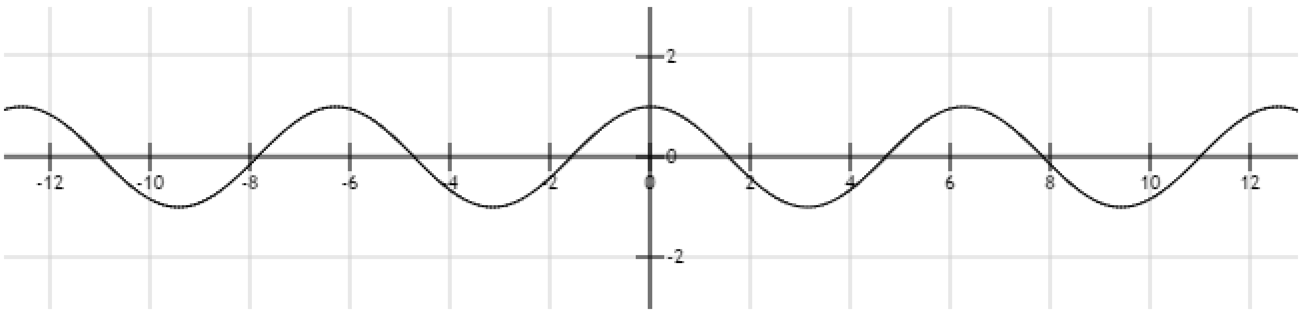
Fourier Transform

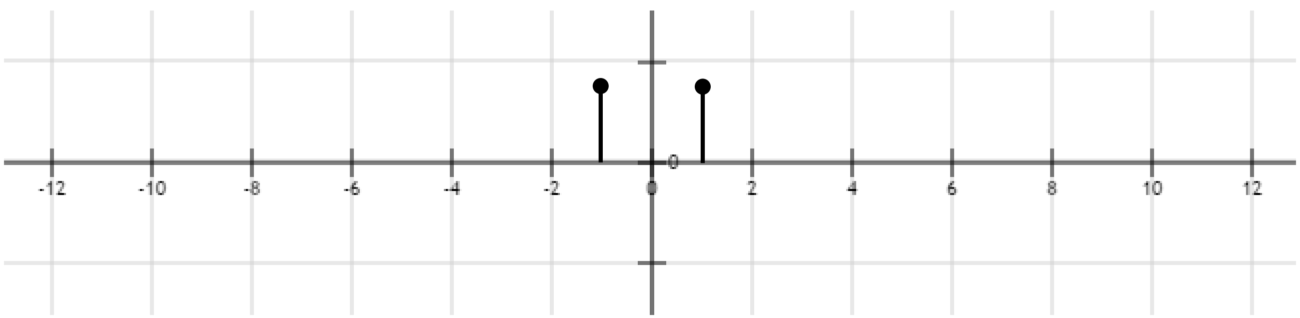
Cosine wave

f(t)



$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega$$

F(ω)



$$e^{-i\omega t} = \cos(\omega t) - i \sin(\omega t)$$

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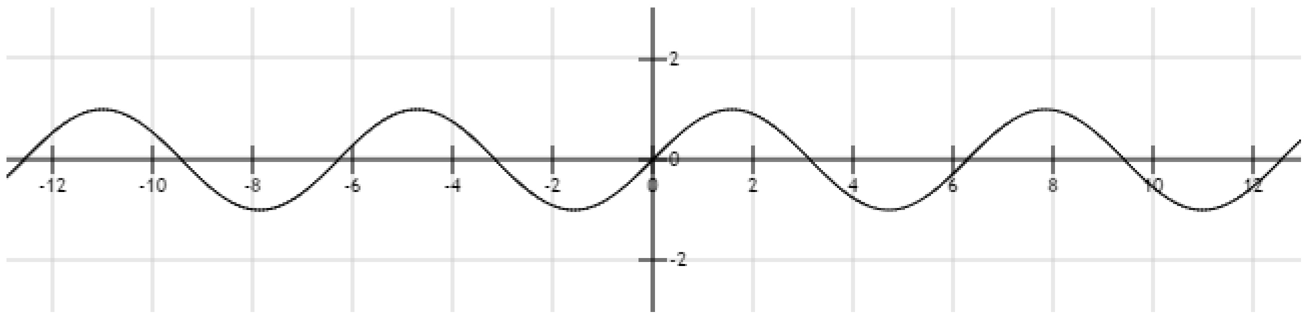
Fourier Transform

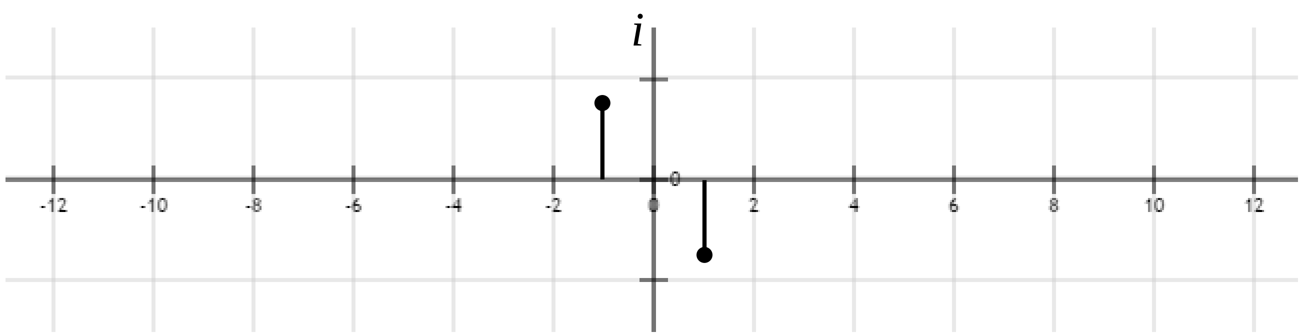
Sine wave

f(t)



$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega$$

F(ω)

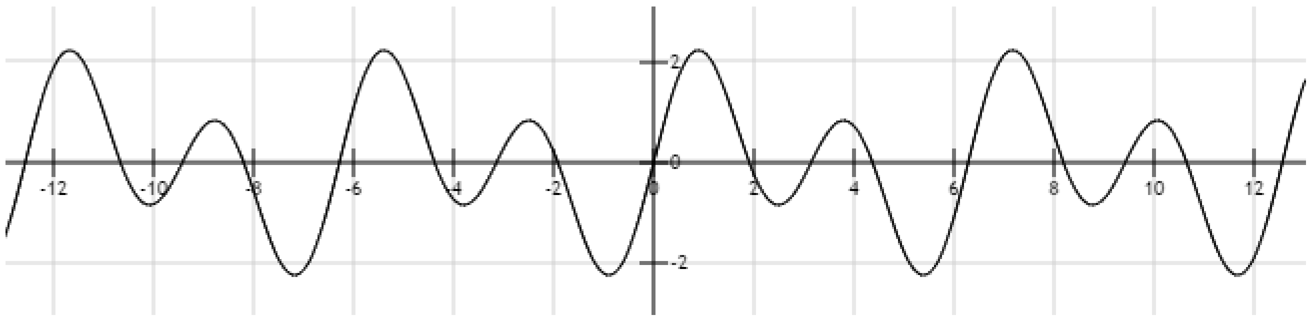


$$e^{-i\omega t} = \cos(\omega t) - i \sin(\omega t)$$

Fourier Transform

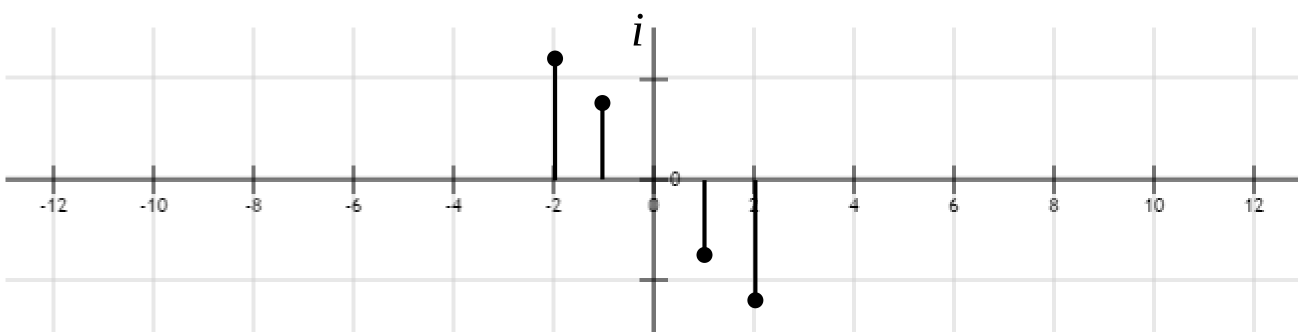
sin(t) + 1.5 * sin(2t)

f(t)



$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega$$

F(ω)



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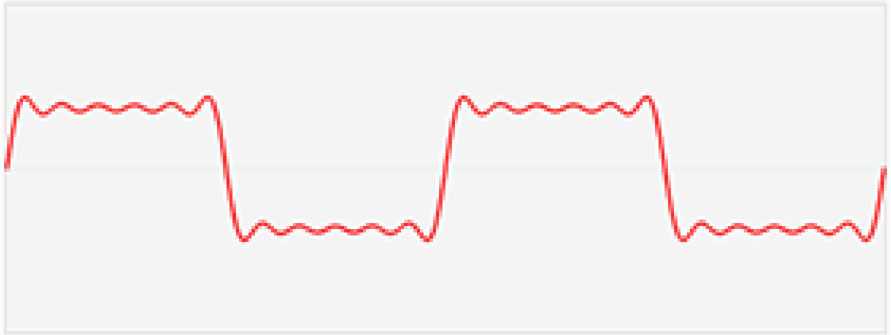
Periodic square (rect, box) function

Approximated by 6 Fourier components



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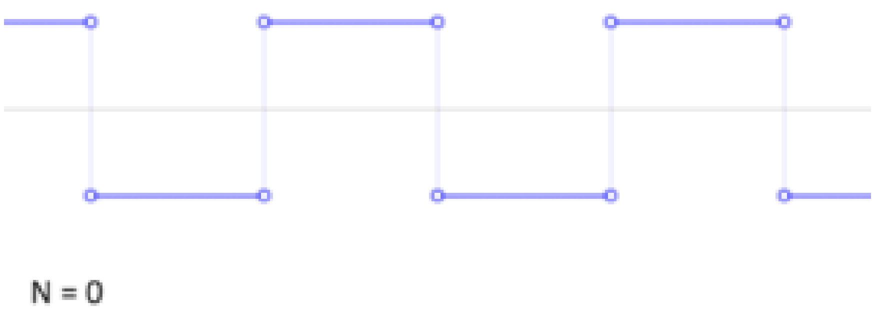
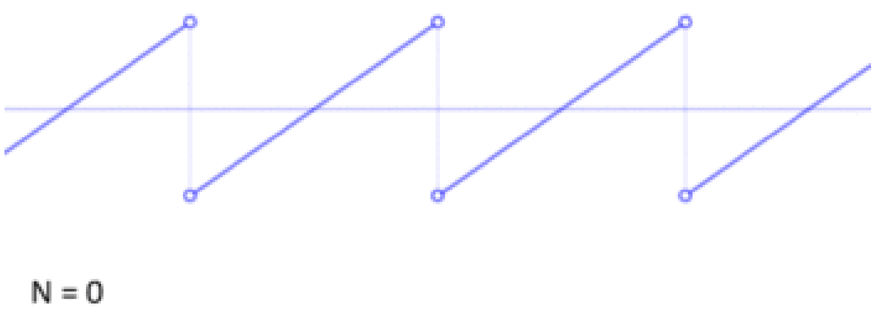
Approximating by Fourier series

Square wave and saw tooth





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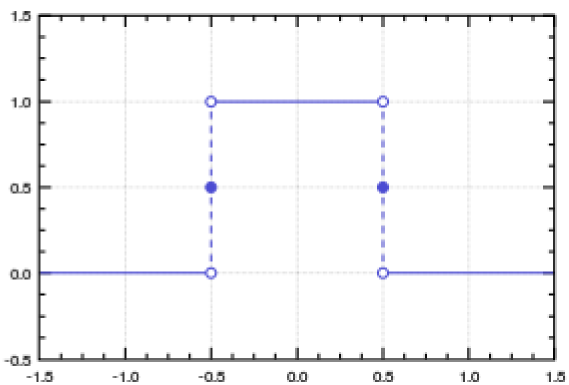
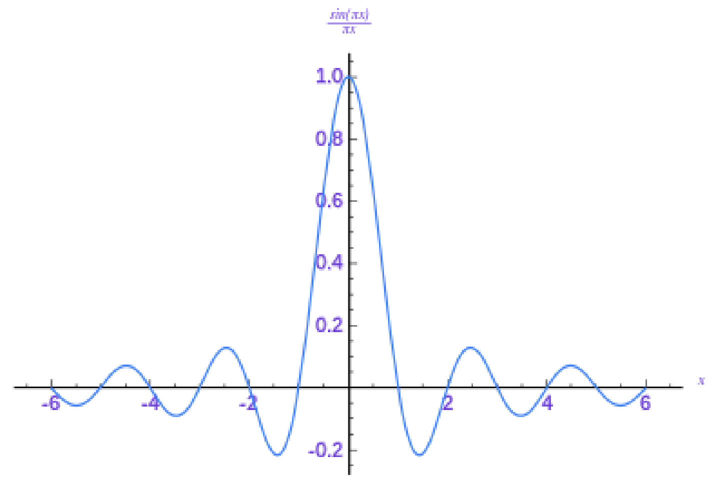
Playground

- http://bl.ocks.org/jinroh/7524988

Fourier transform of a square

rect function, box function





$$\int_{-\infty}^{\infty} \mathrm{rect}(t) \cdot e^{-i2\pi f t}\, dt = \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(\pi f)$$

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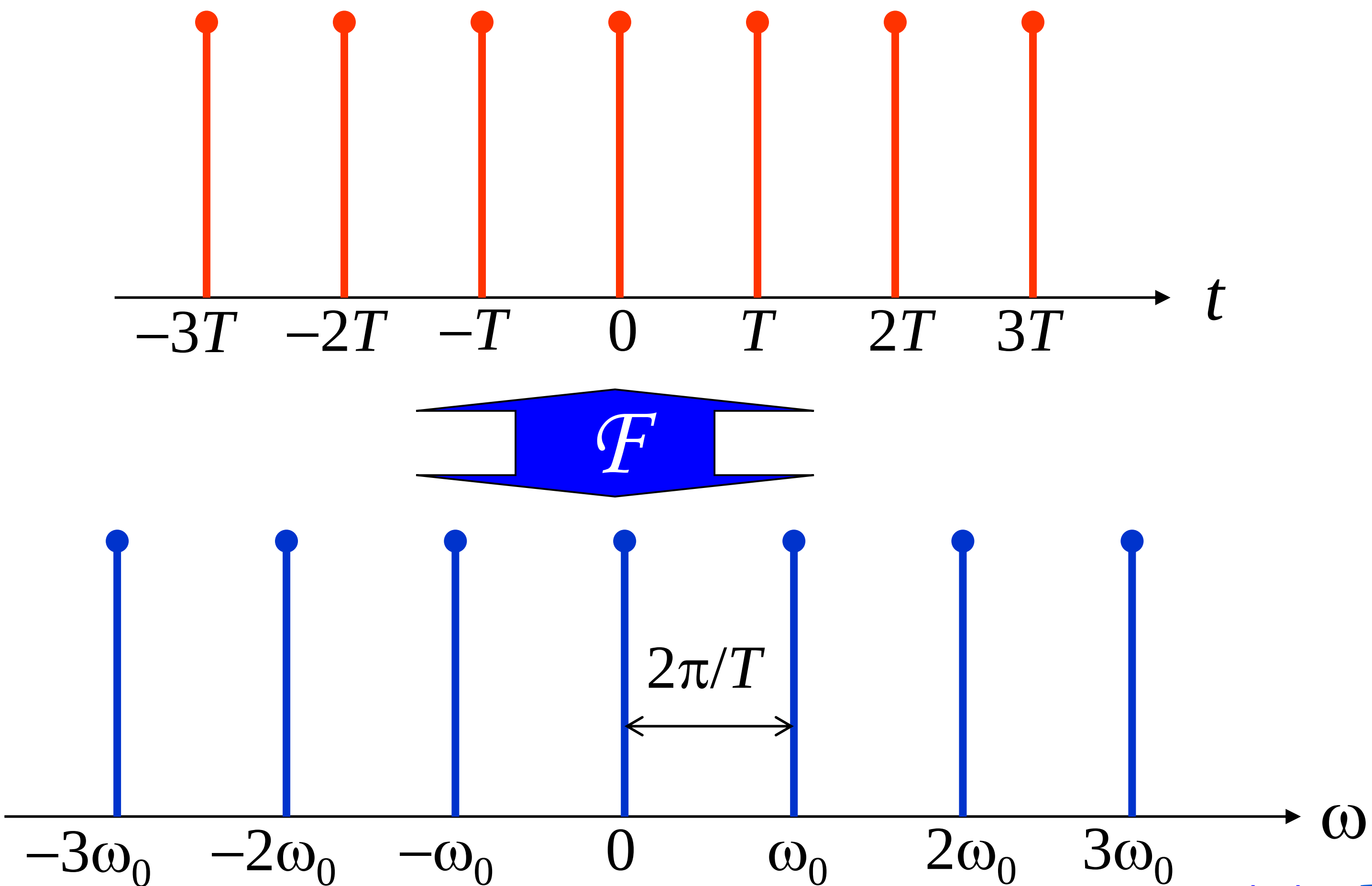
Sampling

Dirac train, pulse train

Also known as: impulse train, pulse train, Dirac comb



proof

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Pulse train example

* Fourier Transform Continuous time signals (FTC)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \;\multimap\; \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

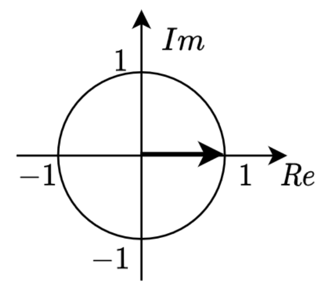
* Let us look at an example of an FTC pair: pulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n \cdot T) \multimap S(\omega) = \sum_{n=-\infty}^{\infty} e^{-j\omega nT}$$

* What will a pulse train look like in frequency domain?
* Look at $S(\omega)$, what happens when $\omega = 0$?

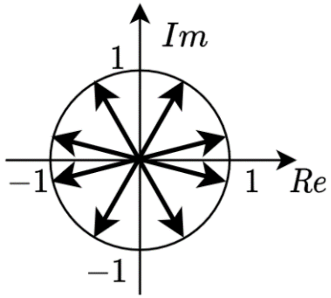
Pulse train example

* $S(\omega) = \sum_{n=-\infty}^{\infty} e^{-j\omega nT}$
* Look at $S(\omega)$, what happens when $\omega = 0$?
* The phasor will be 1 and the sum goes to infinity
* $S(0) = \sum_{n=-\infty}^{\infty} e^{-j\cdot 0\cdot nT} = \sum_{n=-\infty}^{\infty} 1$
* Now what happens when ω is slightly larger than 0?



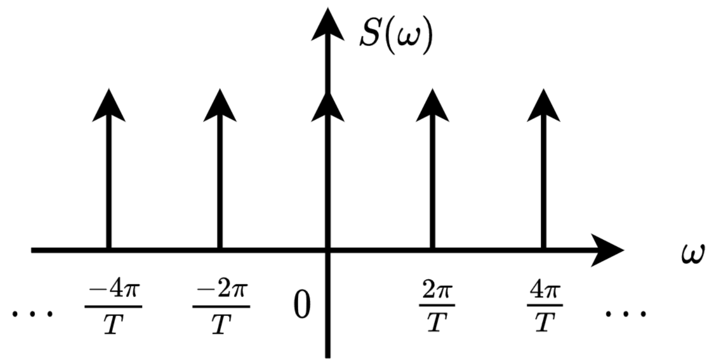
Pulse train example

* Now what happens when ω is slightly larger than 0?
* $S(\omega) = \sum_{n=-\infty}^{\infty} (e^{-j\omega T})^n$
* An infinite amount of phasors will rotate around the unit circle, but for each phasor there is another phasor that is shifted with exactly π
* These phasors will cancel each other out, summing to 0
* $S(\omega) = \sum_{n=-\infty}^{\infty} (e^{-j\omega T})^n = 0$, with $0 < \omega < \frac{2\pi}{T}$
* Now what happens if $\omega = \frac{2\pi}{T}$?



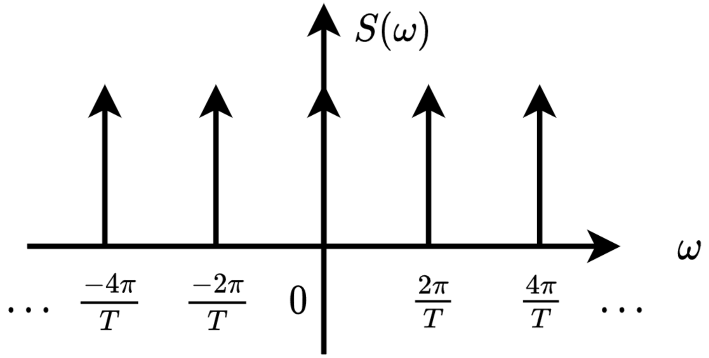
Pulse train example

* Now what happens if $\omega = \frac{2\pi}{T}$?
* $S\left(\frac{2\pi}{T}\right) = \sum_{n=-\infty}^{\infty}(e^{-j\frac{2\pi}{T}T})^n = \sum_{n=-\infty}^{\infty}(e^{-j2\pi})^n = \sum_{n=-\infty}^{\infty} 1 = S(0)$
* In frequency domain, a pulse train will be a pulse train of infinitely high and infinitely narrow pulses spaced $\frac{2\pi}{T}$ apart

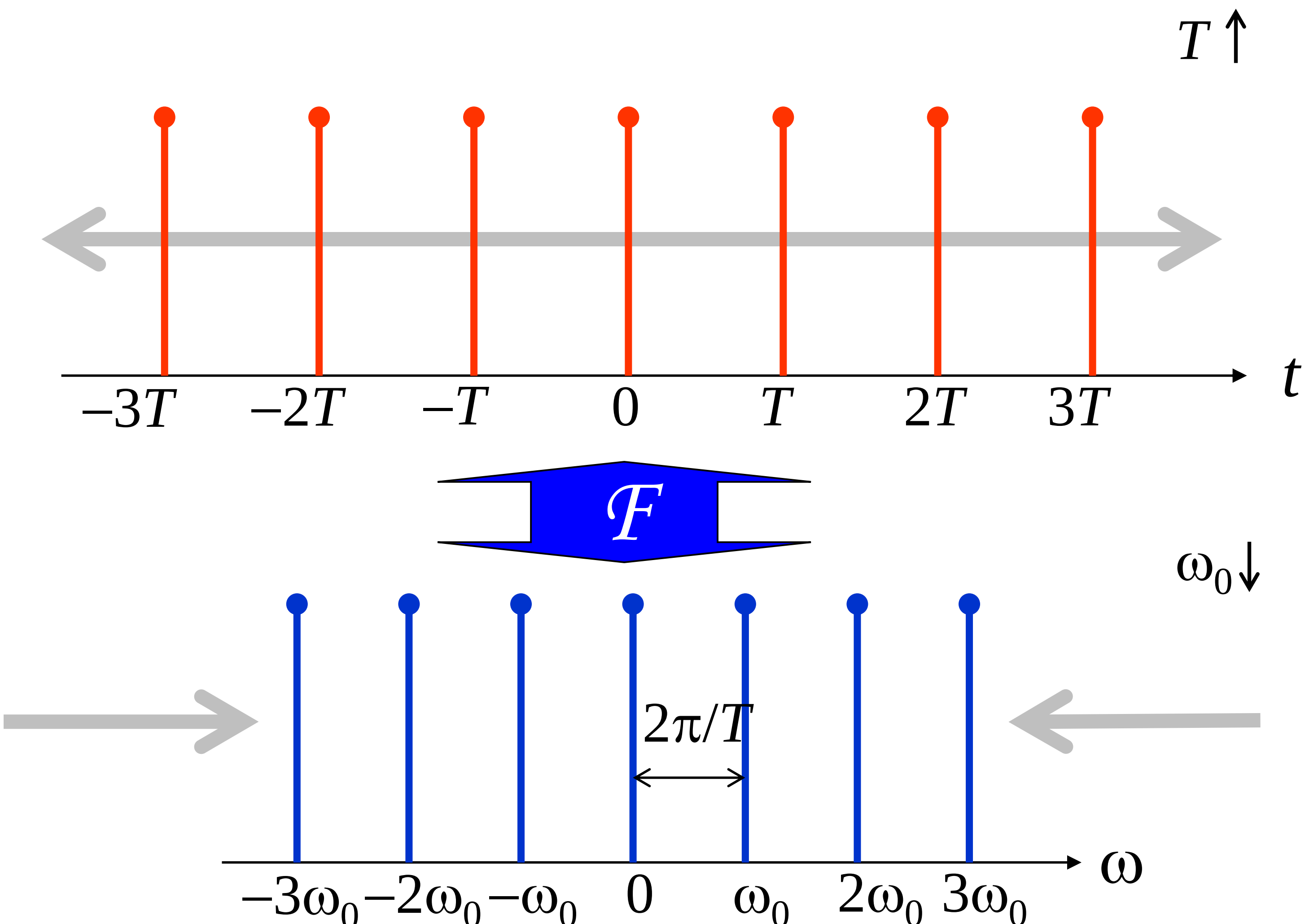


Pulse train example

* Note that in the time domain the pulses of a pulse train are a distance T apart, but in the frequency domain they are a distance $\frac{2\pi}{T}$ apart
* This means that if we lower the sample frequency, the time domain pulses will move further apart, but the frequency domain pulses move closer together



Dirac train, pulse train



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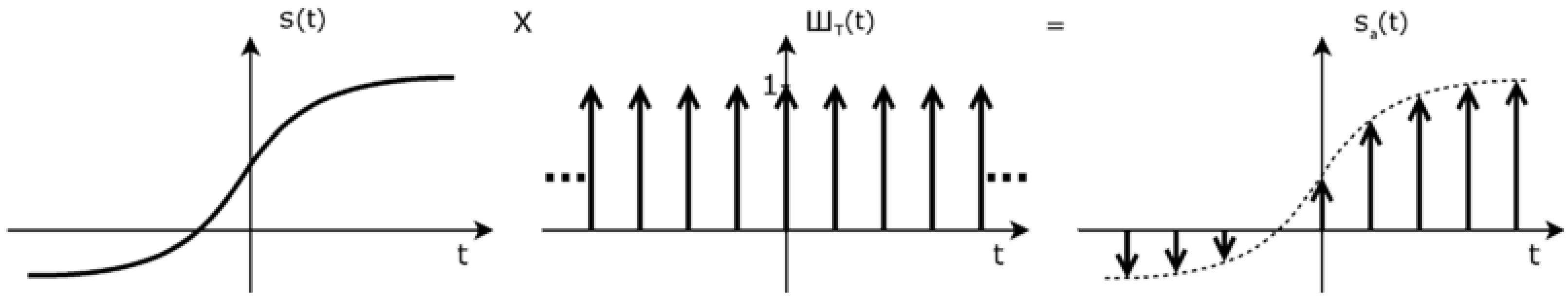
Sampling

= multiplication with Dirac train



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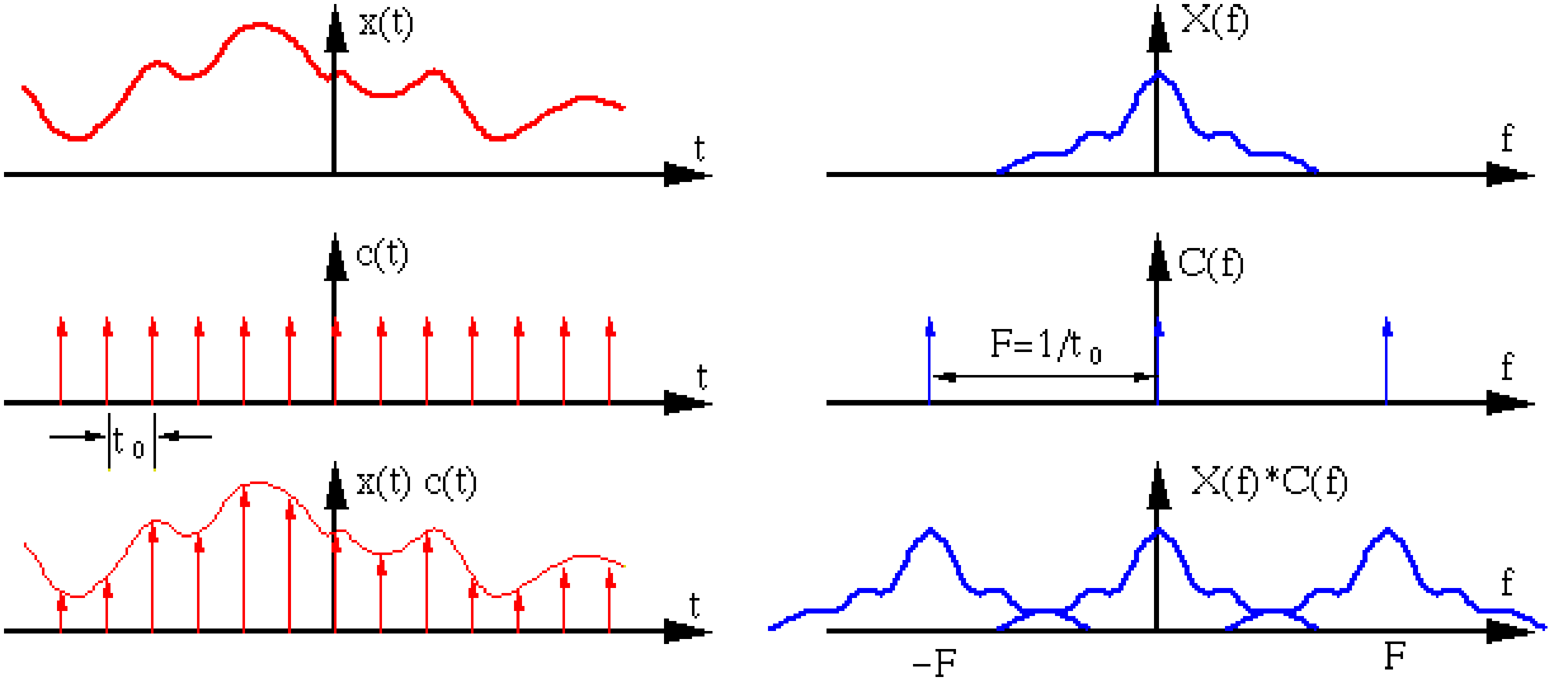
Sampling

Fourier space



Nyquist criterion: $f > 2 \cdot f_{max}$

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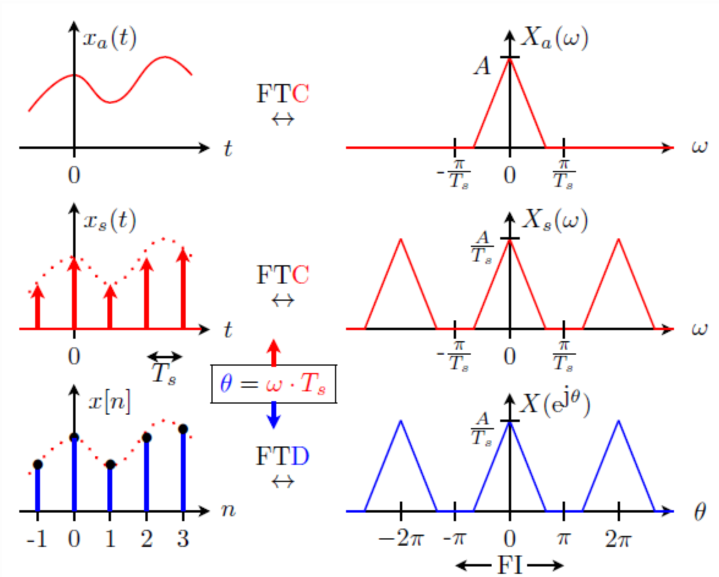
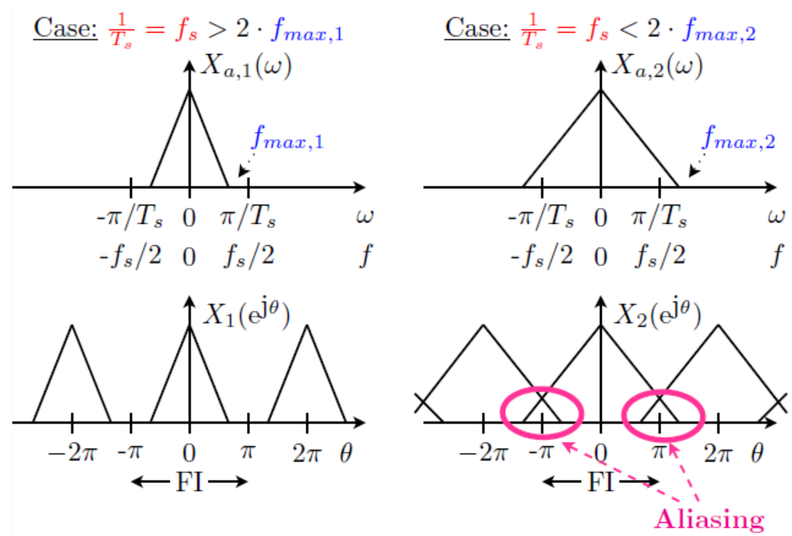
Shannon theorem

C-to-D:



Aliasing:

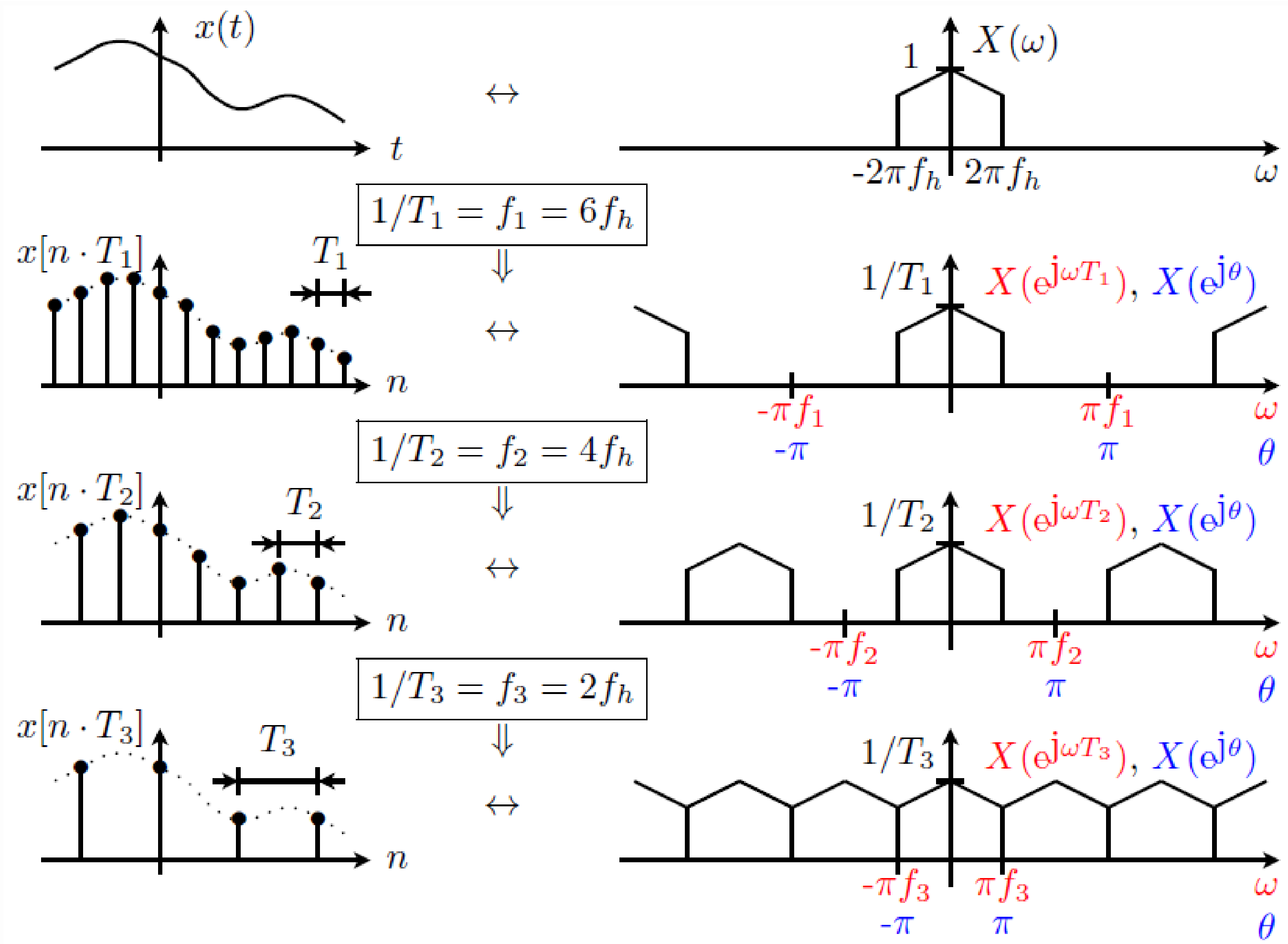


Shannon Sampling Theorem

Continuous-time signal $x_a(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x_a(t)|_{t=n \cdot T_s}$, if samples are taken at a rate $f_s = 1/T_s$, that is greater than $2 f_{max}$

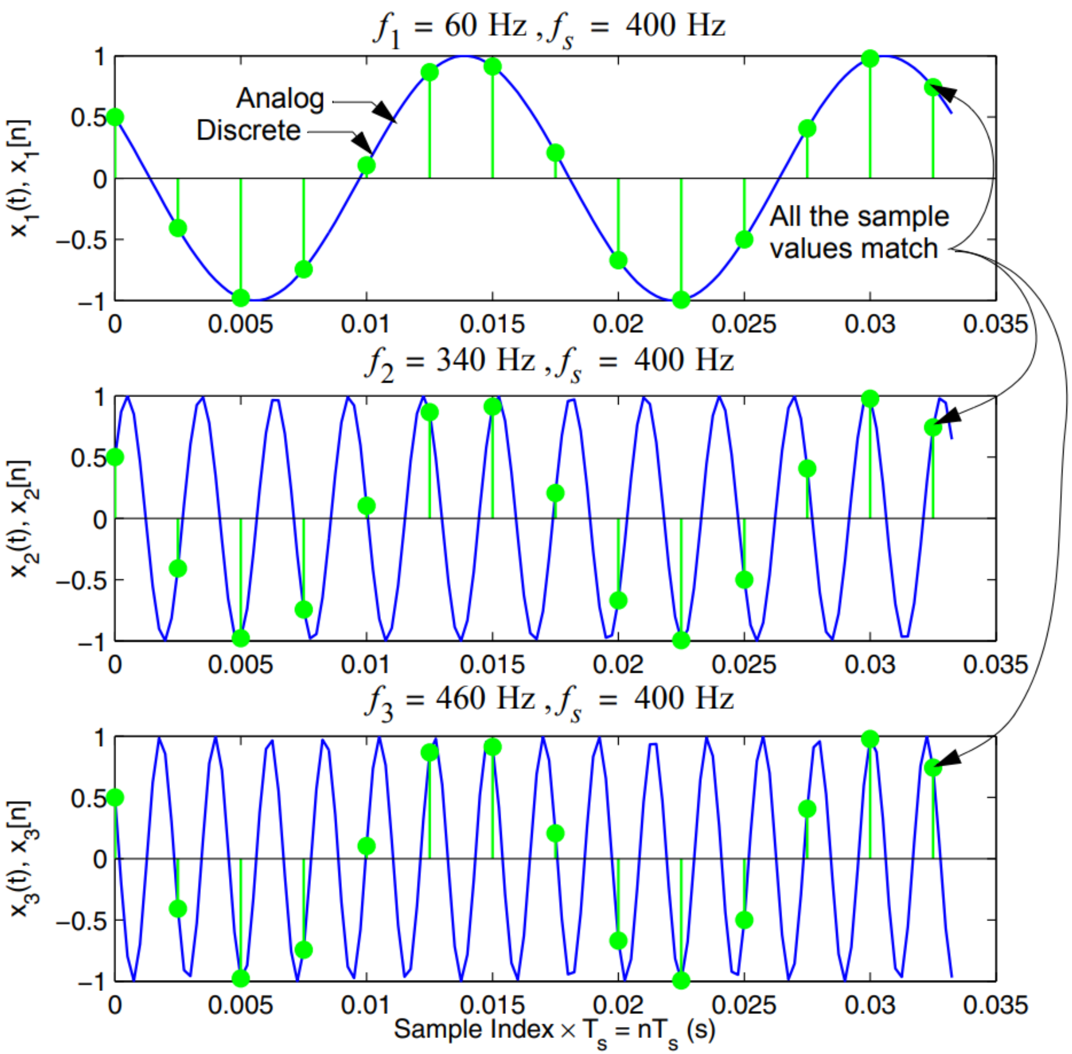
Sampling rate

What happens if we change sample rate?



Sampling rate & Nyquist frequency

[link]



$x_1(t) = \cos(2\pi \cdot 60 \cdot t + \pi/3)$

$x_2(t) = \cos(2\pi \cdot 340 \cdot t - \pi/3)$

$x_3(t) = \cos(2\pi \cdot 460 \cdot t + \pi/3)$

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Sampling rate & Nyquist frequency

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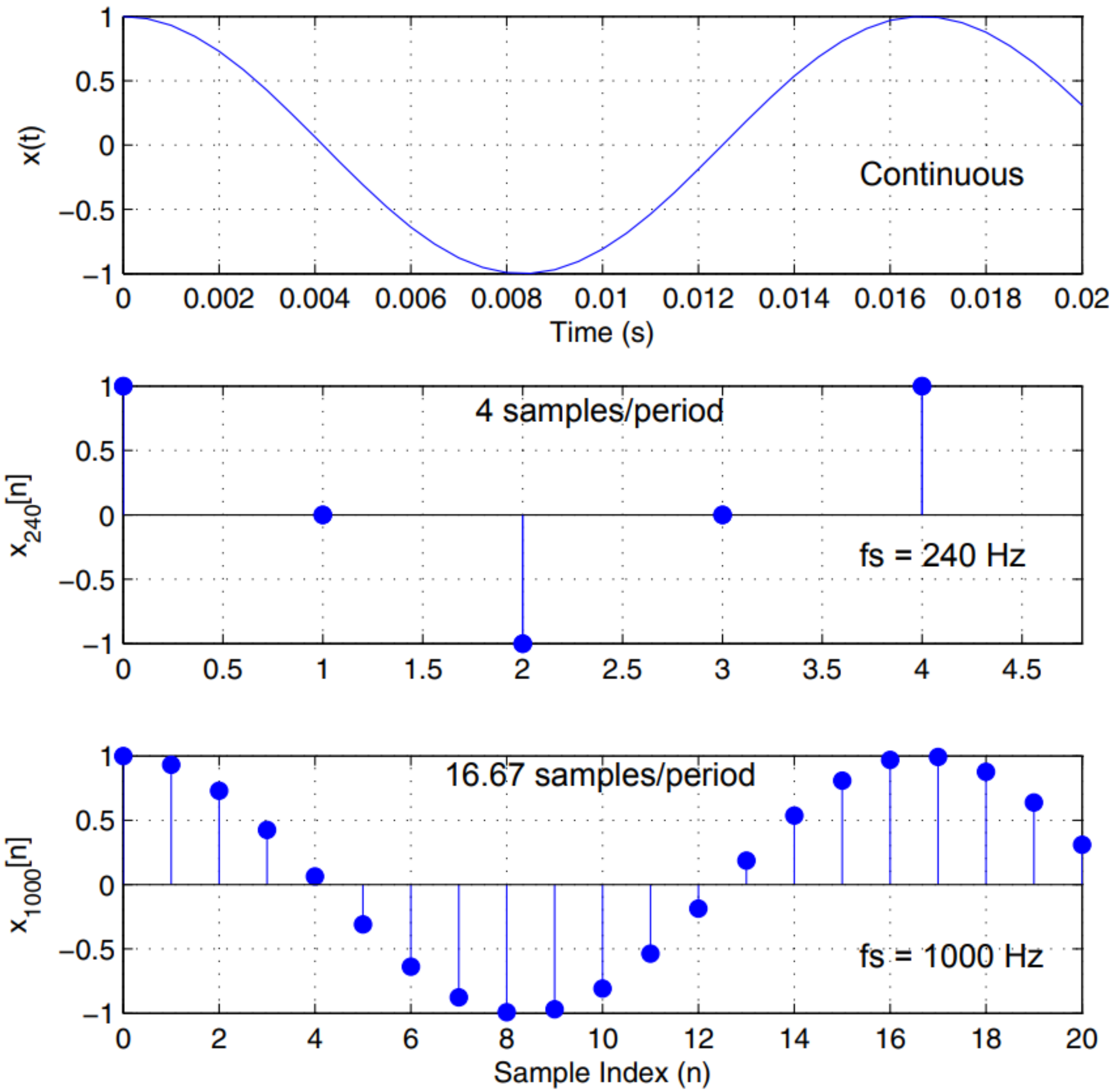

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Playground

- http://rest-term.com/labs/html5/fft.html

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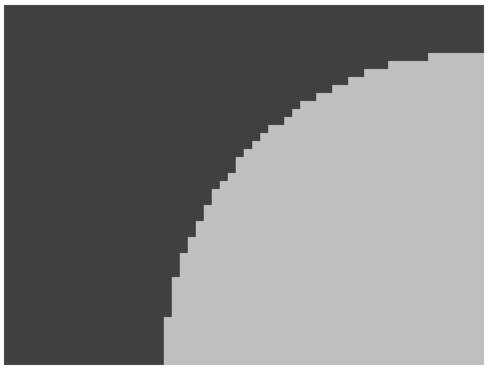
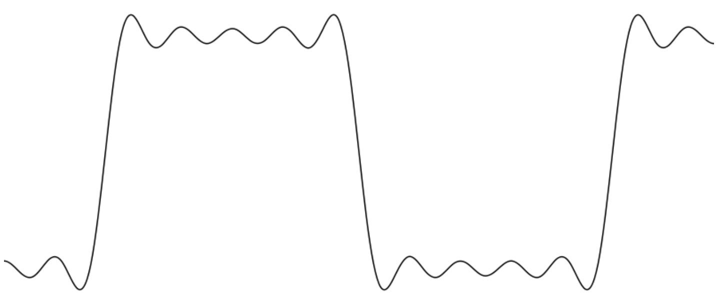
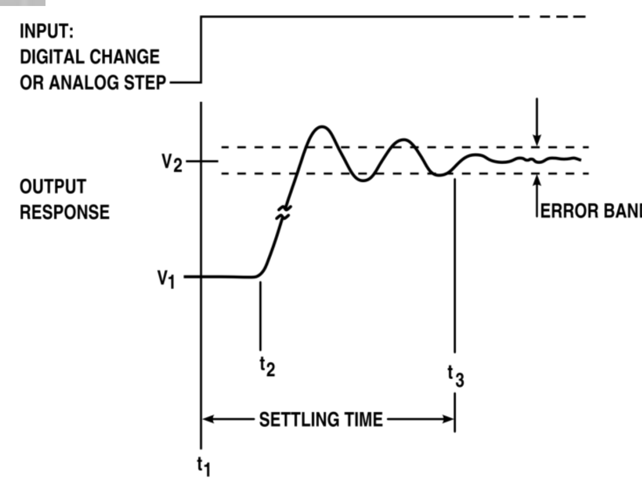
“Ideal sampling”

Not so ideal for non-bandwidth limited signals

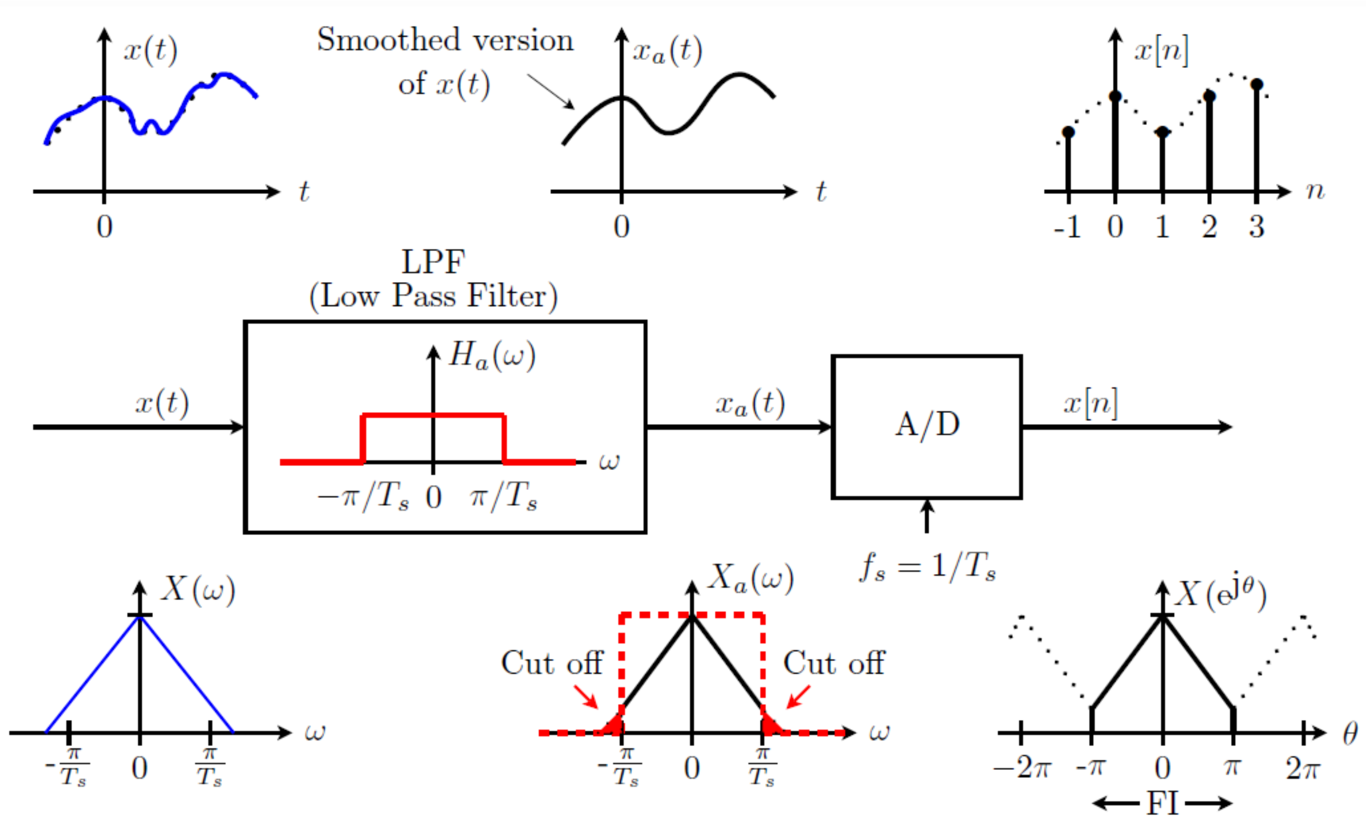






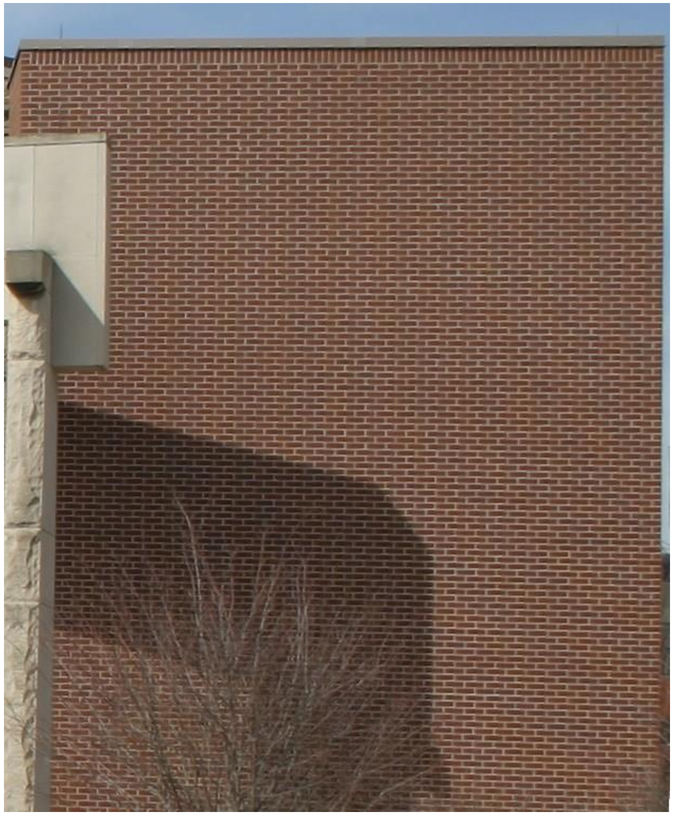
Sampling of natural signals

Not bandwidth limited signals



Aliasing, low pass filter

Without low pass



With low pass

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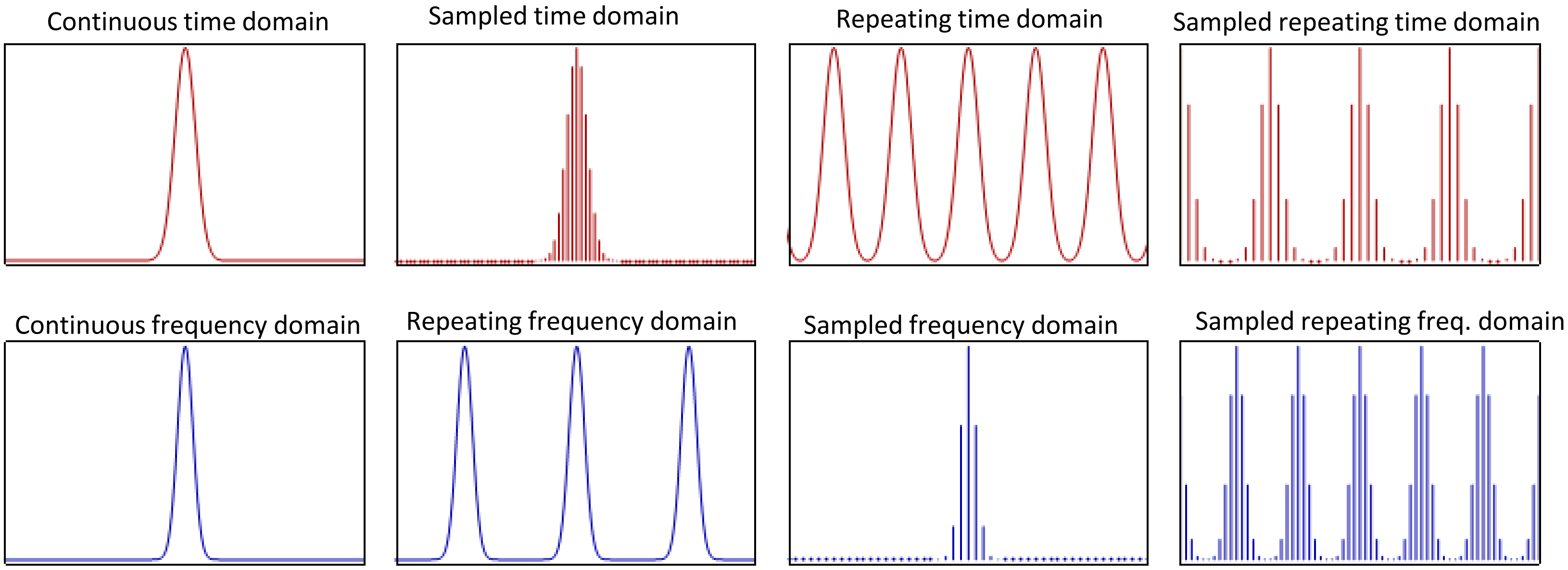
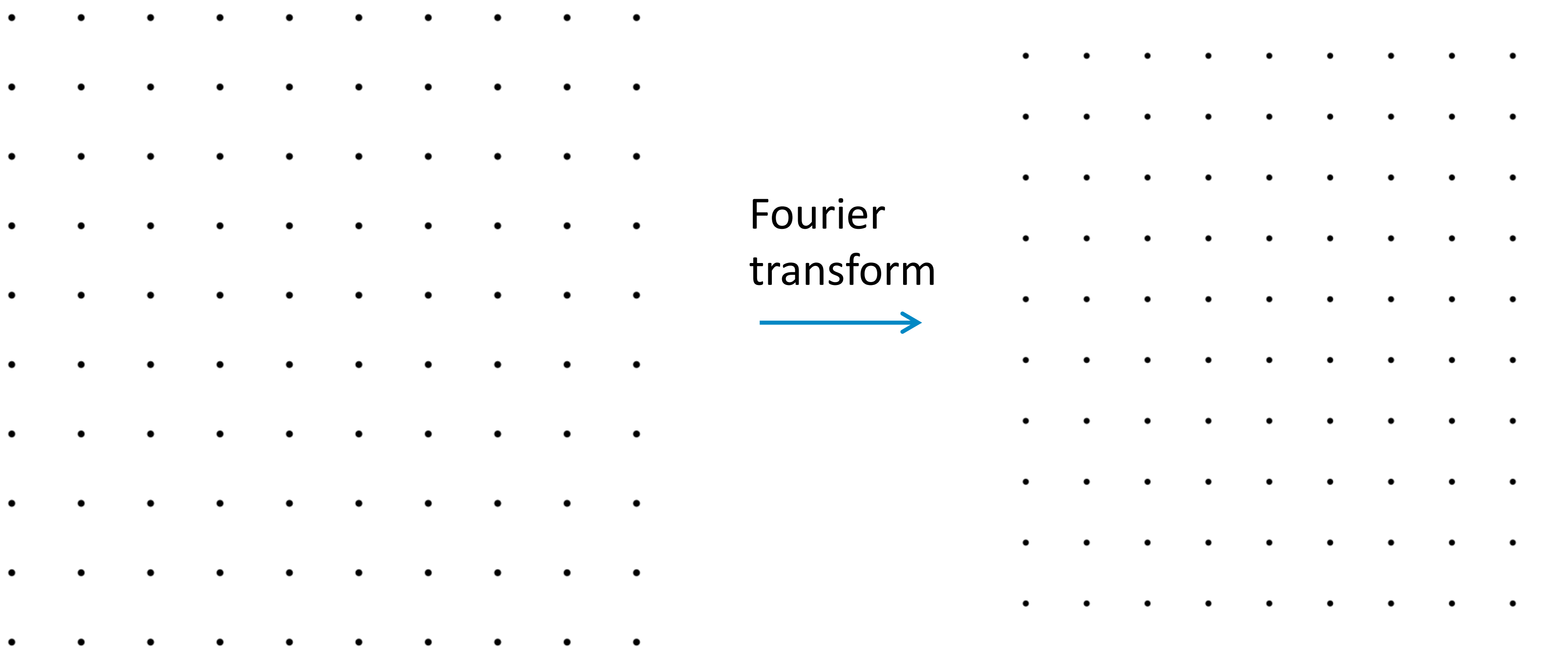
Discrete Fourier Transform



Sampling in higher dimensions

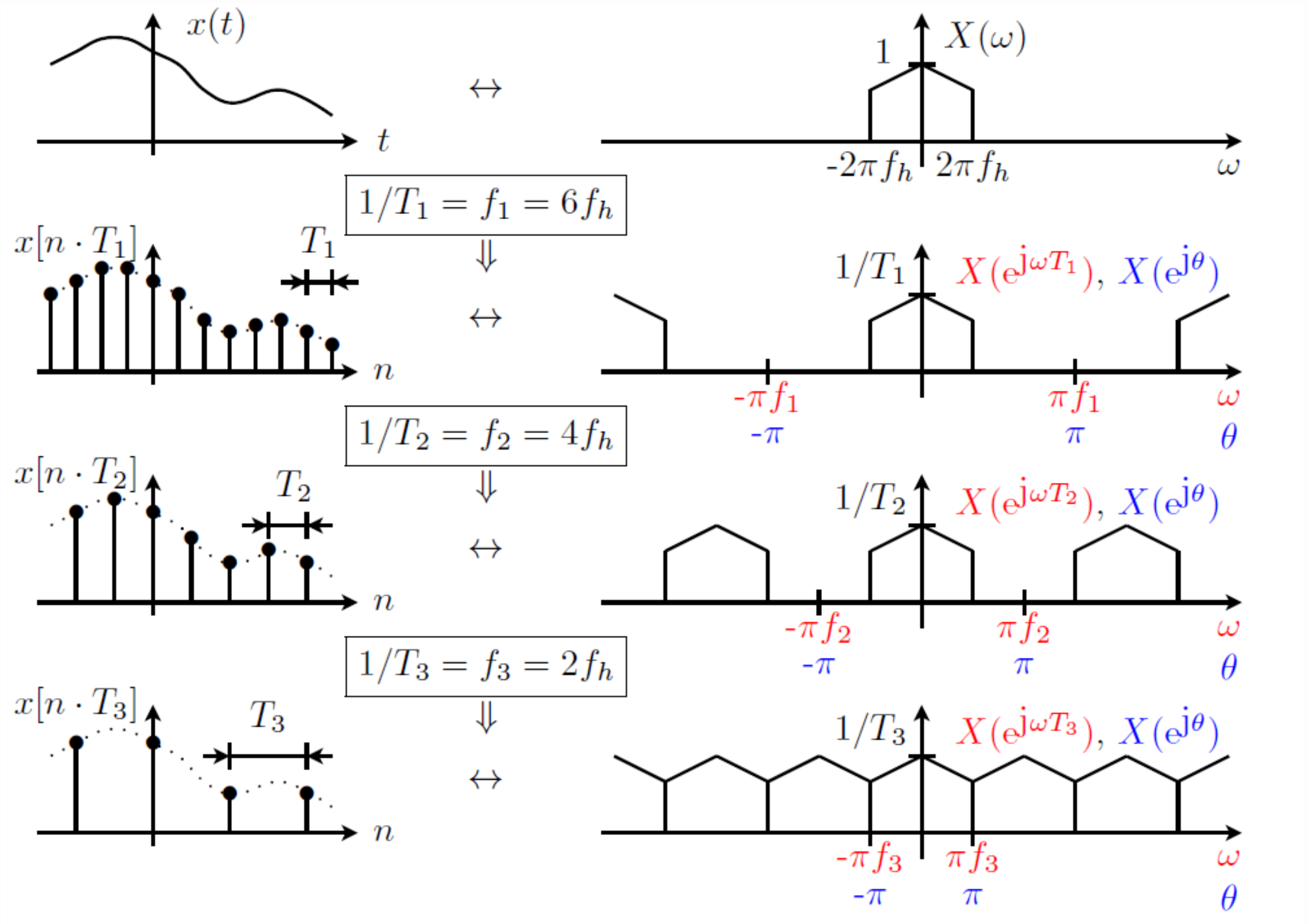
2D, 3D



Sample rate conversion

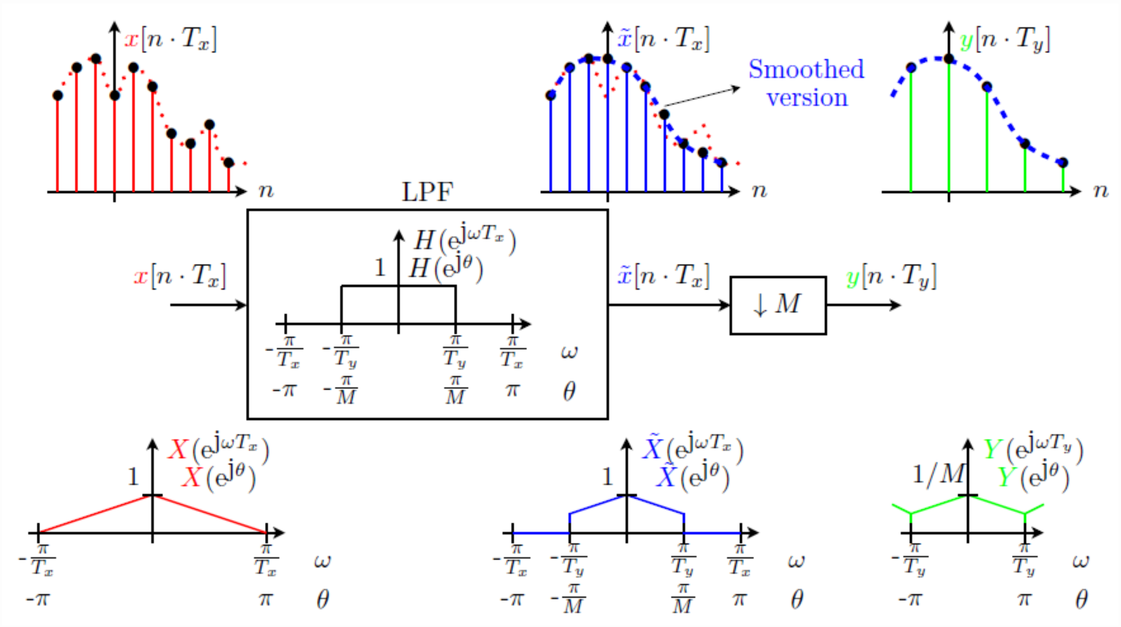
Sample rate conversion

What happens if we change sample rate?



Prevent aliasing artifacts when down-sampling

Decimator by integer factor M:

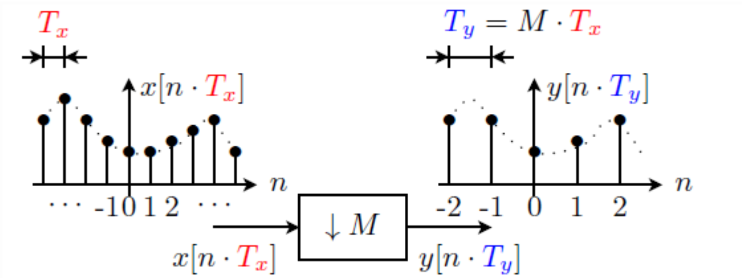


Notes:

- Prevent aliasing by LPF in front of SRD $\Rightarrow$ Decimator
- SRD is <u>not</u> LTI

Sample Rate Decrease

Sample Rate Decrease (SRD) by integer factor M (box $\downarrow M$):



$$y[n \cdot T_y] = x[n \cdot (M \cdot T_x)]$$

$$X(e^{j\theta}) = \frac{1}{T_x} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\theta}{T_x} - k\frac{2\pi}{T_x}\right)$$

$$Y(e^{j\theta}) = \frac{1}{T_y} \sum_{r=-\infty}^{\infty} X_a\left(\frac{\theta}{T_y} - r\frac{2\pi}{T_y}\right)$$

Split last sum with $r = p + k \cdot M$ and use $T_y = M \cdot T_x$

$$Y(e^{j\theta}) = \frac{1}{M} \sum_{p=0}^{M-1} \left\{ \frac{1}{T_x} \sum_{k=-\infty}^{\infty} X_a\left(\frac{(\theta - p\cdot 2\pi)}{M \cdot T_x} - k \cdot \frac{2\pi}{T_x}\right) \right\} \Rightarrow$$

$$\boxed{Y(e^{j\theta}) = \frac{1}{M} \sum_{p=0}^{M-1} X(e^{j\left(\frac{\theta}{M} - p\cdot\frac{2\pi}{M}\right)})}$$

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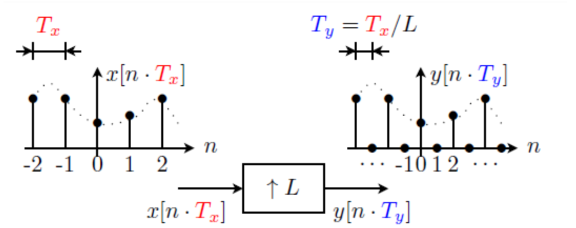
Upsampling

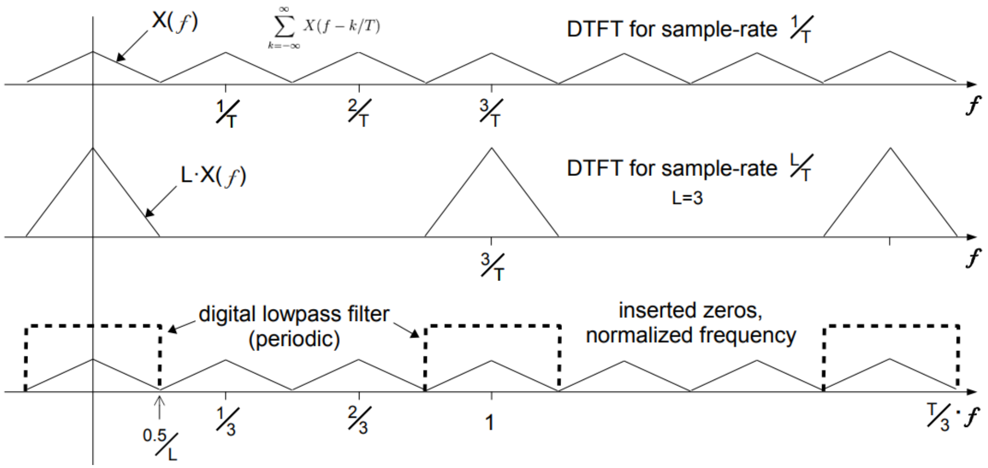
Sample Rate Increase (SRI) by integer factor L



Algorithm:

1. Expansion: Create a sequence $x_L[n]$, comprising the original samples $x[n]$ separated by $L - 1$ zeros. A notation for this operation is: $x_L[n] = x[n]_{\uparrow L}$
2. Interpolation: Smooth out the discontinuities with a lowpass filter, which replaces the zeros.

$$y[j + nL] = \sum_{k=0}^{K} x[n-k] \cdot h[j + kL], \;\; j = 0, 1, \ldots, L-1,$$



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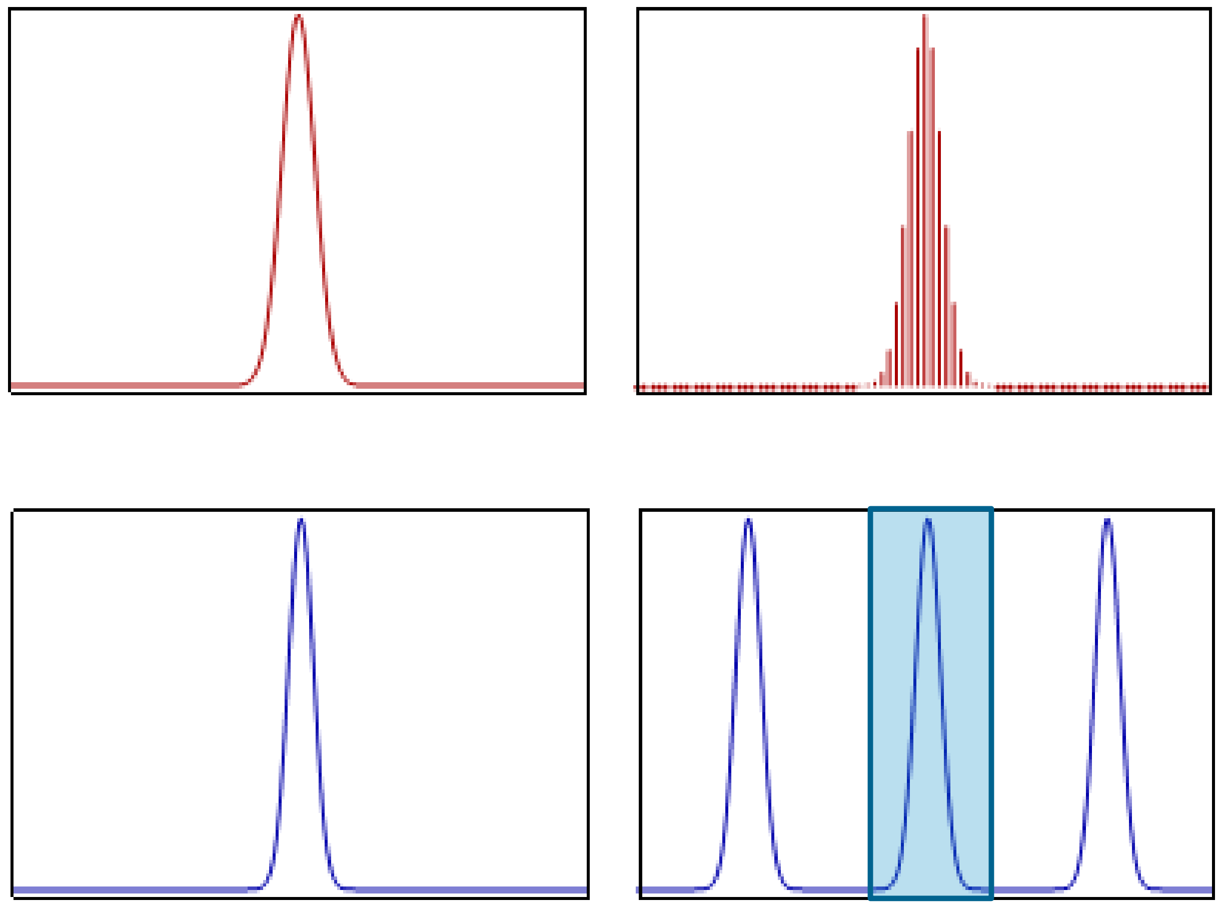
Ideal interpolation

Reconstruct continuous signal from samples



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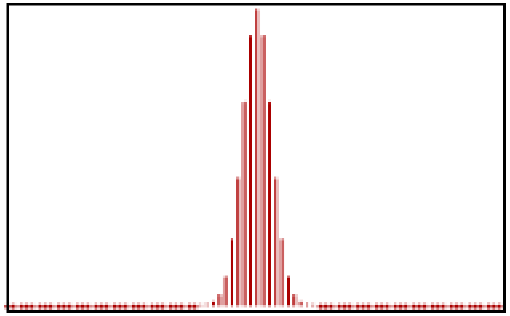
Multiply frequency domain by rect

$$x(t) = \left(\sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t - nT) \right) * \operatorname{sinc}\left(\frac{t}{T}\right) = \sum_{n=-\infty}^{\infty} x[n] \cdot \operatorname{sinc}\left(\frac{t - nT}{T}\right)$$



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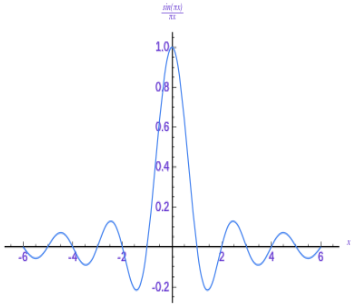

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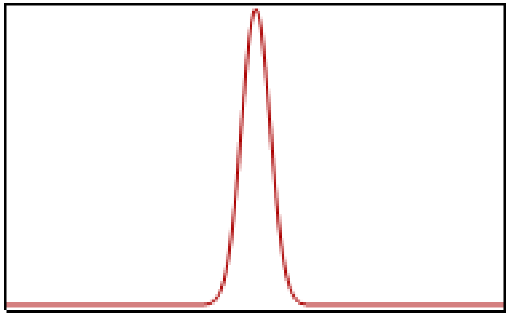
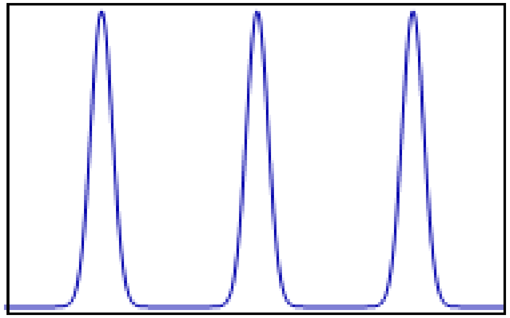
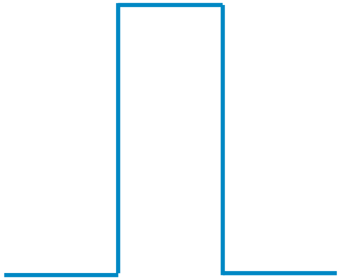




x



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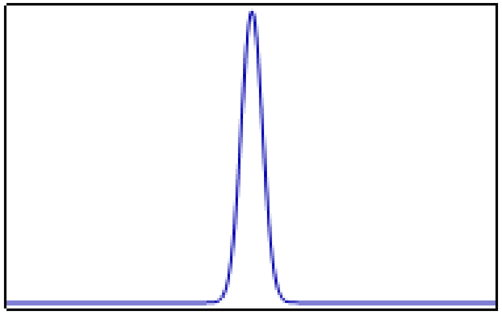

Digital-to-Analog Conversion

Convert samples to pulses: $x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$

Ideal LPF: $H_r(\omega) = \begin{cases} T_s & |\omega| \leq \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases} \quad \circ\!\!-\!\!\circ \quad h_r(t) = \dfrac{\sin(\frac{\pi}{T_s} \cdot t)}{\frac{\pi}{T_s} \cdot t}$

Filter $x_s(t)$ to obtain $x_a(t)$:

$$
\begin{aligned}
x_a(t) &= x_s(t) * h_r(t) = \left(\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \right) * h_r(t) \\
&= \sum_{n=-\infty}^{\infty} x[n] \left(\delta(t - nT_s) * h_r(t) \right) = \sum_{n=-\infty}^{\infty} x[n] \cdot h_r(t - nT_s)
\end{aligned}
$$

$\Rightarrow$ Interpolation formula (time domain):

$$
\boxed{x_a(t) = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\sin(\frac{\pi}{T_s} \cdot (t - nT_s))}{\frac{\pi}{T_s} \cdot (t - nT_s)} \right)}
$$

Note: Refer to Shannon sampling theorem

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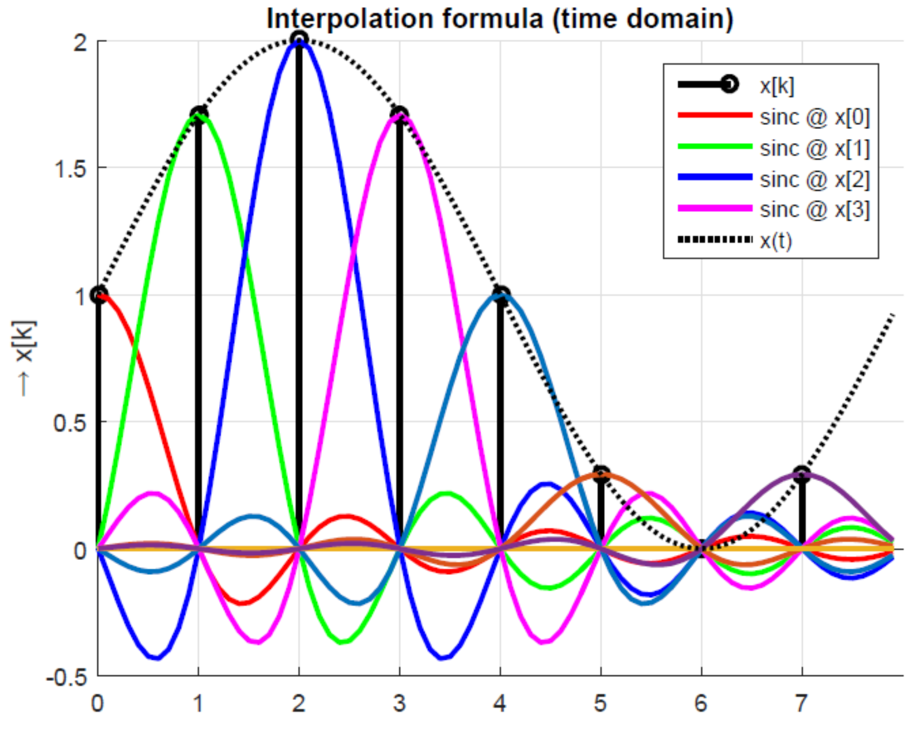
Digital-to-Analog Conversion

Interpretation of interpolation formula (time domain):

$$x_a(t) \;=\; \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\sin(\frac{\pi}{T_s} \cdot (t - nT_s))}{\frac{\pi}{T_s} \cdot (t - nT_s)} \right)$$



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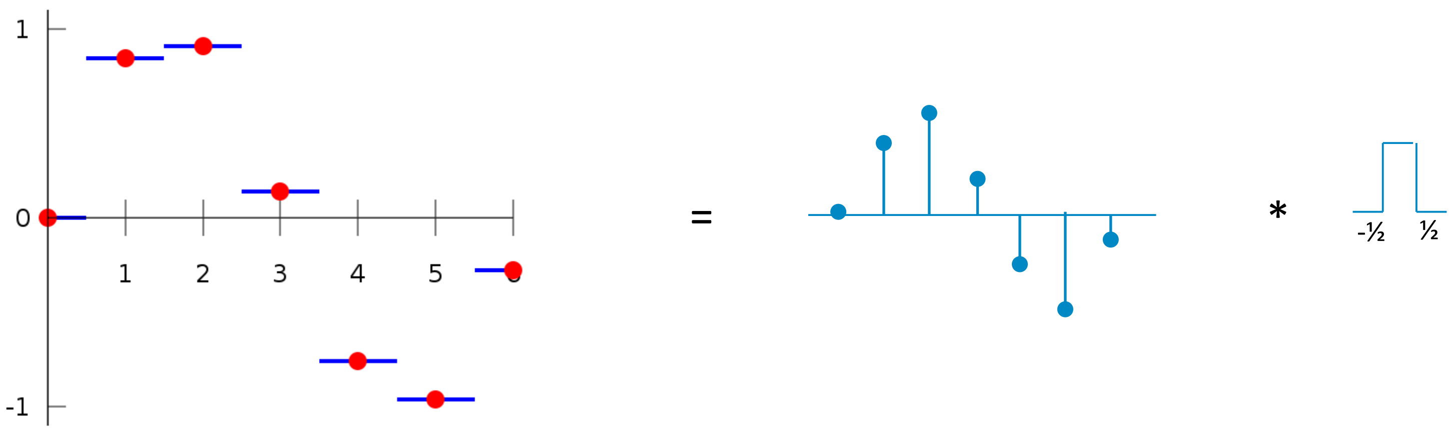
Nearest neighbor, Linear, and Cubic interpolation

(Not LTI)

Nearest Neighbor

interpolation



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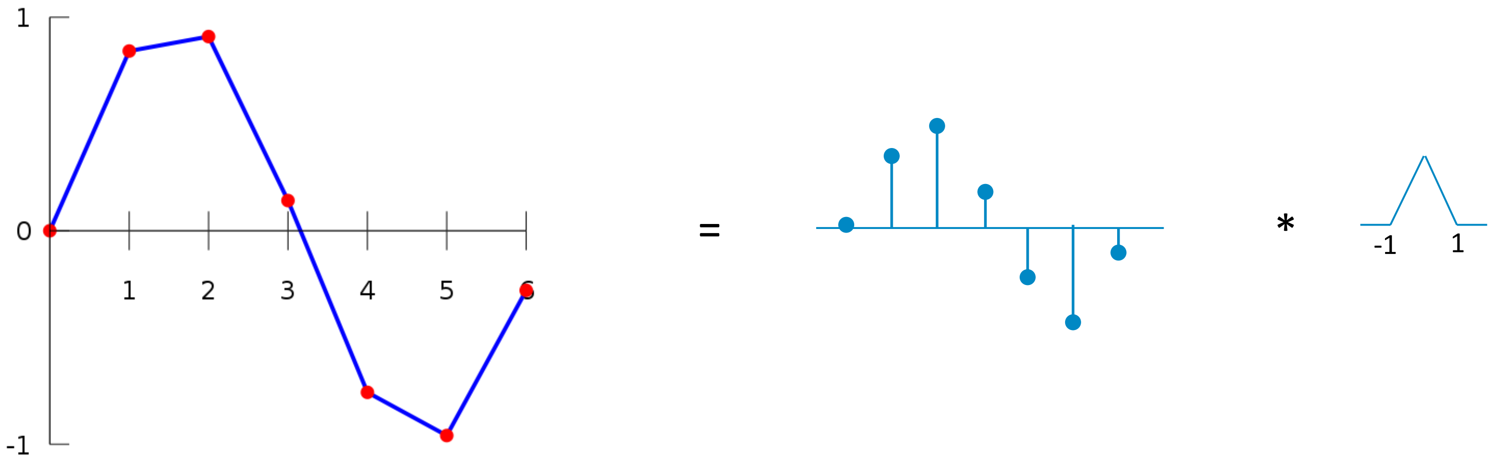
Linear

interpolation



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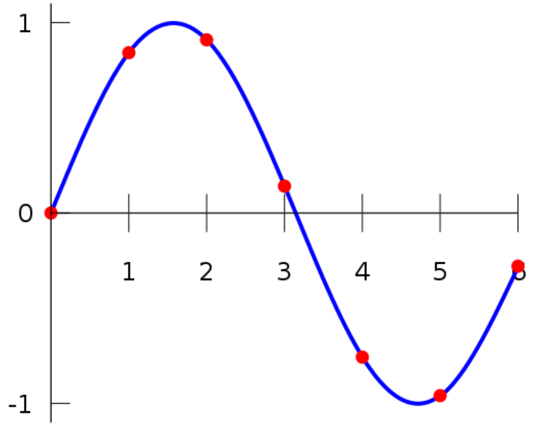
Cubic

interpolation



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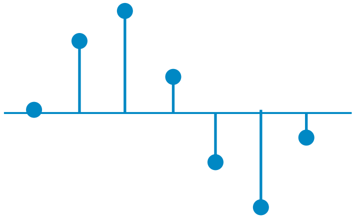

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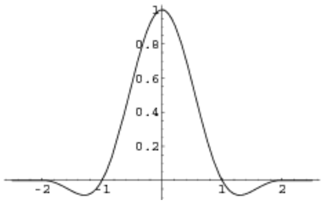

Lanczos kernel

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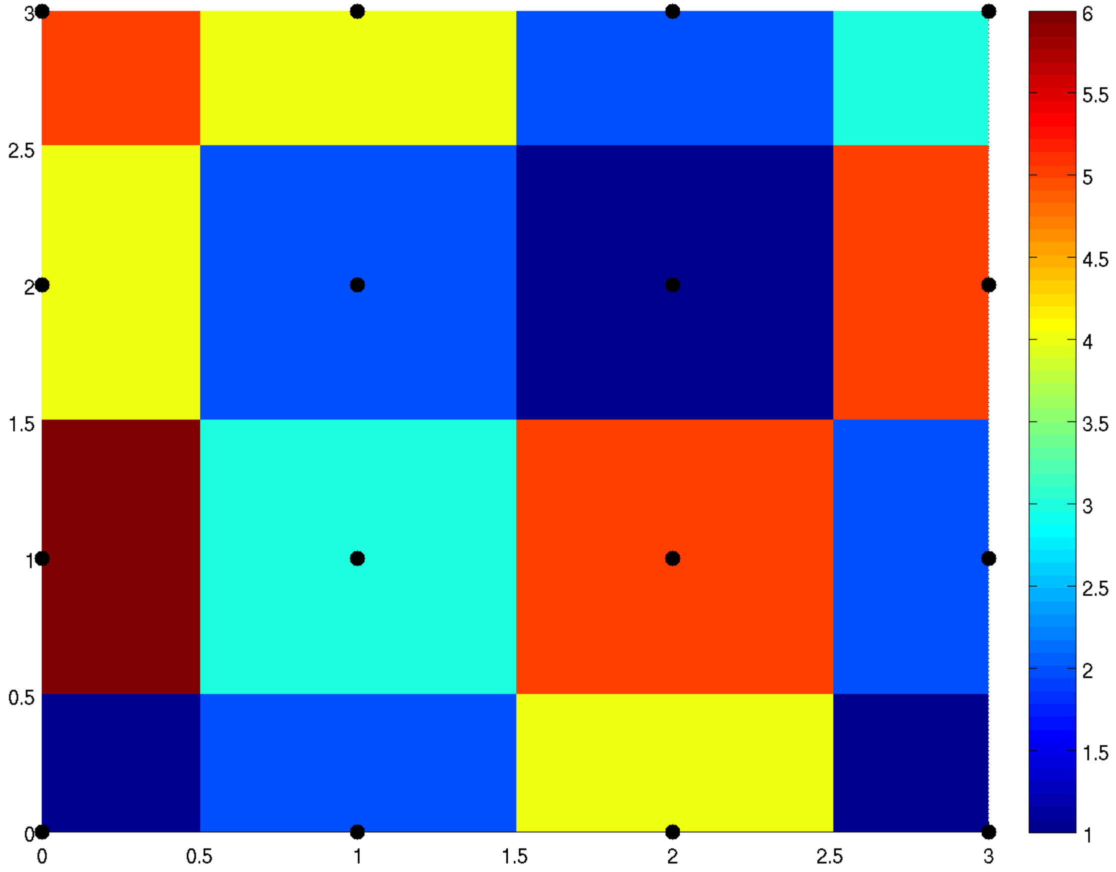
2D Nearest neighbor



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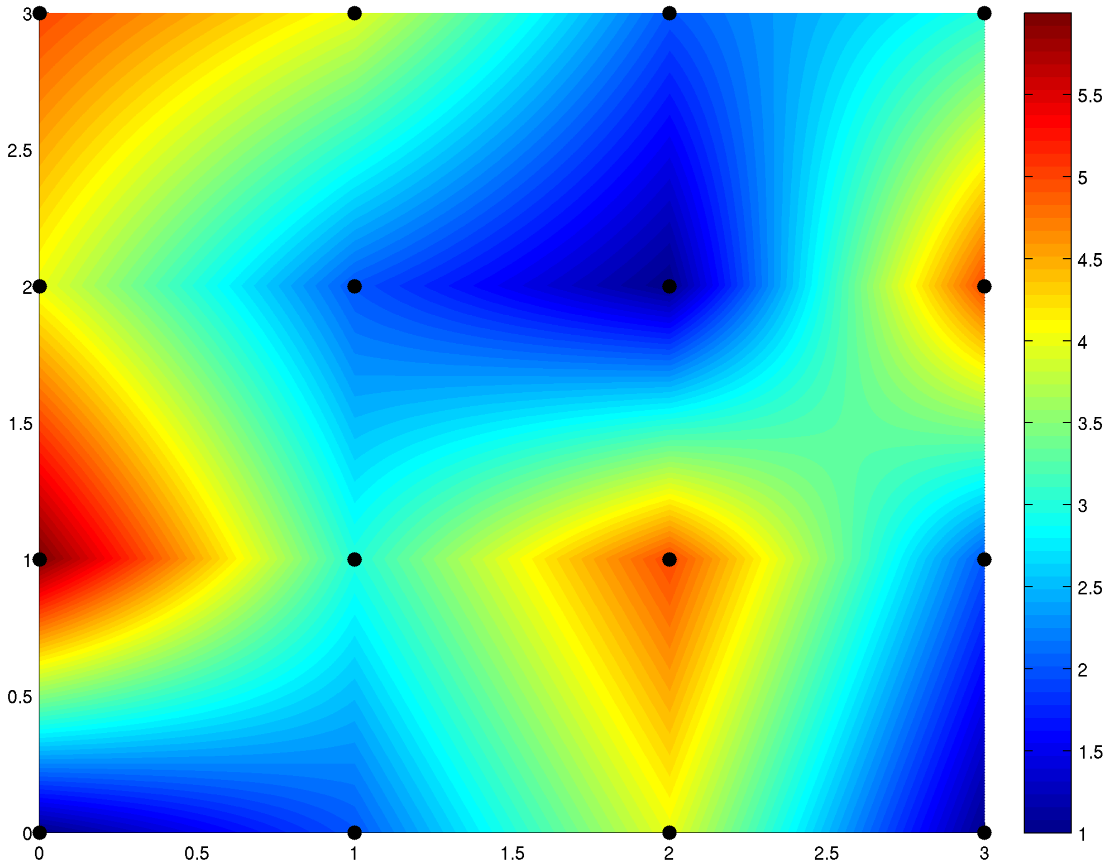
2D Linear



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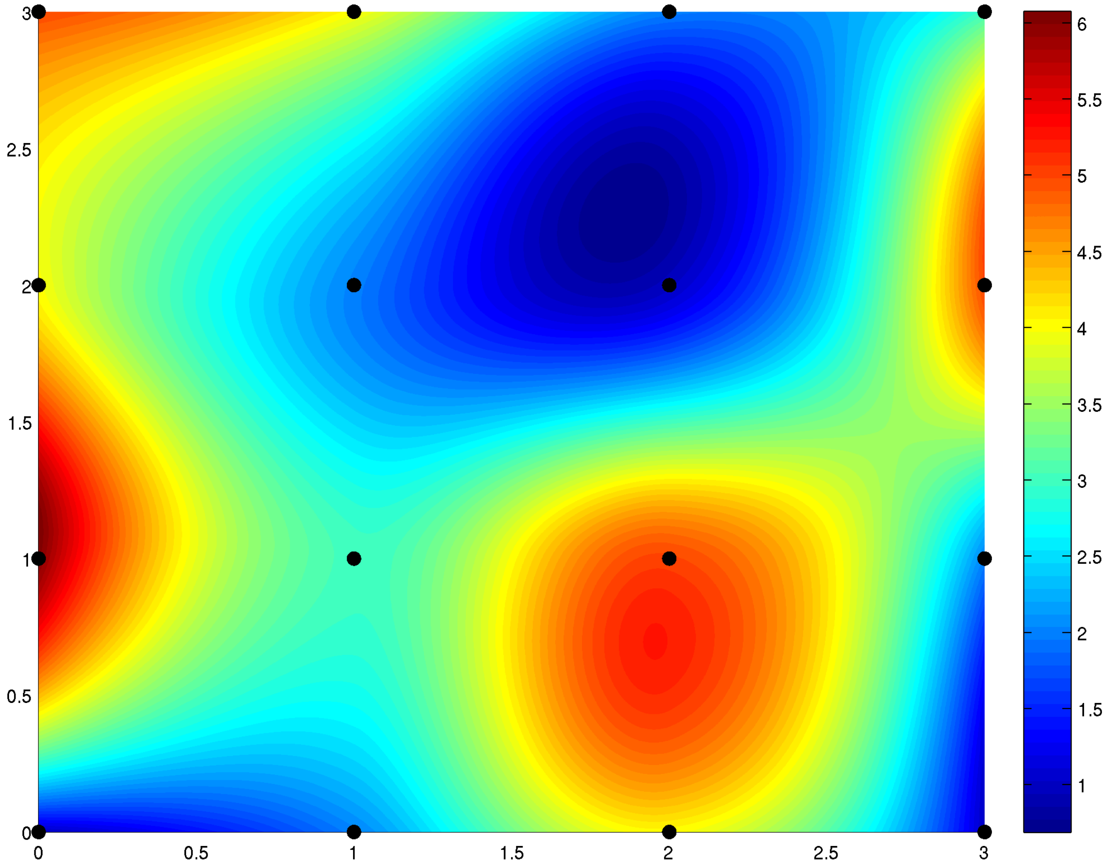
2D Cubic



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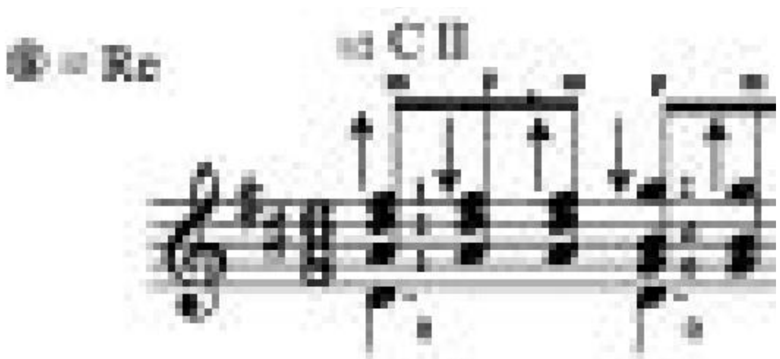
Nearest neighbor vs Cubic



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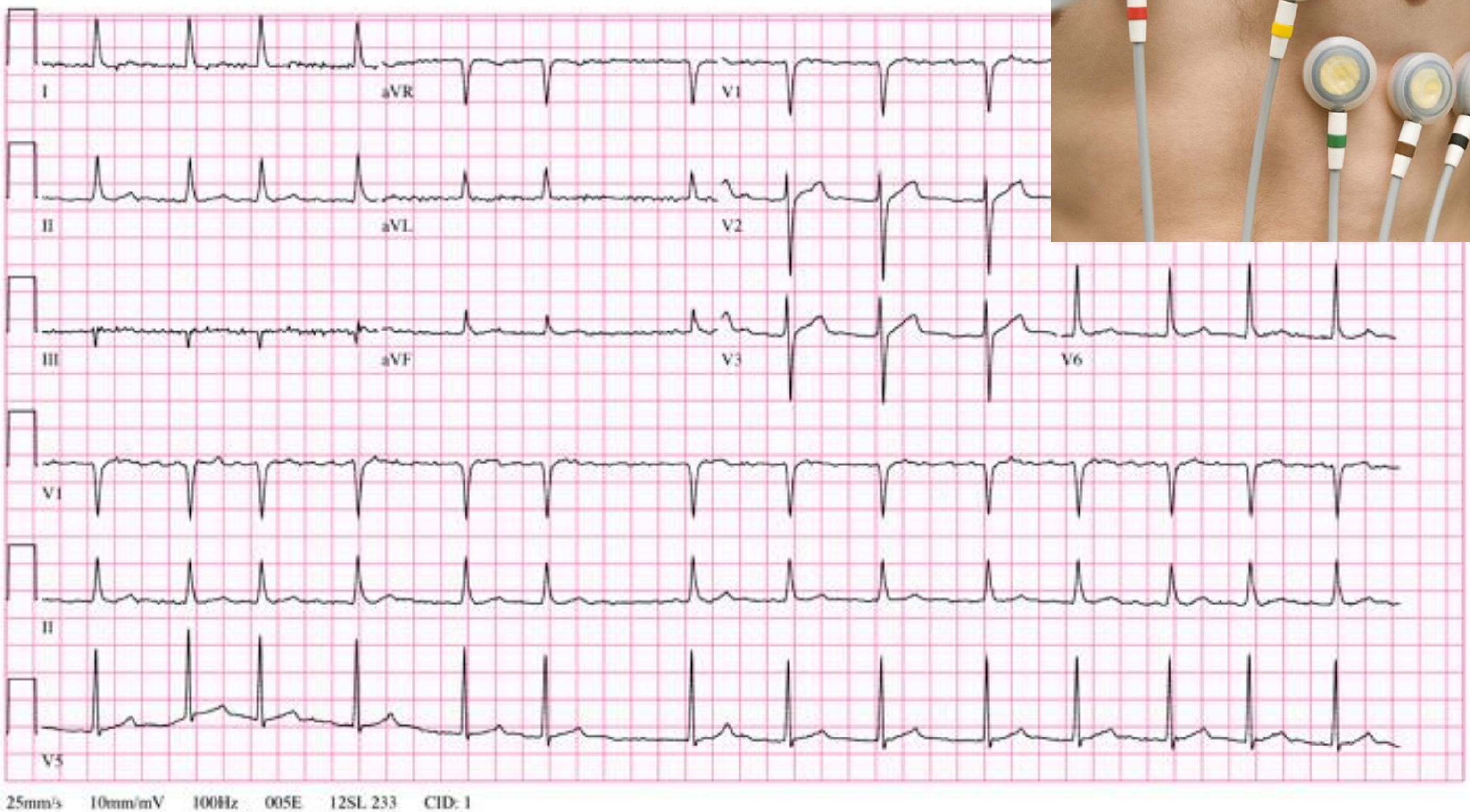
Fast high quality interpolation

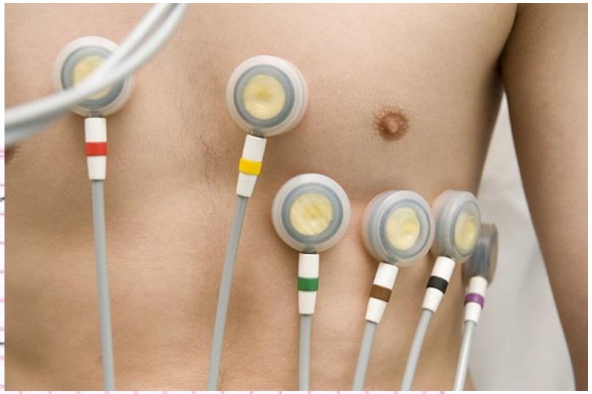
- Ideal interpolation: convolution with sinc
  - ☺ Ideal – high quality for bandwidth limited signals
  - ☹ Infinite support
  - ☹ Slow decay
- Nearest neighbor & linear interpolation
  - ☺ Simple to implement
  - ☺ Hardware acceleration
  - ☹ Low quality, smoothing effect

- Conclusion:
  - we would like best of both worlds
  - high quality (close as possible), fast interpolation

Real world sampling

Electrocardiogram - ECG





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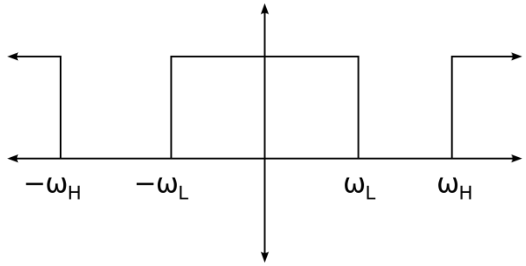
50 Hz or 60 Hz AC power interference

How to remove the 50 or 60 Hz interference?

- Suppose we have a sampled ECG signal, sampled at e.g. 1000 Hz
- The discrete Fourier transform of the sampled signal will show the interference only around the 50 or 60 Hz mark
- A Band-stop filter can remove exactly that part



'ideal' band-stop filter

- In practice it may be preferable to smooth the edges, to prevent undesirable aliasing effects.

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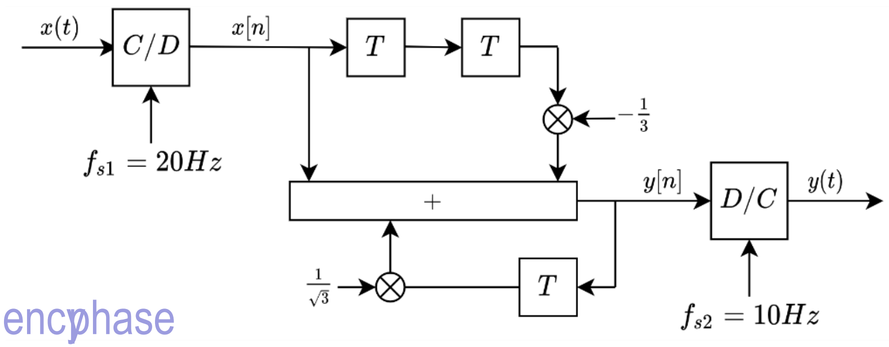
Example

Example



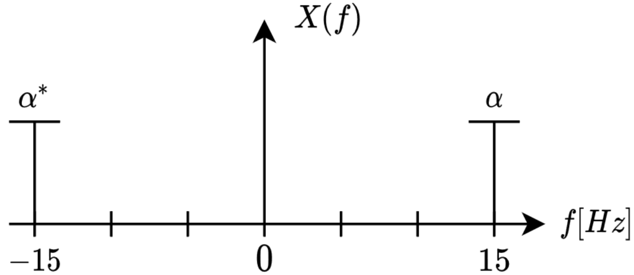
amplitude frequency phase

- $x(t) = 3\sin(30\pi t + \frac{\pi}{3})$
- Find $y(t)$

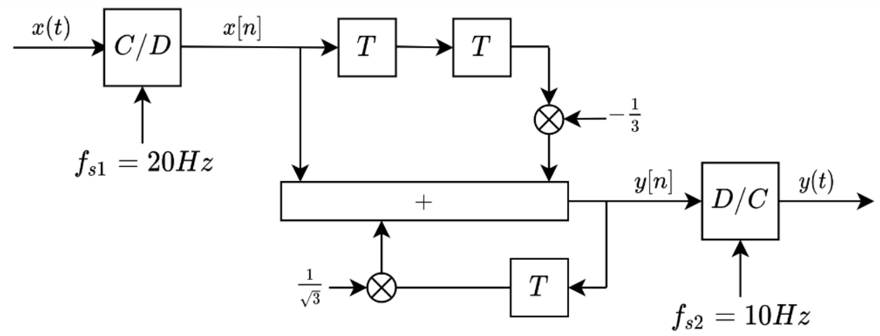
Example

* We look at the spectral content of $x(t)$
* $x(t) = 3\sin(30\pi t + \frac{\pi}{3}) = \frac{3}{2j} \cdot e^{j\frac{\pi}{3}} \cdot e^{j30\pi t} + \frac{-3}{2j} \cdot e^{-j\frac{\pi}{3}} \cdot e^{-30j\pi t}$
* Let us write $\alpha = \frac{3}{2j} \cdot e^{j\frac{\pi}{3}}$ amplitude and phase
* Now $x(t) = \alpha e^{j30\pi t} + \alpha^* e^{-j30\pi t} = \alpha e^{j2\pi \cdot 15t} + \alpha^* e^{-j2\pi \cdot 15t}$

* Now we know that the spectrum will have a delta pulse with weight α at $f = 15\ Hz$ and a delta pulse with weight α^* at $f = -15Hz$



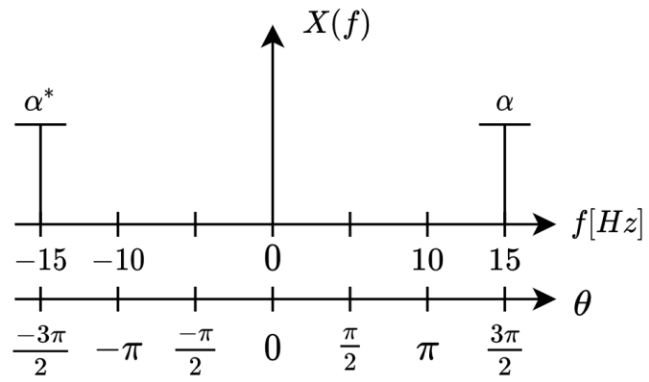
Example



* $x(t) = \alpha e^{j2\pi\cdot 15t} + \alpha^* e^{-j2\pi\cdot 15t}$
* Now we look at what happens when we convert this continuous-time signal to a discrete-time signal
* The C/D converter runs at $f_{s1} = 20\ Hz$
* The fundamental interval will be between $-10\ Hz$ and $10\ Hz$
* Now we convert frequency to relative frequency through f_{s1}
* $\theta = \omega \cdot T_s = 2\pi \cdot 15 \cdot \frac{1}{20} = \frac{3\pi}{2}$
* Let us have a look at the spectrum again

Example

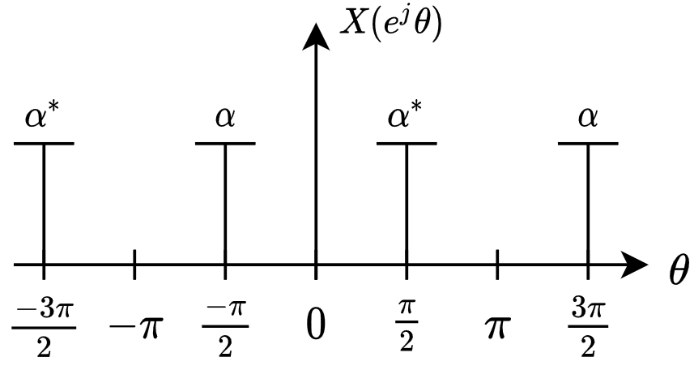
- We can observe that the spectral content is outside of the fundamental interval $[-\pi, \pi]$
- When we sample, we convolve the time-domain signal with a pulse train, which causes the spectral content to repeat itself every 2π
- So the figure below will change



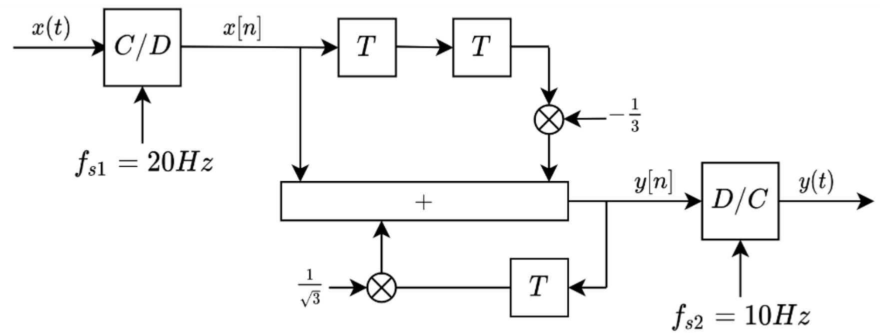
Example

- By adding or subtracting 2π, we find the pulses inside the fundamental interval
- We notice that the delta pulses crossed $\theta = 0$, which we see as a π phase shift
- We find our expression for $x[n]$:

$$x[n] = x_a(t)|_{t=n\cdot T_s} = 3\sin(\frac{\pi}{2}n - \frac{2\pi}{3})$$



Example



* Now that we have found an expression for $x[n]$, we will equate it to $y[n]$ through the difference equation of the LTI system above

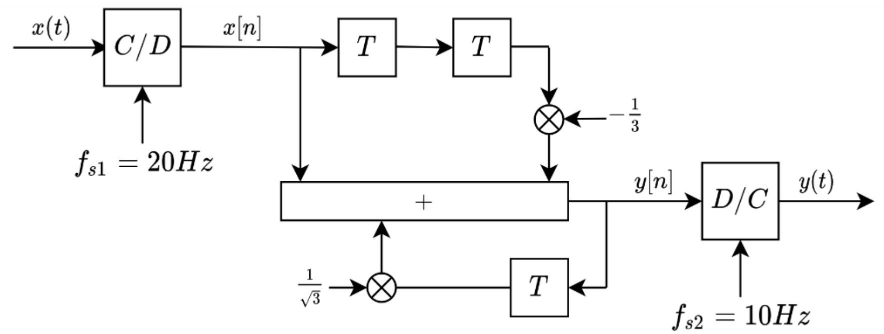
$$y[n] = x[n] - \frac{1}{3}x[n-2] + \frac{1}{\sqrt{3}}\, y[n-1]$$

* We solve this in frequency domain through the FTD:

$$Y\left(e^{j\theta}\right) = X\left(e^{j\theta}\right) - \frac{1}{3}e^{-j2\theta}X\left(e^{j\theta}\right) + \frac{1}{\sqrt{3}}\, e^{-j\theta}Y\left(e^{j\theta}\right)$$

$$\Rightarrow\ Y\left(e^{j\theta}\right)\left(1 - \frac{1}{\sqrt{3}}e^{-j\theta}\right) = X(e^{j\theta})(1 - \frac{1}{3}e^{-j2\theta})$$

Example



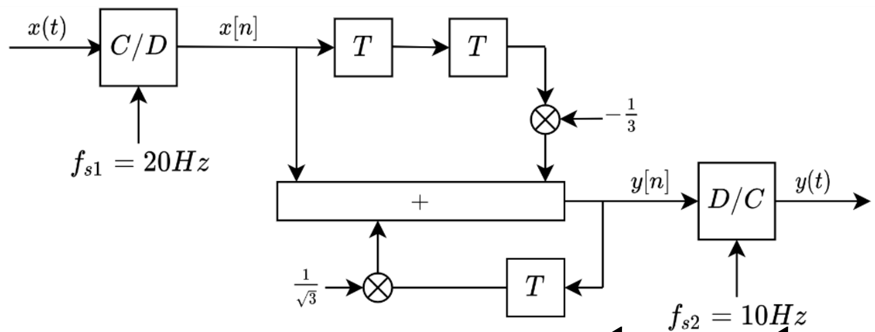
- We will find the frequency response $H(e^{j\theta})$:

$$Y(e^{j\theta})\left(1-\frac{1}{\sqrt{3}}e^{-j\theta}\right) = X(e^{j\theta})\left(1-\frac{1}{3}e^{-j2\theta}\right)$$

$$H(e^{j\theta}) = \frac{Y(e^{j\theta})}{X(e^{j\theta})} = \frac{1-\frac{1}{3}e^{-j2\theta}}{1-\frac{1}{\sqrt{3}}e^{-j\theta}}$$

- We know the system is LTI, so it will not change the frequency of our signal, but only the phase and magnitude
- Therefore, we evaluate the frequency response of our signal: $H(e^{j\theta})|_{\theta=\pi/2}$

Example

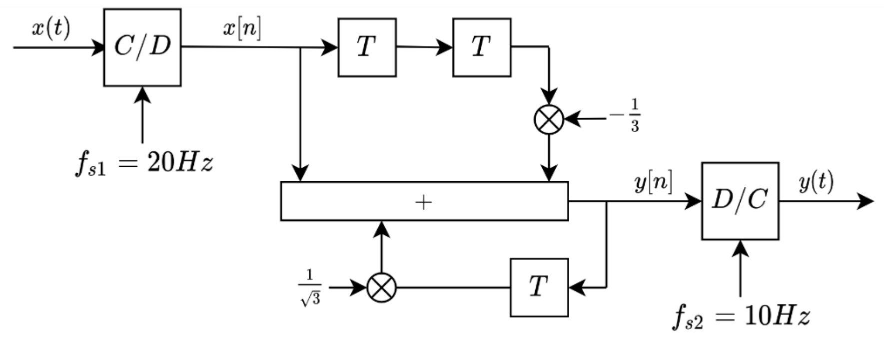


$$H(e^{j\theta})\Big|_{\theta=\pi/2} = \frac{1-\frac{1}{3}e^{-j\pi}}{1-\frac{1}{\sqrt{3}}e^{-j\frac{\pi}{2}}} = \frac{1+\frac{1}{3}}{1+\frac{1}{\sqrt{3}}j}\cdot\frac{1-\frac{1}{\sqrt{3}}j}{1-\frac{1}{\sqrt{3}}j} = \frac{1\frac{1}{3}-1\frac{1}{3\sqrt{3}}j}{1\frac{1}{3}} = 1-\frac{1}{\sqrt{3}}j$$

$$\sqrt{1^2+\left(\frac{-1}{\sqrt{3}}\right)^2} = \frac{2}{\sqrt{3}},\ \operatorname{atan}(-\tfrac{1}{\sqrt{3}}/1) = -\frac{\pi}{6},\ \Rightarrow H(e^{j\theta})|_{\theta=\pi/2} = \frac{2}{\sqrt{3}}e^{-j\frac{\pi}{6}}$$

- $\left|H(e^{j\theta})|_{\theta=\frac{\pi}{2}}\right| = \frac{2}{\sqrt{3}},\ \angle\left\{H(e^{j\theta})|_{\theta=\frac{\pi}{2}}\right\} = -\frac{\pi}{6}$
- Now we know what the system does to the magnitude and phase for a frequency of $\theta = \pi/2$

Example



* Now we apply the effects of the frequency response to find $y[n]$:

$$y[n] = 3 \cdot \frac{2}{\sqrt{3}} \sin(\frac{\pi}{2}n - \frac{2\pi}{3} - \frac{\pi}{6}) = \frac{6}{\sqrt{3}} \sin(\frac{\pi}{2}n - \frac{5\pi}{6})$$

* Now we convert this discrete-time signal back to a continuous-time signal through $\omega = \theta f_{s2}$
* $y(t) = \frac{6}{\sqrt{3}} \sin(5\pi t - \frac{5\pi}{6})$

Conclusions

Conclusions

- Basics: Fourier Transform, Convolution, …
- Sampling theory
- Ideal interpolation
- Ideal upsampling
- Nearest neighbor, linear interpolation
- Generalized approximating interpolation

Acknowledgements:
This lecture content is partially based on the slides by dr. Piet Sommen, and on content taken from Wikipedia and other public sources, indicated by the hyperlinks.

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