Digital Signal Processing Fundamentals (5ESC0)

Z-transform

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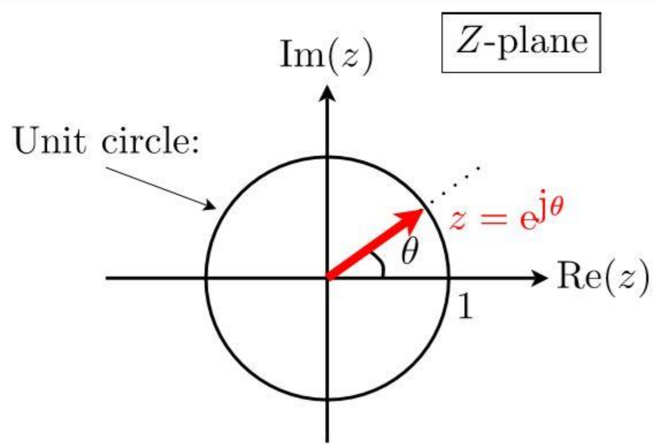
Definition

- Z-transform is performed in the discrete time domain, it is the counterpart of the Laplace transform, which is in the continuous time domain
- Z-transform is a transform in a complex domain
  - Z-domain, with real and imaginary axes
- Z-transform can be defined as

$$\boxed{\textbf{ZT:} \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

Definition

- Fourier transform for discrete time series (FTD) is Z-transform calculated on the unit circle



- Substituting z by $e^{j\theta}$ we obtain the Fourier transform and the other way around:

$$\text{FTD} \equiv \text{ZT evaluated on unit circle: } X(z)\big|_{z=\mathrm{e}^{\mathrm{j}\theta}} = X(\mathrm{e}^{\mathrm{j}\theta})$$

Definition

- FTD

$$\text{FTD:} \quad X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\theta}$$

- Z-transform

$$\boxed{\text{ZT:} \quad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

- Why do we need Z-transform?
  - It allows to perform system analysis through the system function (poles and zeros)

Example 1

Z-transform

$$\text{ZT:} \quad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

If the input signal x is a delta pulse, then $X(z) = 1$:

Explanation:

for $n = 0$ we get $X(z) = \delta(0) z^{-0} = 1$, because $\delta(0) = 1$

for other n we will have $\delta(n) = 0$.

So we can write

$$x_1[n] = \delta[n] \quad \circ\!\!-\!\!\circ \quad X_1(z) = 1$$

Example 1

ZT: $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Input signal $x[n] = \delta(n - k)$, what is $X(z)$?

Here the delta pulse is shifted over k samples, therefore we will have a delta pulse on the position $n = k$.

For $n = k$ we get $X(z) = \delta(0)z^{-k} = z^{-k}$, because $\delta(0) = 1$

for other n we will have $\delta(n) = 0$.

So we can write

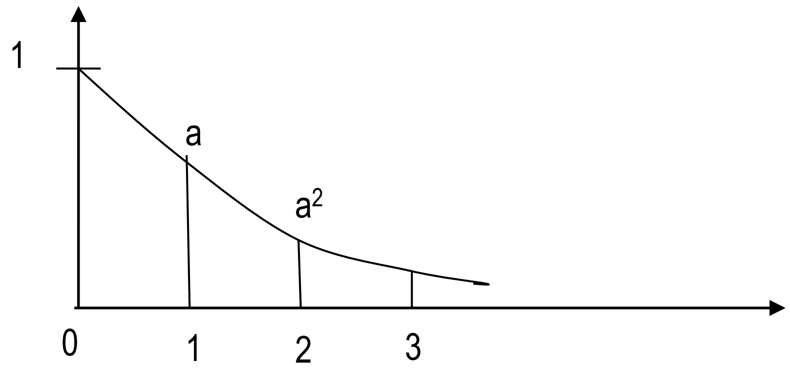
$$x_2[n] = \delta[n - k] \quad \circ\!\!-\!\!\circ \quad X_2(z) = z^{-k}$$

Example 1

Let us consider one more example

Input signal $x[n] = a^n u[n]$, what is $X(z)$?

For $a = \frac{1}{2}$, $x[n]$ is on the plot below:



Example 1

Observing the plot above, we can present $x[n]$ as follows:

$$x[n] = a^n u[n] = \delta[n] + a\delta[n-1] + a^2\delta[n-2] + \cdots$$

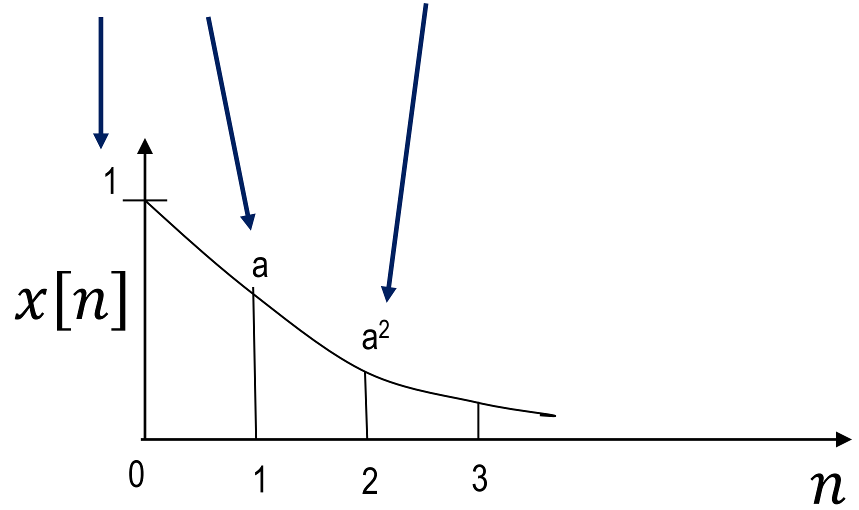
Z-transform is a linear transform, so we can perform it element by element:

$$X(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

In this example we can see that we obtain in Z-domain a polynomial description of a function from which we can see sample values in the digital domain

Example 1

$$X(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots$$



Note: If $X(z)$ writes as polynomial function with terms z^{-1}, then factor of term z^{-i} equals value of $x[n]$ at time $n = i$

Example 1

Summary of this example:

$$x_3[n] = (a)^n u[n] \quad \circ\!\!-\!\!\circ \quad X_3(z) = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} \cdots$$

We can see the step function $u[n]$ in the expression for $x_3[n]$, so the summation for Z-transform can start from 0 in this case instead of $-\infty$:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \underbrace{\sum_{n=0}^{\infty} (az^{-1})^n}_{} = \underbrace{\frac{1}{1 - az^{-1}}}_{}$$

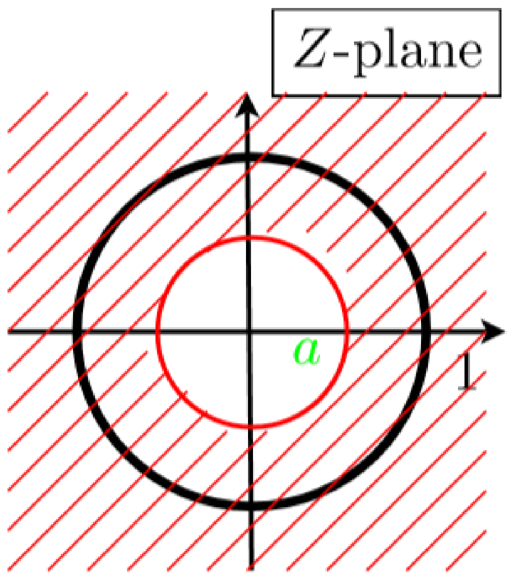
this is a geometric series and can be written as

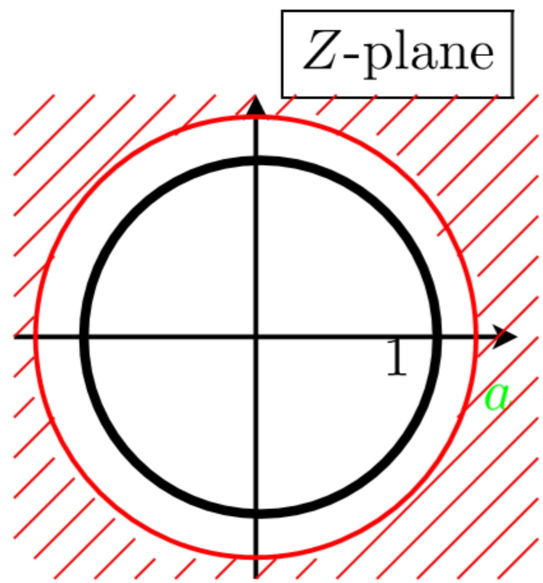
Region of Convergence

The expression above for geometric series is only true when $|az^{-1}| < 1$. It can be rewritten as $|z| > |a|$.

This can be put on the Z-plane –
it is ROC (Region of Convergence):

$a < 1$





$a > 1$

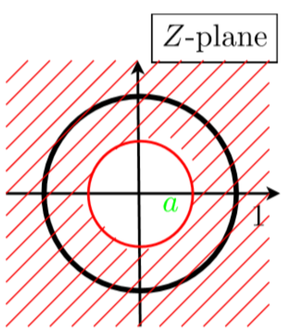
Region of Convergence

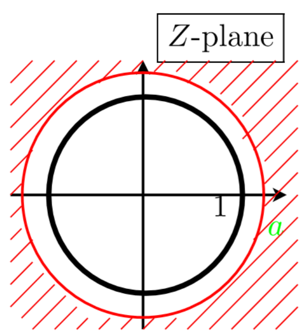
Why is ROC important?

We can calculate the Z-transform of a digital signal, but we are often interested in Fourier transform for discrete time series (FTD):

If the unit circle lies inside the ROC for Z-transform, then we can calculate FTD for such a signal

$a < 1$





$a > 1$

FTD converges <u>iff</u> unit circle in ROC

Example 2

We will find Z-transform of the sequence below:

$$x_4[n] = -(a)^n u[-n-1]$$

The step function $u[-n-1] = 1$ when $(-n-1) \geq 0$,
this can be rewritten as $n \leq -1$,
so we n set the summation in the Z-transform of this sequence in the interval $(-\infty; -1)$:

$$X(z) = -\sum_{-\infty}^{-1} a^n z^{-n} = -\sum_{-\infty}^{-1} (az^{-1})^n = -\sum_{p=1}^{\infty} (az^{-1})^{-p} = \cdots$$

We can replace n by $-p$, so $p = -n$

Let's put the minus inside

Example 2

$$-\sum_{p=1}^{\infty}(az^{-1})^{-p} = -\sum_{p=1}^{\infty}(a^{-1}z)^{p}$$

There is a formula for the series $\sum_{n=0}^{\infty} i^n$,
but our first value of p is equal to 1 and not to zero,

we can change it by adding the term for $p = 0$;
this term equals to 1, because $(a^{-1}z)^0 = 1$,
so we also have to subtract 1:
in this way we do not change our series – we add 1 and subtract 1

Example 2

$$-\sum_{p=1}^{\infty}(a^{-1}z)^p = -\left(\sum_{p=0}^{\infty}(a^{-1}z)^p - 1\right) = 1 - \frac{1}{1-a^{-1}z}$$

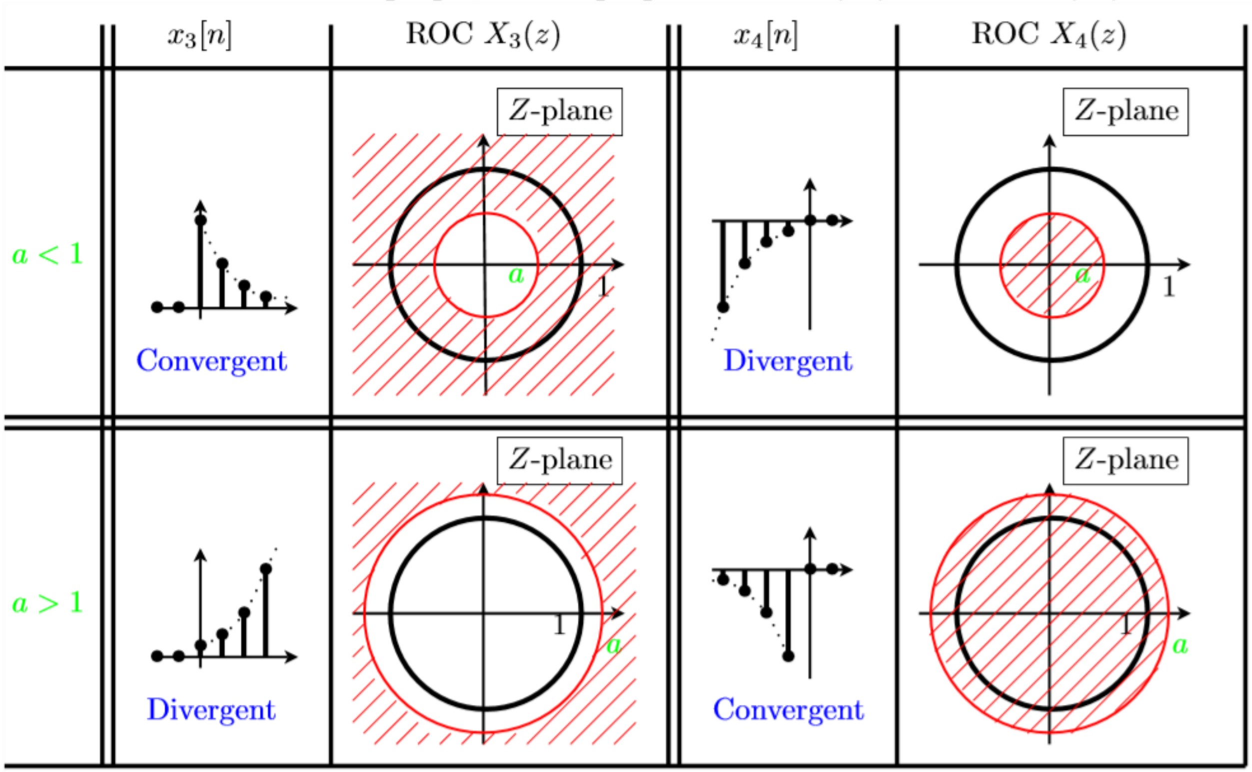
This step is only valid when $|a^{-1}z| < 1$ or $|z| < |a|$

From the expression on the right hand side above we can derive the final formula for the Z-transform in this example:

$$X(z) = \frac{1}{1-az^{-1}}$$

ROC differences

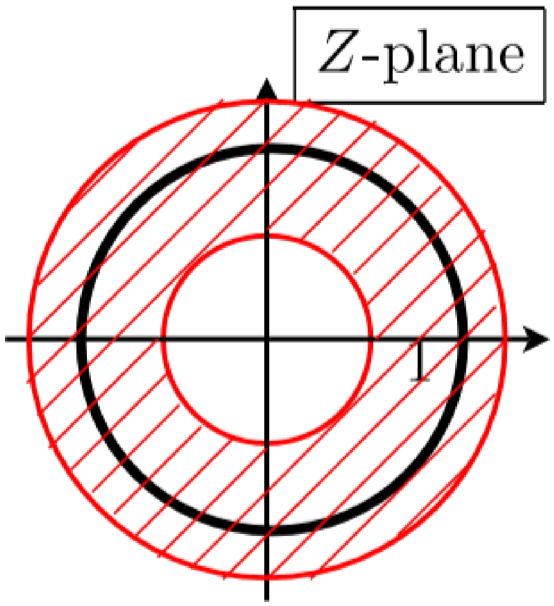
Thus while $x_3[n] \neq x_4[n] \Rightarrow X_4(z) = X_3(z)$ but **different** ROC!



FTD converges <u>iff</u> unit circle in ROC

ROC differences

Region of convergence may also be in a ring,
for example, $\frac{1}{2} < z < 2$



Z-transforms of delta pulses

We have already considered several simple but very useful examples:

Signal in time domain $x[n]$	Z-transform $X(z)$
$\delta[n]$	1
$\delta[n-k]$	z^{-k}
$a\delta[n+1] + b\delta[n] + c\delta[n-1]$	$az + b + cz^{-1}$, a, b, c – parameters

Z-transforms of delta pulses

Explanation for the formula in the last line of the table above

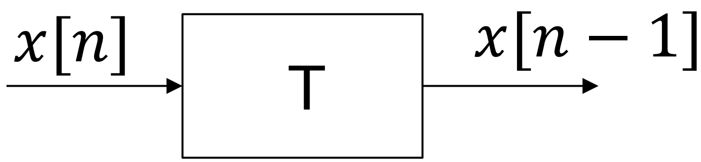
If $x[n] = \delta[n+1]$, we can also write
$x[n] = \delta[n-(-1)] = \delta[n-k]$, if $k = -1$;
then $X(z) = z^{-k} = z^{1}$

If $x[n] = \delta[n-1]$, we can also write
$x[n] = \delta[n-k]$, if $k = 1$;
then $X(z) = z^{-1}$

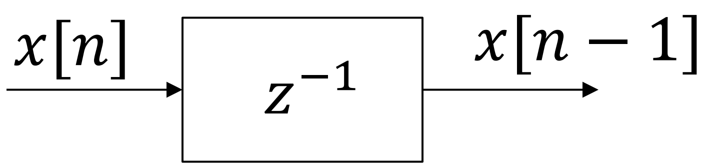
We can see that z^{-1} is related to a delay $[n-1]$

Z-transform and delay

In a flowchart of a system we can indicate a delay by T



A delay can also be indicated by z^{-1}:



Common Z-transform pairs

Sequence	Z-transform	ROC
$\delta[n]$	1	all z
$\delta[n-i]$	z^{-i}	$z \neq 0, \infty$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > a$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < a$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > a$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < a$
$a^n \cos(n\theta_0) u[n]$	$\frac{1-az^{-1}\cos(\theta_0)}{1-2az^{-1}\cos(\theta_0)+a^2z^{-2}}$	$\|z\| > a$
$a^n \sin(n\theta_0) u[n]$	$\frac{az^{-1}\sin(\theta_0)}{1-2az^{-1}\cos(\theta_0)+a^2z^{-2}}$	$\|z\| > a$
$a^n \sin(n\theta_0 + \phi) u[n]$	$\frac{\sin(\phi)+az^{-1}\sin(\theta_0-\phi)}{1-2az^{-1}\cos(\theta_0)+a^2z^{-2}}$	$\|z\| > a$

Example: common Z-transform pair (1/2)

Find the Z-transform of the following sequence:

$$x[n] = \left(\frac{1}{3}\right)^n \cos(n\omega_0)u[n]$$

<u>Solution</u>

Use the Z-transform table on previous slide to find the pair we need:

$$X(z) = \frac{1 - \frac{1}{3}\cos(\omega_0)z^{-1}}{1 - \frac{2}{3}\cos(\omega_0)z^{-1} + \frac{1}{9}z^{-2}}$$

Example: common Z-transform pair (2/2)

So the solution is

$$X(z) = \frac{1 - \frac{1}{3}\cos(\omega_0)z^{-1}}{1 - \frac{2}{3}\cos(\omega_0)z^{-1} + \frac{1}{9}z^{-2}},$$

with a region of convergence $|z| > \frac{1}{3}$ which we can find in the table for Z-transform pairs

Does FTD converge for this signal? Why?

Properties of Z-transform

Linearity	:	$a \cdot x[n] + b \cdot y[n]$	$\circ\!-\!\circ$	$a \cdot X(z) + b \cdot Y(z)$
Shifting property	:	$x[n - n_0]$	$\circ\!-\!\circ$	$z^{-n_0} X(z)$
Time reversal	:	$x[-n]$	$\circ\!-\!\circ$	$X(z^{-1})$
Multiply by exponential	:	$a^n \cdot x[n]$	$\circ\!-\!\circ$	$X(a^{-1} z)$
Convolution theorem	:	$y[n] = x[n] * h[n]$	$\circ\!-\!\circ$	$Y(z) = X(z) \cdot H(z)$
Conjugation	:	$x^*[n]$	$\circ\!-\!\circ$	$X^*(z^*)$
Derivative	:	$nx[n]$	$\circ\!-\!\circ$	$-z \dfrac{\mathrm{d}X(z)}{\mathrm{d}z}$

Example 3

We will find Z-transform of the sequence below:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u(-n-1)$$

The sequence above is a sum of two sequences:

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

and

$$x_2[n] = -2^n u[-n-1]$$

Example 3

We remember from the common transform pairs:

sequence $a^n u[n]$ has Z-transform equal to $\frac{1}{1-az^{-1}}$

with ROC $|z| > a$

So for the first sequence $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

we obtain the following result in Z-domain:

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}\ , \quad |z| > \frac{1}{2}$$

Example 3

We also remember from the common transform pairs:

sequence $-a^n u[-n-1]$ has Z-transform equal to $\frac{1}{1-az^{-1}}$ with ROC $|z| < a$

So for the first sequence $x_2[n] = -2^n u(-n-1)$

we obtain the following result in Z-domain:

$$X_2(z) = \frac{1}{1-2z^{-1}}\,, \quad |z| < 2$$

Example 3

Therefore, the Z-tranform of the provided sequence $x[n]$ will be

$$X(z) = X_1(z) + X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} = \frac{2-\frac{5}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})}$$

With a region of convergence (ROC) $\frac{1}{2} < |z| < 2$

In this case ROC is the set of all points that are in the ROC of both $X_1(z)$ and $X_2(z)$

Example 4

Find the Z-tranform of the convolution of two following sequences: $x[n] = \alpha^n u[n]$ and $h[n] = \delta[n] - \alpha\delta[n-1]$.

The Z-transform of the first sequence $x[n] = \alpha^n u[n]$ is

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

The Z-transform of the second sequence is

$$H(z) = 1 - \alpha z^{-1}$$

Example 4

We know that the convolution in the time domain is multiplication in the Z-domain.

Therefore we can find the Z-transform of the convolution of the given sequences by multipying their Z-transforms:

$$Y(z) = X(z)H(z) = \frac{1}{1-\alpha z^{-1}}(1 - \alpha z^{-1}) = 1.$$

Inverse Z-transform

There are several ways for calculating the inverse z-transform

If we can decompose the expression for Z-transform into parts whose transform pairs are in the table above,
then the inverse Z-transform is straightforward.

For example, let's perform the inverse Z-transform for

$$x(z) = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

Inverse Z-transform

What is $x[n]$ for $x(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$?

We know that the sequence $x[n] = a^n u[n]$ has the Z-transform

$X(z) = \frac{1}{1-az^{-1}}$ $|a| < 1$

So the answer is $x[n] = \left(\frac{1}{3}\right)^n u[n]$,

in this case $a = \frac{1}{3}$

Inverse Z-transform

Let's consider partial fraction expansion for inverse Z-transform

Assume that we are given $X(z) = \frac{5-2z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$ and we have to find $x[n]$.

The denominator here is a second order polynomial,
we can find its zeros and rewrite

$$X(z) = \frac{5-2z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{3}z^{-1}}$$

Inverse Z-transform

To get the common nominator for the sum above, we can write

$$A_1\left(1-\frac{1}{3}z^{-1}\right)+A_2\left(1-\frac{1}{2}z^{-1}\right)=5-2z^{-1},$$

A_1 and A_2 are constant values.

This can be rewritten as

$$\underbrace{(A_1+A_2)}_{\text{constant}}-\underbrace{\left(\frac{A_1}{3}+\frac{A_2}{2}\right)}_{\text{another constant}}z^{-1}=\underbrace{5}_{\text{constant}}-\underbrace{2}_{\text{another constant}}z^{-1}$$

Inverse Z-transform

We can write it as

$$A_1 + A_2 = 5$$

$$\frac{A_1}{3} + \frac{A_2}{2} = 2$$

From this system of equations we obtain

$$A_1 = 3;\ A_2 = 2$$

Inverse Z-transform

Now we can rewrite the expression for the provided Z-transform

$$X(z) = \frac{3}{1 - \frac{1}{2} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}}$$

So we can deduce the signal in the time domain:

$$x[n] = 3\left(\frac{1}{2}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

Inverse Z-transform

There is also another method we can use in the same example

Let's recall the initial Z-transform function:

$$\mathrm{X}(z) = \frac{5 - 2z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

We have rewritten it as

$$\mathrm{X}(z) = \frac{5 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - \frac{1}{3}z^{-1}}$$

Inverse Z-transform

The roots of the polynomial in the denominator of this function are $\frac{1}{2}$ and $\frac{1}{3}$.

Let's multiply both sides of the equation

$$\frac{5-2z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}=\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-\frac{1}{3}z^{-1}}$$

by $\left(1-\frac{1}{2}z^{-1}\right)$.

Inverse Z-transform

We will then obtain

$$\frac{5-2z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)} = A_1 + \frac{A_2}{1-\frac{1}{3}z^{-1}}\underbrace{\left(1-\frac{1}{2}z^{-1}\right)}_{\text{this is zero if } z=\frac{1}{2}}$$

If $z = \frac{1}{2}$ (one of the roots of the polynomial of denominator), then

$A_1 = \frac{5-2z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)} = \frac{5-4}{1-\frac{2}{3}} = 3$; and similarly we can find $A_2 = 2$.

Inverse Z-transform

Z-transforms that are rational functions of z:

$$X(z) = \frac{b_0+b_1z^{-1}+\cdots+b_Nz^{-N}}{1+a_1z^{-1}+\cdots+a_Mz^{-M}} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1+\sum_{k=1}^{M} a_k z^{-k}} =$$

$$= b_0 \frac{\prod_{k=1}^{N}(1-\beta_k z^{-1})}{\prod_{k=1}^{M}(1-\alpha_k z^{-1})}$$

For $M > N$ and simple roots, thus $\alpha_i \neq \alpha_k$ for $i \neq k$, we obtain:

$$\boxed{X(z) = \sum_{k=1}^{M} \frac{A_k}{1-\alpha_k z^{-1}} \quad \circ\!-\!\circ \quad x[n] = \sum_{k=1}^{M} A_k (\alpha_k)^n u[n]}$$

with coefficients $A_k = \left[(1-\alpha_k z^{-1})X(z)\right]\Big|_{z=\alpha_k}$

Summary

We considered the definition of Z-transform,
its properties

region of convergence and its properties,

inverse Z-transform

and examples