1. **Double Integrator:**
Many real-life dynamics (e.g. satellite dynamics) can be characterized by a double integrator transfer function

$$G(s) = \frac{1}{s^2}.$$

This transfer function will be used next for understanding the root locus principles and how the parameters of a controller influence closed-loop stability. To this end, consider the standard closed-loop system with the open-loop transfer function $D(s)G(s)$ and the closed-loop transfer function

$$\frac{D(s)G(s)}{1 + D(s)G(s)}.$$

Consider also the following controller types:

(a) Proportional control: $D(s) = k_{\mathrm{P}}$;

(b) Proportional and integral control: $D(s) = k_{\mathrm{P}} + k_{\mathrm{I}}\frac{1}{s}$;

(c) Proportional and derivative control: $D(s) = k_{\mathrm{P}} + k_{\mathrm{D}}\, s$.

Recall that asymptotic stability in the s-plane corresponds to having all the closed-loop poles in the left half plane (LHP), excluding the imaginary axis.

By drawing the root locus, determine which of the above controller types can stabilize the double-integrator system. Follow the following steps:

i. Take the parameter $K \in \mathbb{R}^+$ as follows:
   (a) $k_{\mathrm{P}} = K$;
   (b) $k_{\mathrm{P}} = k_{\mathrm{I}} = K$;
   (c) $k_{\mathrm{P}} = k_{\mathrm{D}} = K$.

ii. Derive the characteristic equation $1 + KL(s) = 0$, knowing that $KL(s) = D(s)G(s)$.

iii. For each case (i.e. case (a),(b), and (c)), sketch the corresponding root locus.

iv. Assess closed-loop stability (that is, assess whether there exist some $K > 0$ that makes the closed-loop system stable for each controller type). Verify your Root Locus with MATLAB; observe how the closed-loop poles change by varying the gain K in the SISO tool.

Solution:

(a) We get the following characteristic equation:
$$1 + K\frac{1}{s^2} = 0, \text{ which implies that } L(s) = \frac{1}{s^2}.$$
We get the following Root Locus:

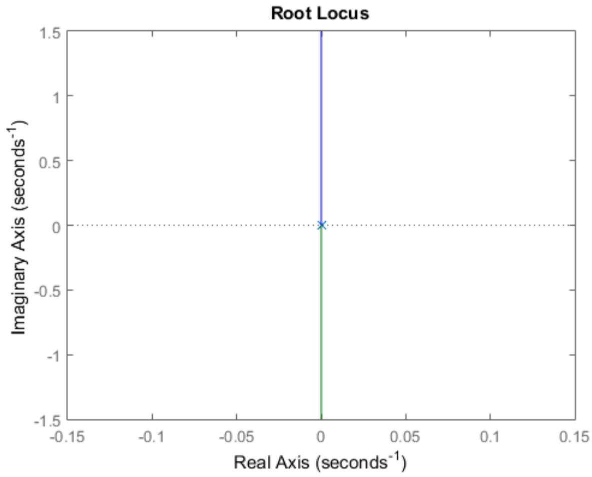


Figure 1: $L(s) = \frac{1}{s^2}$.

Since there is no value for $K > 0$ such that the poles lie in the left half plane (excluding the imaginary axis), we can conclude that the controller will not stabilize the system.

(b) We get the following characteristic equation:

$$1 + K\frac{s+1}{s^3} = 0, \text{ which implies that } L(s) = \frac{s+1}{s^3}.$$

We get the following Root Locus:

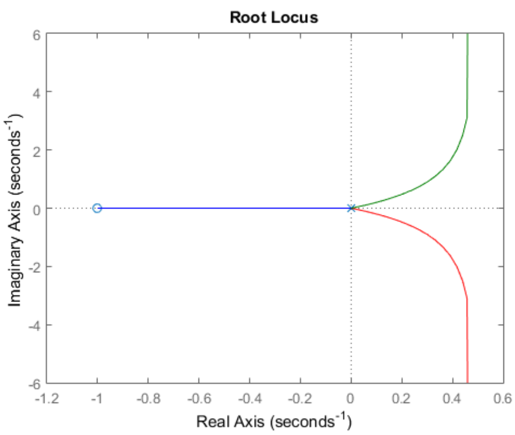


Figure 2: $L(s) = \frac{s+1}{s^3}$.

Since there is no value for $K > 0$ such that the poles lie in the left half plane (excluding the imaginary axis), we can conclude that the controller will not stabilize the system.

(c) We get the following characteristic equation:

$$1 + K\frac{s+1}{s^2} = 0, \text{ which implies that } L(s) = \frac{s+1}{s^2}.$$

We get the following Root Locus:

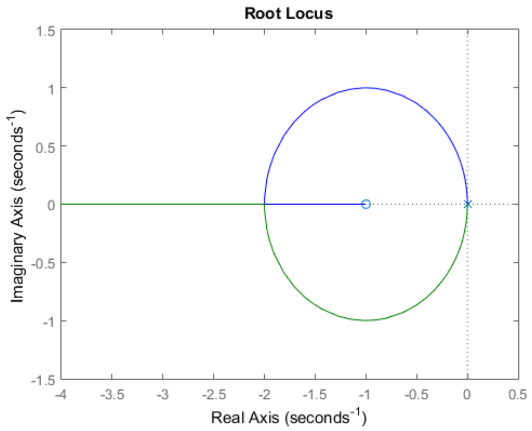


Figure 3: $L(s) = \frac{s+1}{s^2}$.

Since the poles lie in the left half plane for any $K > 0$ we can conclude that the controller can stabilize the system.

The fact that the branches will form a circle can be analyzed from the closed-loop transfer function

$$G(s) = \frac{KL(s)}{1 + KL(s)} = \frac{K\frac{s+1}{s^2}}{1 + K\frac{s+1}{s^2}} = \frac{K(s+1)}{s^2 + Ks + K}.$$

The poles of the closed-loop system are the roots of $s^2 + Ks + K = 0$, which are given by

$$p_{1,2} = \frac{-K \pm \sqrt{K^2 - 4K}}{2}.$$

From this equation we can see that the poles of the system are real valued when $K^2 - 4K \geq 0$, which is true for $K \geq 4$. For the value $K = 4$ we get $p_{1,2} = -2$ and therefore we can conclude that the poles depart from the real axis in the origin and will arrive on the real axis in $(-2, 0)$. If we fill in the value $K = 2$, we get $p_{1,2} = -1 \pm j$.

Additionally, we know that poles will depart form and arrive on the real axis at maximum angles and the Root Locus will be symmetric in the real axis. This implies that the 2 poles will move in exactly the opposite direction (i.e. they make a 180° angle with respect to each other) and since the resulting Root Locus should be symmetric they will do so at an angle of $\pm 90°$ with respect to the real axis.

2. **Root Locus and Time-Domain Specifications:**
For the feedback system shown in Figure 4, find the value of the gain K that yields dominant closed-loop poles with a damping ratio $\zeta \approx 0.5$.

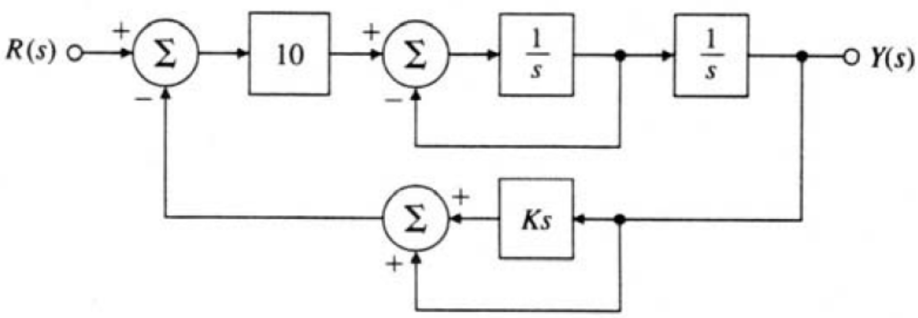


Figure 4: Feedback system for Problem 3.

Follow the following steps:

- Calculate the closed-loop transfer function from R to Y, and bring it to the form $\frac{L_0(s)}{1+KL(s)}$ for some rational function $L(s)$.
- Follow the usual steps for sketching the Root Locus for the determined $L(s)$.
- Compute the angle θ required for the damping ratio $\zeta = 0.5$. (Recall from the specifications of second-order transfer functions that $\zeta = \sin(\theta)$). Draw the damping ratio lines on the complex plane. Based on your sketch, check whether there would be any intersection(s) of the root locus and the damping ratio lines.
- Use MATLAB (`sisotool`) to verify your sketch. Determine: (i.) the numerical values of the poles at intersection points, and (ii.) the gain K required to meet the specification $\zeta \approx 0.5$.

Solution:
First we have to find the root locus form, based on the scheme block. The closed-loop transfer function is

$$G(s) = \frac{10}{s^2 + s + 10Ks + 10} = \frac{\frac{10}{s^2+s+10}}{1 + K\frac{10s}{s^2+s+10}},$$

therefore the characteristic equation is

$$1 + K\frac{10s}{s^2 + s + 10} = 0,$$

and we can define

$$L(s) = \frac{10s}{s^2 + s + 10}.$$

Next we have to draw the root locus for $L(s)$ and $K > 0$, see Figure 5.

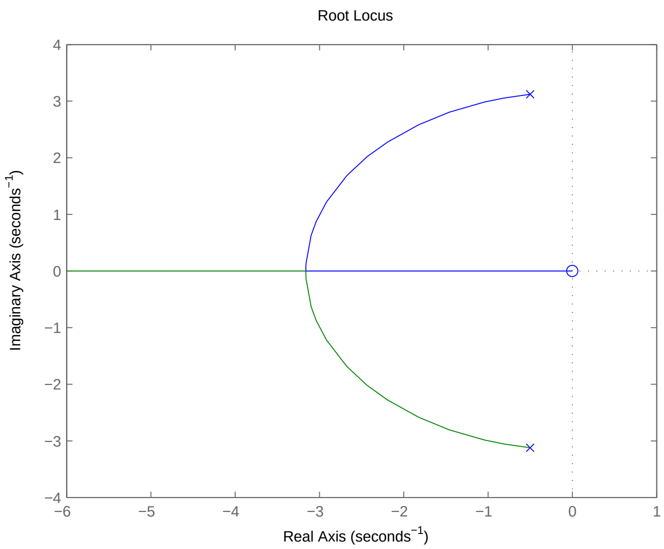


Figure 5: Root Locus for $L(s)$ and $K > 0$ in Problem 3.

The poles of the closed-loop system can be determined by calculating the roots given by the characteristics equation $s^2 + (1 + 10K)s + 10 = 0$, which are given as

$$p_{1,2} = \frac{-(1 + 10K) \pm \sqrt{(1 + 10K)^2 - 40}}{2}$$

These poles are real valued when $(1 + 10K)^2 - 40 \geq 0$, which is true for $K \geq 0.53$. For $K = 0.53$ we get that $p_{1,2} = -3.16$. Next we need to find K for which $\zeta = 0.5$, which implies $\sin\theta = 0.5$ and $\theta = 30°$. For this we draw a line on the Root Locus plot for $\theta = 30°$ and we look at the intersection of the Root Locus with this line. To satisfy the specification, the poles of the closed-loop system are given by the intersection points. Take the approximate value of the complex pair of poles from the Root Locus plot, either estimate by hand the values on the real and imaginary axes, either use the cursor in MATLAB. We get that the numerical values of the poles are: $-1.58 \pm 2.73j$. We can plug these values into the characteristic equation to compute the value for K by using the formula

$$K = \frac{1}{|L(s_0)|},$$

where s_0 is one of the complex poles. It results that $K = 0.216$, which can be verified with the MATLAB functions `rlocus(G)` and `rlocfind(G)`.

For the closed-loop system we get that $t_s = \frac{4.6}{\sigma} = 2.91$, $t_r = \frac{1.8}{\omega_n} = 0.57$ and $M_P = 15\%$.

3. **Root Locus:**

Part 1: Consider a variable K and the listed choices of $L(s)$. Determine (i.) their root loci on the real axis, (ii.) the centers of asymptotes α's, and (iii.) the corresponding asymptotes, (iv.) compute the departure and arrival angles for the poles and zeros respectively. Based on (i.) - (iv.), draw sketches of the loci. Verify your results using MATLAB.

(a) $L(s) = \dfrac{s+1}{s-1}$

(b) $L(s) = \dfrac{s+2}{s(s+1)(s+5)(s+10)}$

(c) $L(s) = \dfrac{s+2}{s(s-1)(s+6)^2}$

Solution:

(a) $L(s) = \dfrac{s+1}{s-1}$

i. First, identify the number of poles, n, and zeros, m, of the system. Here, there is $n = 1$ pole and $m = 1$ zero.

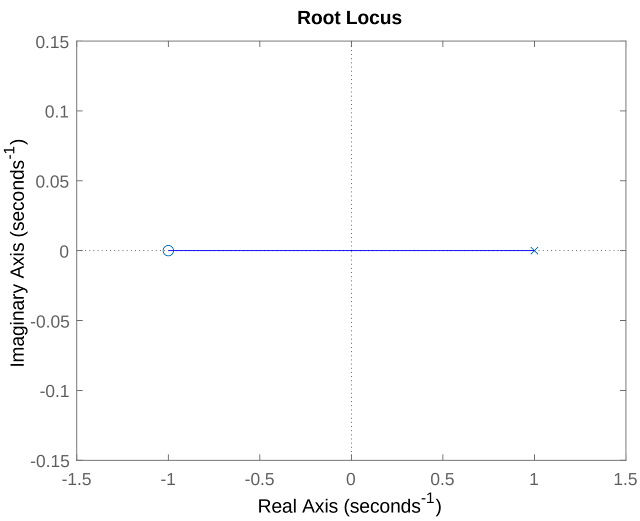


Figure 6: $L(s) = \frac{s+1}{s-1}$.

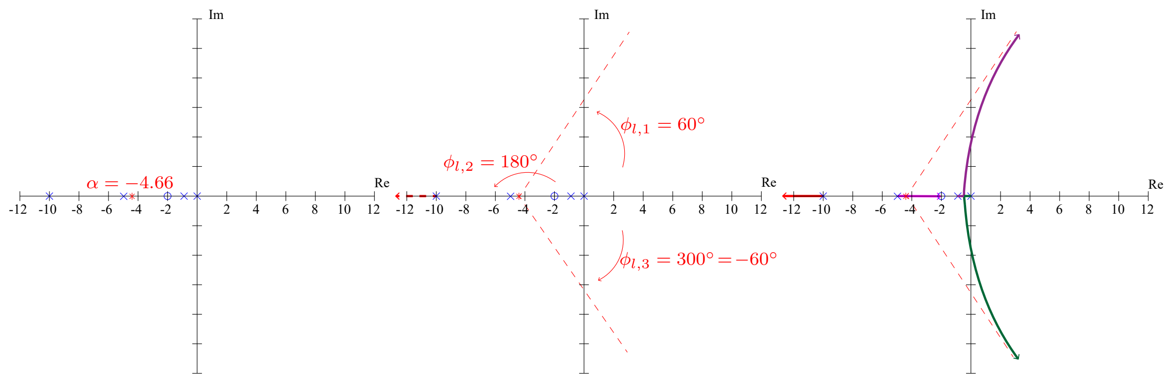
(b) $L(s) = \dfrac{s+2}{s(s+1)(s+5)(s+10)}$

i. First, identify the number of poles, n, and zeros, m, of the system. Here, there are $n = 4$ poles and $m = 1$ zero. See the blue drawing in Figure 7(a).

ii. Next, compute the center of the asymptotes, α, by using Rule 3 from the Franklin book.

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{(0-1-5-10)-(-2))}{3} = \frac{-14}{3} = -4.66$$

We have indicated the center in Figure 7(a) with a red asterisk.



(a) step i. & ii. (b) step iii. (c) step iv. & v.

Figure 7: Solution of Exercise 2(a) (with intermediate steps)

iii. Now, compute the angle of the lines with which the branches of the loci are asymptotic, e.g., ϕ_l, using Rule 3 from the book

$$\phi_l = \frac{180° + 360°(l-1)}{n-m} \qquad l = 1, 2..., n-m,$$

where m are the number of poles and n are the number of zeros. In this case, we have $l \in \{1, 2, 3\}$ and so if we plug these into the equation we can identify the following three angles

$$\phi_{l,1} = \frac{180}{3}|_{l=1} = 60° \qquad \phi_{l,2} = \frac{540}{3}|_{l=2} = 180° \qquad \phi_{l,3} = \frac{900}{3}|_{l=3} = 300°.$$

See Figure 7(b) to combine these elements.

iv. Compute the corresponding departure angles, $\phi_{dep,l}$, of the poles using Rule 4 from the book:

$$\phi_{dep,l} = \sum \psi_i - \sum \phi_i - 180° - 360°(l-1)$$

which leads to

$$\begin{aligned}
\phi_{dep,1} &= 0 - 0 - 180° = -180° \\
\phi_{dep,2} &= 0 - 180° - 180° = -360° \\
\phi_{dep,3} &= 180° - 3 \cdot 180° - 180° = 540°
\end{aligned}$$

v. Finally, use the information gathered above as well as the remaining rules to guide your plot for the root locus. Figure 7(c) shows how the information is combined to form a rough sketch of the root locus diagram. Using matlab, e.g., Figure 8, we can verify that our sketch is actually quite close to the produced plot!

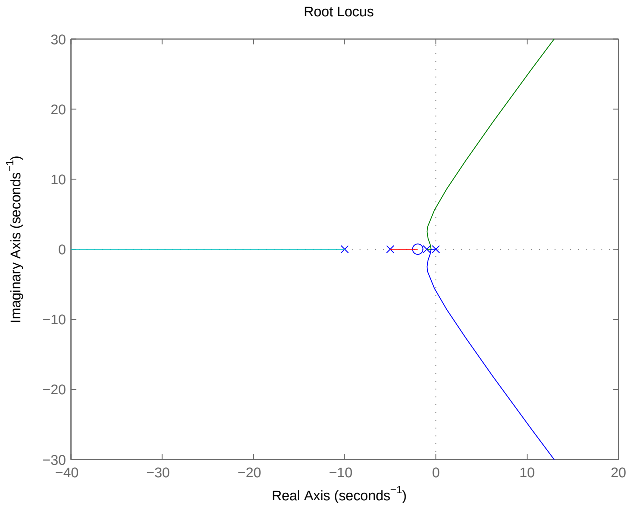
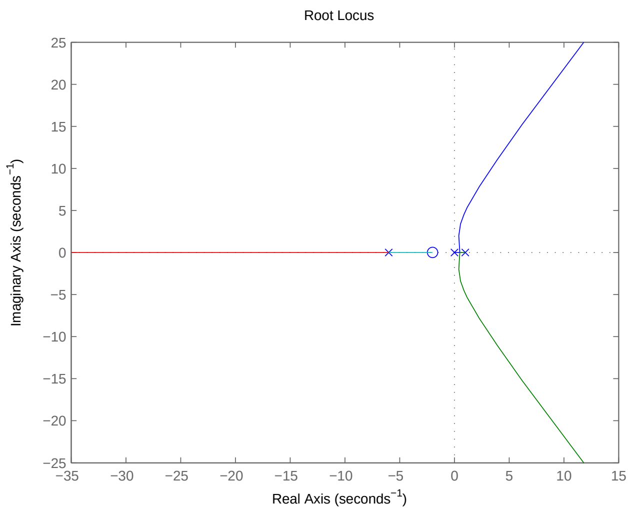


Figure 8: $L(s) = \frac{s+2}{s(s+1)(s+5)(s+10)}$, $\alpha = -4.67$.



(c)

Figure 9: $L(s) = \frac{s+2}{s(s-1)(s+6)^2}$, $\alpha = -3$.

Part 2 (Exam Level Question): Again consider the drawing rules for the root locus. Draw the root loci for each of the following listed $L(s)$. Verify your results using MATLAB. Compare the loci of L_1 and L_2 for each sub-question and explain how the loci of L_1 and L_2 differ despite their similar transfer function.

(a) $L_1(s) = \dfrac{(s+2)(s+20)}{s(s+1)(s+5)(s+10)}$ and $L_2(s) = \dfrac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$

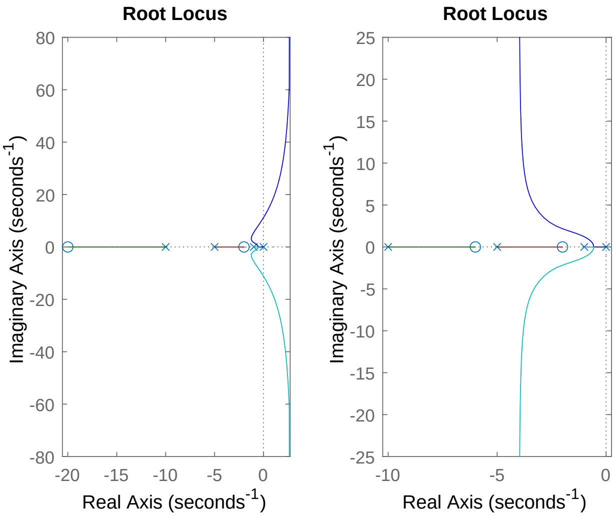
(b) $L_1(s) = \dfrac{1}{s^2(s^2+1)}$ and $L_2(s) = \dfrac{1}{s^2(s^2-1)}$

(c) $L_1(s) = \dfrac{s^2+1}{s(s^2+4)}$ and $L_2(s) = \dfrac{s^2+4}{s(s^2+1)}$

(d) $L_1(s) = \dfrac{s^2+4s+8}{s^4(s+8)}$ and $L_2(s) = \dfrac{s^2+4s+8}{s^3(s+8)}$

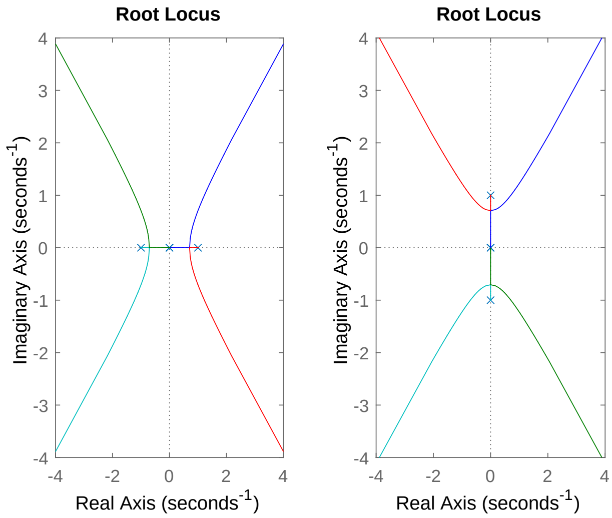
(e) $L_1(s) = \dfrac{s-3}{s^3(s+4)}$ and $L_2(s) = \dfrac{3-s}{s^3(s+4)}$.

Solution:



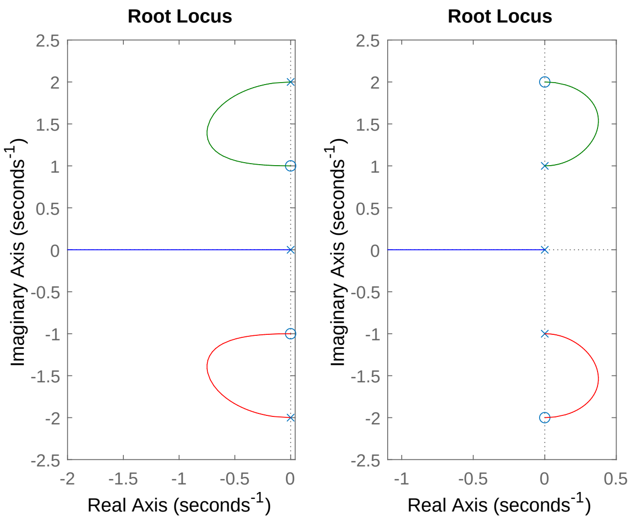
(a)

Figure 10: $L_1(s) = \dfrac{(s+2)(s+20)}{s(s+1)(s+5)(s+10)}$ and $L_2(s) = \dfrac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$



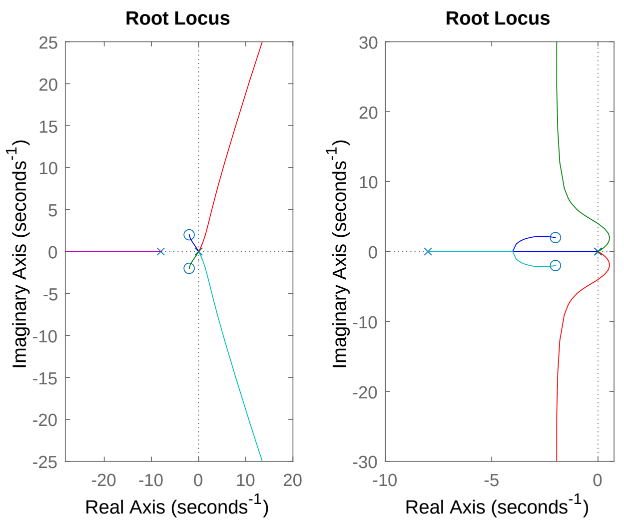
(b)

Figure 11: $L_2(s) = \dfrac{1}{s^2(s^2-1)}$ and $L_1(s) = \dfrac{1}{s^2(s^2+1)}$



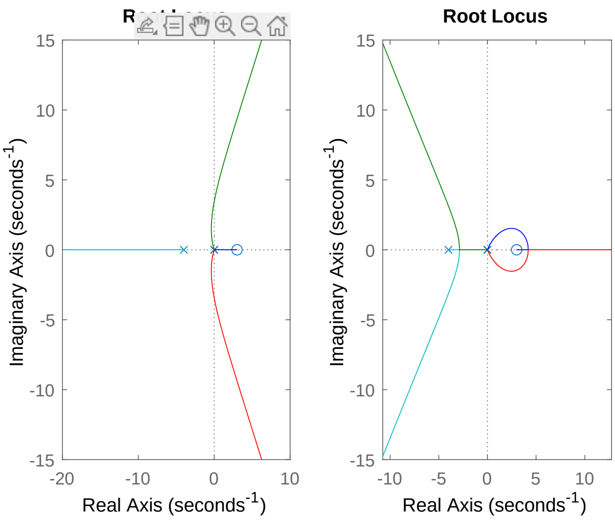
(c)

Figure 12: $L_1(s) = \dfrac{s^2+1}{s(s^2+4)}$ and $L_2(s) = \dfrac{s^2+4}{s(s^2+1)}$



(d)

Figure 13: $L_1(s) = \dfrac{s^2+4s+8}{s^4(s+8)}$ and $L_2(s) = \dfrac{s^2+4s+8}{s^3(s+8)}$

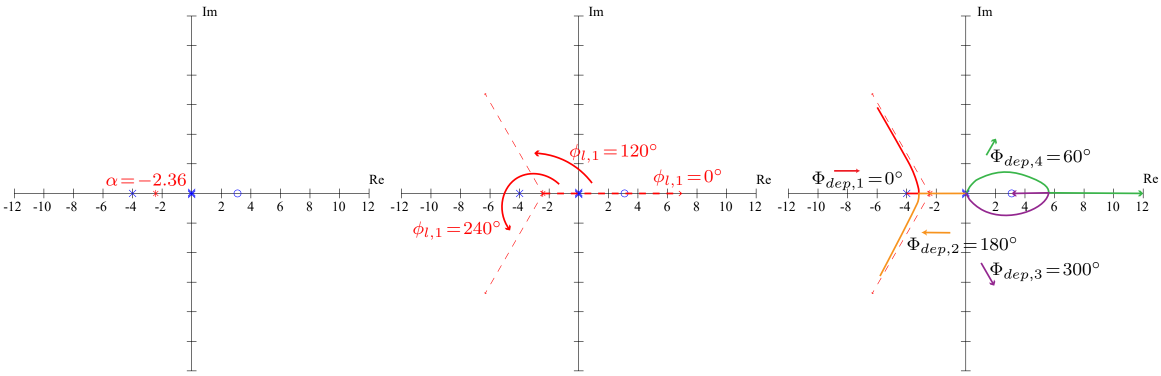


(e)

Figure 14: $L_1(s) = \dfrac{s-3}{s^3(s+4)}$ and $L_2(s) = \dfrac{3-s}{s^3(s+4)}$

Detailed solution for $L_2(s)$

$$L(s) = \frac{3-s}{s^3(s+4)}$$



(a) step i. & ii. (b) step iii. (c) step iv. & v.

Figure 15: Solution of Exercise 2(e) (with intermediate steps)

i. First note that in this case we have a zero in the right-half plane, e.g., a non-minimum phase system. As before, we identify the number of poles, n, and zeros, m, of the system. Here, there are $n = 4$ poles and $m = 1$ zero. See Figure 15(a). In order to indicate multiple poles at $s = 0$, multiple crosses are used.

ii. Next, compute the center of the asymptotes, $\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{(0+0+0-4)-(3))}{3} = \frac{-7}{3} \approx -2.36$

iii. Due to the RHP zero, we apply rules for a 'Negative root locus' (See Section 5.6.1 'Rules for Plotting Negative Root Locus' of the book). The angles of the asymptotes, ϕ_l, are computed using

$$\phi_l = \frac{360°(l-1)}{n-m} \qquad l = 1, 2..., n-m,$$

due to the system being non-minimum phase. In this case, we have $l \in \{1, 2, 3\}$, leading to the following angles

$$\phi_{l,1} = \frac{0}{3}|_{l=1} = 0° \qquad \phi_{l,2} = \frac{360}{3}|_{l=2} = 120° \qquad \phi_{l,3} = \frac{720}{3}|_{l=3} = 240°.$$

See Figure 15(b).

iv. For the singular pole at $p_1 = -4$, we use Rule 4 for the negative root locus

$$\phi_{dep,l} = \sum \psi_i - \sum \phi_i - 360°(l-1),$$

leading to $\phi_{dep,1} = 0° - 0° - 0° = 0°$. Given that we have $q = 3$ poles at the origin, $p_{2,3,4} = 0$, we use Rule 5 for the negative root locus

$$\phi_{dep,l} = \frac{180° + 360°(l-1)}{q}$$

which gives

$$\phi_{dep,2} = \frac{180° + 360°}{3}|_{l=2} = 180°$$
$$\phi_{dep,3} = \frac{180° + 720°}{3}|_{l=3} = 300°$$
$$\phi_{dep,4} = \frac{180° + 1028°}{3}|_{l=4} = 420°.$$

v. Finally, use the information gathered above as well as the remaining rules to draw the loci. Figure 15(c) shows how the information is combined to form a rough sketch of the root locus. Compared your sketch to the plot from matlab in Figure 16.

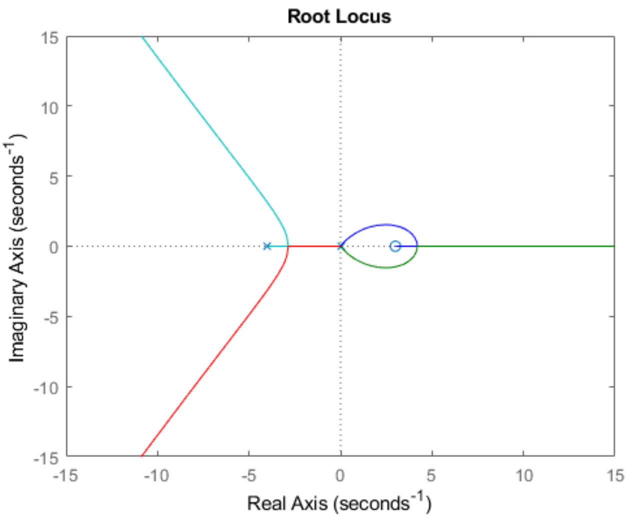


Figure 16: $L(s) = \frac{3-s}{s^3(s+4)}$, $\alpha = -2.33$.