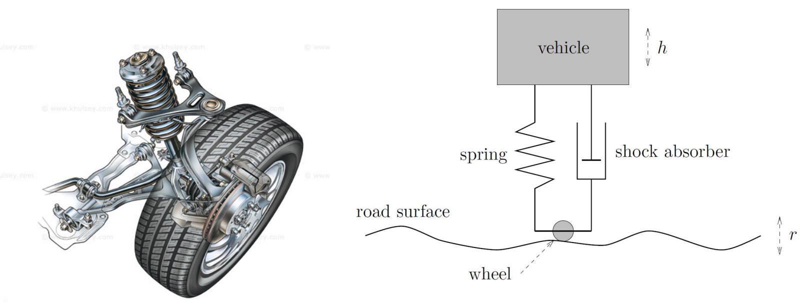
1. **Vehicle suspension system:**
A vehicle suspension is modeled as a spring and a shock absorber between the vehicle and the wheel, which is assumed to remain in contact with the road. The position of the vehicle at a given time $t$, is denoted by $h(t)$; while the height of the road surface at the wheel at time $t$ is denoted by $r(t)$. This is illustrated below.




(a) (b)

Figure 1: Suspension system.

From the Newton's second law of motion, we obtain the following ordinary differential equation (ODE):

$$m\,\ddot{h}(t) = -k\,(h(t) - r(t)) - b\left(\dot{h}(t) - \dot{r}(t)\right), \tag{1}$$

where $m \in \mathbb{R}$ is the vehicle mass, $k \in \mathbb{R}$ is the spring stiffness, and $b \in \mathbb{R}$ is the mechanical resistance of the shock absorber. For this exercise, these parameters assume the values:

$$m = 1,\ k = 2,\ b = 3.$$

Address the following questions:

(a) Find the transfer function $G(s)$ which maps the road height $r(s)$ to the vehicle height $h(s)$. Consider applying a corresponding Laplace transform with the initial conditions $h(0) = 0$, $\dot{h}(0) = 0$ and $r(0) = 0$.

(b) Compute the poles and zeros of the obtained transfer function $G(s)$ and explain whether you think the system is stable or not. Verify using MATLAB (Hint: the basic commands are `tf`, `pole`, `zero`).

(c) Compute the autonomous response of the system (i.e. the homogeneous solution of the differential equation when $r(t) = 0$). Consider using the Laplace transform method with the initial condition $h(0) = 1$, $\dot{h}(0) = 0$.
(Hint: see Sections 3.1.3 and 3.1.4 of the book)

(d) Compute the forced response of the system for the reference signal $r(t) = 2e^{-2t}\mathbb{1}(t)$, with zero initial conditions and where $\mathbb{1}(t)$ is defined as

$$\mathbb{1}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Verify your result with the MATLAB command `lsim`. Also try to replicate this result in MATLAB/SIMULINK (Hint: some helpful blocks are `Transfer Fcn`, `clock`, `exp`, `gain`, `scope`).

**Solution:**

(a) The differential equation is equivalent to

$$m\ddot{h}(t) + b\dot{h}(t) + kh(t) = b\dot{r}(t) + kr(t).$$

By taking the Laplace transform, the transfer function from $r$ to $h$, with $\mathcal{L}\{r(t)\} = R(s)$ and $\mathcal{L}\{h(t)\} = H(s)$, reads as

$$G(s) = \frac{H(s)}{R(s)} = \frac{bs + k}{ms^2 + bs + k} = \frac{3s + 2}{s^2 + 3s + 2}.$$

(b) The poles of the system are given by the values of $s$ that render the denominator to zero, e.g.

$$s^2 + 3s + 2 = 0.$$

Hence, this system has two poles, namely $s = -1$ and $s = -2$. This can be verified using the MATLAB commands

```
num = [3 2];
den = [1 3 2];
G = tf(num,den);
p = pole(G)
```

The zeros of the system are given by the values of $s$ that render then numerator to zero. Hence, this system has one zero, namely $s = -\frac{2}{3}$.This can be verified using the MATLAB command

```
z = zero(G)
```

Since all the poles are placed in the Left Half of the complex Plane (LHP) we can conclude that the system is stable.

(c) The homogeneous differential equation is:

$$\ddot{h}(t) + 3\dot{h}(t) + 2h(t) = 0.$$

By applying the Laplace transform and using the initial conditions we obtain

$$s^2H(s) - s + 3sH(s) - 3 + 2H(s) = 0 \Leftrightarrow H(s) = \frac{s+3}{s^2+3s+2}.$$

By using partial fraction decomposition, we have that

$$H(s) = \frac{-1}{s+2} + \frac{2}{s+1}.$$

Therefore the corresponding function in the time domain is

$$h(t) = \left(-e^{-2t} + 2e^{-t}\right)\mathbb{1}(t).$$

(d) We have that

$$G(s) = \frac{3s+2}{s^2+3s+2},\ R(s) = \frac{2}{s+2}.$$

Therefore,

$$H(s) = G(s)R(s) = \frac{6s+4}{(s+2)^2(s+1)} = \frac{2}{s+2} + \frac{8}{(s+2)^2} + \frac{-2}{s+1},$$

and the resulting function in the time domain is:

$$h(t) = \left(2e^{-2t} + 8te^{-2t} - 2e^{-t}\right)\mathbb{1}(t).$$

MATLAB implementation:

```
t = 0:0.01:5;
r = 2*exp(-2*t).*ones(1,length(t));
figure(2); hold on;
plot(t, r, 'b');

h_forced1 = lsim(G, r, t);
figure(2);
plot(t, h_forced1, 'r' );

h_forced2 = (2*exp(-2*t)-2*exp(-t)+8*t.*exp(-2*t)).*
ones(1,length(t));
figure(2);
plot(t, h_forced2, 'k');
```

An example of the MATLAB/SIMULINK implementation is shown in Figure 2.

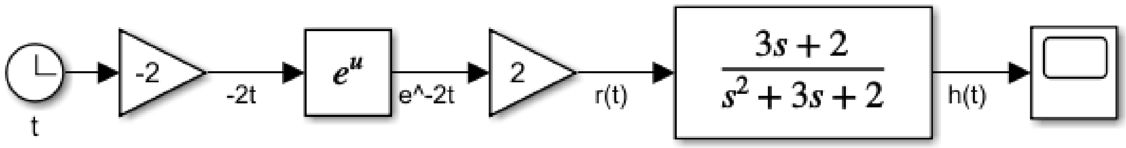



Figure 2: Simulink implementation of exercise 1-d

2. **Second order dynamics:**
Consider the transfer function of a second-order system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\omega_n \zeta s + \omega_n^2}.$$

Sketch the region in the $s$-plane where the poles can be placed so that the system would meet the following specifications:

$$\begin{aligned} t_r &\leq 0.6\,s\,, \\ M_p &\leq 17\%\,, \\ t_s &\leq 9.2\,s. \end{aligned}$$

**Solution:**
First we write the specifications in terms of the $s$-plane characteristics, $\omega_n$, $\zeta$ (see relations in the Franklin book Edition 7 - Chapter 3.4):

$$\omega_n \geq \frac{1.8}{t_r}$$

$$\zeta \geq 0.5,$$

from Figure 3.24 in the Franklin book, and

$$\sigma \geq \frac{4.6}{t_s}.$$

We get that $\omega_n \geq \frac{1.8}{0.6}$ and that $\omega_n \geq 3$, therefore the poles should lie outside the circle of radius 3, centered in the origin.

By $\zeta = \sin(\theta) \geq 0.5$, we have that $\theta \geq 30°$ and $\theta \leq 150°$. Thus, the poles should lie inside the cone centered in 0, whose ray makes an angle of 30° with the imaginary axis.

From $\sigma \geq \frac{4.6}{9.2}$ we have that $\sigma \geq 0.5$, which implies that the poles should lie in the left of the vertical line which passes through the point $(-0.5, 0)$. The resulting region where the poles could be placed is given by the intersection of the three regions from above. See Figure 3.

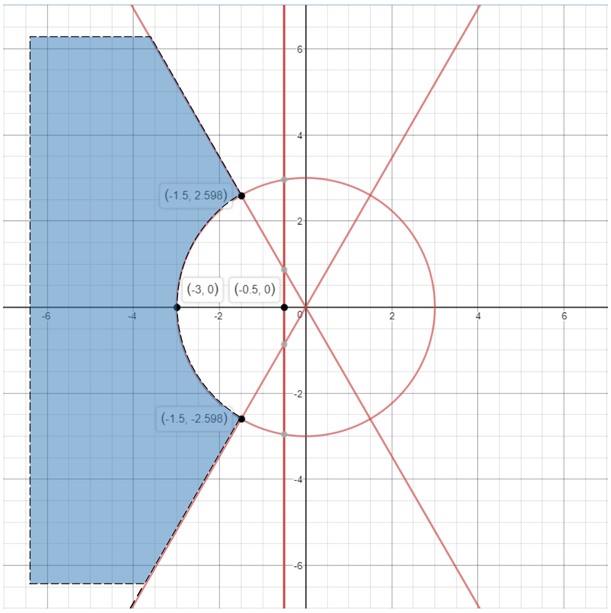



Figure 3: $s$-plane feasible region.

3. **Low Pass Filter**
Consider the following circuit, which is a low-pass filter containing an inductor, $L$, and a resistor, $R$ and a time-varying current $i(t)$.

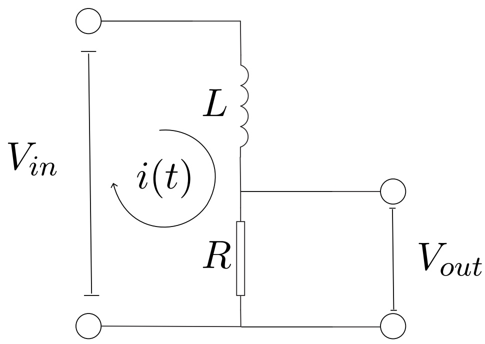



Figure 4: Low Pass Filter Circuit

(a) Assuming zero initial conditions, find a transfer function $H(s)$ which maps the input voltage $V_{in}$ to the output voltage $V_{out}$. (Hint: Use the voltage divider formula to get a differential equation in terms of $i(t)$ and $\frac{di(t)}{dt}$)

(b) Is the system strictly proper, proper or improper?

(c) Can we make any claims about the stability of this system?

(d) Given that $R = 3\ \Omega$ and $L = 10\ H$, what are the poles and zeros of the system?

(e) Assuming zero initial conditions, compute the impulse response of the system.

**Solution:**

(a) From the voltage divider formula, we can deduce the following time domain transfer

$$H(t) = \frac{V_{out}(t)}{V_{in}(t)} = \frac{Ri(t)}{Ri(t) + L\frac{di(t)}{dt}} \tag{2}$$

Applying the LaPlace transform on the equation above yields

$$\mathcal{L}(H(t)) = H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{RI(s)}{RI(s) + LsI(s)} = \frac{I(s)}{I(s)}\frac{R}{R + Ls} = \frac{R}{sL + R} \tag{3}$$

(b) To find out whether the system is (strictly) proper, we need to look at the degree of the polynomial in the numerator and denominator. Since the degree of the denominator is one and the numerator polynomial has degree zero, the system is strictly proper, i.e., the denominator polynomial is of higher degree than the numerator polynomial.

(c) The poles of the system are given by calculating the roots of the denominator of the transfer function. In this case, there is only one pole defined at $s = -\frac{R}{L}$. Since both $L$ and $R$ are positive real values, the poles of this system will be in the left half plane (LHP). Hence the system is stable.

(d) From before, there are no Zeros and the only pole is defined by

$$p_1 = -\frac{R}{L} = -\frac{3}{10} = -0.3 \tag{4}$$

(e) Given the transfer function

$$H(s) = \frac{R}{sL + R} \tag{5}$$

substituting the values for $L$ and $R$, we get

$$H(s) = \frac{3}{10s + 3} \tag{6}$$

In order to translate this function back into the time domain, we can re-arrange it to better resemble a common form seen in a table of LaPlace transforms:

$$H(s) = 3\,\frac{1}{10s + 3} = \frac{3}{10}\,\frac{1}{s + \frac{3}{10}} \tag{7}$$

which can then be transformed into the time domain using the inverse Laplace function

$$F(s) = \frac{1}{s - a} \quad \rightarrow \quad \mathcal{L}^{-1}(F(s)) = f(t) = e^{at}, \tag{8}$$

resulting in the autonomous response

$$h(t) = \frac{3}{10}e^{-0.3t}. \tag{9}$$

4. **Closed-loop transfer functions:**
A DC motor speed control scheme is shown in Figure 5, where $Y$ represents the motor speed, $V_{\mathrm{a}}$ the armature voltage and $W$ is the load torque (disturbance). Assume that the armature voltage is computed using a PI control law, that is, $D(s) = k_{\mathrm{P}} + k_{\mathrm{I}}\frac{1}{s}$.

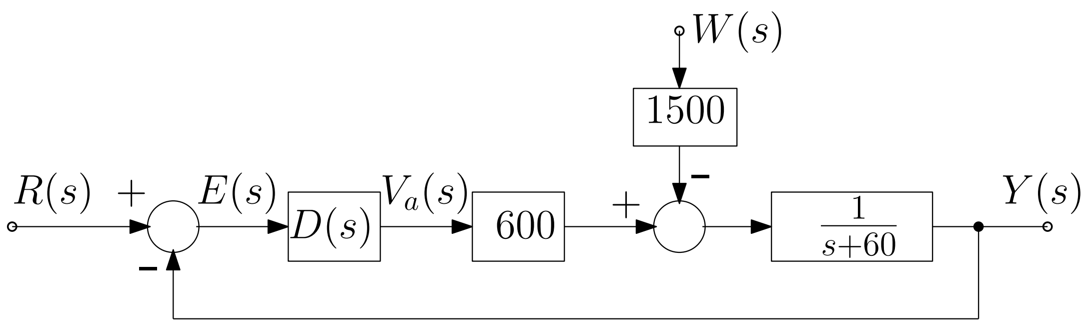



Figure 5: DC motor speed control scheme.

(a) Compute the transfer function from $R$ to $Y$ (in terms of $k_{\mathrm{P}}$, $k_{\mathrm{I}}$, and $s$) when $W = 0$.

(b) Compute numerical values for $k_\mathrm{P}$ and $k_\mathrm{I}$ such that the closed-loop system has a pair of complex conjugate poles at $-60 \pm 60j$.

(c) Compute the system type with respect to disturbance rejection.
(*Hint:* Consider the transfer function from input $W(s)$ to output $E(s)$ (see Franklin book Edition 7 section 4.2.2)).

(d) Suppose that the amplification of the armature voltage $V_\mathrm{a}$ is reduced by half (i.e., from 600 to 300). What will be the effect on the *steady-state* errors for the cases of (i.) a unit-step input disturbance and (ii.) a ramp input disturbance? Justify your answers by calculating the errors.

**Solution:**

(a) If $W(s) = 0$, then we can write

$$Y(s) = \frac{600}{s+60} D(s) E(s)$$

where $D(s) = k_\mathrm{P} + k_\mathrm{I}\frac{1}{s}$ and $E(s) = R(s) - Y(s)$. Hence

$$\begin{aligned}
Y(s) &= \frac{600}{s+60}\left(k_\mathrm{P} + k_\mathrm{I}\frac{1}{s}\right) R(s) - \frac{600}{s+60}\left(k_\mathrm{P} + k_\mathrm{I}\frac{1}{s}\right) Y(s) \\
&= \frac{\frac{600}{s+60}\left(k_\mathrm{P} + k_\mathrm{I}\frac{1}{s}\right)}{1 + \frac{600}{s+60}\left(k_\mathrm{P} + k_\mathrm{I}\frac{1}{s}\right)} R(s) \\
&= \frac{600k_\mathrm{P} + 600k_\mathrm{I}\frac{1}{s}}{s + 60 + 600k_\mathrm{P} + 600k_\mathrm{I}\frac{1}{s}} R(s) \\
&= \frac{600k_\mathrm{P}s + 600k_\mathrm{I}}{s^2 + (60 + 600k_\mathrm{P})s + 600k_\mathrm{I}} R(s)
\end{aligned}$$

(b) Since the roots are $\lambda_1 = -60 + 60j$ and $\lambda_2 = -60 - 60j$, then

$$\begin{aligned}
s^2 + (60 + 600k_\mathrm{P})s + 600k_\mathrm{I} &= (s - \lambda_1)(s - \lambda_2) = (s + 60 - 60j)(s + 60 + 60j) \\
&= (s+60)^2 + 60^2 = s^2 + 120s + 2 \cdot 60^2,
\end{aligned}$$

hence $k_\mathrm{P} = 0.1$ and $k_\mathrm{I} = 12$.

(c) For disturbance rejection we look at the transfer from disturbance $W(s)$ to tracking error $E(s)$. Following the same procedure as for exercise a) but this time with $R(s) = 0$ we get

$$\begin{aligned}
E(s) &= -Y(s) \\
&= -\frac{600}{s+60} D(s)E(s) + \frac{1500}{s+60} W(s) \\
&= \frac{\frac{1500}{s+60}}{1 + \frac{600}{s+60} D(s)} W(s) \\
&= \frac{1500}{s + 60 + 600D(s)} W(s) \\
&= \frac{1500}{s + 60 + 600\left(k_\mathrm{P} + k_\mathrm{I}\frac{1}{s}\right)} W(s) \\
&= \frac{1500s}{s^2 + (60 + 600k_\mathrm{P})s + 600k_\mathrm{I}} W(s)
\end{aligned}$$

To assess the system type with respect to disturbance rejection, we use the Final Value Theorem (FVT) (see Chapter 4.2 in the Franklin book, Edition 7):

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1500s^2}{s^2 + (60 + 600k_\mathrm{P})s + 600k_\mathrm{I}} W(s).$$

We take $W(s) = \frac{1}{s^{k+1}}$ and therefore

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{1500s^2}{s^2 + (60 + 600k_{\mathrm{P}})s + 600k_{\mathrm{I}}} \frac{1}{s^{k+1}} \\ &= \lim_{s\to 0} \frac{1500}{s^2 + (60 + 600k_{\mathrm{P}})s + 600k_{\mathrm{I}}} \frac{1}{s^{k-1}}.\end{aligned}$$

Note that for $k = 1$, $\lim_{t\to\infty} e(t) = \frac{5}{2k_{\mathrm{I}}}$ (constant), which gives type-1 system with respect to disturbance rejection.

(d) Note that changing the amplification of $V_{\mathrm{a}}$ does not change the system type, but only the error amplitude. Indeed, the transfer function now becomes

$$\frac{E(s)}{W(s)} = \frac{1500s}{s^2 + (60 + 300k_{\mathrm{P}})s + 300k_{\mathrm{I}}},$$

and if we go through the FVT again, we notice that the system type remains the same, that is, type 1. For a type-1 system, a step disturbance will be completely rejected at steady state, and therefore $e_{ss} = 0$ when $w(t) = \mathbb{1}(t)$.

If $w(t)$ is a ramp, i.e., $w(t) = t$, then $W(s) = \frac{1}{s^2}$, and therefore

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s \frac{1500s}{s^2 + (60 + 300k_{\mathrm{P}})s + 300k_{\mathrm{I}}} \frac{1}{s^2} = \frac{1500}{300k_{\mathrm{I}}} = \frac{5}{k_{\mathrm{I}}}.$$

This means that the error doubles in comparison to the case when the amplification of $V_{\mathrm{a}}$ was twice higher.

In summary, the effects are: none for step disturbance; the error is doubled for ramp disturbance.