Introduction

Ch 1 Introduction

Womp.... Womp.

Ch 2 Analog vs Digital communication

- Fourier Transform: $X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi f t)\, dt$ forward

  $x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi f t)\, df$ inverse.

linearity: $F\{a x_1(t) + b x_2(t)\} = aX_1(f) + bX_2(f)$

Convolution: $F\{x_1(t) * x_2(t)\} = X_1(f) \cdot X_2(f)$.

Real signals: $X(f) = X^*(-f)$ if $x(t) \in \mathbb{R}$

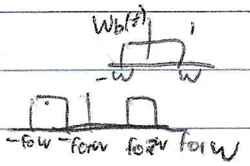
Linear-Time Invariant (LTI) Systems.

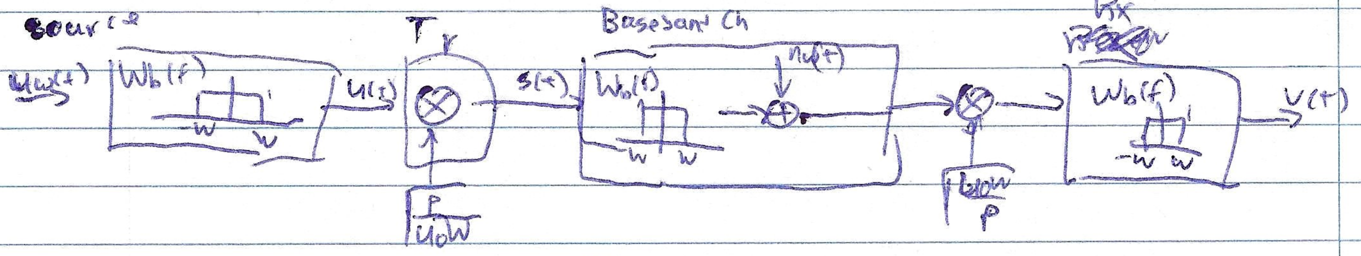
impulse response $h(t)$: $y(t) = x(t) * h(t)$; Freq domain: $Y(f) = X(f) \cdot H(f)$.

- Parsevals Theorem: Energy: $\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$.
- $\mathcal{M} \rightarrow$ set. ; Bayes Rule: $Pr\{Y=y \mid X=x\} = \dfrac{Pr\{X=x \mid Y=y\}\, Pr\{Y=y\}}{Pr\{X=x\}}$

- PMF: $Pr\{X=x\}$; Law Total Pr: $Pr\{Y=y\} = \sum_x Pr\{X=x, Y=y\}$
- PDF: $\int_{r \in \mathbb{R}} p_R(r)\, dr = 1$

- White Gaussian Noise (WGN)
  - zero mean: $E[N_w(t)] = 0$
  - Power spectral density (PSD): $S_{N_w}(f) = N_0/2$
  - Autocorrelations: $R_{N_w}(t,s) = N_0/2\, \delta(t-s)$
- Wideband AWGN: $r(t) = s(t) + n_w(t)$
  - Baseband AWGN: $r(t) = s(t) * W_b(t) + n_w(t)$
  - Passband AWGN: $r(t) = s(t) * W_p(t) + n_w(t)$





Power distribution $d(t) \triangleq v(t) - u(t) = \sqrt{N_0W/P}\, n_w(t) * W_b(t)$

Variance distortion $E[D^2(t)] = \frac{N_0W}{P} \frac{N_0}{2} 2W = N_0W \frac{N_0W}{P}$

Mean Square Distortion: $D = E[D^2(t)]$

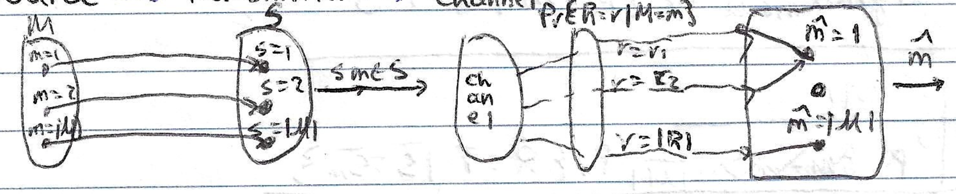
$SDR \triangleq \dfrac{E[u^2(t)]}{E[D^2(t)]} = P/N_0W$ for DSB-SC

Channel Processing Receivers.

Ch 3 - Decision Rules for DIDO Channels.

$Pr\{M=m\}$; $Pr\{R=r|S=s\}$; estimate $\rightarrow \hat{m}=f(r)\in\mathcal{M}$

Source $\xrightarrow{m}$ Transmitter $\xrightarrow{s_m\in S}$ DIDO Channel $\xrightarrow{r\in R}$ Receiver $\xrightarrow{\hat{m}}$ Dest.



decision rule

$P_e \triangleq Pr\{\hat{M}\neq M\}$ → optimal receiver minimizes → $P_c \triangleq Pr\{\hat{M}=M\}=1-P_e$ → $f(r)$

$P_c = Pr\{M=\hat{M}\}$ → $\hat{M}=f(R)$ → $P_c=\sum_{m\in\mathcal{M}}\sum_{r\in R:f(r)=m} Pr\{M=m, R=r\}$.

$= \cdots Pr\{R=r|M=m\}\,Pr\{M=m\} = Pr\{M=m, R=r\}$

ML

Example Tx signals: $s\in S=\{s_1,s_2\}$ ($|M|=2$); Rx signals: $r\in R=\{a,b,c\}$

$P\{M=m\}$ • A-priori: $P\{M=1\}=0.4$, $P\{M=2\}=0.6$

$s_1=1,\ s_2=2$ • Condional: $P\{R=a|S=s_1\}=0.5$; $P\{R=b|S=s_1\}=0.4$; ~~even~~ $Pr\{R=c|S=s_1\}=0.1$

$Pr\{R=a|S=s_2\}=0.1$; $Pr\{R=b|S=s_2\}=0.3$; $Pr\{R=c|S=s_2\}=0.6$.

decision probability • Decision: $r\in\{a,b\}\rightarrow m=1$ ($f(r)=1$); $r\in\{c\}\rightarrow m=2$ ($f(r)=2$).

decision variables • Joint ~~prob~~ (PMF): $P(M=m, R=r)=P(M=m)\cdot P(R=r|S=s_m$ or $M=m)$

m	$Pr\{M=m,R=a\}$	$Pr\{M=m,R=b\}$	$Pr\{M=m,R=c\}$
1	0.2	0.16	0.04
2	0.06	0.18	0.36

$P_c=0.2+0.16+0.36=0.72$; $P_e=0.28$

• Max Likelihood detection: $P_c=0.2+0.18+0.36=0.74$. $0.74>0.72$.

MAP

$$\boxed{\hat{m}^{MAP}(r)\triangleq \underset{m\in\mathcal{M}}{\arg\max}\ Pr\{M=m|R=r\} \rightarrow \max Pr}$$

PMF probability MASS Func

A-posteriori Probability: $P\{M=m|R=r\}=\dfrac{P\{M=m,R=r\}=P\{M=m\}\cdot P\{R=r|S=s_m\}}{P\{R=r\}}$.

Bayes' Rule → Pr of M transmitted based on received R.

$Pr\{R=a\}=0.2+0.06=0.26$ → $Pr\{M=1,R=a\}+Pr\{M=2,R=a\}$

$Pr\{R=b\}=0.34$; $Pr\{R=c\}=0.4$

$Pr\{M=1,R=a\}/Pr\{R=a\}=0.2/0.26=0.77=20/26$ ...

m	$Pr\{M=m\|R=a\}$	$Pr\{M=m\|R=b\}$	$Pr\{M=m\|R=c\}$
1	20/26	16/34 ~~4/10~~	4/40
2	6/26	18/34 ~~36/40~~	36/40

(MAP) → $P_c=20/26+\cancel{18/34}+\cancel{36/40}$

A-Priori: $Pr\{M=m\}$; A-Posteriori: $Pr\{M=m|R=r\}$

• uniform a-priori: $Pr\{M=m\}=1/|M|$ → $Pr\{M=m,R=r\}=\frac{1}{|M|}Pr\{R=r|S=s_m\}$

$$\boxed{\hat{m}^{ML}(r)\triangleq \underset{m\in\mathcal{M}}{\arg\max}\ Pr\{R=r|M=m\} \rightarrow \max Pr}$$

ML If ~~R~~ ~~MA~~ $Pr\{M=m\}=1/2$ → equal $P_c^{ML}=1/2\underbrace{(0.5+0.4)}_{\text{conditional.}}+1/2(0.6)=0.75$

MAP vs ML summary

Decision	MAP
Variable	$Pr\{M=m\} Pr\{\bar{R}=\bar{r} \mid \bar{S}=\bar{s}_m\}$
Rule	$\text{argmax}_{m \in \mathcal{M}} Pr\{M=m \mid \bar{R}=\bar{r}\}$

Decision	ML
Variable	$Pr\{\cancel{M=m}\}\ \frac{1}{\lvert M \rvert} Pr\{\bar{R}=\bar{r} \mid \bar{S}=\bar{s}_m\}$
Rule	$\text{argmax}_{m \in \mathcal{M}} Pr\{\bar{R}=\bar{r} \mid M=m\}$

ML

1. A-Priori:

m	$Pr\{M=m\}$
1	0.4
2	0.6

2. conditional Pr

m	$Pr\{R=a \mid S=s_m\}$	$Pr\{R=b \mid S=s_m\}$	$Pr\{R=c \mid S=s_m\}$
1	0.5	0.4	0.1
2	0.1	0.3	0.6

3. $f(r)=1 \quad r \in \{a,b\} \qquad f(r)=2 \quad r \in \{c\}$

4. Joint Pr

m	$Pr\{M=m, R=a\}$	$Pr\{M=m, R=b\}$	$Pr\{M=m, R=c\}$
1	0.2	0.16	0.04
2	0.6	0.18	0.36

5. $P_c = 0.72$ //

MAP → have to know output → harder to implement even though better.

1. $Pr\{R=a\} = Pr\{M=1, R=a\} + Pr\{M=2, R=a\} = 0.26$.
   $Pr\{R=b\} = 0.34$; $Pr\{R=c\} = 0.4$.
2. A-posteriori Pr
   Bayes eq: $Pr\{M=m \mid \bar{R}=\bar{r}\} = \dfrac{Pr\{M=m, \bar{R}=\bar{r}\}}{Pr\{\bar{R}=\bar{r}\}}$
   $Pr\{M=1 \mid \bar{R}=a\} = 0.2/0.26$

m	$Pr\{M=m \mid R=a\}$	$Pr\{M=m \mid R=b\}$	$Pr\{M=m \mid R=c\}$
1	20/26	16/34	4/40
2	6/26	18/34	36/40

3. $P_c = 0.2 + 0.18 + 0.36 = 0.74$ //

Equal A-Priori → ML

1. $Pr\{M=m\} = 0.5$ 2. $f(r)=1 \quad r \in \{a,b\} \qquad f(r)=2 \quad r \in \{c\}$
3. $P_c = \frac{1}{2}(0.5+0.4) + \frac{1}{2}(0.6) = 0.75$

Channel Processing Receivers

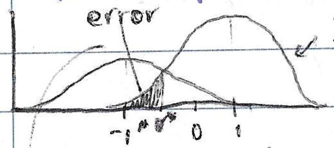
$\Pr\{M=m \mid \vec{R}=\vec{r}\} = \frac{\Pr\{M=m\} p_{\vec{R}}(\vec{r} \mid M=m)}{p_{\vec{R}}(\vec{r})} \to 1$

Ch 4 Decision Rules for DICO Channels

- Decision Variables DICO: $P(M=m, R=r) = P(M=m) \cdot p_R(r \mid S=s_m)$
  $= P(M=m) \cdot p_R(r \mid M=m)$
- Max A-Posteriori (MAP) Decision Rule: $\hat{m}_{MAP}(r) = \underset{m \in M}{\arg\max} P\{M=m \mid R=r\}$
  Bayes rule: $= \underset{m \in M}{\arg\max} P(M=m) \cdot p_R\{r \mid M=m\}$
- Max Likelyhood (ML) Decision Rule: $\hat{m}_{ML} = \underset{m \in M}{\arg\max} p_R\{r \mid M=m\} \to$ all equal a-priori
- Scalar Additive Noise Channel AGN: $r = s + n$

Gaus distribution $p_N = 1/\sqrt{2\pi\sigma^2} \exp\left(-\frac{n^2}{2\sigma^2}\right) = \frac{r}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r-s_m)^2}{2\sigma^2}\right)$

- Conditional Probability Density AGN Channel: $p_R(r \mid S=s_m) = 1/\sqrt{2\pi\sigma^2} \exp\left(-\frac{(r-s_m)^2}{2\sigma^2}\right)$
- MAP rule for Binary-input AGN (bi-AGN) channel: $M \in \{1,2\}$
  $\hat{m} = 1$ if: $P(M=1) \cdot p_R(r \mid S=s_1) \geq P(M=2) p_R(r \mid S=s_2)$
- MAP Threshold: $r^* = \sigma^2/(s_1-s_2) \ln\left(\frac{P(M=2)}{P(M=1)}\right) + \frac{s_1+s_2}{2}$ $\quad \hat{m} = \begin{cases} 1 & r \geq r^* \\ 2 & \text{else.} \end{cases}$
- ML Thr, equiprobable: $P(M=1) = P(M=2) = 0.5$; $r^* = \frac{s_1+s_2}{2}$
- MAP Example: $\Pr\{M=1\} = 3/4$, $s_1 = +1$, $\Pr\{M=2\} = 1/4$, $s_2 = -1 \to r^* = -\frac{\ln(3)}{2} = -0.549$



$\frac{3/4}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r-1)^2}{2\sigma^2}\right)$

$\frac{1/4}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r+1)^2}{2\sigma^2}\right)$

- Similar for ML but both $\frac{1}{\sqrt{2\pi\sigma^2}}$ even if $\Pr\{M=1\} = \Pr\{M=2\} = 1/2$

- If $r^* = 0 \to P_e = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(r-1)^2}{2\sigma^2}\right) dr.$

- Q-function: tail (Pr) of Gaussian Random: $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{a^2}{2}\right) da$
  $Q(x) = 1 - Q(-x)$ and $Q(x) + Q(-x) = 1$
- Error (Pr) for bi-AGN Channel: $P_e = P(M=1)\, \underbrace{Q\left(\frac{s_1 - r^*}{\sigma}\right)}_{\Pr\{R < r^* \mid M=1\}} + P(M=2)\, \underbrace{Q\left(\frac{r^* - s_2}{\sigma}\right)}_{\Pr\{R > r^* \mid M=2\}}$
  - if Equally likely: $P(M=1) = P(M=2) = 0.5 \to \boxed{P_e = Q\left(\frac{s_1 - s_2}{2\sigma}\right)}$
- Decision Region: $I_m = \{r \in \mathbb{R}^N : f(r) = m\}$ $\quad \mathbb{R}^N \to$ observation space
- Example MAP vs ML bi-AGN
  $s_1 = 1$, $s_2 = -1$, $\sigma^2 = 1$, $\Pr\{M=1\} = 3/4$, $\Pr\{M=2\} = 1/4$
  $P_e = 3/4\, Q\left(1 + \frac{\ln 3}{2}\right) + 1/4\, Q\left(-\frac{\ln 3}{2} + 1\right) = 0.1270$ // MAP
  - ML: $Q\left(\frac{2}{2}\right) = 0.1587$ //

- PDF of Noise Vector: $p_N(n) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\|n\|^2}{2\sigma^2}\right)$
- PDF but vector: $\vec{r} = \vec{s}_m + \vec{n} \to p_R(\vec{r} \mid \vec{S} = \vec{s}_m) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\|\vec{r} - \vec{s}_m\|^2}{2\sigma^2}\right)$

!! decision rule based on distance from hyper plane so if $s_1 = A$, $s_2 = -A \to$ its same //

squared Euclidean distance only optimal if a-priori msgs are equiprobable. $\|\bar{r}-\bar{s}_m\|^2$ $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

- Decision Rules for AGN Vector Channels:
  - MAP: $\hat{m}_{MAP}(\bar{r}) = \arg\min_{m\in M}\{\|\bar{r}-\bar{s}_m\|^2 - 2\sigma^2 \ln \Pr\{M=m\}\}$
  - ML: $\hat{m}_{ML}(\bar{r}) = \arg\min_{m\in M}\{\|\bar{r}-\bar{s}_m\|^2\}$
- Error (Pr) in AGN vector channel: $P_I = Q\left(\frac{\Delta}{\sigma}\right)$ — Δ distance from signal point to decision hyperplane; σ^2 variance of noise components

- Upper Bound on Error Probability (ML)

$$P_e \le \sum_{m\in M} 1/|M| \sum_{m'\in M, m'\neq m} Q\left(\frac{\Delta_{m'm}}{\sigma}\right)$$

$\Delta_{m'm} = \frac{\|\bar{s}_{m'}-\bar{s}_m\|}{2}$ ← ML

distance to hyper plane

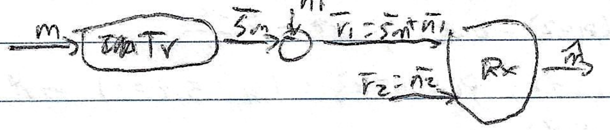
σ^2 = per dimension noise variance

Can be used for MAP as well

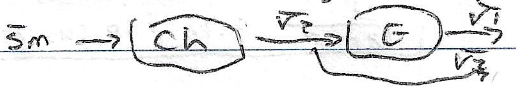
- Theorem of Irrelevance:

Output $\bar{r}_2$ of multivector channel irrelevant (not effect Pe) if for all $\bar{r}_1, \bar{r}_2$

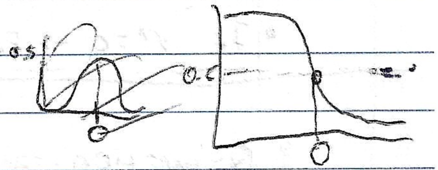
$p_{\bar{R}_2}(\bar{r}_2 \mid \bar{S}=\bar{s}_m, \bar{R}_1=\bar{r}_1\}$ ~~not~~ depend on m $\rightarrow = p_{\bar{R}_2}(\bar{r}_2) = p_{\bar{N}_2}(\bar{r}_2)$



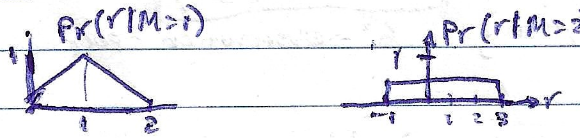
- Theorem of Revertability:
  - Min Pe not effected by reversability at output of channel



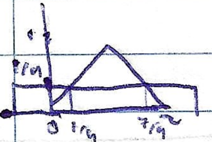
random decision: $P_e = 1/2 \rightarrow 50\% \rightarrow Q(0) =$



MAP or ML $\rightarrow \frac{P(r\mid s_2)}{P(r\mid s_1)} \gtrless 1$



$\begin{cases} \hat{m}=2 & r<1/4 \\ \hat{m}=1 & 1/4<r<7/4 \\ \hat{m}=2 & r>7/4 \end{cases}$



$\Pr\{M=1\}\Pr\{r<1/4 \,\|\, r>7/4 \mid M=1\}$ ①
$+\Pr\{M=2\}\Pr\{1/4<r<7/4 \mid M=2\}$ ②
1/2

①: $2\cdot\int_0^{1/4} x\,dx = 1/16$ ②: $1/4\cdot\int_{1/4}^{7/4} dx = 3/8$

$1/16\cdot\frac{1}{2} + 3/8\cdot 1/2 = 7/32 = P_e$ //

MAP //

SNR

Noise spectral density:
~~constant~~ noise power per unit. frequency (W/Hz)
- Power distribution of noise across diff frequencies

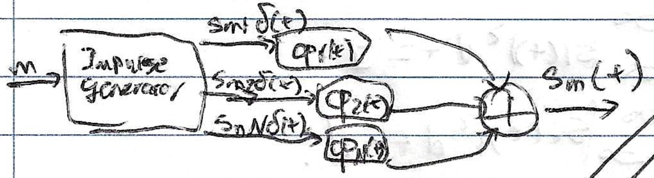
Channel Processing and Receivers

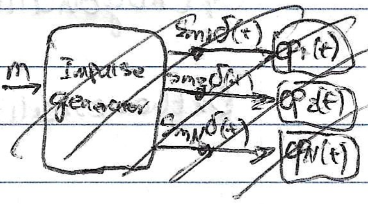
Ch5 Waveform Channels

- Waveform energy: $E_x = \int_{-\infty}^{\infty} x^2(t)\,dt$
- Orthogonality: $\int_{-\infty}^{\infty} \varphi_i(t)\varphi_j(t)\,dt = \begin{cases} E_i, & i=j \\ 0, & i \neq j \end{cases}$
- Orthonormality: $E_i = 1$; waveforms → orthonormal.
- Signal Synthesis: $\boxed{s_m(t) = \sum_{i=1}^{N} s_{mi}\varphi_i(t)}$, $m \in \mathcal{M} = \{1, 2, \dots |\mathcal{M}|\}$.

  where $\boxed{\varphi_i(t)}$ orthonormal basis function.

  ↳ building block waveforms





~~Gram~~ Schmidt → arbitrary signals to orthonormal basis.

1) $\varphi_1(t) = s_1(t)/\sqrt{E_1}$, $s_{11} = \sqrt{E_1}$
2) $\theta_2(t) = s_2(t) - s_{21}\varphi_1(t)$; $s_{21} = \int_{-\infty}^{\infty} s_2(t)\varphi_1(t)\,dt$.
3) $\varphi_2(t) = \theta_2(t)/\sqrt{E_{\theta 2}}$; $s_{22} = \int_{-\infty}^{\infty} s_2(t)\varphi_2(t)\,dt$.
4) $\theta_m(t) \triangleq s_m(t) - \sum_{i=1}^{m-1} s_{mi}\varphi_i(t)$

   $s_{mi} = \int_{-\infty}^{\infty} s_m(t)\varphi_i(t)\,dt$; $i = 1, \dots, m-1$.

5a) If $\theta_m(t) = 0$ stop or: $\to \sum_{i=1}^{m-1} s_{mi}\varphi_i(t)$.

5b) $\theta_m(t)$ ~~mean~~ $\neq 0$ → $\varphi_m(t) = \theta_m(t)/\sqrt{E_{\theta m}}$ → step 4.

$P_e = Q\left(\frac{d/2}{\sigma}\right)$

- Signal in vector space:

  $s_m(t) = \sum_{i=1}^{N} s_{mi}\varphi_i(t) \to \bar{s}_m = (s_{m1}, s_{m2}, \dots s_{mN})$

  ↳ each waveform → corresponds to a vector with N coefficients

$d = \sqrt{(s_{11}-s_{21})^2 + (s_{12}-s_{22})^2}$

$\sigma = \sqrt{\frac{N_0}{2}}$

- Signal space: $\{\bar{s}_1, \bar{s}_2, s_{|\mathcal{M}|}\}$ → signal constellation, set of vectors.

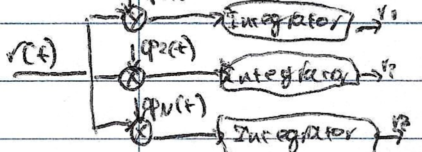
- no noise $r(t) = s_m(t) \to \int_{-\infty}^{\infty} s_m(t)\varphi_i(t)\,dt = \int_{-\infty}^{\infty}\left(\sum_{j=1}^{N} s_{mj}\varphi_j(t)\right)\varphi_i(t)\,dt$.

  $\bar{r}_i = \bar{s}_m + \bar{n}$

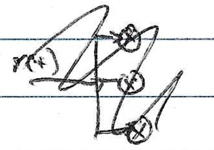
- noise $r(t) = s_m(t) + n_w(t) \to r_i = \int_{-\infty}^{\infty} r(t)\varphi_i(t)\,dt = \int_{-\infty}^{\infty} s_m(t)\varphi_i(t)\,dt + \int_{-\infty}^{\infty} n_w(t)\varphi_i(t)\,dt$

- Noise signal space: Joint PDF of relevant noise: $p_{\bar{N}}(\bar{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\bar{n}\|^2}{N_0}\right)$ spherically symmetric

- irrelevant noise: noise in signal space relevant, all other dimensions not.



- SNR in vector space $= \frac{\|\bar{s}\|^2}{N_0}$



dimensionality = 1 → when all ~~...~~ signals 1 building blok if φ_1, φ_2 → dimensionality = 2

1 dimensional mean (x) not (x,y)
so no S_{11}, S_{21} only S_{11})

Orthogonality → $\int_{-\infty}^{\infty} S_1(t) S_2(t) dt = 0$

Orthonormality → $E_1 = \int_{-\infty}^{\infty} |S_1(t)|^2 dt = 1$

$E_2 = \int_{-\infty}^{\infty} |S_2(t)|^2 dt = 1$.

$P_{avg}(\vec{a}) = \sum_{m \in M} Pr\{M = m\} \|\vec{s}_m - \vec{a}\|^2 = E[\|\vec{S} - \vec{a}\|^2]$

basically the eucleean dean distance sum ~~of~~ is Parg.

antipodal minimize Pe.

Channel Processing Receivers

Ch 6 - Receiver Implementation, Matched Filters

<u>vector representation of signals</u>

- signal representation: $f(t)=\sum_{i=1}^{N} f_i \varphi_i(t)$, $g(t)=\sum_{i=1}^{N} g_i\varphi_i(t)$.
- vector product: $(f\cdot g)=\sum_{i=1}^{N} f_i g_i$
- Norm and Distance: $\|f\|^2=(f\cdot f)=\sum_{i=1}^{N} f_i^2$; $\|f-g\|^2=\|f\|^2+\|g\|^2-2(f\cdot g)$
- correlation as dot prod: $\int_{-\infty}^{\infty} f(t)g(t)dt=(f\cdot g)$

Optimum Receiver

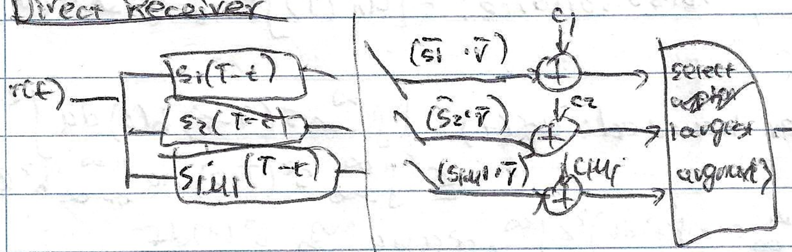
$\hat{m}_{MAP}(\bar{r})=\arg\max_{m\in M}\{(\bar{r}\cdot\bar{s}_m)+c_m\}$; $c_m=N_0/2\ \ln\Pr\{\mathcal{M}=m\}-E_m/2$

$E_m=\int_{-\infty}^{\infty} s_m^2(t)dt$.

! • Decision Criterion: minimize metric: $\|r-s_m\|^2-N_0\ln\Pr\{\mathcal{M}=m\}$

$\hookrightarrow \|\bar{r}\|^2-2(\bar{r}\cdot\bar{s}_m)+\|\bar{s}_m\|^2$

Direct Receiver



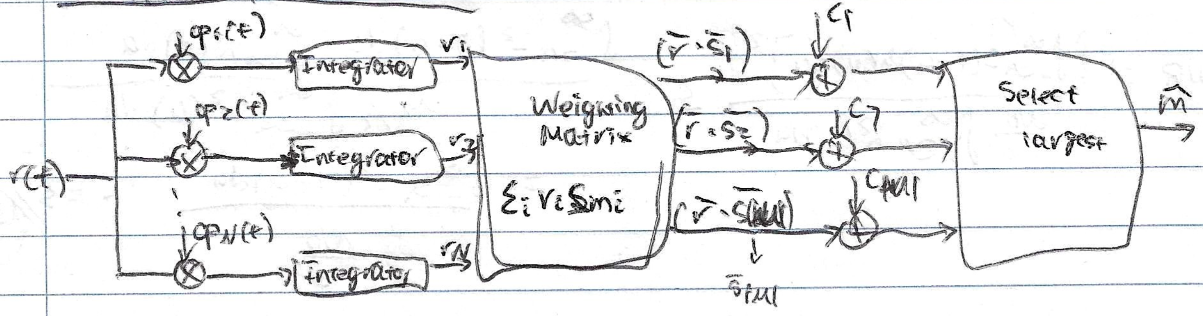
sample $t=T$.

- no need for weighting matrix
- harder to implement
- filters matched to signals

$M\gg N$.

$\int_{-\infty}^{\infty} r(a)s_m(T-t+a)da \overset{t=T}{=} \int_{-\infty}^{\infty} r(a)s_m(a)da=(\bar{s}_m\cdot\bar{r})$

Correlation Receiver



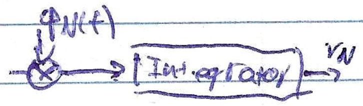
- recovered signal components.

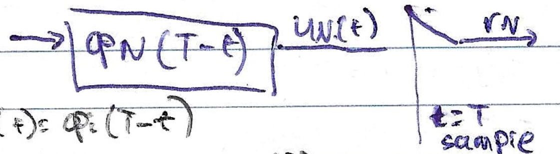
$r_i=\int_{-\infty}^{\infty} r(t)\varphi_i(t)dt$, ; $(\bar{r}\cdot\bar{s}_m)=\sum_i r_i s_{m,i}$ → matrix rep of decisions: $(\bar{r}\cdot\bar{s}_1)$, $(\bar{r}\cdot\bar{s}_2)$, $(\bar{r}\cdot\bar{s}_{|M|})$

$$\begin{bmatrix}(r\cdot s_1)\\(r\cdot s_2)\\\vdots\\(r\cdot s_{|M|})\end{bmatrix}=\begin{bmatrix}s_{11}&s_{12}&\cdots&s_{1N}\\s_{21}&s_{22}&\cdots&s_{2N}\\\vdots&\vdots&\ddots&\\s_{|M|1}&s_{|M|2}&\cdots&s_{|M|N}\end{bmatrix}\begin{bmatrix}r_1\\r_2\\\vdots\\r_N\end{bmatrix}$$

Matched Filter Receiver

Same as Correlation Receiver but instead of:





matched filter $t=T$

$h_i(t)=\varphi_i(T-t)$

$u_i(t)=r(t)*h_i(t)=\int_{-\infty}^{\infty} r(a)h_i(t-a)da=\int_{-\infty}^{\infty} r(a)\varphi_i(T-t+a)da$ ⇒

for $t=T \rightarrow u_i(T)=\int_{-\infty}^{\infty} r(a)\varphi_i(a)da=r_i$ //

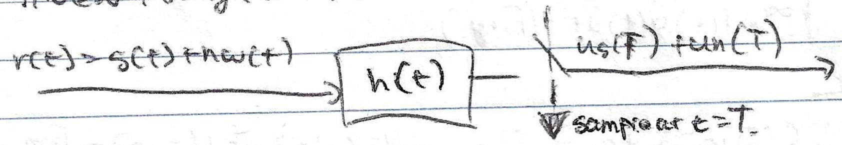
Instead of multipliers/Int filters; samplers → easier practically for analog signals.

~~Matched Filter Receiver~~

~~Same as correlation receiver but instead of~~

Signal to Noise Ratio

AWGN Through cine^r Filter



$u(T) = \int_{-\infty}^{\infty} r(T-a)\,h(a)\,da = u_s(T) + u_n(T)$

$u_s(T) \triangleq \int_{-\infty}^{\infty} s(T-a)\,h(a)\,da \; ; \; u_n(T) \triangleq \int_{-\infty}^{\infty} n_w(T-a)\,h(a)\,da$

$SNR \triangleq \frac{u_s^2(T)}{E[u_n^2(T)]}$; noise variance: $E[u_n^2(T)] = \frac{N_o}{2}\int_{-\infty}^{\infty} h^2(a)\,da$.

- signal energy is upper bounded: $u_s^2(T) = \left[\int_{-\infty}^{\infty} s(T-a)\,h(a)\,da\right]^2 \le \int_{-\infty}^{\infty} s^2(T-a)\,da \int_{-\infty}^{\infty} h^2(a)\,da$

  Schwarz inequality: $\left(\int_{-\infty}^{\infty} a(t)b(t)\,dt\right)^2 \le \int_{-\infty}^{\infty} a^2(t)\,dt \int_{-\infty}^{\infty} b^2(t)\,dt$

Max attained SNR: $SNR \le \frac{2E_s}{N_o}$! Matched Filter Reciver minimizes P_e, max SNR

$$SNR = \frac{\left[\int_{-\infty}^{\infty} s(T-a)h(a)\,da\right]^2}{\frac{N_o}{2}\int_{-\infty}^{\infty} h^2(a)\,da} \le \frac{\int_{-\infty}^{\infty} s^2(T-a)\,da \int_{-\infty}^{\infty} h^2(a)\,da}{N_o/2 \int_{-\infty}^{\infty} h^2(a)\,da}$$

$$= \frac{\int_{-\infty}^{\infty} s^2(T-a)\,da}{\frac{N_o}{2}} = E_s / (N_o/2)$$

Channel Processing:
Transmitters

Ch7 Signal Energy Considerations, Orthogonal Signals

Binary Signaling

$\lim_{|M| \to \infty} E[S] = 0$

- energy/avg energy: $E_{sm} = \int_0^T s_m^2(t)dt = \|s_m\|^2$

$$E_{av} = \sum_{m \in M} P(M=m) E_{sm} = E[\|S\|^2]$$

- Binary Orthogonal Signaling: signal vectors for orthogonal signals.

$s_1 = (\sqrt{E_s}, 0)$, $s_2 = (0, \sqrt{E_s})$, $\sigma^2 = N_0/2 \to \sigma = \sqrt{\frac{N_0}{2}}$

$P_e^{orth} = Q(\sqrt{E_s/N_0})$ AWGN: $Q(\frac{d}{2\sigma})$

- Binary antipodal signaling: signal vectors for antipodal signals.

$s_1 = (\sqrt{E_s}, 0)$, $s_2 = (-\sqrt{E_s}, 0)$; $P_e^{antip} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$

~~Non-Binary Orthogonal Signaling~~ / capacity results.

- Orthogonal signal sets:

$E_{av} = E_s$

- signal representation: $s_m = \sqrt{E_s}\,\varphi_m$ $m \in M$

$\varphi_1 = (1,0,0)$, $\varphi_2 = (0,1,0)$, $\varphi_3 = (0,0,1)$ $|M| = 3$

- Optimum Receiver for orthogonal signals: decision rule:

$\hat{m} = \arg\max_{m \in M} (\langle \bar{r} \cdot s_m \rangle) \Rightarrow \arg\min_{m \in M} \{\|\bar{r} - s_m\|^2\} = \arg\max_{m \in M} \{r_m\}$

$$P_c = \int_{-\infty}^{\infty} P_{R_1}(a|M=1) P(\hat{M}=1|M=1, R_1=a)\, da.$$

$$P_e \le \begin{cases} 2\exp\left(-\left[\sqrt{E_b/N_0} - \sqrt{\ln 2}\right]^2 \log_2 M\right) & \text{if } \ln 2 < \frac{E_b}{N_0} < 4\ln 2 \\ 2\exp\left(-\left[E_b/(2N_0) - \ln 2\right] \log_2 M\right) & \text{if } E_b/N_0 \ge 4 \ln 2 \end{cases}$$

- Channel Capacity. wideband capacity:

$$R = P_s / E_b \le P_s / N_0 \ln 2 \left(\frac{\text{bits}}{\text{sec}}\right) = C$$

$P_s \to$ transmitter power

- $SDR = P/N_0 W$, Shannon capacity: $C = B \log_2(1 + SNR)$

---

lets say you want 0.001 Pe ↓ ~~orthogonal~~ antipodal will achieve it at lower energy.

- Achieve same Pe → orthogonal signaling requires $\boxed{2\times}$ energy of antipodal signaling (a 3dB disadvantage).
- orthogonal: $s_1 = (\sqrt{E_s}, 0)$, $s_2 = (0, \sqrt{E_s})$ → Euclidean dist: $\sqrt{2E_s}$ $Q\left(\sqrt{\frac{E_s}{N_0}}\right)$ - $Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$
- antipodal: $s_1 = (-\sqrt{E_s}, 0)$, $s_2 = (\sqrt{E_s}, 0)$ → euclidean dist: $2\sqrt{E_s}$
- For AWGN channels → Pe depends on Euclidean distance.

$10\log_{10}(2) = 3\text{dB}$.

---

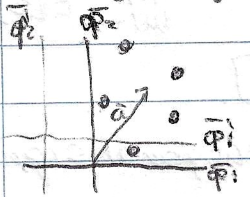
to achieve same Pe orthogonal $2\times$

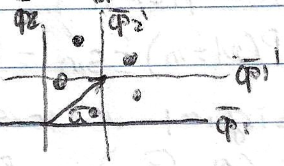
- but orthogonal easier to implement on coherent detection → since receiver doesnt need signal phase (better for practical systems)

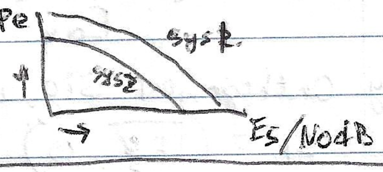
$Q\left(\frac{\sqrt{2E_s}}{2\sqrt{\frac{N_0}{2}}}\right)$ $Q\left(\frac{\sqrt{2E_s}}{\sqrt{2}\sqrt{N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

$Q\left(\frac{2\sqrt{E_s}}{2\sqrt{\frac{N_0}{2}}}\right)$ $Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$

• Translating signal structure closer to origin minimizes average signal energy requwired, keeping decision regions same doesnt effect Pe !







Pe is same

• Orthogonal (FSK) waveforms

$|\mathcal{M}| = \{1,2\}$, $Pr\{m=1\} = Pr\{m=2\} = 1/2$

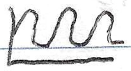
$s_1(t) = \sqrt{2E_s}\sin(\cancel{2\pi}10\pi t)$ $0 \le t < 1$

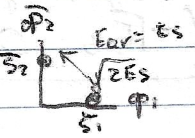
$s_2(t) = \sqrt{2E_s}\sin(12\pi t)$ $0 \le t < 1$

$\bar{s}_1 = (\sqrt{E_s}, 0)$

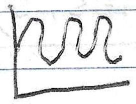
$\bar{s}_2 = (0, \sqrt{E_s})$

$\varphi_1(t) = \frac{s_1(t)}{\sqrt{E_s}}$





$\varphi_2(t) = \frac{s_2(t)}{\sqrt{E_s}}$



antipodal (PSK) waveforms

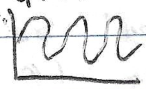
$|\mathcal{M}| = \{1,2\}$ $Pr\{M=1\} = Pr\{M=2\} = 1/2$

$s_1(t) = \sqrt{2E_s}\sin(10\pi t)$ $0 \le t < 1$

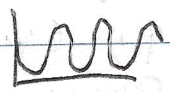
$s_2(t) = -\sqrt{2E_s}\sin(10\pi t)$ $0 \le t < 1$

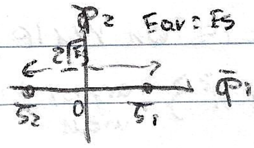
$\bar{s}_1 = (\sqrt{E_s}, 0)$, $\bar{s}_2 = (-\sqrt{E_s}, 0)$

$\varphi_1(t) = s_1(t)/\sqrt{E_s}$

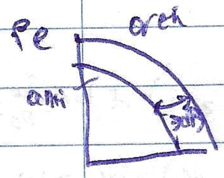


$s_2(t)/\sqrt{E_s} = -\varphi_1(t)$





AGN Vector Channel → Pr that noise pushes signal to wrong hyperplane $P_I = Q\left(\frac{\Delta}{\sigma}\right)$ Δ → dist point + hyperplane

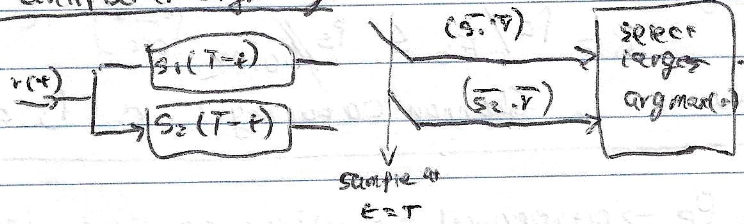


→ Pe doesnt change but for antipodal → same Pe for lower energy can be achieved

orotations → no change Pe → ~~but~~ but save energy.

Receives of antipodal signaling

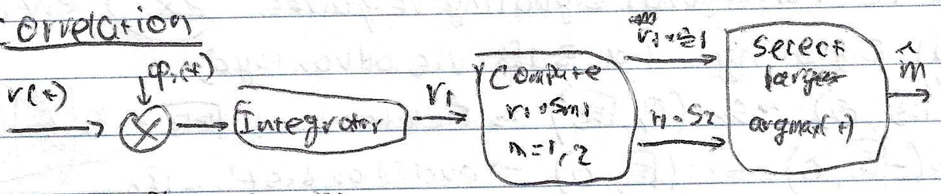
Direct



$E_b \triangleq \frac{E_s}{\log_2|\mathcal{M}|}$

energy per transmitted bit of info.

Correlation



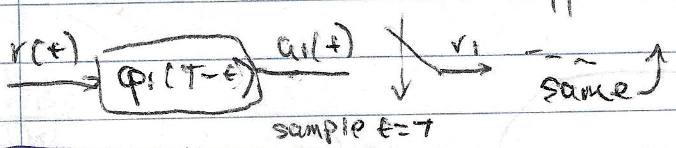
$E_{av} = \sum_{i=1}^{N} Pr\{M=m_i\}\|\bar{s}_m\|^2$

$= \frac{1}{2}(0)^2 + \frac{1}{2}(2)^2 = 2$

assuming $\bar{a} = 0$. origin.

Matched Filter Receiver

||



antipodal minimize Pe//

Channel Processing: Transmitters.

$A = \frac{h \cdot b}{2}$ area of triangle

Ch 8 Message Sequence, Bandwidth

Bit-by bit signaling

$E_{avg} = E_s = P_s T$ Joule

- Transmission Rate: $R = \frac{\log_2 |M|}{T}$ bits/second $\tau = \frac{T}{2^K}$
- Energy per bit: $E_b = \frac{E_s}{\log_2 |M|} = \frac{P_s}{R}$ Joule/bit.
- signal representation: transmitted signal: $s(t) = \sum_{i=1}^{K} (-1)^{b_i+1} p(t-(i-1)\tau)$ K > pulses
- ~~signal representation~~: normalized waveforms: $\varphi_i(t) = \frac{p(t-(i-1)\tau)}{\sqrt{E_b}}$
- signal as vector: $s_m = \sqrt{E_b}((-1)^{b_1+1}, (-1)^{b_2+1}, \ldots (-1)^{b_N+1})$

signal dimensionality: $N = K$. $\frac{p(t)}{\sqrt{E_b}}$

- $P_c = \left(1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)^K$; $P_e = 1 - P_c$

Block Orthogonal Signaling $E_b/N_0 = (1+\epsilon)^2 \ln 2$.

$R = \frac{K}{T}$; orthogonal signal: $s_m(t) = \sqrt{E_s}\,\varphi(t-(m-1)\tau)$, $m = 1, \ldots, 2^K$

$P_e \le 2\exp\left(-\left[\sqrt{\frac{E_b}{N_0}} - \sqrt{\ln 2}\right]^2 \log_2 |M|\right) = 2\exp(-\epsilon^2 RT \ln 2)$; ϵ energy efficiency

- Channel Capacity: $C_\infty = P_s/(N_0 \ln 2)$ bit/sec
- $R < C_\infty$ achievable with low P_e increasing T.
- Dimensionality / Bandwidth: num orthogonal dimensions: $N \le 2WT$ where W is bandwidth and T, time duration.

- Stream of messages → $s_m(t)$ to OLECT → equiprobable

T 2T 3T

- bit/bit signaling: → K binary digits: b_1, b_2, b_K in T, $|M| = 2^K$, $R = \frac{\log_2 |M|}{T} = \frac{K}{T}$

geometric: $s(t) = \sum_{i=1}^{K} (-1)^{b_i+1} \sqrt{E_b}\,\varphi_i(t)$. $\{S = \{s_1(t) s_2(t) \ldots\}$

$s_m = \sqrt{E_b}((-1)^{b_1+1}, (-1)^{b_2+1}, \ldots (-1)^{b_N+1})$

1101 0

$2\sqrt{E_s}$ φ_1 $K = N = 1$ $|M| = 2$

- optimum receiver: $\hat{m} = 1$ if $r_i < 0$.

no error if $r_i = -\sqrt{E_b} + n_i < 0$ for all $i = 1, \ldots K$

$2\sqrt{E_s}$ φ_1 → $K = N = 2$ $|M| = 4$.

- $K = RT$ and $E_b = P_s/R$

$P_e = 1 - \left(1 - Q\left(\sqrt{\frac{2P_s}{RN_0}}\right)\right)^{RT}$

- $T = 1/R \Rightarrow K = 1 \Rightarrow P_e = Q\left(\sqrt{\frac{2P_s}{RN_0}}\right)$

P_e decreased by increasing P_s decreasing R

- reliability can't increase by inc T, for bit bit signaling

$P_e \le 2\exp(-\epsilon RT \ln 2)$

$R = \frac{1}{(1+\epsilon)^2} \frac{P_s}{N_0 \ln 2}$

- reliability increase by increasing P_s and decreasing R (bit by bit sig) → but can with orthogonal

increasing code word T.

Signaling	Dimensions per block	Dimensions per sec
Bit by Bit	$K = RT$	$K/T = R$
Block Orthogonal	$2^K = 2^{RT}$	$2^K/T = 2^{RT}/T$

- Block orth signaling: Pros / cons.
  - large reliability in $c(T)$ (not bit-bit)
  - num dimensions explode ~~w~~ per inc in T.
  - Channel with finite B can't accomodate all dimensions

Dimensionality Theorem

1. $\int_{-W}^{+W} |\Phi_i(f)|^2 df \geq 1$.
2. $\varphi_i(t) = 0$ for all outside ~~b~~ t $[0, T]$

~~Max~~ number of orthogonal waveforms upper bounded by $2WT$.

- $2^k/T \to (2k+1)/T$.

- Increasing P_s, lowering R → bit/bit and orthogonal
- Increasing codeword T → orthogonal (reliability)
- Only up to C_∞ → achieve reliability.

~~non frequency limited spectrums~~ freq

- here time limited not freq-limited

Ch10 → freq limited

Channel Processing : Transmitters.

Ch 9 - Capacity of the Baseband and Wideband Channels

Channel Capacity of AGN Vector Channel.

- Rate per dimension: $R_N = \frac{\log_2 |M|}{N} < C_N \rightarrow C_N = \frac{1}{2}\log_2\left(\frac{E_N + N_0/2}{N_0/2}\right)$ bits/dimension

$\boxed{P_e \rightarrow 0 \text{ if } R_N < C_N}$

- Channel capacity of baseband AWGN: $C = W\log_2\left(1 + \frac{P_s}{WN_0}\right)$ bits/sec
$\rightarrow$ scales linearly with bandwidth.
- Power limited capacity: $C_\infty = \frac{P_s}{N_0 \ln 2}$
- bandlimited : $\rightarrow C = W\log_2(SNR) \rightarrow C = \frac{P_s}{N_0 \ln 2}$
- Sphere hardening: mean/variance of square norm of normalized Gaus vector: $E[\|G'\|^2] = \sigma_g^2$, $var[\|G'\|^2] = \frac{2\sigma_g^4}{N}$
- Error Prob in random code Gen: $R_N = C_N - \delta$
$\lim_{N\to\infty} P_e \le \lim_{N\to\infty} 2^{-\delta N} = 0$
- Digital Comm: capacity binary symmetric channels (BSC): $C = 1 - H(p)$
Shannon Theorem reliable comm: $R_C \le C \Rightarrow P_e \rightarrow 0$
$\rightarrow$ sufficiently long codes//

- Module objective: Reliable transmission $\rightarrow (P_e \rightarrow 0)$ at certain rate (R) increasing T. when (W) bandwidth (P_s) transmission Power $\rightarrow$ (Fixed)

- block orthogonal signaling $\rightarrow$ reliable comm rates
if $R < \frac{P_s}{N_0 \ln 2}$.

Channel Processing: Transmitters. K k

Chapter 8 → time limited → bandwidth unlimited

Ch 10 – Pulse Transmission.

$\bar{x}_a(t) \rightarrow \boxed{p(t)} \rightarrow \bar{s}_a(t)$

where $\bar{a} = (a_0, a_1, \ldots a_{K-1})$

$$\bar{x}_a(t) = \sum_{k=0}^{K-1} a_k \delta(t-kT)$$

$$\bar{s}_a(t) = \sum_{k=0}^{K-1} a_k p(t-kT)$$

→ linear Filter.

$T_s = \frac{1}{R_s}$

- building blocks → impulses → non-time-limited
- finite bandwidth → ISI created. → infinite-time.

- can have binary/non-binary (Pulse Amplitude Modulation) PAM.

0 1 ↑ -1 +1 00 01 11 10 -3 -1 +1 +3.

$R = m = \log_2(M)$ → $M = 4$ → $\log_2(M) = 2$ bits/symbol, 2 bits.

trans rate

- ISI $\xrightarrow{\text{solved}}$ Nyquist Criteria.

$$\int_{-\infty}^{\infty} p(t-kT)\,p(t-k'T)\,dt = \begin{cases} 1 & k = k' \\ 0 & k \neq k' \end{cases}$$

$h(t) = p(t) * p(-t) \rightarrow H(f) = P(f)P^*(f) = |P(f)|^2$

$$= p(t) * p(-t)\big|_{t=kT} = h(kT) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

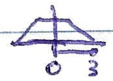
$|P(f)|^2$ → because $p(t)$ → real

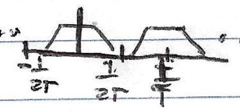
- Time Domain Restriction on $p(t)$ → Zero Forcing (ZF)

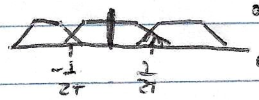
so nyquist in f-domain $= \frac{1}{T}\sum_{m=-\infty}^{\infty} |P(f+m/T)|^2 = 1$ for all f

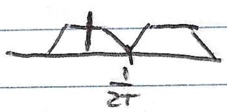
$\boxed{|P(f+m/T)|^2 = H(f+m/T)}$

Orthonormal pulses/Bandwidth

Bandwidth:  0 3, W=3

- $W < 1/(2T)$  $H(f) \neq |P(f)|^2$

- $W > 1/(2T)$ 
  - excess bandwidth → ISI → higher transmission capacity
  - Orthonormal but to min ISI → ~~sq~~ root raised cosine.

- $W = \frac{1}{2(T)}$ 
  - $p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t/T)}{\pi t/T}$; $T=1$ → sinc pulses.
  - $P(f) = \begin{cases} \sqrt{T} & |f| < 1/(2T) \\ 0 & |f| > 1/(2T) \end{cases}$
  - orthogonal → no ISI → unrealistic...

PAM Transiever.

$\bar{a} \rightarrow \boxed{\delta(t)} \xrightarrow{x_a(t)} \boxed{p(t)} \xrightarrow{s_a(t)} \oplus \xrightarrow{\bar{s}_a(t)} \boxed{p(T_p - t)} \xrightarrow{u(t)}$ sample at $t = T_p + kT$ → r_k

$n_w(t)$ → potential, channel

$x(t) \rightarrow \boxed{p(T_p - t)} \rightarrow u(t)$

$$u(t) = \int_{-\infty}^{\infty} x(\tau)\,p(\tau - t + T_p)\,d\tau$$

$$r_k = \sum_{k'=0}^{K-1} a_{k'} \underbrace{\int_{-\infty}^{\infty} p(\tau - k'T)\,p(-kT+\tau)\,d\tau}_{a_k} + \underbrace{\int_{-\infty}^{\infty} n_w(\tau)\,p(-kT+\tau)\,d\tau}_{n_w}$$

0 unless $k' = k$

freq limited
time unlimited

bits/symbol $\rightarrow \log_2(M)$

$R = \text{rate} = \frac{\text{bits/symb}}{T} = \text{bits/sec}$

nyquist bandwidth: $\frac{1}{2T} = B$

- Power of the noise that will remain after filtering $P_{nw} = \frac{N_0}{2} W$

- sampling receiver easier than match filter receiver

- output noise power $\int_{-\infty}^{\infty} \frac{N_0}{2} |P(f)|^2 df$ → band limited

$\int_0^{T/4} T$ insted $\int_0^{T/2} T$

$\left(\frac{N_0}{2}\right) 2 \int_0^{1/2T} T\, df$

$= \frac{N_0}{2}$ Watts

Channel Processing: Transmitters

~~Pass~~ Ch 11 - Pass-Band Channels

Passband Transmission / Capacity:

- bandwidth of baseband: $BW \geq \frac{1}{2T}$. $T \rightarrow$ symbol period
- ~~passband channel capacity: $C_N = \frac{1}{2}\log_2\left(1+\frac{P_s}{N_0 W}\right)$ bits per dimension~~

- ~~Passband Channel Capacity~~:

$$C_N = \frac{1}{2}\log_2\left(1+\frac{P_s}{N_0 W}\right) \text{ bits/dimension.}$$

$$C = 2W\log_2\left(1+\frac{P_s}{2N_0 W}\right) \text{ bits/sec.}$$

2x

- Baseband channel capacity:

$$C = W\log_2\left(1+\frac{P_s}{N_0 W}\right)$$

Quadrature Multiplexing.

transmitted signal: $s_m(t) = s_c(t)\sqrt{2}\cos(2\pi f_0 t) + s_s(t)\sqrt{2}\sin(2\pi f_0 t)$

- orthogonality proof: $\int_{-\infty}^{\infty} \varphi_{c,i}(t)\varphi_{c,i'}(t)\,dt = \delta_{i,i'}$

$$\int_{-\infty}^{\infty} \varphi_{s,j}(t)\varphi_{s,j'}(t)\,dt = \delta_{j,j'}$$

$$\boxed{\int_{-\infty}^{\infty} \varphi_{c,i}(t)\,\varphi_{s,j}(t)\,dt = 0}$$

Matched / Correlation Receiver:

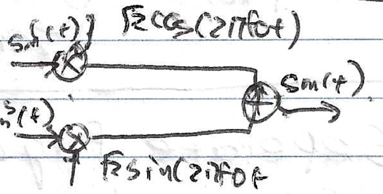
$$r_i = \int_{-\infty}^{\infty} r(t)\,\varphi_i(t)\,dt$$

$$\hat{m}_{MAP} = \arg\max_{m\in\mathcal{M}} \{(r\cdot s_m) + c_m\}$$

$$c_m = \frac{N_0}{2}\ln \Pr(M=m) - \frac{E_m}{2}$$

Quadrature multiplexing model.

$s_m(t) = s_m^c(t)\sqrt{2}\cos(2\pi f_0 t) + s_m^s\sqrt{2}\sin(2\pi f_0 t)$



↳ $\varphi_i(t), \psi_j(t) \rightarrow$ building block waveforms

$s_m^c(t) = \sum_{i=1}^{N_c} s_{mi}^c \varphi_i(t)$ and $s_m^s(t) = \sum_{j=1}^{N_s} s_{mj}^s \psi_j(t)$

↳ transmitted waveform is addition of the two.

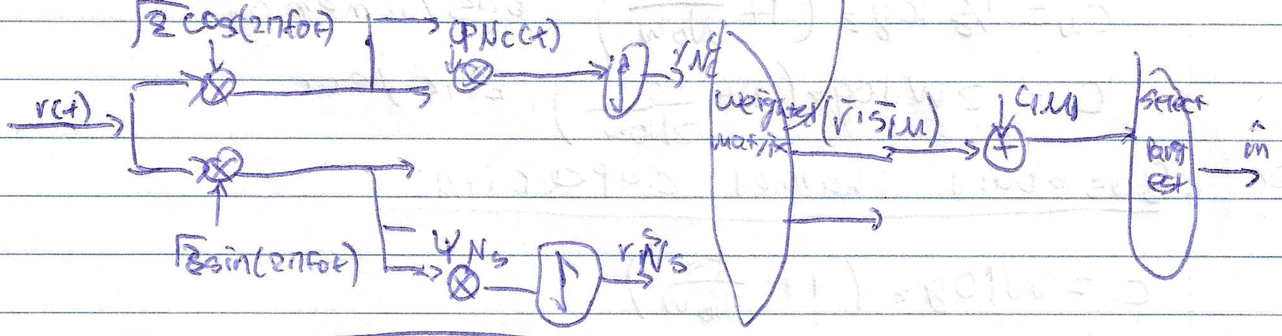
Quadrature optimum receiver.

$$\int_{-\infty}^{\infty} r(t)\varphi_{c,i}(t) = \int_{-\infty}^{\infty} r(t)\sqrt{2}\cos(2\pi f_0 t)\varphi_i(t)\,dt = r_i^c \;;\; i = 1,2,\dots,N_c.$$

$$\int_{-\infty}^{\infty} r(t)\psi_{s,j}(t)\,dt = \int_{-\infty}^{\infty} r(t)\sqrt{2}\sin(2\pi f_0 t)\,\psi_j(t)\,dt = r_j^s \quad i = 1,2,\dots N_s$$

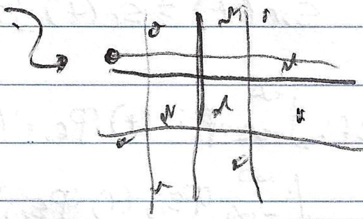
$$(\bar{r}\cdot\bar{s}_m) = \sum_{i=1}^{N_c} r_i^c s_{mi}^c + \sum_{j=1}^{N_s} r_j^s s_{mj}^s$$

Correlation ~~RX~~ Quadrature Receiver.



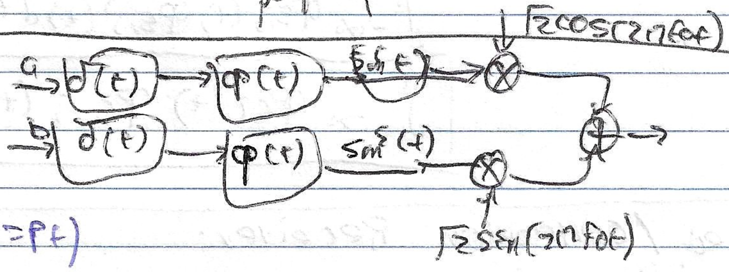
$$N_c > N_s = N \rightarrow \varphi_i(t) = \psi_i(t) \rightarrow s_m(t) = \sum_{i=1}^{N} \varphi_i(t)\left(s_{mi}^c\sqrt{2}\cos(2\pi f_0 t) + s_{mi}^s\sqrt{2}\sin(2\pi f_0 t)\right)$$

• if $s(t-\Delta) + n_w(t)$



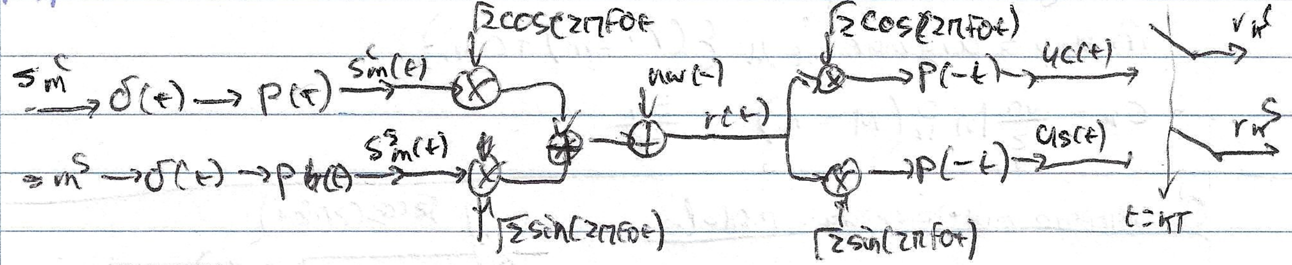
shifted in phase due to time delay

Normal QAM:



• Serial QAM: $\varphi(t) = p(t)$

PA

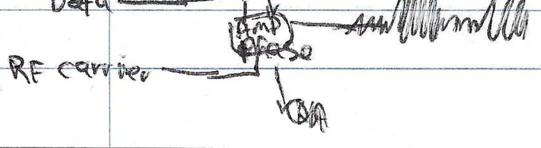


! See Seperate Page on Quadrature

64QAM ↑ 16QAM → noise tolerance.

QPSK = 01 symbols
16QAM = 0001 symbols
64QAM = 001011 symbols.

QAM modulator.



Channel Processing : Transmitters.

Ch 12 - Random Carrier-Phase

- Coherent vs Incoherent Communication
  - incoherent → no knowledge of θ phase means there is only cosine but can be decomposed to have sine component:
    $\cos(a-b) = \cos a \cos b + \sin a \sin b$
- antipodal won't work for incoherent reception.

→ best but unrealistic: coherent antipodal.

Coherent Signal Model: Quadrature-multiplexed Transmitted Waveform.
N or 2N dimension.

$s_m(t) = s_m^b(t)\sqrt{2}\cos(2\pi f_0 t - \theta)$.

$= s_m^b(t)\cos(\theta)\sqrt{2}\cos(2\pi f_0 t) + s_m^b(t)\sin(\theta)\sqrt{2}\sin(2\pi f_0 t)$.

$= \sum_{i=1}^{N} s_{mi}^c \varphi_i(t)\sqrt{2}\cos(2\pi f_0 t) + \sum_{i=1}^{N} \varphi_i(t)\sqrt{2}\sin(2\pi f_0 t)$

$= \sum_{i=1}^{N} s_{mi}^c \varphi_{ci}(t) + \sum_{i=1}^{N} s_{mi}^s \psi_{s,i}(t)$

where $\bar{s}_m^c = (s_{m1}, s_{m2}, \ldots s_{mN})\cos(\theta) = \bar{s}_m\cos(\theta)$.

$\bar{s}_m^s = (s_{m1}, s_{m2}, \ldots s_{mN})\sin(\theta) = \bar{s}_m\sin(\theta)$ //

Incoherent Receiver:

- decision rule: $\hat{m}_{MAP}(r) = \arg\max_{m\in M}\left\{I_0\left(\frac{2X_m}{N_0}\right)\exp\left(-\frac{E_m}{N_0}\right)\right\}$

$X_m = \sqrt{(r_c\cdot s_m)^2 + (r_s\cdot s_m)^2}$; $I_0(x)$ → zero order modified Bessel funct E_m = message energy.

- Equal energy signals: $\hat{m}_{MAP}(r) = \arg\max_{m\in M}\{(r_c\cdot s_m)^2 + (r_s\cdot s_m)^2\}$.
- ~~Error prob~~ →

Incoherent: $P_e^{in} = \frac{1}{2}\exp\left(-\frac{E_s}{2N_0}\right)$

Coherent: $P_e^{orth} = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

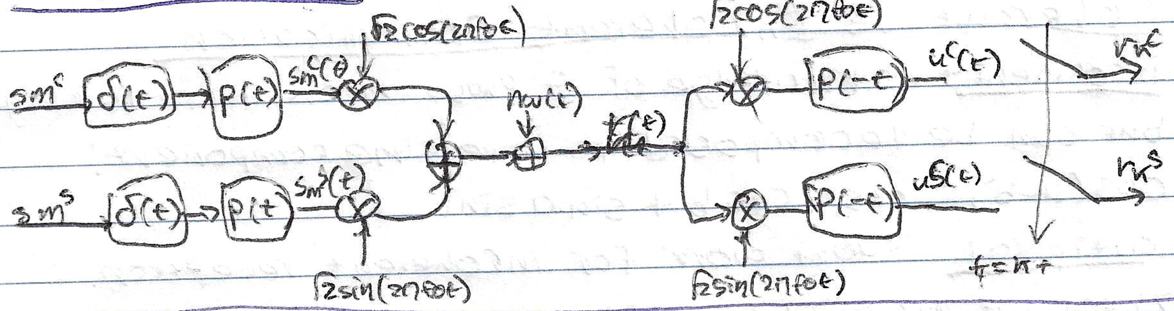
- Envelope detection: $X_m = \sqrt{(u_c^m(T))^2 + (u_s^m(T))^2}$

$u_c^m(T)$ correlation of $r(t)$ with $\cos(2\pi f_0 t)$ ↳ $u_s^m(T)$ correlation of $r(t)$ with $\sin(2\pi f_0 t)$

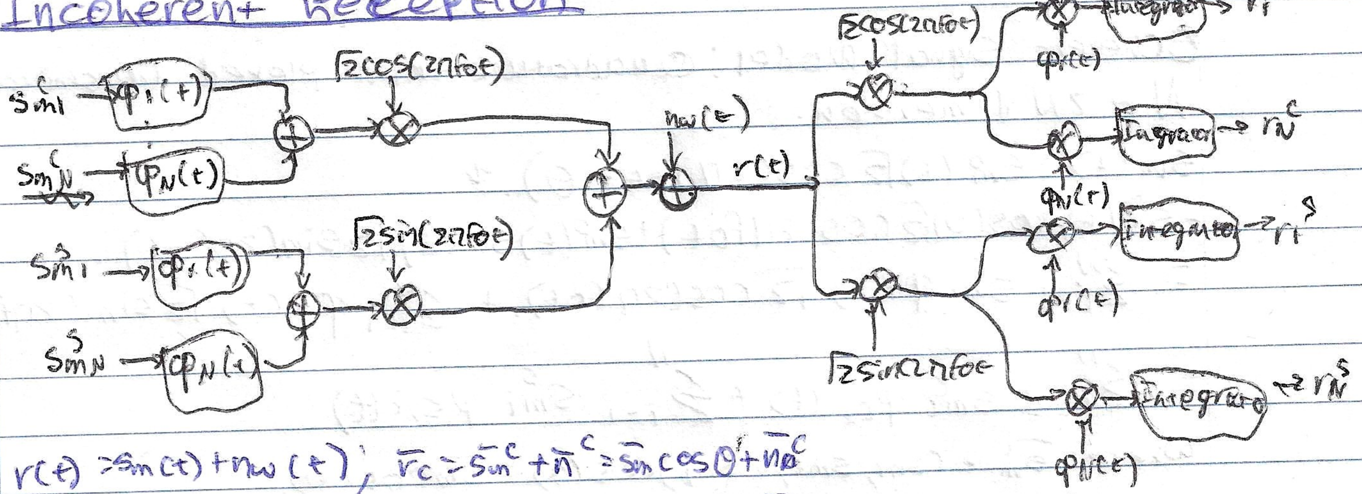
Wideband AWGN: $r(t) = s(t) + n_w(t)$.

Baseband AWGN: $r(t) = s(t) * w_b(t) + n_w(t)$.

Coherent Reception



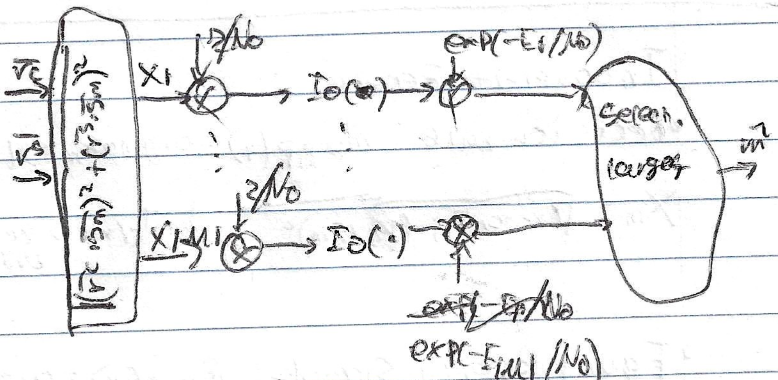
Incoherent Reception



$r(t) = s_m(t) + n_w(t)$; $\bar{r}_c = \bar{s}_m^c + \bar{n}^c = \bar{s}_m \cos\theta + \bar{n}^c$

; $\bar{r}_s = \bar{s}_m^s + \bar{n}^s = \bar{s}_m \sin\theta + \bar{n}^s$

General Structure of Incoherent Receiver

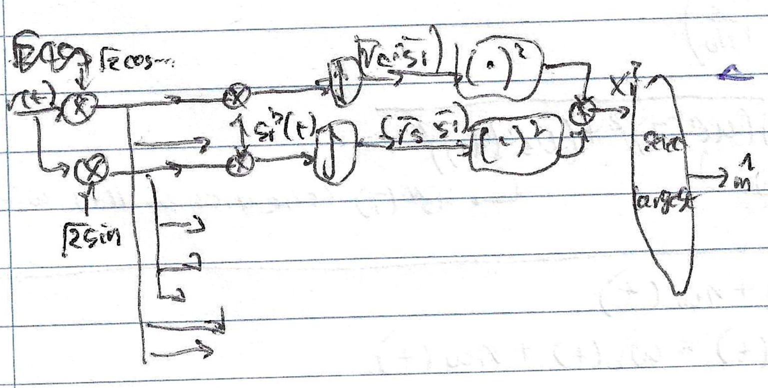


$$\hat{m}^{MAP}(\bar{r}) = \underset{m \in M}{\mathrm{argmax}} \left\{ I_0\left(\frac{2X_m}{N_0}\right) \exp\left(-\frac{E_m}{N_0}\right) \right\}$$

if equal energy

$$\hookrightarrow \underset{m \in M}{\mathrm{argmax}} \left\{ (\bar{r}^c \cdot \bar{s}_m)^2 + (\bar{r}^s \cdot \bar{s}_m)^2 \right\}$$

→ much simpler



The equal energy receiver is also ~~easier~~ easier

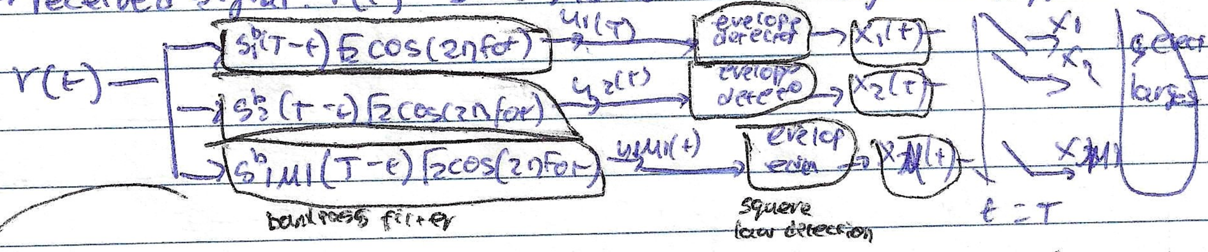
correlation/Direct Random phase receiver

Ch 12 – Cont – Random Carrier Receiver

Envelope Detection: Receiver structure, same energy:

$\hat{m}_{map}(\vec{r}) = \arg\max_{m \in \mathcal{M}} \{X_m^2\} \rightarrow X_m \triangleq \sqrt{(\vec{r}^c \cdot \vec{s}_m)^2 + (\vec{r}^s \cdot \vec{s}_m)^2}$

↳ received signal: $r(t) = s_m^b(t)\sqrt{2}\cos(2\pi f_0 t - \theta) + n_w(t)$.



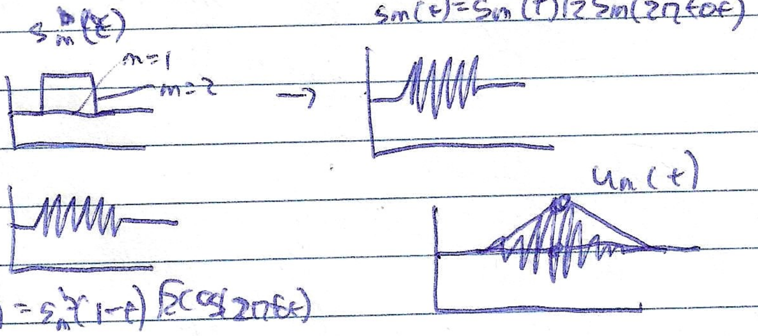
• Output of filters: $u_m(t) = u_m^c(t)\cos(2\pi f_0 t) + u_m^s(t)\sin(2\pi f_0 t)$

$u_m^c(t)$ ↓ baseband; $\cos(2\pi f_0 t)$ high freq; $u_m^s(t)$ ↓ baseband; $\sin(2\pi f_0 t)$ high freq

$\int_{-\infty}^{\infty} r(a)\sqrt{2}\cos(2\pi f_0 a)\underbrace{s_m^b(T-t+a)}_{\text{baseband}}da$ $\quad$ $\int_{-\infty}^{\infty} r(a)\sqrt{2}\sin(2\pi f_0 a)\underbrace{s_m^b(T-t+a)}_{\text{baseband}}da$

$X_m(t) = \sqrt{(u_m^c(t))^2 + (u_m^s(t))^2}$

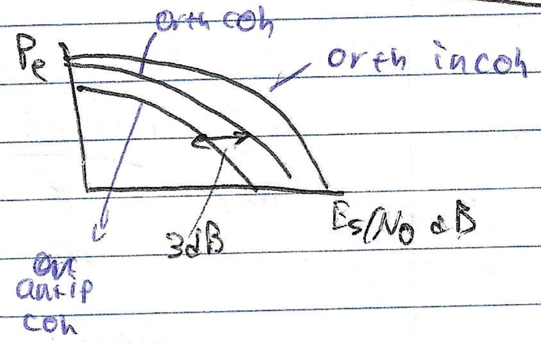
• $\theta = \pi/2,\ T = 1$



$P_e^{in/orthog} = \frac{1}{2}\exp\left(-\frac{E_s}{2N_0}\right)$

$P_e^{orth/coh} = Q\left(\sqrt{\frac{E_s}{N_0}}\right) \leq \frac{1}{2}\exp\left(-\frac{E_s}{2N_0}\right)$

$P_e^{antip/coh} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$



incoherent cheaper easier but worse SNR
3dB loss

~~01~~ ~~11~~

Invited Lecture - Ch 13

FSO: (Free Space optics) vs RF (Radio Frequency Communication.

- FSO → 75 dB more gain for same antenna size, power, distance.
- Receive power. $P_R = \cancel{\eta P_T} \; \eta P_T \cdot \left(\frac{\pi D_T}{\lambda}\right)^2 \left(\frac{\pi D_R}{\lambda}\right)^2 \left(\frac{\lambda}{4\pi R}\right)^2$

  $P_{Receive} \propto 1/\lambda^2 \propto f^2$
- Due to high transmission gains; high directivity : FSO: LPD, LPI

  Low Probability Detection, Interception.
- Diffraction-limited spot sized : $\omega(R) = \omega_0 \sqrt{1 + \left(\frac{\lambda R}{\pi \omega_0^2}\right)^2} = \frac{\lambda R}{\pi \omega_0}$
- high throughput, security, low interfer, low energy per bit, license free.
- DSP design hard, availability (weather limited), location (direct line sight), eye safety.

Moonshot FSO link to moon: 0.01nW avg power received; single photon detector;

NIR wavelength 1550nm; given 256PPM; photon efficiency?

ph $E = h \cdot f$; such that we have $P_{tx} / (h \cdot f)$ ph/sec.

- M-PPM → M timeslots per symbol : send $\log_2(M)$ ~~in/s~~ bits per symbol.
- 1 ph per symbol → $\log_2(M)$ bits per symbol → $1/\log_2(M)$ ph/bit.
- P_{tx} = .001nW → 7.8E7 ph/sec. → M=256 → 623 Mbps → 0.125 ppb

---

Calculate throughput:

multiply X

- A: num of ~~channels~~ optical channels
- B: Baudrate per channel
- C: Modulation Format.
- D: FEC & coding rate.
- E: polarizations
- F: Transmission success factor.
- G: Over head, header and protocols.

- Solutions for fading and diversity methods.
  - Temporal, ~~div~~ spatial, site, frequency diversity
    - Temporal → resend message
    - spatial → multiple receivers
    - site → diff place
    - frequency → diff freq

---

free space optical ~~power~~ varying than fibre optic since travels through air.

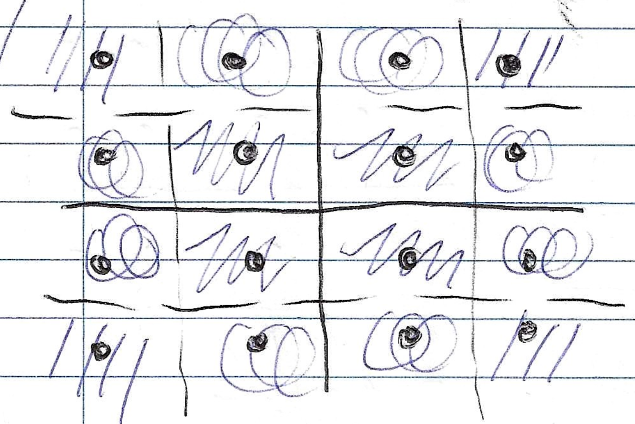
horizontal vertical

$P_c = (1-2Q)(1-2Q) = 1 - 4Q + 4Q^2$

↳ 2 moves horizontally, 2 moves vertically

$P_e = 1 - P_c = 4Q - 4Q^2$

Understanding Quadrature



Union Bound

neighbours

$|||| \; P_e \leq 4Q\left(\frac{d}{2\sigma}\right)$ ↑ → ← ↓ 4

$\circledcirc \; P_e \leq 3Q\left(\frac{d}{2\sigma}\right)$ ← ↑ ↓ 3.

$/// \; P_e \leq 2Q\left(\frac{d}{2\sigma}\right)$ ← ↓ 2.

~~Exact Pe~~

~~$P_e = 4Q\left(\frac{d}{2\sigma}\right) - 4\left(\frac{d}{2\sigma}\right)^2$ ↖ ↗ ↘ ↙ 4~~

~~$P_e = 3Q\left(\frac{d}{2\sigma}\right) - 2\left(\frac{d}{2\sigma}\right)^2$ ↖ ↙~~

Exact Pe xx

always the same.

$|||| \; P_e = 4Q\left(\frac{d}{2\sigma}\right) - 4Q^2\left(\frac{d}{2\sigma}\right)$ ↗ ↖ ↘ ↙ 4

$\circledcirc \; P_e = 3Q\left(\frac{d}{2\sigma}\right) - 2Q^2\left(\frac{d}{2\sigma}\right)$ ↖ ↙ 2.

$/// \; P_e = 2Q\left(\frac{d}{2\sigma}\right) - Q^2\left(\frac{d}{2\sigma}\right)$ ↘ 1.

not the correct way to look at it see 2024 Exam → and other Paper.

Serial QAM upper Bound total Pe:

$$P_e \leq \frac{4}{16} \cdot 2Q\left(\frac{d}{2\sigma}\right) + \frac{8}{16} \cdot 3Q\left(\frac{d}{2\sigma}\right) + \frac{4}{16} \cdot 4Q\left(\frac{d}{2\sigma}\right) = \frac{48}{16} Q\left(\frac{d}{2\sigma}\right)$$

Think of it This way: $\boxed{\sum_{m \in M} \frac{1}{|M|} \sum_{m' \in M, m' \neq m} Q\left(\frac{d}{2\sigma}\right)}$

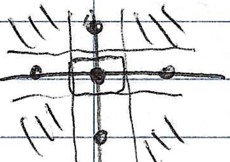
$$P_e = \frac{1}{16}\left(4\left(4Q\left(\frac{d}{2\sigma}\right)\right) + 4\left(2Q\left(\frac{d}{2\sigma}\right)\right) + 8\left(3Q\left(\frac{d}{2\sigma}\right)\right)\right)$$

~~4 neigh~~

4 symbols | that have 4 neighbours

4 symbols that have 2 neighbours

~~Exact Error Prob~~

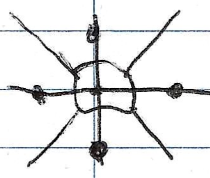


- Center exact Pe → 4 neighbours (4 useless neighbours ////)

↳ $4Q\left(\frac{d}{2\sigma}\right) - 4Q^2\left(\frac{d}{2\sigma}\right)$

dist for → ⊕

~~upper bound base on union bound~~



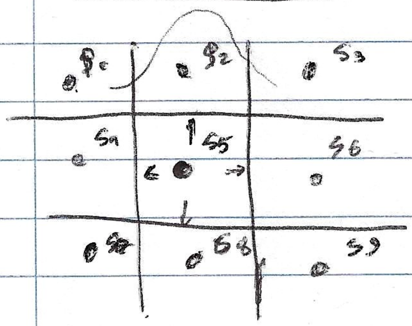
$$P_e = \frac{1}{16}\left(1\left(4Q\left(\frac{d_A}{2\sigma}\right)\right) + 4\left(1Q\left(\frac{d_B}{2\sigma}\right) + 2Q\left(\frac{d_C}{2\sigma}\right)\right)\right)$$

$\boxed{! \; B = A}$ ↳ $P_e = \frac{1}{16}\left(8Q\left(\frac{d_A}{2\sigma}\right) + 8Q\left(\frac{d_C}{2\sigma}\right)\right)$

union Bound of average error probability.

Exact Pe calculation

Exact Probability of Error



≥ horizontal ≥ vertical.

$S5 \rightarrow P_c = (1-2Q)(1-2Q)$

$P_c = 4Q^2 - 4Q + 1.$

$P_e = 1 - P_c = \cancel{4Q^2 + 4Q}$

$= 1 - (4Q^2 - 4Q + 1)$

$= 4Q - 4Q^2$

The distances are relative to up/down in this case both same, both $\sqrt{2}$. $Q(\sqrt{2})$