EINDHOVEN UNIVERSITY OF TECHNOLOGY

Communication Theory (5ETB0)
Module 9.1

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Module 9.1

Presentation Outline

Motivation and Objective

Motivation

- **Dimensionality Theorem**: $\approx 2W$ dimensions per second
- **Block-orthogonality signaling:**
  - Requires a lot of dimensions per second
  - Can give $P_{\mathrm{e}} \to 0$
- **Bit-by-bit signaling:**
  - Requires as many dimensions as there are bits to be transmitted
  - Can only made reliable ($P_{\mathrm{e}} \to 0$) by increasing the transmitter power P_s or by decreasing the rate R

Module Objective

Can we achieve reliable transmission ($P_{\mathrm{e}} \to 0$) at certain rate R by increasing T, when both the bandwidth W and available power P_s are fixed?

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Presentation Outline

Channel Capacity of AGN Vector Channel (1/2)

Capacity of AGN Vector Channel

For the AGN vectorial channel, there exist for N large enough, sets of $|\mathcal{M}|$ vectors, $\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{|\mathcal{M}|}$ where $\|\underline{s}_m\|^2 \approx NE_N$ for all $m = 1, 2, \ldots, |\mathcal{M}|$ and where $P_\mathrm{e} \approx 0$ as long as

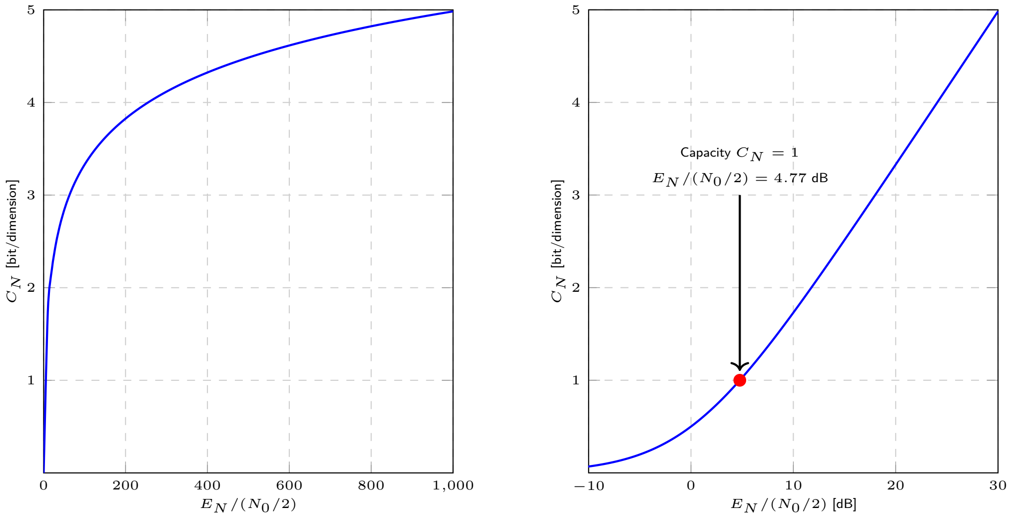
$$R_N = \frac{\log_2 |\mathcal{M}|}{N} < C_N \triangleq \frac{1}{2} \log_2 \left(\frac{E_N + N_0/2}{N_0/2} \right) \quad \left[\frac{\text{bit}}{\text{dimension}} \right]$$

where all vectors have length N, are equiprobable, and where E_N is the available energy per dimension.

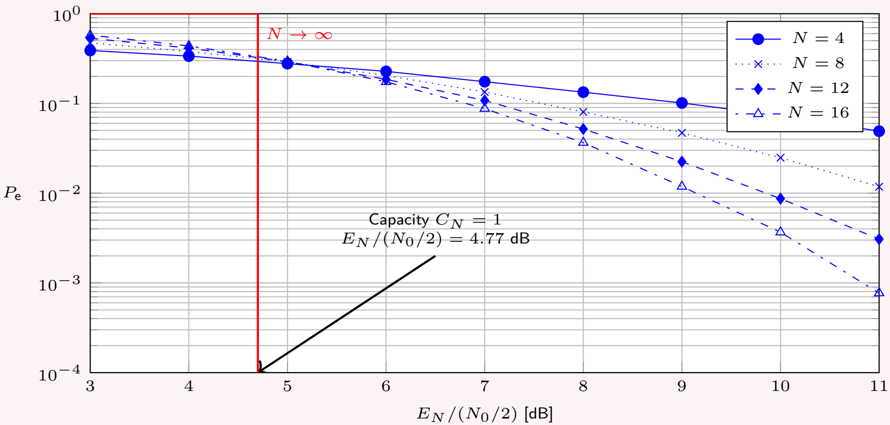
Comments

- Fundamental limit in terms of rate
- Elements in the vectors (codewords) are i.i.d. Gaussian random variables
- Vectors (codewords) are on the shell of a multidimensional sphere
- Converse: It can also be shown that P_e cannot be small if the rate per dimension $R_N > C_N$

Channel Capacity of AGN Vector Channel (2/2)



Example: Randomly Generated Codes



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Presentation Outline

Part I Motivation

Part II Channel Capacity of AGN Vector Channel

Part III Channel Capacity Baseband AWGN Channel

Channel Capacity of Baseband AWGN Channel

Capacity of Baseband AWGN Channel

For a waveform channel with spectral noise density $N_0/2$, frequency bandwidth W, and available transmitter power P_s, the capacity in bit per second is

$$C \triangleq W \log_2\left(1 + \frac{P_s}{W N_0}\right) \left[\frac{\text{bit}}{\text{second}}\right].$$

Thus reliable communication $(P_\mathrm{e} \to 0)$ is possible for rates R in bit per second smaller than C, while rates larger than C are not realizable with $P_\mathrm{e} \to 0$.

Questions and Comments

Q1: What is more benefitial: Bandwidth or power?
Answer: Linear growth with bandwidth. Use bandwidth when available
C1: The argument of the logarithm is $1 + \text{SNR}$
C2: We will study two extreme cases: $\text{SNR} \ll 1$ and $\text{SNR} \gg 1$

Vectorial AGN vs. Baseband AWGN Channels

Are they the same?

$$C = W \log_2\left(1 + \frac{P_s}{W N_0}\right) \left[\frac{\text{bit}}{\text{second}}\right], \text{ vs. } C_N = \frac{1}{2}\log_2\left(\frac{E_N + N_0/2}{N_0/2}\right) \left[\frac{\text{bit}}{\text{dimension}}\right]$$

$$\begin{aligned}\frac{C}{2W} &= \frac{1}{2}\log_2\left(1 + \frac{P_s}{W N_0}\right) \left[\frac{\text{bit}}{\text{dimension}}\right]\\ &= \frac{1}{2}\log_2\left(1 + \frac{E_s}{T W N_0}\right) \left[\frac{\text{bit}}{\text{dimension}}\right]\end{aligned}$$

But $E_N = E_s/N$ and $T \approx N/(2W)$, and thus,

$$\frac{E_s}{T W N_0} = \frac{N E_N}{N/2 N_0}$$

$$\frac{C}{2W} = \frac{1}{2}\log_2\left(1 + \frac{E_N}{N_0/2}\right) \left[\frac{\text{bit}}{\text{dimension}}\right] = \frac{1}{2}\log_2\left(\frac{N_0/2 + E_N}{N_0/2}\right) \left[\frac{\text{bit}}{\text{dimension}}\right]$$

Channel Capacity of the Wideband AWGN Channel

Capacity of the Wideband AWGN Channel (Power-Limited)

The capacity C_∞ of the **wideband AWGN channel** with power spectral density $N_0/2$, when the transmitter power is P_s, is given by

$$C_\infty = \frac{P_s}{N_0 \ln 2}$$

Derivation of Wideband AWGN Channel Capacity

$$\begin{aligned}
\lim_{W\to\infty} W \log_2\left(1 + \frac{P_s}{WN_0}\right) &= \lim_{W\to\infty} \frac{\log_2\left(1+\frac{P_s}{WN_0}\right)}{1/W} \\
&= \lim_{W\to\infty} \frac{P_s}{N_0} \frac{-1/W^2}{\left(1+\frac{P_s}{WN_0}\right)\ln(2)} \frac{1}{-1/W^2}
\end{aligned}$$

where we used $\frac{d}{dx}\log_2(x) = \frac{1}{x\ln(2)}$.

Relations between Capacities and SNR

Capacity of AWGN Channel at High-SNRs (Bandwidth-Limited)

The capacity of the AWGN is $C = W\log_2(1+\mathsf{SNR})$, where where $\mathsf{SNR} \triangleq P_s/(WN_0)$. When $\mathsf{SNR} \gg 1$, the capacity can be approximated as $W\log_2(\mathsf{SNR})$.

Power-limited and Bandwidth-limited Regimes

We can can distinguish between two cases.

$$C \approx \begin{cases} P_s/(N_0 \ln 2) & \text{if SNR} \ll 1, \\ W\log_2(\mathsf{SNR}) & \text{if SNR} \gg 1 \end{cases}$$

- The case $\mathsf{SNR} \ll 1$ is called the **power-limited** regime. There is enough bandwidth.
- When $\mathsf{SNR} \gg 1$ we speak about **bandwidth-limited** channels.

Summary Module 9.1

Take Home Messages

- Capacity of the AGN vector channel
- Capacity of the baseband AWGN channel
- Power-limited and bandwidth-limited regimes
- Performance of random codes

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