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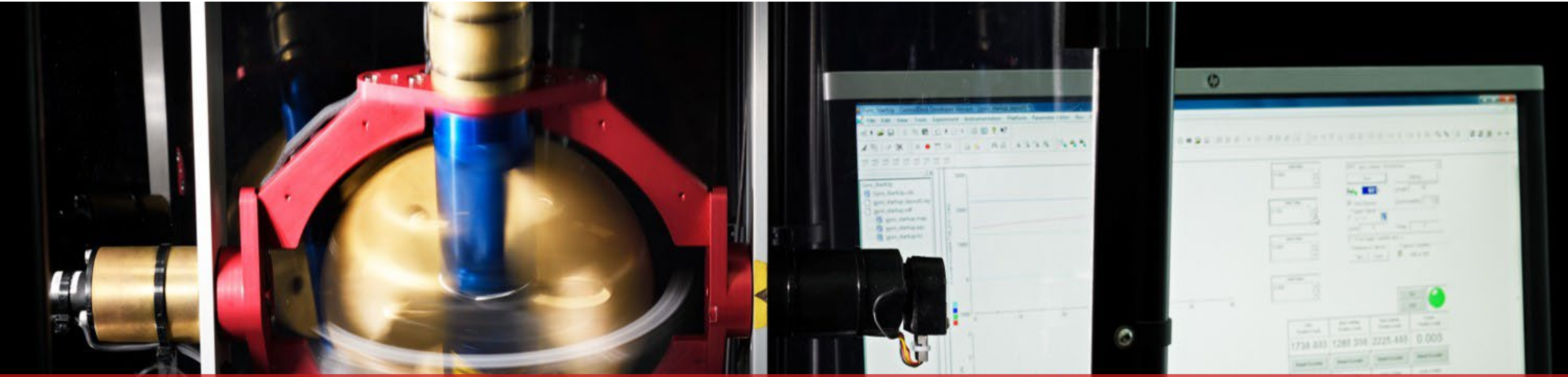


# Control Systems (5ESD0)
# Lecture 3: Frequency Response Functions and Bode Diagrams

Tijs Donkers

# Today's Menu

**Learning objective: Understand the relevance of frequency-domain control-design methods and represent a frequency-response function using a Bode plot**

- Part 1: Overview of the frequency-response design method

- Part 2: Frequency response functions
  - Representing frequency response functions using Bode diagrams
  - Examples on sketching Bode diagrams

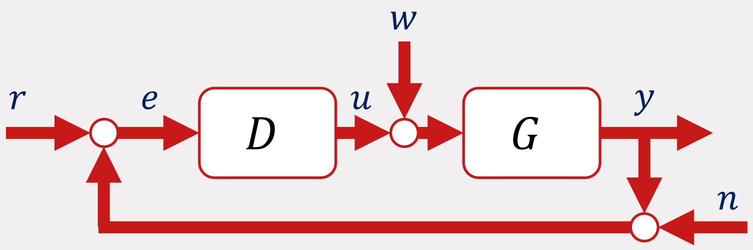
- Part 3: Quiz on Bode Diagrams

**Deadline registration for lab groups is Friday 13 Sept EOB**

# Part 1: Overview Frequency-Response Design Method

# Overview Frequency-Response Design Method

Main objective: **Design** controllers/compensators for closed-loop system




Using the open-loop frequency response function $D(j\omega)G(j\omega)$

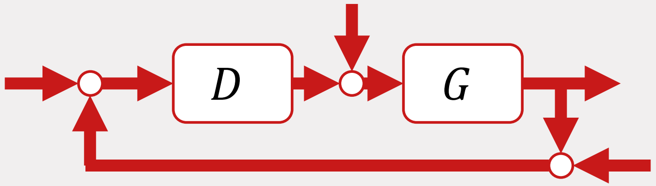
Map properties from open-loop $DG$ to closed-loop $\frac{DG}{1+DG}$

- Stability: Nyquist plot (module 4)
- Steady-state errors (module 5)
- Map time domain requirements of closed-loop to open-loop frequency response (module 5)
- Fundamental limitations (module 6)

# Overview Frequency-Response Design Method

Prerequisite for doing controller/compensator design:

- We specify what $DG$ should look like
- We (approximately) know $G$
- To design $D$…




This is works if we can easily manipulate the frequency response functions (FRF):

- The Bode diagram is a convenient representation of the frequency response
- Composition of FRFs: addition of log curves in the frequency domain

# Part 2: Frequency Response Functions (Section 6.1)

# Frequency Response Functions (Section 3.1 and 6.1)

- A sinusoidal input $u(t) = Ae^{j\omega t}$ applied to an LTI system leads (in steady-state) to a sinusoidal output with the same frequency:

$$y(t) = \int_0^t h(\tau)u(t-\tau)\mathrm{d}\tau$$

Steady-state (i.e., $t$ large):

$$= \int_0^\infty h(\tau)Ae^{j\omega(t-\tau)}\mathrm{d}\tau = \underbrace{\int_0^\infty h(\tau)Ae^{-j\omega\tau}\mathrm{d}\tau}_{H(j\omega)}\ \underbrace{Ae^{j\omega t}}_{u(t)} = H(j\omega)u(t)$$

- The Frequency Response Function (FRF) can be obtained from the transfer function by evaluating at $s = j\omega$



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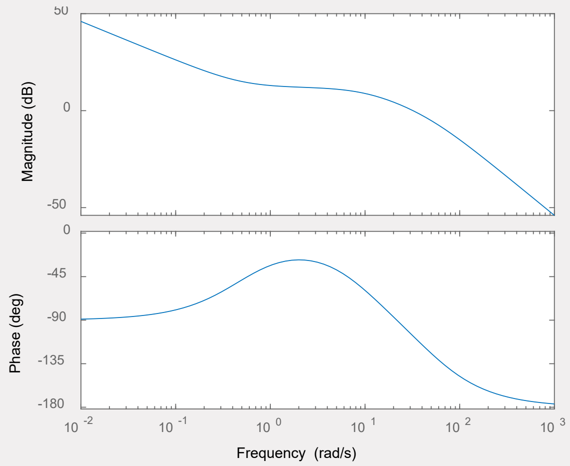
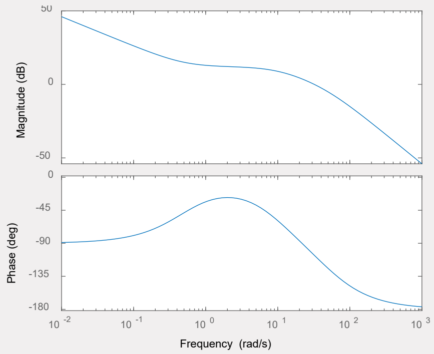
# Bode Diagrams

- The function $H(j\omega)$ produces a complex number for every (real) $\omega$
  - This complex number can be represented by a magnitude and an angle
  - Interpretation: amplification and phase shift

- Bode diagram representation of FRF
  - Magnitude/gain: $|H(j\omega)| = \sqrt{Re\big(H(j\omega)\big)^2 + Im\big(H(j\omega)\big)^2}$
  - dB is the $20\log_{10}|H(j\omega)|$
  - Phase: $\angle H(j\omega) = \arctan\left(\frac{Im(H(j\omega))}{Re(H(j\omega))}\right)$



# Why are Bode diagrams convenient?




Important takeaways about sketching Bode diagrams:

- Bode form of FRF: $KG(j\omega) = K_0\,(j\omega)^n \dfrac{(j\omega\tau_1+1)(j\omega\tau_2+1)\dots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\dots}$

- Composition is adding the magnitude plot and the phase plot, because

$$\log|KG(j\omega)| = \log K_0 + n\log|j\omega| + \log|j\omega\tau_1+1| - \log|j\omega\tau_a+1| + \cdots$$

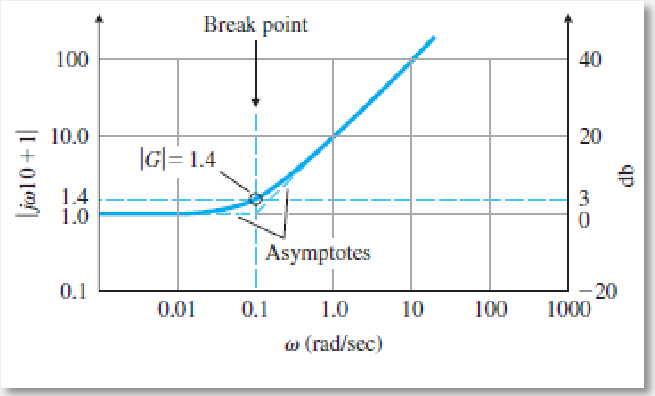
$$\angle KG(j\omega) = n\angle j\omega + \angle(j\omega\tau_1+1) - \angle(j\omega\tau_a+1) + \cdots$$

- Allows for sketching Bode diagrams by considering
  - Class 1: $K_0\,(j\omega)^n$
  - Class 2: $(j\omega\tau+1)^{\pm 1}$
  - Class 3: $\left(\left(\dfrac{j\omega}{\omega_n}\right)^2 + 2\zeta\dfrac{j\omega}{\omega_n} + 1\right)^{\pm 1}$

# Frequency Response Functions

Class 1: $K_0\ (j\omega)^n$

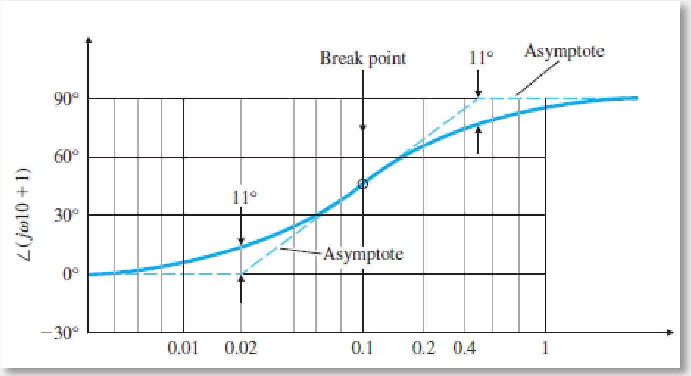
- $20\log|K_0\ (j\omega)^n| = 20\log K_0 + 20n\log|\omega|$
- $\angle K_0\ (j\omega)^n = \angle K_0 + \angle(j\omega)^n = \angle K_0 + n\cdot 90^0$




Class 2: $(j\omega\tau + 1)^{\pm 1}$

- $|j\omega\tau + 1| \approx \begin{cases} 1\ (0\mathrm{dB}) & \text{for } \omega\tau \ll 1 \\ \sqrt{2}\ (3\mathrm{dB}) & \text{for } \omega\tau = 1 \\ |\omega\tau| & \text{for } \omega\tau \gg 1 \end{cases}$

- $\angle(j\omega\tau + 1) \approx \begin{cases} 0^o & \text{for } \omega\tau \ll 1 \\ 45^o & \text{for } \omega\tau = 1 \\ 90^o & \text{for } \omega\tau \gg 1 \end{cases}$

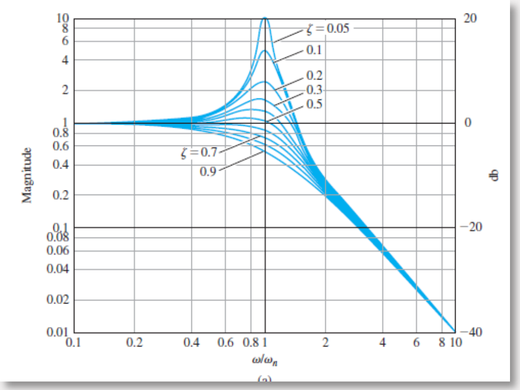


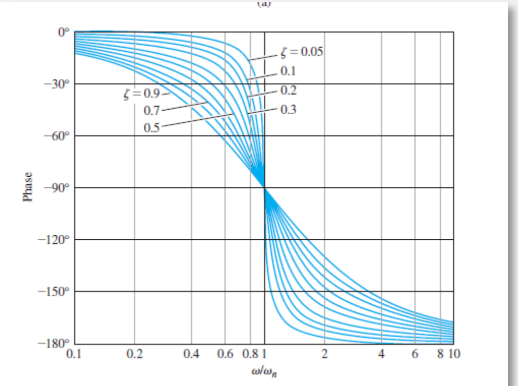
# Frequency Response Functions

Class 3: $\left(\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right)^{\pm 1}$

- $$\left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right| \approx \begin{cases} 1\ (0\text{dB}) & \text{for } \omega \ll \omega_n \\ 2\zeta & \text{for } \omega = \omega_n \\ \left|\frac{\omega}{\omega_n}\right|^2 & \text{for } \omega \gg \omega_n \end{cases}$$

- $$\angle\left(\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right) \approx \begin{cases} 0^o & \text{for } \omega \ll \omega_n \\ 90^o & \text{for } \omega = \omega_n \\ 180^o & \text{for } \omega \gg \omega_n \end{cases}$$



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# Frequency Response Functions

Important consideration about sketching Bode diagrams:

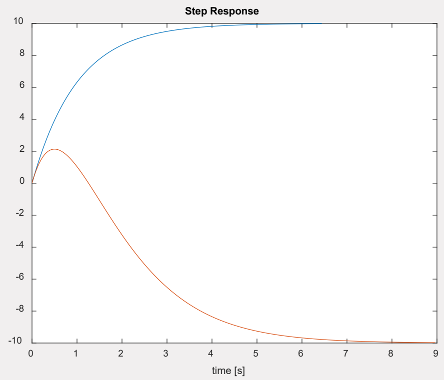
- Asymptote $\log\left|\frac{1}{1+j\omega\tau_a}\right| \approx \log\left|\frac{1}{j\omega\tau_a}\right| = -\log|j\omega\tau_a| = -\log(\omega\tau_a) = -\log(\omega) - \log(\tau_a)$
  - Linear function with slope -1 on log-log scale: 20dB per decade!

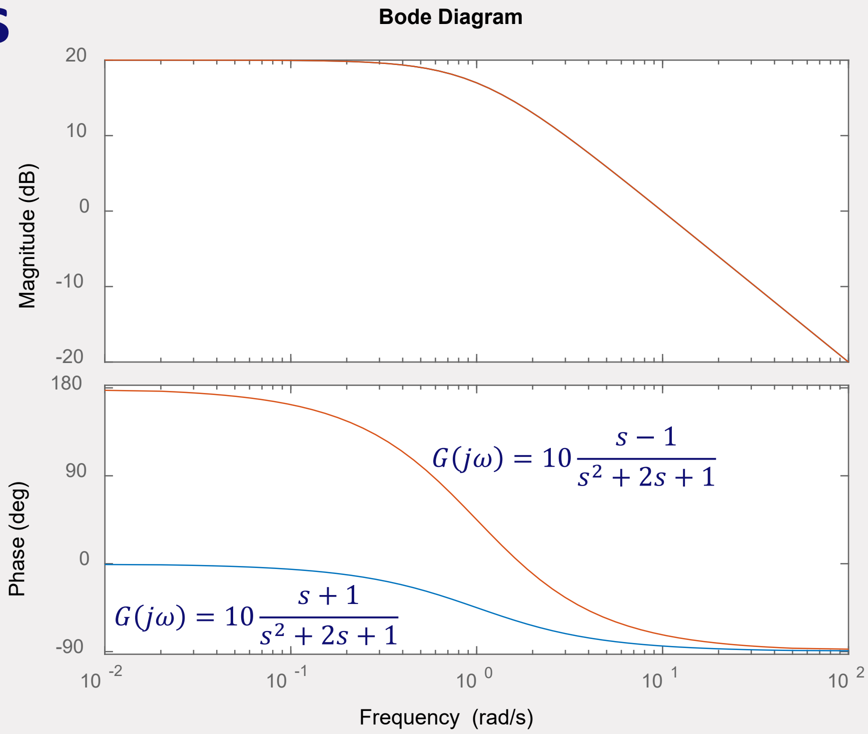
- Bode phase-gain relation (section 6.5 of book) for stable minimum-phase systems
  - If asymptote of gain has slope $n$ ($20 \times n$ dB per decade), phase has asymptote $\mathrm{n} \times 90°$

# Frequency Response Functions

For systems with right half plane zero (and poles):

- The gain rules still apply
- The phase change is larger than for systems with poles and zeros in the LHP:
  - Non-minimum phase systems
  - (Notice the step response: Module 6)



# Summary Bode Sketching

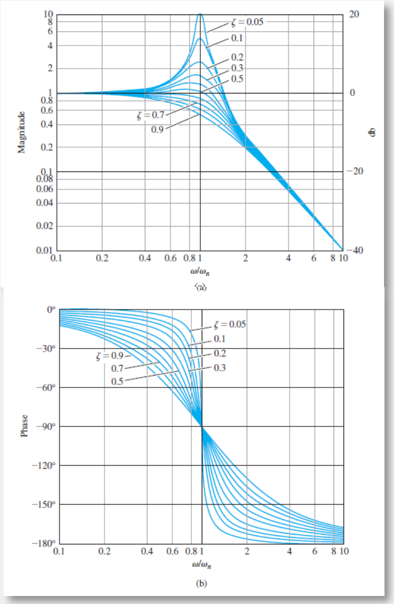
1. Write transfer function in Bode form $KG(j\omega) = K_0\,(j\omega)^n \dfrac{(j\omega\tau_1+1)(j\omega\tau_2+1)\ldots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\ldots}$

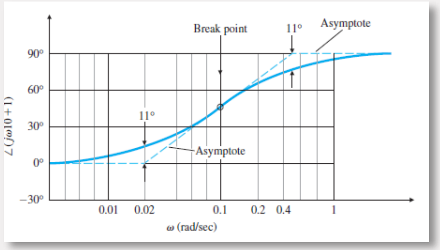
For the magnitude plot:

2. Draw the asymptote though $K_0$ at $\omega = 1$ with slope $n$
3. Compute break points $\omega = 1/\tau_i$ and order them for small to large. Start at low frequencies and change the slope with $\pm n$ for $n^{\text{th}}$-order terms ($n = 1{,}2{,}\ldots$ poles: $-$, zeroes: $+$)
4. At the break points,
   - For 1st-order terms, the magnitude is 3db above/below the asymptote
   - For 2nd-order terms, the magnitude $\approx \pm 20\log_{10}(2\zeta)$ in dB (see also the graph)

For the phase plot:

5. Start low-frequency phase a $n \times 90°$
6. The asymptotic phase is according to Bode-phase gain relation $n \times 90°$
7. For first-order terms: create additional points at 5 times and 1/5 times the break point
8. Break points far apart: the phase reaches the asymptotes. Break points close: they don't



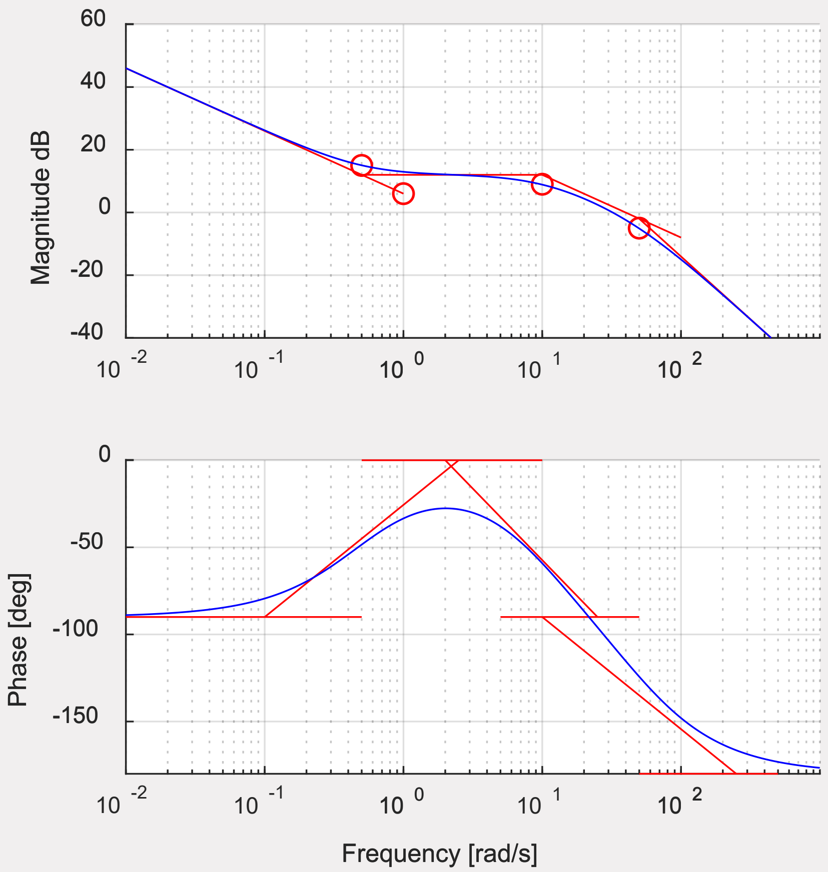



# Example on Sketching

1. Transfer function $KG(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$

   Bode form: $KG(j\omega) = 2(j\omega)^{-1} \frac{\left(\frac{j\omega}{0.5}+1\right)}{\left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}$

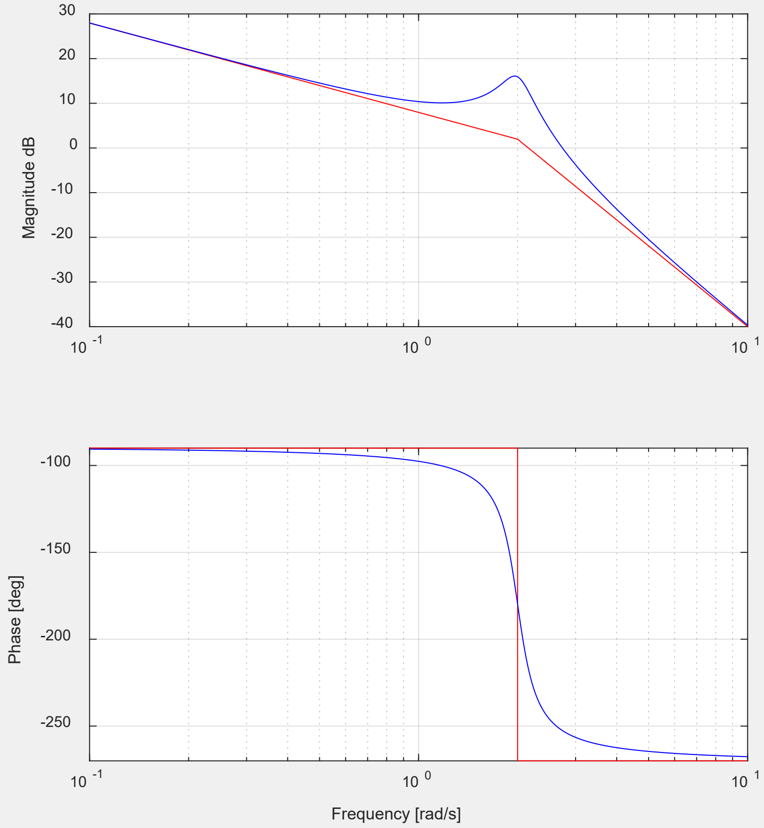
2. Slope $n = -1$ and $|KG(j)| = 2$ which is 6dB
3. Break points at 0.5 (zero), 10 (pole) and 50 (pole). Slopes increase with zeros, decrease with poles
4. Magnitude is 3db above/below at the break point
5. Phase is $-90°$ at low frequencies
6. Phase is $n \times 90°$
7. Compute extra points
8. The phase reaches the asympt $\omega \to 0$ and $\omega \to \infty$



# Example on Sketching

1. Transfer function $KG(s) = \frac{10}{s(s^2+0.4s+4)}$

   Bode form: $KG(j\omega) = \frac{10}{4}(j\omega)^{-1}\frac{1}{\left(\frac{j\omega}{2}\right)^2+2\cdot 0.1\cdot\frac{j\omega}{2}+1}$
2. Slope $n = -1$ and $|KG(j)| = 2.5$ which about 8dB
3. Break point at 2 rad/sec: slope goes to -3
4. Magnitude is $\frac{1}{2\zeta} = 5$ (=14dB) above asymptote (one octave -6db, so asymptote is at 2dB)
5. Phase is $-90°$ at low frequencies
6. Phase goes to $-270°$
7. Sketch transition (limited accuracy can be reached)
8. The phase reaches the asympt $\omega \rightarrow 0$ and $\omega \rightarrow \infty$



# Part 3: Quiz

# Quiz – Question 1/4

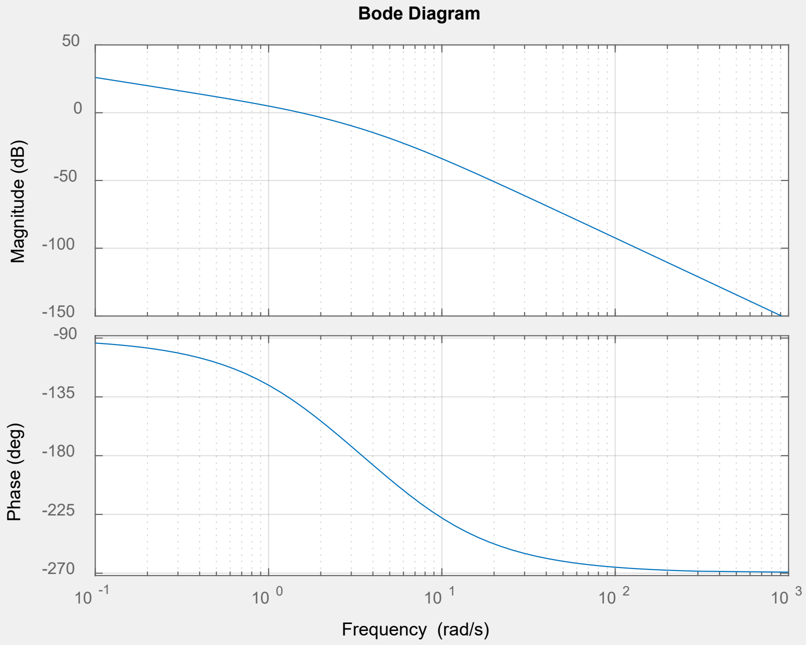
What is the transfer function $G(s)$ represented by the Bode plot below?

a) $G(s) = \dfrac{24}{s(s+2)(s+6)}$

b) $G(s) = \dfrac{24}{s^2(s+6)}$

c) $G(s) = \dfrac{24}{s(s+2)}$

d) $G(s) = \dfrac{24}{(s+2)^2+6}$



# Quiz – Question 2/4

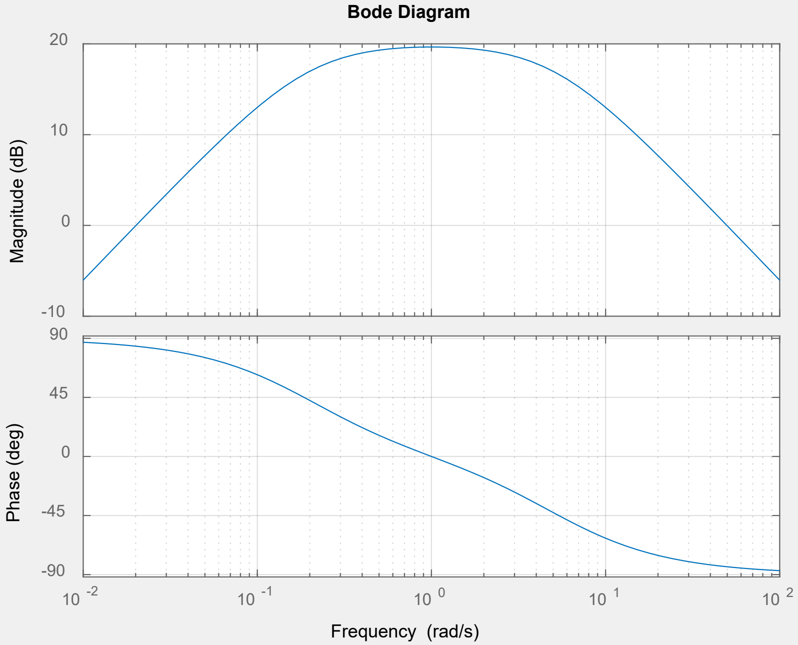
What is the transfer function $G(s)$ represented by the Bode plot below?

a) $G(s) = \dfrac{20}{s(s+0.2)(s+5)}$

b) $G(s) = \dfrac{20s}{(s+0.2)(s+5)}$

c) $G(s) = \dfrac{50}{(s+0.2)(s+5)}$

d) $G(s) = \dfrac{50s}{(s+0.2)(s+5)}$



# Quiz – Question 3/4

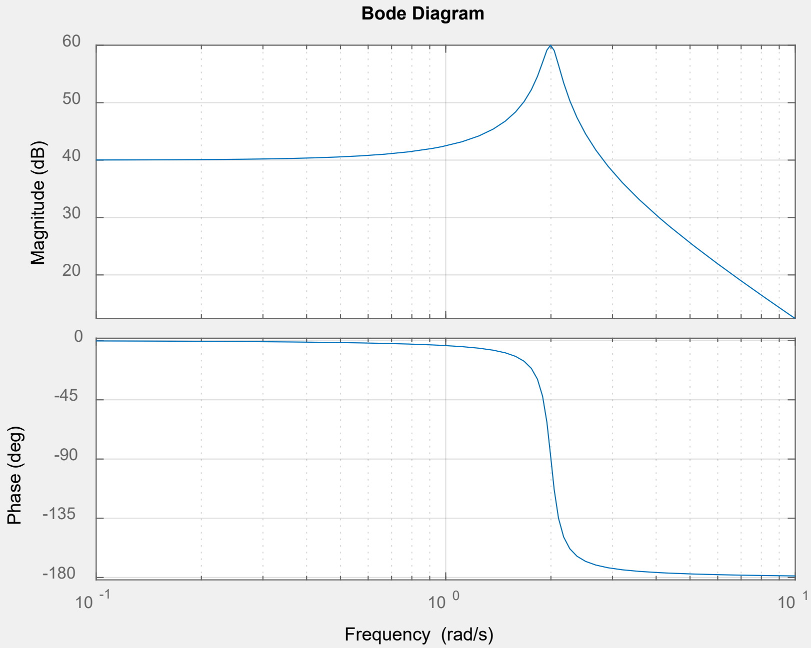
What is the transfer function $G(s)$ represented by the Bode plot below?

a) $G(s) = \dfrac{900}{(s+3)(s-3)}$

b) $G(s) = \dfrac{400}{(s+2)^2}$

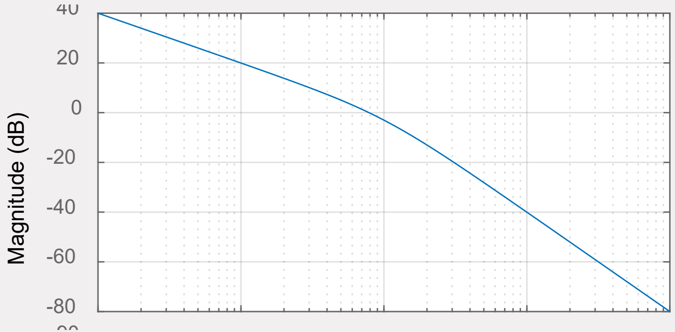
c) $G(s) = \dfrac{900}{s^2+0.2s+9}$

d) $G(s) = \dfrac{400}{s^2+0.2s+4}$

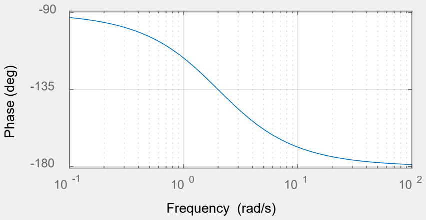


# Quiz – Question 4/4

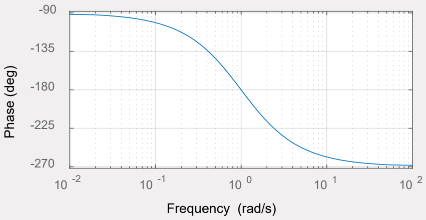
Given this magnitude plot of a stable minimum-phase plant:




What is the corresponding phase plot?




a)

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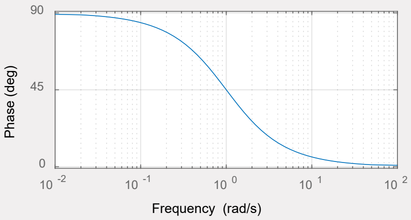
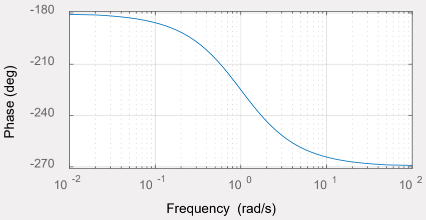



c)




d)

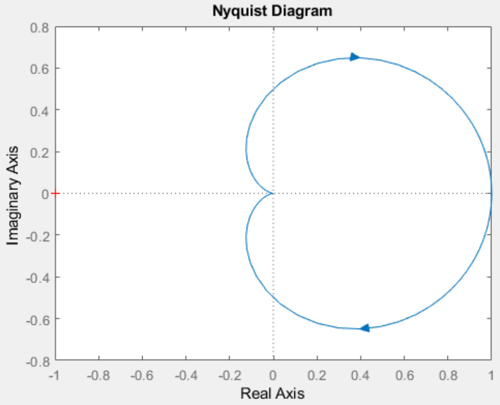
# Wrap up

**Learning objective: Understand the relevance of frequency-domain control-design methods and represent a frequency-response function using a Bode plot**

- Part 1: Overview of the frequency-response design method
  - Design closed-loop controller using open-loop FRF
  - Stability/time-domain performance maps to open-loop FRF (module 5)
  - Bode plots are a graphical tool used for design using FRF
- Part 2: Bode diagrams
  - Sketching Bode diagrams creates insight in system's frequency response
  - Procedure based on decomposition into three classes of functions
- Part 3: Quiz on Bode Diagrams

# Suggested Reading, Exercises and Preview

- (Re)read Section 6.1 and Section 6.5 of the textbook
- Practice with Exercise Set 3
- Watch video material / read book of Module 4 before Friday
  - Nyquist stability criterion




- Preview:
  - Module 4: Stability using the open-loop FRF
  - Module 5: Translate design requirements to desired frequency-frequency
  - Module 5: Shape the loop gain DG for a given G using D
  - Module 6: Fundamental limitations