Question 3.2

Free space: $k = k_0$, $Z = Z_0 = Y^{-1}$

[redacted] $V_s(0) = V_s^+(0) + V_s^-(0)$ [redacted] $\Rightarrow V_s(z) = V_s^+(0)e^{-jkz} + V_s^-(0)e^{jkz}$

$$I_s(z) = Y\left(V_s^+(0)e^{-jkz} - V_s^-(0)e^{jkz}\right)$$

a.) $V_s^+(0) = 1$, $V_s^-(0) = \Gamma = -\frac{1}{2} \Rightarrow V_s(z) = e^{-jkz} - \frac{1}{2}e^{jkz} = e^{-jkz}\left(1 - \frac{1}{2}e^{2jkz}\right)$

$\Rightarrow |V_s(z)| = \left|1 - \frac{1}{2}e^{2jkz}\right| \in \left[\frac{1}{2}, \frac{3}{2}\right] \Rightarrow VSWR = 3 = \frac{3}{2} \Big/ \frac{1}{2}$

c.) $$|V_s(z)|^2 = \left(e^{-jkz} + \Gamma e^{jkz}\right)\left(e^{jkz} + \Gamma^* e^{-jkz}\right) = 1 + |\Gamma|^2 + \underbrace{(\Gamma e^{-2jkz}) + (\Gamma e^{2jkz})^*}_{\text{oscillates between } -2|\Gamma| \text{ and } 2|\Gamma|}$$

$|V_s(z)|^2$ oscillates between

$1 + |\Gamma|^2 - 2|\Gamma| = (1 - |\Gamma|)^2$ and $1 + |\Gamma|^2 + 2|\Gamma| = (1 + |\Gamma|)^2$

$$\Rightarrow \frac{\max|V_s|}{\min|V_s|} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = VSWR$$

b.) $|\Gamma| = \sqrt{85/765} = \frac{1}{3} \Rightarrow VSWR = \frac{\frac{4}{3}}{\frac{2}{3}} = 2$

d.) $$Z_{eff}(z) = \frac{V_s(z)}{I_s(z)} = \frac{V_s^+(0)\left(e^{-jkz} + \Gamma e^{jkz}\right)}{V_s^+(0)\,Y\left(e^{-jkz} - \Gamma e^{jkz}\right)} = Z\,\frac{1 + \Gamma e^{2jkz}}{1 - \Gamma e^{2jkz}}$$

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Question 3.2, the sequel

e.) $V^+(t-\frac{z}{c}) = \frac{1}{2}(V+ZI)$ and $V^-(t+\frac{z}{c}) = \frac{1}{2}(V-ZI)$ can be computed. However $Z_{eff}(z)$ is only meaningful for time-harmonic signals.

f.) If the minus sign had not been present in Eq.(1.29), then we would have $Z_{eff} = Z - Z_0 \;\; \forall z$ (for all), which would render it impossible to realise any of the one-port circuit components except for a circuit element with impedance "Z"

Question 3.3 $\langle \vec{S} \rangle_T = \frac{1}{2}\mathrm{Re}(\vec{S}_s) = \frac{1}{2}\mathrm{Re}(V_s I_s^*)\vec{a}_z$

$\underbrace{\text{a) \& b)} \Rightarrow V_s = V_s^+ + V_s^-,\ V_s^+ = 1,\ V_s^- = \Gamma}$ $\quad$ c) & d) $\Rightarrow V_s^+ = 1+\Gamma,\ V_s^- = 0$

$\langle \vec{S} \rangle_T = \frac{1}{2}\mathrm{Re}\left[Y_0(1+\Gamma)(1-\Gamma^*)\right] = \frac{Y_0}{2}(1-|\Gamma|^2)\vec{a}_z$; $\langle \vec{S} \rangle_T = \frac{Y_0}{2}(1+|\Gamma|^2+2\mathrm{Re}(\Gamma))\vec{a}_z$

↳ free space $\quad$ Because $\Gamma - \Gamma^*$ is imaginary

a.) $\frac{Y_0}{2}(1-\frac{1}{4})\vec{a}_z = \frac{3}{8Z_0}\vec{a}_z$ $\qquad$ b.) $\frac{1}{2Z_0}\left(1-\frac{(7+6j)(7-6j)}{(27+6j)(27-6j)}\right) = \frac{4}{9Z_0}$

c.) $\frac{1}{2Z_0}|1-\frac{1}{2}|^2\vec{a}_z = \frac{1}{8Z_0}$ $\qquad$ d.) $\frac{5}{9Z_0} - \frac{5}{17Z_0} = \frac{40}{153Z_0}$

In a.) & b.), the time average of the power of the sum of the two fields is equal to the sum of the time averages of the powers of the two fields, because the two fields are counter-propagating waves, which do not interact. In c.) & d.) this is not the case. ($|1-\frac{1}{2}|^2 = \frac{1}{4} \neq 1+(\frac{1}{2})^2$)

Question 3.4

Let $\cos k(z_2-z_1) = c$; $\sin k(z_2-z_1) = s$

$\Rightarrow \cos k(z_1-z_2) = c$; $\sin k(z_1-z_2) = -s$

$\Rightarrow c^2+s^2 = 1$

$$\Rightarrow T(z_2,z_1)T(z_1,z_2) = \begin{pmatrix} c & -jZs \\ -jYs & c \end{pmatrix}\begin{pmatrix} c & jZs \\ jYs & c \end{pmatrix} = \begin{pmatrix} c^2+s^2 & jZ(cs-sc) \\ jY(-sc+cs) & c^2+s^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Question 3.5

$$\frac{d\vec{v}}{dt} = \lim_{\Delta t\to 0} \frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}$$

$$\vec{v}(t+\Delta t) = \frac{q}{m}\int_{t'=t_0}^{t+\Delta t} e^{-\nu(t+\Delta t-t')}\vec{E}(\vec{r}(t'),t')\,dt'$$

Use $\int_{t_0}^{t+\Delta t} = \int_{t_0}^{t} + \int_{t}^{t+\Delta t}$ and $e^{-\nu(t+\Delta t-t')} = e^{-\nu(t-t')}e^{-\nu\Delta t}$

$$\Rightarrow \frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t} = \frac{q}{m}\int_{t_0}\underbrace{\left(\frac{e^{-\nu\Delta t}-1}{\Delta t}\right)}_{-\nu+O(\Delta t)} e^{-\nu(t-t')}\vec{E}\,dt'$$

$$+\frac{q}{m}\boxed{\frac{1}{\Delta t}\int_{t'=t}^{t+\Delta t}} \underbrace{e^{-\nu(t-t')}}_{1+O(t-t')}\underbrace{e^{-\nu\Delta t}}_{(1-\nu\Delta t)=1+O(\Delta t)}\vec{E}\,dt' = (-\nu)\vec{v}(t)+\frac{q}{m}\vec{E}$$

$$\Rightarrow \left(\frac{d}{dt}+\nu\right)\vec{v} = \frac{q}{m}\vec{E}$$

Question 3.6

$$\vec{J} = \rho\vec{v} = qN_e\vec{v} = \frac{q^2N_e(\vec{r},t)}{m}\int_{t'=t_0=-\infty}^{t} e^{-\nu(t-t')}\vec{E}(\vec{r}(t'),t')\,dt'$$

(take into account all of history).

Let $\tau = t-t'$ $\quad d\tau = -dt'$

$$\Rightarrow \vec{J} = \frac{q^2N_e(\vec{r},t)}{m}\int_{\tau=\infty}^{0} e^{-\nu\tau}\underbrace{\vec{E}(\vec{r}(t-\tau),t-\tau)}_{\vec{E}(t-\tau)}(-d\tau)$$

$$= \int_{\tau=0}^{\infty}\underbrace{\frac{q^2N_e(\vec{r},t)e^{-\nu\tau}}{m}}_{\sigma(\vec{r},\tau)}\vec{E}(t-\tau)\,d\tau$$

Question 3.7

Collision process very fast $\Rightarrow e^{-\nu\tau}$ decays so rapidly that $\vec{E}$ may be considered constant for $\tau\in[0,\tau_{decay}]$ where τ_{decay} is of the order of $\frac{1}{\nu}$ (or several times $\frac{1}{\nu}$)

$$\Rightarrow \int_{\tau=0}^{\infty}\cdots d\tau \approx \vec{E}(t)\int_{\tau=0}^{\infty}\frac{q^2N_e}{m}e^{-\nu\tau}d\tau = \vec{E}(t)\underbrace{\frac{q^2N_e}{m\nu}}_{\sigma^0(\vec{r})}$$

Electromagnetics II

Question 3.8

a.) $T(0,\ell)=\begin{pmatrix} c & jsZ\\ jsY & c\end{pmatrix}$ $\quad T(0,-\ell)=\begin{pmatrix} c & -jsZ\\ -jsY & c\end{pmatrix}$

Boundary condition at $z=\ell$

b.) $\lim\limits_{z\downarrow 0}\begin{pmatrix}V_s(z)\\ I_s(z)\end{pmatrix} = T(0,\ell)\begin{pmatrix}V_s(\ell)\\ I_s(\ell)\end{pmatrix} = \begin{pmatrix}c & jsZ\\ jsY & c\end{pmatrix}\begin{pmatrix}0\\ I_\ell\end{pmatrix} = \begin{pmatrix}jsZ\\ c\end{pmatrix} I_\ell$

c.) $\lim\limits_{z\uparrow 0}\begin{pmatrix}V_s(z)\\ I_s(z)\end{pmatrix} = T(0,\ell)\begin{pmatrix}0\\ I_{-\ell}\end{pmatrix} = \begin{pmatrix}-jsZ\\ c\end{pmatrix} I_{-\ell}$

d.) Across $z=0$, we have $\vec{a}_z\times\left(\vec{E}_s|_{z\downarrow 0}-\vec{E}_s|_{z\uparrow 0}\right)=\vec{0} \Rightarrow V_s(0^+)=V_s(0^-)$

$\Rightarrow jsZ\,I_\ell = -jsZ\,I_{-\ell} \Rightarrow I_{-\ell} = -I_\ell$

e.) $\vec{a}_z\times\left(\vec{H}_s|_{z\downarrow 0}-\vec{H}_s|_{z\uparrow 0}\right) = -\vec{a}_x \underbrace{I_s|_{z\downarrow 0}}_{cI_\ell} + \vec{a}_x \underbrace{I_s|_{z\uparrow 0}}_{cI_{-\ell}} = \vec{J}_{sS}$

$\Rightarrow \vec{J}_{sS} = -2c\,I_\ell\,\vec{a}_x \quad \left(I_\ell = -\frac{1}{2c}\,\vec{a}_x\cdot\vec{J}_{sS}\right)$

f.) $2k\ell = \pi + 2n\pi \Rightarrow k\ell = \frac{\pi}{2} + n\pi \Rightarrow \begin{cases} c=\cos(k\ell)=0\\ s=\sin(k\ell)=(-1)^n\\ \vec{J}_{sS}=\vec{0}\end{cases}$

g.) $k\ell=\frac{\pi}{2}$