Communication Theory (5ETB0)
Module 4.2

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Module 4.2

Presentation Outline

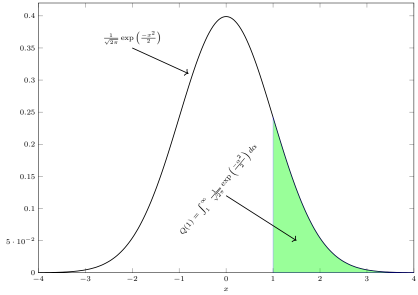
The Q-function

The Q-function

Is a function of $x \in (-\infty, \infty)$:

$$Q(x) \triangleq \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\alpha^2}{2}\right) d\alpha$$

It is the probability that a Gaussian random variable with mean 0 and variance 1 takes a value larger than x.



Useful Property and Three Questions

$$Q(x) = 1 - Q(-x) \implies Q(x) + Q(-x) = 1$$

$$\text{Q1: } Q(0) = ? \quad \text{Q2: } Q(-\infty) = ? \quad \text{and} \quad \text{Q3: } Q(+\infty) = ? \tag{1}$$

Error Probability for bi-AGN Channel

Error Probability for bi-AGN Channel

The error probability of a scalar bi-AGN can be expressed as:

$$
\begin{aligned}
P_{\mathsf{e}} &= \sum_{m \in \mathcal{M}} \Pr\{\hat{M} \neq M | M = m\} \Pr\{M = m\} \\
&= \Pr\{M = 1\} \Pr\{R < r^* | M = 1\} \\
&\qquad + \Pr\{M = 2\} \Pr\{R \geq r^* | M = 2\} \\
&= \Pr\{M = 1\} Q\left(\frac{s_1 - r^*}{\sigma}\right) + \Pr\{M = 2\} Q\left(\frac{r^* - s_2}{\sigma}\right)
\end{aligned}
$$

Error Probability: Derivation

Error Probability for bi-AGN Channel end Equally Likely Messages

Error Probability for Equally Likely Messages

In this case the optimum values is $r^* = 0.5(s_1 + s_2)$ (ML rule), which gives

$$\begin{aligned} P_{\mathrm{e}} &= \frac{1}{2}\Pr\{R < r^* | M = 1\} + \frac{1}{2}\Pr\{R \geq r^* | M = 2\} \\ &= \frac{1}{2}Q\left(\frac{s_1 - r^*}{\sigma}\right) + \frac{1}{2}Q\left(\frac{r^* - s_2}{\sigma}\right) \\ &= \frac{1}{2}Q\left(\frac{s_1 - s_2}{2\sigma}\right) + \frac{1}{2}Q\left(\frac{s_1 - s_2}{2\sigma}\right) \\ &= Q\left(\frac{s_1 - s_2}{2\sigma}\right). \end{aligned}$$

Two Questions

Assume the messages are not equally likely:

- Q1: Can we still use $r^* = 0.5(s_1 + s_2)$?
- Q2: Will the error probability above change?

Example: MAP vs. ML for bi-AGN

Example in Module 4.1

For $s_1 = 1$, $s_2 = -1$, and $\sigma^2 = 1$ with $\Pr\{M = 1\} = 3/4$ and $\Pr\{M = 2\} = 1/4$, the minimum probability of error (achieved by MAP) becomes:

$$
\begin{aligned}
P_{\mathsf{e}} &= \frac{3}{4} Q\left(1 + \frac{\ln 3}{2}\right) + \frac{1}{4} Q\left(-\frac{\ln 3}{2} + 1\right) \\
&\approx 0.75 \cdot Q(1.5493) + 0.25 \cdot Q(0.4507) \\
&\approx 0.75 \cdot 0.0607 + 0.25 \cdot 0.3261 \\
&\approx 0.1270.
\end{aligned}
$$

The ML rule would give

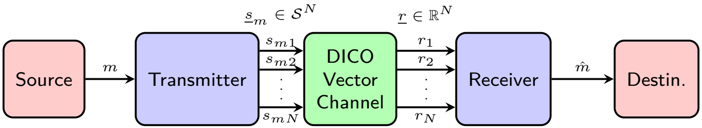
$$
P_{\mathsf{e}} = Q\left(\frac{2}{2}\right) \approx 0.1587
$$

Does this make sense?

Module 4.2

Presentation Outline

Vector Channels



Definitions

- Source: Produces a *message* $m \in \mathcal{M} \triangleq \{1, 2, \ldots, |\mathcal{M}|\}$ with probability $\Pr\{M = m\}$ for $m \in \mathcal{M}$. The r.v. is M
- Transmitter: Sends a *signal* $\underline{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN}) \in \mathcal{S}^N$ if message m is to be transmitted. The random vector is $\underline{S}$
- DICO Vector Channel: Produces output $\underline{r} \in \mathbb{R}^N$ (random vector is $\underline{R}$) with probability density function $p_{\underline{R}}(\underline{r}|\underline{S} = \underline{s}_m) = p_{\underline{R}}(\underline{r}|M = m)$
- Receiver: Forms an *estimate* $\hat{m}$ by observing the received channel output $\underline{r} \in \mathbb{R}^N$ using a mapping $\hat{m} = f(\underline{r}) \in \mathcal{M}$. The r.v. is $\hat{M}$

Decision Variables, MAP and ML

Decision Variables for DICO Vector Channel

The **decision variables** are
$\Pr\{M = m, \underline{R} = \underline{r}\} = \Pr\{M = m\} p_{\underline{R}}(\underline{r}|\underline{S} = \underline{s}_m) = \Pr\{M = m\} p_{\underline{R}}(\underline{r}|M = m)$.

MAP decision rule

The maximum a-posteriori probability (MAP) decision rule is

$$\hat{m}^{\mathsf{MAP}}(\underline{r}) \stackrel{\Delta}{=} \operatorname*{argmax}_{m \in \mathcal{M}} \Pr\{M = m | \underline{R} = \underline{r}\} \tag{2}$$

$$= \operatorname*{argmax}_{m \in \mathcal{M}} \Pr\{M = m\} p_{\underline{R}}(\underline{r}|M = m). \tag{3}$$

ML decision rule

The maximum likelihood (ML) decision rule is

$$\hat{m}^{\mathsf{ML}}(\underline{r}) \stackrel{\Delta}{=} \operatorname*{argmax}_{m \in \mathcal{M}} p_{\underline{R}}(\underline{r}|M = m) \tag{4}$$

Module 4.2

Presentation Outline

Decision Regions

Decision region for vector channel

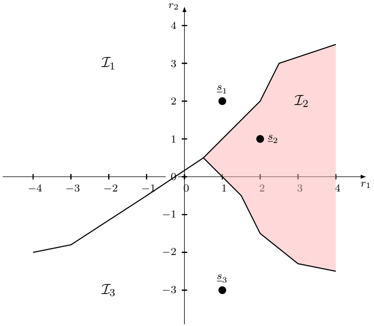
In DICO channels, thresholds define intervals. For DICO vector channels, we need to talk about decision regions

Decision region for vector channel

Given the decision rule $f(\cdot)$ we can write

$$\mathcal{I}_m \triangleq \{\underline{r} \in \mathbb{R}^N : f(\underline{r}) = m\}.$$

$\mathcal{I}_m$ is called the **decision region** that corresponds to message $m \in \mathcal{M}$.



Summary Module 4.2

Take Home Messages

- Q-functions are important to compute error probabilities in the AGN channel
- Detection in vector channels is determinted by *decision regions*
- Error probability in vector channels depend on the channel and detection rule

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