Communication Theory (5ETB0)
Module 4.3

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Module 4.3

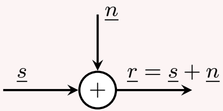
Presentation Outline

AGN Vector Channel

AGN Vector Channel

The AGN vector channel is

$$\underline{r} = \underline{s} + \underline{n},$$



where $\underline{n} \stackrel{\Delta}{=} (n_1, n_2, \ldots, n_N)$ is an N-dimensional noise vector, independent of the signal vector $\underline{S}$, and composed by independent, identically distributed zero-mean Gaussian random variables.

The joint PDF of the noise vector is given by

$$p_{\underline{N}}(\underline{n}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n_i^2}{2\sigma^2}\right) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N} n_i^2\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\|\underline{n}\|^2}{2\sigma^2}\right)$$

The AGN Vector Channel: A Matlab Example

Conclusions from Example

- Vector AGN noise can be interpreted as multidimensional noise balls
- AGN vector channel can be seen as multiple noise balls centered at $\underline{s}_m$
- Decisions will have to be done in N-dimensional space

Decision Rules for AGN Vector Channel

MAP decision rule for AGN Vector Channel

The conditional PDF for the AGN Vector channel

$$p_{\underline{R}}(\underline{r}|\underline{S}=\underline{s}_m) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\|\underline{r}-\underline{s}_m\|^2}{2\sigma^2}\right)$$

The MAP decision rule is

$$\hat{m}^{\mathsf{MAP}}(\underline{r}) \triangleq \operatorname*{argmin}_{m\in\mathcal{M}} \left\{\|\underline{r}-\underline{s}_m\|^2 - 2\sigma^2 \ln \Pr\{M=m\}\right\}$$

ML decision rule for AGN Vector Channel

The ML decision rule

$$\hat{m}^{\mathsf{ML}}(\underline{r}) \triangleq \operatorname*{argmin}_{m\in\mathcal{M}} \left\{\|\underline{r}-\underline{s}_m\|^2\right\}$$

MAP Derivation

Detailed Derivation

$$
\begin{aligned}
\hat{m}^{\mathsf{MAP}}(\underline{r}) &= \operatorname*{argmax}_{m\in\mathcal{M}} \Big\{\Pr\{\underline{R}=\underline{r}, \underline{S}=\underline{s}_m\}\Big\} = \operatorname*{argmax}_{m\in\mathcal{M}} \Big\{\log\Pr\{\underline{R}=\underline{r}, \underline{S}=\underline{s}_m\}\Big\} \\
&= \operatorname*{argmax}_{m\in\mathcal{M}} \Big\{\log\Pr\{M=m\} p_{\underline{R}}(\underline{r}|\underline{S}=\underline{s}_m)\Big\} \\
&= \operatorname*{argmax}_{m\in\mathcal{M}} \left\{\log\Pr\{M=m\} \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\|\underline{r}-\underline{s}_m\|^2}{2\sigma^2}\right)\right\} \\
&= \operatorname*{argmax}_{m\in\mathcal{M}} \left\{\log\Pr\{M=m\} - \log(2\pi\sigma^2)^{N/2} - \frac{\|\underline{r}-\underline{s}_m\|^2}{2\sigma^2}\right\} \\
&= \operatorname*{argmax}_{m\in\mathcal{M}} \left\{\log\Pr\{M=m\} - \frac{\|\underline{r}-\underline{s}_m\|^2}{2\sigma^2}\right\} \\
&= \operatorname*{argmin}_{m\in\mathcal{M}} \left\{\frac{\|\underline{r}-\underline{s}_m\|^2}{2\sigma^2} - \log\Pr\{M=m\}\right\} \\
&= \operatorname*{argmin}_{m\in\mathcal{M}} \Big\{\|\underline{r}-\underline{s}_m\|^2 - 2\sigma^2\log\Pr\{M=m\}\Big\}
\end{aligned}
$$

Decision Rules for AGN Vector Channel

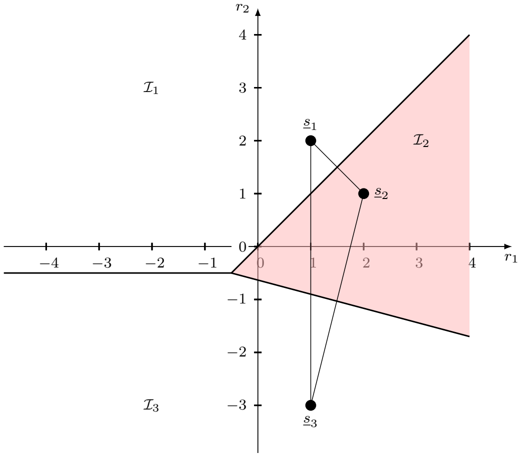
ML decision rule for AGN Vector Channel

- In one dimension (DICO Channel) the optimum threshold was half way between s_1 and s_2
- In N dimensions (DICO Vector Channel) the rule is

$$\hat{m}^{\mathsf{ML}}(\underline{r}) \triangleq \operatorname*{argmin}_{m\in\mathcal{M}} \left\{ \|\underline{r} - \underline{s}_m\|^2 \right\}$$

- For two signals $\underline{s}_1$ and $\underline{s}_2$, this rule corresponds to a hyperplane

ML Decision rule for a 3-signal system



ML Decision Regions: A Matlab Example

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Presentation Outline

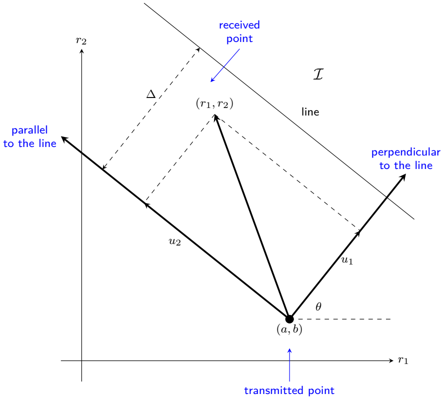
Error Probability: Key Result

AGN vector channel

For the AGN vector channel, the probability that the noise pushes a signal to the wrong side of a hyperplane is

$$P_{\mathcal{I}} = Q\left(\frac{\Delta}{\sigma}\right),$$

where Δ is the distance from the signal-point to the hyperplane and σ^2 is the variance of each noise component.



Error Probability Analysis for ML

Upper Bound on Error Probability (ML)

Average Error Probability: $P_{\mathrm{e}} = \sum_{m\in\mathcal{M}} \Pr\{M=m\} P_{\mathrm{e}}^{m}$

Union bound:

$$P_{\mathrm{e}}^{1} = \Pr\left\{ \bigcup_{m\in\mathcal{M}, m\neq 1} (\|\underline{R}-\underline{s}_m\| \leq \|\underline{R}-\underline{s}_1\|)|M=1 \right\}$$

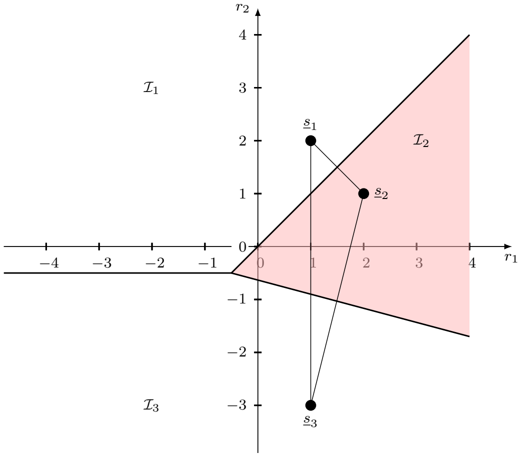
$$\leq \sum_{m\in\mathcal{M}, m\neq 1} \Pr\{\|\underline{R}-\underline{s}_m\| \leq \|\underline{R}-\underline{s}_1\||M=1\}$$

$$P_{\mathrm{e}} \leq \sum_{m\in\mathcal{M}} \frac{1}{|\mathcal{M}|} \sum_{m'\in\mathcal{M}, m'\neq m} \Pr\{\|\underline{R}-\underline{s}_{m'}\| \leq \|\underline{R}-\underline{s}_m\||M=m\}$$

Final Result: AGN channel with per-dimension noise variance σ^2

$$P_{\mathrm{e}} \leq \sum_{m\in\mathcal{M}} \frac{1}{|\mathcal{M}|} \sum_{m'\in\mathcal{M}, m'\neq m} Q\left(\frac{\Delta_{m'm}}{\sigma}\right), \quad \Delta_{m'm} = \frac{\|\underline{s}_{m'}-\underline{s}_m\|}{2}$$

Upper Bound: Geometric Interpretation



Who Cares?

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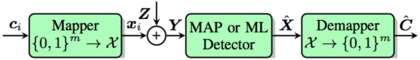
Asymptotic Comparison of ML and MAP Detectors for Multidimensional Constellations

Alex Alvarado, *Senior Member, IEEE*, Erik Agrell, *Senior Member, IEEE*, and Fredrik Brännström, *Member, IEEE*



II. Preliminaries

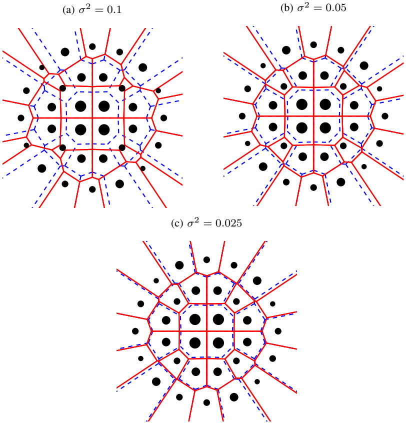
A. System Model

The system model under consideration is shown in Fig. 1. We consider the discrete-time, real-valued, N-dimensional, AWGN channel

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{Z}, \tag{1}$$

where the transmitted symbol $\boldsymbol{X}$ belongs to a discrete constellation $\mathcal{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\}$ and $\boldsymbol{Z}$ is an N-dimensional vector, independent of $\boldsymbol{X}$, whose components are independent and identically distributed Gaussian random variables with zero mean and variance σ^2 per dimension. The conditional channel transition probability is

$$f(\boldsymbol{y}|\boldsymbol{x}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\|\boldsymbol{y}-\boldsymbol{x}\|^2}{2\sigma^2}\right). \tag{2}$$



(a) $\sigma^2 = 0.1$

(b) $\sigma^2 = 0.05$

(c) $\sigma^2 = 0.025$

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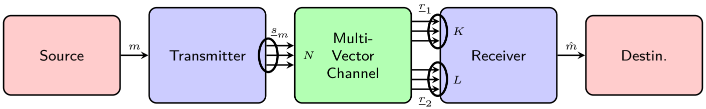
Presentation Outline

Multi-Vector Channels



Importance

- This model includes for example what is called spatial diversity, i.e., then the transmitter and receiver use multiple antennas (MIMO systems). Used in modern WiFi routers, mobile phones, etc.

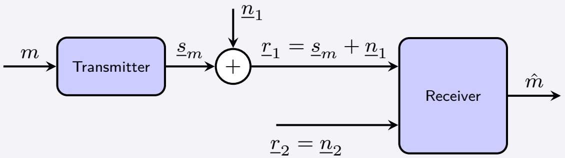
- Theorem of irrelevance: When can we discard $\underline{r}_2$ without affecting performance?

Theorem of Irrelevance

Theorem of Irrelevance (Theorem 4.6)

The output $\underline{r}_2$ of a multi-vector channel is irrelevant (does not affect P_e) if, for all $\underline{r}_1$ and $\underline{r}_2$, the value of $p_{\underline{R}_2}(\underline{r}_2|\underline{S} = \underline{s}_m, \underline{R}_1 = \underline{r}_1)$ does not depend on the message m.

Example 4.5

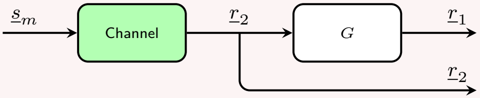


$$p_{\underline{R}_2}(\underline{r}_2|\underline{S} = \underline{s}_m, \underline{R}_1 = \underline{r}_1) = p_{\underline{R}_2}(\underline{r}_2) = p_{\underline{N}_2}(\underline{r}_2)$$

Theorem of Reversibility

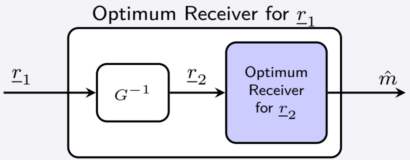
Theorem of Reversibility (Theorem 4.7)

The minimum attainable probability of error is not affected by the introduction of a reversible operation at the output of a channel.



Alternative View

A receiver for $\underline{r}_1$ can be built by first recovering $\underline{r}_2$ from $\underline{r}_1$



Summary Module 4.3

Take Home Messages

- Detection in vector channels is determinted by *decision regions*
- For the AGN vector channel: Euclidean distances!
- Theorems of irrelevance and reversibility let us formally discard certain observations

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