Electromagnetics II - 5EPB0

COURSE MATERIAL THAT REPLACES THE BULK OF CHAPTERS 10–12 OF THE BOOK
ENGINEERING ELECTROMAGNETICS BY W.H. HAYT JR. AND J.A. BUCK (H&B)

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Introduction

This reader covers half of the second BSc course on electromagnetism, *called Electromagnetics II*, and predominantly deals with the properties and applications of electromagnetic (EM) fields that comprise waves in one spatial dimension and time.

A good introduction to a scientific article contains a *review*, a clear *claim*, and an *agenda*.[1] This is not a scientific article, and it is impossible to acknowledge even a fraction of the researchers who have contributed to our knowledge of electromagnetism in a comprehensive review, so I won't bother to try. Instead, I shall aim at positioning the basic concepts, ideas and techniques discussed in this reader within the context of theoretical and applied electromagnetism. Of course, I shall also sketch how this course material sits within the BSc courses *Electromagnetics I* and *II*.

Actually, to understand my motivation for writing this reader, it is useful to retrace our steps back to the first rung of the academic ladder. Every electrical engineering student has been subjected to the joys of circuit theory from week one, which is the bread and butter of the electrical engineer.

In circuit theory, spatial dimensions are neglected, implying that all network elements are lumped, and the connections between the elements have no length. The Kirchhoff voltage and currents laws are then used to derive the equations that govern the electrical behaviour of the circuit. Hence, the state quantities in circuit theory depend on time, but are effectively zero-dimensional (0-D) in space.

The world around us is more complex than that described by circuit theory. Firstly, voltages and currents are inextricably linked to the electric and magnetic fields surrounding the circuit elements and wires via Maxwell's equations, which you have encountered in *Electromagnetics I*. In that course you have learnt a lot about static electric and magnetic fields in three spatial dimensions, and you have seen a glimpse of plane-wave propagation, but overall, the link with circuit theory was not very strong.

[1] To be more specific, a good introduction to a journal or conference article has a short paragraph aimed at positioning the topic in hand within a wider context. This is part of a longer text containing a review of what has already been achieved in the pertaining field of research. The *review* culminates in (or is followed by) the statement of the problem, i.e., the (hopefully glaring) gap left by the scientific predecessors, and the subsequent *claim* that the research reported in the paper comprises progress in solving the problem. If the research consists of a multitude of identifiable logical stages, it is useful to provide an *agenda*, i.e., a brief description the contents.

Undoubtedly, a clearly defined claim is the most important constituent of the introduction, and may determine whether the reader continues reading. Please consult the document "Scrutiny of the introduction" by Jon Claerbout, available at http://sepwww.stanford.edu/sep/prof/Intro.html if you want to learn how to write an effective introduction. That web-page also contains a link to an on-line version of the "Scrutiny of the abstract" by Kenneth K. Landes, which is great source for improving one's ability to write a good abstract.

This reader covers the first half of the course *Electromagnetics II*. In this course, the space-time dependence of fields takes centre stage. The first half of the course deals with one-dimensional (1-D) space-time wavefields. I wrote this reader because I wanted to make the link between circuit theory and EM fields as clear as possible. Further, I wanted to demonstrate how to interference effect may be used in the design of practical devices, in particular frequency filters.

So, you might wonder why we need to depart from the safe zero-dimensional (0-D) confines of circuit theory? Simply put, the answer is the finite speed of light. Let us consider one of the perfect electric conductors that connect two lumped elements. In circuit theory one just assumes that any change in time of the voltage on the conductor occurs simultaneously along the entire conductor. Likewise, the current entering on one end of the conductor is thought to come out immediately (without delay) at the other end. For slowly varying signals (or wide pulses) these may be reasonable assumptions.

However, if the signals generated in the circuit are short pulses, then the situation is more complex, and depends on the length of the two conductors that connect two (lumped) elements in one circuit (or two lumped circuits). If the travel time of pulsed signals propagating from one end to the other is of the order of the pulse width or longer, then 0-D circuit theory will fail, and the functionality of the combined circuit will be affected. For time-harmonic signals, this means that if the signals in the circuit vary slowly with time (low frequencies), the time delay will only result in a small phase shift that will hardly be noticed. Conversely, if the distances bridged by two conductors between two circuits (or circuit elements) exceeds about one tenth of the wavelength of the dominant signals [2], then 0-D circuit theory will fail. In case the pair of conductors are long in that sense, they must be analysed as a transmission line, along which 1-D space-time voltages and currents may propagate.

In *Electromagnetics I* you have learnt that a current flowing through a wire must be accompanied by a magnetic field encircling that wire via the Ampére(-Maxwell) law, and that the electric field between conductors is associated with the voltage difference between those conductors via the Faraday-Henry law, provided one considers (quasi-)static fields.[3] What confuses quite a few students is that *under certain conditions* we may still speak of (and work with) voltages for fields that vary in space and time. This is the case for *transverse electromagnetic (TEM) waves* propagating along transmission lines, and these are exactly the type of waves studied in *transmission-line (TL) theory*.

The simplest kind of TEM waves happen to be uniform plane waves (no conductors required). As a consequence, the plane-wave behaviour in the direction of propagation may be described by exactly the same mathematical formulae that apply to TEM waves along transmission lines. In the early 1950s, the scientist and engineer Nathan Marcuvitz and the theoretical physicist Julian Schwinger[4] recognised that it was beneficial to treat TEM waves (including plane waves) and other modes of propagation along waveguiding structures in one uniform formalism involving a description of EM fields in terms of voltage and current amplitudes and transverse vector fields that account for the waveguide geometry. In the case of TEM waves, these voltage and current amplitudes can directly be linked to the voltages and currents in circuit theory.

[2] Wavelength is wave speed divided by frequency (or $\lambda = c/f$ for short).

[3] $\nabla \times \mathbf{E} = 0 \rightarrow E = -\nabla V$

[4] Schwinger shared the 1965 Nobel prize in physics with Richard Feynman.

You can learn more about the Marcuvitz-Schwinger formalism in the MSc course Wavefield Representations. In this reader, I shall adopt some of the underlying ideas in order to treat plane waves and more general TEM waves on an equal footing regarding terminology and notation.[5]

This reader consists of four chapters, (roughly) corresponding to the first four weeks of the course *Electromagnetism II*. In Chapter 1, we explore the most elementary types of electromagnetic wavefields, i.e., waves in time and one spatial dimension. The resulting electric and magnetic fields may be represented by voltage and current amplitudes that depend on the longitudinal cartesian coordinate z and time t, times constant transverse vectors. The voltage amplitude satisfies a two-way wave equation with arbitrary counter-propagating waves as its solutions. The associated current amplitudes are proportional to their counterpropagating voltage counterparts through a characteristic wave admittance (or conversely, a characteristic impedance), and a *sign* that depends on the direction of propagation. Subsequently, we shall consider more general transverse electromagnetic waves that may be guided along transmission lines, but are otherwise indistinguishable from plane waves as regards their propagation behaviour. Next, we shall show that the termination of a transmission line by a resistive load will cause TEM waves incident on that load to be partially absorbed and partially reflected.

In Chapter 2, we shall demonstrate that plane waves, normally incident on an interface between two half spaces filled with different instantaneously reacting homogeneous media will also result in a reflected wave. However, in that case a plane wave will be transmitted into the other half space, rather than being absorbed in a load. This is equivalent to the reflection and transmission of TEM waves that occurs when a TEM wave is incident on a junction between two transmission lines with the same geometry (but filled with different materials). Multiple reflection and transmission occurs when there are two or more interfaces or junctions, or two loads terminating a finite transmission line at both ends. These situations may effectively be analysed with the aid of bounce diagrams.

TEM-wave propagation along transmission lines involves transverse vector fields that are no longer constant, but instead may depend on the transverse coordinates. In Chapter 2, we shall express the transverse vector fields in terms of scalar potentials that satisfy Laplace's equation in the transverse plane, supplemented with the boundary conditions at the surfaces of the cylindrical perfect electric conductors that demarcate the transverse cross-section of a transmission line. The potentials will be normalised such that the voltage and current amplitudes of the EM field are consistent with the "real" voltages and currents that determine the electric state in circuit theory. We shall also review the electromagnetic power balance and apply it to TEM-wave problems.

In Chapters 3 and 4, we shall consider time-harmonic electromagnetic fields, which will facilitate the description of the interaction of EM fields with matter through the so-called constitutive relations for linear time-invariant (LTI) media with relaxation. In Chapter 3, we shall also examine the exchange of energy for time-harmonic EM fields (time-averaged over one period). Single-frequency sines and cosines are easy to add. This will allow us to analyse and design quite sophisticated 1-D EM structures, e.g., the anti-reflection

[5]Conversely, I am convinced that it is illogical, confusing and inefficient to introduce an artificial separation between transmission line theory and the theory of plane EM waves by using different terminology and notations for these phenomena. Sadly, this is the case in almost all of the textbooks my colleagues and I have encountered, including Engineering Electromagnetics by Hayt and Buck.

coating and the Bragg reflector in Chapter 4, by means of the transfer-matrix formalism introduced Chapter 3, or equivalently, via back-propagation of the effective impedance. The effective impedance is closely connected to the effective reflection coefficient, and the voltage standing-wave ratio, and their relation will be visualised in Chapter 4 by means of the Smith Chart. Further, we shall discuss the pros and cons of using scattering matrices as opposed to transfer matrices. We conclude the reader with a discussion of the phase and group speeds (velocities) and pulse dispersion of EM waves in dispersive media. That discussion is also relevant for waveguide mode dispersion.

The second half of the course will be taught from the book by Hayt and Buck (H&B). Time-harmonic plane waves in arbitrary directions, plane-wave polarisation, and inhomogeneous plane waves will be discussed, as well as reflection and transmission of plane waves at oblique incidence, refraction (Snell's law), Fresnel reflection and transmission coefficients, Brewster angle, critical angle, and skin effect. We shall analyse waveguide modes, e.g., TM and TE modes, and the TEM wave as a special case of a TM mode. Cavities will be introduced as closed waveguides. Simple dielectric waveguides will also be on the menu. Regarding the field emitted by an electric dipole, we shall distinguish between the near and the far field, and discuss radiation, basic antenna parameters, the magnetic dipole, and duality. The course will conclude with a description of the rainbow from an EM point of view, i.e., we shall show why rainbows are polarised, how refraction, reflection and basic diffraction applies to water droplets, and demonstrate that rainbows can also be enjoyed inside the classroom.

Chapter 1

Week 1 — Waves in time and one spatial dimension

Maxwell's equations (reminder)

In general, the electromagnetic field depends on all three spatial coordinates and on time, i.e., the respective electric and magnetic field quantities $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{r}, t)$ (with $[\mathbf{E}] = \text{V/m}$ and $[\mathbf{H}] = \text{A/m}$) satisfy Maxwell's equations, which in their most general form can be written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{K}, \tag{1.1a}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}. \tag{1.1b}$$

where $\mathbf{D}$, $\mathbf{B}$, $\mathbf{J}$, and $\mathbf{K}$ denote the electric and magnetic flux densities and the electric and magnetic current densities, respectively.[1] In this chapter we shall first consider electromagnetic fields in free space[2], implying that there are no charges and hence no current densities either. However, magnetic currents will briefly return in the next chapter.

Maxwell's main innovation was that he added the displacement current density term $\partial \mathbf{D}/\partial t$ to what used to be called Ampères law, and is nowadays referred to as the law of Ampère-Maxwell. In Maxwell's time, the constitutive relations between the flux densities and the fields in free space had already been found to be

$$\mathbf{B} = \mu_0 \mathbf{H} \quad \text{and} \quad \mathbf{D} = \varepsilon_0 \mathbf{E}. \tag{1.2}$$

In view of these constitutive relations, Maxwell's introduction of the displacement current density in Eq. (1.1b), meant that the two equations couple for time-varying field quantities, implying that the electric and magnetic fields become inextricable linked.[3] Maxwell recognised that the two coupled equations in free space admit simple source-free solutions

[1]One might feel uneasy at the introduction of a magnetic current, and hence a magnetic charge, which has never been observed experimentally. However, *equivalent* (sometimes referred to as *notional* or *virtual*) magnetic current sources play an important role in the development of mathematical and computational methods for electromagnetic fields, so we feel justified in sneaking it in.

[2]synonymous with empty space, or vacuum

[3]The link between $\mathbf{E}$ and $\mathbf{H}$ disappears in the static limit, for which $\partial \mathbf{D}/\partial t = 0$ and $\partial \mathbf{B}/\partial t = 0$.

called plane waves. Moreover, the resulting wave speed was very close to the speed of light measured in 1862 by Léon Foucault using a rotating mirror. From this, in 1865, Maxwell came to the radical conclusion that light must be an electromagnetic wave.

Before the theory of electromagnetic waves was introduced, scholars had developed a decent understanding of acoustic waves that may propagate through a fluid medium. Thinking in terms of analogies, scientists have long believed that for electromagnetic waves and hence light to propagate, there had to be a luminiferous aether. One of the arguments for this was that in view of the dimensions of the electric and magnetic fields, respectively, there just *had* to be virtual charges to support the spatial variations of the voltage and current amplitudes. With the advent of Einstein's theory of relativity and Ockham's razor[4] it slowly dawned on researchers that electromagnetic waves could happily travel through vacuum and the concept of an aether was gradually abandoned.[5]

Below, we shall derive the properties of the simplest non-trivial solutions to Maxwell's equations, viz., *uniform plane waves*.

1.1 Uniform Plane Waves

1.1.1 Fields that are independent of x and y

A *uniform plane wave*[6] is a source-free solution of Maxwell's equations that only depends on a single spatial cartesian coordinate and on time. Although physics does not favour any specific coordinate, it is customary to choose the z-direction, i.e.,

$$\mathbf{E} = \mathbf{E}(z,t) \qquad \text{and} \qquad \mathbf{H} = \mathbf{H}(z,t). \tag{1.3}$$

The curl operator is a vectorial linear differential operator involving partial derivatives with respect to x, y and z, operating on the components of a vector field. Since the fields under consideration are assumed independent of x and y, and sources are assumed absent, Maxwell's equations for uniform plane waves reduce to

$$\nabla \times \mathbf{E} = -\mathbf{a}_x \frac{\partial E_y}{\partial z} + \mathbf{a}_y \frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{1.4a}$$

$$\nabla \times \mathbf{H} = -\mathbf{a}_x \frac{\partial H_y}{\partial z} + \mathbf{a}_y \frac{\partial H_x}{\partial z} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{1.4b}$$

[4]Whatever we do not need, we should discard.

[5]Although free space (or empty space) is uneventful in classical physics, the absence of particles in a certain volume of space leads to small vacuum fluctuations in quantum physics, which first became manifest in the observation in 1947 of the so-called Lamb shift in the energy levels of the hydrogen atom. Through a back-of-the-envelope calculation, Hans Bethe demonstrated shortly after that the Lamb shift may be accurately predicted upon considering quantum vacuum fluctuations. These vacuum fluctuations arise from quantum interactions between charged particles and the vacuum, which results in an uncertainty relation between the number of photons and the electromagnetic field. In the classical, macroscopic, theory of electromagnetism discussed in this course, such small quantum effects may safely be ignored.

[6]The adjective *uniform* suggests that there are other types of plane waves. Indeed, *non-uniform plane waves* also play an important role in electromagnetism. They will be treated in the second part of the course in the context of post-critical refraction resulting from oblique incidence of plane waves the interface between two dielectric half spaces.

from which we infer that

$$\frac{\partial H_z}{\partial t} = 0 \quad \text{and} \quad \frac{\partial E_z}{\partial t} = 0. \tag{1.5}$$

Owing to the causality principle, we may assume, without loss of generality, that for uniform plane waves we have[7]

$$H_z = 0 \quad \text{and} \quad E_z = 0. \tag{1.6}$$

This means that the plane wave must be a *transverse electromagnetic field*, i.e., a vectorial wavefield for which the directions of electric and magnetic field vectors are orthogonal to the direction of propagation[8]

The y-component of Eq. (1.4a) and the x-component of Eq. (1.4b) read

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t}, \tag{1.7a}$$

$$-\frac{\partial H_y}{\partial z} = \varepsilon_0 \frac{\partial E_x}{\partial t}, \tag{1.7b}$$

respectively. This indicates that E_x and H_y are coupled, in the sense that a change in the one causes a change in the other and vice versa. Likewise, the x-component of Eq. (1.4a) and the y-component of Eq. (1.4b) read

$$-\frac{\partial E_y}{\partial z} = -\mu_0 \frac{\partial H_x}{\partial t}, \tag{1.8a}$$

$$\frac{\partial H_x}{\partial z} = \varepsilon_0 \frac{\partial E_y}{\partial t}, \tag{1.8b}$$

respectively. Hence E_y and H_x are coupled as well.

Here, we should emphasise that Eqs. (1.7) and (1.8) are mutually uncoupled and lead to independent plane-wave solutions of Maxwell's equations. These solutions comprise two independent *field polarisations*, and any linear combination of such solutions is also a solution of Maxwell's equations. In fact, by rotating the field associated with a solution to Eq. (1.7) by an angle of $\pi/2$ (90°) in the positive direction around the z-axis[9], which amounts to $E_x \to E_y$ and $H_y \to -H_x$, we obtain a solution to Eq. (1.8). This is quite

[7]If the z-components of the electromagnetic field associated with a uniform plane wave that is independent of x and y exist at all, then Eq. (1.5) implies that they must be static (independent of time). Upon invoking $\nabla \cdot \varepsilon_0 \mathbf{E} = \varepsilon_0 \partial E_z/\partial z = 0$ and $\nabla \cdot \mu_0 \mathbf{H} = \mu_0 \partial H_z/\partial z = 0$, we conclude that E_z and H_z must be constant throughout space-time. Such constant field components can play no role in electromagnetism since Maxwell's equations provides a relation between the partial derivatives of the field quantities. The causality principle is a more fundamental concept, stating that any (electromagnetic) field must have been generated by some source distribution, however long ago. Before that time the field did not exist. This completely rules out the existence of static fields introduced in *Electromagnetics I*, albeit that electrostatic and magnetostatic fields may still serve as a very useful approximation to slowly varying fields. Whichever viewpoint you adopt, $E_z(z,t)$ and $H_z(z,t)$ must vanish identically for uniform electromagnetic plane waves in the z-direction.

[8]By contrast, an acoustic wavefield in a fluid (or a gas), characterised by a coupled pressure and particle velocity field, is a longitudinal field, i.e., the particles move back and forth in the direction of wave propagation. Elastic wavefields (or elastodynamic fields), characterised by the particle velocity field and a stress tensor field, may be decomposed into a longitudinal part and a transverse part (an earthquake may trigger a strong Rayleigh surface wave, which consists of a mixture of longitudinal and transverse wave components).

[9]positive in the sense of the right-hand rule, where the right-hand thumb points in the direction of the z-axis, and the fingers indicate the direction of rotation.

a relief since our choice for the x-y-axes should be irrelevant to the physics of a uniform plane wave in the z-direction.[10] Below, we shall investigate an alternative approach to representing the electromagnetic fields that not only removes the direct link to the arbitrary dependence of the choice of the x-y-axes, but may also be readily generalised so as to describe general TEM waves.

1.1.2 Voltage and current amplitudes

In order to avoid having to perform calculations for a fixed, specific polarisation in a particular frame of reference as regards the x-y-axes, we introduce an alternative representation for the electromagnetic field, viz.,

$$\mathbf{E}(z,t) = V(z,t)\mathbf{e}_\mathrm{t}, \tag{1.9a}$$
$$\mathbf{H}(z,t) = I(z,t)\mathbf{h}_\mathrm{t}, \tag{1.9b}$$

in which $V(z,t)$ [V] and $I(z,t)$ [A] are the scalar voltage and current amplitudes of the plane wave, and

$$\mathbf{e}_\mathrm{t} = e_x\mathbf{a}_x + e_y\mathbf{a}_y, \tag{1.10a}$$
$$\mathbf{h}_\mathrm{t} = h_x\mathbf{a}_x + h_y\mathbf{a}_y \tag{1.10b}$$

are *constant transverse field vectors*[11]. In this course, a vector is called transverse (denoted through a subscript $_\mathrm{t}$) if it is orthogonal to a specific direction, in this case the z-direction.

Upon substituting Eqs. (1.9) and (1.10) into Maxwell's equations, we arrive at

$$\nabla\times\mathbf{E} = -\frac{\partial V}{\partial z}\left(e_y\mathbf{a}_x - e_x\mathbf{a}_y\right) = \mu_0\frac{\partial I}{\partial t}\left(-h_x\mathbf{a}_x - h_y\mathbf{a}_y\right). \tag{1.11a}$$
$$\nabla\times\mathbf{H} = -\frac{\partial I}{\partial z}\left(h_y\mathbf{a}_x - h_x\mathbf{a}_y\right) = \varepsilon_0\frac{\partial V}{\partial t}\left(e_x\mathbf{a}_x + e_y\mathbf{a}_y\right), \tag{1.11b}$$

The two polarisations described in Eqs. (1.7) and (1.8) correspond to the choices

$$V = E_x,\ I = H_y, \quad e_x = 1 = h_y, \quad e_y = 0 = -h_x, \tag{1.12}$$

and

$$V = E_y,\ I = -H_x, \quad e_y = 1 = -h_x, \quad e_x = 0 = h_y, \tag{1.13}$$

respectively.

We may deconstruct Eq. (1.11) into a relation between the constant transverse vectors

$$\mathbf{h}_\mathrm{t} = \begin{pmatrix} h_x \\ h_y \\ 0 \end{pmatrix} = \begin{pmatrix} -e_y \\ e_x \\ 0 \end{pmatrix} = \mathbf{a}_z\times\mathbf{e}_\mathrm{t}, \tag{1.14}$$

[10] Although we can analyse the two field polarisations independently, it can be useful, especially for time-harmonic waves, to consider linear combinations of the two solutions. Depending of the phase difference between the two components, this may result in linear polarisation in an arbitrary transverse direction, or circular or elliptic polarisation. All will be explained in the second part of this course.

[11] The unit of the transverse field vectors $\mathbf{e}_\mathrm{t}$ and $\mathbf{h}_\mathrm{t}$ is m^{-1}. For uniform plane waves along the z-direction, it is customary to choose the normalisation $\int(\mathbf{e}_\mathrm{t}\times\mathbf{h}_\mathrm{t})\cdot\mathbf{a}_z\,\mathrm{d}A = 1$, where the integral is computed over a unit area part of the cross-sectional plane (z constant).

and a system of coupled first-order partial differential equations

$$-\frac{\partial V}{\partial z} = \mu_0 \frac{\partial I}{\partial t}, \tag{1.15a}$$

$$-\frac{\partial I}{\partial z} = \varepsilon_0 \frac{\partial V}{\partial t}, \tag{1.15b}$$

which relate the voltage and current amplitudes, and are known as the *transmission-line equations* for uniform plane waves in free space.

Through Eqs. (1.9), (1.14) and (1.15), we have arrived at a unified approach to describe both polarisations. Moreover, it helps to distinguish between the wave propagation behaviour through the coupled equations for V and I and the geometric behaviour through the relation between $\mathbf{e}_\mathrm{t}$ and $\mathbf{h}_\mathrm{t}$.

To describe waves along transmission lines, we shall generalise the uniform-plane-wave assumptions in Eq. (1.9) by assuming that the transverse vectors could be functions of the transverse coordinates, i.e., $\mathbf{e}_\mathrm{t} = \mathbf{e}_\mathrm{t}(x, y)$ and $\mathbf{h}_\mathrm{t} = \mathbf{h}_\mathrm{t}(x, y)$ rather than constant vectors. This is the so-called *transverse electromagnetic wave* (TEM-wave) *ansatz*.

Before all that, we still have to demonstrate that the solutions to Eq. (1.15) actually represent waves travelling in the positive or negative z-directions.

1.1.3 Wave equations and their solutions

Let us try to convert the coupled system of first-order partial differential equations for the voltage and current amplitudes to a single second-order partial differential equation for V only. This can be achieved by subjecting Eqs. (1.15a) and (1.15b) to partial derivatives with respect to z and t, respectively, and eliminating I from the resulting equations according to

$$\left.\begin{array}{rcl} \frac{\partial}{\partial z}\left(-\frac{\partial V}{\partial z}\right. & = & \left.\mu_0 \frac{\partial I}{\partial t}\right) \\ \mu_0 \frac{\partial}{\partial t}\left(-\frac{\partial I}{\partial z}\right. & = & \left.\varepsilon_0 \frac{\partial V}{\partial t}\right) \end{array}\right\} \Rightarrow -\frac{\partial^2 V}{\partial z^2} = \mu_0 \frac{\partial}{\partial z}\left(\frac{\partial I}{\partial t}\right) = \mu_0 \frac{\partial}{\partial t}\left(\frac{\partial I}{\partial z}\right) = -\varepsilon_0 \mu_0 \frac{\partial^2 V}{\partial t^2}, \tag{1.16}$$

where we have made used of the fact that ε_0 and μ_0 are constants. The result of Eq. (1.16) comprises a second-order partial differential equation for V.

To analyse this equation, let us introduce the constant $c_0 = 1/\sqrt{\varepsilon_0 \mu_0}$, and apply the algebraic rule $(a^2 - b^2) = (a-b)(a+b) = (a+b)(a-b)$ to the second-order derivatives, i.e.,

$$0 = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) V \tag{1.17a}$$

$$= \left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right) V \tag{1.17b}$$

$$= \left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right) V. \tag{1.17c}$$

Here, Eq. (1.17a) is called a two-way wave equation, while the term between the parentheses is the corresponding two-way wave operator. We shall argue that the operators

$$\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t} \tag{1.18}$$

are the one-way wave operators for *forward* (in the direction of increasing z) and *backward* (in the direction of decreasing z) propagating waves, respectively. These waves are also referred to as *progressive* and *regressive* waves, respectively.

Now, suppose that we are able to solve

$$\left(\frac{\partial}{\partial z}+\frac{1}{c_0}\frac{\partial}{\partial t}\right)V=0 \quad \text{or} \quad \left(\frac{\partial}{\partial z}-\frac{1}{c_0}\frac{\partial}{\partial t}\right)V=0, \tag{1.19}$$

then, any superposition (linear combination) of such solutions would also satisfy the two-way wave equation on account of Eqs. (1.17b) and (1.17c), respectively.

The one-way wave equations tell us that $\partial V/\partial z$ must be proportional to $\partial V/\partial t$, i.e., up to a multiplicative constant they are the same functions. It is then reasonable to try a solution for $V(z,t)$ as a function $f(\tau)$ of a single variable $\tau=\tau(z,t)=t-pz$, which, for p constant, is a linear combination of t and z. So, let us introduce $f'(\tau)=\mathrm{d}f/\mathrm{d}\tau$ and invoke the chain rule of differentiation, resulting in

$$\frac{\partial f}{\partial t}=\frac{\partial\tau}{\partial t}\frac{\mathrm{d}f}{\mathrm{d}\tau}=\frac{\partial(t-pz)}{\partial t}f'=f' \tag{1.20a}$$

$$\frac{\partial f}{\partial z}=\frac{\partial\tau}{\partial z}\frac{\mathrm{d}f}{\mathrm{d}\tau}=\frac{\partial(t-pz)}{\partial z}f'=-pf'. \tag{1.20b}$$

From Eqs. (1.19) and (1.20) we infer that if we let $p=1/c_0$, the *any function* $V^+(\tau)=V^+(t-z/c_0)$ would satisfy the one-way wave equation for progressive waves (the one on the left in Eq. (1.19)). Likewise, for $p=-1/c_0$, *any (other) function* $V^-(t+z/c_0)$ would satisfies the one-way wave equation for regressive waves (the one on the right in Eq. (1.19)).

The *general solution* to the two-way wave equation in Eq. (1.17a) is the superposition (linear combination) of the two solutions to the one-way wave equations given by

$$V(z,t)=V^+(t-z/c_0)+V^-(t+z/c_0). \tag{1.21}$$

To see that $V^+(t-z/c_0)$ represents a wave propagating in the positive z-direction, consider a thought experiment with an observer, let us call her Jane, who samples the wave at time t_1 and position z_1, and measures a voltage amplitude $V^+(t_1-z_1/c_0)$. If Jane stays put at z_1 she would see the wave rushing past as time progresses from t_1 to $t_2=t_1+\Delta t$. However, she could also choose to move forward at the speed[12] c_0. In particular, at time t_2, she would find herself at $z_2=z_1+c_0\Delta t$, where she would measure

$$V^+\left(t_2-\frac{z_2}{c_0}\right)=V^+\left(t_1+\Delta t-\frac{z_1+c_0\Delta t}{c_0}\right)=V^+\left(t_1-\frac{z_1}{c_0}\right). \tag{1.22}$$

Hence, c_0 is the wavespeed in free space. Analogously, in order to "surf" along the wave associated with $V^-(t+z/c_0)$, an observer would have to move backward (in the negative z-direction) with speed c_0.

[12]We say that waves travel at a certain *speed*, which, as far as terminology in concerned, is more precise than the velocity of a wave (used in the book by H&B). In general one reserves the term velocity for a vectorial quantity that describes speed *and direction* of a particle (or a point of reference). Further, I am aware it is impossible for Jane to actually travel at the speed of light.

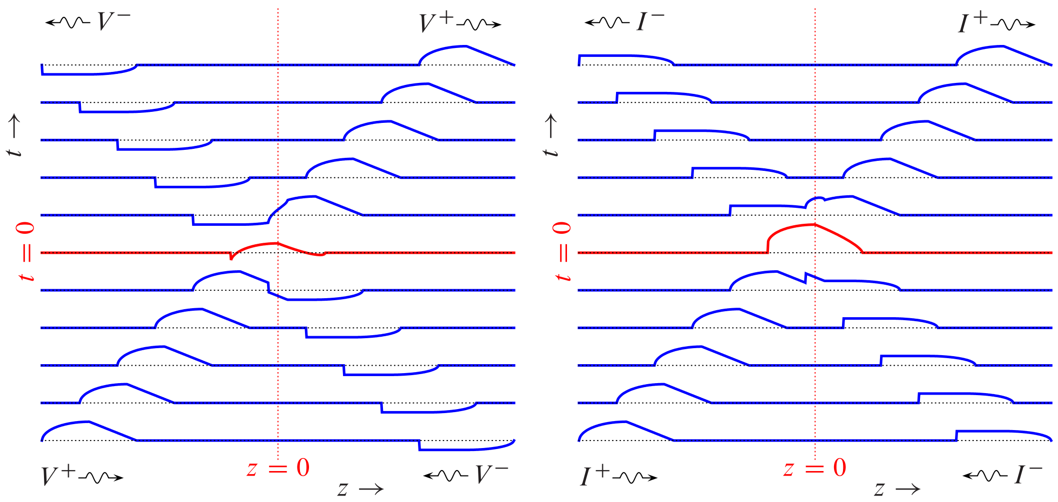


Figure 1.1: The voltage (left) and current (right) amplitudes of counterpropagating plane waves with different pulse shapes. We have chosen the coordinate system such that the waves "meet" at $z = 0$, and fully overlap at $t = 0$.

The families of lines $t - z/c_0 = \text{constant}^+$, and $t + z/c_0 = \text{constant}^-$ are straight lines in the z-t-plane, called *characteristic curves*. Along these curves the respective voltage amplitudes $V^+(t - z/c_0)$ and $V^-(t + z/c_0)$ remain constant. In the bounce diagrams introduced in Section 2.1.3 such characteristic curves are used to keep track of the timing of waves bouncing back and forth due to interfaces between different segments of transmission lines, or a stack of plane layers in the case of plane waves.

What about the current amplitude of the magnetic field? With reference to Eq. (1.15a), we have $\partial I/\partial t = -(1/\mu_0)\partial V/\partial z$. Hence, the current amplitudes for the respective forward and backward propagating waves satisfy

$$\frac{\partial I^\pm}{\partial t} = \pm \frac{1}{\mu_0 c_0} \frac{\partial V^\pm}{\partial t}, \tag{1.23}$$

implying that up to an irrelevant additive constant (see Footnote [7]), we have

$$I^\pm(t \mp z/c_0) = \pm \frac{V^\pm(t \mp z/c_0)}{Z_0} \qquad \text{with} \quad Z_0 = \mu_0 c_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{1.24}$$

denoting the *(plane) wave impedance* in free space. Please note that the signs in the equations are connected in that we either take the *upper signs* for *progressive waves*, or the *lower signs* for *regressive waves*.

In Figure 1.1, we have depicted the voltage and current amplitudes associated with two counterpropagating electromagnetic waves with different pulse shapes. Observe that the pulse signatures associated with the progressive and regressive waves have been chosen different, and that these counterpropagating waves do not interact (other than adding up on collision). We shall analyse this scenario in further detail once we have derived the transmission-line equations for transmission lines. In choosing the symbol Z_0 to denote the plane-wave impedance in free space, we are following in the footsteps of many predecessors.[13]

In Chapter 3, we shall discuss wave interactions in linear time-invariant (LTI) media, which, in case relaxation effects play a role, will prompt us to consider time-harmonic waves (or, equivalently the frequency domain). However for homogeneous *instantaneously reacting LTI media* — also referred to non-dispersive (or lossless) media — the constitutive relations may simply be written as

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \text{and} \qquad \mathbf{B} = \mu \mathbf{H} \tag{1.25}$$

with constant permittivity $\varepsilon \geq \varepsilon_0$, and permeability $\mu \geq \mu_0$. Note that these constitutive equations are the same for those in vacuum given by Eq. (1.2), except that the subscripts $_0$ have been dropped.

The reason for introducing material media here is that it helps us clean up the notation a little bit. Below, we shall prefer to use the symbols

$$c = \frac{1}{\sqrt{\varepsilon\mu}} \leq c_0 \qquad \text{and} \qquad Z = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{Y} \tag{1.26}$$

for the plane-wave wavespeed, wave impedance and wave admittance in a non-specific *instantaneously reacting LTI* medium (which includes vacuum as a special limiting case). Upon considering a stack of plane layers, this notation allows us to reintroduce subscripts to indicate that the medium properties and hence the wavespeeds, impedances, and admittances differ from layer to layer, e.g., Z_1, $Z_2, \ldots$. If one of the layers happens to be vacuum, we just reintroduce the subscript $_0$.

The simplest generalisation of plane waves is to consider transverse electromagnetic waves that can propagate along transmission lines.[14] As stated before, this will lead to a set of transmission line equations for the propagation along the positive or negative z-directions. For consistency, we shall again denote the speed of propagation and the transmission-line wave impedance (also known as the *characteristic impedance*) as c and Z, respectively. Often, one denotes the characteristic impedance of a transmission line as Z_0 or Z_c. To my mind this just clutters the notation, especially in case we are considering a concatenation of different transmission line with different properties.[15]

Before we explore this further, let us summarise the results obtained so far.

1.1.4 Summary of uniform plane waves

Let us summarise what we have discovered so far. The electromagnetic field associated with uniform plane waves in the positive and negative z-directions in free space were found to be

$$\mathbf{E} = V(z,t)\mathbf{e}_\mathrm{t} = \left[V^+\left(t - \frac{z}{c}\right) + V^-\left(t + \frac{z}{c}\right)\right]\mathbf{e}_\mathrm{t}, \tag{1.27a}$$

$$\mathbf{H} = I(z,t)\mathbf{h}_\mathrm{t} = \left[I^+\left(t - \frac{z}{c}\right) + I^-\left(t + \frac{z}{c}\right)\right]\mathbf{h}_\mathrm{t}, \tag{1.27b}$$

[13] In the book by H&B the (plane) wave impedance in free space, Z_0 is denoted as η_0. This is unfortunate, because it suggests a substantial difference between the (plane) wave impedance and the characteristic impedance of a transmission line to be introduced shortly. Although there are differences between plane waves and TEM waves along transmission lines, we would rather emphasise the commonalities.

[14] or more accurately, along cylindrical waveguides with two disjoint metal conductors.

[15] I am sure you'll agree that Z_1, $Z_2, \ldots$ comprises a cleaner notation than $Z_{0;1}$, $Z_{0;2}, \ldots$.

in which

$$c = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad Z = \mu c = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{Y}, \tag{1.28}$$

denote the wave speed and impedance, respectively, while

$$I^{\pm} = \pm Y V^{\pm} \quad \Longleftrightarrow \quad V^{\pm} = \pm Z I^{\pm} \tag{1.29}$$

relate the voltage and current amplitudes of the respective waves in the positive and negative z-directions. Further, the constant transverse vectors $\mathbf{e}_\mathrm{t} \perp \mathbf{a}_z$ and $\mathbf{h}_\mathrm{t} \perp \mathbf{a}_z$ are related via

$$\mathbf{h}_\mathrm{t} = \mathbf{a}_z \times \mathbf{e}_\mathrm{t} \quad \Rightarrow \quad \mathbf{h}_\mathrm{t} \perp \mathbf{e}_\mathrm{t}, \tag{1.30}$$

Finally, we would like to point out that the minus sign in the relation $V^- = -ZI^-$ (or $I^- = -YV^-$) is the most important sign in the course (without it, circuit theory, physics, and life as we know it would not be possible).

Thinking about that minus sign, it may seem confusing at first because the physics of wave propagation is certainly the same for waves in the positive or negative z-directions. However, we have chosen a right-handed system of vectors $\{\mathbf{e}_\mathrm{t}, \mathbf{h}_\mathrm{t}, \mathbf{a}_z\}$, meaning that $\mathbf{e}_\mathrm{t} \times \mathbf{h}_\mathrm{t}$ points in the direction of $\mathbf{a}_z$. As demonstrated in Chapter 2, the Poynting vector (the instantaneous power density) for a forward propagating uniform plane wave $\mathbf{S} = \mathbf{E}^+ \times \mathbf{H}^+ = Z(I^+)^2 \mathbf{e}_\mathrm{t} \times \mathbf{h}_\mathrm{t}$ points in the direction of $\mathbf{a}_z$.

For a uniform plane wave in the negative z-direction, we would have to change the direction (flip the sign) of either $\mathbf{E}$ or $\mathbf{H}$. This could be achieved through the substitution $\mathbf{h}_\mathrm{t} \to -\mathbf{h}_\mathrm{t}$ (or $\mathbf{e}_\mathrm{t} \to -\mathbf{e}_\mathrm{t}$), while keeping the same voltage and current amplitudes. This would be an unusual and awkward choice. We follow standard practice by keeping the right-handed system of vectors $\{\mathbf{e}_\mathrm{t}, \mathbf{h}_\mathrm{t}, \mathbf{a}_z\}$ fixed, while flipping the sign of the voltage *or* current amplitude.

Plane waves may seem at first a bit too simple and restrictive to be useful, especially because of the invariance of the fields with respect to the transverse directions. Nothing could be further from the truth. It turns out that through superposition, plane waves can be used as building blocks for the synthesis of (almost) any type of wavefield, e.g., the modes in a rectangular waveguide consist of a congruence of four plane waves, while the spherical wave generated by a dipole may be constructed via a continuous superposition of plane waves.

The book by H&B is inconsistent in notation as regards forward and backward propagating waves. In Chapter 10, the authors mostly use the superscripts $^{\pm}$ introduced above. In Chapter 11, the superscript $^{+}$ is dropped for forward propagating waves, whereas a superscript $'$ indicates backward propagating waves. In Section 10.4, the authors use subscripts $_f$ and $_b$ to denote the respective forward and backward propagating waves.

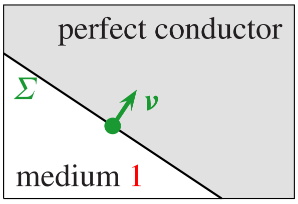
We shall adhere to the notation involving the superscripts $^{\pm}$, (or, when dealing with incident, reflected and transmitted waves at an interface, we shall use superscripts $^{\mathrm{i}}$, $^{\mathrm{r}}$, and $^{\mathrm{t}}$, respectively).

1.2 TEM-wave propagation along transmission lines

1.2.1 Boundary conditions at the surface of a perfect electric conductor (reminder)

Below, we shall broaden the scope of our analysis to allow for the presence of two disjoint cylindrical perfect electric conductors (PECs).[16] Perfect electric conductors have boundaries. In *Electromagnetics I* you have learnt that an electric field inside a perfect electric conductor, or tangent to its smooth perfectly conducting surface would result in an infinite current inside that PEC or along its surface, and hence infinite energy. This is *not allowed*, and hence the components of the electric field that are tangent to the PEC must vanish identically. Via Maxwell's equations, the magnetic field must also vanish inside the PEC.

However, the magnetic field *can* have tangential components at the surface of the PEC, and hence the jump in value from those tangential components to zero upon crossing the boundary into the PEC must be accounted for by a surface current density $\mathbf{J}_S$. Let us follow convention, and choose the symbol $\boldsymbol{\nu}$ (bold version of the Greek letter "nu") to denote the normal to the boundary Σ of the PEC, and pointing away from medium 1 into the PEC[17]. Then, the boundary conditions at the PEC surface read



$$\boldsymbol{\nu} \times \mathbf{E}_1 = \mathbf{0}, \tag{1.31a}$$
$$\boldsymbol{\nu} \times \mathbf{H}_1 = -\mathbf{J}_S, \tag{1.31b}$$
$$\boldsymbol{\nu} \cdot \mathbf{B}_1 = 0, \tag{1.31c}$$
$$\boldsymbol{\nu} \cdot \mathbf{D}_1 = -\rho_S. \tag{1.31d}$$

where the subscript $_1$ indicates that the point of observation on the PEC surface is approached from medium 1, and ρ_S denotes the surface density of electrical charge.

In case no confusion can arises, we may just as well omit the subscript $_1$, and write $\boldsymbol{\nu} \times \mathbf{E} = 0$ at the PEC surface Σ. Alternatively, we may express this by decomposing the electric field at Σ into a tangential and a normal part, according to $\mathbf{E} = \mathbf{E}_{\mathrm{tg}} + \mathbf{E}_{\mathrm{normal}}$, and demanding that $\mathbf{E}_{\mathrm{tg}} = 0$ at Σ.

1.2.2 Transmission Lines

In Section 1.1, we have derived generic expressions for uniform plane waves that propagate in the positive or negative z-directions in free space.

Now, suppose we put two perfectly conducting parallel infinite plates at the levels $x = 0$ and $x = d$, thus creating a layer of free space enclosed by the respective boundaries Σ_1 with unit normal $\boldsymbol{\nu} = -\mathbf{a}_x$ and Σ_2 with unit normal $\boldsymbol{\nu} = \mathbf{a}_x$ (see Figure 1.2). The polarisation of a plane wave propagating in the z-direction in free space is characterised by the direction of the electric-field vector $\mathbf{E} = V\mathbf{e}_{\mathrm{t}}$. If we were to transplant this plane-wave solution into the parallel-plate setting, the boundary condition $\boldsymbol{\nu} \times \mathbf{E}$ at $x = 0$ and $x = d$ would be violated if $\mathbf{e}_{\mathrm{t}}$ had a component in the y-direction. However, if $\mathbf{e}_{\mathrm{t}} \parallel \mathbf{a}_x$, then the

[16] Note that a cylindrical structure is defined as a structure that is invariant in (at least) one spatial directions. Hence, two parallel plates form a cylindrical structure. A co-axial cable is an example of a *circularly* cylindrical structure.

[17] The symbol $\mathbf{a}_{N12}$ used in the book by H&B is highly unusual, and overcomplicated if you ask me

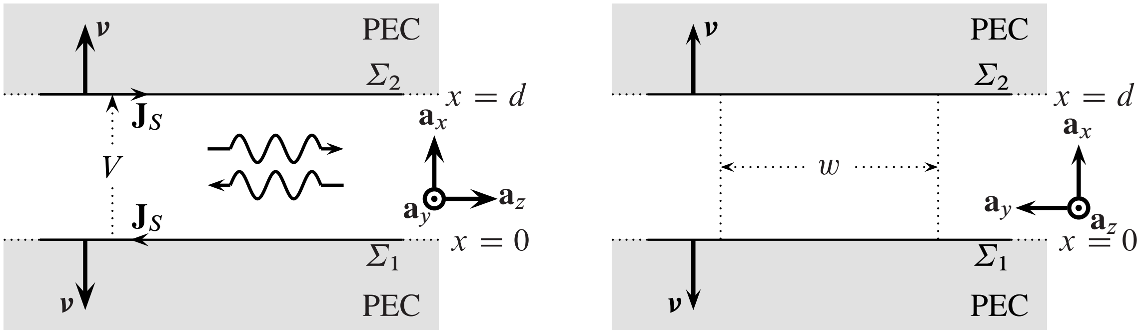


Figure 1.2: A longitudinal section (left) and a cross-section (right) of a parallel-plate waveguide with perfectly conducting (PEC) walls.

plane wave solution restricted to the region in between the parallel PEC plates *does* satisfy Maxwell's equations *with* boundary conditions. The surface current densities $\mathbf{J}_S$ ensure that the magnetic field $\mathbf{H} = H_y \mathbf{a}_y$ jumps from finite values in between the plates to zero in the PECs.

This is a special case of a transverse electromagnetic (TEM) wave in the z-direction, so called because the z-components of the electromagnetic field vanish. It turns out that *all cylindrical waveguides with two or more disjoint conductors in a homogeneous isotropic background admit TEM-wave solutions*, e.g., the co-axial cable depicted in Figure 1.3. Clearly, it is impossible to fit a portion of a plane-wave inside the co-axial cable with its

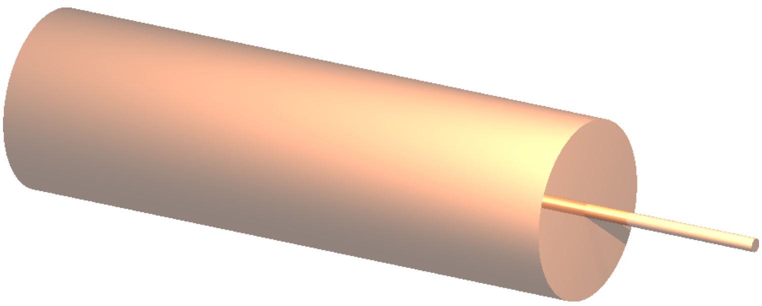


Figure 1.3: The coaxial cable

circularly cylindrical boundaries without violating the boundary conditions for $\mathbf{E}$ on most of the PEC boundaries. Hence, to find the TEM-wave solutions, we would have to relax the condition that $\mathbf{e}_t$ and $\mathbf{h}_t$ are constant vector fields. Before we explore that idea, let us first give some thought to the voltage and current amplitude scaling.

1.2.3 Alternative voltage and current amplitude scaling for plane waves

Let us return to the uniform plane wave *ansatz*, which was made explicit in Eq. (1.9) and resulted in Eq. (1.11). That equation may be rewritten in a more compact and pre-structured form as

$$
\begin{aligned}
-\nabla \times \mathbf{E} &= -\frac{\partial V}{\partial z}\,\mathbf{a}_z \times \mathbf{e}_\mathrm{t} \\
&= \mu \frac{\partial I}{\partial t}\,\mathbf{h}_\mathrm{t}, && (1.32a) \\
\nabla \times \mathbf{H} &= -\frac{\partial I}{\partial z}\,\mathbf{h}_\mathrm{t} \times \mathbf{a}_z \\
&= \varepsilon \frac{\partial V}{\partial t}\,\mathbf{e}_\mathrm{t}, && (1.32b)
\end{aligned}
$$

with $\mathbf{e}_\mathrm{t}$ and $\mathbf{h}_\mathrm{t}$ denoting *constant transverse vectors*. From this, we extracted the relation between $\mathbf{e}_\mathrm{t}$ and $\mathbf{h}_\mathrm{t}$ in Eq. (1.14) and the transmission-line equations in Eq. (1.15). However, we might equally well have chosen to split Eq. (1.32) in a different way, e.g.,

$$
\begin{aligned}
-\nabla \times \mathbf{E} &= -\frac{\partial V}{\partial z} \quad \mathbf{a}_z \times \mathbf{e}_\mathrm{t} \\
&= L \frac{\partial I}{\partial t} \quad \frac{\mu}{L}\mathbf{h}_\mathrm{t}, && (1.33a) \\
\nabla \times \mathbf{H} &= -\frac{\partial I}{\partial z} \quad \mathbf{h}_\mathrm{t} \times \mathbf{a}_z \\
&= C \frac{\partial V}{\partial t} \quad \frac{\varepsilon}{C}\mathbf{e}_\mathrm{t}, && (1.33b)
\end{aligned}
$$

in which we recognise (and have literally separated off) an alternative pairing of (red) scalar terms and (blue) transverse vectorial terms, resulting in the *transmission-line equations*

$$
-\frac{\partial}{\partial z}\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & L \\ C & 0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} V \\ I \end{pmatrix}, \tag{1.34}
$$

the associated two-way wave equation

$$
\left(\frac{\partial^2}{\partial z^2} - LC\frac{\partial^2}{\partial t^2}\right) V = 0, \tag{1.35}
$$

and the equations that relate the transverse vectors

$$
\begin{aligned}
\mathbf{h}_\mathrm{t} &= \frac{L}{\mu}\mathbf{a}_z \times \mathbf{e}_\mathrm{t}, && (1.36a) \\
\mathbf{e}_\mathrm{t} &= \frac{C}{\varepsilon}\mathbf{h}_\mathrm{t} \times \mathbf{a}_z. && (1.36b)
\end{aligned}
$$

We have cast Eq. (1.35) in the form of a system of first-order partial differential equations for a *voltage-current vector*. Further, we have introduced *constants* L and C that — *for consistency (given the chosen pairing)* — have to satisfy

$$
\mathbf{h}_\mathrm{t} = \frac{L}{\mu}\mathbf{a}_z \times \left(\frac{C}{\varepsilon}\mathbf{h}_\mathrm{t} \times \mathbf{a}_z\right) = \mathbf{h}_\mathrm{t} = \frac{LC}{\mu\varepsilon}\mathbf{h}_\mathrm{t} \quad \Rightarrow \quad LC = \mu\varepsilon = \frac{1}{c^2}. \tag{1.37}
$$

Hence, the effect of introducing the new scaling leaves the speed of light in a homogeneous non-dispersive medium $c = (LC)^{-1/2} = (\mu\varepsilon)^{-1/2}$ unaffected,[18] but allows for an adjustment of the characteristic impedance (the voltage to current amplitude ratio) of a wave travelling in the positive z-direction. Since L and C are constants with respective units $[L] = \mathrm{VsA^{-1}m^{-1}} = \mathrm{H/m}$ and $[C] = \mathrm{AsV^{-1}m^{-1}} = \mathrm{F/m}$, we shall refer to L and C as the inductance and capacitance per unit length of a transmission line, respectively.

For plane waves, there is no reason whatsoever to make any other choice than $L = \mu$ and $C = \varepsilon$. However, *for transmission lines* there are compelling reasons for letting the values of L and C follow from the voltage and current normalisations described in Chapter 2, but we are getting ahead of ourselves.

1.2.4 Transverse electromagnetic waves

For transmission lines other than the parallel-plate waveguide, we have to break free from the confines of the plane-wave *ansatz*. Beyond the simple case of plane waves, the next level of complexity involves keeping $\mathbf{e}_\mathrm{t}$ and $\mathbf{h}_\mathrm{t}$ transverse, but no longer constant, i.e.,

$$\mathbf{E}(\mathbf{r},t) = V(z,t)\mathbf{e}_\mathrm{t}(x,y), \tag{1.38a}$$

$$\mathbf{H}(\mathbf{r},t) = I(z,t)\mathbf{h}_\mathrm{t}(x,y). \tag{1.38b}$$

Upon substituting Eq. (1.38) into Maxwell's equations, we find that this *so-called* transverse electromagnetic (TEM) wave *ansatz* leads to

$$\begin{aligned}
-\nabla\times\mathbf{E} &= -\frac{\partial V}{\partial z}\,\mathbf{a}_z\times\mathbf{e}_\mathrm{t} - V\,\overbrace{\nabla\times\mathbf{e}_\mathrm{t}}^{\left(\frac{\partial e_y}{\partial x}-\frac{\partial e_x}{\partial y}\right)\mathbf{a}_z}\\
&= L\frac{\partial I}{\partial t}\,\frac{\mu}{L}\mathbf{h}_\mathrm{t} + 0\,\mathbf{a}_z,
\end{aligned}\tag{1.39a}$$

$$\begin{aligned}
\nabla\times\mathbf{H} &= -\frac{\partial I}{\partial z}\,\mathbf{h}_\mathrm{t}\times\mathbf{a}_z + I\,\overbrace{\nabla\times\mathbf{h}_\mathrm{t}}^{\left(\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}\right)\mathbf{a}_z}\\
&= C\frac{\partial V}{\partial t}\,\frac{\varepsilon}{C}\mathbf{e}_\mathrm{t} + 0\,\mathbf{a}_z.
\end{aligned}\tag{1.39b}$$

Hence, *in addition to* the transmission line equations in Eq. (1.34) we *also* have to solve a new set of partial differential equations for $\mathbf{e}_\mathrm{t}$ and $\mathbf{h}_\mathrm{t}$, viz.,

$$\nabla\times\mathbf{e}_\mathrm{t} = \mathbf{0}, \tag{1.40a}$$

$$\nabla\times\mathbf{h}_\mathrm{t} = \mathbf{0}, \tag{1.40b}$$

in conjunction with the boundary conditions at the surfaces of the PEC conductors, and Eq. (1.36), that relates $\mathbf{e}_\mathrm{t}$ to $\mathbf{h}_\mathrm{t}$ and to the medium properties. This shall be referred to as the *transverse field problem*.

[18] as it should, since the speed of light is non-negotiable; in fact it has been fixed in the International System of Units to be 299792458 [m/s].

We have now arrived at a crossroads. We have seen that electromagnetic plane waves are a special case of electromagnetic TEM-waves. TEM waves propagate at the speed of light in the homogeneous non-dispersive medium under consideration, denoted as c. Hence, once we have determined characteristic impedance $Z = \sqrt{L/C} = 1/Y$, then Eqs. (1.27) and (1.29) are the general propagating-wave solutions for $V(z,t)$ and $I(z,t)$ to the transmission-line equations given by Eq. (1.34).

The transverse field problem will be tackled in Section 2.2, where we shall also explain the choice for the normalisation (or scaling) of V to $\mathbf{e}_\mathrm{t}$ and I to $\mathbf{h}_\mathrm{t}$, which will finally fix the values of L and C for the transmission line at hand. For now, we shall assume that L and C have already been determined, which allows us to focus on the propagation and reflection of TEM waves.

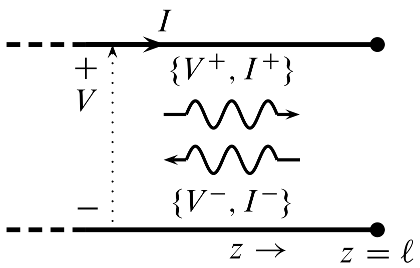


Figure 1.4: Abstract depiction of a semi-infinite transmission line.

Since we have effectively disentangled the description of the wave propagation along the transmission line from the transverse problem, it is useful to introduce an abstract depiction of a transmission line. The actual transmission line that sustains the TEM waves may be any two-conductor transmission line, e.g., a coaxial waveguide, a parallel-plate waveguide, or a microstrip line. In Figure 1.4 we have depicted a semi-infinite transmission line with $z \leq \ell$. Here, we have highlighted the fact that the voltage and current amplitudes V and I consists of progressive and regressive constituents. The dashed portions of the transmission line indicate that the line extends to $-\infty$ in the z-direction, whereas the solid black disks indicate that the line terminates at $z = \ell$.

1.3 Wave reflection

Below we shall first reverse engineer the wave composition (superposition) of two counterpropagating waves, which will allow us to decompose wavefields (plane waves or TEM waves along a transmission line) at an arbitrary position along the z-axis in terms of its progressive and regressive counterparts. Then we shall relate the ratio of the voltage and the current amplitudes to the voltage amplitudes of the progressive and regressive waves, which gives us the reflection coefficient for simple passive (resistive, short circuit, or open) terminations of transmission lines, but will also allow us to solve more complicated cases involving capacitative or inductive loads.

1.3.1 Wave decomposition

Recall that through Eqs. (1.27) and (1.29) we have explicity constructed the respective voltage and current amplitudes $V(z,t)$ and $I(z,t)$ in terms of its progressive and regressive (forward and backward propagating) wave constituents. This superposition is also referred to as *wave composition*.

Often, we want to achieve the reverse, i.e., to perform *wave decomposition*. To this end, one could inspect the voltage graphs on the left in Figure 1.1 for all z and for various instances in time. From this it should be straightforward to disentangle the progressive and regressive voltage components. However, if we only had voltage amplitude data for all z at a single instant in time, or for all t at a single z-location, then such a decomposition would be *impossible*.

Here, knowledge of both V and I comes to the rescue. To see this, let us suppose that both the voltage and current amplitudes are known at a certain point $z = z_1$ and instant in time $t = t_1$. Then, the voltage amplitudes $V^+(t_1 - z_1/c_0)$ and $V^-(t_1 + z_1/c_0)$ of the respective progressive and regressive waves simply follow from Eqs. (1.27) and (1.29) by considering the voltage-current amplitude vector according to

$$\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} V^+ + V^- \\ I^+ + I^- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ Y & -Y \end{pmatrix} \begin{pmatrix} V^+ \\ V^- \end{pmatrix}, \tag{1.41}$$

which may be inverted to yield

$$\begin{pmatrix} 1 & 1 \\ Y & -Y \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & Z \\ 1 & -Z \end{pmatrix} \Rightarrow \begin{pmatrix} V^+ \\ V^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} V + ZI \\ V - ZI \end{pmatrix}. \tag{1.42}$$

This amounts to wave decomposition.

1.3.2 Termination of transmission lines and reflection of TEM waves

Let us return to the example of two counterpropagating plane waves depicted in Figure 1.1. In that figure the set of traces are snapshots at different instants in time. Although these general wave solutions are valid anywhere in space, we may also choose to restrict the domain under consideration to a certain (possibly semi-infinite) interval of the z-axis, which for plane wave would amount to a layer (or a half space), and for transmission lines would amount to a (possibly semi-infinite) transmission line segment. For the sake of the discussion, we shall stick to transmission-line segments. In any case, in addition to specifying progressive and regressive waves solutions, we must now also specify boundary conditions where the segment ends.[19]

In Figure 1.5, we have depicted the situation at $z = 0$. In particular, we have depicted the voltage amplitudes of the progressive and regressive voltage amplitudes (left) the total voltage and current amplitudes (middle), and the ratio between the voltage and current

[19]Strictly speaking, in order to solve any *specific* TEM-wave problem along an infinite transmission line (or a plane-wave problem in infinite space), we would also have to *specify* boundary conditions for $z \to \pm\infty$. For instance, if we specify that waves should not arrive from $z \to \infty$, then the particular source-free solution in free space would have to exclude $V^-(t + z/c)$. In the boundary value problems occurring in physics, such boundary conditions are called singular boundary conditions.

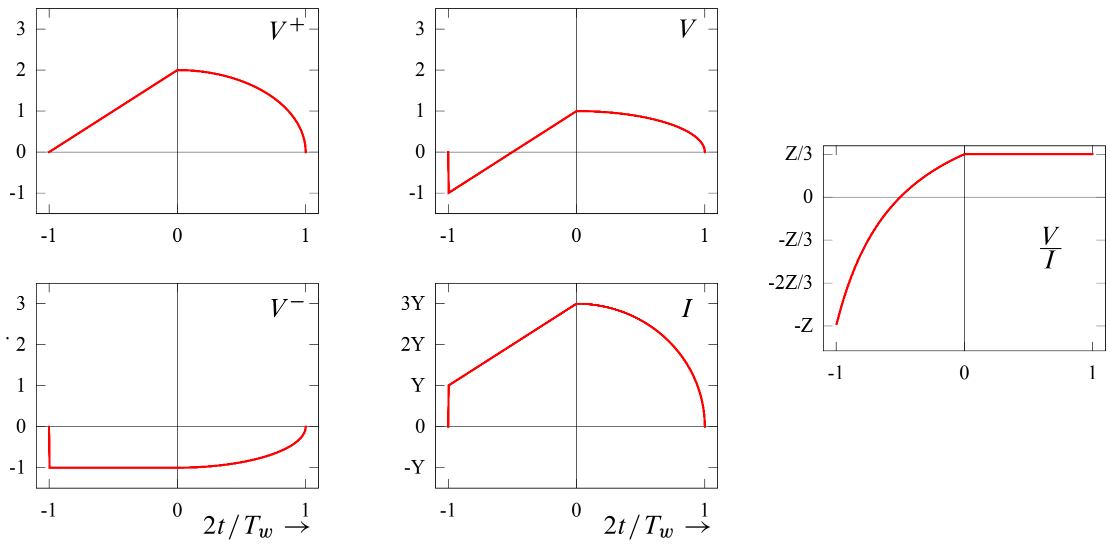


Figure 1.5: The counterpropagating plane (or TEM) wave amplitudes at $z = 0$ pertaining to the example depicted in Figure 1.1. The progressive (top left) and regressive (bottom left) voltage amplitudes combine to give $V(0,t) = V^+ + V^-$ (top middle) and $I(0,t) = Y_0(V^+ - V^-)$ (bottom middle). For $t > 0$ the signals V^+ and V^- have been chosen proportional, viz. $V^- = -V^+/2$, and hence the ratio V/I (right) is constant for $t > 0$.

amplitudes (right). We have chosen the times signatures of the progressive and regressive waves such that for $t > 0$ the signals V^+ and V^- are proportional.

In this case, $V^-(t) = -V^+(t)/2$ for $t > 0$. As a consequence, the ratio

$$\frac{V(0,t)}{I(0,t)} = \frac{V^+(t) + V^-(t)}{I^+(t) + I^-(t)} = \frac{1}{Y}\frac{V^+(t) + V^-(t)}{V^+(t) - V^-(t)} \overset{t>0}{=} \frac{1}{Y}\frac{1 + \left(-\frac{1}{2}\right)}{1 - \left(-\frac{1}{2}\right)} \overset{t>0}{=} \frac{Z}{3} \tag{1.43}$$

is constant for $t > 0$, as shown in the graph on the right in Figure 1.5. Note that for $t < 0$ the signals V^+ and V^- are not proportional, and hence the corresponding ratio V/I is not constant for $t < 0$.

Now, suppose that the signals V^+ and V^- had been proportional for all t, say that $V^-(t) = \Gamma V^+(t)$ with Γ a real constant. Then, we would have arrived at

$$\frac{V(0,t)}{I(0,t)} = \frac{V^+(t) + \Gamma V^+(t)}{Y V^+(t) - \Gamma Y V^+(t)} = Z\,\frac{1+\Gamma}{1-\Gamma}, \tag{1.44}$$

which for $-1 \le \Gamma < 1$ can take any positive value.

A constant ratio between a voltage and a current should have rung a bell by now. It is the voltage-current relation that characterises a *resistor*. In fact, we have just arrived at a solution to a specific reflection problem, namely that of a TEM wave that is incident from $z = -\infty$ and propagates along a semi-infinite transmission line with wave impedance Z towards $z = 0$. At $z = 0$ the transmission is terminated through an impedance Z_L

(where L stands for load), which induces a reflected wave with voltage amplitude $V^-(t) = \Gamma V^+(t)$, such that

$$Z_L = Z\,\frac{1+\Gamma}{1-\Gamma} \quad \longleftrightarrow \quad \Gamma = \frac{Z_L - Z}{Z_L + Z}. \tag{1.45}$$

Hence Γ is called the (voltage) reflection coefficient.

This elementary example belies the complexity of the underlying scattering problem. In the real world it is not so straightforward to manufacture a purely resistive load. Apart from simple reflection, more intricate scattering could also take place, which may result in mode conversion and radiation.[20] Further, a load may not be purely resistive. The idea of a resistive/reactive termination of a transmission line is explored in Question 1.5. Expanding on the idea of a TEM wave reflecting at a simple termination of a transmission line, Section 2.1 is devoted to TEM waves bouncing back and forth in cascades of transmission lines, or equivalently, plane waves undergoing multiple reflection and transmission in stacks of plane homogeneous layers with different material properties. Although it is illuminating to analyse resistive/reactive terminations, and more complicated reflection/transmission problems in the time domain, the analysis simplifies upon considering time-harmonic TEM (or plane) waves, which will be discussed in Chapters 3 and 4.

Questions

Question 1.1
A plane wave characterised by $\mathbf{E}^\mathrm{i} = V^\mathrm{i}(z,t)\mathbf{a}_x$ is travelling in the negative z-direction in a lossless half space $z > 0$ with plane wave impedance Z towards a perfectly conducting plate at $z = 0$. This results in a reflected wave characterised by $\mathbf{E}^\mathrm{r} = V^\mathrm{r}(z,t)\mathbf{a}_x$.

Sketch the following field and source vectors in a single figure in the plane $z = 0$:
$\mathbf{E}^\mathrm{i}/Z$, $\mathbf{E}^\mathrm{r}/Z$, $\mathbf{H}^\mathrm{i}$, $\mathbf{H}^\mathrm{r}$ and the surface current density $\mathbf{J}_S$.
Make sure that the relative amplitudes are drawn to scale.

Question 1.2 A uniform plane wave is travelling in free space in the positive z direction with a voltage amplitude at $z = 0$ given by $V^+(t) = [U(t) - U(t-T)]\ t/T$ [V]. Here, $U(t)$ and T denote the Heaviside step function, and the rise time of the pulse, respectively. Another plane wave is travelling in the negative z-direction with a voltage amplitude at $z = 0$ given by $V^-(t) = -U(t)\,t/T + U(t-T)\,(t/T - 1)$ [V].

a) Sketch the total voltage and current amplitudes side by side as a function of t, at $z = -cT$, $z = 0$, $z = cT$.

b) Sketch the total voltage and current amplitudes side by side as a function of z for $t = -T$, $t = 0$, and $t = T$.

c) If the plane wave travelling in the negative z-direction were characterised by a voltage amplitude at $z = 0$ given by $V^-(t) = -[U(t) - U(t-T)]\ t/T$ [V], then sketch the total voltage and current amplitudes side by side as a function of t, at $z = 0$.

[20] You will learn the basics regarding waveguide modes and elementary dipole radiation in the second half of this course.

d) In the latter example, compute the impedance $Z = V/I$ at $z = 0$, and try to interpret what the result signifies.

Question 1.3

The transmission line equations comprise a so-called hyperbolic system of partial differential equations with progressive and regressive propagating waves as their solutions. Electrical engineers often prefer to think in terms of (passive) circuits instead of wave equations. To accommodate for that trail of thought, one may discretise the transmission-line equations, and consider a so-called ladder network. Suppose that we have a transmission-line segment of length ℓ, chopped up into many segments of length Δz that are so short that mathematically, we may replace the derivative $\mathrm{d}f(z)/\mathrm{d}z$ of a function $f(z)$ by a finite-difference approximation $(f(z + \Delta z) - f(z))/(\Delta z)$.

a) In order to obtain a set of equations involving lumped electric-circuit elements, apply the finite-difference approximation to the transmission-line equations for a short transmission-line segment in between z and $z + \Delta z$. (Hint: eventually multiply by Δz.)

b) Sketch the equivalent circuit that describes the part of a ladder network in between z and $z + \Delta z$. Clearly indicate the current flowing in at z and out at $z + \Delta z$.

c) Imagine that our transmission lines were lossy. How would you adapt your ladder network to describe that? (Hint: think about lumped elements that dissipate power in circuit theory.)

Question 1.4

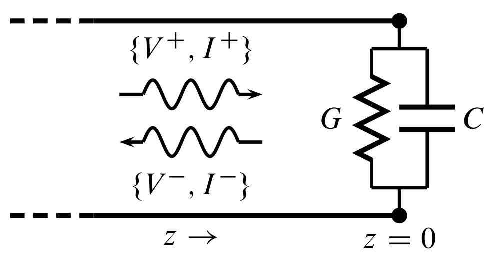
Our analysis of counterpropagating waves that has inducted us into reflection problems has left two questions to be answered.

a) First of all, what happens if $V^-(t)$ is proportional to a time-delayed copy of $V^+(t)$, with time delay T, i.e., $V^-(t) = \Gamma V^+(t - T)$? (Hint: try shifting the point at which the transmission line is terminated to $z = \ell$, and determine the voltage to current ratio at $z = \ell$. Subsequently, determine ℓ such that that ratio becomes constant.)

b) Secondly, one may consider $|\Gamma| > 1$. This case may be interpreted as a reflection problem for a semi-infinite transmission line in two different ways, i.e., we may either regard the progressive wave as the incident wave, or the regressive one. Determine and discuss the associated load impedances.

Question 1.5

Consider a two-conductor waveguide that is aligned with the z-axis and operates as a transmission line (see the figure on the right). The TEM-wave impedance and the speed of propagation are Z and c, respectively. The waveguide is terminated by a resistive/reactive load at $z = 0$. This resistive/reactive load may be characterised by a conductance G parallel to a capacitance C.



An incident TEM wave propagates in the positive z-direction. Its voltage amplitude is given by $V^+(t) = U(t)$, for $z \leq 0$, with $U(t)$ denoting the unit step function. The incident wave arrives at the load at time $t_0 = 0$. This results in the excitation of a reflected wave with voltage amplitude $V^-(t)$ for $z \leq 0$. Because the load is not purely resistive, the pulse signature of the reflected wave will be different from the step-function signature of the incident wave, and hence we can no longer work with a simple reflection coefficient.

a) Express the current amplitudes $I^+(t)$ and $I^-(t)$ in terms of $V^+(t)$ and $V^-(t)$, respectively.

b) Apply the voltage-current relation at $z = 0$ to derive the differential equation with respect to time for $V^-(t)$.

c) Give the time constant, τ, associated with the homogeneous solution.[21]

Since we do not intend to draw too heavily on circuit theory, let us provide the solution for $V^-(t)$ up to a few constants, i.e.,

$$V^-(t) = \left(A + B\, e^{-t/\tau}\right) U(t)$$

d) Determine A, and B.

e) Determine the total voltage and current amplitudes $V(0,t)$ and $I(0,t)$ for $t \downarrow 0$, and give an interpretation of the result.

f) Determine the total voltage and current amplitudes $V(0,t)$ and $I(0,t)$ for $t \to \infty$, and give an interpretation of the result.

g) Determine the reflected voltage amplitude $V^-(t + z/c)$ for $z < 0$.

[21] In a homogeneous first-order constant coefficient linear differential equation in time, $(\tau \frac{\mathrm{d}}{\mathrm{d}t} + 1)\, f_{\mathrm{hom}}(t) = 0$, the parameter τ is called the time constant.

Chapter 2

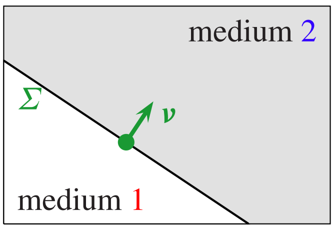
Week 2 — TL-parameters and reflection and transmission of 1-D waves

2.1 Practical applications of waves in time and one spatial dimension

2.1.1 Boundary conditions for penetrable media (reminder)

In *Electromagnetics I* you have learnt that the tangential components of both the electric and magnetic field must be continuous across a source-free interface between two penetrable media. Let us refer to the penetrable media as medium 1 and medium 2. Again, we assume that the interface under consideration is smooth, which implies that upon zooming in on a sufficiently small neighbourhood about an arbitrary point of observation on the interface, we may regard that interface as locally flat.

Sometimes, we would like to model the action of electric current sources, which can be accounted for by the presence of an electric surface current density $\mathbf{J}_S$ on the interface. Let us follow convention, and let $\boldsymbol{\nu}$ denote the normal to the boundary Σ that points away from medium 1 into medium 2. Then, the boundary conditions at the PEC surface read



$$\boldsymbol{\nu} \times \mathbf{E}_2 - \boldsymbol{\nu} \times \mathbf{E}_1 = \mathbf{0}, \tag{2.1a}$$
$$\boldsymbol{\nu} \times \mathbf{H}_2 - \boldsymbol{\nu} \times \mathbf{H}_1 = \mathbf{J}_S, \tag{2.1b}$$
$$\boldsymbol{\nu} \cdot \mathbf{B}_2 - \boldsymbol{\nu} \cdot \mathbf{B}_1 = 0, \tag{2.1c}$$
$$\boldsymbol{\nu} \cdot \mathbf{D}_2 - \boldsymbol{\nu} \cdot \mathbf{D}_1 = \rho_S. \tag{2.1d}$$

where the subscripts $_1$ and $_2$ indicate that the point of observation on the interface is approached from medium 1 and medium 2, respectively.

Note that Eqs. (2.1a) and (2.1b) imply that in the absence of sources the tangential components of both $\mathbf{E}$ and $\mathbf{H}$ are continuous across a penetrable interface. We shall exploit the transverse field continuity below to solve reflection-transmission (Rx-Tx) problems.

2.1.2 Cascades of plane dielectric layers or transmission lines

In Section 1.3, we considered a transmission line with characteristic impedance Z, terminated by a load with resistivity Z_L at some point, say $z = 0$. We discovered that a progressive TEM-wave propagating along the line towards the load will be reflected at the load, causing a regressive wave with voltage amplitude $V^-(t) = \Gamma V^+(t)$ with $\Gamma = (Z_L - Z)/(Z_L + Z)$.

Below, we shall demonstrate that plane waves, normally incident on an interface at $z = 0$ between two half spaces filled with different instantaneously reacting homogeneous media will also result in a reflected wave. However, in that case a plane wave will be transmitted into the other half space, rather than being absorbed in a load.

So, let us consider two half spaces containing different homogeneous instantaneously reacting LTI media, separated by a source-free interface at $z = 0$. The half spaces with the different media are labelled as 1 ($z < 0$) and 2 ($z > 0$), with wavespeeds and wave impedances c_1, Z_1, c_2, and Z_2, respectively. Further, let us assume that a uniform plane wave propagating in the positive z-direction is incident on that interface. As one might expect, the incident field may be partially reflected and partially transmitted (see Figure 2.1). This is problem is equivalent to the reflection and transmission of TEM waves that occurs

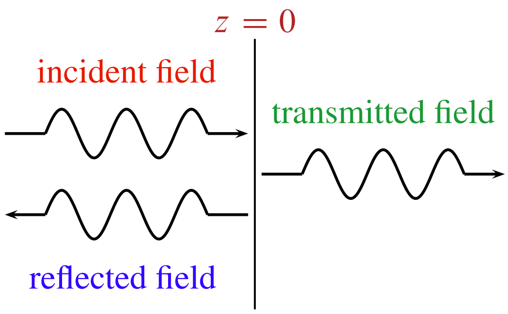


Figure 2.1: Time-domain plane wave at normal incidence on an interface, and the reflected and transmitted plane waves.

when a TEM wave is incident on a junction between two transmission lines with the same conductor geometry, but filled with (or embedded in) different materials.

Above, we have established that the tangential components of the electromagnetic field vectors are continuous across source-free interfaces. Let us write $\mathbf{E} = V\mathbf{e}_\mathrm{t}$ and $\mathbf{H} = I\mathbf{h}_\mathrm{t}$. In this reader, we shall always choose the transverse vectors ($\mathbf{e}_\mathrm{t}$ and $\mathbf{h}_\mathrm{t}$) to be the same on either side of the source-free interface. Since $\boldsymbol{\nu} = \mathbf{a}_z$, the relevant boundary conditions in Eq. (2.1) may be cast in the following form

$$\left[\lim_{z\downarrow 0} V(z,t) - \lim_{z\uparrow 0} V(z,t)\right] \mathbf{a}_z \times \mathbf{e}_\mathrm{t} = \mathbf{0}, \tag{2.2a}$$

$$\left[\lim_{z\downarrow 0} I(z,t) - \lim_{z\uparrow 0} I(z,t)\right] \mathbf{a}_z \times \mathbf{h}_\mathrm{t} = \mathbf{0}. \tag{2.2b}$$

Since $\mathbf{a}_z \perp \mathbf{e}_\mathrm{t}$ and $\mathbf{a}_z \perp \mathbf{h}_\mathrm{t}$, we conclude that for TEM waves (which include plane waves at normal incidence), the voltage and current amplitudes must be continuous across source-free interfaces.

Now, let us apply the composition relation given by Eq. (1.41) to the voltage-current vectors on either side of the interface. Further, for $z \uparrow 0$, let us denote the progressive incident voltage-current vector through a superscript $^{\mathrm{i}}$, and the regressive reflected voltage-current vector through a superscript $^{\mathrm{r}}$. Likewise, for $z \downarrow 0$, let us denote the progressive transmitted voltage-current vector through a superscript $^{\mathrm{t}}$. Then, the continuity of the voltage-current vector across the interface may be expressed as

$$\begin{pmatrix} V^{\mathrm{i}} + V^{\mathrm{r}} \\ I^{\mathrm{i}} + I^{\mathrm{r}} \end{pmatrix}_{z=0} = \begin{pmatrix} V^{\mathrm{t}} \\ I^{\mathrm{t}} \end{pmatrix}_{z=0}. \tag{2.3}$$

Now, it is important to recognise that in medium 2 there is only wave propagation in one direction (the positive z-direction), because the transmitted wave is generated at $z = 0$ as a result of the incident wave. Hence, the transmitted current amplitude is related to the transmitted voltage amplitude via $V^{\mathrm{t}}(t) = +Z_2 I^{\mathrm{t}}(t)$. Owing to the continuity of the voltage-current vector, the voltage to current ratio at the other side of the interface must also be equal to Z_2, and hence the reflection problem is equivalent to that of a semi-infinite transmission line with characteristic impedance Z_1 and wavespeed c_1, terminated through a load with impedance Z_2. Further, the reflected voltage amplitude at $z = 0$ is related to the incident one via $V^{\mathrm{r}}(t) = \Gamma V^{\mathrm{i}}(t)$. We have solved this reflection problem in Section 1.3, and hence we already know that $\Gamma = (Z_2 - Z_1)/(Z_2 + Z_1)$.

Because we also want to determine the transmitted wavefield, we introduce the voltage transmission coefficient T at $z = 0$ through $V^{\mathrm{t}}(t) = T V^{\mathrm{i}}(t)$. Application of the composition relation yields

$$\begin{pmatrix} 1 + \Gamma \\ (1 - \Gamma) Y_1 \end{pmatrix} V^{\mathrm{i}}|_{z=0} = \begin{pmatrix} T \\ T Y_2 \end{pmatrix} V^{\mathrm{i}}|_{z=0}. \tag{2.4}$$

Upon using $1/Y = Z$, and dividing voltage by current, we obtain

$$\frac{1 + \Gamma}{(1 - \Gamma) Y_1} = \frac{1}{Y_2} \Rightarrow (1 + \Gamma) Z_1 = (1 - \Gamma) Z_2 \Rightarrow \Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{2.5}$$

$$1 + \Gamma = T \Rightarrow T = \frac{2 Z_2}{Z_2 + Z_1} \tag{2.6}$$

An alternative way of solving for the reflection and transmission coefficients is to multiply Eq. (2.4) by the decomposition matrix

$$\begin{pmatrix} 1 & 1 \\ Y_1 & -Y_1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix}, \tag{2.7}$$

associated with medium 1, which leads to

$$\begin{pmatrix} 1 \\ \Gamma \end{pmatrix} V^{\mathrm{i}}|_{z=0} = \frac{1}{2} \begin{pmatrix} 1 + Z_1/Z_2 \\ 1 - Z_1/Z_2 \end{pmatrix} T V^{\mathrm{i}}|_{z=0}, \tag{2.8}$$

from which T, and subsequently Γ may readily be determined.

Multiple reflection and transmission occurs when there are two or more interfaces or junctions, or two loads terminating a finite transmission line at both ends. These situations

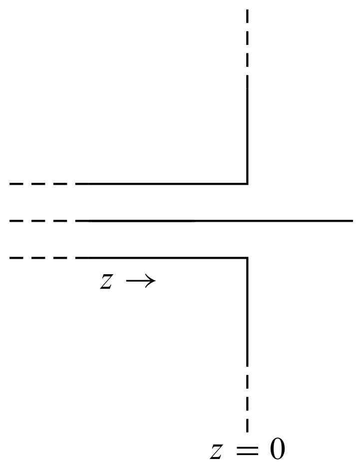


Figure 2.2: Longitudinal section of a semi-infinite coaxial cable, with the outer rim connected to an infinite screen, and the inner conductor protruding beyond the screen.

may effectively be analysed with the aid of bounce diagrams, to be discussed in the next section.

In the real world, transmission lines are not infinite, and hence will always have to be terminated at both ends in some way. Let us consider three types of terminations of the coaxial cable, i.e., the short circuit, the resistive load and the open circuit, respectively.

- For the coaxial cable, the short circuit is the easiest termination to realise in practice, since a plane PEC cap that connects the inner and outer conductors and shields the interior of the coaxial cable from the environment, should act as a very good short circuit.

- By connecting a (semi-infinite) coaxial cable with wave impedance Z_1 to another semi-infinite coaxial cable of the same cross section (but filled with a different instantaneously reacting material with wave impedance Z_2), one would obtain the ideal real (resistive) constant $Z_L = Z_2$ termination. Since it is certainly awkward (if not impossible) to use a terminating coaxial cable that is semi-infinite, one may attempt to replace it by a finite segment of coaxial cable, as long as it has the same real impedance, while being lossy at the same time. This would have to involve both electric and magnetic losses in equal measure and is hard to achieve in practice for the entire frequency band.

- It may seem easy to create an open-circuit termination by cutting a transmission line and leaving it open. However, this will give rise to electromagnetic radiation, and does not correspond to a perfect open circuit in circuit theory. In fact, suppose that we insert one end of a coaxial cable into a hole in a perfectly conducting screen, and that the rim of the outer conductor makes perfect electric contact with the perimeter of the hole in the screen. Further, let us assume that the inner conductor protrudes some distance[1] beyond the plane in which the screen is located. Then, we would have

[1]A distance that is considerably larger than the distance between the inner and outer conductors of the cable

created an effective quarter-wavelength monopole antenna, depicted in Figure 2.2. Like most dipole-like antennas, this antenna is effectively narrow band, with resonance (a resistive load) occurring if the protruding inner conductor is slightly shorter than a quarter of the wavelength of the incident wave. For slightly lower (higher) frequencies, such antennas become slightly capacitive (inductive).

2.1.3 Bounce diagrams

A bounce diagram is a graphical tool designed to facilitate the bookkeeping of multiple TEM-wave (or plane-wave) reflections and transmissions.

We shall describe how a bounce diagram works through the analysis of a simple transmission line circuit depicted in Figure 2.3. A voltage source V_g with internal impedance

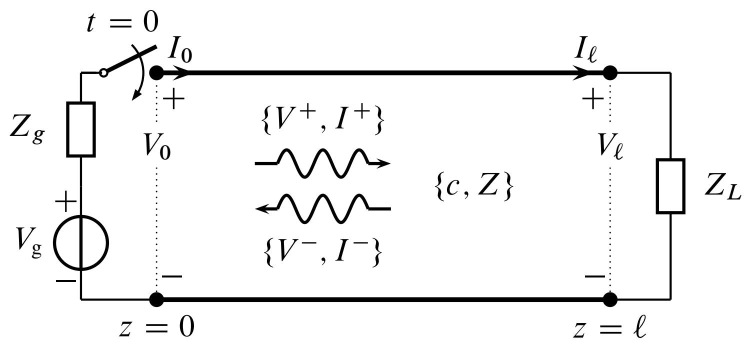


Figure 2.3: A simple transmission-line circuit

Z_g is connected to a transmission line of length ℓ via a switch that is switched on at $t = 0$. Unless otherwise specified, we shall assume that the voltage source is a battery, delivering a constant voltage. The wavespeed and the characteristic impedance of the transmission line are denoted as c and $Z = 1/Y$, respectively. Let us discuss the chain of events following the time $t = 0$, when the connection between the source and the transmission line is switched on.

- Immediately after $t = 0$, a progressive TEM wave with voltage amplitude V_1^+ starts to propagate along the transmission line towards the load with impedance Z_L, located at $z = \ell$. Do not worry about the subscript $_1$. It is a label that we have introduced with the benefit of foresight.

  Because of the finite speed of light, the TEM wave is oblivious to the termination, i.e., it only senses the characteristic impedance of the line, and hence the associated current amplitude of the TEM wave is $I_1^+ = Y V_1^+ = V_1^+/Z$.

- At $z = 0$, the voltage $V_1^+(t)$, which is also referred to as the injection current, $V_{\text{inj}}(t)$, follows from simple voltage division, i.e.,

$$V_1^+ = \frac{Z}{Z + Z_g} V_g. \tag{2.9}$$

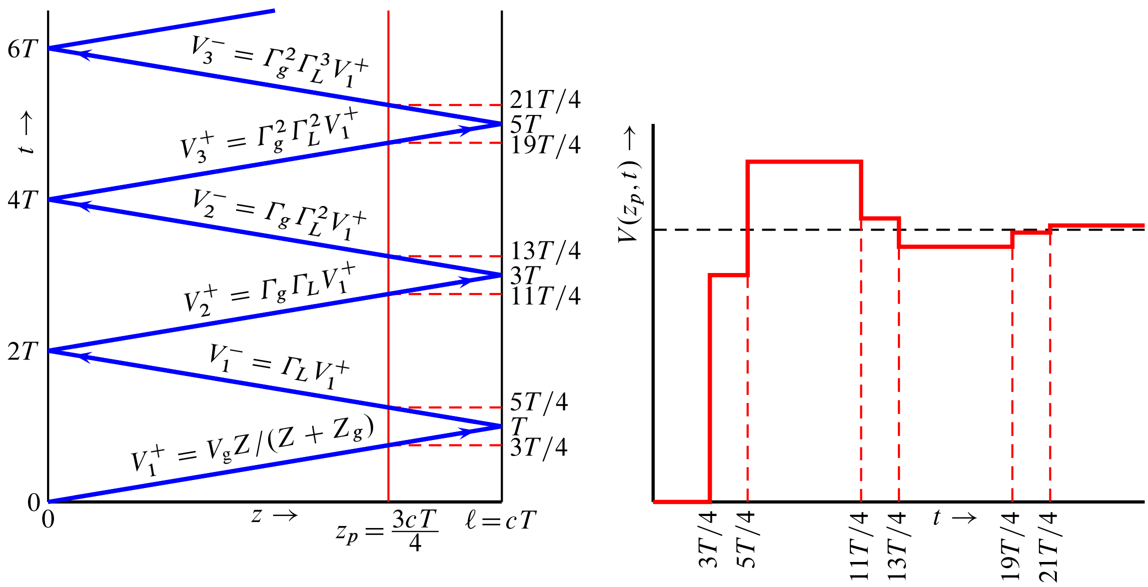


Figure 2.4: Bounce diagram for the transmission-line circuit problem depicted in Figure 2.3, and the voltage amplitude recorded by a probe at $z_p = 3cT/4$.

- If the line has not been terminated properly at $z = \ell$, the TEM wave that arrives at the load at time $t = T = \ell/c$ will be partially reflected, resulting in a regressive TEM wave with voltage amplitude $V_1^-(t + \ell/c) = \Gamma_L V_1^+(t - \ell/c)$, propagating towards the voltage source with impedance Z_g at $z = 0$. The reflection coefficient has already been found to be $\Gamma_L = (Z_L - Z)/(Z_L + Z)$.

- So, how could we analyse the possible partial reflection of the regressive wave from time $t = 2T$? Well, the voltage source V_g itself acts as a short circuit for any signals other than the one it keeps generating, and hence, all we need is the reflection coefficient $\Gamma_g = (Z_g - Z)/(Z_g + Z)$. The resulting progressive reflected TEM wave has voltage amplitude

$$V_2^+(t) = \Gamma_g V_1^-(t) = \Gamma_g \Gamma_L V_1^+(t - 2\ell/c). \tag{2.10}$$

  Hence, this TEM wave reflection takes place from $t = 2T = 2\ell/c$, and the subscript $_2$ indicates that the voltage amplitude V_2^+ represents the second progressive TEM wave that will propagate down the line.

- Conceptually, this should be simple enough to digest. Aside from the timing issue, the voltage and current amplitudes of the successive reflections are readily determined.

  The bounce diagram depicted in Figure 2.4 is a graphical representation of this multiple scattering process that contains both amplitude and timing information. It consists of a set of *characteristic curves* in the z-t-plane along which the wavefronts of the successive progressive waves (positive flanks, V_n^+) and regressive waves (negative flanks, V_n^-) propagate.

- Since we have assumed that the voltage source is a battery that delivers a constant voltage, and the switch was activated at $t = 0$, all the TEM waves propagating back and forth have a scaled and time-delayed Heaviside step function pulse signature.

  Suppose that we want to determine the electric field by recording the voltage amplitude using a (non-interfering) probe at $z = z_p = 3\ell/4$. The probe should measure the voltage amplitude depicted on the right in Figure 2.4. That figure has been constructed from the bounce diagram by following the vertical red solid line in the bounce diagram in the upward time direction, and, upon crossing a *characteristic curve*, adding the corresponding voltage amplitude of the pertaining progressive or regressive wave.

- If one omits the timing aspects, it is a straightforward matter to determine the voltage and current amplitudes of the progressive waves that have reflected $2n$ times, with n integer (starting at $n = 0$). In particular, we have

$$V^+_{n+1} = (\Gamma_g \Gamma_L)^n \frac{Z}{Z + Z_g} V_g, \tag{2.11}$$

$$I^+_{n+1} = (\Gamma_g \Gamma_L)^n \frac{1}{Z + Z_g} V_g. \tag{2.12}$$

  Likewise, the voltage and current amplitudes of the regressive waves that have reflected $2n + 1$ times, with n integer (starting at $n = 0$) are found to be

$$V^-_{n+1} = \Gamma_L (\Gamma_g \Gamma_L)^n \frac{Z}{Z + Z_g} V_g, \tag{2.13}$$

$$I^-_{n+1} = -\Gamma_L (\Gamma_g \Gamma_L)^n \frac{1}{Z + Z_g} V_g. \tag{2.14}$$

- In principle, this process of successive reflections will go on forever. Thanks to the finite travel time and the fact that the magnitudes of the reflection coefficients do not exceed unity, convergence will generally occur.

  If we assume that $|\Gamma_L \Gamma_g| < 1$, convergence will occur, implying that for $t \to \infty$ a steady-state is reached for both the progressive and regressive TEM waves. The steady-state progressive and regressive voltage $V^\pm_{\text{steady}} = \lim_{t\to\infty} V^\pm(z,t)$ and current $I^\pm_{\text{steady}} = \lim_{t\to\infty} I^\pm(z,t)$ amplitudes follows through application of the geometric series[2] and elementary algebra, i.e.,

$$V^+_{\text{steady}} = \frac{1}{1 - \Gamma_g \Gamma_L} \frac{Z}{Z + Z_g} V_g = \frac{Z_L + Z}{2(Z_L + Z_g)} V_g, \tag{2.15}$$

$$I^+_{\text{steady}} = Y V^+(z,t)|_{t\to\infty} = \frac{Z_L/Z + 1}{2(Z_L + Z_g)} V_g, \tag{2.16}$$

$$V^-_{\text{steady}} = \frac{\Gamma_L}{1 - \Gamma_g \Gamma_L} \frac{Z}{Z + Z_g} V_g = \frac{Z_L - Z}{2(Z_L + Z_g)} V_g, \tag{2.17}$$

$$I^-_{\text{steady}} = -Y V^-(z,t)|_{t\to\infty} = -\frac{Z_L/Z - 1}{2(Z_L + Z_g)} V_g. \tag{2.18}$$

[2]The geometric series is given by $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$ for $|x| < 1$.

- It should come at no surprise that the steady-state total voltage and current amplitudes of the TEM-wave electromagnetic fields,

$$V_{\text{steady}} = \lim_{t\to\infty} V(z,t) = V^{+}_{\text{steady}} + V^{-}_{\text{steady}} = \frac{Z_L}{Z_L + Z_g} V_g, \tag{2.19}$$

$$I_{\text{steady}} = \lim_{t\to\infty} I(z,t) = I^{+}_{\text{steady}} + I^{-}_{\text{steady}} = \frac{1}{Z_L + Z_g} V_g \tag{2.20}$$

  are independent of the characteristic impedance Z of the transmission line, since the steady-state situation just amounts to a battery connected by PEC wires to two series resistors. Note that the steady state is still composed of progressive and regressive wave constituents.

In case the configuration consists of cascades of plane dielectric layers or transmission lines, the associated reflection/transmission problems for the corresponding uniform plane or TEM waves can also readily be analysed with the aid of bounce diagrams.

Let us now turn our attention to the determination of the transmission-line parameters by analysing the transverse field distribution.

2.2 The transverse field distribution in transmission lines

The transmission-line parameters L and C will be uniquely determined once we apply the proper voltage and current normalisation, which involves solving the transverse field problem.

Recall that we had deferred the treatment of the partial differential equations (cf. Eq. (1.40))

$$\nabla \times \mathbf{e}_{\mathrm{t}} = 0 \quad \text{and} \quad \nabla \times \mathbf{h}_{\mathrm{t}} = 0. \tag{2.21}$$

According to the so-called Helmholtz decomposition, any sufficiently smooth and rapidly decaying vector field $\mathbf{F}$ may be written as $\mathbf{F} = -\nabla\Phi + \nabla \times \mathbf{A}$. Hence, upon writing

$$\mathbf{e}_{\mathrm{t}} = -\nabla_{\mathrm{t}}\Phi(x,y) \qquad \text{with } \nabla_{\mathrm{t}} = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y}, \tag{2.22}$$

we infer that Eq. (1.40a),

$$\nabla \times \mathbf{e}_{\mathrm{t}} = -\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial y} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial x} = 0 \tag{2.23}$$

is automatically satisfied. Obviously, any constant multiplying the gradient of a scalar 2-D field would have performed equally well. In particular, upon inspecting Eq. (1.36a), we may conveniently choose $\mathbf{h}_{\mathrm{t}} = (L/\mu)\nabla_{\mathrm{t}}\Psi(x,y)$. As a consequence, $\nabla \times \mathbf{h}_{\mathrm{t}} = 0$ is also satisfied automatically, and Φ and Ψ are related by

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \tag{2.24a}$$

$$\frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}, \tag{2.24b}$$

which are known as the Cauchy-Riemann equations.[3]

Elimination of Ψ from Eq. (2.24) leads to Laplace's equation for Φ (Ψ satisfies the same equation),

$$\nabla_{\mathrm{t}}^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \tag{2.25}$$

which you have encountered before in *Electromagnetics I*.

Although Eqs. (1.34), (1.36), (2.22) and (2.25) are the equations we would like to solve, we still have to supplement the Laplace equation for Φ with the appropriate boundary conditions, which involves voltage normalisation, and still have to fix the value of C relative to L, which involves current normalisation. The voltages and currents will be normalised such that they satisfy the quasi-static Kirchhoff laws, and may hence be identified with the voltages and current definitions used in circuit theory. This is essential for the description of the interaction of electronic circuit devices that are connected by tranmission lines. The reason for this is that it it allows us to design electronic circuits using lumped (0-dimensional) elements as long as the devices are much smaller than the smallest relevant wavelength, while accounting for transverse electromagnetic propagation effects in the communication between the devices.

Recall that the tangential components of the electric field must vanish on the surface of a perfect electric conductor. Hence, $\mathbf{e}_{\mathrm{t}} = -\nabla_{\mathrm{t}}\Phi$ must be perpendicular to the PEC surface, or, in other words, the gradient of Φ must point in the direction of the normal $\boldsymbol{\nu}$ to the surface (or in the direction of $-\boldsymbol{\nu}$). Since Φ does not change in any direction perpendicular to its gradient, it must be constant on a PEC.

If the configuration were to consist of only one cylindrical conductor on which Φ is constant, say Φ_0, then, $\Phi = \Phi_0$ everywhere would be the unique solution to Laplace's equation *with boundary conditions*, and $\mathbf{e}_{\mathrm{t}} = \mathbf{0}$ would be the corresponding *trivial* transverse vector field. As a consquence, for a non-trivial solution for Φ to exist, we must have at least two disjoint (unconnected in the transverse plane) cylindrical PECs. A configuration with three or more disjoint PECs is called a multi-conductor waveguide.[4]

We shall restrict ourselves to two-conductor transmission lines with perfectly conducting surfaces Σ_1 and Σ_2. In such a configuration, we want to establish the appropriate voltage and current normalisations. The voltage normalisation will give a condition for the normalised potential difference between Φ on Σ_1 and Σ_2. The current normalisation will fix C relative to L.

First, let us consider voltage normalisation. In Figure we have depicted the cross-section of a two-conductor waveguide. The path $\mathcal{C}_e$ connects an arbitrary point O on the outer conductor to an arbitrary point Q on the inner conductor. Kirchhoff's voltage law in circuit theory amounts to *(quasi)-statically neglecting* the time rate of change of the magnetic flux density in the Faraday-Henry equation, i.e.,

$$\nabla \times \mathbf{E}\big|_{\frac{\partial \mathbf{B}}{\partial t}=\mathbf{0}} = \mathbf{0} \Longleftrightarrow \mathbf{E} = -\nabla V_{\mathrm{circuit}}, \tag{2.26}$$

[3]A complex function $\Xi = \Phi + \mathrm{j}\Psi$ of a complex variable $\zeta = x + \mathrm{j}y$ is analytic (differentiable) on a certain domain of the complex ζ-plane if and only if its respective real and imaginary parts $\Phi = \Phi(x, y)$ and $\Psi = \Psi(x, y)$, regarded as functions on the 2-D (x, y)-plane, satisfy the Cauchy-Riemann equations.

[4]Often referred to as a multi-conductor transmission line, which is slightly sloppy, since a transmission line is an ideal abstraction of a physical waveguide that is thought to support (quasi)-TEM waves only.

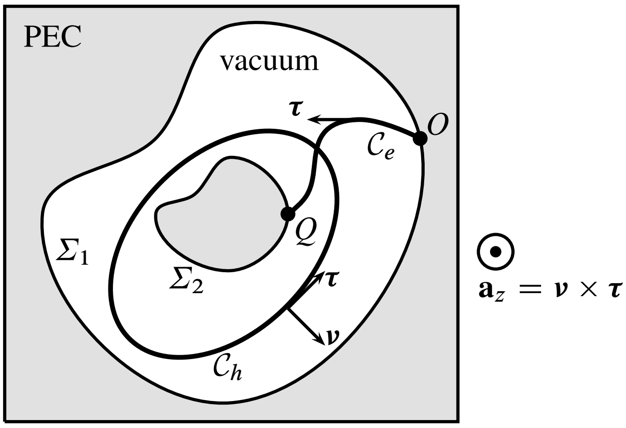


Figure 2.5: Cross-section of a two-conductor waveguide. The path $\mathcal{C}_e$ connects an arbitrary point O on the outer conductor to an arbitrary point Q on the inner conductor. The path $\mathcal{C}_h$ is an arbitrary closed path that circumnavigates the inner conductor once in a counter-clockwise fashion.

where V_{circuit} denotes the voltage defined in circuit theory.[5]

Voltage normalisation amounts to identifying the circuit theory voltage (potential difference), V_{circuit} in the cross-sectional plane with the TEM-wave voltage amplitude. So, let us consider the TEM-wave electric field by combining Eqs. (1.9) and (2.22)

$$\mathbf{E} = -V(z,t)\nabla_{\text{t}}\Phi(x,y) \tag{2.27}$$

in the contour-integral representation for $V_{\text{circuit}}(z,t)$ along the path from O to Q in the cross-sectional plane. This leads to

$$V_{\text{circuit}}(z,t) = \int_O^Q -\mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = V(z,t)\int_O^Q \nabla_{\text{t}}\Phi\cdot \mathrm{d}\boldsymbol{\ell} \overset{\text{desired}}{=} V(z,t), \tag{2.28}$$

where the last equality enforces the equivalence of the two voltages. Hence, voltage normalisation amounts to

$$\Phi|_{\mathbf{r}\in\Sigma_2} - \Phi|_{\mathbf{r}\in\Sigma_1} = 1.$$

Without loss of generality[6], we may set $\Phi|_{\mathbf{r}\in\Sigma_1} = 0$, thus fully defining the Laplace problem for the normalised potential Φ,

$$\nabla_{\text{t}}^2\Phi = 0 \qquad \text{with } \Phi|_{\mathbf{r}\in\Sigma_2} = 1 \text{ and } \Phi|_{\mathbf{r}\in\Sigma_1} = 0. \tag{2.29}$$

Current normalisation amounts to identifying the total current flowing through the inner conductor[7] I_{circuit} with the TEM-wave current amplitude. So, let us consider the TEM-wave magnetic field

$$\mathbf{H} = -I(z,t)\frac{L}{\mu}\mathbf{a}_z \times \nabla_{\text{t}}\Phi(x,y) \tag{2.30}$$

[5] Although the TEM-wave electromagnetic fields under consideration are certainly not quasi-static, the transverse fields are static in a cross-sectional plane that co-moves with the wave at the speed of light.

[6] as far as the resulting fields are concerned

[7] or actually, “along the inner conductor”, in the case of a perfect conductor.

in the surface-integral representation for $I_{\text{circuit}}(z,t)$,

$$I_{\text{circuit}}(z,t) = \int_{\mathcal{S}} \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \oint_{\mathcal{C}_h = \partial\mathcal{S}} \mathbf{H} \cdot \mathrm{d}\boldsymbol{\ell}$$
$$= I(z,t)\frac{L}{\mu}\oint_{\mathcal{C}_h} (-\mathbf{a}_z \times \nabla_{\mathrm{t}}\Phi) \cdot \boldsymbol{\tau}\, \mathrm{d}\ell$$
$$= I(z,t)\frac{L}{\mu}\oint_{\mathcal{C}_h} -\boldsymbol{\nu} \cdot \nabla_{\mathrm{t}}\Phi\, \mathrm{d}\ell \overset{\text{desired}}{=} I(z,t), \tag{2.31}$$

where we have used $D_z = 0$ in the Ampère-Maxwell equation, and $\mathrm{d}\boldsymbol{\ell} = \boldsymbol{\tau}\,\mathrm{d}\ell$. Hence, upon employing $LC = \mu\varepsilon$, current normalisation leads to

$$C = \varepsilon \oint_{\mathcal{C}_h} -\boldsymbol{\nu} \cdot \nabla_{\mathrm{t}}\Phi\, \mathrm{d}\ell. \tag{2.32}$$

At this point, we have almost completely analysed wave propagation in one spatial dimension and time. The only topic left to be considered before we turn our attention to time-harmonic fields in Chapters 3 and 4 is the exchange of energy in the electromagnetic field.

2.3 The electromagnetic power balance

The electromagnetic force acting on a charged particle moving with velocity $\mathbf{v}$ in free space is given by

$$\mathbf{F} = q\mathbf{E} + q\mu_0\mathbf{v} \times \mathbf{H}. \tag{2.33}$$

While the particle is moving, the electromagnetic field carries out *mechanical* work on it. In the time interval $t_0 < t < t_0 + \Delta t$, with Δt so short that the velocity of the particle has hardly changed at all, the displacement of the particle is $\mathbf{v}\Delta t$, and the work carried out by the electromagnetic force is

$$W_{\text{mech}}(t_0 + \Delta t) - W_{\text{mech}}(t_0) = \mathbf{F} \cdot \mathbf{v}\Delta t + o(\Delta t) = q\mathbf{E} \cdot \mathbf{v}\Delta t + o(\Delta t), \tag{2.34}$$

in which o denotes Landau's little-order symbol.[8] Note that the Lorentz force does not carry out work on the particle, since $\mathbf{v} \times \mathbf{H}$ is orthogonal to the particle displacement.

Employing the usual definition of the derivative with respect to time, we find the following expression for the *power* delivered to the particle by the electromagnetic field.

$$P_{\text{diss}} = \frac{\partial W_{\text{mech}}}{\partial t} = \lim_{\Delta t \downarrow 0} \frac{W(t + \Delta t) - W(t)}{\Delta t} = q\mathbf{E} \cdot \mathbf{v}. \tag{2.35}$$

For a "cloud" of individual charged particles with charge q_i and velocity $\mathbf{v}_i$, we would have

$$P_{\text{diss}} = \sum_i q_i \mathbf{E} \cdot \mathbf{v}_i. \tag{2.36}$$

[8] A function $f(h)$ of h is $o(h^\lambda)$ with $\lambda \geq 0$, if $\lim_{h \to 0} |f(h)/h^\lambda| = 0$.

Subsequently, let us consider a *continuous charge distribution* with charge density $\rho(\mathbf{r}, t)$. An elementary volume $\mathrm{d}v$ would then carry a charge $\rho\,\mathrm{d}v$ and the power delivered by the field would be given by

$$P_{\text{diss}} = \int_{\mathcal{V}_\infty} \mathbf{E} \cdot \mathbf{v}\,\rho\,\mathrm{d}v = \int_{\mathcal{V}_\infty} \mathbf{E} \cdot \mathbf{J}\,\mathrm{d}v. \tag{2.37}$$

where $\mathcal{V}_\infty$ denotes the volume $\mathbb{R}^3$, or equivalently, all of space.

One may wonder where the energy goes as electromagnetic power is delivered to the system of charged particles. Through the mechanical work, the energy is converted into heat, so although energy must be conserved in the universe, it can be lost from the electromagnetic field. Hence, the subscript "$_{\text{diss}}$", which stands for *dissipated* does not constitute an "energy crisis".

Having identified Eq. (2.37) as a dissipated power, we may wonder whether we could derive a power balance through inspection, by trying to link a volume integral of $\mathbf{E}\cdot\mathbf{J}$ to Maxwell's equations. Taking a closer look at the Ampère-Maxwell equation (cf. Eq. (1.1b)), it stands to reason to first evaluate the scalar product of the Ampère-Maxwell equation and $\mathbf{E}$, which yields

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}. \tag{2.38}$$

A sense of symmetry (or fairness if you will) suggests that since we have just evaluated the scalar product of one of Maxwell's equations and $\mathbf{E}$, we should also consider the scalar product of the Faraday-Henry equation (cf. Eq. (1.1a)) of Section 9.3) and $\mathbf{H}$. This results in

$$\mathbf{H} \cdot \nabla \times \mathbf{E} = -\mathbf{H} \cdot \mathbf{K} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{2.39}$$

See Footnote [1] to learn more about the magnetic current density that we have just sneaked in. From the vector identity $\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}$, we infer that we should subtract Eq. (2.39) from Eq. (2.38), resulting in

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{K} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{2.40}$$

Subsequently, let us integrate the result over a volume $\mathcal{V}$ with boundary $\mathcal{S} = \partial\mathcal{V}$, and use Gauss' theorem. This leads to

$$-\int_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}) \cdot \mathrm{d}\mathbf{S} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{K})\ \mathrm{d}v + \int_{\mathcal{V}} \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} \right) \mathrm{d}v, \tag{2.41}$$

which comprises an all-encompassing incarnation of Poynting's theorem.

As mentioned before, the constitutive relations for instantaneously reacting LTI media may be written as $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ with ε, μ constant. If we use the vector counterpart of $f\,\partial f/\partial t = (1/2)\partial f^2/\partial t$. This leads to

$$-\int_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}) \cdot \mathrm{d}\mathbf{S} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{K})\ \mathrm{d}v + \frac{\partial}{\partial t}\frac{1}{2}\int_{\mathcal{V}} (\varepsilon\mathbf{E} \cdot \mathbf{E} + \mu\mathbf{H} \cdot \mathbf{H})\ \mathrm{d}v, \tag{2.42}$$

which is Poynting's theorem for instantaneously reacting media.

Poynting's theorem is also known as the electromagnetic power balance. For further interpretation, one should observe that the first integral on the right-hand side in Eqs. (2.41) or (2.42) represents the dissipated power in the volume $\mathcal{V}$. Hence, the surface integral (*including the minus sign*) on the left-hand side must represent a power as well. In fact, it is the total power flowing *into* $\mathcal{V}$ through the surface $\mathcal{S}$. The vectorial space-time-domain quantity $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is called the Poynting vector and represents the (surface) density of instantaneous electromagnetic power flow. Finally, the derivative with respect to time of the second integral on the right-hand side in Eq. (2.42) must also represent a power, and hence the integral

$$W_{\text{em}} = \frac{1}{2} \int_{\mathcal{V}} (\varepsilon \mathbf{E} \cdot \mathbf{E} + \mu \mathbf{H} \cdot \mathbf{H}) \, \mathrm{d}v \tag{2.43}$$

represents the stored electromagnetic energy.

We may take the development of Poynting's theorem (either Eqs. (2.41) or (2.42)) one step further by explicitly decomposing the current densities into source constituents denoted by a subscript $_0$, and constituents strictly associated with dissipation according to $\mathbf{J} \to \mathbf{J}_0 + \mathbf{J}$ and $\mathbf{K} \to \mathbf{K}_0 + \mathbf{K}$. Subjecting Eq. (2.42) to this decomposition results in

$$\begin{aligned} -\int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J}_0 + \mathbf{H} \cdot \mathbf{K}_0) \, \mathrm{d}v = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{K}) \, \mathrm{d}v + \frac{\partial}{\partial t} \frac{1}{2} \int_{\mathcal{V}} (\varepsilon \mathbf{E} \cdot \mathbf{E} + \mu \mathbf{H} \cdot \mathbf{H}) \, \mathrm{d}v \\ + \int_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}) \cdot \mathrm{d}\mathbf{S}, \end{aligned} \tag{2.44}$$

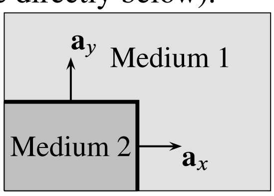
where the source terms on the left-hand side deliver positive power to the system, which, in view of the minus sign implies that $\mathbf{E} \cdot \mathbf{J}_0$ and $\mathbf{H} \cdot \mathbf{K}_0$ must be negative. This is typical for source (active) current densities. The three terms on the right-hand side represent the dissipated power, the time rate of change of the stored electromagnetic energy and the power radiation away from the volume $\mathcal{V}$, respectively.

We have derived Eq. (2.42) from Eq. (2.41) upon assuming instantaneously reacting LTI media. In Eq. (70) of Section 11.3 in the book, Hayt & Buck arrive at a form of Poynting's theorem that is valid for instantaneously reacting LTI media, but is *wrong* for the more general case of LTI media with relaxation.

Questions

Question 2.1

Non-smooth boundaries or interfaces are not often encountered in basic textbooks on electromagnetism. In this exercise, we shall try to reveal why that is the case. Consider a two-dimensional domain containing two material media with different electromagnetic properties (see the figure directly below).

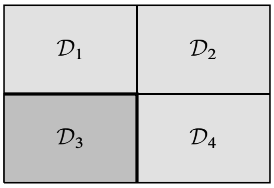


Medium 1: $\varepsilon = \varepsilon_1$, $\mu = \mu_1$.

Medium 2: $\varepsilon = \varepsilon_2$, $\mu = \mu_2$.

Note that $\varepsilon_1 \neq \varepsilon_2$, $\mu_1 \neq \mu_2$.

For the determination of the electromagnetic field in non-canonical configurations one has to devise numerical models. Often this entails having to discretise both the computational domain and Maxwell's equations. Adopting a crude and naive strategy, we chop up the domain into little cells (cubes, or in this 2-D case, squares), in which the field quantities are considered to be *constant*. Further, we assume that upon crossing the interface between adjoining cells with the same material properties a field quantity hardly changes at all, and that for all intents and purposes that field quantity may be regarded as constant across the pertaining adjoining cells. However, upon crossing an interface between two cells with different material properties, we must impose the *boundary conditions* for the field quantities. In other words, in the first instance, we disregard the field equations and focus on the boundary conditions.



In the figure directly above, we have subdivided the domain into four the sub-domains $\mathcal{D}_i$, $i = 1, 2, 3, 4$. Suppose that we know the y-component $H_y = H_y^{(1)}$ of the magnetic field strength $\mathbf{H}$ in the sub-domain $\mathcal{D}_1$.

a) Given $H_y^{(1)}$ in $\mathcal{D}_1$, determine the y-component $H_y = H_y^{(2)}$ of the magnetic field strength $\mathbf{H}$ in the sub-domain $\mathcal{D}_2$.

b) Given $H_y^{(1)}$ in $\mathcal{D}_1$, determine the y-component $H_y = H_y^{(3)}$ of the magnetic field strength $\mathbf{H}$ in the sub-domain $\mathcal{D}_3$.

c) Based on your result for $H_y^{(2)}$ in $\mathcal{D}_2$, determine the y-component $H_y = H_y^{(4)}$ of the magnetic field strength $\mathbf{H}$ in the sub-domain $\mathcal{D}_4$.

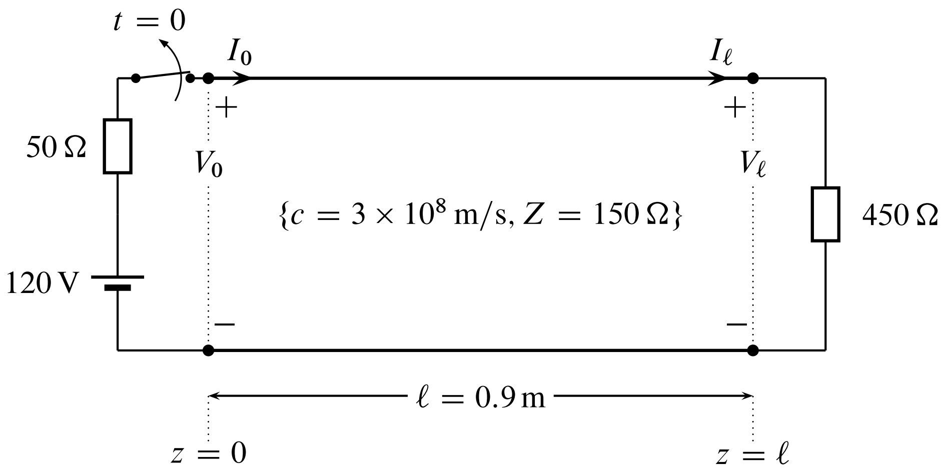
d) Based on your result for $H_y^{(3)}$ in $\mathcal{D}_3$, determine the y-component $H_y = H_y^{(4)}$ of the magnetic field strength $\mathbf{H}$ in the sub-domain $\mathcal{D}_4$.

The two results that you have found for $H_y^{(4)}$ are inconsistent (or you have done something wrong). This is due to the fact that the field will become singular.

What causes the discrepancy (bonus question)?

- The two line segments comprising the interface between the two material media?
- The right angle where these two line segments meet?
- Both the line segments and the right angle?

Question 2.2



A 120 V battery with internal impedance $50\,\Omega$ has been connected forever to a 0.9 m long transmission line via a closed switch. Hence, a steady state has certainly been reached.

a) Determine the steady state total voltage and current amplitudes, V_{steady} and I_{steady}, respectively.

b) Nevertheless, this steady state may be decomposed into forward and backward propagating waves. The characteristic impedance of the transmission line is $150\,\Omega$. Determine the respective steady-state TEM-wave voltage and current amplitudes $V^{\pm}_{\mathrm{steady}}$ and $I^{\pm}_{\mathrm{steady}}$ of these forward and backward propagating waves.

At $t = 0$, the switch is *opened* (switched off). As a consequence, in addition to the steady-state voltage and current amplitudes, a *compensating* TEM wave, with voltage amplitude $V^{+}(0, t) = V_{\mathrm{comp}}U(t)$ at $z = 0$, starts to propagate along the transmission line towards the $450\,\Omega$ load[9].

c) Discuss the *total* current and/or voltage amplitudes at $z = 0$ after $t = 0$.

d) Determine the compensating voltage amplitude V_{comp} by considering an arbitrary moment in time just after the switch has been activated, but before possible reflections from the end of the line have returned.

e) Because the line has not been terminated properly at either end, the compensating waves that arrive at the terminations will be partially reflected. Determine the reflection coefficients at $z = 0$ and $z = \ell$, which we shall call Γ_g and Γ_L, respectively.

In principle, this process of successive reflections will go on forever. Thanks to the finite travel time and the fact that the moduli of the reflection coefficients do not exceed unity, convergence will occur.

f) Determine the voltage and current amplitudes of the *compensating* forward propagating wave constituents that have reflected $2n$ times, with n integer (starting at $n = 0$). For simplicity, you may omit the step functions that describe the time delay due to the finite travel time.

g) Determine the voltage and current amplitudes of the *compensating* backward propagating wave constituents that have reflected $2n + 1$ times, with n integer (starting at $n = 0$). For simplicity, you may again omit the step functions that describe the time delay due to the finite travel time.

[9] This amounts to a linear superposition of field constituents

h) Sketch the bounce diagram for the *compensating* waves for $0 < t < 15\,\text{ns}$, and sketch the *total* voltage amplitude $V(z, t)$ at $z = 0.6\,\text{m}$ for $0 < t < 15\,\text{ns}$.

i) For $t \to \infty$ a new steady-state is reached for the *compensating* forward and backward compensating propagating waves. Give the steady-state voltage and current amplitudes $V^{\pm}(z, t)|_{t\to\infty}$ and $I^{\pm}(z, t)|_{t\to\infty}$ of these *compensating* forward and backward propagating waves.

Question 2.3

Consider a circularly cylindrical coaxial cable, consisting of an inner perfect conductor of radius a, and an outer perfect conductor of radius b. Because of the circular symmetry, it makes sense to work in circular cylindrical coordinates (ρ, φ, z). In this coordinate system, Laplace's equation reads

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)\Phi = 0 \tag{2.45}$$

a) Argue whether Φ depends on φ or not.

b) Give the general solution for $\rho\dfrac{\partial\Phi}{\partial\rho}$.

c) Give the general solution for Φ.

d) Give the solution for Φ that satisfies the boundary conditions in Eq. (2.29), thus applying the voltage normalisation.

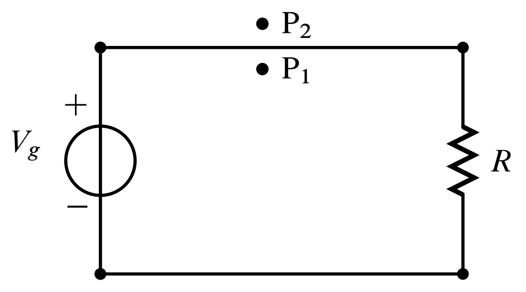
e) Apply the current normalisation to determine C and L.

f) Determine the characteristic impedance $Z = \sqrt{L/C}$.

With reference to Footnote[3], we note that in this case $\Xi(\zeta) = \ln(b/\zeta)/\ln(b/a)$. This complex function analogue in the complex $\zeta = x + jy$ plane is consistent with the answer that you should have given at Question 2.3d).

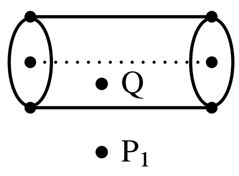
Question 2.4

As depicted on the right, a voltage source (say a battery) is connected via two good (but not perfect) conductors to a light-bulb. In such a static field situation (the current is stationary, but the fields are independent of time), we may assume that $\mathbf{E} = -V\nabla\Phi$. Here, Φ represents the normalised static potential.



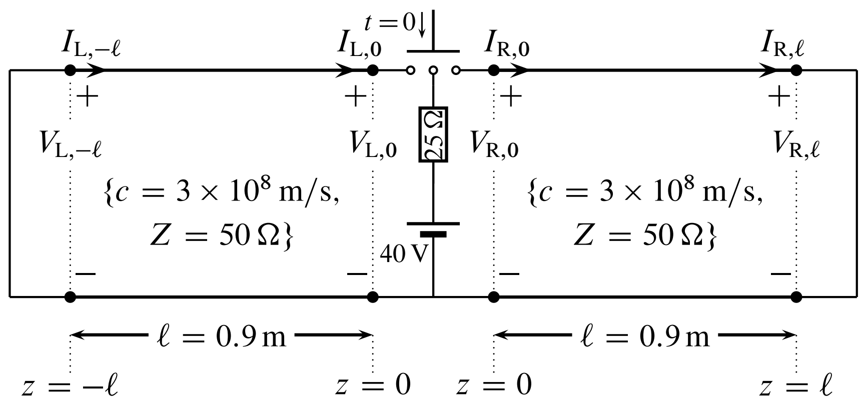
a) Sketch $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{S}$ in the plane $z = z_0$ for the points P_1 and P_2 just below and above the top wire, respectively.

Let us examine the top wire more closely. As indicated on the right, the wire has a circular cross section. (Observe that the direction of the current density follows from the sign of the voltage.)



b) Sketch **E**, **H** and **S** in the plane $z = z_0$ for the points Q and P_1 just on the inside and the outside of the wire surface. Comment on the direction of the Poynting vector.

Question 2.5



A 40 V battery with internal impedance 25 Ω is connected half-way along a transmission line of length $2\ell = 1.8$ m via a switch that is activated (closed) at $t = 0$ (see figure above). The transmission line is terminated at both ends (at $z = \pm\ell$) by short circuits. The wavespeed along, and characteristic impedance of the transmission line are 3×10^8 m/s and $Z = 50\,\Omega$, respectively. To distinguish between the transmission-line sections to the left ($z < 0$) and to the right ($z > 0$) of the source (at $z = 0$), we use the subscripts $_{\mathrm{L}}$ and $_{\mathrm{R}}$, respectively.

a) For $t \to \infty$ a symmetric steady state is reached. Determine both the respective steady state total voltage and current amplitudes, $V_{\mathrm{L,steady}}(z,t)$ and $I_{\mathrm{L,steady}}(z,t)$ to the left of the source,

   and

   the TEM-wave voltage and current wave amplitudes $V^{\pm}_{\mathrm{L,steady}}(z,t)$ and $I^{\pm}_{\mathrm{L,steady}}(z,t)$ of the corresponding progressive and regressive propagating waves. (Hint: apply wave decomposition).

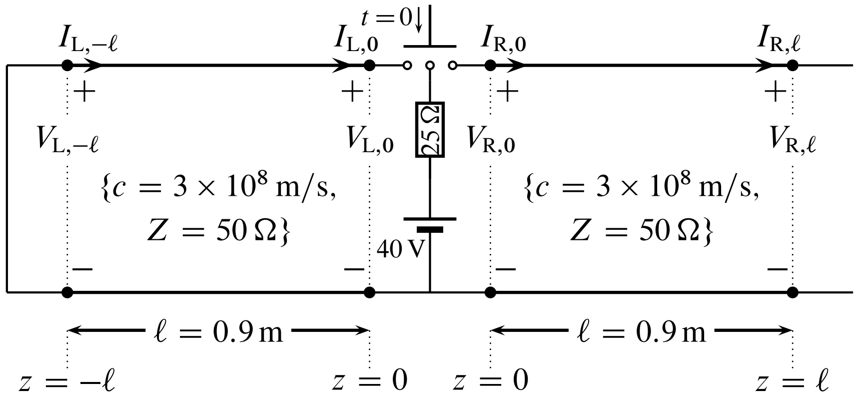
b) Give the voltage reflection coefficients at $z = -\ell$ and $z = \ell$.

   For a progressive wave constituent that arrives at $z = 0$ from the left (from $z < 0$), use the continuity of the voltage amplitude and Kirchhoff's current law at $z = 0$ to determine the associated voltage reflection and transmission coefficients, Γ and T, respectively.

   Also determine the voltage reflection and transmission coefficients for a regressive wave constituent that arrives at $z = 0$ from the right (from $z > 0$).

c) Sketch the (voltage) bounce diagram and the *total* voltage at $z = 0.6$ m for $0 < t < 18$ ns.

Next, let us consider a slight modification to the configuration in that the short circuit at $z = \ell$ is replaced by an "open" termination (see figure below).



d) For $t \to \infty$ a steady state is reached. Determine both the respective steady state total voltage and current amplitudes, $V_{\mathrm{R,steady}}(z,t)$ and $I_{\mathrm{R,steady}}(z,t)$ to the right of the source,

and

the TEM-wave voltage and current wave amplitudes $V^{\pm}_{\mathrm{R,steady}}(z,t)$ and $I^{\pm}_{\mathrm{R,steady}}(z,t)$ of the corresponding progressive and regressive waves.

e) Sketch the *total* voltage at $z = 0.6\,\mathrm{m}$ for $0 < t < 18\,\mathrm{ns}$.

Chapter 3

Week 3 — Time-harmonic wave fields

3.1 Time-harmonic fields

3.1.1 Linear Time-Invariant Field-Matter Interaction

Now that we have explored electromagnetic field solutions in free space and instantaneously reacting media both through uniform plane waves and TEM waves, it is time to turn our attention to the interaction of electromagnetic fields with matter that does not react instantaneously due to inertia. This topic can be investigated in its own right on many different levels, ultimately involving nonlinear quantum optics and quantum mechanics.

We aim for a more modest objective. The book by H&B discusses the classical Lorentz model for the bulk interaction of atoms with the electromagnetic field. This is a model for a dielectric medium that interacts with electromagnetic fields in a linear, time-invariant (LTI) and isotropic manner.

Actually, many materials belong to the class of LTI media (to a satisfactory degree of approximation). The associated constitutive relations between the current or flux densities and the fields can be described on a generic systems level involving relaxation functions.

For example, let us consider the current density generated in a non-perfect conductor under the influence of an electromagnetic field. The constitutive relation between the electric field $\mathbf{E}(\mathbf{r}, t)$ and the resulting current density $\mathbf{J}(\mathbf{r}, t)$ reads

$$\mathbf{J}(\mathbf{r}, t) = \int_0^\infty \sigma(\mathbf{r}, t')\mathbf{E}(\mathbf{r}, t - t')\, \mathrm{d}t', \tag{3.1}$$

in which $\sigma(\mathbf{r}, t)$ denotes the conduction relaxation function. From the specific form of Eq. (3.1), we infer that this constitutive relation is

- causal; the current density $\mathbf{J}(\mathbf{r}, t)$ at time t is a consequence of the electric field values $\mathbf{E}(\mathbf{r}, t - t')$ at the present time t (for $t' = 0$), and those in the past (for $t' > 0$);

- linearly reacting; this is easy to verify: if two arbitrary electric fields $\mathbf{E}_1(\mathbf{r}, t)$ and $\mathbf{E}_2(\mathbf{r}, t)$ generate the electric current densities $\mathbf{J}_1(\mathbf{r}, t)$ and $\mathbf{J}_2(\mathbf{r}, t)$, then, the linear combination $A_1\mathbf{E}_1(\mathbf{r}, t) + A_2\mathbf{E}_2(\mathbf{r}, t)$, with A_1 and A_2 constant, will generate the electric current density $A_1\mathbf{J}_1(\mathbf{r}, t) + A_2\mathbf{J}_2(\mathbf{r}, t)$;

- time-invariant; this means that if an arbitrary electric field $\mathbf{E}(\mathbf{r}, t)$ generates the electric current density $\mathbf{J}(\mathbf{r}, t)$, then, the same electric field applied with a time delay T, i.e., $\mathbf{E}(\mathbf{r}, t - T)$, will generate the delayed electric current density $\mathbf{J}(\mathbf{r}, t - T)$;

- isotropic (from the Greek for "the same for all directions"); this means that if an arbitrary electric field $\mathbf{E}(\mathbf{r}, t)$ generates the electric current density $\mathbf{J}(\mathbf{r}, t)$, then, that electric field rotated over an arbitrary but fixed angle φ, $\mathcal{R}_\varphi \mathbf{E}(\mathbf{r}, t)$ generates the electric current density $\mathcal{R}_\varphi \mathbf{J}(\mathbf{r}, t)$, rotated over the same angle.

The converse is also true, i.e., if a material medium *reacts*[1] isotropically, linearly and in a time invariant fashion, then the corresponding constitutive relations between the flux and/or current densities and the field quantities is of the form of Eq. (3.1). So, if we are not interested in material characterisation, we may just work with *ad hoc* provided relaxation functions.

A final remark concerns the limiting case of instantaneously reacting (no inertia or relaxation effects) media for which the relaxation function is a constant times a delta function in time,[2] e.g., $\sigma(\mathbf{r}, t) = \sigma^0(\mathbf{r})\delta(t)$. Because charged particles have mass and therefore exhibit inertia, instantaneous reaction can only be an idealised case (just like perfect conductivity does not occur in practice either). However, if a material reacts so fast relative to the characteristic pulse width of the electromagnetic field, then that material may be regarded as instantaneously reacting for all intents and purposes. This is what we have assumed in Chapters 1 and 2 to justify working with waves may propagate in material media without distortion. In Chapter 4, we shall analyse pulse propagation in media that are not instantaneously reacting.

3.1.2 The frequency-domain

Although we could work with relaxation integrals in the space-time-domain Maxwell's equations, those pesky integrals do spoil the fun of it. However, the relaxation integrals are convolution integrals. From theory of systems, we know that upon applying a Fourier transformation with respect to time convolution, integrals become products.

In particular, we write

$$\mathbf{E}_\mathrm{s}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, t) e^{-\mathrm{j}\omega t}\, \mathrm{d}t = \int_{t=t_0}^{\infty} \mathbf{E}(\mathbf{r}, t) e^{-\mathrm{j}\omega t}\, \mathrm{d}t, \tag{3.2a}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \mathbf{E}_\mathrm{s}(\mathbf{r}, \omega) e^{\mathrm{j}\omega t}\, \mathrm{d}\omega = \mathrm{Re}\left[\frac{1}{\pi} \int_{\omega=0}^{\infty} \mathbf{E}_\mathrm{s}(\mathbf{r}, \omega) e^{\mathrm{j}\omega t}\, \mathrm{d}\omega \right], \tag{3.2b}$$

where, for the second identity in Eq. (3.2b), we have used $\mathbf{E}_\mathrm{s}(\mathbf{r}, -\omega) = \mathbf{E}_\mathrm{s}^*(\mathbf{r}, \omega)$, which is a direct consequence of the fact that all physical space-time quantities are real quantities.

Now, let us apply the Fourier transformation to Eq. (3.1). This leads to

$$\mathbf{J}_\mathrm{s} = \sigma \mathbf{E}_\mathrm{s}, \tag{3.3}$$

[1] implying causality

[2] limiting case of instantaneously reacting media, the lower limit, 0, of the integral in Eq. (3.1) is sometimes replaced by 0^-, which means infinitesimally less than 0. This is done to make explicit that in the case of an instantaneously reacting medium, the whole support of the delta function should be taken into account, and *not just half* of it, which those who insist of the symmetry of $\delta(t)$ about $t = 0$ might do.

in which the complex conductivity $\sigma = \sigma(\mathbf{r}, \omega)$ is a complex function of frequency, defined through

$$\sigma(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \sigma(\mathbf{r}, t) e^{-\mathrm{j}\omega t}\, \mathrm{d}t = \sigma' - \mathrm{j}\sigma'' \qquad \begin{cases} \sigma' = \sigma'(\mathbf{r}, \omega) = \mathrm{Re}(\sigma), \\ \sigma'' = \sigma''(\mathbf{r}, \omega) = -\mathrm{Im}(\sigma). \end{cases} \tag{3.4}$$

Graphs in which σ' and σ'' are displayed as a function of the radial frequency ω are called *dispersion characteristics*.

Application of the Fourier transformation to $\partial\mathbf{E}/\partial t$ results in

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial \mathbf{E}}{\partial t} e^{-\mathrm{j}\omega t}\, \mathrm{d}t &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}[\mathbf{E} e^{-\mathrm{j}\omega t}]\, \mathrm{d}t + \mathrm{j}\omega \int_{-\infty}^{\infty} \mathbf{E} e^{-\mathrm{j}\omega t}\, \mathrm{d}t \\ &= \left[\mathbf{E}(\mathbf{r}, t) e^{-\mathrm{j}\omega t}\right]_{-\infty}^{\infty} + \mathrm{j}\omega \mathbf{E}_{\mathrm{s}} = \mathrm{j}\omega \mathbf{E}_{\mathrm{s}}, \end{aligned} \tag{3.5}$$

where we have assumed that $\mathbf{E} \to 0$ for $t \to \pm\infty$. The techniques we used in the derivation of Eq. (3.5) is known as *integration by parts*.[3] Hence, in the space-time domain a partial derivative with respect to time corresponds to a scalar multiplication by $\mathrm{j}\omega$ in the space-frequency domain.[4]

In addition to conductors, we may also have to contend with dielectric and magnetic materials or combinations for which we again assume isotropy, linearity, time-invariance, and of course causality. Ultimately, Maxwell's equations in the space-frequency domain may be cast in the following form.

$$\nabla \times \mathbf{E}_{\mathrm{s}} + \mathrm{j}\omega\mu \mathbf{H}_{\mathrm{s}} = -\mathbf{K}_{s0}, \tag{3.6a}$$

$$\nabla \times \mathbf{H}_{\mathrm{s}} - \mathrm{j}\omega\varepsilon \mathbf{E}_{\mathrm{s}} = \mathbf{J}_{s0}. \tag{3.6b}$$

Here, the impressed electric and magnetic current densities $\mathbf{J}_{s0}$ and $\mathbf{K}_{s0}$ are considered given. Further we have introduced $\varepsilon = \varepsilon_0 \varepsilon_{\mathrm{r}}$ and $\mu = \mu_0 \mu_{\mathrm{r}}$, in which the *complex relative permittivity*[5] ε_{r} and the *complex relative permeability* μ_{r} are complex functions of ω.[6]

In view of our description of the conduction relaxation function for conductors, we might have expected a term $(\mathrm{j}\omega\varepsilon + \sigma)\mathbf{E}_{\mathrm{s}}$ in the left-hand side of Eq. (3.6b). Indeed, it is quite common for a material to behave both as a dielectric and as a conductor. However, at a single frequency it is almost impossible to distinguish between the so-called displacement current $\mathrm{j}\omega\varepsilon\mathbf{E}_{\mathrm{s}}$ and the current density $\sigma\mathbf{E}$ associated with free charges that due tot the time-harmonic field also bob up and down (or back and forth). Mathematically, for a fixed frequency both $\mathrm{j}\omega\varepsilon$ and σ are just complex numbers that manifest themselves through their sum $\mathrm{j}\omega\varepsilon + \sigma$. Unless, we have a

[3] It used to be part of the standard repertoire of secondary-school mathematics.

[4] The reason why the exponential function $\exp(\mathrm{j}\omega t)$ is so useful in electromagnetism, and in many other branches of science is that it is an eigenfunction of the operator $\partial/\partial t$ with $\mathrm{j}\omega$ the corresponding eigenvalue.

[5] Sometimes ε_{r} is referred to as the dielectric constant. This is old-fashioned and somewhat sloppy use of language since ε_{r} often depends on frequency and therefore is not constant

[6] To picture a dielectric, one might imagine the relative time-harmonic displacements of clouds of negatively charged light electrons relative to positively charged heavy nuclei set in a lattice or some amorphous grid. This so-called electric polarisation generates an electromagnetic field of its own, which in turn modifies the polarisation, etc. The total electromagnetic field may be decomposed into a primary field, which would be present in the absence of matter, and a secondary field.

decent model of the underlying physics, the sum can not be decomposed. Hence, we combine the total effect into a single term $\mathrm{j}\omega\varepsilon\mathbf{E}$, in which the effect of the free charges is silently understood.

3.1.3 Time-harmonic waves

There is a subtle difference between the description of the electromagnetic field in the space-frequency domain, and the description of complex time-harmonic (called sinusoidal in the book by H&B) electromagnetic fields that we shall consider for the rest of the course.[7]

First of all, the algebra is exactly the same, i.e., the derivative with respect to time is replaced by a factor $\mathrm{j}\omega$. However, in the case of the transformation from the space-frequency-domain quantities to their space-time-domain counterparts, we have to apply the inverse Fourier transformation, e.g., via Eq. (3.2b). Since the unit of $\mathrm{d}\omega$ is s^{-1}, the unit of $\mathbf{E}_\mathrm{s}$ must be $\mathrm{Vs/m}$. On the other hand, for time-harmonic waves with angular frequency ω_0, we directly write $\mathbf{E} = \mathrm{Re}(\mathbf{E}_\mathrm{s}(\omega_0)e^{\mathrm{j}\omega_0 t})$ and so $\mathbf{E}_\mathrm{s}$ would have the familiar unit $\mathrm{V/m}$.

The connection between the two descriptions is that the time-harmonic electric field can be regarded as a space-frequency-domain electric field of the form

$$\pi\left[\mathbf{E}_\mathrm{s}(\omega_0)\delta(\omega-\omega_0) + \mathbf{E}_\mathrm{s}^*(\omega_0)\delta(\omega+\omega_0)\right], \tag{3.7}$$

where the delta functions take on the dimension of the reciprocal of their arguments, i.e., $\delta(\omega-\omega_0)$ has unit s, and $\mathbf{E}_\mathrm{s}$ has the familiar unit $\mathrm{V/m}$.

Whenever we state that we consider time-harmonic waves, it makes sense to simplify notation by omitting the subscript $_0$ in the symbol for the angular frequency, i.e., $\omega_0 \to \omega$, and hence

$$\mathbf{E} = \mathrm{Re}\left[\mathbf{E}_\mathrm{s}(\omega)e^{\mathrm{j}\omega t}\right]. \tag{3.8}$$

Before we can experience the delights offered by time-harmonic fields in the analysis electromagnetic waves, let us investigate the consequences of considering time-harmonic fields for the electromagnetic power balance.

3.2 The electromagnetic power balance for time-harmonic waves

In pursuit of a physical description of the exchange of energy, we have derived the power balance for time-domain electromagnetic fields. However, we often encounter time-harmonic field problems, for which the time-domain power terms oscillate rapidly (at twice the frequency of the fields themselves). It is often impossible to observe such fast oscillations, due to the slow(ish) response of measurement devices. However, it turns out to be quite easy to measure the time-averaged power, or the associated time-averaged power density

$$\langle\mathbf{S}\rangle_T = \frac{1}{T}\int_{t_0}^{t_0+T} \mathbf{S}\,\mathrm{d}t, \tag{3.9}$$

[7] except in Section 4.3, where we shall analyse pulse propagation in dispersive media.

where $T = 2\pi/\omega$ denotes the period of a time-harmonic signal.

So, let us consider time-harmonic fields with $\mathbf{E} = \mathrm{Re}(\mathbf{E}_\mathrm{s} e^{\mathrm{j}\omega t})$ and $\mathbf{H} = \mathrm{Re}(\mathbf{H}_\mathrm{s} e^{\mathrm{j}\omega t})$. Since $\mathrm{Re}(\mathbf{E}_\mathrm{s} e^{\mathrm{j}\omega t}) = (\mathbf{E}_\mathrm{s} e^{\mathrm{j}\omega t} + \mathbf{E}_\mathrm{s}^* e^{-\mathrm{j}\omega t})/2$, we have

$$\begin{aligned}
\mathbf{S} &= \frac{1}{4}\left(\mathbf{E}_\mathrm{s} e^{\mathrm{j}\omega t} + \mathbf{E}_\mathrm{s}^* e^{-\mathrm{j}\omega t}\right) \times \left(\mathbf{H}_\mathrm{s} e^{\mathrm{j}\omega t} + \mathbf{H}_\mathrm{s}^* e^{-\mathrm{j}\omega t}\right) \\
&= \frac{1}{4}\left(\mathbf{E}_\mathrm{s} \times \mathbf{H}_\mathrm{s}^* + \mathbf{E}_\mathrm{s}^* \times \mathbf{H}_\mathrm{s}\right) + \frac{1}{4}\left(\mathbf{E}_\mathrm{s} \times \mathbf{H}_\mathrm{s} e^{2\mathrm{j}\omega t} + \mathbf{E}_\mathrm{s}^* \times \mathbf{H}_\mathrm{s}^* e^{-2\mathrm{j}\omega t}\right) \\
&= \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_\mathrm{s} \times \mathbf{H}_\mathrm{s}^*\right) + \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_\mathrm{s} \times \mathbf{H}_\mathrm{s} e^{2\mathrm{j}\omega t}\right). \qquad (3.10)
\end{aligned}$$

Note that the second term on the right-hand side in the last line of Eq. (3.10) oscillates twice as fast as the time-harmonic fields themselves. Hence, its contribution to the time averaging in Eq. (3.9) vanishes, implying that

$$\langle \mathbf{S} \rangle_T = \frac{1}{2}\mathrm{Re}\left(\mathbf{S}_\mathrm{s}\right), \qquad (3.11)$$

where $\mathbf{S}_\mathrm{s} = \mathbf{E}_\mathrm{s} \times \mathbf{H}_\mathrm{s}^*$ denotes the complex Poynting vector.

The complex Poynting vector is the outer (exterior) product of the frequency-domain electric field and the complex conjugate of the frequency-domain magnetic field, and strictly applies to time-harmonic fields of a fixed frequency. It is *NOT* the frequency-domain counterpart of the time-domain Poynting vector associated with arbitrary time-domain electromagnetic field quantities, because it is not the Fourier transform with respect to time of $\mathbf{E} \times \mathbf{H}$ (that quantity would involve the convolution of $\mathbf{E}_\mathrm{s}$ and $\mathbf{H}_\mathrm{s}$ in the frequency domain).

To derive Poynting's theorem for time-domain quantities, we set out to construct an equation involving $\mathbf{E} \cdot \mathbf{J}$, with the aid of Maxwell's equations. That equation contained the term $\nabla \cdot (\mathbf{E} \times \mathbf{H})$, and any divergence of a vector quantity always suggests integration over a volume. We may derive the time-averaged power balance for time-harmonic fields following a similar procedure. This is the objective of the exercise below

3.3 Time-harmonic waves in one spatial dimension

3.3.1 Time-harmonic voltage amplitudes in instantaneously reacting media

Let us first consider instantaneously reacting media (or vacuum), and analyse the total time-harmonic voltage amplitude $V(z,t)$ consisting of two counterpropagating waves for which $V^{\pm}(t) = |A^{\pm}|\cos(\omega t + \phi^{\pm})$, i.e.,

$$\begin{aligned}
V(z,t) &= |A^+|\mathrm{Re}\left(e^{\mathrm{j}\omega t + \mathrm{j}\phi^+ - \mathrm{j}\omega z/c}\right) + |A^-|\mathrm{Re}\left(e^{\mathrm{j}\omega t + \mathrm{j}\phi^- + \mathrm{j}\omega z/c}\right) \\
&= \mathrm{Re}\left[e^{\mathrm{j}\omega t}\left(\underbrace{\underbrace{\underbrace{|A^+|e^{\mathrm{j}\phi^+}}_{V_{\mathrm{s}0}^+} e^{-\mathrm{j}\omega z/c}}_{V_\mathrm{s}^+(z)} + \underbrace{\underbrace{|A^-|e^{\mathrm{j}\phi^-}}_{V_{\mathrm{s}0}^-} e^{\mathrm{j}\omega z/c}}_{V_\mathrm{s}^-(z)}}_{V_\mathrm{s}(z)}\right)\right]. \qquad (3.12)
\end{aligned}$$

So, we may solve time-harmonic electromagnetic plane-wave (and more general TEM-wave problems) by computing the complex voltage amplitudes (or voltage phasors) $V_s^{\pm}(z)$ of the progressive and regressive waves, as long as we multiply the resulting V_s by $\exp(j\omega t)$ and finally take the real part.

Since, the procedure of the multiplication of V_s by $\exp(j\omega t)$ and the restriction of the final time-harmonic solution to its real part takes place at the very end, and is the same for all the field constituents involved, we might just as well regard the complex result V_s of our computations as our final result, and omit the last two trivial steps.

The fabulous property of the exponential function that $\exp(a + b) = \exp(a)\exp(b)$, for arbitrary complex numbers a and b has the following consequences.

- From Eq. (3.12) we infer that if, in a *homogeneous* medium, we know $V_{s0}^{+} = V_s^{+}(0)$, then $V_s^{+}(z)$ simply follows from a multiplication by $\exp(-j\omega z/c)$, which is therefore referred to as a propagation factor. Likewise, if we know $V_{s0}^{-} = V_s^{-}(0)$, then $V_s^{-}(z)$ simply follows from a multiplication by $\exp(j\omega z/c)$. This means that if we can *decompose* a time-harmonic wavefield at a certain point along the z-axis into its progressive and regressive constituents, then we may readily compute those progressive and regressive constituents at another point along the z-axis. Subsequent wave *composition* then gives the time-harmonic wavefield at that other point. We shall make extensive use of this feature in the remaining part of this reader.

- Suppose that we want to compute the superposition of two progressive (or two regressive) time-harmonic waves with different amplitudes, and different time delays (and hence different phases). Then, in the phasor domain we merely have to add up the complex amplitudes of those time-harmonic waves, which makes light work of problems involving multiple reflections and transmissions.

3.3.2 Time-harmonic uniform plane waves

Let us now construct the electromagnetic field solutions associated with time-harmonic uniform plane waves in the positive and negative z-directions in a general (not instantaneously reacting) homogeneous isotropic LTI medium.

To this end, we observe that the left-hand sides of Maxwell's equations for time-harmonic signals, Eq. (3.6), are essentially the same as those for instantaneously reacting media, provided we replace the real constants $\{\mu, \varepsilon\}$ by the complex ones. With this simple modification, the field solutions follow from Eq. (1.27) for uniform plane waves (or Eq. (1.38) in case of more general TEM waves) upon assuming that all complex time signals are proportional to $\exp(j\omega t)$, e.g.,

$$\text{we replace} \quad e^{j\omega(t \mp z/c)} \quad \text{by} \quad e^{j(\omega t \mp kz)}, \tag{3.13}$$

in which

$$k = \beta - j\alpha = \omega\sqrt{\varepsilon\mu} \quad (\text{or } \omega\sqrt{LC}) \tag{3.14}$$

denotes the wavenumber associated with a plane wave (or a transmission line).[8] Note that for real ε and μ, we had found that $1/\sqrt{\varepsilon\mu} = c$ is a wavespeed. This is no longer true for complex ε and μ. Further, α must be positive in a lossy medium.

[8] Sometimes β is referred to as the phase constant. I disapprove since β often depends on frequency and therefore is not constant. It is not a phase either.

Analogously, we write

$$\text{and} \quad Z = \frac{1}{Y} = \sqrt{\frac{\mu}{\varepsilon}} \quad (\text{or } \sqrt{\frac{L}{C}}), \tag{3.15}$$

which denotes the complex plane-wave impedance (or the characteristic impedance in case of a transmission line).

In the phasor domain, the electromagnetic field associated with plane (or TEM) waves in the z-direction is written as

$$\mathbf{E}_{\mathrm{s}} = V_{\mathrm{s}}(z)\,\mathbf{e}_{\mathrm{t}} = \left[V_{\mathrm{s}}^{+}(z) + V_{\mathrm{s}}^{-}(z)\right]\,\mathbf{e}_{\mathrm{t}}, \tag{3.16a}$$

$$\mathbf{H}_{\mathrm{s}} = I_{\mathrm{s}}(z)\,\mathbf{h}_{\mathrm{t}} = \left[I_{\mathrm{s}}^{+}(z) + I_{\mathrm{s}}^{-}(z)\right]\,\mathbf{h}_{\mathrm{t}}, \tag{3.16b}$$

with

$$V_{\mathrm{s}}^{\pm} = V_{\mathrm{s0}}^{\pm} e^{\mp \mathrm{j}kz} = \pm Z I_{\mathrm{s}}^{\pm} \quad \Longleftrightarrow \quad I_{\mathrm{s}}^{\pm} = I_{\mathrm{s0}}^{\pm} e^{\mp \mathrm{j}kz} = \pm Y V_{\mathrm{s}}^{\pm} \tag{3.17}$$

Further, $\mathbf{e}_{\mathrm{t}} \perp \mathbf{a}_z$ and $\mathbf{h}_{\mathrm{t}} \perp \mathbf{a}_z$ remain the same transverse vectors,[9] related by

$$\mathbf{h}_{\mathrm{t}} = \mathbf{a}_z \times \mathbf{e}_{\mathrm{t}} \quad \Rightarrow \quad \mathbf{h}_{\mathrm{t}} \perp \mathbf{e}_{\mathrm{t}}. \tag{3.18}$$

The corresponding real time-domain electromagnetic field quantities and voltage and current amplitudes simply follow from

$$\mathbf{E}(z,t) = \mathrm{Re}\left(\mathbf{E}_{\mathrm{s}}(z)e^{\mathrm{j}\omega t}\right) \quad \text{and} \quad \mathbf{H}(z,t) = \mathrm{Re}\left(\mathbf{H}_{\mathrm{s}}(z)e^{\mathrm{j}\omega t}\right), \tag{3.19}$$

and

$$V(z,t) = \mathrm{Re}\left(V_{\mathrm{s}}(z)e^{\mathrm{j}\omega t}\right) \quad \text{and} \quad I(z,t) = \mathrm{Re}\left(I_{\mathrm{s}}(z)e^{\mathrm{j}\omega t}\right), \tag{3.20}$$

respectively.

3.3.3 Counterpropagating time-harmonic waves

Before turning our attention to the formal transfer of complex waves from one constant z-plane to another, and reflection/transmission problems let us investigate counterpropagating time-harmonic waves. The voltage and current amplitudes are superpositions of the one-way propagating voltage amplitudes and current amplitude constituents, i.e.,

$$V_{\mathrm{s}}(z) = V_{\mathrm{s}}^{+}(z) + V_{\mathrm{s}}^{-}(z) = V_{\mathrm{s0}}^{+} e^{-\mathrm{j}kz} + V_{\mathrm{s0}}^{-} e^{\mathrm{j}kz} \tag{3.21}$$

$$I_{\mathrm{s}}(z) = I_{\mathrm{s}}^{+}(z) + I_{\mathrm{s}}^{-}(z) = \frac{V_{\mathrm{s0}}^{+}}{Z} e^{-\mathrm{j}kz} - \frac{V_{\mathrm{s0}}^{-}}{Z} e^{\mathrm{j}kz} \tag{3.22}$$

with

$$k = \beta - \mathrm{j}\alpha = \omega\sqrt{\varepsilon\mu} \quad (\text{or } \omega\sqrt{LC}) \quad \text{and} \quad Z = \sqrt{\frac{\mu}{\varepsilon}} \quad (\text{or } \sqrt{\frac{L}{C}}), \tag{3.23}$$

where k denotes the wavenumber, and Z denotes the plane-wave impedance (or the characteristic impedance in case of a transmission line).[10] In a lossy medium, α must be positive.

[9] constant for plane waves, functions of x and y in the more general transmission-line setting, in which case $\mathbf{E}$ and $\mathbf{H}$ are functions of $\mathbf{r}$ and t.

[10] Sometimes β is referred to as the phase constant. I disapprove since β often depends on frequency and therefore is not constant. It is not a phase either.

In the book by H&B, the characteristic impedance of a transmission line is denoted as Z_0. In case of multi-segmented transmission lines labelled $1, 2, \ldots$, the corresponding characteristic impedances are then denoted as $Z_{01}, Z_{02}, \ldots$. I regard the subscript 0 as unnecessary clutter. More importantly, I would like to treat transmission-line and uniform plane-wave problems at *one fell swoop*, and hence have chosen to use exactly the same notation for the two topics. Finally, it is customary to denote the (plane) wave impedance in free space as Z_0, and I would like to avoid confusion in that regard as well.

In this subsection, we shall assume that the medium (or the transmission line) is lossless, implying that the wavenumber $k = \beta - \mathrm{j}\alpha$ is purely real ($\alpha = 0$) and positive, and is related to the wavelength λ via $k = \beta = 2\pi/\lambda$. Further, we assume that $|V_s^-(z)| \leq |V_s^+(z)|$, i.e., the progressive wave carries at least as much power as the regressive one. As a consequence, we may write $V_{s0}^- = \varGamma V_{s0}^+$ with $|\varGamma| \leq 1$, and hence,

$$V_s = V_{s0}^+ \left(e^{-\mathrm{j}kz} + \varGamma\, e^{+\mathrm{j}kz} \right) \quad \text{and} \quad I_s = \frac{V_{s0}^+}{Z} \left(e^{-\mathrm{j}kz} - \varGamma\, e^{+\mathrm{j}kz} \right) \tag{3.24}$$

Using $|ab| = |a||b|$, we conclude that

$$|V_s| = |V_{s0}^+| \left| e^{-\mathrm{j}kz} \right| \left| 1 + |\varGamma|\, e^{\mathrm{j}\arg(\varGamma) + 2\mathrm{j}kz} \right|. \tag{3.25}$$

Hence, upon letting z run over a half-wavelength interval, $\exp(\mathrm{j}\arg(\varGamma) + 2\mathrm{j}kz)$ traces out the unit circle in the complex plane, from which we infer that $|V_s|$ assumes the maximum and minimum values

$$|V_s|_{\max} = |V_{s0}^+|\, (1 + |\varGamma|) \quad \text{and} \quad |V_s|_{\min} = |V_{s0}^+|\, (1 - |\varGamma|)\,, \tag{3.26}$$

respectively. The voltage *standing wave ratio* for $V_s = V_{s0}^+ e^{-\mathrm{j}kz} + \varGamma V_{s0}^+ e^{\mathrm{j}kz}$ is defined as

$$V_{\mathrm{SWR}} = \frac{|V_s(z)|_{\max}}{|V_s(z)|_{\min}} = \frac{1 + |\varGamma|}{1 - |\varGamma|}. \tag{3.27}$$

This quantity plays an important role in the analysis of reflection/transmission problems. In carefully prepared waveguides, the voltage standing wave ratio may be probed using a sliding probe.

Another quantity that will frequently occur below, is the effective impedance at a certain point on the z-axis, i.e.,

$$Z_{\mathrm{eff}}(z) = \frac{V_s(z)}{I_s(z)} \tag{3.28}$$

For the counterpropagation waves under consideration, we have

$$Z_{\mathrm{eff}}(z) = Z\, \frac{1 + \varGamma\, e^{+2\mathrm{j}kz}}{1 - \varGamma\, e^{+2\mathrm{j}kz}}, \tag{3.29}$$

which, for arbitrary $Z > 0$, real z and complex $|\varGamma| \leq 1$ may take any value $\mathrm{Re}(Z_{\mathrm{eff}}) \geq 0$.

The mapping between the normalised effective impedance $\hat{Z}_{\mathrm{eff}}(z) = Z_{\mathrm{eff}}(z)/Z$ and $\varGamma$ will be investigated in more detail in Section 4.2.

3.4 The transfer matrix

3.4.1 Wave decomposition

In view of Eqs. (3.16) and (3.17), we may express the complex voltage and current amplitudes in terms of the complex voltage amplitudes V_s^+ and V_s^- associated with the respective progressive and regressive waves acccording to

$$\begin{pmatrix} V_s \\ I_s \end{pmatrix} = \begin{pmatrix} V_s^+ + V_s^- \\ I_s^+ + I_s^- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ Y & -Y \end{pmatrix} \begin{pmatrix} V_s^+ \\ V_s^- \end{pmatrix}. \tag{3.30}$$

which may be inverted to yield

$$\begin{pmatrix} 1 & 1 \\ Y & -Y \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & Z \\ 1 & -Z \end{pmatrix} \Rightarrow \begin{pmatrix} V_s^+ \\ V_s^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & Z \\ 1 & -Z \end{pmatrix} \begin{pmatrix} V_s \\ I_s \end{pmatrix}. \tag{3.31}$$

This wavefield decomposition shall prove very useful for the transfer of complex voltage and current amplitudes, and in reflection/transmission problems discussed below.

3.4.2 Wavefield decomposition in a homogeneous layer

Now, let us consider a homogeneous layer (or a uniform section of a transmission line), which includes the section $[z_1, z_2]$ of the real z-axis. Further, let us assume that at the level $z = z_2 = z_1 + \ell$ with $\ell \geq 0$, $V_{s2} = V_s(z_2)$ and $I_{s2} = I_s(z_2)$ are known.

Through Eq. (3.30), we have demonstrated that V_s and I_s are *composed* of one-way propagating wave constituents. Hence, we may use Eq. (3.31) to *decompose* the complex field amplitudes into their forward and backward propagating complex voltage wave constituents.

Subsequently, we may use Eq. (3.17) to transfer the forward and backward propagating complex voltage wave constituents from $z = z_2$ to $z = z_1$. On the one hand, the regressive wave amplitudes $V_{s2} = V_s(z_2)$ and $YI_{s2} = I_s(z_2)$ may be propagated in the negative z-direction for a distance ℓ, through multiplication by a factor $\exp(-jk\ell)$. On the other hand, the forward propagating wave at z_2 has arrived there from $z = z_1$, and may be *back-propagated* in the negative z-direction for a distance ℓ, through multiplication by a factor $\exp(jk\ell)$ (cf. Figure 3.1).

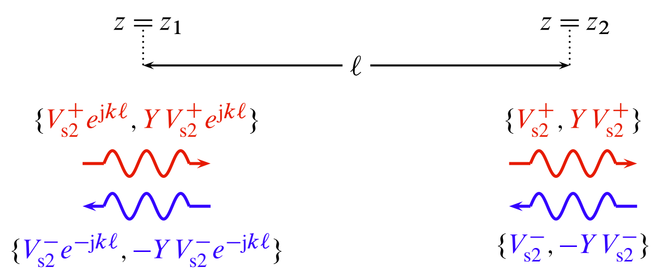


Figure 3.1: Backward propagation of the progressive voltage and current amplitudes, and forward propagation of the regressive voltage and current amplitudes from z_2 to z_1.

The wavefield composition at z_2 just described may be expressed as

$$\begin{pmatrix} V_{s2} \\ I_{s2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ Y & -Y \end{pmatrix} \begin{pmatrix} V_{s2}^{+} \\ V_{s2}^{-} \end{pmatrix}. \tag{3.32}$$

The wavefield decomposition at $z = z_2$ amounts to inverting Eq. (3.32), i.e.,

$$\begin{pmatrix} V_{s2}^{+} \\ V_{s2}^{-} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & Z \\ 1 & -Z \end{pmatrix} \begin{pmatrix} V_{s2} \\ I_{s2} \end{pmatrix}.$$

After (back-)propagation from $z_2 = z_1 + \ell$ to z_1, and subsequent wavefield composition at z_1, we arrive at

$$\begin{pmatrix} V_{s1} \\ I_{s1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ Y & -Y \end{pmatrix} \begin{pmatrix} e^{jk\ell} & 0 \\ 0 & e^{-jk\ell} \end{pmatrix} \begin{pmatrix} V_{s2}^{+} \\ V_{s2}^{-} \end{pmatrix}. \tag{3.33}$$

Combining the results leads to

$$\begin{pmatrix} V_{s1} \\ I_{s1} \end{pmatrix} = \mathsf{T}(z_1, z_2) \begin{pmatrix} V_{s2} \\ I_{s2} \end{pmatrix}, \tag{3.34}$$

in which

$$\begin{aligned} \mathsf{T}(\underbrace{z_2 - \ell}_{z_1}, z_2) &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ Y & -Y \end{pmatrix} \begin{pmatrix} e^{jk\ell} & 0 \\ 0 & e^{-jk\ell} \end{pmatrix} \begin{pmatrix} 1 & Z \\ 1 & -Z \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{jk\ell} & e^{-jk\ell} \\ Ye^{jk\ell} & -Ye^{-jk\ell} \end{pmatrix} \begin{pmatrix} 1 & Z \\ 1 & -Z \end{pmatrix} = \begin{pmatrix} \cos(k\ell) & jZ \sin(k\ell) \\ jY \sin(k\ell) & \cos(k\ell) \end{pmatrix}. \end{aligned}$$

Note that the condition that ℓ be positive may be relaxed. In particular, upon interchanging z_1 and z_2, which amounts to replacing ℓ by $-\ell$, we arrive at

$$\begin{pmatrix} V_{s1} \\ I_{s1} \end{pmatrix} = \mathsf{T}(z_1, z_2) \begin{pmatrix} V_{s2} \\ I_{s2} \end{pmatrix} = \mathsf{T}(z_1, z_2)\mathsf{T}(z_2, z_1) \begin{pmatrix} V_{s1} \\ I_{s1} \end{pmatrix},$$

implying that $\mathsf{T}(z_1, z_2)$ and $\mathsf{T}(z_2, z_1)$ are each others inverse.

In Chapter 4, we shall turn our attention to reflection/transmission problems pertaining to composite structures, in which the shall combine the transfer matrix for homogeneous layers with the continuity of the voltage and current amplitudes across interfaces to construct the transfer matrix for composite structures.

3.5 Reflection & transmission of time-harmonic waves

3.5.1 Reflection & transmission at a single interface

Let us consider a reflection/transmission problem for a single interface between half spaces or transmission lines. A wave propagating in the positive z-direction, characterised by a

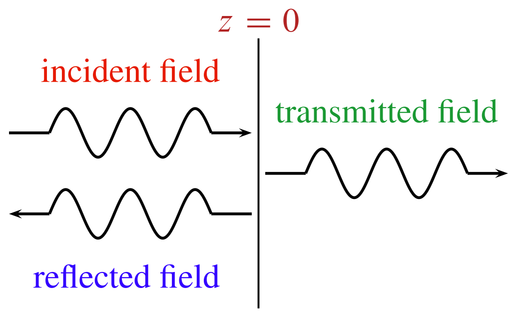


Figure 3.2: Plane wave, normally incident from region 1 ($z < 0$) on an interface at $z = 0$, the reflected plane wave in region 1, and the transmitted plane wave in region 2 ($z > 0$).

voltage and current amplitude vector $(V_s^i, I_s^i)^T$, a wavenumber k_1 and a plane-wave or characteristic impedance Z_1 is incident from $z < 0$ on the interface at $z = 0$, where it is partially reflected and partially transmitted. For $z > 0$, the transmitted wave, characterised by a voltage and current amplitude vector $(V_s^t, I_s^t)^T$, propagates in the positive z-direction with wavenumber k_2 and plane-wave or characteristic impedance Z_2. (Alternatively, in case of a transmission line, the line may be terminated at $z = 0$ by a load impedance Z_2.) Finally, a reflected wave, characterised by a voltage and current amplitude vector $(V_s^r, I_s^r)^T$, is excited at $z = 0$ and propagates in the negative z-direction (cf. Figure 3.3)

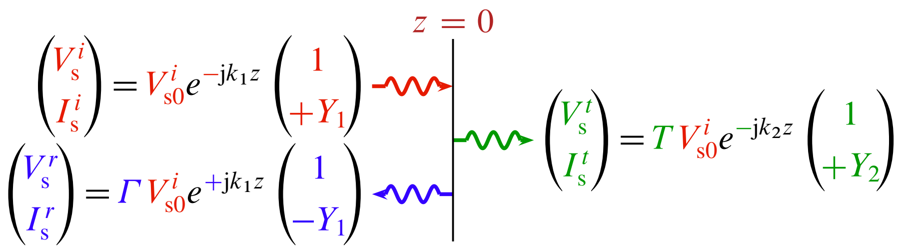


Figure 3.3: Reflection/transmission problem for a single interface at $z = 0$ between region 1 ($z < 0$) and region 2 ($z > 0$).

We are particularly interested in the voltage reflection and transmission coefficients, $\Gamma = V_{s1}^r / V_{s1}^i$ and $T = V_{s1}^r / V_{s1}^i$, respectively. To determine these coefficients, we

- use the continuitity across $z = 0$ of the respective voltage and current amplitudes $V_s = V_{s0}$ and $I_s = I_{s0}$,
- decompose the wavefields on either side of $z = 0$ into their one-way propagating constituents.

Upon invoking the continuity of V_s and I_s at $z = 0$, and noting that the wavefield at the left of the interface is *composed* of forward and backward propagating wave constituents, while the wavefield on the right of the interface is *composed* of a forward propagating wave only, we may write

$$\begin{pmatrix} 1 & 1 \\ Y_1 & -Y_1 \end{pmatrix} \begin{pmatrix} V_{s0}^i \\ \Gamma V_{s0}^i \end{pmatrix} = \begin{pmatrix} V_{s0} \\ I_{s0} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ Y_2 & -Y_2 \end{pmatrix} \begin{pmatrix} T V_{s0}^i \\ 0 \end{pmatrix} \tag{3.35}$$

Next, we multiply by the *decomposition matrix* pertaining to the medium on the left, and divide by V_{s0}^i. This leads to

$$\begin{pmatrix} 1 \\ \Gamma \end{pmatrix} = \frac{1}{2} T \begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix} \begin{pmatrix} 1 \\ Y_2 \end{pmatrix} \Rightarrow T = \frac{2Z_2}{Z_2 + Z_1}, \quad \Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}. \tag{3.36}$$

The reflection coefficient relates the voltage amplitude of the reflected wave at $z = 0$ to that of the incident wave through $V_{s0}^r = \Gamma V_{s0}^i$.

3.5.2 A different plane of reference

We have derived reflection coefficient that relates the voltage amplitude of the incoming wave at the interface between two media to the voltage amplitude of the reflected wave. For simplicity, we have assumed that that the interface is located at $z = 0$.

Sometimes it is deemed useful to define another reference plane, say at $z = -\ell$, and to define the reflection coefficient, due to the reflection at $z = 0$, at the new reference plane through

$$\Gamma_{\mathrm{eff}}|_{z=-\ell} = \frac{V_s^r(-\ell)}{V_s^i(-\ell)} = \frac{V_{s0}^r e^{-\mathrm{j}k_1\ell}}{V_{s0}^i e^{\mathrm{j}k_1\ell}} = \Gamma e^{-2\mathrm{j}k_1\ell}. \tag{3.37}$$

Now, let us apply the transfer matrix to the reflection & transmission problem at a single interface discussed above. We have used the continuity of V_s and I_s in Eq. (3.35), i.e.,

$$\begin{pmatrix} V_{s0} \\ I_{s0} \end{pmatrix} = \begin{pmatrix} 1 \\ Y_2 \end{pmatrix} T V_{s0}^i = \begin{pmatrix} Z_2 \\ 1 \end{pmatrix} Y_2 T V_{s0}^i. \tag{3.38}$$

Application of the transfer matrix in medium 1 from $z = 0$ to $z = -\ell$ yields the voltage and current amplitude vector at $z = -\ell$

$$\begin{pmatrix} V_s(-\ell) \\ I_s(-\ell) \end{pmatrix} = \begin{pmatrix} Z_2 \cos(k_1\ell) + \mathrm{j}Z_1 \sin(k_1\ell) \\ \mathrm{j}Y_1 Z_2 \sin(k_1\ell) + \cos(k_1\ell) \end{pmatrix} Y_2 T V_{s0}^i. \tag{3.39}$$

The associated effective impedance in the reference plane $z = -\ell$, is found to be

$$Z_{\mathrm{eff}}(-\ell) = \frac{V_s(-\ell)}{I_s(-\ell)} = Z_1 \frac{Z_2 \cos(k_1\ell) + \mathrm{j}Z_1 \sin(k_1\ell)}{Z_1 \cos(k_1\ell) + \mathrm{j}Z_2 \sin(k_1\ell)}. \tag{3.40}$$

An alternative (but equivalent) way of deriving Eq. (3.40) is

$$\begin{aligned}
Z_{\mathrm{eff}}(-\ell) = \frac{V_s(-\ell)}{I_s(-\ell)} &= Z_1 \frac{V_{s0}^i e^{\mathrm{j}k_1\ell} + \Gamma V_{s0}^i e^{-\mathrm{j}k_1\ell}}{V_{s0}^i e^{\mathrm{j}k_1\ell} - \Gamma V_{s0}^i e^{-\mathrm{j}k_1\ell}} \\
&= Z_1 \frac{1 + \Gamma e^{-2\mathrm{j}k_1\ell}}{1 - \Gamma e^{-2\mathrm{j}k_1\ell}} \qquad (3.41\mathrm{a}) \\
&= Z_1 \frac{e^{\mathrm{j}k_1\ell} + \Gamma e^{-\mathrm{j}k_1\ell}}{e^{\mathrm{j}k_1\ell} - \Gamma e^{-\mathrm{j}k_1\ell}} = Z_1 \frac{(1+\Gamma)\cos(k_1\ell) + \mathrm{j}(1-\Gamma)\sin(k_1\ell)}{(1-\Gamma)\cos(k_1\ell) + \mathrm{j}(1+\Gamma)\sin(k_1\ell)} \\
&= Z_1 \frac{Z_2 \cos(k_1\ell) + \mathrm{j}Z_1 \sin(k_1\ell)}{Z_1 \cos(k_1\ell) + \mathrm{j}Z_2 \sin(k_1\ell)} \qquad (3.41\mathrm{b}) \\
&= Z_1 \frac{Z_2 + \mathrm{j}Z_1 \tan(k_1\ell)}{Z_1 + \mathrm{j}Z_2 \tan(k_1\ell)}. \qquad (3.41\mathrm{c})
\end{aligned}$$

Here, each of the three labelled representations for $Z_{\text{eff}}(-\ell)$ on the right-hand sides has its own merits. Firstly, Eq. (3.41a) is the most compact expression, and ties in very well with the Smith Chart, briefly discussed in Subsection 4.2.2. Secondly, Eq. (3.41b) which we have already derived from the transfer matrix formalism, is perhaps the most transparent in that it relates impedances to impedances, and clearly shows the oscillating behaviour (provided that $Z_2 \neq Z_1$ of course). Finally, Eq. (3.41c) is more compact than Eq. (3.41b), but some caution should be issued because in the lossless case (real k_1), it introduces spurious singularities through $\tan(k_1\ell)$.

Questions

Question 3.1
To arrive at the time-averaged power balance for time-harmonic fields, we know we have to work with $\mathbf{S}_\text{s}$. In particular, we would have to expand $\nabla \cdot (\mathbf{E}_\text{s} \times \mathbf{H}_\text{s}^*)$ with the aid of a vector identity, and then employ Eq. (3.6) and its complex conjugate to derive the desired result. This is what you have to do in this exercise. (Hint: it is not difficult, but for the complex conjugation you have to be careful to identify every quantity that may be complex.)

Question 3.2
A time-harmonic uniform plane wave is travelling in free space in the positive z direction with a (complex) voltage amplitude at $z = 0$ given by $V_\text{s}^+ = 1$. Another plane wave is travelling in the negative z-direction with a (complex) voltage amplitude at $z = 0$ given by $V_\text{s}^- = \varGamma$. The counterpropagating waves result in an electromagnetic field that are in part standing waves and in part propagating waves.

a) Assume that $\varGamma = -1/2$. Compute the ratio between the maximum and minimum voltage magnitudes $|V_\text{s}(z)|_{\max}/|V_\text{s}(z)|_{\min}$
b) Assume that $\varGamma = -(7 + 6\text{j})/(27 + 6\text{j}))$. Again, compute the ratio between the maximum and minimum voltage magnitudes $|V_\text{s}(z)|_{\max}/|V_\text{s}(z)|_{\min}$.
c) For an arbitrary complex $\varGamma$, compute the ratio between the maximum and minimum voltage magnitudes $|V_\text{s}(z)|_{\max}/|V_\text{s}(z)|_{\min}$.
d) For an arbitrary complex $\varGamma$, compute the complex impedance $Z_{\text{eff}}(z) = V_\text{s}(z)/I_\text{s}(z)$. Also, evaluate $Z_{\text{eff}}(0)$ explicitly.
e) Can $Z_{\text{eff}}(z)$ be evaluated directly from $V(z,t)$ and $I(z,t)$? If so, how?
f) Explain why the minus sign in the relation between V^- and I^- in Eq. (1.29) is so important.

Question 3.3
For the superposition of time-harmonic plane waves propagating in the positive and negative z-directions with respective voltage amplitudes $V_\text{s}^+ = 1$ and $V_\text{s}^- = \varGamma$, studied in Question 3.2, compute the time-average of the Poynting vector for

a) $\varGamma = -1/2$,
b) $\varGamma = -(7 + 6\text{j})/(27 + 6\text{j}))$.

Now, suppose that the total wavefield consists of a superposition of two plane waves that *both* travel in the positive z-direction, i.e., $V_s^+ = 1 + \Gamma$ and $V_s^- = 0$. Compute the time-average of the Poynting vector for

c) $\Gamma = -1/2$,

d) $\Gamma = -(7 + 6\mathrm{j})/(27 + 6\mathrm{j}))$.

Comment on the equality or difference of the power of a sum of fields versus the sum of powers of fields.

Question 3.4

Use trigonometric identities to demonstrate that the matrix product $\mathsf{T}(z_2, z_1)\mathsf{T}(z_1, z_2)$ evaluates to the identity matrix I, implying that $\mathsf{T}(z_1, z_2)$ and $\mathsf{T}(z_2, z_1)$ are indeed each others inverses.

Question 3.5

If we neglect mechanical forces other than friction, then the equation of motion for a charged particle in the presence of many other charged particles states that the total electromagnetic force brings about a change in the momentum of particle, according to

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{E} + \mu_0 q\mathbf{v} \times \mathbf{H} - \nu m\mathbf{v},$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the impressed (external) electromagnetic fields, while the force $-\nu m\mathbf{v}$ accounts for friction. Further, ν denotes the effective collision frequency. This is the most elementary form that still captures a flavour of the internal interaction between charged particles. For $|\mathbf{v}| \ll c_0$ we may neglect the term involving $\mathbf{H}$ In that case, the following equation ensues

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu\right)\mathbf{v} = \frac{q}{m}\mathbf{E},$$

in which $\mathbf{E}$ is the electric field strength as experienced by the particle at time t (and hence at the location $\mathbf{r}(t)$). Through substitution, verify that the particle velocity $\mathbf{v}$ is given by

$$\mathbf{v}(t) = \frac{q}{m}\int_{t_0}^{t} e^{-\nu(t-t')}\mathbf{E}\left(\mathbf{r}(t'), t'\right)\mathrm{d}t',$$

where $t = t_0$ is the time at which the particle was accelerated by the electric field for the first time.

Question 3.6

Now consider a conductor with electron density $N_e(\mathbf{r})$ with every electron carrying am electric charge q. Determine the relationship between $\mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ by using the expression for $\mathbf{v}(t)$ in Question 3.5. Also derive the relaxation function $\sigma(\mathbf{r}, t)$ in terms of the physical parameters introduced here and above.

Question 3.7

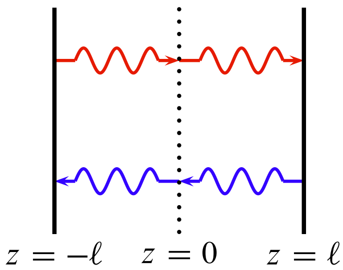
Finally, let us assume that the electric field does depend on time, but that the collision process is so fast that only the field values about $t' = t$ contribute to the integral in Question 3.5. Show that in that case, a good approximation would be

$$\mathbf{J}(\mathbf{r}, t) = \sigma^0(\mathbf{r})\mathbf{E}(\mathbf{r}, t),$$

in which the conductor with conductivity $\sigma^0(\mathbf{r})$ is considered to act instantaneously.

Question 3.8

Let us consider a *cavity*, consisting of two perfectly electrically conducting parallel plates at $z = \pm\ell$. A source, located in the plane $z = 0$ may be characterised by a uniform impressed electric current density, $\mathbf{J}_s(z) = \mathbf{J}_{sS}\delta(z)$ with, as yet, unknown ampltitude and direction. The wavefield in the lossless medium between the plates and the source plane consist of plane-wave constituents that propagate in both positive and negative z-directions, as sketched in the figure on the right.



The electric and magnetic fields are given by $\mathbf{E}_s = V_s\mathbf{a}_x$ and $\mathbf{H}_s = I_s\mathbf{a}_y$ respectively. The current amplitudes of the magnetic field on $z = -\ell$ and $z = \ell$ are denoted as $I_{-\ell}$ and I_ℓ, respectively. Using the transfer matrix $\mathsf{T}(z_1, z_2)$ one may express the voltage and current amplitudes at $z = z_1$ in terms of the voltage and current amplitudes at $z = z_2$. To simplify notation, let us introduce the symbols $c = \cos(k\ell)$ and $s = \sin(k\ell)$.

a) Give the transfer matrices $\mathsf{T}(0, \ell)$ and $\mathsf{T}(0, -\ell)$.
b) Use the appropriate transfer matrix to express $\lim_{z\downarrow 0} V(z)$ and $\lim_{z\downarrow 0} I(z)$ in terms of I_ℓ.
c) Use the appropriate transfer matrix to express $\lim_{z\uparrow 0} V(z)$ and $\lim_{z\uparrow 0} I(z)$ in terms of $I_{-\ell}$.
d) Use the boundary condition for $\mathbf{E}_s$ to express $I_{-\ell}$ in terms of I_ℓ.
e) Use the boundary condition for $\mathbf{H}_s$ to express $\mathbf{J}_{sS}$ in terms of I_ℓ (or the other way around).
f) What happens for $2k\ell = \pi(1 + 2n)$, with $n = 0, 1, 2, \ldots$?
g) Suppose that $I_\ell = 1/(\mathrm{j}Z)$ and $2k\ell = \pi$. Sketch $V(z)$ for $z \in [-\ell, \ell]$.

Chapter 4

Week 4 — Practical applications of time-harmonic 1-D waves

4.1 Practical examples in one spatial dimension

In this chapter, we shall apply the transfer-matrix formalism introduced in Chapter 3 to a few typical practical applications involving reflection/transmission (Rx/Tx) and interference of waves in composite structures.

4.1.1 Anti-reflection structures

Let us consider a simple composite structure, i.e., a single homogeneous layer, sandwiched between two homogeneous half-spaces (or a transmission line segment connecting two semi-infinite transmission-line segments). The associated reflection/transmission problem is depicted in Figure 4.1 Here, it is important to note that there is no incident wave from

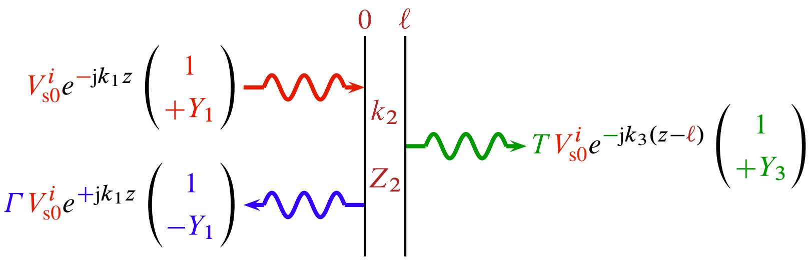


Figure 4.1: Reflection/transmission problem involving a homogeneous layer in Region 2 with medium properties $\{k_2, Z_2\}$, sandwiched between two homogeneous half spaces in Regions 1 and 3 with medium properties $\{k_1, Z_1\}$ and $\{k_3, Z_3\}$, respectively.

$z > \ell$. As a consequence, we know that the ratio between the voltage and the current amplitudes for $z > \ell$ is that of a progressive wave in Region 3, i.e., $Z_3 = 1/Y_3$.

We employ the transfer matrix of the layer to bridge the continuity relations of V_s and I_s across the interfaces at $z = \ell$ and $z = 0$, and the wavefield composition in Region 1 in terms of forward and backward propagating waves.

Since the exponential functions in Region 3 at $z = \ell$ and in Region 1 at $z = 0$ evaluate to 1, this leads to

$$\begin{pmatrix} V_{s0} \\ I_{s0} \end{pmatrix} = \mathsf{T}(0,\ell) \begin{pmatrix} V_{s\ell} \\ I_{s\ell} \end{pmatrix} \tag{4.1a}$$

$$\begin{pmatrix} (1+\Gamma)V_{s0}^i \\ (Y_1 - Y_1\Gamma)V_{s0}^i \end{pmatrix} = \mathsf{T}(0,\ell) \begin{pmatrix} TV_{s0}^i \\ Y_3 T V_{s0}^i \end{pmatrix}. \tag{4.1b}$$

Next, we multiply Eq. (4.1b) by the decomposition matrix of medium 1, and divide the result by V_{s0}^i, resulting in

$$\begin{pmatrix} 1 \\ \Gamma \end{pmatrix} = \frac{T}{2} \begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix} \mathsf{T}(0,\ell) \begin{pmatrix} 1 \\ Y_3 \end{pmatrix}. \tag{4.2}$$

Given the transfer matrix

$$\mathsf{T}(0,\ell) = \begin{pmatrix} \cos(k_2\ell) & \mathrm{j}Z_2\sin(k_2\ell) \\ \mathrm{j}Y_2\sin(k_2\ell) & \cos(k_2\ell) \end{pmatrix},$$

we may solve Eq. (4.2) for the reflection and transmission coefficients Γ and T, respectively.

$\frac{1}{4}\lambda$ transformer

A special case arises when the layer thickness ℓ is adjusted to the local wavelength $\lambda_2 = 2\pi/k_2 = 2\pi/(\omega\sqrt{\varepsilon_2\mu_2})$, according to

$$\ell = \frac{1}{4}\lambda_2 \Rightarrow k_2\ell = \frac{\pi}{2} \Rightarrow \mathsf{T}(0,\ell) = \begin{pmatrix} 0 & \mathrm{j}Z_2 \\ \mathrm{j}Y_2 & 0 \end{pmatrix}$$

As a consequence, Eq. (4.2) becomes

$$V_{s0}^i \begin{pmatrix} 1 \\ \Gamma \end{pmatrix} = \frac{\mathrm{j}TV_{s0}^i}{2} \begin{pmatrix} Z_2Y_3 + Z_1Y_2 \\ Z_2Y_3 - Z_1Y_2 \end{pmatrix}.$$

Hence, there is no reflection if

$$\frac{Z_2}{Z_3} = \frac{Z_1}{Z_2} \Rightarrow Z_2 = \sqrt{Z_1Z_3},$$

which, together with the condition that ℓ is a quarter wavelength, comprises the $\frac{1}{4}\lambda$ transformer, for which

$$\Gamma = 0 \quad \text{and} \quad T = -\mathrm{j}\sqrt{\frac{Z_3}{Z_1}}.$$

Examination of the quarter-wavelength transformer ($Z_2 = \sqrt{Z_1Z_3}$) at other frequencies yields

$$\Gamma = \frac{\cos(k_2\ell)(Z_3 - Z_1)}{\cos(k_2\ell)(Z_3 + Z_1) + 2\mathrm{j}\sin(k_2\ell)\sqrt{Z_1Z_3}}$$

$$T = \frac{2Z_3}{\cos(k_2\ell)(Z_3 + Z_1) + 2\mathrm{j}\sin(k_2\ell)\sqrt{Z_1Z_3}}.$$

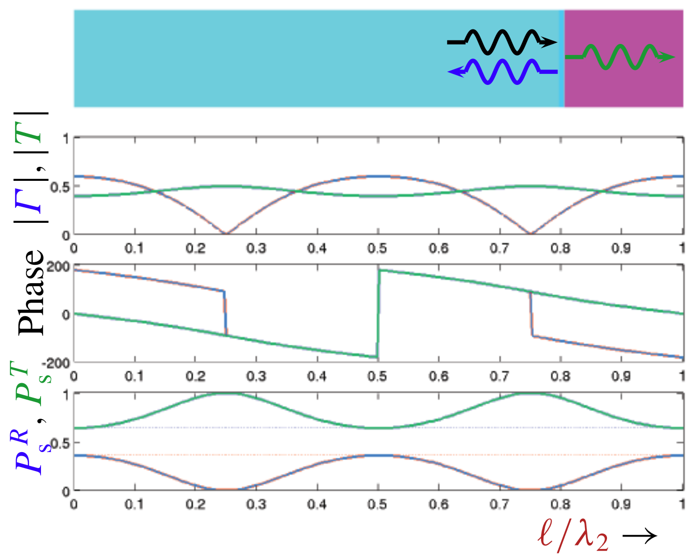


Figure 4.2: The frequency behaviour (in terms of the wavelength in Region 2) of a $\frac{1}{4}\lambda$ transformer. The middle graphs, labelled Phase, depict $\arg(\Gamma)$ and $\arg(T)$. The symbols P^R and P^T stand for the respective time-averaged powers of the reflected and transmitted waves, relative to the time-averaged incident power.

This frequency behaviour is depicted in Figure 4.2 in terms of the wavelength in Region 2. Note that for $\ell = \frac{1}{4}\lambda_2$ the reflection vanishes, implying that all the power is transferred. This property recurs at higher frequencies, specifically, when $\ell = \frac{1+2m}{4}\lambda_2$, $m \in \mathbb{N}$. Such quarter-wavelength transformers have applications in optics (anti-reflection coatings), and in microwave engineering for matching different transmission lines, or transmission lines to antennas.

4.1.2 Bragg reflector

We may also design composite structures to behave as almost perfect reflectors in certain frequency bands. In particular, we consider periodic composite structures, consisting of a periodic arrangement of two different types of layers with medium properties distinguished by labels a and b, respectively. The pile of layers *babababababababab* is sandwiched between two half spaces with medium properties $\{k_a, Z_a\}$.

In our analysis, which is very similar to that in Subsection 4.1.1, we again go from right to left, following the downward pointing arrows in Figure 4.3. Firstly, we use the freedom in adjusting the complex incident field voltage amplitude V_{s0}^i, to choose $V_s^t = T V_{s0}^i = 1$. Secondly, given the fixed ratio $I_s^t = Y V_s^t = Y$, we employ the continuity of $\mathbf{f}_s = \mathbf{f}_s = (V_s^t\ I_s^t)^T$ and the transfer matrix: $\mathbf{f}_s \rightarrow \mathsf{T}_b\mathbf{f}_s$ to arrive at the second arrow from the right. Thirdly, we employ the continuity of $\mathbf{f}_s = \mathbf{f}_s$, and the transfer matrix: $\mathbf{f}_s \rightarrow \mathsf{T}_a\mathsf{T}_b\mathbf{f}_s$ to arrive at the third arrow, et cetera. Finally, we arrive at the transfer matrix for the entire stack of layers, viz., $\mathbf{f}_s \rightarrow \mathsf{T}_b(\mathsf{T}_a\mathsf{T}_b)^9\mathbf{f}_s$. Employing the continuity across the leftmost

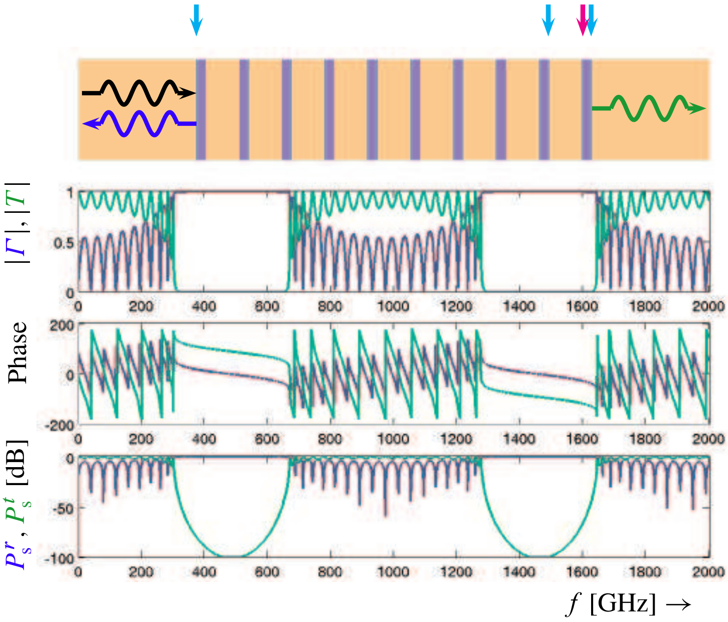


Figure 4.3: Bragg reflector exhibiting frequency bands, in which waves are almost completely reflected. The middle graphs, labelled Phase, depict $\arg(\Gamma)$ and $\arg(T)$. The symbols P^R and P^T stand for the respective time-averaged powers of the reflected and transmitted waves, relative to the time-averaged incident power.

interface, and the composition relation $\mathbf{f}_s = \mathbf{f}^i_s + \mathbf{f}^r_s$, we arrive at

$$\begin{pmatrix} 1/T + \Gamma/T \\ Y/T - Y\Gamma/T \end{pmatrix} = \mathsf{T_b}(\mathsf{T_a}\mathsf{T_b})^9 \begin{pmatrix} 1 \\ Y \end{pmatrix}$$

which allows us to solve for the reflection and transmission coefficients Γ and T, respectively.

4.1.3 Transfer and scattering matrices for composite structures

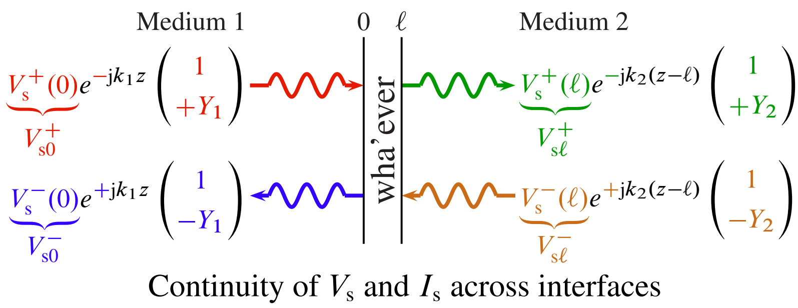


Figure 4.4: General composite structure by which we may relate the transfer matrix to the scattering matrix.

As an alternative to the transfer-matrix formalism, one often employs the scattering-matrix formalism. To explore the relation between the two formalisms, let us consider the relation between the voltage and current amplitudes at either side of a general composite structure (cf. Figure 4.4)

$$\begin{pmatrix} V_{s0} \\ I_{s0} \end{pmatrix} = \mathsf{T}(0,\ell) \begin{pmatrix} V_{s\ell} \\ I_{s\ell} \end{pmatrix} = \begin{pmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} \\ \mathsf{T}_{21} & \mathsf{T}_{22} \end{pmatrix} \begin{pmatrix} V_{s\ell} \\ I_{s\ell} \end{pmatrix}. \tag{4.3}$$

To derive the scattering matrix formalism for composite structures, we employ wavefield composition and decomposition, i.e.,

$$\begin{pmatrix} V_{s0}^{+} \\ V_{s0}^{-} \end{pmatrix} = \underbrace{\frac{1}{2}\begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix}}_{\text{Decomposition in medium 1}} \begin{pmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} \\ \mathsf{T}_{21} & \mathsf{T}_{22} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 \\ Y_2 & -Y_2 \end{pmatrix}}_{\text{Composition in medium 2}} \begin{pmatrix} V_{s\ell}^{+} \\ V_{s\ell}^{-} \end{pmatrix}. \tag{4.4}$$

A little algebra (Question 1) leads to scattering matrix formalism

$$\begin{pmatrix} V_{s0}^{-} \\ V_{s\ell}^{+} \end{pmatrix} = \begin{pmatrix} \mathsf{S}_{11} & \mathsf{S}_{12} \\ \mathsf{S}_{21} & \mathsf{S}_{22} \end{pmatrix} \begin{pmatrix} V_{s0}^{+} \\ V_{s\ell}^{-} \end{pmatrix} \tag{4.5}$$

in which the voltage-wave amplitudes of the outgoing waves are expressed in those of the incoming waves.

Although the transfer and scattering matrix formalisms are equivalent for the 1-D plane-wave (or TEM-wave) problems discussed in this chapter, the two formalisms may be generalised for the analysis of more complex situations often encountered in practice, which warrants the brief discussion below.

Transfer vs scattering matrix formalism

We have explored the *transfer* and *scattering matrix* formalisms for *two-port* systems, e.g., plane waves at normal incidence onto a stack of homogeneous layers (or TEM waves in case of a concatenation of uniform transmission-line segments)

- The *transfer matrix* formalism is easier to evaluate algebraically, in that the *transfer matrix* of a stack of layers is the product of the transfer matrices of the individual layers.
- Because the *scattering matrix* formalism expresses what comes out in terms of what comes in, it naturally leads to a *computationally stable formalism*.
- The *scattering matrix* formalism may readily be extended for *multi-port* (up to continuously infinite) systems arising from 3-D electromagnetic wave problems.
- A similar extension of the *transfer matrix* formalism may become *computationally unstable*.

Transfer matrix formalism vs effective impedance transfer

We have seen that reflection/transmission problems involving composite structures, with a progressive wave incident from the left, can be solved by starting at the back-end of the structure, beyond which we must have a progressive wavefield characterised by a constant wave impedance (or characteristic impedance in case of transmission lines), which is the effective impedance at the back-end of the structure.

Subsequently, we employed the continuity of V_s and I_s and transfer matrices to work our way back to the front of the structure, where the resulting effective impedance is directly related to the reflection coefficient.

We could have circumvented the use of transfer matrices, and have directly transferred the effective impedance by nesting Eq. (3.41b) (or Eq. (3.41c)) from the back-end of the structure to the front end. This is the method advocated in the book by H&B, and there is nothing wrong with that, if one is only interested in the reflection coefficient. To compute the transmission coefficient one would still have to transfer the fields from the front to the back.

4.2 Reflection analysis

4.2.1 The relation between Γ and the normalised effective impedance

In Eq. (3.41), we have introduced the *effective impedance* $Z_{\text{eff}}(z) = V_s(z)/I_s(z)$ at $z < 0$, due to a reflection coefficient Γ at $z = 0$. Below, we investigate the mapping between the *normalised effective impedance*

$$\hat{Z}_{\text{eff}}(z) = \frac{Z_{\text{eff}}(z)}{Z} = \frac{1 + \Gamma\, e^{+2\mathrm{j}kz}}{1 - \Gamma\, e^{+2\mathrm{j}kz}} \tag{4.6}$$

and Γ. This mapping and its inverse

$$\Gamma = e^{-2\mathrm{j}kz}\, \frac{\hat{Z}_{\text{eff}}(z) - 1}{\hat{Z}_{\text{eff}}(z) + 1} \tag{4.7}$$

are examples of Möbius transformations. In Appendix 4.A. we argue that

> a Möbius transformation maps lines in one complex plane to lines or circles in another complex plane, and circles in one complex plane to circles or lines in another complex plane.

For convenience, we write the normalised effective impedance at $z = 0$ as

$$\hat{Z}_{\text{eff}0} = \hat{Z}_{\text{eff}}(0) = r + \mathrm{j}x \quad \text{with } r, x \in \mathbb{R}. \tag{4.8}$$

From

$$\Gamma|_{r=0} = \frac{\mathrm{j}x - 1}{\mathrm{j}x + 1} \tag{4.9}$$

it is easy to see that the line $r = \mathrm{Re}(\hat{Z}_{\text{eff}0}) = 0$ maps to the unit circle in the complex Γ-plane. Since $\hat{Z}_{\text{eff}0} = 1$ (matched load) maps to $\Gamma = 0$, we infer that the half plane

$\mathrm{Re}(\hat{Z}_{\mathrm{eff0}}) \geq 0$ (corresponding to a passive system) maps to the unit disk $|\Gamma| \leq 1$ in the complex Γ-plane.

As mentioned above, this connection between lines and circles is no coincidence, but rather a direct consequence of Möbius transformations. For electrical engineers, the ramifications for the so-called Smith Chart, discussed below, are more important than the details of the elementary algebraic operations associated with Möbius transformations.

4.2.2 The Smith Chart

The Smith Chart represents the unit disk in the complex Γ-plane,

and in this disk displays the images of straight lines in the complex $\hat{Z}_{\mathrm{eff0}}$-plane, with $\hat{Z}_{\mathrm{eff0}} = \hat{Z}_{\mathrm{eff}}(0)$. In particular, for $\hat{Z}_{\mathrm{eff0}} = r + \mathrm{j}x$, we may express Γ in terms of the real parameters r and x (at $z = 0$) as

$$\Gamma = \frac{r - 1 + \mathrm{j}x}{r + 1 + \mathrm{j}x} = 1 - \frac{2}{r + 1 + \mathrm{j}x} \tag{4.10}$$

For $r = 0$, the Möbius transformation from $\hat{Z}_{\mathrm{eff0}}$ maps the line $\mathrm{j}x$, $x \in \mathbb{R}$ to the unit circle in the complex Γ-plane.

For $r \geq 0$ fixed, the Möbius transformation maps vertical lines in the complex $\hat{Z}_{\mathrm{eff0}}$-plane, $r + \mathrm{j}x$ with $x \in \mathbb{R}$ to circles that pass vertically through the real Γ-axis at $\Gamma = 1$, and are contained within the unit disk (e.g., the blue circle for $r = 2$ in Figure 4.5).

For x fixed, the Möbius transformation maps line segments $r + \mathrm{j}x$ with $r \geq 0$ to circular arcs that touch the real Γ axis at $\Gamma = 1$ and run inside the unit disk to a point on its boundary (e.g., the green circular arc for $x = 1/2$ in Figure 4.5).

Finally, shifting the reference plane back from $z = 0$ to $z = -\ell$ along a lossless transmission line with wavenumber $k = \beta \in \mathbb{R}$ amounts to a counterclockwise rotation about the origin of the complex Γ-plane from $\Gamma|_{z=0}$ to $\Gamma|_{z=-\ell} = \Gamma \exp(-2\mathrm{j}k\ell)$, which is illustrated by the red circular arc in Figure 4.5). For $\ell = \frac{1}{4}\lambda$ $(2k\ell = \pi)$ that circular arc would be a semi-circle.

The Smith Chart also allows for the design of so-called stubs — short-circuited transmission line segments of a certain length, connected in parallel at a certain z-level to a multi-segment transmission line configuration — by which impedance matching between disparate transmission lines and/or loads, can be achieved for a frequency of choice. The scale of the circles and circular arcs representing curves of constant r and constant x, respectively, used to be sufficiently fine for microwave engineers to employ the Smith Chart as a design tool. In the age of ubiquitous computing, the role of the Smith Chart has changed to that of a graphical design aid. For further details, see Section 10.10 of the book by H&B.

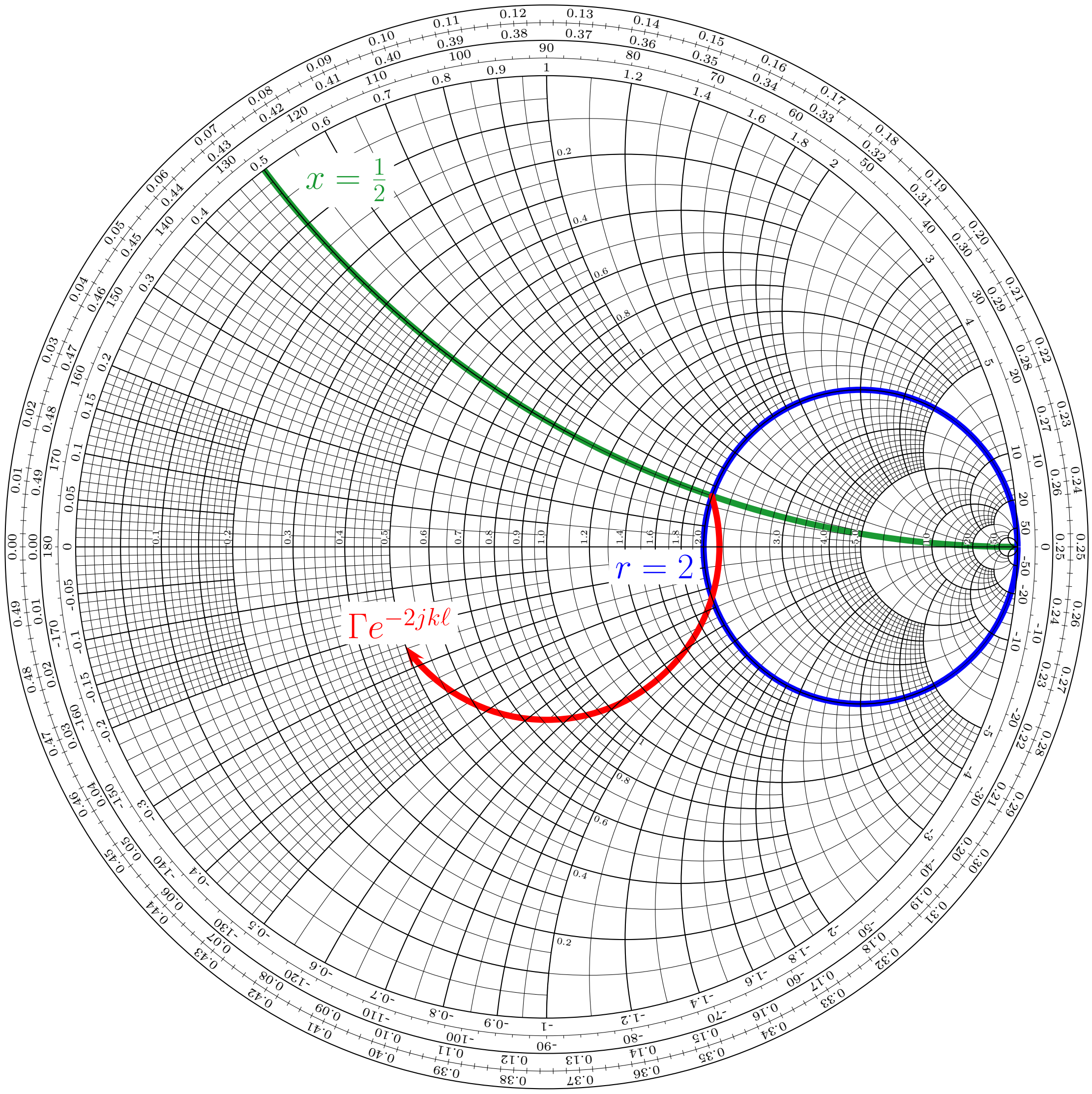


Figure 4.5: Smith Chart with a blue circle of constant $\mathrm{Re}(\hat{Z}_{\mathrm{eff0}}) = r = 2$, a green arc of constant $\mathrm{Im}(\hat{Z}_{\mathrm{eff0}}) = x = \frac{1}{2}$, and a red arc showing the effect of shifting the reference plane from $z = 0$ to $z = -\ell$. (I adapted the pstricks version created by Germi Camps.)

4.3 Pulse propagation in dispersive media

Finally, we turn our attention to the propagation of a modulated pulsed plane wave in a homogeneous LTI medium.[1] In particular, we are interested in the so-called phase and group velocities.

Let us consider the frequency-domain complex voltage amplitude of a progressive wave

$$V_{\mathrm{s}}^{+}(z,\omega) = V_{\mathrm{s0}}^{+} e^{-\mathrm{j}kz} = |V_{\mathrm{s0}}^{+}| e^{-\alpha z - \mathrm{j}\beta z + \mathrm{j}\arg(V_{\mathrm{s}}^{+})}, \tag{4.11}$$

[1] or a modulated pulsed TEM wave propagating along a waveguide with an LTI filling. In fact, the concepts discussed in this section are also applicable to other modes of propagation along closed waveguides and dielectric ones (such as optical fibres).

where it should be noted that $|V_{s0}^{+}|$, α, β, and $\arg(V_s^+)$ are real functions of the radial frequency. In practice, V_{s0}^{+} (the complex voltage amplitude at $z = 0$) is often a narrow-band signal about a centre frequency ω_0, which, upon performing the inverse Fourier transformation back to the space-time domain, means that only a very limited part of the frequency spectrum is involved. The complex wavenumber $k = \beta - \mathrm{j}\alpha$ determines the properties of the pulse propagation, implying that for the narrow-band signals under consideration, it suffices to zoom in on the frequency behaviour of α and β about ω_0.

The most extreme example of this is the time-harmonic wave for which the *frequency-domain complex voltage* is given by $V_s^+(z,\omega) = \pi V_{s0}^{+}(\omega_0)[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]e^{-\mathrm{j}kz}$. The resulting time-harmonic wave is given by

$$V^+(z,t) = |V_{s0}^{+}(\omega_0)|e^{-\alpha_0 z}\cos\left(\omega_0 t - \beta_0 z + \arg(V_s^+)\right). \tag{4.12}$$

where $\alpha_0 = \alpha(\omega_0)$ and $\beta_0 = \beta(\omega_0)$. The propagation properties of this time-harmonic wave are incorporated in the phase

$$\psi^+ = \omega_0\left(t - \frac{\beta_0}{\omega_0}z\right) + \arg(V_s^+). \tag{4.13}$$

The *phase speed*[2] v_{ph} is the speed at which a point with constant phase travels as time progresses. From Eq. (4.13), we infer that the phase speed of a time-harmonic wave is given by

$$v_{\mathrm{ph}} = \frac{\omega_0}{\beta_0}. \tag{4.14}$$

Further, α_0 is responsible for the attenuation of the time-harmonic wave.

Unfortunately, a simple time-harmonic wave is useless for carrying information from A to B. However, information can be transfered through modulated signals, so one may wonder whether at what speed that information will travel. To investigate this, let us assume that the space-time-domain voltage amplitude at $z = 0$ of the progressive wave is the amplitude modulated pulse

$$V_0^+(t) = f(t)\cos(\omega_0 t), \tag{4.15}$$

in which

- ω_0 is the *carrier-wave* (angular) *frequency*,
- $f(t)$ is the slowly varying hull (slow compared to the factor $\cos(\omega_0 t)$).

The Fourier transform of the amplitude-modulated pulse is given by

$$\begin{aligned} V_{s0}^{+}(\omega) &= \int\limits_{t=-\infty}^{\infty} f(t)\cos(\omega_0 t)e^{-\mathrm{j}\omega t}\,\mathrm{d}t = \frac{1}{2}\int\limits_{t=-\infty}^{\infty} f(t)\left[e^{-\mathrm{j}(\omega-\omega_0)t} + e^{-\mathrm{j}(\omega+\omega_0)t}\right]\mathrm{d}t \\ &= \frac{1}{2}f_s(\omega-\omega_0) + \frac{1}{2}f_s(\omega+\omega_0), \end{aligned} \tag{4.16}$$

where $f_s(\omega)$ is the frequency-domain counterpart of the $f(t)$. An example of an amplitude modulated pulse and its spectrum is depicted in Figure 4.6.

[2]more often (unfortunately) referred to as the *phase velocity*.

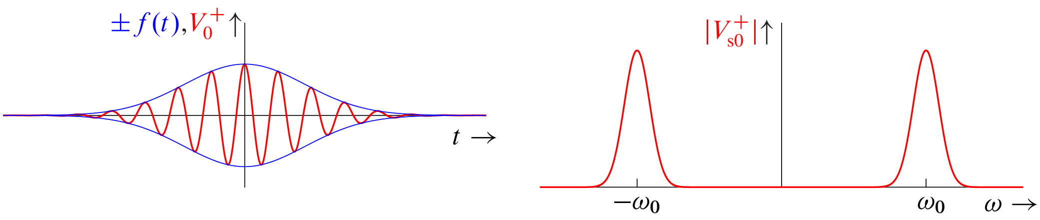


Figure 4.6: An amplitude modulated pulse (left) and the corresponding spectrum (right).

Next, we would like to derive a physically cogent expression for the real space-time-domain voltage amplitude at $z > 0$. To this end, we first employ the standard two-sided inverse Fourier transformation (cf. Eq. (3.2b)) according to

$$V_0^+(z,t) = \frac{1}{4\pi} \int_{\omega=-\infty}^{\infty} e^{\mathrm{j}\omega t - \mathrm{j}\beta z - \alpha z} \, [f_\mathrm{s}(\omega - \omega_0) + f_\mathrm{s}(\omega + \omega_0)] \, \mathrm{d}\omega. \tag{4.17}$$

Of course, this integral may be split into two integrals, involving $f_\mathrm{s}(\omega - \omega_0)$ and $f_\mathrm{s}(\omega + \omega_0)$, respectively. Upon subjecting the latter integral to a gentle massage, we may rewrite it as the complex conjugate of the former integral, implying that

$$V^+(z,t) = \mathrm{Re}\left[\frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} e^{\mathrm{j}\omega t - \mathrm{j}\beta z - \alpha z} \, f_\mathrm{s}(\omega - \omega_0) \, \mathrm{d}\omega \right]. \tag{4.18}$$

If you want to know the details of the derivation of Eq. (4.18), please consult the subsection below, otherwise you may proceed with the one after that.

Some simplifying manipulations

To simplify matters, we would first like to get rid of the constituent involving $f_\mathrm{s}(\omega + \omega_0)$. In Subsection 3.1.2, we had already used the fact that the electric field in the space-time domain is a real quantity in order to demonstrate that $\mathbf{E}_\mathrm{s}(\mathbf{r}, -\omega) = \mathbf{E}_\mathrm{s}^*(\mathbf{r}, \omega)$. By the same token, we have

$$\begin{aligned} \{\mathbf{H}_\mathrm{s}, \mathbf{D}_\mathrm{s}, \mathbf{B}_\mathrm{s}, \mathbf{J}_\mathrm{s}, \mathbf{K}_\mathrm{s}\}(\mathbf{r}, -\omega) &= \{\mathbf{H}_\mathrm{s}^*, \mathbf{D}_\mathrm{s}^*, \mathbf{B}_\mathrm{s}^*, \mathbf{J}_\mathrm{s}^*, \mathbf{K}_\mathrm{s}^*\}(\mathbf{r}, \omega) \\ &\Rightarrow \{\varepsilon, \mu\}(\mathbf{r}, -\omega) = \{\varepsilon^*, \mu^*\}(\mathbf{r}, \omega) \end{aligned} \tag{4.19}$$

which in turn implies that

$$\begin{aligned} \beta(-\omega) - \mathrm{j}\alpha(-\omega) &= (-\omega)\sqrt{\varepsilon(-\omega)\mu(-\omega)} \\ &= -(\omega\sqrt{\varepsilon(\omega)\mu(\omega)})^* = -\beta(\omega) - \mathrm{j}\alpha(\omega). \end{aligned} \tag{4.20}$$

Hence, β and α are odd and even real functions of ω, respectively. As a consequence, we may write

$$\begin{aligned}
&\int\limits_{\omega=-\infty}^{\infty} e^{\mathrm{j}\omega t-\mathrm{j}\beta(\omega)z-\alpha(\omega)z}\, f_{\mathrm{s}}(\omega+\omega_0)\,\mathrm{d}\omega \\
&\overset{\omega=-w}{=} -\int\limits_{w=\infty}^{-\infty} e^{-\mathrm{j}wt-\mathrm{j}\beta(-w)z-\alpha(-w)z}\, f_{\mathrm{s}}(-w+\omega_0)\,\mathrm{d}w \\
&= \int\limits_{w=-\infty}^{\infty} e^{-\mathrm{j}wt+\mathrm{j}\beta(w)z-\alpha(w)z}\, f_{\mathrm{s}}^{*}(w-\omega_0)\,\mathrm{d}w \\
&\overset{w\to\omega}{=} \left[\int\limits_{\omega=-\infty}^{\infty} e^{\mathrm{j}\omega t-\mathrm{j}\beta(\omega)z-\alpha(\omega)z}\, f_{\mathrm{s}}(\omega-\omega_0)\,\mathrm{d}\omega\right]^{*}.
\end{aligned} \tag{4.21}$$

Hence, Eq. (4.17) may indeed be rewritten as Eq. (4.18).

We could have saved ourselves some trouble by observing that for band-limited signals $f_{\mathrm{s}}(\omega+\omega_0)$ practically vanishes for $\omega > 0$, implying that in the computation of $V^{+}(z,t)$ it plays no role as long as we employ the one-sided inverse Fourier transformation (cf. Eq. (3.2b)). Similarly, $f_{\mathrm{s}}(\omega-\omega_0)$ practically vanishes for $\omega < 0$, so it does not make any difference if we subsequently replace the lower limit of integration from 0 to $-\infty$.

A first-order approximation

We are but a stone's throw away from the final expression. Without loss of generality, let us first change (shift) the parameter of integration in Eq. (4.18) from ω to Ω with $\omega = \omega_0+\Omega$. This leads to

$$V^{+}(z,t) = \mathrm{Re}\left[\frac{e^{\mathrm{j}\omega_0 t}}{2\pi}\int\limits_{\Omega=-\infty}^{\infty} e^{\mathrm{j}\Omega t-\mathrm{j}\beta(\omega_0+\Omega)z-\alpha(\omega_0+\Omega)z}\, f_{\mathrm{s}}(\Omega)\,\mathrm{d}\Omega\right]. \tag{4.22}$$

Since we are considering narrow-band signals (the hull $f(t)$ varies slowly compared to $\cos(\omega_0 t)$), we note that $f_{\mathrm{s}}(\Omega)$ only differs significantly from zero for Ω (relatively) small. Hence, it makes sense to use a Taylor expansion of α and β about ω_0, i.e.,

$$\alpha(\omega_0+\Omega) = \alpha_0 + \Omega\alpha_0' + o(\Omega), \tag{4.23}$$

$$\beta(\omega_0+\Omega) = \beta_0 + \Omega\beta_0' + o(\Omega), \tag{4.24}$$

in which we have introduced the symbols

$$\alpha_0 = \alpha(\omega_0),\ \beta_0 = \beta(\omega_0),\ \alpha_0' = \frac{\mathrm{d}\alpha}{\mathrm{d}\omega}|_{\omega=\omega_0},\ \text{and}\ \beta_0' = \frac{\mathrm{d}\beta}{\mathrm{d}\omega}|_{\omega=\omega_0}, \tag{4.25}$$

and we have used Landau's little-order symbol $o(\Omega)$ (see Footnote 8 of Chapter 3).

We can now be a bit more precise about what we mean by narrow-band signals. We assume that the maximum $|\Omega|$ for which $f_{\mathrm{s}}(\Omega)$ yields a significant contribution is still small enough that

$$\Omega\beta_0' \ll \beta_0 \quad \text{and} \quad \Omega\alpha_0' \ll \alpha_0, \tag{4.26}$$

and that we may safely neglect the higher-order terms in the Taylor expansions. Then, Eq. (4.22) reduces to

$$V^{+}(z,t) \approx \mathrm{Re}\left[e^{\mathrm{j}(\omega_0 t - \beta_0 z) - \alpha_0 z} \frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} e^{\mathrm{j}\Omega(t-\beta_0' z) - \Omega\alpha_0' z} f_{\mathrm{s}}(\Omega)\,\mathrm{d}\Omega \right]. \tag{4.27}$$

Finally, one should realise that if the attenuation due to $e^{-\Omega\alpha_0' z}$ were significant, then Eq. (4.26) would imply that the signal would already have attenuated so much due to $e^{-\alpha_0 z}$ that it probably is of little use. In other words $\alpha \approx \alpha_0$ is good enough. This is not the case for β. In fact, for optical fibres $\beta_0 z$ may exceed values of the order of 10^{10}, and although $\Omega\beta_0' z$ is significantly smaller, it will still play an important role. For single-mode fibres, one often has to consider dispersion and even dispersion slope involving a third-order Taylor expansion for β about ω_0 (two extra terms). Here, we stick to a first-order expansion for β, but neglect α_0', which leads to

$$\begin{aligned} V^{+}(z,t) &\approx \mathrm{Re}\left[e^{\mathrm{j}(\omega_0 t - \beta_0 z) - \alpha_0 z} \overbrace{\frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} e^{\mathrm{j}\Omega(t-\beta_0' z)} f_{\mathrm{s}}(\Omega)\,\mathrm{d}\Omega}^{f(t-\beta_0' z)\in\mathbb{R}} \right] \\ &= e^{-\alpha_0 z} \cos(\omega_0 t - \beta_0 z)\, f(t-\beta_0' z). \end{aligned} \tag{4.28}$$

This pulse consists of two factors, i.e.,

- the factor $e^{-\alpha_0 z} \cos(\omega_0 t - \beta_0 z)$, which is the so-called carrier wave that propagates with phase speed $v_{\mathrm{ph}} = \omega_0/\beta_0$ and attenuates as if it were a time-harmonic wave at the single carrier-wave frequency ω_0,
- the factor $f(t-\beta_0' z)$, which is the hull that contains the information that was sent from the location $z = 0$.

Within the context of this approximation, we have demonstrated that information may travel without distortion at a speed

$$v_{\mathrm{g}} = v_{\mathrm{g}}(\omega_0) = \frac{1}{\frac{\mathrm{d}\beta}{\mathrm{d}\omega}\big|_{\omega=\omega_0}} = \frac{\mathrm{d}\omega}{\mathrm{d}\beta}\Big|_{\omega=\omega_0}, \tag{4.29}$$

which is called the *group speed*[3] at the carrier-wave frequency.

If the bandwidth were wide enough that the group speed would vary significantly within that bandwidth, then the dispersion effects mentioned above must be taken into account as well.

[3]more often (unfortunately) referred to as the *group velocity*.

4.A The Möbius transformation

Möbius transformations are simple rational transformations from one complex plane to another. Let us consider a general complex variable[4] $z = x + \mathrm{j}y$, and let $c \neq 0$, then

$$w = \frac{az+b}{cz+d} = \frac{a}{c} + \left(\frac{bc-ad}{c}\right)\frac{1}{cz+d}, \quad a, b, c, d \in \mathbb{C} \tag{4.30}$$

$$\iff z = \frac{-dw+b}{cw-a} = -\frac{d}{c} + \left(\frac{bc-ad}{c}\right)\frac{1}{cw-a} \tag{4.31}$$

Let us analyse the three elementary algebraic operations featuring in the Möbius transformations $z \leftrightarrow w$.

M1 Multiplication by $c = |c|e^{\mathrm{j}\arg(c)}$ in $w = cz$ amounts to a scaling by $|c|$ and a counterclockwise rotation by an angle $\arg(a)$. The images of a line and a circle in the complex z-plane are a rotated and scaled line and circle in the complex w-plane, respectively (see the top diagram in Figure 4.7).

M2 Addition of d in $w = z + d$ amounts to a translation by d, depicted in the second diagram from the top in Figure 4.7. Note that a combination of multiplication and addition in $w = cz + d$ maps the line and circle in the top-left diagram to the second-from-the-top-right diagram in Figure 4.7.

M3 Finally, and perhaps most surprisingly, the complex operation $w = 1/z$ maps lines and circles in the complex z-plane to lines and circles (*not respectively*) in the complex w-plane.

In particular, a line in the complex z-plane that *does not* pass through the origin maps to a circle in the complex w-plane that *does* pass through the origin, and vice versa.

A line in the complex z-plane that *does* pass through the origin maps to a line in the complex w-plane that *does* pass through the origin.

Finally, a circle in the complex z-plane that *does not* pass through the origin maps to a circle in the complex w-plane that *does not* pass through the origin.

Hence, the Möbius transformation sends lines in one complex plane to lines or circles in another complex plane, and circles in one complex plane to circles or lines in another complex plane.

[4]with apologies for the reusing the variables $\{x, y, z\}$; it is the standard notation, and will only be used in this way in this subsection to investigate the properties of Möbius transformation

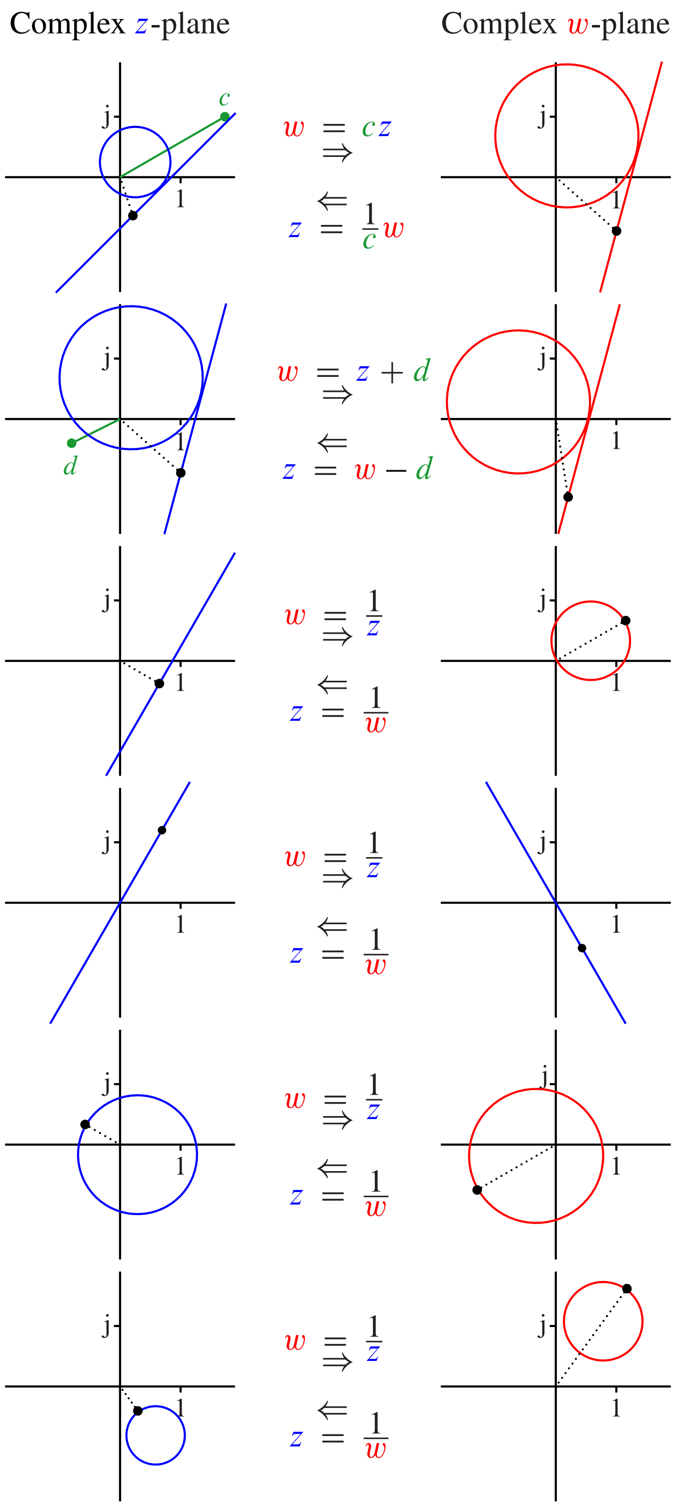


Figure 4.7: The mapping properties of lines and circles under the action of Möbius transformations. The black dots in adjacent complex z-plane and complex w-plane diagrams depict z-w pairs.

Questions

Question 4.1

Express the matrix elements of the scattering matrix in Eq. (4.5) in terms of the matrix elements of the transfer matrix and the characteristic impedances and/or admittances of medium 1 and 2. To save on ink and paper, express the matrix elements of the scattering matrix in those of the matrix A, given by

$$\begin{pmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix}\begin{pmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} \\ \mathsf{T}_{21} & \mathsf{T}_{22} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ Y_2 & -Y_2 \end{pmatrix},$$

and after that, compute the matrix elements of A.

Question 4.2

Prove that for constant $r \geq 0$ in Eq. (4.10) a vertical line $r + \mathrm{j}x$ in the complex $\hat{Z}_{\text{eff0}}$-plane with $x \in \mathbb{R}$ maps to a circle in the complex Γ-plane that passes vertically through the real Γ-axis at $\Gamma = 1$, and is contained within the unit disk.

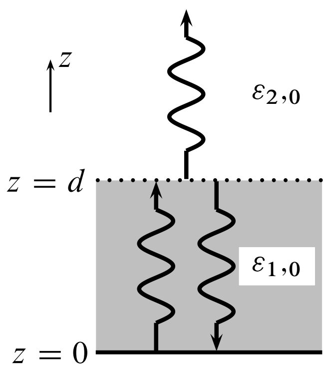
Question 4.3

Prove that for constant x in Eq. (4.10) a horizontal line segment $r + \mathrm{j}x$ in the complex $\hat{Z}_{\text{eff0}}$-plane with $r \geq 0$ maps to a circlular arcs that touches the real Γ axis at $\Gamma = 1$ and runs inside the unit disk to a point on its boundary.

Question 4.4

The figure below is a sketch of an open one-dimensional cavity. The cavity may be regarded as being one-dimensional because we only consider plane-wave propagation in the positive or negative z-directions. The dielectric media in the cavity, $z \in [0, d]$, and above it are instantaneously reacting, homogeneous and lossless with $\mu = \mu_0$ throughout. The indices of refraction in the two regions are given by $n_1 = \sqrt{\varepsilon_{\text{r};1}}$ and $n_2 = \sqrt{\varepsilon_{\text{r};2}}$. At $z = 0$, a perfect electric conductor is located. An electric current source density $\mathbf{J}_{\text{s}0}(z) = \mathbf{J}_{\text{s}S}\delta(z - d)$, with surface current source density $\mathbf{J}_{\text{s}S} = J_{\text{s}S}\mathbf{a}_x$, generates a time-harmonic electromagnetic field with angular frequency ω, consisting of plane waves that propagate upwards and downwards.

At the interface $z = 0$, total reflection occurs, while the waves incident on the interface $z = d$ from below are partially reflected and partially transmitted. The corresponding electric and magnetic fields in the two regions $(0, d)$ and (d, ∞) may be cast in the following form



$$\mathbf{E}_\text{s}(z) = V_\text{s}(z)\mathbf{a}_x \quad \text{and} \quad \mathbf{H}_\text{s}(z) = I_\text{s}(z)\mathbf{a}_y,$$

in which

$$V_\text{s}(z) = \begin{cases} -\mathrm{j}Z_1 I_\text{s}(0)\sin(k_1 z) & \text{for } 0 < z < d, \\ V_\text{s}(d)e^{-\mathrm{j}k_2(z-d)} & \text{for } d \leq z. \end{cases}$$

where $I_\text{s}(0)$ is an as yet unknown current amplitude. The other quantities are introduced below. Above, we have made use of the fact that in source-free regions the fields may alternatively be represented as linear combinations of exponential functions, or as linear combinations of sines and cosines. Thus we have already made sure that the boundary conditions at $z = 0$ are satisfied.

a) Express the wavenumbers k_1 and k_2, the wave impedances Z_1 en Z_2 and the wave admittances Y_1 en Y_2 in terms of the wavenumber, k_0, and wave impedance, Z_0, in vacuum and the refractive indices n_1 and n_2.

b) Give the transmission line equations for $V_s(z)$ and $I_s(z)$ in the regions $0 < z < d$ and $d < z$. Here, you are asked to use $\mu_0 = k_0 Z_0/\omega$ and $\varepsilon_0 = k_0/(Z_0\omega)$ to express $\omega L = \omega\mu$ and $\omega C = \omega\varepsilon$ in terms of k_0, Z_0, n_1 and n_2.

c) Use one of the transmission-line equations to determine the current amplitude $I_s(z)$ of the magnetic field in the two regions in terms of the quantities introduced above.

d) Give the boundary conditions at the level $z = d$. Use these boundary conditions to deduce that

$$I_s(0) = \frac{J_{sS}}{\cos(k_0 n_1 d) + \mathrm{j}\dfrac{n_2}{n_1}\sin(k_0 n_1 d)}.$$

Although this equation expresses the initially unknown complex quantity $I_s(0)$ in terms of the known source strength J_{sS}, it is easier to perform the calculations below in terms of $I_s(0)$.

e) Compute the complex Poynting vector for $z > d$.

Now, consider a generator domain $\mathcal{V}_{\text{source}}$ consisting of a part of the cavity that is restricted to unit area in the transverse directions, e.g., the region $0 < x < 1$, $0 < y < 1$, $0 < z \leq d$ (including the surface current source density at $z = d$). The domain $\mathcal{V}_{\text{source}}$ is bounded by the closed surface $\mathcal{S}_{\text{source}}$. Often, one is interested in the so-called quality factor $Q = \omega\langle W_{\text{em}}\rangle_{\mathrm{T}}/\langle P_{\mathrm{r}}\rangle_{\mathrm{T}}$ of a cavity.[5] In the lossless non-dispersive configuration under consideration, the electromagnetic field energy averaged per period may be calculated using

$$\langle W_{\text{em}}\rangle_{\mathrm{T}} = \frac{1}{4}\int_{\mathcal{V}_{\text{source}}} \mu_0|\mathbf{H}_s|^2 + \varepsilon_1|\mathbf{E}_s|^2\,\mathrm{d}V.$$

f) Compute the quality factor.

[5] One usually considers a mode of the cavity, rather than the field generated by a source, but that scenario is not considered here.