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# Digital Signal Processing Fundamentals (5ESC0)

## Fourier Analysis


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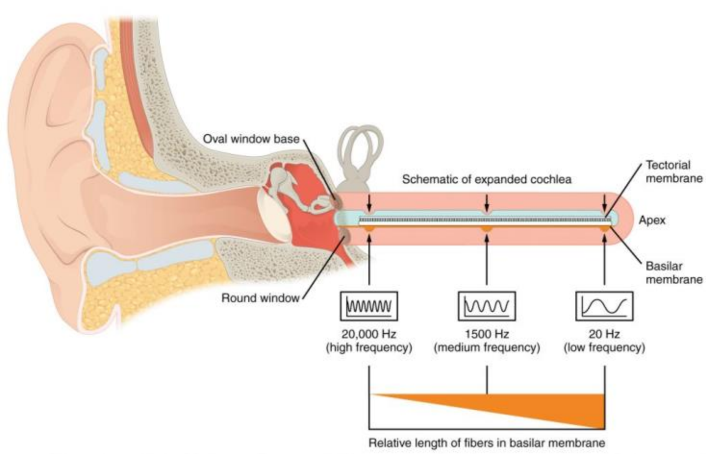
# The frequency domain



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# Notation

* The book's notation:

  *Book* : $\omega \cdot T_s = 2\pi f \cdot \frac{1}{f_s}$,

  where $f, \omega =$ Absolute frequency

* In the slides we will use:

  Slides : $\theta = 2\pi \left(\frac{f}{f_s}\right)$,

  where $\theta =$ Normalized frequency

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# Fourier analysis

* Fourier representation of signals plays an important in both continuous-time and discrete-time signal processing
* It maps signals into **another "domain"** in which we can manipulate them, perform filtering
* Fourier representation is useful due to one of its properties: **convolution operation is mapped to multiplication**
* Fourier transform provides a different way to interpret signals and systems

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# Continuous time Fourier transform

* We will start with the continuous time Fourier transform and then continue in the digital domain

* Fourier Transform Continuous time signals (FTC):

* $X(f)$ via $\omega = 2\pi f$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \;\circ\!\!-\!\!\circ\; x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Example: FTC of pulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \;\circ\!\!-\!\!\circ\; S(\omega) = \sum_{n=-\infty}^{\infty} e^{-j\omega nT}$$



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# Fourier series

* Only valid for periodic signals: $x(t) = x(t + T_0)$
* If the signal is periodic, it can be described by a sum of weighted exponents

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T_0}nt}$$

* To find the weights $c_n$ we look at the frequency components present in the signal and integrate them over one period and then we normalize:

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-j\frac{2\pi}{T_0}nt}\, dt$$

# Fourier series example

**Note: Difference FTC and FS for periodic signals**

$x(t) = \cos(2\pi F_0 t)$, where $F_o = \frac{1}{T_0}$

From Euler's expression we know: $\cos(2\pi F_0 t) = \frac{1}{2}(e^{j2\pi F_0 t} + e^{-j2\pi F_0 t})$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \frac{1}{2}(e^{j2\pi F_0 t} + e^{-j2\pi F_0 t})\, e^{-j2\pi F_0 t n} dt$$

$$= \frac{1}{2T_0}\int_{-T_0/2}^{T_0/2} e^{j2\pi F_0 t(1-n)} + e^{-j2\pi F_0 t(n+1)}\, dt$$

$$= \frac{1}{2T_0}\left(\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi F_0 t(1-n)}\, dt + \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi F_0 t(n+1)}\, dt\right)$$





# Fourier series example

$$c_n = \frac{1}{2T_0}\left(\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi F_0 t(1-n)}\, dt + \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi F_0 t(n+1)}\, dt\right)$$

We will work out one of the two integrals above, since the approach is the same.

$$\frac{1}{2T_0}\left(\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi F_0 t(1-n)}\, dt\right)$$

If $n = 1$, the exponent equals $e^{j0} = 1$ and we integrate 1 over one period, which yields $\frac{1}{2T_0}\left(t\Big|_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\right) = \frac{1}{2T_0}\left(\frac{T_0}{2} - -\frac{T_0}{2}\right) = \frac{1}{2}$

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# Fourier series example

$$\frac{1}{2T_0}\left(\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi F_0 t(n-1)}\, dt\right)$$

If $n \neq 1$, this equals

$$\frac{1}{2T_0}\left(\frac{1}{j2\pi F_0(n-1)} e^{j2\pi F_0 t(n-1)}\right)\Big|_{-\frac{T_0}{2}}^{\frac{T_0}{2}}$$

$$= \frac{1}{2T_0}\frac{1}{j2\pi F_0(n-1)}\left(\underbrace{e^{j2\pi\frac{1}{T_0}\frac{T_0}{2}(n-1)} - e^{-j2\pi\frac{1}{T_0}\frac{T_0}{2}(n-1)}}\right)$$

$$\left(e^{j2\pi\frac{1}{T_0}\frac{T_0}{2}(n-1)} - e^{-j2\pi\frac{1}{T_0}(n-1)}\right) = \left(e^{j\pi(n-1)} - e^{-j\pi(n-1)}\right)$$

$$= \cos\big(\pi(n-1)\big) + j\sin\big(\pi(n-1)\big) - \cos\big(-\pi(n-1)\big) - j\sin(-\pi(n-1))$$

Because $\cos(x) = \cos(-x)$ the cosines cancel out and because $n$ is an integer and $\sin(\pm n\pi) = 0$, the term above is equal to 0.



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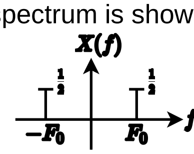
# Fourier series example

$x(t) = \cos(2\pi F_0 t)$, where $F_o = \frac{1}{T_0}$

$$c_n = \frac{1}{2T_0}\left(\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi F_0 t(n-1)}\, dt + \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi F_0 t(n+1)}\, dt\right)$$

* We know that the only nonzero values are at $n = 1$ and $n = -1$
* $c_1 = c_{-1} = \frac{1}{2}$
* The frequency domain spectrum is shown in the figure below



# Special case FTC: pulse train (necessary for Ch3)

Proof of: $\boldsymbol{S(\omega)} = \sum_{n=-\infty}^{\infty} \boldsymbol{e^{-j\omega nT}} \equiv \boldsymbol{P(\omega)} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \boldsymbol{\delta(\omega - k\frac{2\pi}{T})}$

With $P(\omega)$ periodic with period $\frac{2\pi}{T}$ ⇒ we can compute the FS :

$P(\omega) = \sum_{n=-\infty}^{\infty} p_n e^{jn\omega T}$, with $p_n = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{2\pi}{T} \delta(\omega)\, e^{-jTn\omega} d\omega = 1$

$\Rightarrow P(\omega) = \sum_{n=-\infty}^{\infty} 1 e^{jn\omega T} = \sum_{n=-\infty}^{\infty} 1 e^{-jn\omega T} \equiv S(\omega)$

⇒ FTC of pulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \circ\!\!-\!\!\circ\ S(\omega) = \frac{2\pi}{T} \sum_{n-\infty}^{\infty} \delta(\omega - n\frac{2\pi}{T})$$

Pulse train with distance proportional to 2 $\pi$ /T





# Fourier Transform of Discrete-time (FTD) signals

- In this course we work in the digital domain, therefore we will look at the Fourier Transform of discrete-time signals
- Because the time is discrete, the integral becomes a summation

$$X(e^{j\theta}) \overset{FTD}{\triangleq} \sum_{n=-\infty}^{\infty} x[n] e^{-jn\theta}$$

- Notation:
  - $\theta$ is the relative frequency and we use a capital $X$ to denote that we are in the frequency domain
  - The only variable $X$ depends on is $\theta$
  - $\theta$ is continuous, so $X$ is continuous, therefore we use round brackets
  - We could write $X(\theta)$ because $\theta$ is the only variable, but we write $X(e^{j\theta})$ to stress that this is a periodic function

# Fourier Transform of Discrete-time (FTD) signals

- To go back to the discrete-time domain, we use the Inverse Fourier Transform for Discrete-time signals (IFTD)
- Because $X(e^{j\theta})$ is continuous and periodic, we have to integrate over one period:

$$x[n] \overset{IFTD}{\triangleq} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{jn\theta}d\theta$$

- We normalize over one period with the factor $\frac{1}{2\pi}$
- Note: the Fundamental Interval (FI) is usually: $|\theta| \leq \pi$, integrating from $0$ to $2\pi$ also works





# Fourier Transform of Discrete-time (FTD) signals

$$X\left(e^{j\theta}\right) \overset{FTD}{\triangleq} \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta} \circ\!\!-\!\!\circ\; x[n] \overset{IFTD}{\triangleq} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{jn\theta}d\theta$$

- For common signals/ sequences we have FTD pairs that can be used without derivation
- This saves time and possible mistakes
- Example sequence: $\delta[n]$

  FTD of a delta pulse: $\sum_{n=-\infty}^{\infty} \delta[n]e^{-jn\theta} = \delta[0]e^{-j\cdot 0\cdot\theta} = 1$

# Common FTD pairs

$$X(e^{j\theta}) \overset{FTD}{\triangleq} \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta} \circ\!\!-\!\!\circ\; x[n] \overset{IFTD}{\triangleq} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{jn\theta}d\theta$$

| Time Sequence | FTD |
|---|---|
| $\delta[n]$ | $1$ |
| $\delta[n-n_0]$ | $e^{-jn_0\theta}$ |
| $1$ | $2\pi\delta(\theta)$ |
| $e^{jn\theta_0}$ | $2\pi\delta(\theta-\theta_0)$ |
| $a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{1-ae^{-j\theta}}$ |
| $-a^n u[-n-1], \quad \lvert a\rvert > 1$ | $\frac{1}{1-ae^{-j\theta}}$ |
| $\cos(n\theta_0)$ | $\pi\delta(\theta+\theta_0)+\pi\delta(\theta-\theta_0)$ |





# Example: FTD pair

Find FTD of the sequence $x[n] = a^n u[n], \;\; |a| < 1$ .

The FTD of this sequence is

$$X(e^{j\theta}) = \sum_{n=0}^{\infty} a^n e^{-jn\theta} = \sum_{n=0}^{\infty} \left(ae^{-j\theta}\right)^n$$

Using the geometric series, $|a| < 1$, this sum is

$$X(e^{j\theta}) = \frac{1}{1-ae^{-j\theta}} .$$

# FTD properties

- Periodicity : $X(e^{j\theta}) = X(e^{j\theta + l\cdot 2\pi}) \quad l \in \mathbb{N}$
  $X(e^{j\theta})$ is periodic, meaning its behavior repeats every $2\pi$
- Symmetry :

| $x[n]$ | $X(e^{j\theta})$ |
|---|---|
| Real, even | Real, even |
| Real, odd | Imaginary, odd |
| Imaginary, even | Imaginary, even |
| Imaginary, odd | Real, odd |

For FTD, forms of symmetry will hold as listed in the table above





# FTD properties

- Linearity : $ax_1[n] + bx_2[n] \;\circ\!\!-\!\!\circ\; aX_1(e^{j\theta}) + bX_2(e^{j\theta})$
  Additive and homogeneous
- Shifting : $x[n-n_0] \;\circ\!\!-\!\!\circ\; e^{-jn_0\theta} \cdot X(e^{j\theta})$
  Shifting in time domain is a modulation operation in frequency domain
- Time-reversal: $x[-n] \;\circ\!\!-\!\!\circ\; X(e^{-j\theta})$
- Modulation : $e^{jn\theta_0} \cdot x[n] \;\circ\!\!-\!\!\circ\; X(e^{j(\theta-\theta_0)})$
  Modulation in time domain is shifting in frequency domain
  Example using Euler's expression:

$$\cos(n\theta_0)x[n] \;\circ\!\!-\!\!\circ\; \frac{1}{2}X(e^{j(\theta+\theta_0)}) + \frac{1}{2}X(e^{j(\theta-\theta_0)})$$

# FTD properties

* Convolution : $x[n] * y[n] \circ\!\!-\!\!\circ X(e^{j\theta}) \cdot Y(e^{j\theta})$

* Multiplication: $x[n] \cdot y[n] \circ\!\!-\!\!\circ \frac{1}{2\pi}\int_{\varphi=-\pi}^{\pi} X(e^{j\varphi})Y(e^{j(\theta-\varphi)})\, d\varphi$

Because $X(e^{j\theta})$ and $Y(e^{j\theta})$ are continuous, the convolution requires integration. Multiplication in one domain is convolution in the other domain.

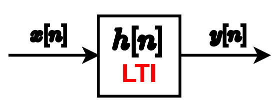
* Parseval : $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} |X(e^{j\theta})|^2\, d\theta$

  The energy in one domain equals the energy in the other domain; a transform does not introduce or use energy.





# Frequency response LTI system




* So far we had an introduction on FTD, now we will use it on an LTI system
* We know that we find the output by convolving the input with the impulse response

$$y[n] = x[n] * h[n] \overset{LTI}{\triangleq} \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

* Now we use a complex exponent with a single frequency $\theta$ as input:

$$x[n] = e^{jn\theta}$$

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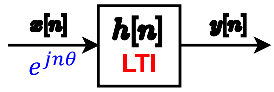
# Frequency response LTI system




* Now we use a complex exponent as input: $x[n] = e^{jn\theta}$

$$y[n] = e^{jn\theta} * h[n] = h[n] * e^{jn\theta} \overset{LTI}{\triangleq} \sum_{k=-\infty}^{\infty} h[k]e^{j(n-k)\theta}$$

* Since the summation has index $k$, we can take $n$ out of the summation

$$\sum_{k=-\infty}^{\infty} h[k]e^{j(n-k)\theta} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-jk\theta}\right) \cdot e^{jn\theta}$$

* Now we notice a product of our input $e^{jn\theta}$ and the part between brackets. The part between brackets does not depend on $n$, but only on the impulse response



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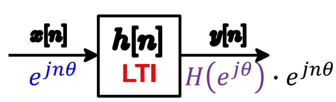
# Frequency response LTI system




* We can write the part between brackets as $H(e^{j\theta})$

$$\left(\sum_{k=-\infty}^{\infty} h[k]e^{-jk\theta}\right) \cdot e^{jn\theta} = H(e^{j\theta}) \cdot e^{jn\theta}$$

* The relation above holds for all $\theta$
* The system's response to a frequency $\theta$ only depends on $h[n]$
* We can conclude that the impulse response $h[n]$ and the frequency response $H(e^{j\theta})$ are an FTD pair:

$$H(e^{j\theta}) = \sum_{k=-\infty}^{\infty} h[k]e^{-jk\theta} \;\circ\!\!-\!\!\circ\; h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})\, e^{jn\theta}\, d\theta$$

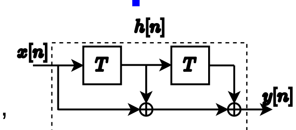
# Properties frequency response

- Complex: $H(e^{j\theta}) = H_r(e^{j\theta}) + jH_I(e^{j\theta}) = |H(e^{j\theta})| \cdot e^{j\varphi(e^{j\theta})}$
  We can write the frequency response as a real and an imaginary part or in polar notation (magnitude and phase)
- Periodicity: $H(e^{j\theta_0}) = H(e^{j(\theta_0 + l \cdot 2\pi)})$ for $l \in \mathbb{N}$
  As denoted by $e^{j\theta}$ in $H(e^{j\theta})$, the frequency response is periodic
- Conjugate symmetry: For real valued $h[k] \Rightarrow H(e^{-j\theta}) = H^*(e^{j\theta})$
  The magnitude is symmetric, but the phase is antisymmetric
- Let us consider an example: $h[n] = \sum_{i=0}^{2} \delta[n-i]$
- Find $H(e^{j\theta})$ and draw the magnitude and phase plots





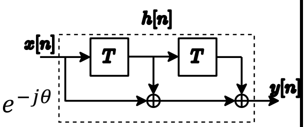
# Frequency response example



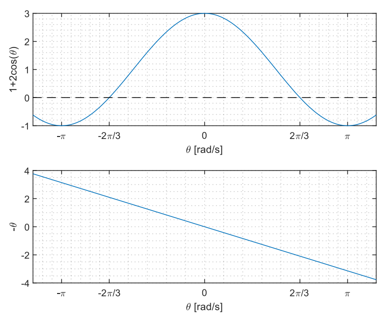

- $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$
- We want to find the frequency response $H(e^{j\theta})$, which is an FTD pair with $h[n]$
- We take the FTD of $h[n]$ and use the known FTD pair of a delayed delta pulse: $\delta[n-n_0] \;\circ\!\!-\!\!\circ\; e^{-jn_0\theta}$
- $h[n] \;\circ\!\!-\!\!\circ\; H(e^{j\theta}) = 1 + e^{-j\theta} + e^{-2j\theta}$
- We can take the factor $e^{-j\theta}$ outside of brackets:
  $$H(e^{j\theta}) = 1 + e^{-j\theta} + e^{-2j\theta} = e^{-j\theta}(e^{j\theta} + 1 + e^{-j\theta})$$
- We can rewrite this to use an Euler expression:
  $$e^{-j\theta}(e^{j\theta} + 1 + e^{-j\theta}) = e^{-j\theta}\left(1 + 2\left(\frac{e^{j\theta} + e^{-j\theta}}{2}\right)\right)$$

# Frequency response example

- $h[n] = \sum_{i=0}^{2} \delta[n-i]$
- $H(e^{j\theta}) = e^{-j\theta}\left(1 + 2\left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right)\right) = (1 + 2\cos\theta)e^{-j\theta}$




- From the complex property we know that $(1 + 2\cos\theta)$ describes the magnitude and $-\theta$ describes the phase
- Now we plot the magnitude and phase as a function of $\theta$








# Frequency response example

- $(1 + 2\cos\theta)$ describes the magnitude and $-\theta$ describes the phase
- In practice, both the magnitude and phase are plotted within the fundamental interval: $|\theta| \leq \pi$
- The magnitude $|H(e^{j\theta})|$ is often taken absolute and the phase $\angle\{H(e^{j\theta})\}$ is limited from $-\pi$ to $\pi$
- When taken absolute, where the magnitude would cross zero the value is made positive. In other words: a phase shift of $\pi$. The phase plot follows this behavior by a $\pi$ phase shift
- In the phase plot, $\pi$ is added or subtracted so that the phase stays within the limits of $-\pi$ to $\pi$ when the magnitude plot crosses zero

# Properties frequency response

- Solving DE:

$$y[n] = -\sum_{k=1}^{p} a_k y[n-k] + \sum_{k=0}^{q} b_k x[n-k]$$

$$\circ\!\!-\!\!\circ \quad Y(e^{j\theta}) = -\sum_{k=1}^{p} a_k e^{-jk\theta} Y(e^{j\theta}) + \sum_{k=0}^{q} b_k e^{-jk\theta} X(e^{j\theta})$$

$$H(e^{j\theta}) = \frac{Y(e^{j\theta})}{X(e^{j\theta})} = \frac{\sum_{k=0}^{q} b_k e^{-jk\theta}}{1 + \sum_{k=1}^{p} a_k e^{-jk\theta}}$$

- A difference equation in frequency domain can be solved with linear algebra

- Convolution: $\quad x[n] * h[n] \;\circ\!\!-\!\!\circ\; X(e^{j\theta}) \cdot H(e^{j\theta})$





# Example FTD

Find the FTD of $x[n] = a^n \sin(n\Theta_0)\, u[n]$

<u>Solution</u>

Express the sinusoid as a sum of two complex numbers using *Euler's formula*:

$$\sin(n\Theta_0) = \frac{1}{2j}\left(e^{jn\Theta_0} - e^{-jn\Theta_0}\right)$$

Therefore, the expression becomes:

$$x[n] = a^n \frac{1}{2j}\left(e^{jn\Theta_0} - e^{-jn\Theta_0}\right) u[n]$$

$$= \frac{1}{2j} a^n e^{jn\Theta_0} u[n] - \frac{1}{2j} a^n e^{-jn\Theta_0} u[n]$$

# Example FTD

Let's take the FTD of the first term which is equal to $\frac{1}{2j}a^n e^{jn\theta_0}u[n]$:

$$X_1(e^{j\theta}) = \frac{1}{2j}\sum_{n=0}^{\infty} a^n e^{jn\theta_0} e^{-jn\theta}$$

Take the common power $n$ outside the bracket and join the powers of $e$ over a single exponential

$$X_1(e^{j\theta}) = \frac{1}{2j}\sum_{n=0}^{\infty}(ae^{-j(\theta-\theta_0)})^n$$

Use the geometric series definition for $\alpha^n u[n]$ where $\alpha = ae^{-j(\theta-\theta_0)}$

$$X_1(e^{j\theta}) = \frac{1}{2j}\frac{1}{1-ae^{-j(\theta-\theta_0)}}$$





# Example FTD

Take the FTD of the second term using the same steps:

$$X_2(e^{j\theta}) = -\frac{1}{2j}\sum_{n=0}^{\infty} a^n e^{-jn\theta_0} e^{-jn\theta}$$

$$= -\frac{1}{2j}\sum_{n=0}^{\infty}(ae^{-j(\theta+\theta_0)})^n$$

$$= -\frac{1}{2j}\frac{1}{1-ae^{-j(\theta+\theta_0)}}$$

The final expression for $X(e^{j\theta}) = X_1(e^{j\theta}) + X_2(e^{j\theta})$ is:

$$X(e^{j\theta}) = \frac{1}{2j}\left(\frac{1}{1-ae^{-j(\theta-\theta_0)}} - \frac{1}{1-ae^{-j(\theta+\theta_0)}}\right)$$

# Filters

* Digital filter is often used to refer to a discrete-time system
* A definition[1] of digital filter ". ..computational process or algorithm by which a sampled signal or sequence of numbers (acting as the input) is transformed into a second sequence of numbers termed the output signal. The computational process may be that of lowpass filtering (smoothing), bandpass filtering, interpolation, the generation of derivatives, etc."
* Filters may be characterized in terms of their system properties, such as linearity, shift-invariance, causality, stability, etc.

[1] System Analysis by Digital Computer. F. F. Kuo and J. F. Kaiser, Eds.. John Wiley and Sons, New York. 1966





# Examples: Averaging filter

| Time domain | ⊶ | Frequency domain |
|---|---|---|
| Averaging filter<br>$h[n] = u[n] - u[n-N]$ | ⊶ | Dirichlet function<br>$H(e^{j\theta}) = e^{-j\frac{N-1}{2}\theta} \cdot \frac{\sin(\frac{N}{2}\theta)}{\sin(\frac{1}{2}\theta)}$ |

* The values of the impulse response $h[n]$ should be multiplied by $1/N$ to obtain the averaging effect
* An array of $N$ delta pulses can be described in the frequency domain by the Dirichlet function
* We will look at the derivation and the magnitude plot

# Examples: Averaging filter

$$h[n] = u[n] - u[n-N] = \sum_{k=0}^{N-1} \delta[n-k]$$

$$\circ\!\!-\!\!\circ \; H(e^{j\theta}) = \sum_{k=0}^{N-1} (e^{-j\theta})^k = \frac{1-e^{-j\theta N}}{1-e^{-j\theta}}$$

We take a factor $\frac{2je^{-j\theta N/2}}{2je^{-j\theta/2}}$ out of brackets to use an Euler equation:

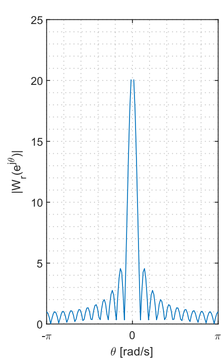
$$\frac{1-e^{-j\theta N}}{1-e^{-j\theta}} = \frac{\left(\dfrac{e^{-\frac{j\theta N}{2}} - e^{\frac{j\theta N}{2}}}{2j}\right)}{\left(\dfrac{e^{j\theta/2} - e^{-j\theta/2}}{2j}\right)} \cdot \frac{2je^{-\frac{j\theta N}{2}}}{2je^{-\frac{j\theta}{2}}} = \frac{\sin(\frac{\theta N}{2})}{\sin(\frac{\theta}{2})} \cdot e^{-\frac{j\theta N}{2} + \frac{j\theta}{2}}$$

$$= \frac{\sin(\frac{\theta N}{2})}{\sin(\frac{\theta}{2})} \cdot e^{-j\theta(\frac{N-1}{2})}$$





# Examples: Averaging filter

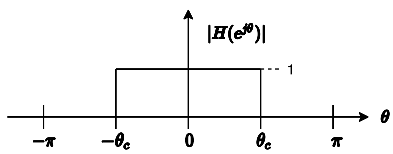



* There are many zero crossings in the magnitude (not in the figure because it is taken absolute)
* The main lobe of the magnitude is quite narrow
* Our time domain block (array of delta pulses) has length 21
* Stretching in one domain is shrinking in the other domain

# Examples: Low Pass Filter

| Time domain | ⊶ | Frequency domain |
|---|---|---|
| Sinc function $\frac{\theta_c}{\pi}(\frac{\sin(\theta_c n)}{\theta_c n})$ | ⊶ | Ideal Low Pass Filter (LPF) $H(e^{j\theta}) = \begin{cases} 1, & \lvert\theta\rvert \leq \theta_c \\ 0, & \text{elsewhere} \end{cases}$ |

* If we want to filter out higher frequencies, we use a LPF
* An ideal Low Pass Filter is shown in the figure below








# Examples: Low Pass Filter

* We can obtain the time-domain function that will give us a Low Pass Filter through the IFTD

$$H(e^{j\theta}) = \begin{cases} 1, & \lvert\theta\rvert \leq \theta_c \\ 0, & \text{elsewhere} \end{cases}$$

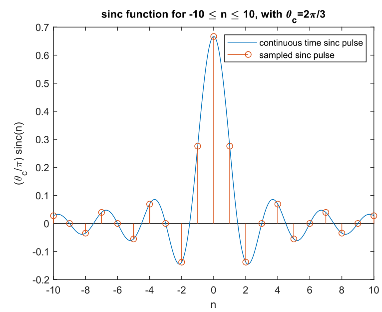
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})\, e^{jn\theta}\, d\theta = \frac{1}{2\pi}\int_{-\theta_c}^{\theta_c} 1\, e^{jn\theta}\, d\theta$$

$$= \frac{1}{2\pi}\left(\frac{1}{jn} e^{jn\theta}\right)_{-\theta_c}^{\theta_c} = \frac{1}{\pi n}\cdot\frac{1}{2j}\left(e^{jn\theta_c} - e^{-jn\theta_c}\right) = \frac{\sin(n\theta_c)}{\pi n}$$

This can be written as: $h[n] = \frac{\theta_c}{\pi}\cdot\frac{\sin(n\theta_c)}{n\theta_c}$, because $\text{sinc}(x) = \frac{\sin(x)}{x}$

# Examples: Low Pass Filter

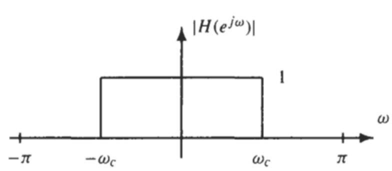
- So the time-domain function that will give us a Low Pass Filter in frequency domain is a sinc function: $h[n] = \frac{\theta_c}{\pi} \cdot \frac{\sin(n\theta_c)}{n\theta_c}$
- The sinc function spans from $-\infty$ to $\infty$, so it is infinite and we cannot make it in practice
- This means the LPF will not be ideal: there will be a ripple and the transition will not be a straight line but a slope.
- On the right, a sinc pulse and the sampled version are shown



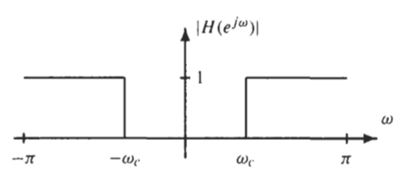




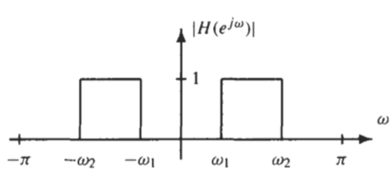

# Ideal filters



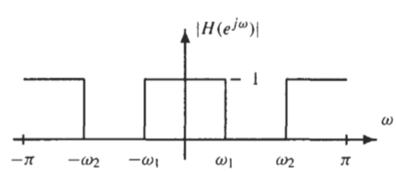

(a) Ideal low-pass filter.




(b) Ideal high-pass filter.




(c) Ideal bandpass filter.




(d) Ideal bandstop filter.

# Example: windowing

* We cannot make infinitely long sequences/ signals. What will the spectrum of these signals look like in practice?
* We will consider an example: $x[n] = \cos(0.28\pi n)$
* We know from the Fourier Series that this will result in a spectrum of two weighted delta pulses
* The cosine function spans infinitely long, so in practice we cannot generate it. We can only generate a part of it

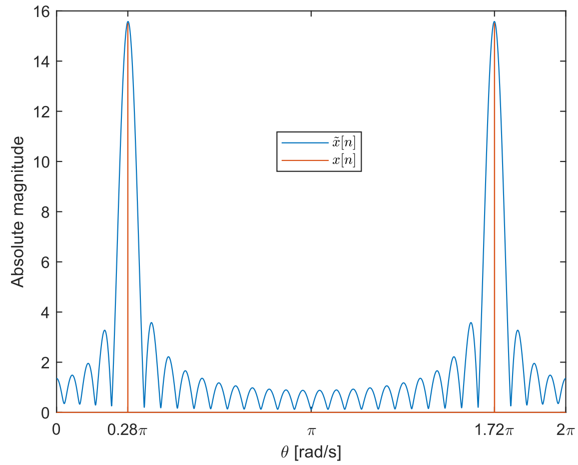




# Example: windowing

* $x[n] = \cos(0.28\pi n)$
* To make it finite, we multiply $x[n]$ with a weight function (or a rectangular window) $w_R[n] = \begin{cases} 1, & n = 0, \dots, N-1 \\ 0, & \text{elsewhere} \end{cases}$
* $w_R[n]$ is an averaging filter, as we have seen before
* Say we name the signal $\tilde{x}[n] = x[n] \cdot w_R[n]$
* Now we observe what happens when we use the FTD on both signals

# Example: windowing

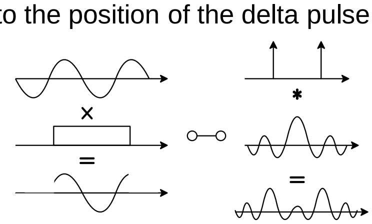







# Example: windowing

- Why do we see this behavior?
- We multiply the time domain cosine with an averaging filter that gives us only the windowed part of the cosine
- In frequency domain this means that we have two delta pulses that are convolved with the Dirichlet function
- Convolving with a delta pulse shifts (modulates) the function you are convolving with to the position of the delta pulse



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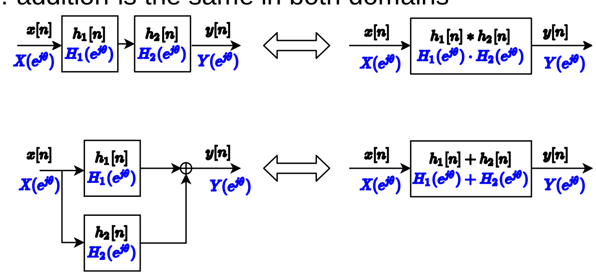
# Interconnection of systems

- Frequency response properties of interconnecting systems
- We can describe interconnecting systems as one system
- Cascaded: convolution in time domain is multiplication in frequency domain
- Parallel: addition is the same in both domains

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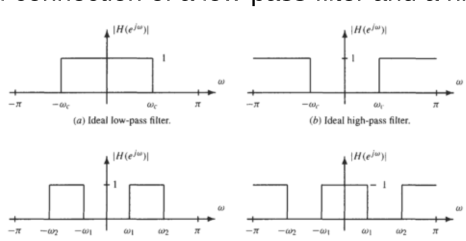
# Example: system connections

- The cascade of a low-pass filter with a high-pass filter may be used to implement a bandpass filter.
- For example, the ideal bandpass filter shown in (c) may be realized by cascading a low-pass filter with a high-pass filter that has lower cutoff frequency
- Similarly, the bandstop filter shown in (d) may be realized with a parallel connection of a low-pass filter and a high-pass filter

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# Summary

* We had an introduction on the Fourier Transform of Continuous time signals (FTC).
* We considered the Fourier Transform of Discrete-time signals (FTD)
  - FTD pairs
  - FTD properties
* We introduced the frequency response
  - The system's response to an input signal $e^{jn\theta}$
  - It forms an FTD pair with the impulse response
  - Frequency response properties
* We looked at some filter examples in time domain and frequency domain: the averaging filter and the low pass filter
* We saw an example of what happens when we window a signal
* We examined interconnection of systems in time domain and frequency domain





# Reference

M. H. Hayes, "Schaum's Outline of Theory and Problems of Digital Signal Processing", McGraw-Hill, 1999; Chapter 2.