EINDHOVEN UNIVERSITY OF TECHNOLOGY

Communication Theory (5ETB0)
Module 3.2

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Module 3.2

Presentation Outline

MAP Detection (1/2)

Motivation MAP Decision Rule

- Recall that

$$P_c = \sum_{r \in \mathcal{R}} \Pr\{R = r, M = f(r)\} \tag{1}$$

- Interpretation: For each column $(R = r)$, decision rule picks a row $(M = m)$.
- This interpretation leads to the upper bound

$$P_c \leq \sum_{r \in \mathcal{R}} \max_{m \in \mathcal{M}} \Pr\{R = r, M = m\} \tag{2}$$

- Upper bound is achieved by $f(r)$ that picks the row that maximizes the joint probability

MAP Detection (2/3)

A Different Interpretation

The optimum receiver is a maximization over $f : \mathcal{R} \to \mathcal{M}$, i.e.,

$$\max_f \{P_c\} = \max_f \sum_{r \in \mathcal{R}} \Pr\{R = r, M = f(r)\} \tag{3}$$

$$= \sum_{r \in \mathcal{R}} \max_{f(r)} \Pr\{R = r, M = f(r)\} \tag{4}$$

$$= \sum_{r \in \mathcal{R}} \max_{f(r)} \Big\{ \underbrace{\Pr\{R = r, 1 = f(r)\}, \ldots, \Pr\{R = r, |\mathcal{M}| = f(r)\}}_{\begin{cases} \Pr\{R = r, M = m\}, & \text{if } f(r) = m \\ 0, & \text{if } f(r) \neq m \end{cases}} \Big\} \tag{5}$$

Thus,

$$\max_f \{P_c\} = \sum_{r \in \mathcal{R}} \max_{m \in \mathcal{M}} \Pr\{R = r, M = m\} \tag{6}$$

$$= \sum_{r \in \mathcal{R}} \max_{m \in \mathcal{M}} \Pr\{M = m | R = r\} \cancel{\Pr\{R = r\}} \tag{7}$$

MAP Detection (3/3)

Decision Variables

For a communication system using a DIDO channel, the joint PMFs

$$\Pr\{M = m, R = r\} = \Pr\{M = m\} \Pr\{R = r | M = m\} \tag{8}$$
$$= \Pr\{M = m\} \Pr\{R = r | S = s_m\} \tag{9}$$

are called the **decision variables**. An optimum receiver uses these variables.

MAP Decision Rule

The maximum a-posteriori probability (MAP) decision rule is

$$\hat{m}^{\mathsf{MAP}}(r) \triangleq \operatorname*{argmax}_{m \in \mathcal{M}} \Pr\{M = m | R = r\} \tag{10}$$

and has two important properties:

- $\Rightarrow$ Maximizes P_{c} $\Rightarrow$ Minimizes P_{e} $\Rightarrow$ optimum receiver!
- Produces the largest decision variable for each r (Bayes' rule)

Example 3.1 Revisited

MAP for Example 3.1

- A-posteriori probabilities:

m	$\Pr\{M=m\|R=a\}$	$\Pr\{M=m\|R=b\}$	$\Pr\{M=m\|R=c\}$
1	$20/26$	$16/34$	$4/40$
2	$6/26$	$18/34$	$36/40$

obtained from $\Pr\{R=a\}=0.26, \Pr\{R=b\}=0.34$, and $\Pr\{R=c\}=0.4$

- Correct probability is $P_\mathsf{c} = 0.2+0.18+0.36=0.74 \Rightarrow P_\mathsf{e}=0.26$
- MAP decision rule coincides with the one that maximizes P_c

Intuition behind MAP decision rule

Maximize the probability that, for a given $R=r$, the chosen message is equal to the transmitted message

$$\hat{m}^{\mathsf{MAP}}(r) \triangleq \operatorname*{argmax}_{m\in\mathcal{M}} \Pr\{M=m|R=r\}$$

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Presentation Outline

Detection with Equally Likely Messages

Probabilities

- A-priori probabilities: $\Pr\{M = m\}$
- A-posteriori probabilities: $\Pr\{M = m|R = r\}$

Uniform a-priori Probabilities

- All messages are equally likely (uniform probability)

$$\Pr\{M = m\} = \frac{1}{|\mathcal{M}|} \text{ for all } m \in \mathcal{M} = \{1, 2, \dots, |\mathcal{M}|\}, \tag{11}$$

- Decision variables are

$$\begin{aligned} \Pr\{M = m, R = r\} &= \Pr\{R = r|S = s_m\} \Pr\{S = s_m\} & (12) \\ &= \frac{1}{|\mathcal{M}|} \Pr\{R = r|S = s_m\} & (13) \end{aligned}$$

ML Detection

ML decision rule

The maximum likelihood (ML) decision rule is

$$\hat{m}^{\mathsf{ML}}(r) \triangleq \operatorname*{argmax}_{m \in \mathcal{M}} \Pr\{R = r | M = m\}$$

A few words on ML

- Name comes from definition
- For equally likely messages:
  - Largest decision variable for each r
  - $\Rightarrow$ Maximizes P_{c} $\Rightarrow$ Minimizes P_{e} $\Rightarrow$ optimum receiver!
- For nonequally likely messages:
  - Can be used (it is simple to implement)
  - Suboptimal

Example 3.1 Re-revisited (1/2)

ML for Example 3.1

- Transition probabilities:

m	$\Pr\{R=a\|S=s_m\}$	$\Pr\{R=b\|S=s_m\}$	$\Pr\{R=c\|S=s_m\}$
1	0.5	0.4	0.1
2	0.1	0.3	0.6

- The maximum-likelihood decision rule is then:

r	a	b	c
$f(r)$	1	1	2

Correct probability is 0.72, lower than MAP (0.74).

- If a-priori probabilities are $\Pr\{M=m\}=1/2$:

$$P_{\mathsf{c}}^{\mathsf{ML}} = \Pr\{\hat{M}^{\mathsf{ML}} = M\} \tag{14}$$

$$= \sum_{m\in\mathcal{M}} \Pr\{\hat{M}^{\mathsf{ML}} = M | M=m\} \Pr\{M=m\} \tag{15}$$

$$= \frac{1}{2}(0.5+0.4) + \frac{1}{2}0.6 = 0.75. \tag{16}$$

Example 3.1 Re-revisited (2/2)

ML for Example 3.1

- Transition probabilities:

m	$\Pr\{R=a \mid S=s_m\}$	$\Pr\{R=b \mid S=s_m\}$	$\Pr\{R=c \mid S=s_m\}$
1	0.5	0.4	0.1
2	0.1	0.3	0.6

- The maximum-likelihood decision rule is then:

r	a	b	c
$f(r)$	1	1	2

- If a-priori probabilities are $\Pr\{M=m\}=1/2$:

$$P_{\mathrm{c}}^{\mathrm{ML}} = \frac{1}{2}(0.5+0.4)+\frac{1}{2}0.6 = 0.75. \tag{17}$$

Three Questions

- Q1: What would the MAP detector give?
- Q2: If the a-priori probabilities are now 0.4 and 0.6, $P_{\mathrm{c}}^{\mathrm{ML}}=0.72$ and $P_{\mathrm{c}}^{\mathrm{MAP}}=0.74$. Does this make sense?
- Q3: Why do we care about ML if it is suboptimal?

Module 3.2

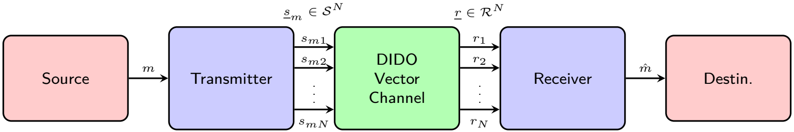
Presentation Outline

MAP and ML for DIDO Channels

Definitions

- Source: Produces a *message* $m \in \mathcal{M} \triangleq \{1, 2, \ldots, |\mathcal{M}|\}$ with probability $\Pr\{M = m\}$ for $m \in \mathcal{M}$. The r.v. is M
- Transmitter: Sends a *signal* $s_m \in \mathcal{S}$ if message m is to be transmitted. The r.v. is S
- Channel: Produces output $r \in \mathcal{R}$ (r.v. is R) with conditional probability $\Pr\{R = r|S = s\}$
- Receiver: Forms an *estimate* $\hat{m}$ by observing the received channel output $r \in \mathcal{R}$ using a mapping $\hat{m} = f(r) \in \mathcal{M}$. The r.v. is $\hat{M}$

MAP and ML Detection for Vectorial Channels



Definitions

- Transmitter: Sends a *signal* $\underline{s}_m \in \mathcal{S}^N$ if message m is to be transmitted. The random vector is $\underline{S}$
- Vector Channel: Produces output $\underline{r} \in \mathcal{R}^N$ (random vector is $\underline{R}$) with conditional probability $\Pr\{\underline{R} = \underline{r} | \underline{S} = \underline{s}\}$
- Receiver: Forms an *estimate* $\hat{m}$ by observing the received channel output $\underline{r} \in \mathcal{R}^N$ using a mapping $\hat{m} = f(\underline{r}) \in \mathcal{M}$. The r.v. is $\hat{M}$

MAP and ML Detection: Summary

MAP Detection

Decision	MAP
Variable	$\Pr\{M = m\}\Pr\{\underline{R} = \underline{r} \mid \underline{S} = \underline{s}_m\}$
Rule	$\operatorname{argmax}_{m\in\mathcal{M}} \Pr\{M = m \mid \underline{R} = \underline{r}\}$

ML Detection

Decision	ML
Variable	$\frac{1}{\lvert\mathcal{M}\rvert}\Pr\{\underline{R} = \underline{r} \mid \underline{S} = \underline{s}_m\}$
Rule	$\operatorname{argmax}_{m\in\mathcal{M}} \Pr\{\underline{R} = \underline{r} \mid M = m\}$

Summary Module 3.2

Take Home Messages

- MAP is the optimal receiver
- ML is sometimes optimal and in general simpler to implement
- Scalar analysis can be generalized to vectorial channels

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