Proposed Solutions EM II (5EPB0) - Plane Waves

1) Plane Waves. Derive the Helmholtz equation (the wave equation) from Maxwell's equations. Verify that the plane wave $\underline{E} = \underline{A}e^{-jkz}$ and its associated magnetic field are solutions of Maxwell's equations.

a.

Assumptions:

- source-free region
- medium is:
  - LTI
  - homogeneous
  - isotropic
  - unbounded

$$\begin{aligned}
\nabla \times \vec{E} &= -j\omega\vec{B} - \vec{k} \\
\nabla \times \vec{H} &= j\omega\vec{D} + \vec{J} \\
\nabla \times \nabla \times \vec{E} &= \nabla \times \left(-j\omega\mu\vec{H}\right) \\
\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2\vec{E} &= -j\omega\mu\nabla + \vec{H} \\
-\nabla^2\vec{E} &= \omega^2\varepsilon\mu\vec{E}
\end{aligned}$$

$$\Rightarrow \nabla^2\vec{E} + k^2\vec{E} = \vec{0}$$

Analogous for $\vec{H}$

b.

$\vec{E} = \vec{A}e^{-jkz}$ with associated magnetic field $\vec{H} = \frac{1}{Z}\vec{a_z} \times \vec{A}e^{-jkz}$

(Faraday):

$$\nabla \times \vec{E} \underset{\substack{\uparrow \\ \text{see slide \#8}}}{=} -jk\vec{a_z} \times \vec{A}e^{-jkz} \underset{\substack{\uparrow \\ k=\frac{\omega}{c}=\frac{\omega\mu}{Z}}}{=} \frac{-j\omega\mu}{Z}\vec{a_z} \times \vec{A}e^{-jkz}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} = -j\omega\vec{B}$$

Analogous for Ampère

2) Plane Waves. Prove that, under the assumption of a plane wave, Maxwell's equations become algebraic.

First steps in slide #8 of lecture 5a:

$$\begin{aligned}\nabla \times \vec{E} &= \nabla e^{-j\vec{k}\cdot\vec{r}} \times \vec{A} = \left[\left(\partial_x \vec{a_x} + \partial_y \vec{a_y} + \partial_z \vec{a_z}\right) e^{-j(k_x x + k_y y + k_z z)}\right] \times \vec{A} \\ &= \left[-jk_x \vec{a_x} e^{-j\vec{k}\cdot\vec{r}} - jk_y \vec{a_y} e^{-j\vec{k}\cdot\vec{r}} - jk_z \vec{a_z} e^{-j\vec{k}\cdot\vec{r}}\right] \times \vec{A} \\ &= -jk \times \underbrace{\vec{A} e^{-j\vec{k}\vec{r}}}_{\vec{E}} = -j\vec{k} \times \vec{E} = \underbrace{-j\omega\mu\vec{H}}_{\text{Faraday's law}}\end{aligned}$$

$$\Rightarrow \quad \vec{k} \times \vec{E} = \omega\mu\vec{H}$$

Analogous with Ampère's law

3) Plane Waves. The electric field of a uniform plane wave travelling in a medium with magnetic permeability $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m is of the form:

$$\underline{E}(\underline{r}, t) = 10 \exp(j(4\pi \times 10^6 t - 4\pi \times 10^{-2} z))\mathbf{a}_x \qquad \text{V/m}\,. \tag{1}$$

Indicate:
a. The direction and sense of propagation.
$+z$ (in the positive $\vec{a_z}$)

b. The type of wave (travelling, evanescent, etc.).
$k \in \mathbb{R} \quad \Rightarrow \quad$ travelling wave

c. If the medium in which it propagates is a lossless or a lossy one.
$k \in \mathbb{R} \quad \Rightarrow \quad$ lossless

And calculate:
d. The frequency f.
$\omega = 4\pi 10^6 \ rad/s = 2\pi f \quad \Rightarrow \quad f = 2 \ MHz$

e. The wavenumber k.
$k = 4\pi 10^{-2} \ rad/m$

f. The wavelength λ.
$k = \frac{2\pi}{\lambda} \quad \Rightarrow \quad \lambda = 50 \ m$

g. The speed of the wave in that medium.
$c = \lambda f \quad \Rightarrow \quad c = 100\ Mm/s\ = \frac{c_0}{3}$

h. The electric permittivity ε.
$c = \frac{1}{\sqrt{\varepsilon\mu}} \quad \Rightarrow \quad \varepsilon = \frac{1}{4\pi}10^{-9}\ F/m$

i. The wave impedance Z.
$Z = \sqrt{\frac{\mu}{\varepsilon}} \quad = \quad 40\pi\ \Omega$

j. The magnetic field $\underline{H}(\underline{r}, t)$.
$\vec{H}(\vec{r}, t) = \vec{H}(\vec{z}, t)$ (we omit from now on the $e^{j\omega t}$ dependence)

Three possible way to do this:
1) Apply Faraday's law $\nabla \vec{E} = -j\omega\mu\vec{H}$
2) Knowing that under PW conditions $\vec{k} \times \vec{E} = \omega\mu\vec{H}$
3) Knowing the physics of PW: $\vec{E} \perp \vec{H}\ \wedge\ \vec{k} \perp \vec{H}\ \wedge\ |\vec{E}| = Z|\vec{H}|$

for 2)

$$\vec{H} = \frac{1}{\omega\mu}\vec{k} \times \vec{E} \quad \text{with } \vec{k} = k\vec{a_z}$$

$$\vec{H} = \frac{10}{\omega\mu}ke^{-jkz}\underbrace{(\vec{a_z} \times \vec{a_x})}_{\vec{a_y}} = \frac{10}{\omega\mu}ke^{-jkz}\vec{a_y} = \vec{H}$$

k. The time-averaged power density.

$$\vec{S_h} = \frac{1}{2}\,\text{Re}\left\{\vec{E} \times \vec{H}^*\right\} = \frac{10^2}{2}\frac{k}{\mu\omega}\vec{a_z}$$

4) Energy considerations of plane waves. Consider a plane wave propagating in an arbitrary direction in a lossless medium.

$$\vec{E} = \vec{A}e^{-j\vec{k}\cdot\vec{r}}; \qquad \vec{H} = \frac{\vec{k} \times \vec{E}}{\omega\mu}$$

a. Prove that for such a plane wave, the electric energy density $w_e = \varepsilon^*\underline{E} \cdot \underline{E^*}$ is identical to the magnetic energy density $w_m = \mu\underline{H} \cdot \underline{H^*}$.

$$w_e = \varepsilon \vec{E} \cdot \vec{E}$$

$$w_m = \mu \vec{H} \cdot \vec{H} = \mu \left(\frac{\vec{k} \times \vec{E}}{\omega\mu} \right) \cdot \left(\frac{\vec{k} \times \vec{E}}{\omega\mu} \right) = \frac{1}{\omega^2 \mu} (\vec{k} \times \vec{E}) \cdot (\vec{k} \times \vec{E}) =$$
$$= \underbrace{\frac{1}{\omega^2\mu}[(\vec{E} \times (\vec{k} \times \vec{E}))] \cdot \vec{k}}_{\text{(A.6)}} = \underbrace{\frac{1}{\omega^2\mu}[(\vec{E} \cdot \vec{E})k - \cancelto{0}{(\vec{E} \cdot \vec{k})\vec{E}}\,] \cdot \vec{k}}_{\text{(A.7)}}$$
$$= \frac{\omega^2 \varepsilon\mu}{\omega^2\mu} \vec{E} \cdot \vec{E} = \varepsilon \vec{E} \cdot \vec{E} = w_e$$

b. Knowing that $w_e = w_m$ and knowing the basic assumptions for the plane wave, make the proper substitutions in the Poynting theorem (power balance) and discuss the results.

$w_e = w_m$ and source-free medium $\Rightarrow$ Poynting theorem reduces to:

$$\oiint \vec{E} \times \vec{H} \cdot d\vec{S} = 0$$

or, equivalently

$$\nabla \cdot (\vec{E} \times \vec{H}) = 0 \qquad \text{(prove!)}$$

Therefore plane waves are <u>solenoidal</u> (no sources or sinks). For <u>any</u> closed surface in space, the net flux of power flow is zero. Power flows in the same direction of propagation (!)

c. Prove that for a plane wave that travels in the $+z$ direction $\mathbf{S} = cw_e\mathbf{a}_z = cw_m\mathbf{a}_z$, with c being the speed of propagation. You can assume a lossless medium. Provide a physical interpretation.

$\omega_m = \mu \vec{H} \cdot \vec{H}; \quad \vec{k} = k\vec{a_z}$

$$\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \left(\frac{\vec{k} \times \vec{E}}{\omega\mu} \right) = \frac{1}{\omega\mu}[(\vec{E} \cdot \vec{E})\vec{k} - \cancelto{0}{(\vec{E} \cdot \vec{k})\vec{E}}\,]$$
$$= \frac{\cancel{\omega}\sqrt{\varepsilon\mu}}{\cancel{\omega}\mu} \vec{E} \cdot \vec{E}\vec{a_z} = \frac{1}{\sqrt{\varepsilon\mu}} \varepsilon \vec{E} \cdot \vec{E} = cw_e\vec{a_z} = cw_m\vec{a_z}$$

Book. 11.5 a) - 11.10 a) - 11.29 - 11.34
odd-numbered problems have solutions in the book

11.10 a

$$\vec{E}_s = |R|[H_{0z}\vec{a_y} - H_{0z}\vec{a_z}]e^{-\alpha x}e^{-j\beta x}e^{j0}$$

11.30 a linear polarization → x and y components are in-phase

$$\Rightarrow \Delta\beta z = m\pi \Rightarrow z = \frac{m\pi}{\Delta\beta} \quad (m \in \mathbb{I})$$

11.30 b Circular polarization → x and y components in quadrature

$$\Rightarrow \Delta\beta z = \frac{(2n+1)\pi}{2} \Rightarrow z = \frac{(2n+1)\pi}{2\Delta\beta} \quad (n \in \mathbb{I})$$

11.30 c

$$H_s(z) = \frac{E_0}{Z}\left(\vec{a_y} - \vec{a_x}e^{-j\Delta\beta z}\right)e^{-j\beta z}$$

$$\langle\vec{S}\rangle = \frac{1}{2}\,\mathrm{Re}\left\{\vec{E}_s \times \vec{H}_s^*\right\} = \frac{E_0^2}{z}\vec{a_z}\ W/m^2$$

11.32 If L is doubled ⇒ the phase difference is π rad.

$$\vec{E}_s(L) = E_0\left(e^{j\beta_x L}\vec{a_x} + e^{-j\beta_y L}\vec{a_y}\right) = E_0 e^{-j\beta_x L}\left(\vec{a_x} + e^{-j(\beta_y - \beta_x)L}\vec{a_y}\right)$$

$$(\beta_y - \beta_x)L = -\pi \quad (\beta_x > \beta_y)$$

the y-component is reversed, the wave polarization is <u>rotated 90°</u>, but still is <u>linearly-polarized</u>

11.34 a Adding the two fields

$$\vec{E}_{tot} = \left[E_{x0}(1 + e^{j\delta})\vec{a_x} + E_{y0}(e^{j\phi} + e^{-j\phi + j\delta})\vec{a_y}\right]e^{-j\beta z}$$

using some well-known trigonometric identities, we can get:

$$\vec{E}_{tot} = \left[E_{x0}e^{j\delta/2}2\cos(\delta/2)\vec{a_x} + E_{y0}e^{j\delta/2}2\cos(\phi - \delta/2)\vec{a_y}\right]e^{-j\beta z}$$

which is linearly polarized.

11.34 b $E_y = 0$ for $2\phi - \delta = \pi$