1. **Linear algebra:**

   (a) Consider the following differential equation

$$\begin{aligned}\dot{x}_1 &= -2x_1 + 3x_2 + u,\\ \dot{x}_2 &= x_1 - 3x_2,\\ \dot{x}_3 &= 2x_1 - 5x_3 + u;\end{aligned}$$

   and define $x = [x_1\ x_2\ x_3]^T$. Write this system in the form $\dot{x} = Ax + Bu$, give the corresponding matrices A and B, and compute the inverse of A.

   (b) For invertible matrices A and B, prove the following identities:
   - $(AB)^{-1} = B^{-1}A^{-1}$,
   - $(AB)^T = B^T A^T$,
   - $A(I + BA)^{-1} = (I + AB)^{-1}A$.

Solution:

(a) We have the state-space representation as:

$$\dot{x} \;=\; Ax + Bu \;=\; \begin{bmatrix} -2 & 3 & 0 \\ 1 & -3 & 0 \\ 2 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u.$$

We have the inverse of A as:

$$A^{-1} \;=\; \frac{\mathrm{adj}(A)}{\det(A)} \;=\; \frac{-1}{15}\begin{bmatrix} 15 & 15 & 0 \\ 5 & 10 & 0 \\ 6 & 6 & 3 \end{bmatrix} \;=\; \begin{bmatrix} -1 & -1 & 0 \\ -0.333 & -0.667 & 0 \\ -0.4 & -0.4 & -0.2 \end{bmatrix}.$$

(b)
- To prove $(AB)^{-1} = B^{-1}A^{-1}$, we have that

$$(AB)^{-1}AB = I \iff (AB)^{-1} = B^{-1}A^{-1} \quad \text{since} \quad B^{-1}A^{-1}AB = I.$$

- To prove $(AB)^T = B^T A^T$, we have that
  I. the (i,j) element of AB is $\sum_{k=1}^{n} a_{ik}\, b_{kj}$,
  II. the (i,j) element of $(AB)^T$ is $\sum_{k=1}^{n} a_{jk}\, b_{ki}$,
  III. the (i,j) element of $B^T A^T$ is $\sum_{k=1}^{n} b_{ki}\, a_{jk}$.
  Observe that II. and III. are equivalent; therefore, $(AB)^T = B^T A^T$.

- To prove $A(I + BA)^{-1} = (I + AB)^{-1}A$, we have

$$\begin{aligned} A(I+BA)^{-1} = (I+AB)^{-1}A &\iff (I+AB)A = A(I+BA)\\ &\iff A + ABA = A + ABA.\end{aligned}$$

  Since the last statement is true the earlier statements hold as well.

2. **Conversion between plant representations:**

   Consider the following plants represented by differential equations:

   i. $\ddot{y} + 2\dot{y} + u = 0$,
   ii. $\ddot{y} + 2\dot{y} + \dot{u} + u = 0$,

iii. $\ddot{y} + 2\dot{y} + \ddot{u} + \dot{u} + u = 0.$

Under the assumption that initial conditions equal to zeroes, do the following exercises—for each case (i. - iii.):

(a) Derive a corresponding state-space model using the simulation diagram approach, then compute its transfer function (from the obtained state-space model representation).

(b) Directly apply Laplace transform to the differential equation and obtain its transfer function. Compare the transfer functions you have obtained from (2a) and (2b).

Solution:

i. (a) We first sketch the simulation diagram which is given in Figure 1.

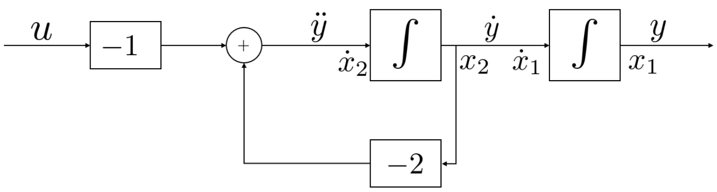

Figure 1: Problem 2: Simulation diagram for case (i.)

We then observe that

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2x_2 - u\\ y &= x_1\end{aligned}$$

With $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$, we have that

$$\begin{cases}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{cases}$$

with

$$A = \begin{bmatrix} 0 & 1\\ 0 & -2\end{bmatrix}, \quad B = \begin{bmatrix} 0\\ -1\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0\end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 \end{bmatrix}.$$

We can compute the transfer function from the state-space representation using

$$G(s) = C\,(sI - A)^{-1}\,B + D = \frac{-1}{s(s+2)}$$

(b) Assume the zero initial conditions. We have

$$(s^2 + 2s)y(s) + u(s) = 0.$$

Therefore, the corresponding transfer function is

$$\frac{y(s)}{u(s)} = \frac{-1}{s(s+2)}.$$

This is the exact same transfer function as the one found in (a).

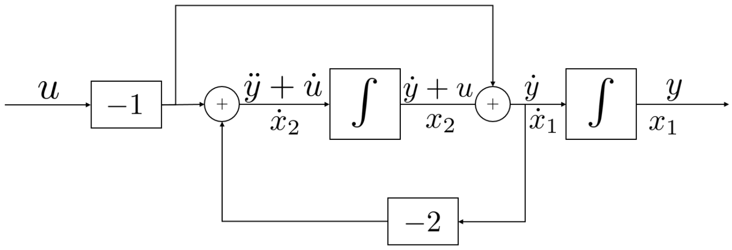

Figure 2: Problem 2: Simulation diagram for case (ii.)

ii. (a) Again, we first sketch the simulation diagram which is given in Figure 2. We then observe that

$$\begin{aligned}\dot{x}_1 &= x_2 - u\\ \dot{x}_2 &= -2\dot{x}_1 - u = -2x_2 + u\\ y &= x_1\end{aligned}$$

With $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$, we have that

$$\begin{cases}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{cases}$$

with

$$A = \begin{bmatrix} 0 & 1\\ 0 & -2\end{bmatrix}, \quad B = \begin{bmatrix} -1\\ 1\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0\end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 \end{bmatrix}.$$

We can compute the transfer function from the state-space representation using

$$G(s) = C\left(sI - A\right)^{-1}B + D = -\frac{(s+1)}{s(s+2)}$$

(b) Assume the zero initial conditions. We have

$$(s^2+2s)y(s) + (s+1)u(s) = 0.$$

Therefore, the corresponding transfer function is

$$\frac{y(s)}{u(s)} = -\frac{(s+1)}{s(s+2)}.$$

iii. (a) Again, we first sketch the simulation diagram which is given in Figure 3. We then observe that

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + u\\ \dot{x}_2 &= -u\\ y &= x_1 - u\end{aligned}$$

With $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$, we have that

$$\begin{cases}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{cases}$$

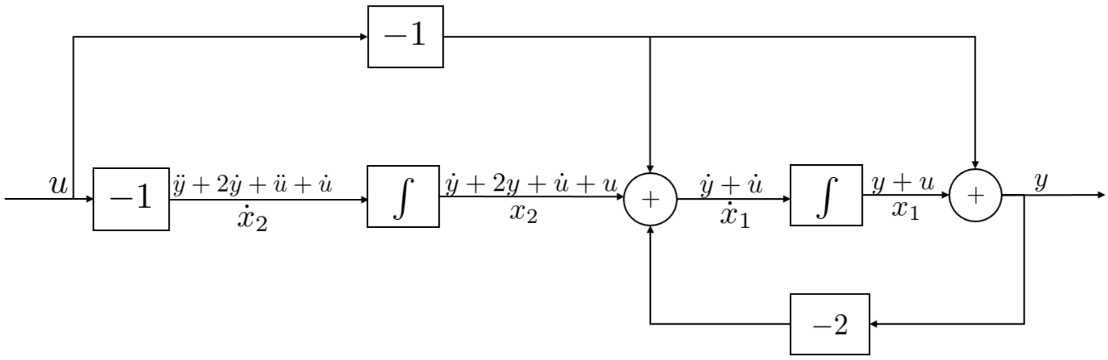

Figure 3: Problem 2: Simulation diagram for case (iii.)

with

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} -1 \end{bmatrix}.$$

We can compute the transfer function from the state-space representation using

$$G(s) = C\,(sI - A)^{-1}\,B + D = -\frac{(s^2 + s + 1)}{s(s+2)}$$

(b) Assume the zero initial conditions, we have

$$(s^2 + 2s)y(s) + (s^2 + s + 1)u(s) = 0.$$

Therefore, the corresponding transfer function is

$$\frac{y(s)}{u(s)} = -\frac{(s^2 + s + 1)}{s(s+2)}$$

3. **Transformation matrices and canonical forms:**

   For case ii. and iii. from problem 2, use the obtained transfer functions (from either (2a) or (2b)) to do the following exercises:

   (a) Derive the corresponding state-space model in the *control canonical* form.

   (b) Derive the corresponding state-space model in the *modal* form.

   (c) For each case, compare the state-space models obtained from (3a) and (3b). Find the similarity transformation matrix T which brings the state-space representation in (3a) to the one in (3b).

   Solution:
   Remember that if we have a transfer function

$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} + b_0$$

We can have a state-space realization in control canonical form given by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where

$$A = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_n \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} b_1 & b_2 & \dots b_n \end{bmatrix}, \quad D = \begin{bmatrix} b_0 \end{bmatrix}$$

and if we have a transfer function

$$G(s) = \frac{c_1}{s + p1} + \cdots + \frac{c_n}{s + p_n} + c_0$$

We can have a state-space realization in modal form given by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where

$$A = \begin{bmatrix} -p_1 & 0 & \dots & 0 \\ 0 & -p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -p_n \end{bmatrix}, \quad B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}, \quad D = \begin{bmatrix} c_0 \end{bmatrix}$$

(Case ii.) We have the plant transfer function as

$$G(s) = -\frac{(s+1)}{s(s+2)} = \frac{-s-1}{s^2+2s}$$

(a) The control canonical form is given by

$$\begin{cases} \dot{x}_c &= A_c x_c + B_c u \\ y &= C_c x_c + D_c u \end{cases}$$

with

$$A_c = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad \text{and} \quad D_c = \begin{bmatrix} 0 \end{bmatrix}.$$

(b) We can perform a partial fraction expansion and have that:

$$G(s) = -\frac{(s+1)}{s(s+2)} = \frac{c_1}{s} + \frac{c_2}{s+2}$$

with $c_1 = -0.5$ and $c_2 = -0.5$.

Therefore we can derive the modal form as:

$$\begin{cases} \dot{x}_m &= A_m x_m + B_m u \\ y &= C_m x_m + D_m u \end{cases}$$

with

$$A_m = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad B_m = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \quad C_m = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \text{and} \quad D_m = \begin{bmatrix} 0 \end{bmatrix}.$$

(c) Both representations have different system matrices, except for the feed-through term $D_c = D_m = 0$. To find the transformation matrix T, compute

$$\begin{bmatrix} B_c & A_c B_c \end{bmatrix} = T \begin{bmatrix} B_m & A_m B_m \end{bmatrix}.$$

We then have

$$T = \begin{bmatrix} B_c & A_c B_c \end{bmatrix} \begin{bmatrix} B_m & A_m B_m \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix},$$

and we check that indeed

$$\begin{aligned} A_m = T^{-1} A_c T &= \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \\ B_m = T^{-1} B_c &= \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \\ C_m = C_c T &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \\ D_m = D_c &= [0]. \end{aligned}$$

(Case iii.) We have the plant transfer function as

$$G(s) = -\frac{(s^2+s+1)}{s(s+2)} = \frac{-s^2-s-1}{s^2+2s} = \frac{s-1}{s^2+2s} - 1$$

(a) The control canonical form is then given as

$$\begin{cases} \dot{x}_c &= A_c x_c + B_c u \\ y &= C_c x_c + D_c u \end{cases}$$

with

$$A_c = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad \text{and} \quad D_c = \begin{bmatrix} -1 \end{bmatrix}.$$

(b) We can perform a partial fraction and have that:

$$G(s) = \frac{s-1}{s^2+2s} - 1 = \frac{c_1}{s} + \frac{c_2}{s+2} - 1.$$

with $c_1 = -0.5$ and $c_2 = 1.5$.

Therefore we can derive the modal form as

$$\begin{cases} \dot{x}_m &= A_m x_m + B_m u \\ y &= C_m x_m + D_m u \end{cases}$$

with

$$A_m = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad B_m = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}, \quad C_m = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \text{and} \quad D_m = \begin{bmatrix} -1 \end{bmatrix}.$$

(c) It can be observed that both representations have the same non-zero feed-through term $D_c = D_m = -1$. To find the transformation matrix T, compute

$$\begin{bmatrix} B_c & A_c B_c \end{bmatrix} = T \begin{bmatrix} B_m & A_m B_m \end{bmatrix}.$$

We then have

$$T = \begin{bmatrix} B_c & A_c B_c \end{bmatrix} \begin{bmatrix} B_m & A_m B_m \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{2}{3} \\ -1 & -\frac{1}{3} \end{bmatrix},$$

and we check that indeed

$$\begin{aligned} A_m = T^{-1} A_c T &= \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \\ B_m = T^{-1} B_c &= \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}, \\ C_m = C_c T &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \\ D_m = D_c &= [-1]. \end{aligned}$$

4. **Determination of poles/zeros from a state-space model**

Consider the state-space model:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

with

$$A = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} -1 \end{bmatrix}.$$

(a) Determine the poles of this system.

(b) Determine the zeros of this system.

(c) Compute the transfer function and verify the poles and zeros found in questions (a) and (b) are correct.

Solution:

(a) The poles of the system are any $s \in \mathbb{C}$ such that:

$$\det(sI - A) = 0 \quad \Longleftrightarrow \quad \begin{vmatrix} s+2 & 0 \\ -1 & s \end{vmatrix} = 0,$$
$$\Longleftrightarrow \quad (s+2)(s) = 0.$$

Therefore, we have the poles at $s = 0, -2$.

(b) The zeros are are any $s \in \mathbb{C}$ such that:

$$\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0 \quad \Longleftrightarrow \quad \begin{vmatrix} s+2 & 0 & -2 \\ -1 & s & 0 \\ 0.5 & -0.5 & -1 \end{vmatrix} = 0,$$
$$\Longleftrightarrow \quad -(s^2 + s + 1) = 0$$

Therefore, we have the zeros at $s = -0.5 \pm \frac{\sqrt{3}}{2} i$.

(c) The transfer function is given by

$$G(s) = C\,(Is - A)^{-1}\,B + D = \frac{-(s^2 + s + 1)}{s(s+2)},$$

which has poles and zeros equal to the obtained answers in questions (a) and (b).

5. **Linearisation:**

(a) The mathematical model of a stick-balancing problem is

$$\ddot{\theta} = \sin(\theta(t)) - f(t)\cos(\theta(t)).$$

Derive a linearized state-space model around $\theta = \dot{\theta} = f = 0$, by setting $x_1 = \theta$, and $x_2 = \dot{\theta} = \dot{x}_1$. Choose $f(t)$ as the input.

(b) The mathematical model of a single-link robotic manipulator with a flexible joint is given by

$$\begin{aligned} I\ddot{\theta}_1(t) + mgl\sin\theta_1(t) + k(\theta_1(t) - \theta_2(t)) &= 0 \\ J\ddot{\theta}_2(t) - k(\theta_1(t) - \theta_2(t)) &= f(t), \end{aligned}$$

where $\theta_1(t), \theta_2(t)$ are the angular positions at time t, I, J are the moments of inertia, m and l are the link mass and length, and k is the link spring constant.

i. Determine the state-space equations by considering $\boldsymbol{x} = [\theta_1 \;\; \dot{\theta}_1 \;\; \theta_2 \;\; \dot{\theta}_2]$.

ii. Linearize the system around the nominal operating point $(\boldsymbol{x}_\mathrm{n}, f_\mathrm{n})$.

Solution:

(a) By setting $x_1 = \theta$, $x_2 = \dot{\theta}$, and $u = f$ we have that $\dot{x} = g(x, u)$, where

$$g(x, u) = \begin{bmatrix} x_2 \\ \sin x_1 - u\cos x_1 \end{bmatrix}$$

We can then linearize around the point $\theta = \dot{\theta} = f = 0$ using a first order Taylor approximation given by

$$\begin{aligned} g(x, u) &\approx g(x^*, u^*) + \nabla_x g(x^*, u^*)(x - x^*) + \nabla_u g(x^*, u^*)(u - u^*) \\ &\approx A_\mathrm{lin}\bar{x} + B_\mathrm{lin}\bar{u} \end{aligned}$$

where

$$A_\mathrm{lin} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_\mathrm{lin} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(b) i. If $x = \begin{bmatrix} \theta_1 & \dot{\theta}_1 & \theta_2 & \dot{\theta}_2 \end{bmatrix}^\top$ then

$$\dot{x} = \begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{mgl\sin(x_1) + k(x_1 - x_3)}{I} \\ x_4 \\ \frac{f + k(x_1 - x_3)}{J} \end{bmatrix}$$

and hence we have $\dot{x} = g(x, f)$, where

$$g(x, f) = \begin{bmatrix} x_2 \\ -\frac{mgl}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) + \frac{1}{J}f \end{bmatrix}$$

ii. Linearizing around a nominal operating point then gives

$$\begin{aligned} g(x, f) &\approx g(x_n, f_n) + \nabla_x g(x_n, f_n)(x - x_n) + \nabla_u g(x_n, f_n)(f - f_n) \\ &\approx A_\mathrm{lin}\bar{x} + B_\mathrm{lin}\bar{u} \end{aligned}$$

assuming our nominal operating point is chosen such that $g(x_n, f_n) = 0$ and where

$$A_\mathrm{lin} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k + mgl\cos(x_{1,n})}{I} & 0 & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix}, \quad B_\mathrm{lin} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}$$

Assuming that the output variable is the link angular position, i.e., $y = x_1$, we have

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

6. **State-space realisation:**
Make a state-space realisation of the form

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases} \tag{1}$$

for the transfer function

$$G(s) = \frac{s+4}{(2s+1)(s+1)}. \tag{2}$$

Give the matrices A, B, C and D. Next, determine whether the state-space model obtained is controllable.

Solution:
Remember that if we have a transfer function

$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} + b_0$$

We can have a state-space realization (in control canonical form) given by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where

$$A = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_n \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} b_1 & b_2 & \dots b_n \end{bmatrix}, \quad D = \begin{bmatrix} b_0 \end{bmatrix}$$

First, we rewrite,

$$G(s) = \frac{s+4}{(2s+1)(s+1)} = \frac{s+4}{2s^2+3s+1} = \frac{\frac{1}{2}s+2}{s^2+\frac{3}{2}s+\frac{1}{2}}. \tag{3}$$

It directly follows that,

$$A = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

The system is controllable if the controllability matrix is full rank. So we must check the rank of $\mathcal{C} = [B \ AB]$.

$$\mathcal{C} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix},$$

which has rank 2, so the state-space system is controllable.

7. **Control canonical form:**
Give the state description matrices in control-canonical form for the following transfer functions:

(a) $G(s) = \frac{1}{2s+1}$.

(b) $G(s) = \frac{6(s/3+1)}{(s/10+1)}$.

(c) $G(s) = \frac{8s+1}{s^2+s+2}$.

(d) $G(s) = \frac{s+7}{s(s^2+2s+2)}$.

Solution:
Remember that if we have a transfer function

$$G(s) = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} + b_0$$

We can have a state-space realization (in control canonical form) given by

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

where

$$A = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_n \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} b_1 & b_2 & \dots b_n \end{bmatrix}, \quad D = \begin{bmatrix} b_0 \end{bmatrix}$$

(a) $G(s) = \frac{1}{2s+1} = \frac{\frac{1}{2}}{s+\frac{1}{2}}$.
$A = -\frac{1}{2},\ B = 1,\ C = \frac{1}{2},\ D = 0$

(b) $G(s) = \frac{6(s/3+1)}{(s/10+1)} = \frac{20s+60}{(s+10)} = 20 - \frac{140}{s+10}$
$A = -10,\ B = 1,\ C = -140,\ D = 20$

(c) $G(s) = \frac{8s+1}{s^2+s+2}$

$$A = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 8 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

(d) $G(s) = \frac{s+7}{s(s^2+2s+2)} = \frac{s+7}{(s^3+2s^2+2s)}$

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

8. **Transfer function from state-space representation:**
For each of the listed state-space models,

i)

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 3 \end{bmatrix} \mathbf{x}.\end{aligned}$$

ii)

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} u, \\ y &= \begin{bmatrix} 4 & -1 \end{bmatrix} \mathbf{x}.\end{aligned}$$

iii)

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 2 & 5 \end{bmatrix} \mathbf{x}.\end{aligned}$$

iv)

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}.$$

v)

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \mathbf{x}.$$

complete the following tasks (and check your result using Matlab):

(a) Determine the poles of this system.

(b) Determine the zeros of this system.

(c) Compute the transfer function and verify the poles and zeros found in questions (a) and (b) are correct.

Solution:

i) (a) The poles of the system are any $s \in \mathbb{C}$ such that:

$$\det(sI - A) = 0 \quad \Longleftrightarrow \quad \begin{vmatrix} s & -1 \\ -7 & s+4 \end{vmatrix} = 0,$$
$$\Longleftrightarrow \quad (s+4)(s) - 7 = 0.$$

Therefore, we have the poles at $s = -2 + \sqrt{11}$, $s = -2 - \sqrt{11}$.

(b) The zeros are are any $s \in \mathbb{C}$ such that:

$$\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0 \quad \Longleftrightarrow \quad \begin{vmatrix} s & -1 & -1 \\ -7 & s+4 & -2 \\ 1 & 3 & 0 \end{vmatrix} = 0,$$
$$\Longleftrightarrow \quad 7s + 27 = 0$$

Therefore, we have the zeros at $s = -\frac{27}{7}$.

(c) The transfer function is given by

$$\begin{aligned}
G &= C(sI - A)^{-1}B + D \\
&= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} s & -1 \\ -7 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 \end{bmatrix} \cdot \frac{1}{s^2 + 4s - 7} \begin{bmatrix} s+4 & 1 \\ 7 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \frac{1}{s^2 + 4s - 7} \cdot \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ 7 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \frac{1}{s^2 + 4s - 7} \cdot [s + 25 \quad 1 + 3s] \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
G(s) &= \frac{s + 25 + 2 + 6s}{s^2 + 4s - 7} = \frac{7s + 27}{s^2 + 4s - 7}
\end{aligned}$$

which has poles and zeros equal to the obtained answers in questions (a) and (b).

ii) (a) The poles of the system are any $s \in \mathbb{C}$ such that:

$$\det(sI - A) = 0 \iff \begin{vmatrix} s-2 & 1 \\ -5 & s-3 \end{vmatrix} = 0, \iff (s-2)(s-3)+5 = 0.$$

Therefore, we have the poles at $s = \frac{5}{2} + i\frac{\sqrt{19}}{2}$, $s = \frac{5}{2} - i\frac{\sqrt{19}}{2}$.

(b) The zeros are are any $s \in \mathbb{C}$ such that:

$$\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0 \iff \begin{vmatrix} s-2 & 1 & -3 \\ -5 & s-3 & 2 \\ 4 & -1 & 0 \end{vmatrix} = 0, \iff 14s - 47 = 0$$

Therefore, we have the zeros at $s = \frac{47}{14}$.

(c) The transfer function is given by

$$G(s) = C\left(Is - A\right)^{-1} B + D = \frac{14s - 47}{s^2 - 5s + 11}$$

which has poles and zeros equal to the obtained answers in questions (a) and (b).

iii) (a) The poles of the system are any $s \in \mathbb{C}$ such that:

$$\det(sI - A) = 0 \iff \begin{vmatrix} s-1 & -2 \\ 3 & s-4 \end{vmatrix} = 0, \iff (s-1)(s-4)+6 = 0.$$

Therefore, we have the poles at $s = \frac{5}{2} + i\frac{\sqrt{15}}{2}$, $s = \frac{5}{2} - i\frac{\sqrt{15}}{2}$.

(b) The zeros are are any $s \in \mathbb{C}$ such that:

$$\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0 \iff \begin{vmatrix} s-1 & -2 & 0 \\ 3 & s-4 & -1 \\ 2 & 5 & 0 \end{vmatrix} = 0, \iff 5s + -1 = 0$$

Therefore, we have the zeros at $s = \frac{1}{5}$.

(c) The transfer function is given by

$$G(s) = C\left(Is - A\right)^{-1} B + D = \frac{5s - 1}{s^2 - 5s + 10}$$

which has poles and zeros equal to the obtained answers in questions (a) and (b).

iv) (a) The poles of the system are any $s \in \mathbb{C}$ such that:

$$\det(sI - A) = 0 \iff \begin{vmatrix} s-1 & -2 \\ 0 & s-3 \end{vmatrix} = 0, \iff (s-1)(s-3) = 0.$$

Therefore, we have the poles at $s = 3, s = 1$.

(b) The zeros are are any $s \in \mathbb{C}$ such that:

$$\det \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = 0 \quad \iff \quad \begin{vmatrix} s-1 & -2 & -1 \\ 0 & s-3 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0,$$
$$\iff \quad 0 = 0$$

Therefore, we have no zeros. Note that the state space model is observable.

(c) The transfer function is given by

$$G(s) = C\left(Is - A\right)^{-1} B + D = 0$$

which has poles and zeros equal to the obtained answers in questions (a) and (b).

iv) (a) The poles of the system are any $s \in \mathbb{C}$ such that:

$$\det(sI - A) = 0 \quad \iff \quad \begin{vmatrix} s-1 & 0 & 0 \\ 0 & s-2 & 0 \\ 0 & 0 & s-3 \end{vmatrix} = 0,$$
$$\iff \quad (s-1)(s-2)(s-3) = 0.$$

Therefore, we have the poles at $s = 1, s = 2, s = 3$.

(b) The zeros are are any $s \in \mathbb{C}$ such that:

$$\det \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = 0 \quad \iff \quad \begin{vmatrix} s-1 & 0 & 0 & -1 \\ 0 & s-2 & 0 & 0 \\ 0 & 0 & s-3 & -1 \\ 1 & 0 & 1 & 0 \end{vmatrix} = 0,$$
$$\iff \quad 2s^2 - 8s + 8 = 0$$

Therefore, we have 2 zeros at $s = 2$.

(c) The transfer function is given by

$$G(s) = C\left(Is - A\right)^{-1} B + D = \frac{2s^2 - 8s + 8}{s^3 - 6s^2 + 11s - 6}$$

which has poles and zeros equal to the obtained answers in questions (a) and (b).

9. **SIR state-space model representation:**
Given is the SIR (Susceptible-Infectious-Recovered) model,

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I. \end{aligned}$$

This model is used in epidemiology to describe the spread of infectious diseases. It categorizes the population into susceptible (S), infectious (I), and recovered (R) individuals. For $\beta = 1$ and $\gamma = 2$, the model can be written as,

$$\begin{aligned} \dot{x}_1 &= -x_1 x_2 \\ \dot{x}_2 &= x_1 x_2 - 2x_1 \\ \dot{x}_3 &= 2x_2 \end{aligned}$$

Linearize this nonlinear model around equilibrium position $x^0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$, and take $y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$, to arrive at the linearized model:

$$\begin{cases} \dot{x}^\delta(t) = Ax^\delta(t) + Bu^\delta(t) \\ y^\delta(t) = Cx^\delta(t) + Du^\delta(t). \end{cases}$$

Solution:
To linearized the nonlinear model we use,

$$A = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} & \frac{\delta f_1}{\delta x_3} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} & \frac{\delta f_2}{\delta x_3} \\ \frac{\delta f_3}{\delta x_1} & \frac{\delta f_3}{\delta x_2} & \frac{\delta f_3}{\delta x_3} \end{bmatrix}$$

For,
$\dot{x}_1 = -x_1 x_2$
$\dot{x}_2 = x_1 x_2 - 2x_1$ this will give $A = \begin{bmatrix} -x_2 & -x_1 & 0 \\ x_2 - 2 & x_1 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$
$\dot{x}_3 = 2x_2$

Since there is no u present in the system, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and D is also equal to zero.

As output y is given to be $y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$,

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$