Markov modeling, discrete-event simulation – Exercises module B4

5XIE0 Computational Modeling

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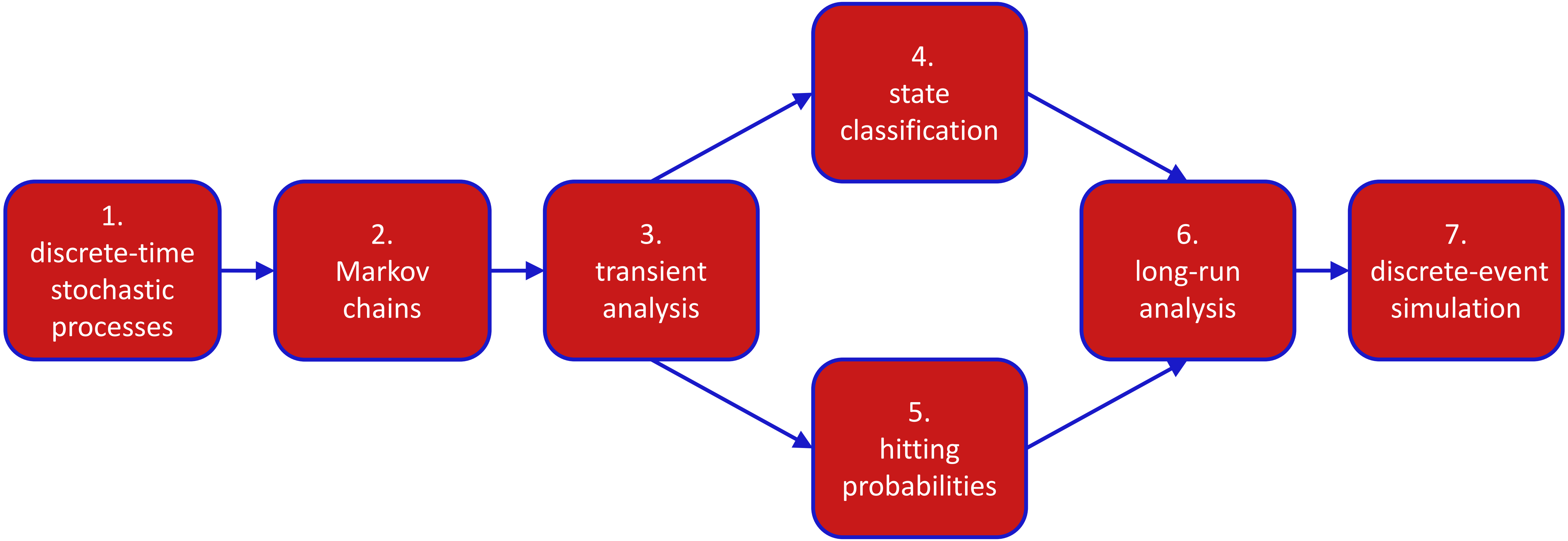
module B - submodules and dependencies



$$G_b = \begin{bmatrix} 1 & -\infty & 2 \\ 1 & -\infty & 2 \\ -\infty & 3 & -\infty \end{bmatrix}$$

B.4 – state classification

state classification – exercises

- Section B.4 in the course notes
  - Exercise B.15 (Classes of a Markov chain)
    - use CMWB (DTMC) to verify your answer
      1. create the model corresponding to the given probability matrix
      2. use ‘View Transition Diagram’ transition diagram to identify classes
      3. select ‘Communication Classes’ to compute the classes
  - Exercise B.16 (State accessibility versus paths)
  - Exercise B.17 (Communicating states – equivalence relation)
- answers are provided in Section B.8 of the course notes

state classification – exercises

- Section B.4 in the course notes
  - Exercise B.18 (Recurrent versus transient classes)
    - use CMWB (DTMC) to check answer
      1. create the model corresponding to the given probability matrix (same as Exercise B.15)
      2. select 'Classify Transient Recurrent' to determine whether states are transient or recurrent
  - Exercise B.19 (Computing return probabilities through paths)
    - possibly use CMWB (DTMC) to create transition graph (same as Exercise B.15)
- answers are provided in Section B.8 of the course notes

state classification – exercises

- Section B.4 in the course notes
  - Exercise B.20 (Periodic versus aperiodic classes)
    - use CMWB (DTMC) to check answer
      1. create the model corresponding to the given probability matrix
      2. select 'Determine Periodicity' to determine whether states are aperiodic or periodic, and in the later case compute the period
      3. select 'Determine MC Type' to obtain information about the Markov chain type
  - Exercise B.21 (Aperiodic states are eventually visited)
    - use CMWB (DTMC) to aid in solving (b)
- answers are provided in Section B.8 of the course notes

Exercise B.15 (Classes of a Markov chain)

Exercise B.15 (Classes of a Markov chain). Consider a Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition probability matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Determine the classes of this Markov chain.

Exercise B.15 (Classes of a Markov chain). The classes are found by considering the individual states. State 2 is accessible from state 1 but not the other way around. Therefore state 1 only communicates with itself and forms class $\{1\}$. State 2 communicates with states 3 and 4, and therefore these states together form class $\{2, 3, 4\}$. State 5 is isolated; no other states are accessible from state 5 and state 5 cannot be accessed by any other state. Therefore state 5 only communicates with itself and forms class $\{5\}$. Together these classes form a partition of the state space. This partition is $\{\{1\}, \{2, 3, 4\}, \{5\}\}$.

Use CMWB (DTMC) to verify your answer

1. create the model corresponding to the given probability matrix
2. use ‘View Transition Diagram’ transition diagram to identify classes
3. select ‘Communication Classes’ to compute the classes

Exercise B.16 (State accessibility versus paths)

Exercise B.16 (State accessibility versus paths). Show that $i \to j$ if and only if there exists a path from i to j.

Exercise B.16 (State accessibility versus paths).

'⇒' Assume $i \to j$. Then by definition $P_{ij}^n > 0$ for some $n \geq 0$. By equation (B.19) we then have that the sum of the probabilities of all paths of length n from i to j is at least 0. Therefore there exists at least on path of length n from i to j.

'⇐' Assume a path exists between i and j. If this path has length n, the probability of all paths of length n from i to j is at least 0. By equation (B.19) we then must have that $P_{ij}^n > 0$ and therefore $i \to j$.

$$\text{A state } j \text{ is } \textit{accessible} \text{ from } i, \text{ written } i \to j, \\ \text{if and only if } P_{ij}^n > 0 \text{ for some } n \geq 0 \tag{B.21}$$

$$\text{A sequence } i_1, i_2, \cdots, i_n \text{ of states (where } n \geq 1) \\ \text{is called a } \textit{path} \text{ if } P_{i_m i_{m+1}} > 0 \text{ for each } m \in \{1, 2, \cdots n-1\} \tag{B.17}$$

$$P(i_1, i_2, \cdots, i_n) = \prod_{m=1}^{n-1} P_{i_m i_{m+1}} \tag{B.18}$$

$$P_{ij}^n = \sum \{P(i, i_1, \cdots, i_{n-1}, j) \mid i, i_1, \cdots, i_{n-1}, j \text{ is a path of length } n\}^a \tag{B.19}$$

[a] In case different paths have the same probability, these probabilities have to be individually accounted for. Therefore this set is a multiset.

Exercise B.17 (Communicating states – equivalence relation)

Exercise B.17 (Communicating states - equivalence relation). Show that relation $\leftrightarrow$ is an equivalence relation (see equation (B.23)).

$$\text{Relation } \leftrightarrow \text{ is an equivalence relation} \tag{B.23}$$

- $i \leftrightarrow i$ (reflexive);
- if $i \leftrightarrow j$ then $j \leftrightarrow i$ (symmetric);
- if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$ (transitive).

Exercise B.17 (Communicating states - equivalence relation). We have to show that $\leftrightarrow$ is reflexive, symmetric and transitive:

- Since $i \to i$, we have by definition $i \leftrightarrow i$. Hence $\leftrightarrow$ is reflexive.
- Assume $i \leftrightarrow j$. By definition we then have $i \to j$ and $j \to i$ and thus also $j \leftrightarrow i$. Hence $\leftrightarrow$ is symmetric.
- Assume $i \leftrightarrow j$ and $j \leftrightarrow k$. Then $i \to j$ and $j \to k$ and thus a path exists from i to j and a path exists from j to k. Concatenation yields a path from i to k and thus $i \to k$. Vice versa we also have $k \to i$ and hence $i \leftrightarrow k$. Hence $\leftrightarrow$ is transitive.

$$\text{A state } j \text{ is } \textit{accessible} \text{ from } i, \text{ written } i \to j, \\ \text{if and only if } P_{ij}^n > 0 \text{ for some } n \geq 0 \tag{B.21}$$

$$\text{States } i \text{ and } j \textit{ communicate}, \text{ written } i \leftrightarrow j, \\ \text{if and only if } i \to j \text{ and } j \to i \tag{B.22}$$

$$P_{ij}^n = \sum \{P(i, i_1, \cdots, i_{n-1}, j) \mid i, i_1, \cdots, i_{n-1}, j \text{ is a path of length } n\}^a \tag{B.19}$$

[a] In case different paths have the same probability, these probabilities have to be individually accounted for. Therefore this set is a multiset.

Exercise B.18 (Recurrent versus transient classes)

Exercise B.18 (Recurrent versus transient classes). Consider a Markov chain with state space $\{1,2,3,4,5\}$ and transition probability matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.
Determine for each of the classes and states whether they are recurrent or transient.

$$\text{A state } i \text{ is said to be } \textit{recurrent} \text{ if and only if } f_{ii} = 1 \tag{B.24}$$

$$\text{A state } i \text{ is said to be } \textit{transient} \text{ if and only if } f_{ii} < 1 \tag{B.26}$$

$$f_{ii} = \sum \begin{array}{l} \{P(i, i_1, \cdots, i_{n-1}, i) \mid i, i_1, \cdots, i_{n-1}, i \text{ is a path} \\ \text{of length } n \geq 1 \text{ such that } i_k \neq i \text{ for all } k = 1, \cdots, n-1\} \end{array} \tag{B.25}$$

$$\text{The properties of recurrence and transience are class properties} \tag{B.27}$$

$$C \text{ is recurrent if and only if for all } i \in C \text{ and } j \in \mathcal{S} \setminus C, P_{ij} = 0 \tag{B.28}$$

Use CMWB (DTMC) to check answer

1. create the model corresponding to the given probability matrix (same as Exercise B.15)
2. select 'Classify Transient Recurrent' to determine whether states are transient or recurrent

Exercise B.18 (Recurrent versus transient classes). Class $\{1\}$ has an outgoing transition to class $\{2,3,4\}$ and is therefore transient. State 1 is therefore also transient. Class $\{2,3,4\}$ is closed (i.e. has no outgoing transitions) and is thus recurrent. Hence each of the states 2, 3 and 4 is recurrent. Class $\{5\}$ is also closed and is therefore recurrent. Hence state 5 is recurrent as well.

Exercise B.19 (Computing return probabilities through paths)

Exercise B.19 (Computing return probabilities through paths). Consider a Markov chain with state space $\{1,2,3,4,5\}$ and transition probability matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(a) State 1 is transient. Show this by computing the return probability f_{11}.

(b) State 2 is recurrent. Show this by computing the return probability f_{22}.

$$f_{ii} = \sum \begin{array}{l} \{P(i, i_1, \cdots, i_{n-1}, i) \mid i, i_1, \cdots, i_{n-1}, i \text{ is a path} \\ \text{of length } n \geq 1 \text{ such that } i_k \neq i \text{ for all } k = 1, \cdots, n-1\} \end{array} \quad \text{(B.25)}$$

Possibly use CMWB (DTMC) to create transition graph (same as Exercise B.15)

Exercise B.19 (Computing return probabilities through paths).

(a) A single path of length ≥ 1 exists from state 1 to state 1 which is path 1, 1. $P(1,1) = \frac{1}{3}$ and hence $f_{11} = \frac{1}{3}$. Since $f_{11} < 1$, state 1 is transient.

(b) Two paths of length ≥ 1 exist from state 2 to state 2 which are paths 2, 2 and 2, 3, 4, 2. $P(2,2) = \frac{1}{2}$ and $P(2,3,4,2) = \frac{1}{2}$ and hence $f_{22} = 1$. Since $f_{22} = 1$, state 2 is recurrent.

Exercise B.20 (Periodic versus aperiodic classes)

Exercise B.20 (Periodic versus aperiodic classes). Consider a Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition probability matrix $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

(a) Determine for each recurrent class whether it is periodic or aperiodic.

(b) Is this Markov chain a unichain?

The *period* of recurrent state i, written $d(i)$, is defined as
$$d(i) = gcd\{n \geq 1 \mid P_{ii}^{(n)} > 0\} \qquad \text{(B.29)}$$

$$d(i) = gcd\{n \geq 1 \mid \text{ a path of length } n \text{ exists from } i \text{ to } i\} \qquad \text{(B.30)}$$

A recurrent state i is called *aperiodic* if $d(i) = 1$ and *periodic* if $d(i) > 1$. (B.31)

Periodicity is a class property;
all states in a recurrent class have the same period (B.32)

Use CMWB (DTMC) to check answer

1. create the model corresponding to the given probability matrix
2. select 'Determine Periodicity' to determine whether states are aperiodic or periodic, and in the later case compute the period
3. select 'Determine MC Type' to obtain information about the Markov chain type

Exercise B.20 (Periodic versus aperiodic classes).

(a) Class $\{1\}$ is transient; for transient classes we have not defined the concept of periodicity. Class $\{2, 3, 4, 5\}$ is recurrent. To determine the period of this class, we can pick any state and only have to consider the paths that do not visit that state in between. For state 3 only two such paths exist, namely $3, 4, 5, 3$ and $3, 4, 2, 3$, both with length 3. Hence the period of class $\{2, 3, 4, 5\}$ equals 3.

(b) The chain has a single recurrent class which is periodic. Hence this chain is a non-ergodic unichain.

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Exercise B.20 (Periodic versus aperiodic classes)

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(a) Determine for each recurrent class whether it is periodic or aperiodic.

(b) Is this Markov chain a unichain?

Markov chain type	unichain	non-unichain
ergodic	a single recurrent class which is aperiodic (and zero or more transient classes)	at least two recurrent classes, all of which are aperiodic (and zero or more transient classes)
non-ergodic	a single recurrent class which is periodic (and zero or more transient classes)	at least two recurrent classes, which are not all aperiodic (and zero or more transient classes)

Table B.1: Markov chain types

Use CMWB (DTMC) to check answer

3. select ‘Determine MC Type’ to obtain information about the Markov chain type

Exercise B.21 (Aperiodic states are eventually visited)

Exercise B.21 (Aperiodic states are eventually visited). Consider a Markov chain with state space $\{1, 2, 3\}$ and transition probability matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$.

(a) Determine the classes of this chain, determine for each class whether it is transient or recurrent and periodic or aperiodic.

(b) Argue that P^4, P^5, P^6, $\cdots$ contain only positive elements.

Exercise B.21 (Aperiodic states are eventually visited).

(a) This chain a single recurrent, aperiodic class. Hence this is an ergodic unichain.

(b) For $P^5, P^6, \cdots$ this follows from equation (B.33), where the number of states (N) equals 3. It states that for $n \geq (3-1)^2 + 1 = 5$, $P_{ij}^n > 0$ for all $i, j \in \{1, 2, 3\}$.

For $n = 4$ it follows from explicitly computing P^4 yielding the matrix $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \\ \frac{5}{16} & \frac{1}{8} & \frac{9}{16} \end{bmatrix} \approx \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.125 & 0.25 & 0.625 \\ 0.312 & 0.125 & 0.562 \end{bmatrix}$ having only positive elements.

$$\text{For any aperiodic class } C \text{ and all states } i, j \in C, \quad P_{ij}^n > 0 \text{ for all } n \geq (N-1)^2 + 1 \tag{B.33}$$

Use CMWB (DTMC) to aid in solving (b)