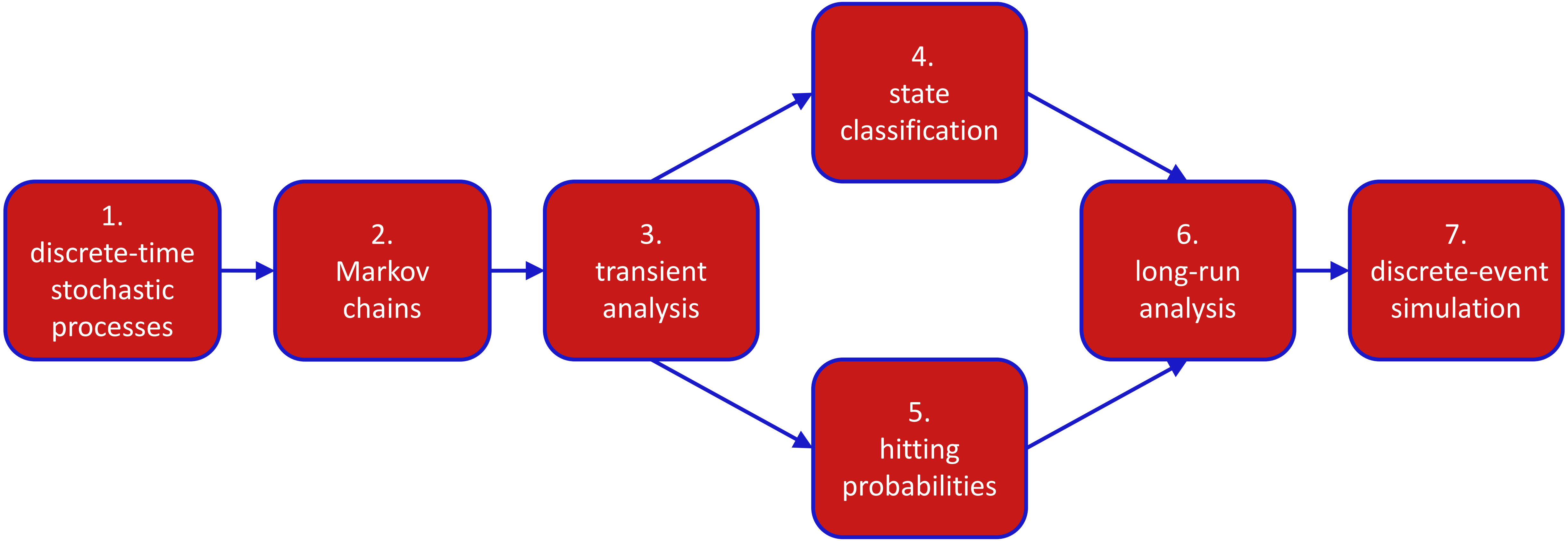
EINDHOVEN UNIVERSITY OF TECHNOLOGY

Markov modeling, discrete-event simulation – Exercises module B2

5XIE0 Computational Modeling

Twan Basten, Marc Geilen, Jeroen Voeten
Electronic Systems Group, Department of Electrical Engineering

module B - submodules and dependencies



$$G_b = \begin{bmatrix} 1 & -\infty & 2 \\ -\infty & 3 & -\infty \end{bmatrix}$$

B.2 – Markov chains

Markov chains – exercises

- Section B.2 in the course notes
  - Exercise B.4 (Transition diagram to matrix)
  - Exercise B.5 (Matrix to transition diagram)
  - Exercise B.6 (Markov chains – dependent and non-identically distributed variables)
  - Exercise B.7 (Gambler's ruin – probability distributions)
    - use CMWB (DTMC) to double check your answers
      1. select 'Create a new DTCM model' in 'General Operations', enter the Gambler's ruin model and save
      2. select 'View Transition Diagram' in 'Operations on Markov chains' to inspect the transition diagram
      3. selected 'Transient Distribution' and enter a number (say 2) of steps to analyze
      4. distribution vectors are provided in 'Analysis Output' pane
  - Exercise B.8 (Markov chains – independent identically distributed variables)
    - use CMWB (DTMC) to double check your answers to (a)
      1. select 'Create a new DTCM model' in 'General Operations' enter the model and save
      2. select 'Transient Distribution' and enter the number of steps to analyze
- answers are provided in Section B.8 of the course notes

Exercise B.4 (Transition diagram to matrix)

Exercise B.4 (Transition diagram to matrix). Consider the transition diagram of the three-state Markov chain depicted in Figure B.4. Give the transition probability matrix of this chain.

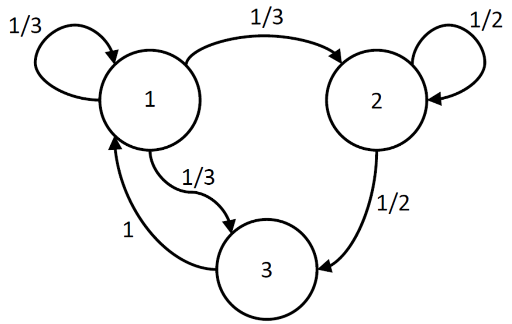


Figure B.4: Transition diagram of three-state Markov chain

Exercise B.4 (Transition diagram to matrix). The transition probability matrix is given by

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

$$P(X_{n+1} = j \mid X_n = i) = P_{ij} \qquad \text{(B.9)}$$

Exercise B.5 (Matrix to transition diagram)

Exercise B.5 (Matrix to transition diagram). Draw the transition diagram corresponding the Markov chain with transition probability matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \end{bmatrix}$$

Exercise B.5 (Matrix to transition diagram). The transition diagram is depicted in Figure B.14.

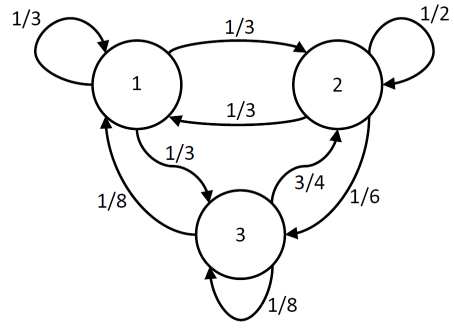


Figure B.14: Transition diagram of three-state Markov chain

Exercise B.6 (Markov chain – dependent and non-identically distributed variables)

Exercise B.6 (Markov chains - dependent and non-identically distributed variables). Consider a Markov chain $X_0, X_1, \cdots$ with two states, 1 and 2. At times $0, 2, \cdots$ the process visits state 1 and at times $1, 3, \cdots$ it visits state 2.

(a) Give the transition probability matrix of this chain and draw the transition diagram.

(b) Give the initial distribution at time 0, i.e. $\pi^{(0)}$.

(c) Determine the distribution at time 1.

(d) Show that X_0 and X_1 are not identically distributed.

(e) Show that X_0 and X_1 are dependent variables.

Exercise B.6 (Markov chains - dependent and non-identically distributed variables).

(a) When the chain is in state 1, it will transition to state 2 with probability 1. Vice versa, when the chain is in state state 2, it will transition to state 1 with probability 1. Hence the transition probability matrix is given $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The corresponding transition diagram is shown in Figure B.15.

(b) At time 0, the chain visits state 0 with probability 1. Therefore $\pi^{(0)} = [1, 0]$.

(c) After visiting state 0, the chain jumps to state 1 with probability 1. At time 1, the chain thus visits state 1 with probability 1 and therefore $\pi^{(1)} = [0, 1]$.

(d) Distributions $\pi^{(0)}$ and $\pi^{(1)}$ are different and therefore X_0 and X_1 are not identically distributed.

(e) Assume X_1 and X_0 are independent. Then for all $i, j \in \{1, 2\}$, $P(X_1 = j \mid X_0 = i) = P(X_1 = j)$. In particular, for $i = 2$ and $j = 2$, we then have $P(X_1 = 2 \mid X_0 = 2) = P(X_1 = 2)$. But $P(X_1 = 2 \mid X_0 = 2) = P_{22} = 0$ and $P(X_1 = 2) = \pi_2^{(1)} = 1$, so we have a contradiction. Therefore X_1 and X_0 are dependent variables.

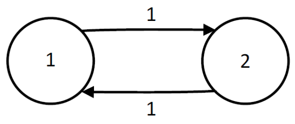


Figure B.15: Transition diagram of two-state Markov chain

$$P(X_{n+1} = j \mid X_n = i) = P_{ij} \quad \text{(B.9)}$$

$$X_n \text{ and } X_m \text{ are independent if} \\ P(X_n = i \mid X_m = j) = P(X_n = i) \text{ for all } i, j \in \mathcal{S} \quad \text{(B.7)}$$

Exercise B.7 (Gambler's ruin – probability distributions)

Exercise B.7 (Gambler's ruin - probability distributions). Consider Markov chain $X_0, X_1, \cdots$ corresponding to transition diagram of the gambler's ruin in Figure B.3 and assume $\pi^{(0)} = [0, 1, 0, 0]$.

(a) Determine $\pi^{(1)}$.

(b) Determine $\pi^{(2)}$.

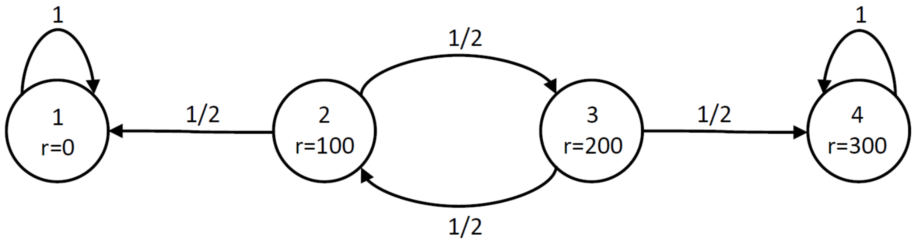


Figure B.3: Transition diagram of the gambler's ruin

Use CMWB (DTMC) to double check your answers

1. select 'Create a new DTCM model' in 'General Operations', enter the Gambler's ruin model and save
2. select 'View Transition Diagram' in 'Operations on Markov chains' to inspect the transition diagram
3. selected 'Transient Distribution' and enter a number (say 2) of steps to analyze
4. distribution vectors are provided in 'Analysis Output' pane

Exercise B.7 (Gambler's ruin - probability distributions).

(a) The chain starts in state 2 (with probability 1). After one transition, it will be in state 1 with probability $\frac{1}{2}$ and in state 3 with probability $\frac{1}{2}$. Therefore $\pi^{(1)} = [\frac{1}{2}, 0, \frac{1}{2}, 0]$.

(b) The probability that the chain is in state 1 at time 2 equals the probability that the chain is in state 1 at time 1 times 1 plus the probability that the chain is in state 2 at time 1 times $\frac{1}{2}$. Hence $\pi_1^{(2)} = \pi_1^{(1)} \cdot 1 + \pi_2^{(1)} \cdot \frac{1}{2} = \frac{1}{2}$. With a similar line of thought we obtain $\pi_2^{(2)} = \frac{1}{4}$, $\pi_3^{(2)} = 0$ and $\pi_4^{(2)} = \frac{1}{4}$. Hence $\pi^{(2)} = [\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}]$.

Exercise B.8 (Markov chains – independent identically distributed variables)

Exercise B.8 (Markov chains - independent identically distributed variables). Consider Markov chain $X_0, X_1, \cdots$ with state-space $\{1, 2\}$, initial distribution $\pi^{(0)} = [\frac{1}{2}, \frac{1}{2}]$ and transition probability matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(a) Show that X_n and X_{n+1} are identically distributed for all $n = 0, 1, \cdots$.

(b) Show that X_n and X_{n+1} are independent for all all $n = 0, 1, \cdots$.

Exercise B.8 (Markov chains - independent identically distributed variables).

(a) Using a similar line of thought as used in Exercise B.7, we find that $\pi^{(n)} = [\frac{1}{2}, \frac{1}{2}]$ for all $n = 0, 1, \cdots$. Therefore X_n and X_{n+1} are identically distributed for all $n = 0, 1, \cdots$.

(b) We have to show that $P(X_{n+1} = j \mid X_n = i) = P(X_{n+1} = j)$ for all $i, j \in \{1, 2\}$. Now $P(X_{n+1} = j \mid X_n = i) = P_{ij} = \frac{1}{2}$ and $P(X_{n+1} = j) = \pi_j^{(n+1)} = \frac{1}{2}$ (for all $i, j \in \{1, 2\}$) from which the result follows.

$$X_n \text{ and } X_m \text{ are independent if} \quad P(X_n = i \mid X_m = j) = P(X_n = i) \text{ for all } i, j \in \mathcal{S} \qquad \text{(B.7)}$$

Use CMWB (DTMC) to double check your answers to (a)

1. select ‘Create a new DTCM model’ in ‘General Operations’ enter the model and save
2. select ‘Transient Distribution’ and enter the number of steps to analyze