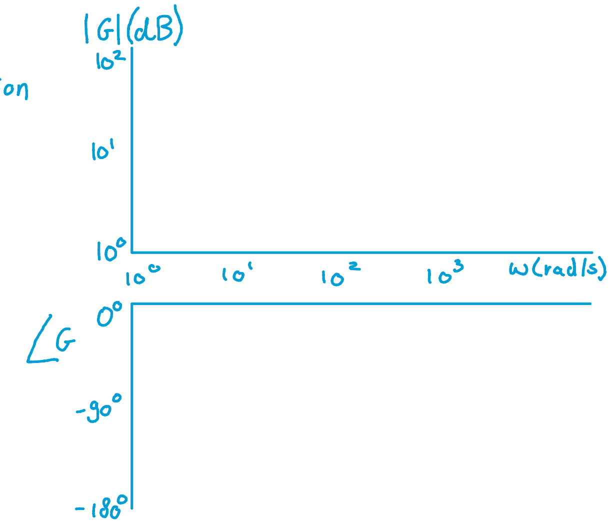
3 - Frequency response - Type 1 and type 2 building blocks

Frequency response analysis

- Frequency response analysis → System identification / Controller design
  - ▷ visualisation with Bode Plots



- Steady state behaviour
  - ▷ only sinusoidal inputs
  - ▷ Math operations
    - addition/subtraction
    - multiplication/division
    - Integrating / derivating

  } change in magnitude & phase, not Frequency

- Polar Form $G(j\omega_0) = \underbrace{M}_{\text{phase}} e^{j\phi}$ } magnitude

- Composite system

$$G(j\omega) = \frac{M_1 e^{j\phi_1} M_2 e^{j\phi_2}}{M_3 e^{j\phi_3} M_4 e^{j\phi_4} M_5 e^{j\phi_5}} = \frac{M_1 M_2}{\underbrace{M_3 M_4 M_5}} e^{j(\phi_1 + \phi_2 - \phi_3 - \phi_4 - \phi_5)}$$

$$|G(j\omega)| = \log(M_1 + M_2 - M_3 - M_4 - M_5)$$

- $s = j\omega$ Frequency dependent $K\,G(j\omega) = K_0 (j\omega)^n \dfrac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\ldots}{(j\omega\underbrace{\tau_a}_{}+1)(j\omega\tau_b + 1)\ldots}$

  Break point Frequencies

- Three classes:
  1. $K_0 (j\omega)^n$
  2. $(j\omega\tau + 1)^{\pm 1}$
  3. $\left[(j\omega/\omega_n)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^{\pm 1}$

Class 1: singularities at the origin

- $KG(s) = K\frac{(s-z_1)(s-z_2)\dots}{(s-p_1)(s-p_2)\dots} \overset{s=j\omega}{\Longleftrightarrow} \underbrace{K_o(j\omega)^n}_{\text{Class 1}} \frac{(j\omega\tau_1+1)(j\omega\tau_2+1)\dots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\dots}$

degree of poles/zeros

$K \neq K_o$!

- At the origin:
  - Pole $1/s$ or s^{-1}, integration
  - Zero s or s^{1}, derivation

  } Laplace

Illustration

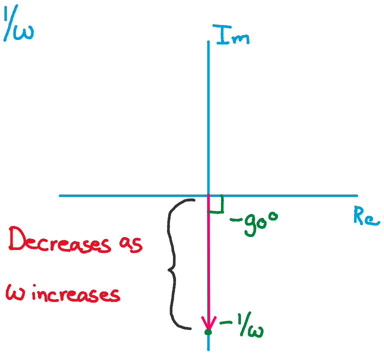
$\frac{1}{s}$

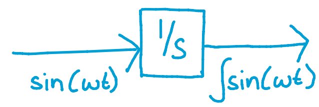
$$\frac{1}{j\omega} = 0 + \frac{1}{\omega}\underbrace{\frac{1}{j}\frac{j}{j}} = 0 - \frac{1}{\omega}j$$

$$\frac{1}{\sqrt{-1}}\cdot\frac{\sqrt{-1}}{\sqrt{-1}} = \frac{\sqrt{-1}}{-1} = -j$$

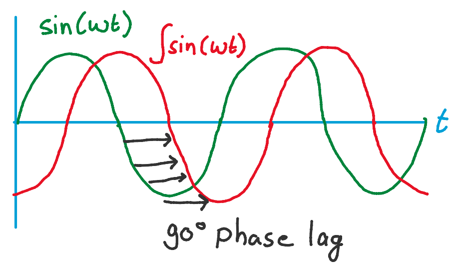
Real : 0

Imag : $-1/\omega$





$$= -\frac{1}{\omega}\cos(\omega t) = -\frac{1}{\omega}\sin(\omega t + \underbrace{\phi}_{-90^\circ})$$

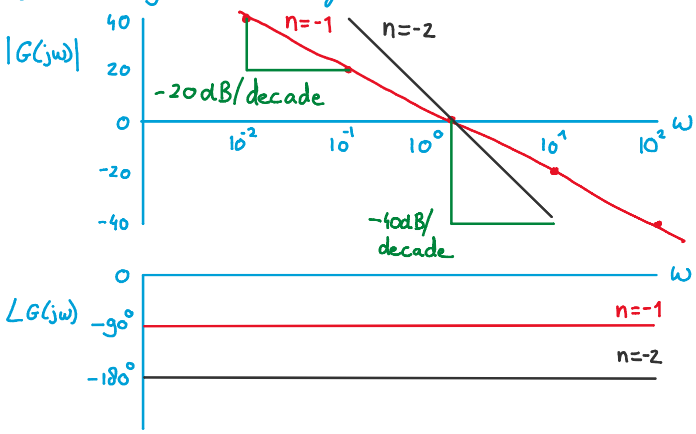


Class 1: in bode form $1/s$

$\omega = 1$: gain $= 1$ $20\log(1) = 0$

$\omega = 10$ gain $= 1/10$ $20\log(\frac{1}{10}) = -20$

$\omega = 100$ gain $= 1/100$ $20\log(1/100)\ -40$

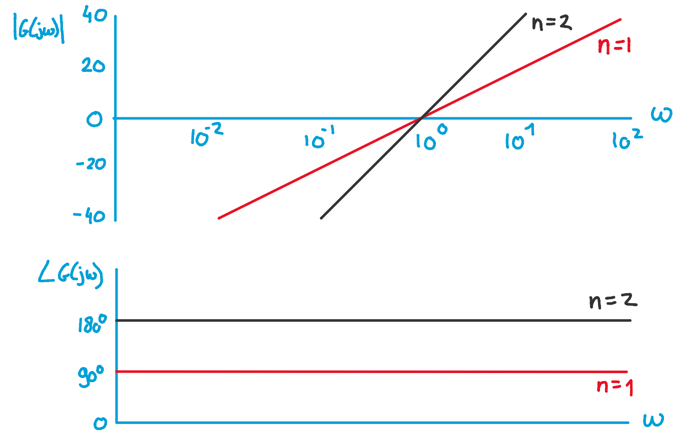


Bode gain-phase relationship

Gain : $n \times 20\,dB/decade$

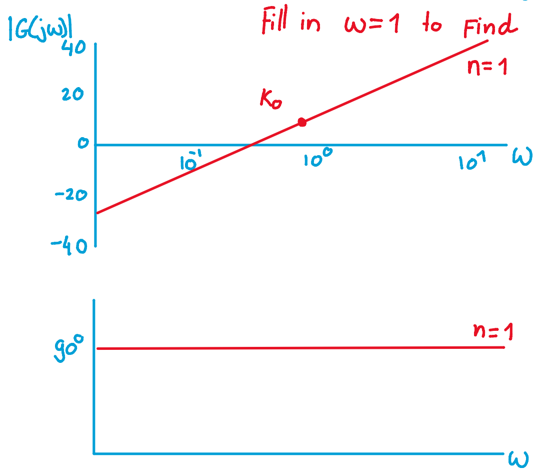
Phase : $n \times 90^\circ$

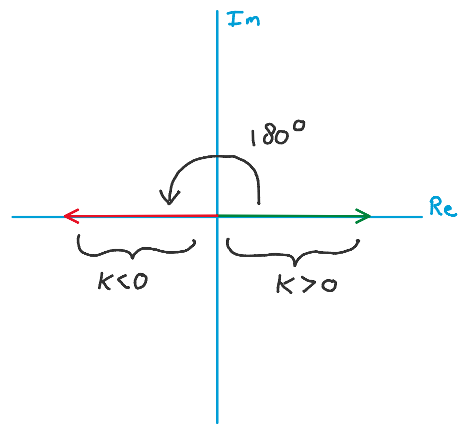
s^n



$K_0 (j\omega)^n$ $\log K_0 |(j\omega)^n| = \log K_0 + n \log(j\omega)$

Fill in $\omega = 1$ to Find K_0

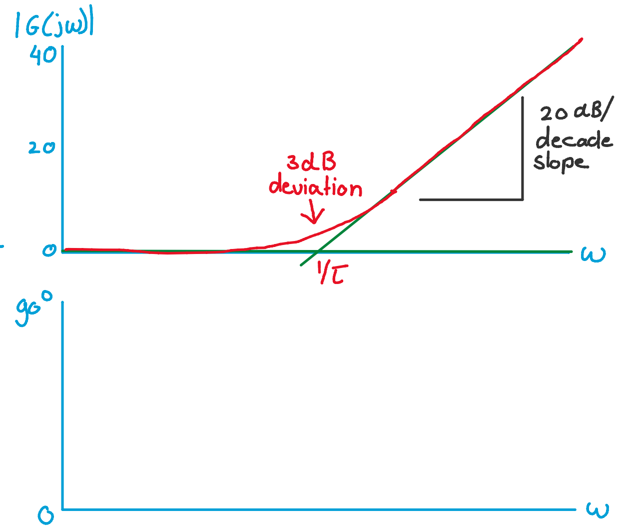




Class 2: First order terms

$(j\omega\tau+1)^{\pm 1}$ $\begin{cases} +1 & \text{Zero} \\ -1 & \text{Pole} \end{cases}$

$$\begin{cases} \omega\tau \ll 1, & j\omega\tau+1 \cong 1 \longrightarrow 0\,\text{dB} \\ \omega\tau \gg 1, & j\omega\tau+1 \cong j\omega\tau \longrightarrow \pm 20\,\text{dB/decade} \end{cases}$$



Break point frequency τ

$\omega = 1/\tau$ Break point

Pole

$$\frac{1}{j\omega\tau+1}\cdot\frac{1-j\omega\tau}{1-j\omega\tau} = \frac{1-j\omega\tau}{1+\omega^2\tau^2}$$

Re: $1/1+\omega^2\tau^2$

Im: $-\omega\tau/1+\omega^2\tau^2$

Gain: $$\left[\left(\frac{1}{1+\omega^2\tau^2}\right)^2+\left(\frac{-\omega\tau}{1+\omega^2\tau^2}\right)^2\right]^{1/2} = \left(\frac{\cancel{1+\omega^2\tau^2}}{(1+\omega^2\tau^2)^{\cancel{2}}}\right)^{1/2}$$

$= -20\log\left(\sqrt{1+\omega^2\tau^2}\right)$ in decibels

Phase: $\arctan\left(\text{Im}/\text{Re}\right) = \arctan(-\omega\tau)$

Zero

$1+j\omega\tau$

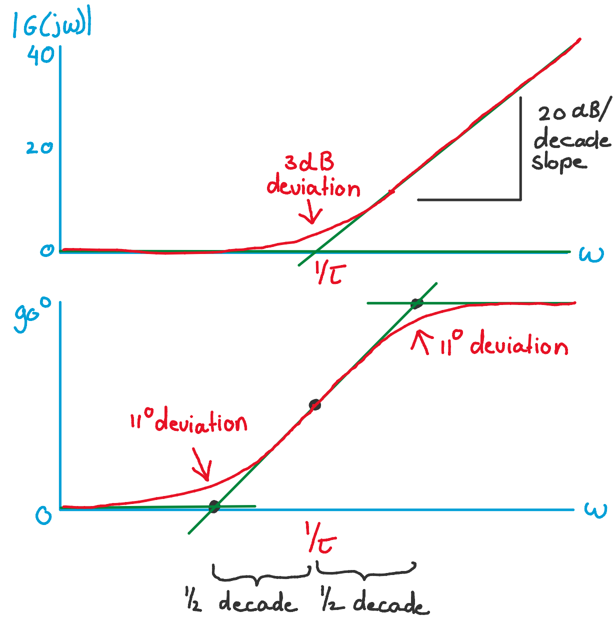
Re: 1

Im: $\omega\tau$

Gain: $\sqrt{1+\omega^2\tau^2} = 20\log\left(\sqrt{1+\omega^2\tau^2}\right)$

phase: $\arctan(\omega\tau)$

$$\begin{cases} \omega\tau \ll 1: & j\omega\tau+1 \cong 1 & , \angle 1 = 0^\circ \\ \omega\tau \gg 1: & j\omega\tau+1 \cong j\omega\tau & , \angle j\omega\tau = 90^\circ \\ \omega\tau \cong 1: & & \angle(j\omega\tau+1) \cong 45^\circ \end{cases}$$



Summary:

- Class 1: $K_o(j\omega_o)^n$
  Class 2: $(j\omega\tau+1)^{\pm 1}$
- Bode gain-phase relationship

  evaluate at $\omega = 1$ rad/s

  gain : $n \times 20$ dB/decade $\longrightarrow$ magnitude through K_o $\log K_o |(j\omega)^n| = \log K_o + n \log(j\omega)$
  phase : $n \times 90°$
- First order systems at break point frequency τ
  $\hookrightarrow$ Assymptotic behaviour used to draw frequency behaviour
- Up next: Class 3 terms & frequency response of poles and zeros in RHP