1. **Discrete-time filter:**
A discrete-time filter is given by:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1 + (1/2)z^{-1}}{(1 - (1/2)z^{-1})(1 + (1/3)z^{-1})},$$

where $u(k)$ and $y(k)$ are the discrete input and output of this filter, respectively.

   (a) Find a difference equation relating $u(k)$ and $y(k)$ (for zero initial conditions).

   (b) Assess whether the filter is stable.

   (c) For a sampling period $T = 1$ sec, find the continuous time poles and compute the natural frequency ω_n and damping coefficient ζ of the poles of $H(z)$. *Hint*: $e^{-j\pi} = -1$.

Solution:

   (a) By

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1 + (1/2)z^{-1}}{(1 - (1/2)z^{-1})(1 + (1/3)z^{-1})},$$

   we derive that

$$Y(z) - \frac{1}{6}z^{-1}Y(z) - \frac{1}{6}z^{-2}Y(z) = U(z) + \frac{1}{2}z^{-1}U(z),$$

   hence

$$y(k) - \frac{1}{6}y(k-1) - \frac{1}{6}y(k-2) = u(k) + \frac{1}{2}u(k-1).$$

   (b) The poles of the filter are at $1/2$ and $-1/3$, both inside the unit circle. Therefore, the filter is stable.

   (c) We have two poles at $z_1 = 1/2$ and $z_2 = -1/3$ in the z-plane.
   Since $z = e^{sT}$, we have the poles at $s_i = \frac{1}{T}\ln(z_i)$ and therefore:

$$s_1 = \frac{1}{T}\ln(1/2) = -0.693/T$$
$$s_2 = \frac{1}{T}\ln(-1/3) = \frac{1}{T}\ln(e^{j\pi}1/3) = (-1.1 \pm 3.14j)/T$$

   in the s-plane, where $T = 1$ is the sampling period.
   Note that the equation $-\frac{1}{3} = e^{sT}$ has complex conjugate solutions. As a result we get a complex conjugate pole pair in the s-domain for the pole at $z_2 = -1/3$.
   Thus, for $z = \frac{1}{2}$, we have $\omega_n = 0.693$ rad/s, $\zeta = 1$, and for $z = \frac{-1}{3}$, $\omega_n = 3.33$ rad/s, $\zeta = 0.33$.

2. **Tustin approximation:**
Consider the usual feedback loop scheme with

$$G(s) = \frac{250}{s((s/10) + 1)}$$

and lag compensator

$$D(s) = \frac{(s/1.25) + 1}{50s + 1},$$

which ensures PM $\simeq 50°$.

(a) Determine an appropriate sampling time.

(b) Determine an equivalent digital realization of the compensator $D(s)$ using the Tustin approximation.

Solution:

(a) For the closed-loop system $G_{\mathrm{cl}} = \frac{D(s)G(s)}{1+D(s)G(s)}$, the bandwidth is approximately 6.5 rad/s. Thus, a safe sampling rate would be

$$\omega_{\mathrm{s}} = 20 \cdot 6.5 = 162.5 \text{ rad/s} ,$$

and $T_s = 2\pi/\omega_{\mathrm{s}} \simeq 0.04$ s.

(b) By using the Tustin approximation, i.e.

$$s \longmapsto \frac{2}{T_s} \cdot \frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T_s} \cdot \frac{z-1}{z+1},$$

we have

$$D_{\mathrm{d}}(z) = \frac{41z-39}{2501z-2499} \simeq \frac{0.0164z-0.0156}{z-0.9992}.$$

3. **Time-delay:**
Consider the transfer function

$$G(s) = 10\,\frac{s+2}{(s+1)(s+5)}\,e^{-2s} = \hat{G}(s)\,e^{-2s}.$$

Note that the term e^{-2s} is due to the presence of a time delay introduced by sampling (ZOH with sampling introduces a delay of $T/2$ seconds).

(a) Sketch the bode diagram of $\hat{G}(s)$ and $G(s)$.

(b) Make a Padé approximation of the delay.

(c) Plot the step response of $G(s)$, $\hat{G}(s)$ and of $G(s)$ with the Padé approximation of the delay. Explain what you see.

(d) Design a stabilizing controller for $\hat{G}(s)$ with $\omega_c \geq 2$ rad/s.

(e) Design a stabilizing controller for $G(s)$ with ω_c as high as possible (Hint: use the Padé approximation of the delay).

Solution:

(a) The bode plot of $\hat{G}(s)$ and $G(s)$ are given in Figures 1 and 2
Note that the magnitude of $\hat{G}(s)$ is equal to the magnitude of $G(s)$ but that the phase of $G(s)$ is significantly less than $\hat{G}(s)$.

(b) A (2,2) Padé approximation of a time delay $e^{-T_{\mathrm{d}}s}$ is given by

$$e^{-T_{\mathrm{d}}s} \approx \frac{1-T_{\mathrm{d}}s/2+\left(T_{\mathrm{d}}s\right)^2/12}{1+T_{\mathrm{d}}s/2+\left(T_{\mathrm{d}}s\right)^2/12}.$$

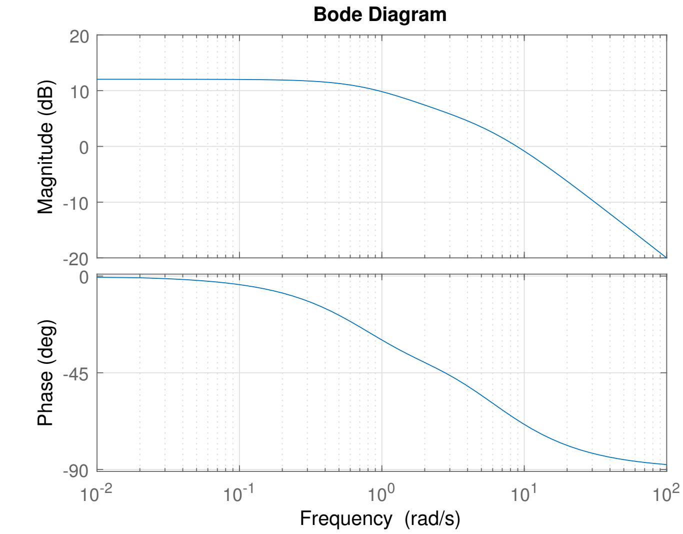

Figure 1: Problem 1: Bode plot of $\hat{G}(s)$.

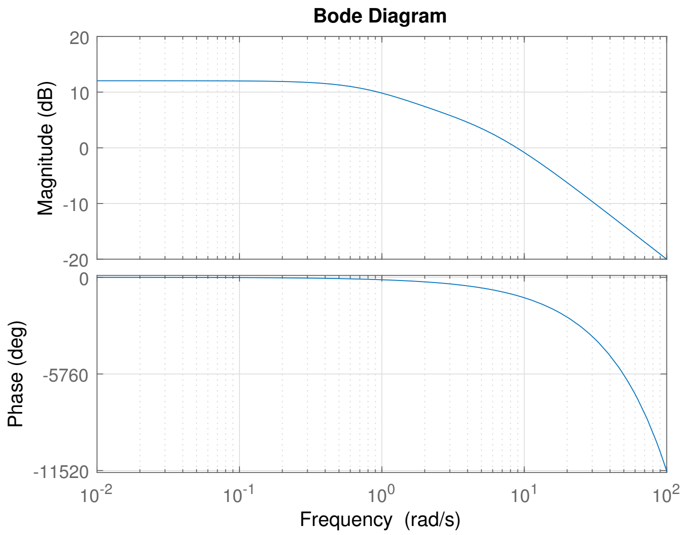

Figure 2: Problem 1: Bode plot of $G(s)$.

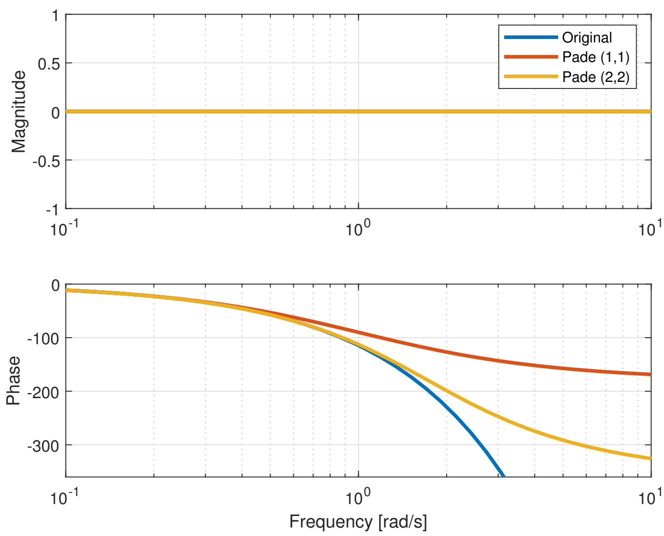

Figure 3: Problem 1: Bode plot of e^{-2s},a (1,1) Padé approximation of the delay and a (2,2) Padé approximation of the delay.

For $T_{\rm d} = 2$ this then gives

$$e^{-2s} \approx \frac{1 - s + s^2/3}{1 + s + s^2/3}.$$

Figure 3 gives the original bode plot of the delay, together with a (1,1) and (2,2) Padé approximation of the delay.

(c) Figure 4 shows the step response of $G(s)$, $\hat{G}(s)$ and of $G(s)$ with the Padé approximation of the delay of order 2 and 10. As expected, for a higher order Padé approximation the resulting step is closer to the exact $G(s)$. Between $G(s)$ and $\hat{G}(s)$ we can clearly see the delay between the steps.

(d) $\hat{G}(s)$ is already a stable system, closing the loop with a gain $K = 1$ will give a stable system with $\omega_c = 8.87$ rad/s. Lowering the gain K to 0.4256 will result in $\omega_c = 2$ rad/s.

(e) We start with sketching the bode diagram for

$$G(s) \approx \hat{G}(s)\frac{1 - s + s^2/3}{1 + s + s^2/3},$$

which gives the bode diagram in Figure 5.

At a frequency of 1.33 rad/s the phase is -180 degrees and the magnitude is equal to 8.878 dB. Using a gain of $K = -8.878$ dB $= 0.3554$ will then result in a stable system without any phase margin and a crossover frequency of 1.33 rad/s. The time delay introduces so much phase lag that even with lead filters this crossover frequency cannot be increased. This shows that time delay introduced by sampling puts a significant limitation on the achievable performance.

4. **Sampling:**

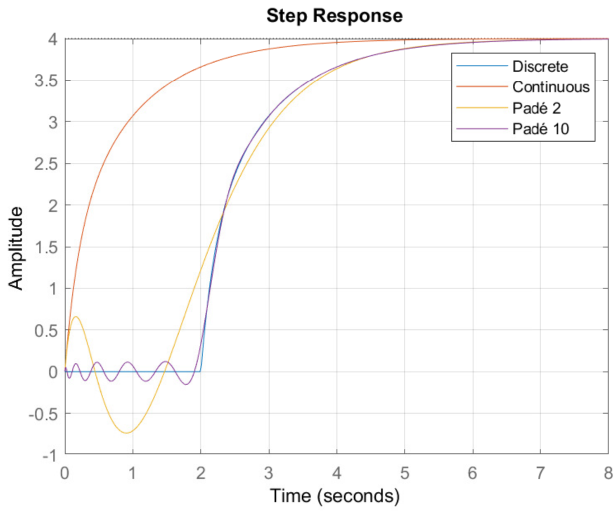

Figure 4: Comparison step response $G(s)$, $\hat{G}(s)$ and $G(s)$with Padé approximated delay

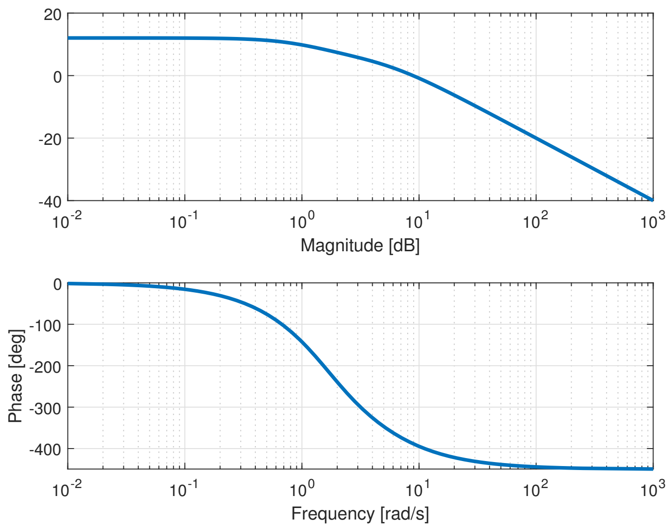

Figure 5: Problem 1: Bode plot of the approximation of $G(s)$.

Consider the system configuration in Figure 6. Use the following two cases for $G(s)$,

i)
$$G(s) = \frac{(s+1)}{s(s+8)}.$$

ii)
$$G(s) = \frac{40(s+2)}{(s+10)(s^2-1.4)}.$$

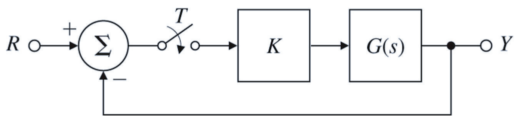

Figure 6: Control system for Problem 3.

For both cases considered of $G(s)$, perform the following tasks:

$$G(s) = \frac{40(s+2)}{(s+10)(s^2-1.4)}.$$

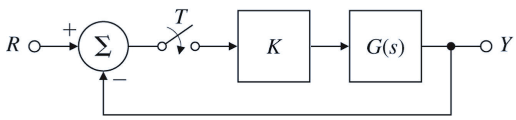

Figure 7: Control system for Problem 3.

(a) Find the transfer function $G(z)$ for $T=1$, assuming that there is a ZOH in front.

(b) Use MATLAB to draw the Root Locus of the open-loop discrete-time transfer function.

(c) Determine the range of K for which the closed-loop system is stable.

(d) Compare your results of part (b) with the case in which an analog controller is used. Determine for which range of the gain K the closed-loop is stable.

Solution:

i)

(a) By using the Partial Fraction Expansion, we get

$$\begin{aligned} G(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\ &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{(s+1)}{s(s+8)}\right\} \\ &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{1}{8s}+\frac{7}{8\,(s+8)}\right\} \\ &= \frac{z-1}{z}\left(-\frac{1}{8}\frac{z}{z-1}+\frac{7}{8}\frac{z}{z-e^{-8}}\right) \\ &= \ldots = \frac{0.2343z-0.1094}{z^2-z+0.0003355}. \end{aligned}$$

This can be verified in MATLAB with the `c2d` command.

(b) The root locus of the closed-loop system is shown in Figure 8.

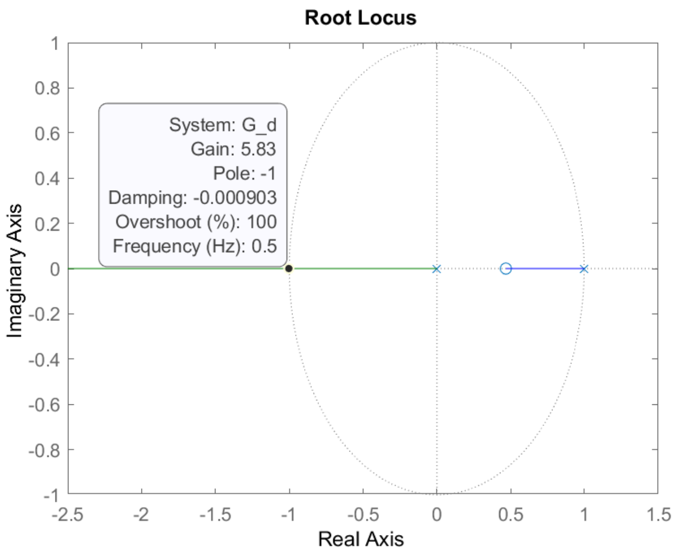

Figure 8: Problem 4: Root locus for $G(z)$.

(c) Since one branch of the Root Locus is outside the unit circle for $K > 5.83$. So the closed-loop system will be stable with any gain $0 < K < 5.83$.

(d) The Root Locus of the continuous-time system is shown in Figure 9. The closed-loop system will be stable for all (positive) K.

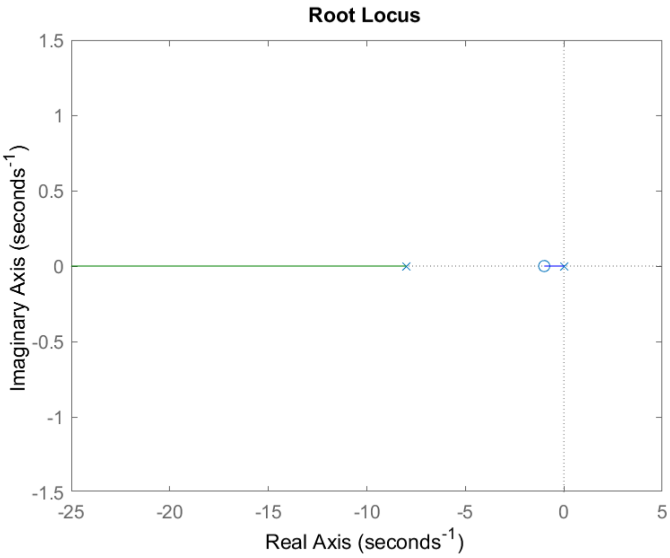

Figure 9: Problem 4: Root locus for $G(s)$.

ii)

(a) By using the Partial Fraction Expansion, we get

$$\begin{aligned}
G(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{\frac{40(s+2)}{s(s+10)(s^2-1.4)}\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{40\left(-\frac{0.1429}{s}+\frac{0.0081}{s+10}+\frac{0.0331}{s+\sqrt{1.4}}+\frac{0.1017}{s-\sqrt{1.4}}\right)\right\} \\
&= 40\frac{z-1}{z}\left(-0.1429\frac{z}{z-1}+0.0081\frac{z}{z-e^{-10}}+0.0331\frac{z}{z-e^{-\sqrt{1.4}}}+0.1017\frac{z}{z-e^{\sqrt{1.4}}}\right) \\
&= \ldots = \frac{7.967(z+0.3023)(z-0.1347)}{(z-3.265)(z-0.3063)(z-0.0000454)}.
\end{aligned}$$

This can be verified in MATLAB with the `c2d` command.

(b) The root locus of the closed-loop system is shown in Figure 10.

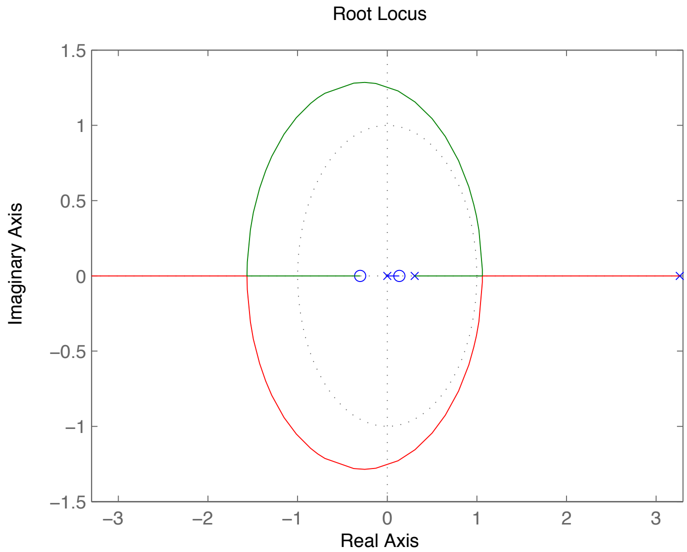

Figure 10: Problem 4: Root locus for $G(z)$.

(c) Since one branch of the Root Locus is always outside the unit circle, the closed-loop system cannot be made stable with any positive gain K.

(d) The Root Locus of the continuous-time system is shown in Figure 11. The closed-loop system will be stable for all $K > 0.175$.

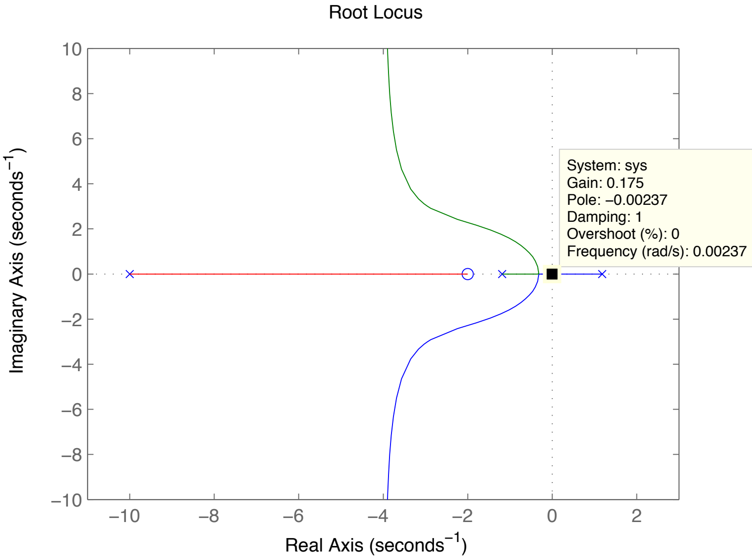

Figure 11: Problem 4: Root locus for $G(s)$.

5. **Discrete-time proportional control:**
We want to control the system

$$G(s) = \frac{1}{(s + 0.1)(s + 3)}$$

with a digital proportional controller having a sampling period of $T_s = 0.1$ seconds. Thus, consider the presence of the ZOH in front of $G(s)$.

(a) Compute the poles of the closed-loop discrete-time system when the proportional gain is $K = 4$.

(b) Plot the step response of the analog and discrete closed-loop systems for a simulation runtime of 10 seconds.

(c) Show what happens to the steady state error if we add a discrete time controller of the form

$$D(z) = \frac{K_I T_s}{z - 1}$$

with $K_I = 0.1$. Increase the runtime to 300 seconds. Explain your results.

Solution:

(a) Using the same methodology as for Problem 4, we obtain the discrete-time transfer function given by

$$G(z) = 0.0045 \frac{z + 0.9019}{(z - 0.7408)(z - 0.99)}.$$

From the root locus of the system in the z-plane we have that the poles are located at

$$z \simeq 0.8564 \pm 0.1278j.$$

Alternatively, we can numerically solve the characteristic equation for $K = 4$ and obtain the same result.

(b) Figure 12 shows a possible simulink configuration. Figure 13 shows the step response for the analog and digital system. From the step responses we can see that the digital system has more overshoot than the analog system due to a reduction of phase through sampling. We also see that there is a steady state error for both the analog and digital system.

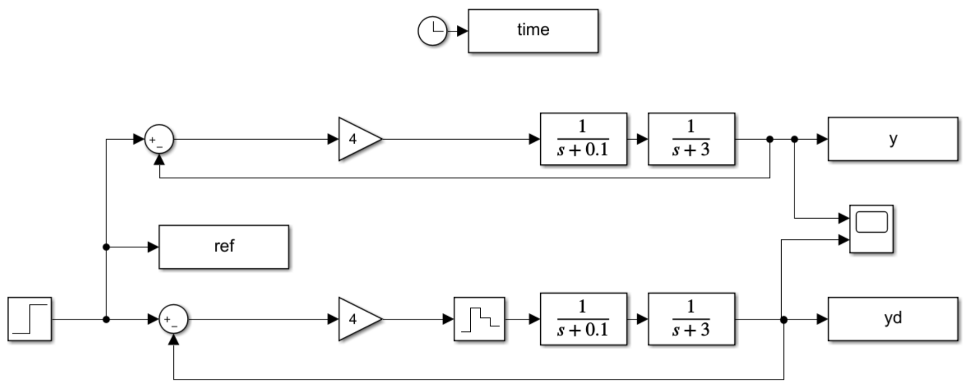

Figure 12: Problem 5: Possible simulink configuration.

(c) If we add the proposed controller and increase the simulation runtime we get the 'step responses shown in Figure 14. We can see that the steady state error disappears. This is due to the fact that the proposed controller is in fact a discrete-time integrator. Hence, by adding it to the system, we now have a system of type 1 in stead of type 0 and hence we can track a step reference without steady state error.

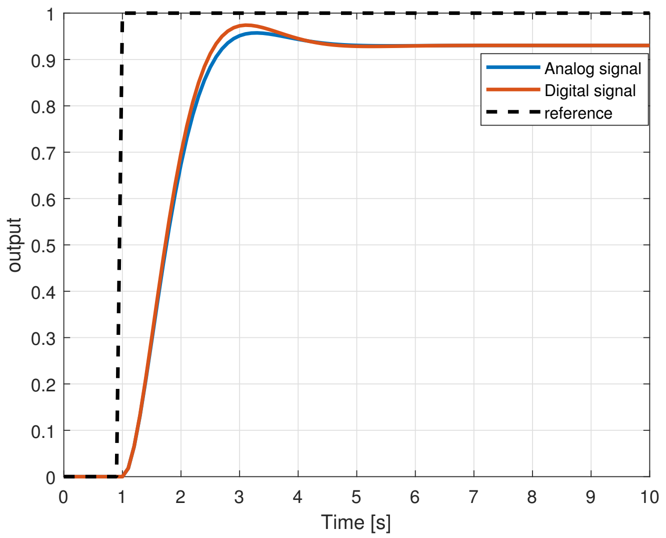

Figure 13: Problem 5: Step response for the analog and digital system.

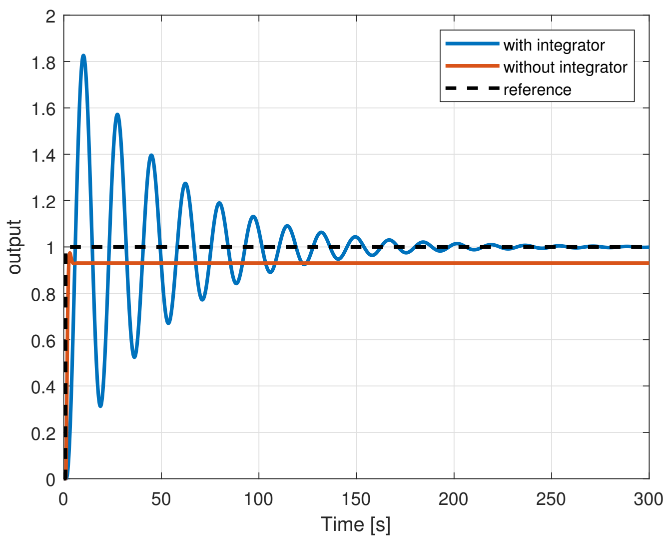

Figure 14: Problem 5: Step response for digital system with and without the proposed controller.