Root Locus drawing rules

Varying K changes dynamic response

$$\frac{D(s)G(s)}{1+D(s)G(s)} = \frac{KL(s)}{\underbrace{1+KL(s)}_{\text{Characteristic equation}}}$$

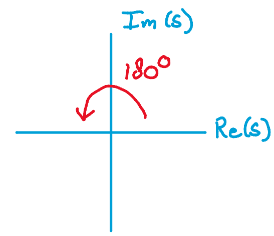
roots of $1+KL(s)=0$ → closed loop poles

$$b(s) = \prod_{i=1}^{m}(s-z_i) \qquad a(s) = \prod_{i=1}^{n}(s-p_i)$$

$$\left.\begin{aligned} &1+KL(s)=0 \\ &1+K\frac{b(s)}{a(s)}=0 \\ &a(s)+Kb(s)=0 \\ &L(s)=-1/K \end{aligned}\right\} \text{equivalent}$$

Key definitions

Definition I: $1+KL(s)=0$ must be satisfied under varying K, $K \in \underbrace{[0,\infty)}_{\mathbb{R}^+}$



$L(s) = -1/K$ $\qquad$ $Re^{st} = Re^{j\omega t}$ ($s=j\omega$ ← phase; ← amplitude)

Definition II: Root locus of $L(s)$ is the set of points in the s-plane where the phase of $L(s)$ is $180°$

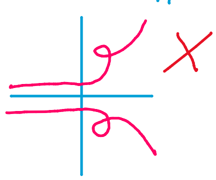
ℓ : integer value

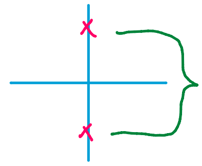
→ $\sum \psi_i - \sum \phi_i = 180° + 360°(\ell-1)$ ← easy to check!

$a(s) = s^n + a_1 s^{n-1} + \dots + a_n = 0$

hard to solve...

LTI system:





Complex conjugate behaviour

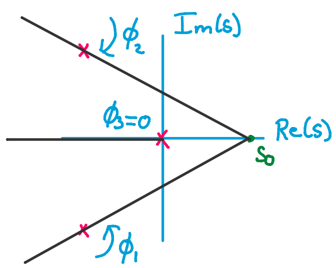
Drawing rules

Rule 1: n branches start at the poles of $L(s)$, m branches end at the zeros of $L(s)$

$$a(s) + K b(s) = 0 \quad \Rightarrow \quad \begin{cases} a(s) = 0 & K = 0 \quad \text{poles } (n) \\ b(s) = 0 & K \to \infty \quad \text{zeros } (m) \end{cases}$$

Strictly proper TF: n−m branches go to ∞

Rule 2: The loci are on the real axis to the left of an odd number of poles and zeros



S_0 does not satisfy Definition II

Rule 3: For large s and K, the loci that go to infinity are asymptotic lines radiating at a fixed angle from a central point, the centre of gravity of the poles.

$$\phi_\ell = \frac{180^\circ + 360^\circ(\ell-1)}{n-m} \qquad \ell = 1, 2, \ldots, n-m$$

loci going to infinity: n−m

angles at which the asymptotes branch out

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$ } only real parts needed!

Complex conjugate pair: $\sum c \pm di = c + di + c - di = 2c$

Centre of gravity of the asymptotes

Intermediate example

$$G(s) = \frac{s+1}{(s-1)(s^2+4s-8)} \qquad m=1 \qquad z_i=-1$$
$$n=3 \qquad p_i = 1,\ -2\pm 2j$$

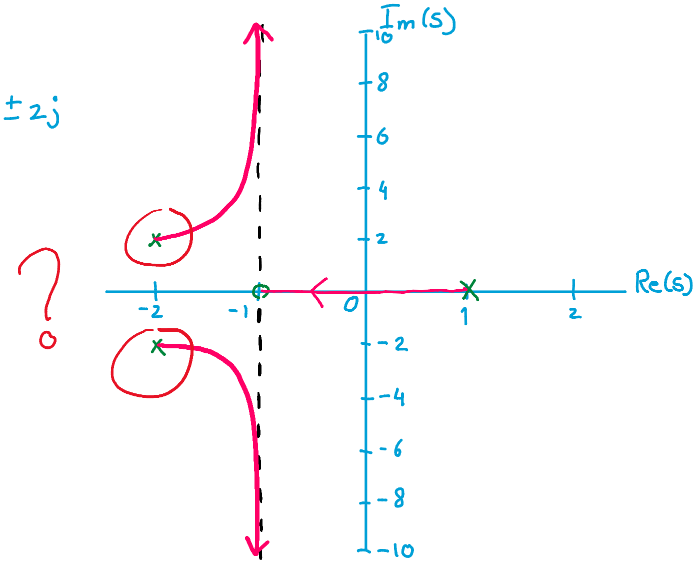
$D(s) = 1$

$1 + D(s)\,G(s) = 1 + kL(s) = 0 \qquad n-m=2$

$$\phi_\ell = \frac{180^\circ + 360^\circ(\ell-1)}{n-m} \qquad \ell = 1, \overset{=2}{n-m}$$

$$\phi_1 = \frac{180^\circ + 0^\circ}{2} = 90^\circ \qquad \phi_2 = \frac{180^\circ + 360}{2} = 270^\circ$$

$$\alpha = \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{1-2-2-(-1)}{3-1} = -1$$



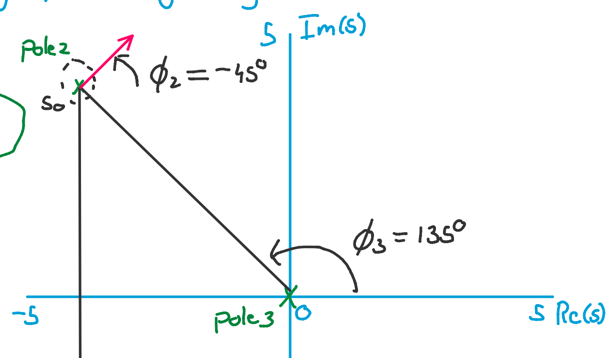
Rule 4: The departure angle of a branch of the locus from a single pole is given by

$$\phi_{dep} = \underbrace{\Sigma \psi_i}_{\text{sum of zeros}} - \underbrace{\Sigma_{i \neq dep} \phi_i}_{\text{sum of poles not considered}} - 180^\circ \quad \text{Definition II}$$

$$q\,\phi_{\ell,dep} = \Sigma \psi_i - \sum_{i \neq \ell, dep} \phi_i - 180^\circ - 360^\circ(\ell-1) \qquad \ell = 1, 2, \ldots, \overset{=1}{q}$$

q: repeated poles

$$q\,\psi_{\ell,arr} = \Sigma \phi_i - \sum_{i \neq \ell, arr} \psi_i + 180^\circ + 360^\circ(\ell-1) \qquad \ell = 1, 2, \ldots, q$$



$$\phi_2 = -90^\circ - 135^\circ - 180^\circ$$
$$= -405^\circ$$
$$= -45^\circ$$

Rule 5: The locus can have multiple roots at points on the locus and the branches approach a point of q roots with angles seperated by $\dfrac{180^\circ + 360^\circ(\ell-1)}{q}$

angles of departure and arrival: Continuation Locus

$$\left.\begin{aligned} G(s) &= \frac{s+2c}{s(s+c)} \\ D(s) &= 1 \end{aligned}\right\} \quad 1 + kL(s) = 1 + k\,\frac{s+2}{s(s+1)} = 0 \quad (=c)$$

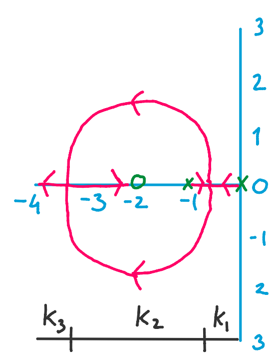
$$r_1, r_2 = -\frac{k+1}{2} \pm \frac{\sqrt{k^2 - 6k + 1}}{2} \qquad \text{Complex when } 0{,}172 < k < 5.828$$

$K = K_1$

$K = K_1 + K_2$

$K = K_1 + K_2 + K_3$

apply Rule 4 !



Summary

* Drawing rules for Root Locus → understanding closed-loop behaviour through its open-loop plant
* Two definitions 1: $1 + kL(s) = 0$ $\quad k \in \mathbb{R}^+$ $\quad$ 2: $\sum \psi_i + \sum \phi_i = 180^\circ + 360^\circ(\ell - 1)$
* Rule 1: Branches from pole to zero or pole to infinity

  Rule 2: Phase requirement, Loci on the real axis to the left of odd numbers of poles and zeros

  Rule 3: Asymptotic behaviour (angles + central point) $\phi_\ell = \dfrac{180^\circ + 360^\circ(\ell - 1)}{n-m}$ $\quad \ell = 1, 2, \ldots, n-m$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$

  Rule 4: Angles of departure and arrival

$q\,\phi_{\ell,dep} = \sum \psi_i + \sum_{i \neq \ell,dep} \phi_i - 180^\circ - 360^\circ(\ell - 1)$ $\quad q\,\psi_{\ell,arr} = \sum_{i \neq \ell,arr} \psi_i + \sum \phi_i + 180^\circ + 360^\circ(\ell - 1)$ $\quad \ell = 1, 2, \ldots, q$

Rule 5: Multiple roots at point on the Locus $\dfrac{180^\circ + 360^\circ(\ell - 1)}{q}$

↳ Continuation locus