Markov modeling, discrete-event simulation – Exercises module B3

5XIE0 Computational Modeling

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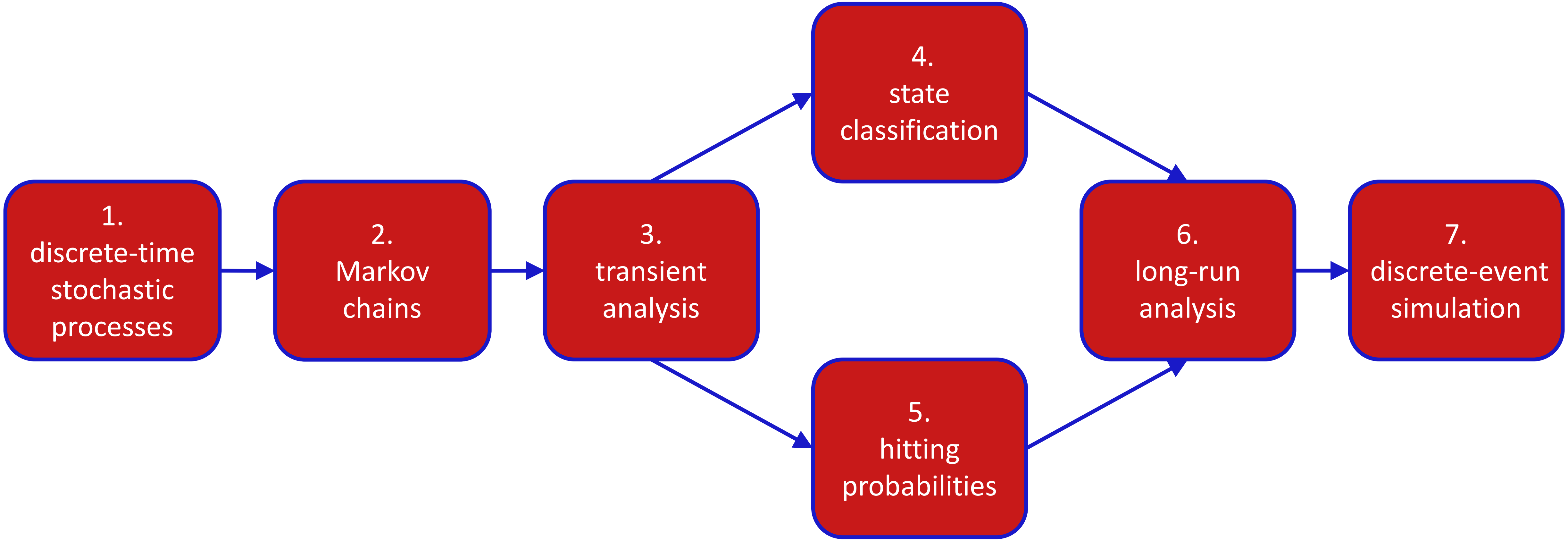
module B - submodules and dependencies



$$G_b = \begin{bmatrix} 1 & \infty & 2 \\ 1 & -\infty & 2 \\ -\infty & 3 & - \end{bmatrix}$$

B.3 – transient analysis

transient analysis– exercises

- Section B.3 in the course notes
  - Exercise B.9 (Probability distributions via matrix algebra)
    - use CMWB (DTMC) to answer (c) and (d)
      1. create the model corresponding to the given probability matrix
      2. select ‘Transient Distribution’ button and enter the number of steps to analyze

- answers are provided in Section B.8 of the course notes

transient analysis – exercises

- Section B.3 in the course notes
  - Exercise B.10 (N-step transition probabilities)
    - use CMWB (DTMC)
      1. create the model corresponding to the given probability matrix
      2. select 'Transient Matrix' and enter the number of steps (n) to compute the n-step transition matrix
  - Exercise B.11 (Gambler's ruin - n-step probabilities and expected reward)
    - use CMWB (DTMC) to answer (a) and (c); for (c)
      1. create a copy of the Gambler's ruin model and adapt the initial distribution
      2. selected 'Transient Rewards' and enter the number of steps to compute the expected reward
  - Exercise B.12 (Gambler's ruin – dependent and non-identically distributed variables)
    - use CMWB (DTMC)
  - Exercise B.13 (Queue in time-slotted communication network – modeling and transient analysis)
    - use CMWB (DTMC)
  - Exercise B.14 (Independent identically distributed variables as Markov chain)
- answers are provided in Section B.8 of the course notes

Exercise B.9 (Probability distributions via matrix algebra)

Exercise B.9 (Probability distributions via matrix algebra). Consider Markov chain $X_0, X_1, \cdots$ with initial distribution $\pi^{(0)} = [1, 0]$ and transition probability matrix

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix}$$

(a) Compute $P(X_1 = 1)$.

(b) Compute $P(X_2 = 1)$.

(c) Approximate $P(X_{20} = 1)$.

(d) Approximate the probability that the Markov chain is in state 1 after a very large number of transitions.

Exercise B.9 (Gambler's ruin - probability distributions via matrix algebra).

(a) $P(X_1 = 1) = \pi_1^{(1)} = (\pi^{(0)}P^1)_1 = ([1,0] \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix})_1 = [\frac{1}{3}, \frac{2}{3}]_1 = \frac{1}{3}$.

(b) $P(X_2 = 1) = \pi_1^{(2)} = (\pi^{(0)}P^2)_1 = ([1,0] \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix}^2)_1 = ([1,0] \cdot \begin{bmatrix} \frac{7}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix})_1$
$= [\frac{7}{9}, \frac{2}{9}]_1 = \frac{7}{9}$.

(c) $P(X_{20} = 1) = \pi_1^{(20)} = (\pi^{(0)}P^{20})_1 = ([1,0] \cdot \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix}^{20})_1 \approx ([1,0] \cdot \begin{bmatrix} 0.60 & 0.40 \\ 0.60 & 0.40 \end{bmatrix})_1$
$= [0.60, 0.40]_1 = 0.60$.

(d) For very large n, $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix}^n \approx \begin{bmatrix} 0.60 & 0.40 \\ 0.60 & 0.40 \end{bmatrix}$. Hence the probability that the Markov chain is in state 1 after a very large number of transitions is (approximately) 0.60.

$$\pi^{(n+1)} = \pi^{(n)}P \tag{B.14}$$

$$\pi^{(n)} = \pi^{(0)}P^n \tag{B.15}$$

Use CMWB (DTMC) to answer (c) and (d)

1. create the model corresponding to the given probability matrix
2. select 'Transient Distribution' button and enter the number of steps to analyze

Exercise B.10 (N-step transition probabilities)

Exercise B.10 (N-step transition probabilities). Consider a Markov chain with transition probability matrix

$$P = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ 1 & 0 \end{bmatrix}$$

(a) Approximate the probability that the chain transits from state 1 to state 2 in precisely 5 steps using the technique of matrix multiplication (see also equation (B.16)).

(b) Approximate the probability that the chain transits from state 1 to state 2 in precisely 5 steps, by summing of the probabilities of paths (see also equation (B.19)).

$$P^n_{ij} = P(X_{m+n} = j \mid X_m = i) \tag{B.16}$$

$$P^n_{ij} = \sum\{P(i, i_1, \cdots, i_{n-1}, j) \mid i, i_1, \cdots, i_{n-1}, j \text{ is a path of length } n\}^a \tag{B.19}$$

[a] In case different paths have the same probability, these probabilities have to be individually accounted for. Therefore this set is a multiset.

Use CMWB (DTMC)

1. create the model corresponding to the given probability matrix
2. select 'Transient Matrix' and enter the number of steps (n) to compute the n-step transition matrix

Exercise B.10 (N-step transition probabilities).

(a) The probability is given by $P^5_{12} = \begin{bmatrix} \frac{1281}{3125} & \frac{1844}{3125} \\ \frac{461}{625} & \frac{164}{625} \end{bmatrix}_{12} = \frac{1844}{3125} \approx \begin{bmatrix} 0.41 & 0.59 \\ 0.74 & 0.26 \end{bmatrix}_{12} = 0.59.$

(b) The probability is computed by summing the probabilities of all paths of from state 1 to state 2 of length 5. There exist five such paths:

- path $1, 1, 1, 1, 1, 2$ with probability $\frac{1}{5}^4 \cdot \frac{4}{5}$
- path $1, 1, 1, 2, 1, 2$ with probability $\frac{1}{5}^2 \cdot \frac{4}{5}^2$
- path $1, 1, 2, 1, 1, 2$ with probability $\frac{1}{5}^2 \cdot \frac{4}{5}^2$
- path $1, 2, 1, 1, 1, 2$ with probability $\frac{1}{5}^2 \cdot \frac{4}{5}^2$
- path $1, 2, 1, 2, 1, 2$ with probability $\frac{4}{5}^3$

Adding the probabilities yields $\frac{1844}{3125} \approx 0.59$, so as expected we obtain the same answer as in (a).

Exercise B.11 (Gambler's ruin – n-step probabilities and expected reward)

Exercise B.11 (Gambler's ruin - n-step probabilities and expected reward). Consider Examples B.4, B.5 and B.6 of the gambler's ruin.

(a) Approximate the probability that the gambler is broke after 100 spins, starting with an initial capital of €200.

(b) Compute the probability that the gambler has €200 cash after 1000 spins, starting with an initial capital of €100.

(c) Assume $\pi^{(0)} = [0, \frac{1}{2}, \frac{1}{2}, 0]$. Approximate the expected amount of cash the gambler is carrying after 16 spins of the wheel.

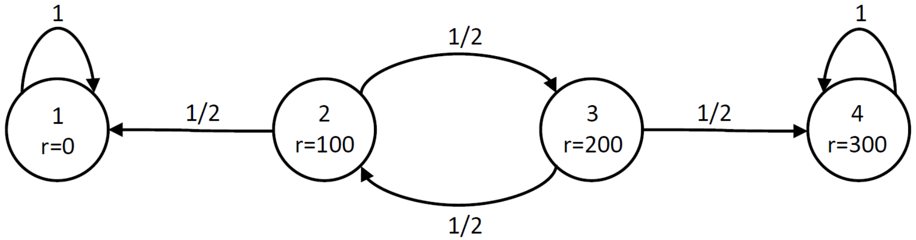


Figure B.3: Transition diagram of the gambler's ruin

Use CMWB (DTMC) to answer (a) and (c); for (c)

1. create a copy of the Gambler's ruin model and adapt the initial distribution
2. selected 'Transient Rewards' and enter the number of steps to compute the expected reward

Exercise B.11 (Gambler's ruin - n-step probabilities and expected reward).

(a) With an initial capital of €200, the Markov chain is in state 3. The gambler is broke in state 1. Therefore the answer is $P^{100}_{31} \approx 0.333$.

(b) Since there exist no paths of length 1000 from state 2 to state 3, this probability is 0.

(c) The expected amount of cash is given by (see equation (B.5)) $\pi^{(16)} r^T =$ { see equation (B.15) } $\pi^{(0)} P^{16} r^T \approx$ €150.

$$P^n_{ij} = P(X_{m+n} = j \mid X_m = i) \qquad \text{(B.16)}$$

$$P^n_{ij} = \sum \{P(i, i_1, \cdots, i_{n-1}, j) \mid i, i_1, \cdots, i_{n-1}, j \text{ is a path of length } n\}^a \qquad \text{(B.19)}$$

[a] In case different paths have the same probability, these probabilities have to be individually accounted for. Therefore this set is a multiset.

Exercise B.12 (Gambler's ruin – dependent and non-identically distributed variables)

Exercise B.12 (Gambler's ruin - dependent and non-identically distributed variables). Consider the Markov chain $X_0, X_1, \cdots$ of the gambler's ruin in Example B.4, where $\pi^{(0)} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$.

(a) Show that X_3 and X_6 are not identically distributed.

(b) Show that X_3 and X_6 are dependent variables.

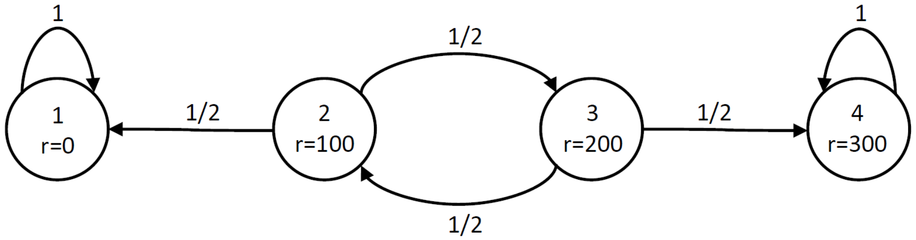


Figure B.3: Transition diagram of the gambler's ruin

Use CMWB (DTMC)

Exercise B.12 (Gambler's ruin - dependent and non-identically distributed variables).

(a) $\pi^{(3)} = \pi^{(0)}P^3 = [\frac{15}{32}, \frac{1}{32}, \frac{1}{32}, \frac{15}{32}] \approx [0.46875, 0.03125, 0.03125, 0.46875]$ and $\pi^{(6)} = \pi^{(0)}P^6 = [\frac{127}{256}, \frac{1}{256}, \frac{1}{256}, \frac{127}{256}] \approx [0.49609375, 0.00390625, 0.00390625, 0.49609375]$. These distributions are different and hence X_3 and X_6 are not identically distributed.

(b) We have to show (see equation (B.7)) that $P(X_6 = j \mid X_3 = i) \neq P(X_6 = j)$ for some $i, j \in \{1, 2, 3, 4\}$. Now $P(X_6 = j \mid X_3 = i) = \{$ see equation (B.16) $\}$ P^3_{ij}

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{5}{8} & 0 & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} & 0 & \frac{5}{8} \\ 0 & 0 & 0 & 1 \end{bmatrix}_{ij} \approx \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.625 & 0.000 & 0.125 & 0.250 \\ 0.250 & 0.125 & 0.000 & 0.625 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix}_{ij}$$

and $P(X_6 = j) = \pi^{(6)}_j$. Clearly $P^3_{ij} \neq \pi^{(6)}_j$ even for all $i, j \in \{1, 2, 3, 4\}$ and hence X_3 and X_6 are dependent.

$$\pi^{(n)} = \pi^{(0)}P^n \tag{B.15}$$

$$X_n \text{ and } X_m \text{ are independent if} \\ P(X_n = i \mid X_m = j) = P(X_n = i) \text{ for all } i, j \in \mathcal{S} \tag{B.7}$$

$$P^n_{ij} = P(X_{m+n} = j \mid X_m = i) \tag{B.16}$$

Exercise B.13 (Queue in time-slotted communication network – modeling and transient analysis)

Exercise B.13 (Queue in time-slotted communication network - modeling and transient analysis). Consider a queue in a time-slotted communication network. The queue can store up to 3 packets. During a time slot a packet is offered to the queue with probability $\frac{1}{3}$ in case the buffer is not full and retrieved from the queue with probability $\frac{1}{4}$ in case the buffer is not empty. Initially, the queue is empty.

(a) Model the behaviour of the queue as a Markov reward model and specify the probability matrix, the initial distribution, and the reward function. Consider as states the queue occupancy at the beginning of a time slot.

(b) Approximate the probability that the queue is full at the beginning of time slot 15.

(c) Assume that the queue contains 2 packets at the beginning of time slot 20. Approximate the probability that the queue is empty at the beginning of time slot 40.

(d) Approximate the expected queue occupancy at the beginning of time slot 5.

Exercise B.13 (Queue in time-slotted communication network - modeling and transient analysis).

(a) We can model the queue by a Markov chain $X_0, X_1, \cdots$ with state-space $\mathcal{S} = \{1, 2, 3, 4\}$ and reward $r : \mathcal{S} \to \mathbb{R}$ such that $r(i) = i-1$, where $r(X_i)$ denotes the number of packets present in the queue at the beginning of time slot i. The probability matrix is given by $P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{6} & \frac{7}{12} & \frac{1}{4} & 0 \\ 0 & \frac{1}{6} & \frac{7}{12} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$, and the initial distribution by $\pi^{(0)} = [1, 0, 0, 0]$. Here P_{23}, for instance, is the probability that the occupancy of the queue is increased from $r(2) = 1$ to $r(3) = 2$. This probability equals the probability that one packet is offered and no packet is retrieved, which is $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$.

(b) The probability that the queue is full at the beginning of time slot 15 is given by $\pi_4^{(15)} = \pi^{(0)} P_4^{15} \approx 0.29162365$.

(c) The answer equals P_{31}^{20}, i.e. the probability that the chain transits from state 3 to state 1 in 20 steps. $P_{31}^{20} \approx 0.109$.

(d) The expected occupancy is given by $\pi^{(5)} r^T = \pi^{(0)} P^5 r^T \approx 1.1189$.

Use CMWB (DTMC)

Exercise B.14 (Independent identically distributed variables as Markov chain)

Exercise B.14 (Independent identically distributed variables as Markov chain). Let $X_0, X_1, \cdots$ be a discrete-time stochastic process with state space $\{1, 2\}$ and independent identically distributed variables.

(a) Show that this process is in fact a Markov chain by construct a transition probability matrix that captures the indented behaviour.

(b) Show that the random variables of the Markov chain $X_0, X_1, \cdots$ defined in (a) are indeed independent and identically distributed.

Exercise B.14 (Independent identically distributed variables as Markov chain).

(a) Assume that $\pi^{(0)} = [p_1, p_2]$ where $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$. Define probability matrix $P = \begin{bmatrix} p_1 & p_2 \\ p_1 & p_2 \end{bmatrix}$.

(b) First note that $P^n = P$ for all $n = 1, 2, \cdots$. From this it follows that $\pi^{(n)} = \pi^{(0)} P^n = [p_1, p_2] P = [p_1, p_2]$ for all $n = 0, 1, \cdots$. Hence X_n and X_m are identically distributed for all $n, m \geq 0$. We still have to show that X_n and X_m are independent when $m \neq n$. For this assume $n = m + k$ for some $k > 0$. We have to show that $P(X_{m+k} = j \mid X_m = i) = P(X_{m+k} = j)$ for all $i, j \in \{1, 2\}$. Now $P(X_{m+k} = j \mid X_m = i) =$ {by using (B.16)} $P^k_{ij} = P_{ij} = \pi_j^{(m+k)} = P(X_{m+k} = j)$ which completes the proof.

$$P^n_{ij} = P(X_{m+n} = j \mid X_m = i) \qquad \text{(B.16)}$$