Appendix A

Formulae Electromagnetics II

A.1 Vector algebra

Let **a**, **b**, **c** and **d** denote arbitrary three-dimensional vectors. They satisfy the following identies

$$\begin{aligned}
&\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a},\\
&\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}),\\
&\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a}),\\
&\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c},\\
&(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}).
\end{aligned}$$

A.2 Differentiation of scalar and vector fields

Let $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ denote unit vectors in a cartesian system of coordinates. Φ is a scalar field, and **A** a vector field, satisfying the following relations

$$\begin{aligned}
\nabla\Phi &= \mathbf{a}_x\frac{\partial\Phi}{\partial x} + \mathbf{a}_y\frac{\partial\Phi}{\partial y} + \mathbf{a}_z\frac{\partial\Phi}{\partial z},\\
\nabla\cdot\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z},\\
\nabla\times\mathbf{A} &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}\\
&= \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right),\\
\Delta\Phi &= \nabla^2\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi.
\end{aligned}$$

A.3 Vector identities

Let us introduce a second scalar field Ψ and a second vector field **B**. We have

$$\begin{aligned}
\nabla(\Phi+\Psi) &= \nabla\Phi + \nabla\Psi,\\
\nabla\cdot(\mathbf{A}+\mathbf{B}) &= \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B},\\
\nabla\times(\mathbf{A}+\mathbf{B}) &= \nabla\times\mathbf{A} + \nabla\times\mathbf{B},
\end{aligned}$$

$$\begin{aligned}
\nabla(\Phi\Psi) &= \Phi\,\nabla\Psi + \Psi\nabla\Phi,\\
\nabla\cdot(\Phi\mathbf{A}) &= \Phi\,\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\Phi,\\
\nabla\times(\Phi\mathbf{A}) &= \Phi\,\nabla\times\mathbf{A} - \mathbf{A}\times\nabla\Phi,
\end{aligned}$$

$$\begin{aligned}
\nabla(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times\nabla\times\mathbf{A} + \mathbf{A}\times\nabla\times\mathbf{B},\\
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B},\\
\nabla\times(\mathbf{A}\times\mathbf{B}) &= (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + \mathbf{A}\nabla\cdot\mathbf{B},
\end{aligned}$$

$$\begin{aligned}
\nabla\times\nabla\times\mathbf{A} &= \nabla\nabla\cdot\mathbf{A} - \nabla^2\mathbf{A},\\
\nabla\cdot\nabla\times\mathbf{A} &= 0,\\
\nabla\times\nabla\Phi &= \mathbf{0},
\end{aligned}$$

$$\begin{aligned}
(\nabla\times\mathbf{A})\times\mathbf{A} &= (\mathbf{A}\cdot\nabla)\mathbf{A} - \tfrac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A}),\\
\nabla_{\mathrm{t}}\cdot(\Phi\nabla_{\mathrm{t}}\Phi) &= \Phi\nabla_{\mathrm{t}}^2\Phi + (\nabla_{\mathrm{t}}\Phi)\cdot(\nabla_{\mathrm{t}}\Phi).
\end{aligned}$$

A.4 Integral relations

For an arbitrary volume $\mathcal{V}$, bounded by a surface $\mathcal{S} = \partial\mathcal{V}$ with unit normal $\mathbf{n}$ pointing outward from $\mathcal{V}$, we have

$$\begin{aligned}
\int_{\mathcal{V}} \nabla\cdot\mathbf{A}\,\mathrm{d}V &= \int_{\mathcal{S}} \mathbf{A}\cdot\,\mathrm{d}\mathbf{S}, \qquad \text{(Gauss' theorem)}\\
\int_{\mathcal{V}} \nabla\Phi\,\mathrm{d}V &= \int_{\mathcal{S}} \Phi\,\mathrm{d}\mathbf{S},\\
\int_{\mathcal{V}} \nabla\times\mathbf{A}\,\mathrm{d}V &= -\int_{\mathcal{S}} \mathbf{A}\times\,\mathrm{d}\mathbf{S}.
\end{aligned}$$

By combining Gauss' law and a relation from Section A.3, one may deduce that

$$\int_{\mathcal{V}} (\Phi\Delta\Psi - \Psi\Delta\Phi)\,\mathrm{d}V = \int_{\mathcal{S}} (\Phi\nabla\Psi - \Psi\nabla\Phi)\cdot\,\mathrm{d}\mathbf{S},$$

(Green's theorem).

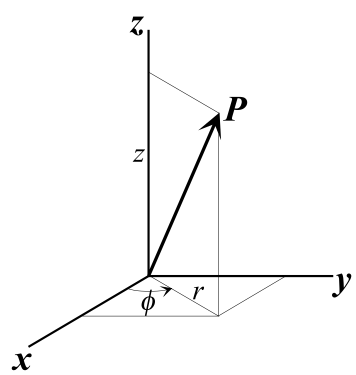
For an orientable surface $\mathcal{S}$ with normal $\mathbf{n}$ bounded by a boundary curve $\mathcal{C} = \partial\mathcal{S}$ with tangent vector $\boldsymbol{\tau}$ and outward normal vector $\boldsymbol{\nu}$, we have

$$\begin{aligned}
\int_{\mathcal{S}} (\nabla\times\mathbf{A})\cdot\,\mathrm{d}\mathbf{S} &= \oint_{\mathcal{C}} \mathbf{A}\cdot\,\mathrm{d}\boldsymbol{\ell}, \qquad \text{(Stokes' theorem)}\\
\int_{\mathcal{S}} \nabla_{\mathrm{t}}\cdot\mathbf{A}\,\mathrm{d}S &= \oint_{\mathcal{C}} \mathbf{A}\cdot\boldsymbol{\nu}\,\mathrm{d}\ell. \qquad \text{(Gauss' theorem in two dimensions)}
\end{aligned}$$

Here, $\boldsymbol{\nu}$ is the normal to the boundary $\mathcal{C}$.

A.5 Coordinate systems

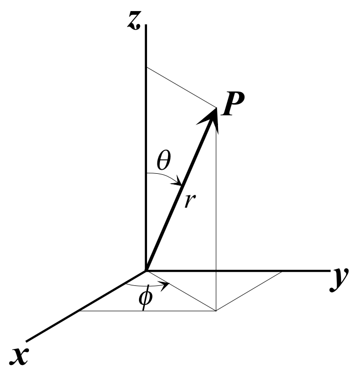
Cylindrical polar coordinates (r, φ, z)



Coordinates: $x = r\cos(\varphi)$, $y = r\sin(\varphi)$, $z = z$.
Line elements: $\mathrm{d}r$, $r\,\mathrm{d}\varphi$, $\mathrm{d}z$.
Vector operators:

$$
\begin{aligned}
\nabla\Psi &= \mathbf{a}_r \frac{\partial \Psi}{\partial r} + \mathbf{a}_\varphi \frac{1}{r}\frac{\partial \Psi}{\partial \varphi} + \mathbf{a}_z \frac{\partial \Psi}{\partial z}, \\
\nabla \cdot \mathbf{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}, \\
\nabla \times \mathbf{A} &= \mathbf{a}_r \left[\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right] + \mathbf{a}_\varphi \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] + \mathbf{a}_z \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial \varphi}\right], \\
\nabla^2\Psi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \varphi^2} + \frac{\partial^2 \Psi}{\partial z^2}.
\end{aligned}
$$

Spherical polar coordinates (r, θ, φ)



Coordinates: $x = r\sin(\theta)\cos(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$, $z = r\cos(\theta)$.
Line elements: $\mathrm{d}r$, $r\,\mathrm{d}\theta$, $r\sin(\theta)\,\mathrm{d}\varphi$.
Vector operators:

$$
\begin{aligned}
\nabla\Psi &= \mathbf{a}_r \frac{\partial \Psi}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial \Psi}{\partial \theta} + \mathbf{a}_\varphi \frac{1}{r\sin(\theta)}\frac{\partial \Psi}{\partial \varphi}, \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}[\sin(\theta)A_\theta] + \frac{1}{r\sin(\theta)}\frac{\partial A_\varphi}{\partial \varphi}, \\
\nabla \times \mathbf{A} &= \mathbf{a}_r \frac{1}{r\sin(\theta)}\left\{\frac{\partial}{\partial \theta}[\sin(\theta)A_\varphi] - \frac{\partial A_\theta}{\partial \varphi}\right\} + \\
&\quad \mathbf{a}_\theta \left[\frac{1}{r\sin(\theta)}\frac{\partial A_r}{\partial \varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right] + \mathbf{a}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right], \\
\nabla^2\Psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 \Psi}{\partial \varphi^2}.
\end{aligned}
$$

A.6 Miscellaneous formulae

Taylor approximations:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4), \\
\sin x &= x - \frac{x^3}{6} + O(x^5), \\
\cos x &= 1 - \frac{x^2}{2} + O(x^4).
\end{aligned}
$$

Geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x| < 1.$$

A.7 Reflection and refraction

$$\theta_1 = \theta_1' \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{k_2}{k_1}$$

$$\Gamma^s = \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_1\cos\theta_2} \qquad \tau^s = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2}$$

$$\Gamma^p = \frac{Z_2\cos\theta_2 - Z_1\cos\theta_1}{Z_2\cos\theta_2 + Z_1\cos\theta_1} \qquad \tau^p = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_2 + Z_1\cos\theta_1}$$

$$\tan^2\theta_{\mathrm{B}} = \frac{\varepsilon_2}{\varepsilon_1} \qquad \sin\theta_{\mathrm{c}} = \frac{\mathrm{n}_2}{\mathrm{n}_1}$$

A.8 Waveguides

$$E_x = \frac{-j\omega\mu}{k_t^2}\partial_y H_z + \frac{1}{k_t^2}\partial_z\partial_x E_z \quad H_y = \frac{1}{k_t^2}\partial_z\partial_y H_z + \frac{-j\omega\varepsilon}{k_t^2}\partial_x E_z$$

$$E_y = \frac{j\omega\mu}{k_t^2}\partial_x H_z + \frac{1}{k_t^2}\partial_z\partial_y E_z \quad H_x = \frac{1}{k_t^2}\partial_z\partial_x H_z + \frac{j\omega\varepsilon}{k_t^2}\partial_y E_z$$

$$= \sqrt{k_t^2 - k^2}$$

Where κ_t is either κ_m, κ_{mp} or κ_{mn} depending on the problem under consideration.

$$\frac{\omega_m}{c} = \frac{m\pi}{a}$$

$$\frac{\omega_{mn}}{c} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\frac{\omega_{mnp}}{c} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

A.9 Antennas

$$\nabla\cdot\mathbf{A} + j\omega\varepsilon\mu\Psi = 0$$

$$G_d = \frac{\overline{S_h}(\theta,\phi)_{\max}}{\overline{P_h}/(4\pi r^2)}$$

$$F_{\text{array}} = \frac{\sin^2\left(\frac{N\Psi}{2}\right)}{\sin^2\left(\frac{\Psi}{2}\right)} \qquad \Psi = \beta d\cos\varphi + \alpha$$

$$R_{\text{rad}} = \frac{1}{|I_0|^2} \quad \oiint_{S_{\text{gen}}} \mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{a}_r\,\mathrm{d}A$$