1. **Design - Lead Compensator:**
Consider the third-order system

$$G(s) = \frac{21000}{s(s+1)(s+70)}.$$

By using the Bode plot sketches, design a lead compensator such that the closed-loop system has PM $\geq 50°$ and $\omega_{\text{bw}} \geq 20$ rad/sec. Verify and refine your design using MATLAB.

solution:
By using the guideline in the appendix at the end of the document, we will design a Lead compensator of the form

$$D(s) = K\frac{1+Ts}{1+\alpha Ts},$$

with the design of the parameters $\alpha \in (0,1), T > 0$, and K.

(a) Step 1 can be skipped as there are no error requirements imposed.

(b) For step 2, we start by analyzing the Bode plots for the open-loop transfer function $G(s)$, see Figure 1. According to step 2 in the guideline, we must pick K such that the open

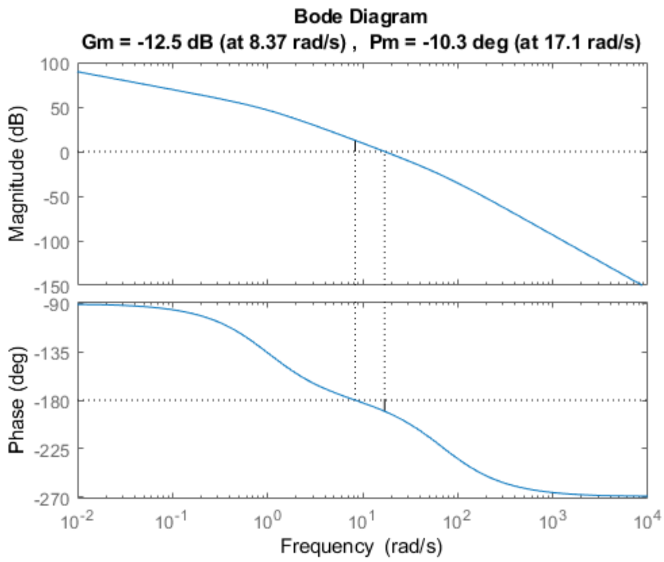

Figure 1: Bode plot for uncompensated system in Problem 1

loop crossover frequency is a factor 2 below the desired closed-loop bandwidth. Given that the crossover frequency, the frequency at which the magnitude plot crosses 0dB, $\omega_c < 20$ rad/s, we must increase this such that the bandwidth requirement is met: $\omega_{bw} \geq 20$ rad/s. The magnitude at 20 rad/s is $|G(20j)| = -2.85$ dB, hence we need to increase our gain by $2.85\text{dB} = 1.3884$.

(c) For Step 3, we evaluate the phase margin (PM) of the uncompensated system using the value for K and any added integrators from step 1 and 2. Since our phase at ω_c is $-193°$ we have PM $= -13°$. The aim is to keep the crossover frequency at 20 rad/s, whilst increasing the phase by at least $63°$ at ω_c.

(d) In step 4, we choose the required phase lead ϕ_{max}. We impose some conservativity by choosing our maximum phase increase $\phi_{max} = 75°$ at $\omega_{max} = 20$rad/s.

(e) For step 5, we use the required phase from step 4. Since

$$\sin(\phi_{max}) = \frac{1-\alpha}{1+\alpha}$$

we have that

$$\alpha = \frac{1-\sin(\phi_{max})}{1+\sin(\phi_{max})}$$

and thus $\alpha = 0.0173$.

(f) For step 6, we position the pole-zero pair by selecting T such that ϕ_{max} occurs at $\omega_{max} = \omega_c = 20$ rad/s. By

$$\omega_{max} T = \frac{1}{\sqrt{\alpha}}$$

we obtain $T = 0.3798$

(g) Finally, we find K such that $|KD(j\omega_c)G(j\omega_c)| = 1$. Specifically, we need an attenuation of 17.6 dB, that is $K = 0.1317$. Resulting in the final Lead compensator

$$D_{lead}(s) = 1.3884 \cdot 0.1317 \frac{1 + 0.3798s}{1 + 0.0173 \cdot 0.3798s}$$

We can verify from the Bode plot in Figure 2 that we meet our specifications using the lead compensator.

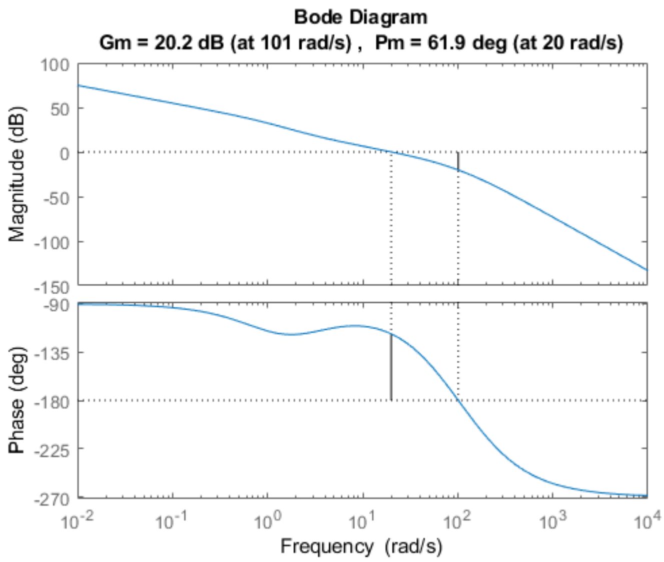

Figure 2: Bode plot for compensated system in Problem 1

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2. **Design - Lag Compensator:**
We can follow the guidelines given in the Appendix (last page). For a system with open-loop transfer function

$$G(s) = \frac{10}{s\left(1 + \frac{s}{1.4}\right)\left(1 + \frac{s}{3}\right)}$$

design a lag compensator with unitary DC gain such that PM $\geq 40°$.

Solution:
We first plot the Bode diagrams for the open-loop transfer function $G(s)$ in Figure 3. We see that PM $\simeq -20°$ with crossover frequency $\omega_c = 3$ rad/s. Moreover, we have GM $= -7.13$ dB at $\omega_{-180} = 2.05$ rad/sec.

From the Bode plot, we note that at frequency $\hat{\omega}_c = 0.5$ rad/sec, we have $|G(j\hat{\omega}_c)| \simeq 25$ dB and $\angle(G(j\hat{\omega}_c)) \simeq -120°$, which is $60°$ above the critical phase $-180°$.

Therefore, we might place the pair pole-zero at low frequencies with the aim of moving the crossover frequency from ω_c to $\hat{\omega}_c = 0.5$ rad/sec, while keeping the phase almost unchanged.

To achieve this, we design a lag compensator

$$D(s) = \frac{1 + sT}{1 + sT\alpha}$$

that has unitary DC gain as $D(0) = 1$, for $\alpha > 1$ and $T > 0$.

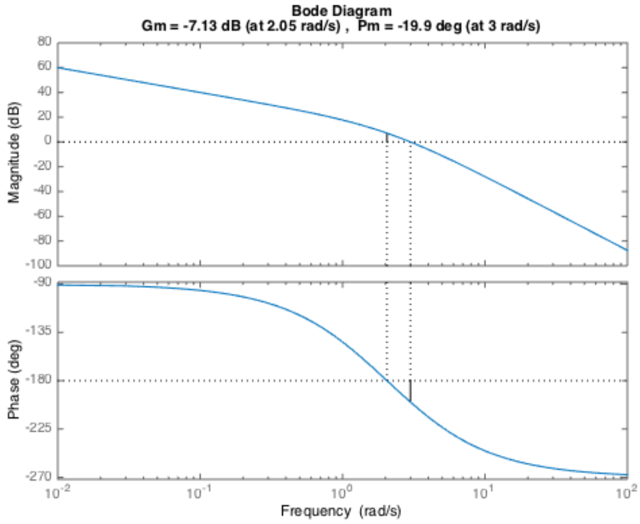


Figure 3: Bode plot for $G(s)$ in Problem 2.

We have to select α for reducing the amplitude of 25 dB, which is equal to the factor 0.0562. Since $\lim_{s\to\infty} D(s) = 1/\alpha$, we impose $\frac{1}{\alpha} = 0.0562$, hence $\alpha = 17.7936$. Then we select T such that the pair pole-zero is 1 decade left to $\hat{\omega}_c$, hence $1/T = 0.05$ and equivalently $T = 20$.

The resulting lag compensator is

$$D(s) = \frac{1 + 20s}{1 + 355.8720s}$$

The Bode plot of $D(s)G(s)$ in figure 4 shows that we achieve PM $= 54.6°$ with crossover frequency 0.521 rad/s.

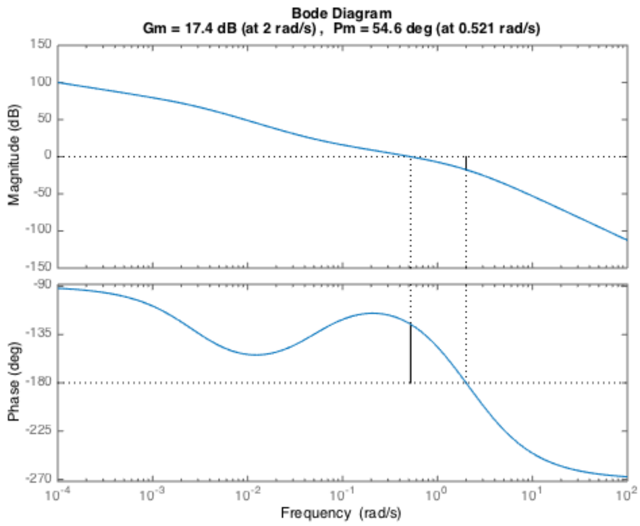


Figure 4: Bode plot for $D(s)G(s)$ in Problem 2.

3. **Pole at the origin:**
Derive the transfer function from the disturbance torque, $T_d(s)$, to the attitude of a spacecraft, $\Theta(s)$, for the system in Figure 5.

Next assume that the torque T_d is constant, that is, $T_d(s) = |T_d|\frac{1}{s}$. Apply the Final Value Theorem for determining whether $\lim_{t\to\infty} \theta(t)$ is nonzero for the following two cases:

a) $D(s)$ has no integral term, i.e., $\lim_{s\to 0} D(s) = A \in \mathbb{R}$;

b) $D(s)$ has an integral term, i.e., $D(s) = \frac{\tilde{D}(s)}{s}$, for some $\tilde{D}(s)$ such that $\lim_{s\to 0} \tilde{D}(s) = A \in \mathbb{R}$.

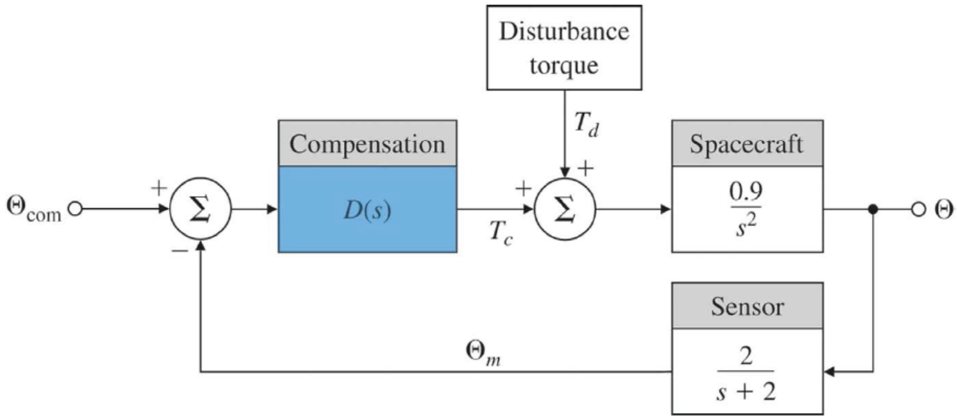


Figure 5: Block diagram, Problem 3.

Solution:
For computing the transfer function from T_d to Θ we have to consider $\Theta_{\text{com}} = 0$. Then we have that

$$\begin{aligned}\Theta(s) &= \frac{0.9}{s^2}\left(T_d(s) + D(s)\left(-\Theta_m(s)\right)\right)\\ &= \frac{0.9}{s^2}T_d(s) - \frac{0.9}{s^2}\frac{2}{s+2}D(s)\Theta(s)\end{aligned}$$

Thus, the transfer function from T_d to Θ is

$$\frac{\Theta(s)}{T_d(s)} = \frac{\frac{0.9}{s^2}}{1 + \frac{0.9}{s^2}\frac{2}{s+2}D(s)}.$$

a) By the FVT, we have

$$\begin{aligned}\lim_{t\to\infty}\theta(t) &= \lim_{s\to 0} s\,\Theta(s)\\ &= \lim_{s\to 0}|T_d|\frac{\frac{0.9}{s^2}}{1 + \frac{0.9}{s^2}\frac{2}{s+2}D(s)}\\ &= \lim_{s\to 0}\frac{|T_d|}{D(s)} = \frac{|T_d|}{A} \neq 0.\end{aligned}$$

Therefore, there will be a steady-state error in $\theta(t)$ for constant disturbance T_d.

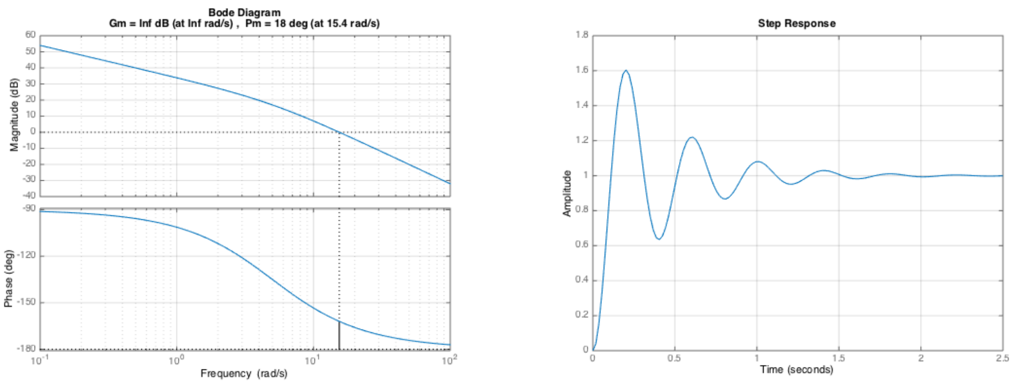
b) By applying the FVT as above, this time with $D(s) = \frac{\tilde{D}(s)}{s}$, we obtain that

$$\lim_{t\to\infty}\theta(t) = \lim_{s\to 0} s\Theta(s) = \lim_{s\to 0}\frac{|T_d|}{D(s)} = \lim_{s\to 0} s\frac{|T_d|}{\tilde{D}(s)} = 0.$$

4. **Reduced overshoot:**
A DC motor with negligible armature inductance is to be used in a position control system. Its open-loop transfer function is given by

$$G(s) = \frac{50}{s\,(1 + s/5)}.$$

The open-loop transfer function has $\omega_c = 15.4$ rad/s, see Figure 6(a). When we close the loop with $D(s) = 1$, we obtain 60% overshoot in the step response, see Figure 6(b).



(a) Bode plot for $G(s)$. (b) Step response of $\frac{G(s)}{1+G(s)}$.

Figure 6: Problem 4.

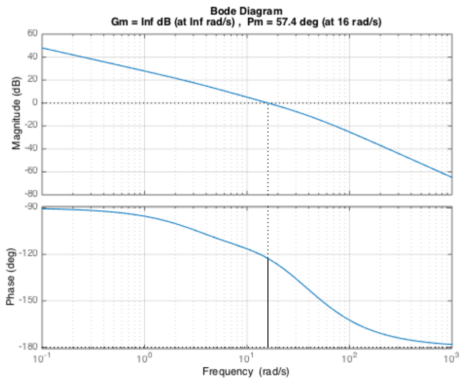
Design a controller such that the step response of the closed-loop maximum overshoot 20%, and crossover frequency larger than 15 rad/s.

Solution:
The specification on the maximum overshoot M_{p} can be translated in terms of phase margin (PM). Specifically, from Lecture 4, it follows that for $M_{\mathrm{p}} \leq 0.2$, we need PM $\geq 50°$. Thus, we can design a lead compensator

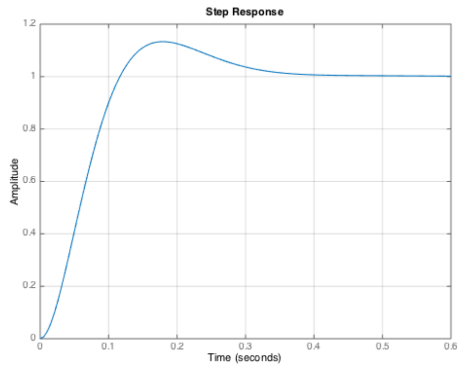
$$D(s) = K\frac{Ts+1}{\alpha Ts+1}$$

for adding $\phi_{\max} = 40°$ of phase at some new crossover frequency, e.g. $\omega_{\max} = 16$ rad/s, that is higher than that of the uncompensated system. Then we have $\alpha = \frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})} = 0.2174$, $T = \frac{1}{\omega_{\max}}\frac{1}{\sqrt{\alpha}} = 0.1340$. Finally, we tune $K = 0.5003$ such that $|D(j\omega_{\max})G(j\omega_{\max})| = 1$.

We obtain the Bode plot and the step response for the compensated open-loop transfer function $D(s)G(s)$ shown in Figures 7(a), 7(b).



(a) Problem 4: Bode plot for $D(s)G(s)$.



(b) Problem 4: Step response of $\frac{D(s)G(s)}{1+D(s)G(s)}$.

Figure 7: Problem 4.

5. **Design - Compensators (Exam Level Question):**
Again consider the feedback control scheme in Figure 8, this time with a plant that has the transfer function:

$$G(s) = \frac{s+2}{(s+4)(s+25)} \tag{1}$$

and consider the following performance specifications:

i) A phase margin of at least 30 degrees.

ii) A crossover frequency of $\omega_c > 100$ rad/sec.

iii) Zero steady-state tracking error $e(t) = r(t) - y(t)$ for a ramp input given by

$$r(t) = \begin{cases} t & \text{for } t > 0, \\ 0 & \text{for } t \leqslant 0, \end{cases} \tag{2}$$

for the situation that $d(t) = n(t) = 0$,

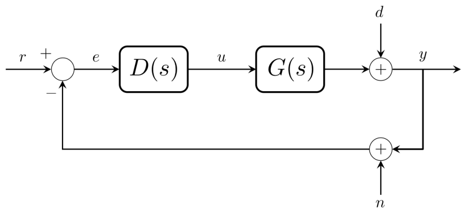


Figure 8: Feedback interconnection.

a) How many integrators need to be added to satisfy requirement iii)? Make a sketch of the corresponding bode diagram of $L = D_1(s)G(s)$, with $D_1 = s^{-n}$

b) Consider now the system $L(s) = D_2(s)D_1(s)G(s)$. How much additional phase and gain does $D_2(s)$ need to provide to satisfy requirement i) and ii)?

c) Design the controller $D(s) = D_1(s)D_2(s)$, such that the system satisfies all requirements.

Solution:

a) For the specified ramp input, in order to ensure zero steady-state tracking error, we need 2 integrators. So $D_1(s) = s^{-2}$. The bode diagram of $D_1(s)G(s)$ is depicted in Figure 9.

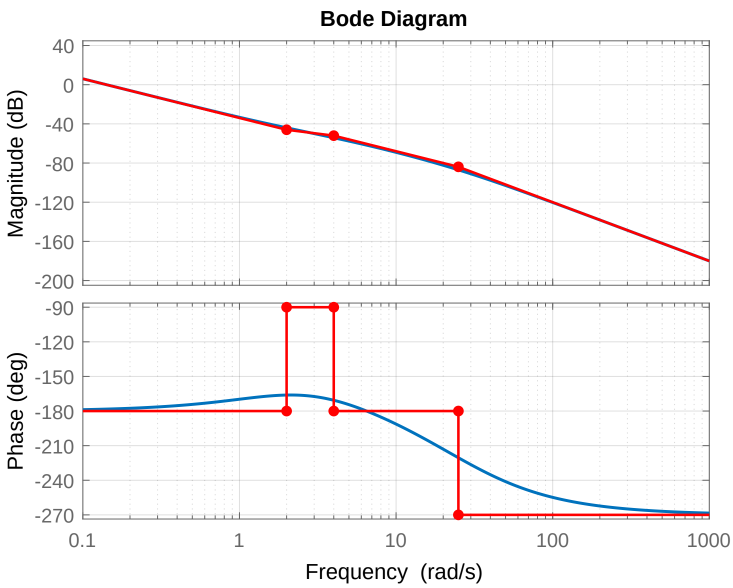


Figure 9: Bode diagram of $D_1(s)G(s)$

b) The transfer function L(s) is given by: $L(s) = D_1(s)G(s) = \frac{s+2}{s^2(s+4)(s+25)}$.
Evaluating the angle at 100 rad/s,

$$\angle L_1(j100) = \angle \frac{100j+2}{(100j)(100j)(100j+4)(100j+25)} = -255,96^\circ \approx -256^\circ$$

there is -256° at $\omega_c = 100$rad/s. We need $-180^\circ + 30^\circ = -150^\circ$at that point. So for additional phase, we require $(-150^\circ) - (-256^\circ) = 106^\circ$. We add 10° for safety, so 116° additional phase.

For the additional gain,

$$|L_1(j100)| = \left| \frac{-100j+2}{(100j)(100j)(10j+4)(100j+25)} \right| = 9,6956 \cdot 10^{-7}$$
$$\approx -120 \text{ dB}$$

Hence we need to add 120 dB, so K would be $10^{\frac{120}{20}} = 10^6$.

c) We have, $D_1(s) = \frac{1}{s^2}$ and we need two lead compensators to reach the required phase margin, so $D_2(s) = \left(\frac{T_d \cdot s+1}{\alpha T_d \cdot s+1}\right)^2 \cdot K$, with $K = 10^{\frac{120}{20}} = 10^6$ as calculated before.
The required additional phase is 106°, 116° for safety with one lead compensator. Since we require two compensators, the additional phase is set to 126° for safety. $126/2 = 63^\circ$ per compensator.

$$\alpha = \frac{1-\sin 63^\circ}{1+\sin 63^\circ} = 0,0576 \quad T_d = \frac{1}{\omega_c \sqrt{\alpha}} = \frac{1}{100\sqrt{0,0576}} = 0,042$$
$$D_2(s) = 10^6 \cdot \left(\frac{0,042 \cdot s + 1}{0,0576 \cdot 0,042 \cdot s + 1}\right)^2$$
$$D(s) = D_1(s) \cdot D_2(s) = \frac{10^6}{s^2} \cdot \left(\frac{0,042 \cdot s + 1}{0,0576 \cdot 0,042 \cdot s + 1}\right)^2$$

6. **Lead-Lag Compensator design:**
Consider the system in Figure 10 with transfer function

$$G(s) = \frac{10}{s(1+\frac{s}{10})}.$$

We wish to design a compensator $D(s)$ that satisfies the following design specifications:

- velocity constant $K_\text{v} = \lim_{s\to 0} sKG(s) = 100$,
- PM $\geq 45°$,
- sinusoidal inputs with frequency of up to 1 rad/sec, to be reproduced with less than 2% error,
- sinusoidal inputs with frequency higher than 100 rad/sec to be attenuated at the output to less than 5% of their input amplitude.

To do so,

i. Draw the Bode diagrams for $G(s)$, choosing the open-loop gain K such that $K_\text{v} = 100$.

ii. Show that for meeting the specifications on sinusoidal inputs, the magnitude plot of the open-loop transfer function $K\,D(s)G(s)$ shall lie outside the shaded regions in Figure 11. Recall the following transfer functions. Input-Output: $\frac{Y(s)}{R(s)} = \frac{KD(s)G(s)}{1+KD(s)G(s)}$; Input-Error: $\frac{E(s)}{R(s)} = \frac{1}{1+KD(s)G(s)}$.

iii. Explain why a lead compensator alone cannot meet the design specifications.

iv. Explain why a lag compensator alone cannot meet the design specifications.

v. Using MATLAB, design a lead-lag compensator for satisfying all the specifications, without altering the previously chosen low-frequency open-loop gain.

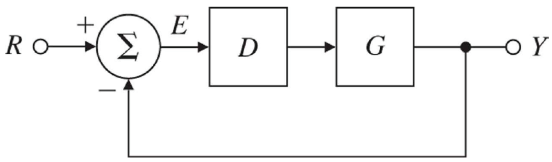


Figure 10: Block scheme for Problem 5.

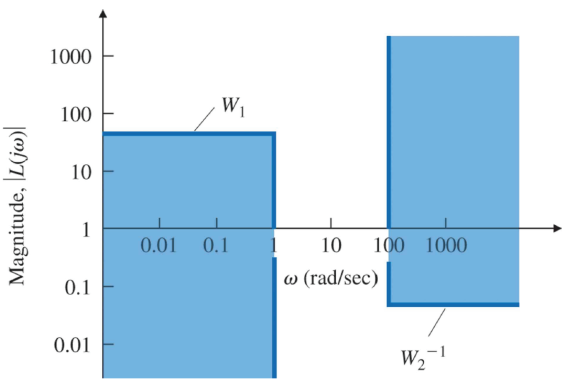


Figure 11: Specifications on sinusoidal inputs for Problem 5 ($L(s) = KD(s)G(s)$).

Solution:

i. We have that $K_\mathrm{v} = \lim_{s\to 0} sKG(s) = 10K = 100$, hence $K = 10$. The Bode plot of $K\,G(s) = 10\,G(s)$ is shown in Figure 12.

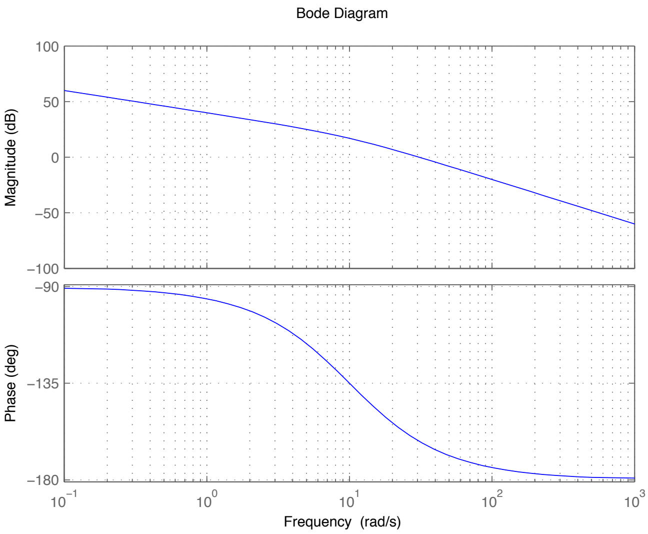


Figure 12: Bode plot of $KG(s)$, Problem 5.

ii. From the specification (iii)—i.e. sinusoidal inputs with frequency of up to 1 rad/sec to be reproduced with less than 2% error with respect to their amplitude, we impose that $|\frac{E(j\omega)}{R(j\omega)}| = |\frac{1}{1+KD(j\omega)G(j\omega)}| < 0.02$ for all $\omega \leq 1$ rad/sec. In order to impose a specification on $|KD(j\omega)G(j\omega)|$, we can make the following approximation. Observe that at low frequencies we can have that $|KD(j\omega)G(j\omega)| \gg 1$, and therefore, $|\frac{1}{1+KD(j\omega)G(j\omega)}| \approx |\frac{1}{KD(j\omega)G(j\omega)}|$. Hence, we need $|KD(j\omega)G(j\omega)| > 50 \quad (\approx 34\text{db})$ for all $\omega \leq 1$ rad/sec.

For the specification (iv)—i.e. sinusoidal inputs with frequency higher than 100 rad/sec to be attenuated at the output to less than 5% of their input amplitude, we impose that $\frac{Y(j\omega)}{R(j\omega)} = |\frac{K\,D(j\omega)G(j\omega)}{1+KD(j\omega)G(j\omega)}| < 0.05$ for all $\omega \geq 100$ rad/sec. Observe that, at high frequencies we can have $|KD(j\omega)G(j\omega)| \ll 1$ and therefore $\left|\frac{K\,D(j\omega)G(j\omega)}{1+KD(j\omega)G(j\omega)}\right| \approx |KD(j\omega)G(j\omega)|$. Hence, we need $|KD(j\omega)G(j\omega)| < 0.05 \quad (\approx -26\text{db})$ for all $\omega \geq 100$ rad/sec. See the illustration of these concepts Figure 11.

iii. A lead compensator may provide a sufficient PM, but it would increase the gain at high frequency, which would violate the high-frequency specification.

iv. A lag compensator could satisfy the PM specifications by lowering the crossover frequency, but it would violate the low-frequency specification.

v. A possible lead-lag compensator which meets all the specifications is

$$K\,D(s) = 10\frac{(\frac{s}{8.52}+1)(\frac{s}{4.47}+1)}{(\frac{s}{22.36}+1)(\frac{s}{0.568}+1)}.$$

The Bode plot of the compensated open-loop transfer function is shown in Figure 13.

A compensator can roughly be designed by hand, using the following steps:

First we observe that the gain of $KG(s)$ is roughly 40 dB at $\omega = 1$, while the requirement from (iii.) implies a minimum gain of roughly 34 dB. Therefore, it is still possible to reduce the gain at this frequency by a value of 6 dB.

The gain of $KG(s)$ is roughly 20 dB at $\omega = 10$ since we have a slope of $-20\ \frac{\text{dB}}{\text{decade}}$. Furthermore, the slope for $\omega \geq 10$ will be $-40\ \frac{\text{dB}}{\text{decade}}$ and therefore at $\omega = 100$ the gain of $KG(s)$ is roughly

-20 dB, while the requirement states that the gain should be below -26 dB. We therefore have to reduce the gain at this frequency by a value of 6 dB.

The slope for $\omega \geq 10$ will be $-40\ \frac{\text{dB}}{\text{decade}}$, or $-12\ \frac{\text{dB}}{\text{octave}}$. Considering that the gain of $KG(s)$ is roughly 20 dB at $\omega = 10$, we therefore expect that ω_c would be about $\frac{20}{12} = 1.667$ octaves away from $\omega = 10$, which gives us an *estimate* for $\omega_c \approx 10 \cdot 2^{1.667} = 31.75$. For ω_c the phase of $KG(s)$ is $-162.5°$, which amounts to a PM of 17.5°.

Using these observations we expect that we need to apply a lead compensator in order to meet the PM requirement of at least 45°. This lead compensator can, for example, be set to have $\alpha = \frac{1}{5}$, and consequently will introduce an increased gain at high frequencies of $\frac{1}{\alpha} = 5$ (≈ 14 dB). Since we already had to reduce the gain at high frequencies by 6 dB, we now expect that a lag compensator has to be used in order to reduce the the gain at high frequencies by $14 + 6 = 20$ dB.

A lag compensator in essence is a "slow" pole, followed by a "faster" zero. Since we get a slope of $-20\ \frac{\text{dB}}{\text{decade}}$ after a pole, we should make the zero 1 decade faster than the pole in order to reduce the gain at high frequencies by 20 dB. We would like to make the pole for this lag compensator as slow as possible, since it will reduce the phase at higher frequencies (and therefore potentially reducing the PM). Because we are allow to reduce the gain by 6 dB (at most) at $\omega = 1$, we cannot make the pole slower than 1 octave before $\omega = 1$, which would be at $\omega = 0.5$. Since we would like some margin, we place it at $\omega = 0.6$ and because the zero will be 1 decade faster, we get a zero at $\omega = 6$. The resulting lag compensator will be:

$$D_{lag}(s) = \frac{\frac{s}{6} + 1}{\frac{s}{0.6} + 1}$$

Now looking at $KD_{lag}(s)G(s)$, we expect $\omega_c \approx 10$, since we reduced the gain at high frequencies by 20 dB. The phase in this point is $-162.5°$, and therefore we would like to use a lead compensator that adds 40° phase at $\omega_{max} = 10$. As a result we get a lead compensator $D_{lead}(s)$ and the full lead-lag compensator now becomes $D(s) = D_{lead}(s)D_{lag}(s)$.

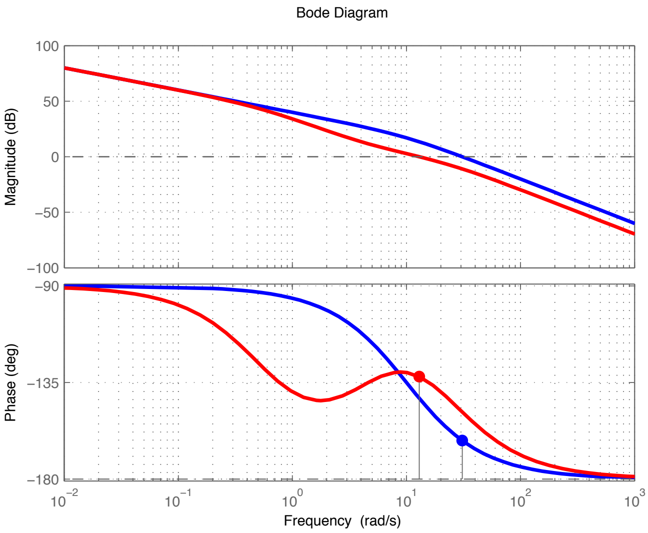


Figure 13: Bode plot of $KD(s)G(s)$, Problem 5.

7. **Compensator design:**
The transfer function of a satellite-attitude control system is

$$G(s) = \frac{0.05(s+25)}{s^2(s^2+0.1s+4)}.$$

Using MATLAB, stabilize the system in closed-loop with GM ≥ 6 dB and PM $\geq 45°$, while keeping the bandwidth as high as possible.

Solution:
From the uncompensated Bode plot, Figure 14, we see that the slope around the crossover frequency is around -40 dB/dec, hence a lead compensator will be required to meet the PM requirement. Also, the requirement of keeping the bandwidth as high as possible implies the use of lead compensator.

Furthermore, the resonant peak needs to be kept below magnitude 1, which implies lowering the gain at the resonance.

By using the lead compensator

$$D(s) = \frac{s+0.06}{s+6},$$

we lower the low frequency gain by a factor of 100, achieve -20 dB/dec slope at the crossover frequency, and lower the gain at the resonance.

The Bode diagrams for the compensated open-loop transfer function are plotted in Figure 15. The Bode diagrams for the uncompensated and compensated systems are shown in Figure 16, in blue and red color, respectively.

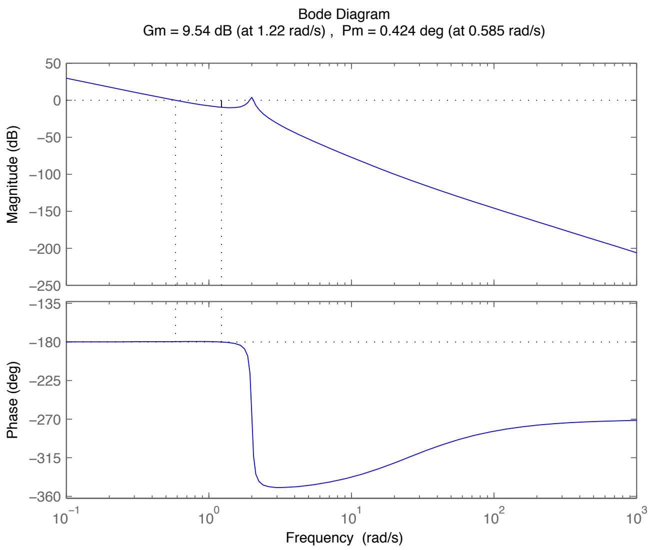


Figure 14: Bode plot of the uncompensated system for Problem 6.

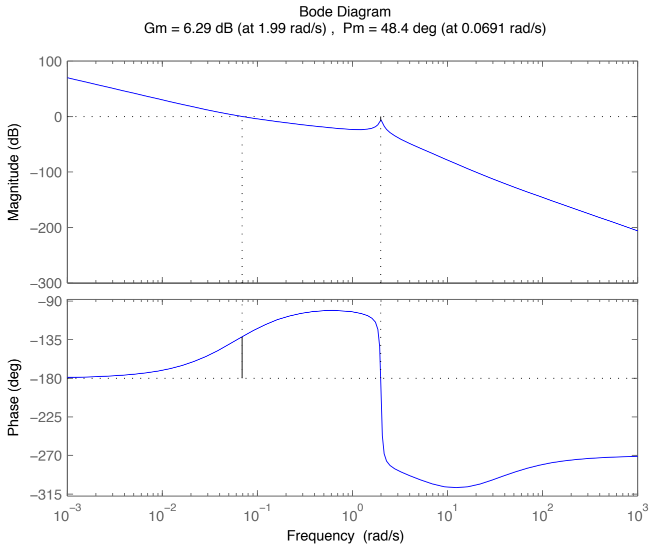


Figure 15: Bode plot of the compensated system for Problem 6.

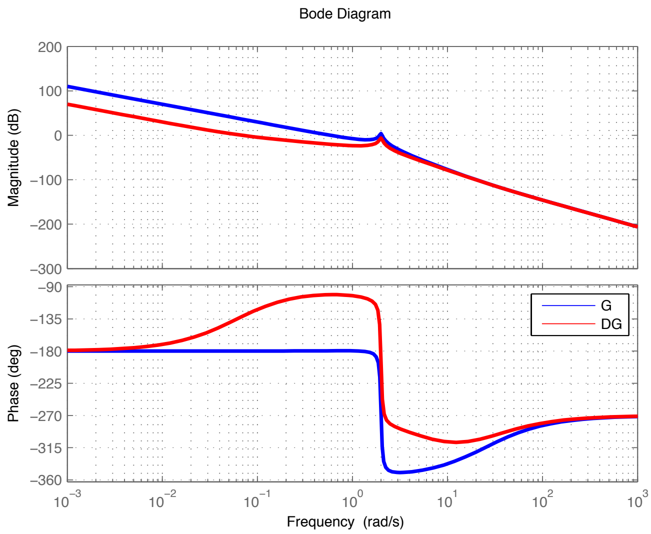


Figure 16: Bode plot for the uncompensated and compensated systems for Problem 6.