Photonics

Scalar wave optics

Helmholtz equation

Monochromatic waves

Reflection and refraction

Interference and interferometers

Light = waves

- 17^{th} century: Christian Huygens

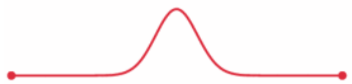
Light is a wave phenomenon

- Simplest description:
  Scalar function $u(\mathbf{r}, t)$
  - explains diffraction
  - explains interference
  - includes the ray theory (approximation $\lambda \to 0$)

The postulates of wave optics

1. Light waves propagate in the free space with the speed of light $c = 3 \times 10^8$ m/s
2. Homogeneous, isotropic, transparent media are characterized by the refractive index $n \geq 1$.
   The speed of light in this medium $v = c/n$
3. A light wave: a scalar function $u(\mathbf{r}, t)$ which satisfies

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{(the wave equation)}$$



4. Each function satisfying the wave equation is a **possible** light wave
5. Superposition: If $u_1(\mathbf{r}, t)$ and $u_2(\mathbf{r}, t)$ are solutions, then a linear combination,

   $au_1(\mathbf{r}, t) + bu_2(\mathbf{r}, t)$, is also a solution

6. The wave function is continuous on the boundary between two media with different refractive indices
7. Wave optics is approximately applicable if $n(\mathbf{r})$ is slowly varying

Intensity and power

- **Intensity** $I(\mathbf{r},t)$:

  $I(\mathbf{r},t) = 2\langle u^2(\mathbf{r},t)\rangle$ [W/m^2]

  average over time $\gg$ 1/optical frequency

- **Optical power** propagating through a surface A:

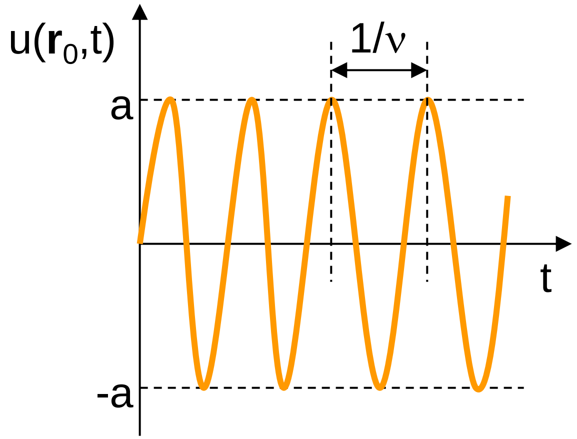
  $P(t) = \int_A I(\mathbf{r},t)\mathrm{d}A$ [Watt]

Monochromatic waves

- Monochromatic wave: harmonic time dependence

$$u(\mathbf{r},t) = a(\mathbf{r})\cos[2\pi\nu t + \phi(\mathbf{r})]$$

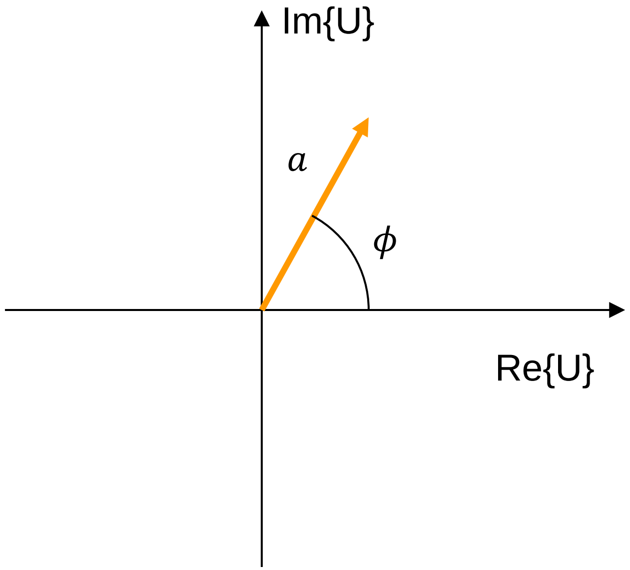
  - $a(\mathbf{r})$: amplitude
  - $\phi(\mathbf{r})$: phase
  - ν: frequency [Hz]
  - $\omega = 2\pi\nu$: angular frequency [rad/s]

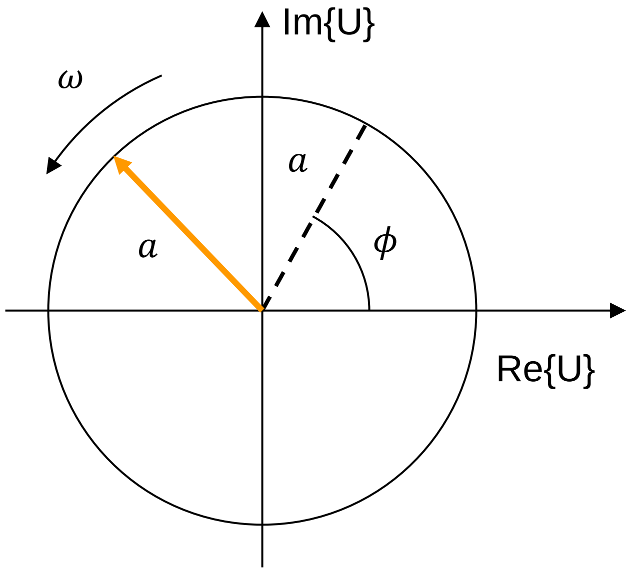


Complex representation

- Wave function $u(\mathbf{r},t) = a(\mathbf{r})\cos[2\pi\nu + \phi(\mathbf{r})]$
- Complex wave function $U(\mathbf{r},t) = a(\mathbf{r})e^{j\phi(\mathbf{r})}e^{j2\pi\nu t}$

  so that $u(\mathbf{r},t) = \mathrm{Re}\{U(\mathbf{r},t)\} = \frac{1}{2}[U(\mathbf{r},t) + U^*(\mathbf{r},t)]$
- Complex amplitude $U(\mathbf{r}) = a(\mathbf{r})e^{j\phi(r)}$
  - $|U(\mathbf{r})| = |u(\mathbf{r})| = a(\mathbf{r})$: the amplitude
  - $\angle U(\mathbf{r}) = \phi(\mathbf{r})$: the phase
- Representation as a vector in a complex plane (= phasor)
  - Argand diagram with $U(\mathbf{r})$
  - Rotating phasor for $U(\mathbf{r},t)$





Helmholtz equation

- Eliminate time dependence in monochromatic waves $U(\mathbf{r},t)$:

$$\underbrace{\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad U(\mathbf{r},t) = a(\mathbf{r})\mathrm{e}^{j\phi(\mathbf{r})}\mathrm{e}^{j2\pi\nu t}}\; = U(\mathbf{r})\mathrm{e}^{j2\pi\nu t}$$

- Helmholtz equation: $\boxed{(\nabla^2 + k^2)U(\mathbf{r}) = 0}$ with $k = \frac{2\pi n\nu}{c} = \frac{n\omega}{c}$

- Wave fronts: $\phi(r) = m \cdot 2\pi$.

- Intensity $I(\mathbf{r}) = |U(\mathbf{r})|^2$

- Simple solutions:
  - plane waves
  - spherical waves

Plane waves

- Complex amplitude

$$U(\mathbf{r}) = A\mathrm{e}^{-j\mathbf{k}\cdot\mathbf{r}} = A\mathrm{e}^{-j\left(k_x x + k_y y + k_z z\right)}$$

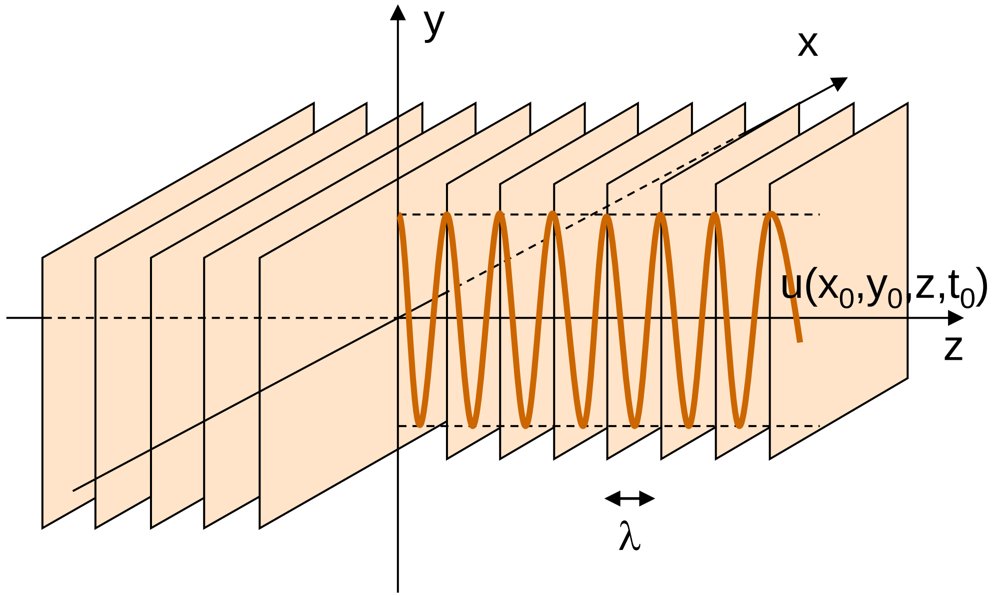
- is a solution if

$$\boxed{k_x^2 + k_y^2 + k_z^2 = \left(\frac{n\omega}{c}\right)^2}$$

- For $k_{x,y,z}$ and n being real it means:

$$\boxed{|\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = k = \frac{n\omega}{c}}$$

- Wave fronts:
  - planes $\perp$ $\mathbf{k}$
  - distance between planes with the same phase: $\lambda = \frac{v}{\nu} = \frac{c}{n\nu}$
- Intensity $I(\mathbf{r}) = |A|^2$



Evanescent plane waves (1)

- Plane wave:

$$U(\mathbf{r}) = A\mathrm{e}^{-j\mathbf{k}\cdot\mathbf{r}} = A\mathrm{e}^{-j\left(k_x x + k_y y + k_z z\right)}$$

$$\boxed{k_x^2 + k_y^2 + k_z^2 = k^2 = \left(\frac{n\omega}{c}\right)^2}$$

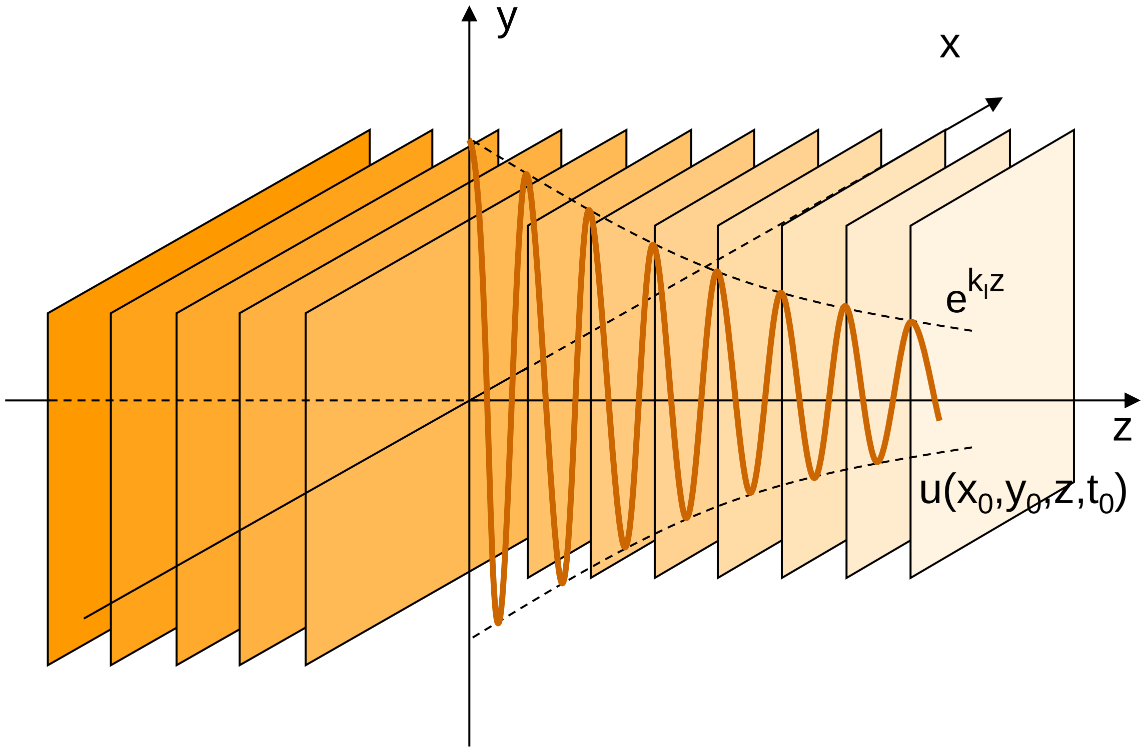
- If $\mathbf{k}$ is complex: $\mathbf{k} = \mathbf{k}_\mathrm{R} + j\mathbf{k}_\mathrm{I}$
  - propagation in the direction $\mathbf{k}_\mathrm{R}$
  - exponential decrease in the direction $\mathbf{k}_\mathrm{I}$.
- Special cases:
  - n is complex and $\mathbf{k}_\mathrm{R} \parallel \mathbf{k}_\mathrm{I}$
  - n is real and $\mathbf{k}_\mathrm{R} \perp \mathbf{k}_\mathrm{I}$

Evanescent plane waves (2)

- If $\mathbf{k}_\mathrm{R} \parallel \mathbf{k}_\mathrm{I} \parallel z$-axis:

$$U(\mathbf{r}) = A\mathrm{e}^{-j(k_\mathrm{R}+jk_\mathrm{I})z}$$
$$U(\mathbf{r}) = A\mathrm{e}^{k_\mathrm{I}z}\mathrm{e}^{-jk_\mathrm{R}z}$$

- decreasing amplitude as the wave propagates

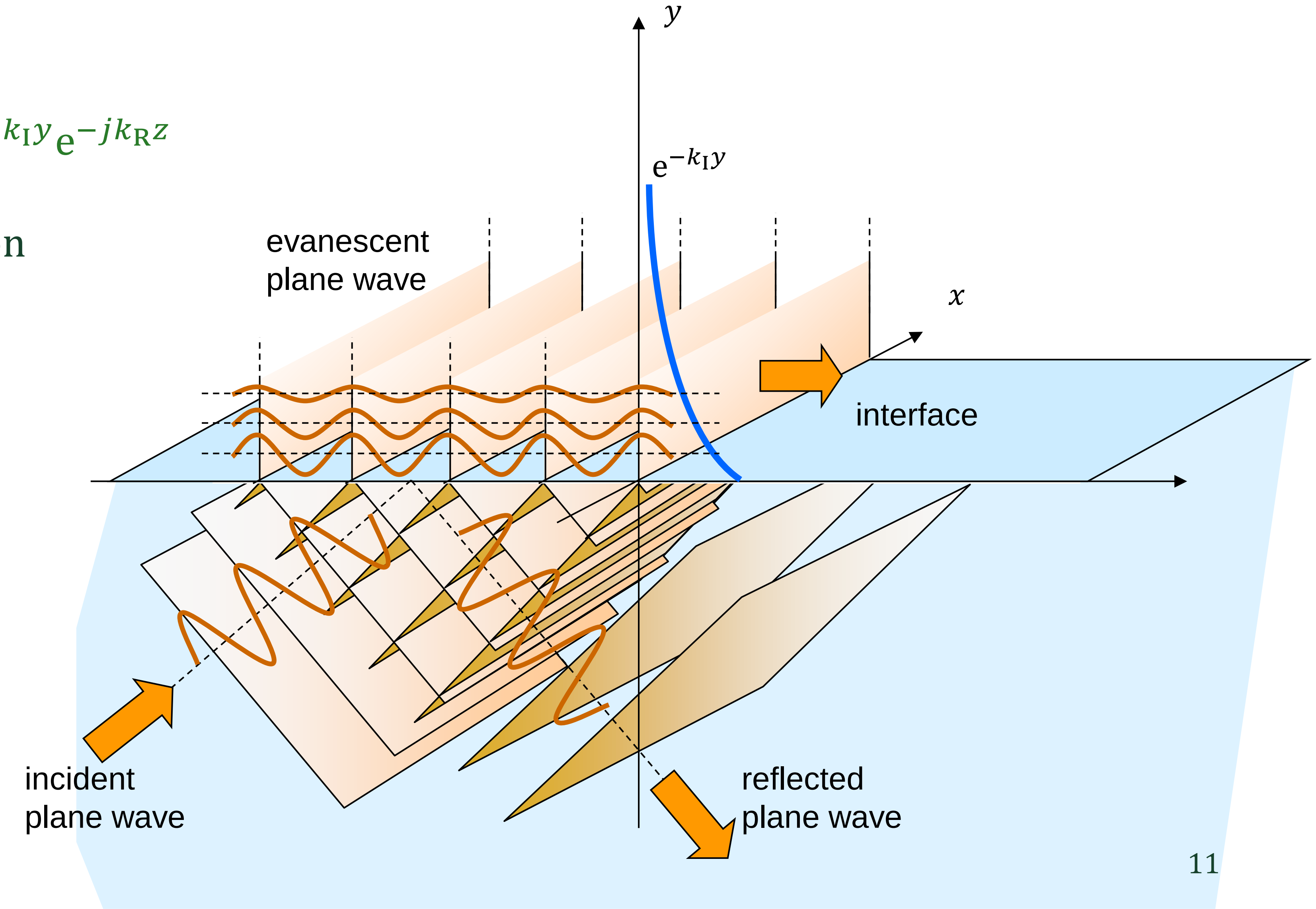


Evanescent plane waves (3)

- If $\mathbf{k}_\mathrm{I} \perp \mathbf{k}_\mathrm{R} \parallel z$-axis:

$$U(\mathbf{r}) = A\mathrm{e}^{-k_\mathrm{I} y}\mathrm{e}^{-jk_\mathrm{R} z}$$

  propagation in the z-direction
  - decrease in the y-direction
- Example: Total internal reflection



Spherical wave fronts

- Complex amplitude:

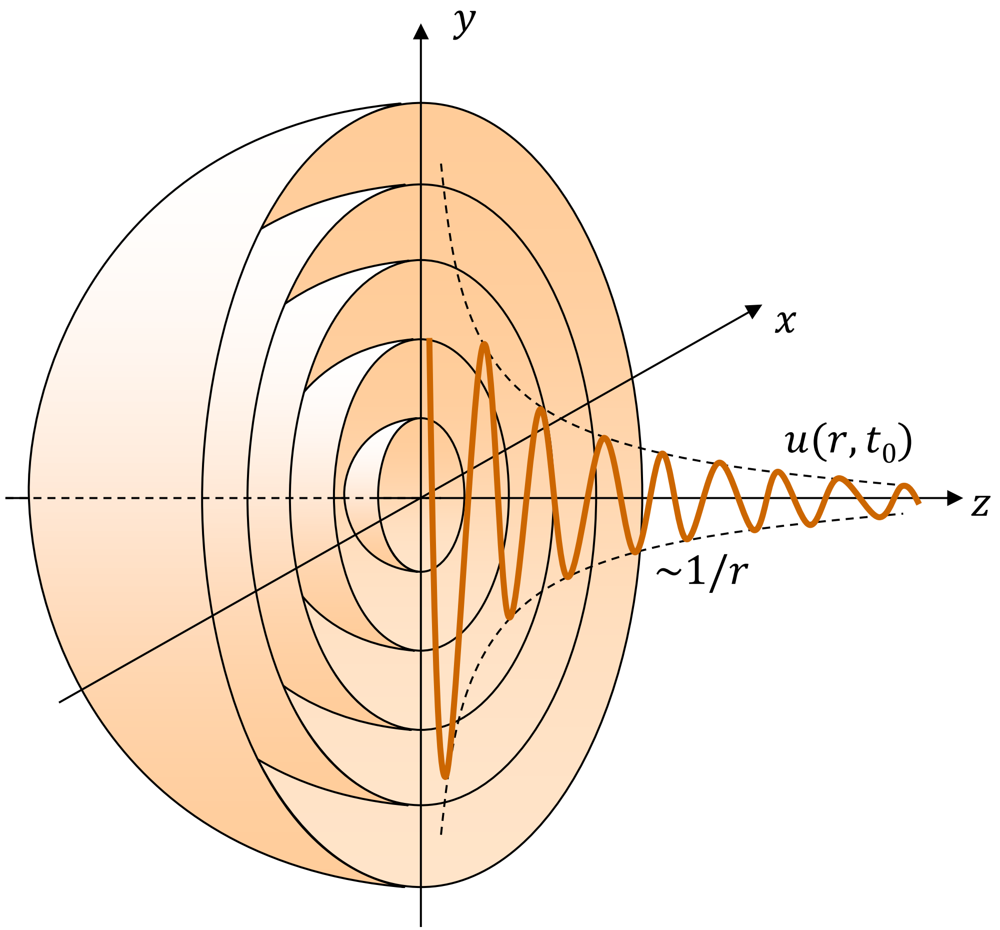
$$U(r) = \frac{A}{r} \mathrm{e}^{-jkr}$$

- Wave fronts
  - Concentric spheres
  - Distance $\lambda = \frac{v}{\nu} = \frac{c}{n\nu}$ (speed of light in material: v; optical frequency: ν)
  - Radial propagation

    $-jkr$: away from the origin

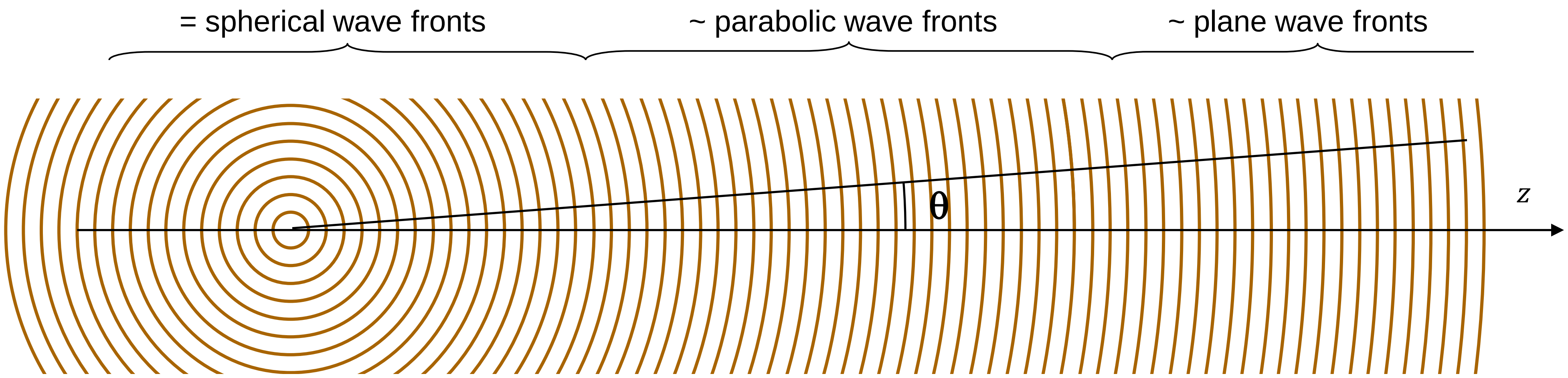
    $+jkr$: to the origin

- Intensity $I(r) = \frac{|A|^2}{r^2}$



Fresnel approximation: parabolic waves

- Behavior of a spherical wave near the z-axis (propagation axis)
  - Taylor expansion for r: $$r \simeq z + \frac{x^2+y^2}{2z}$$
  - Complex amplitude $$U(r) = \frac{A}{z}\underbrace{\mathrm{e}^{-jkz}}_{\text{propagation}}\underbrace{\mathrm{e}^{-jk\frac{x^2+y^2}{2z}}}_{\text{bending of the wave fronts}}$$
  - For large enough z: (quasi-)plane wave fronts

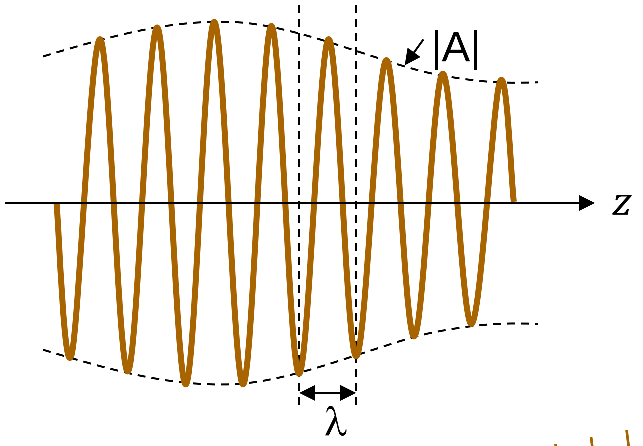


Paraxial waves

- Paraxial wave = wave which propagates at a small angle θ with the propagation axis

  $U(r) = A(r)e^{-jkz}$ whereas the variation of $A(\boldsymbol{r})$ is slow compared to λ:

  - $\dfrac{\partial A}{\partial z} \ll kA$
  - $\dfrac{\partial^2 A}{\partial z^2} \ll k^2 A$

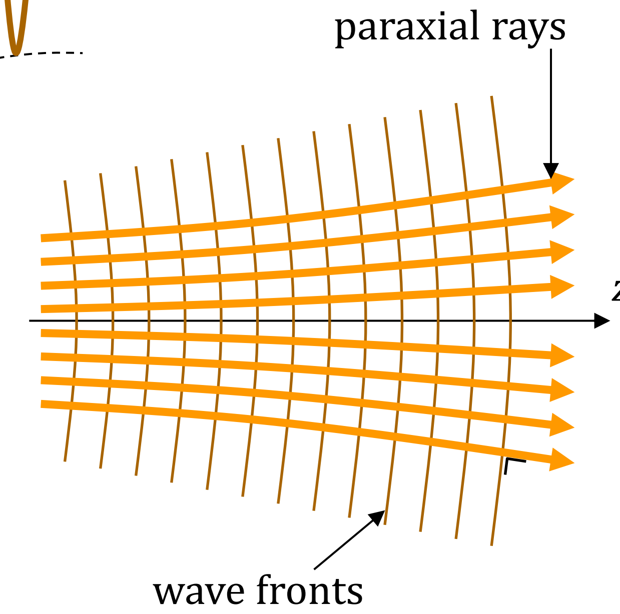


- Paraxial Helmholtz equation:

$$\nabla_T^2 A(r) - 2jk\frac{\partial A(r)}{\partial z} = 0$$

with

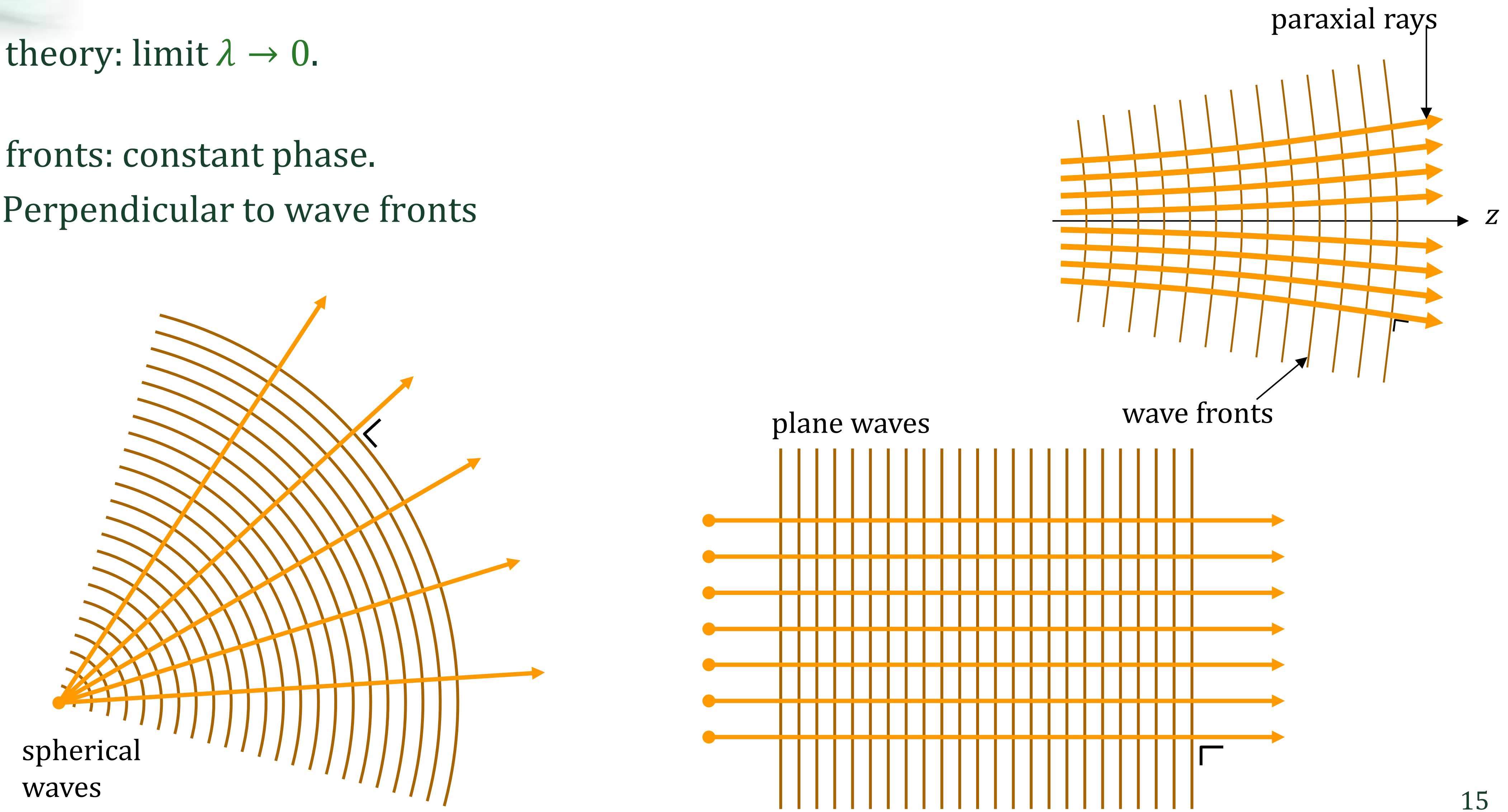
$$\nabla_T^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$



Waves and rays

- Wave theory: limit $\lambda \to 0$.

- Wave fronts: constant phase.
- Rays: Perpendicular to wave fronts



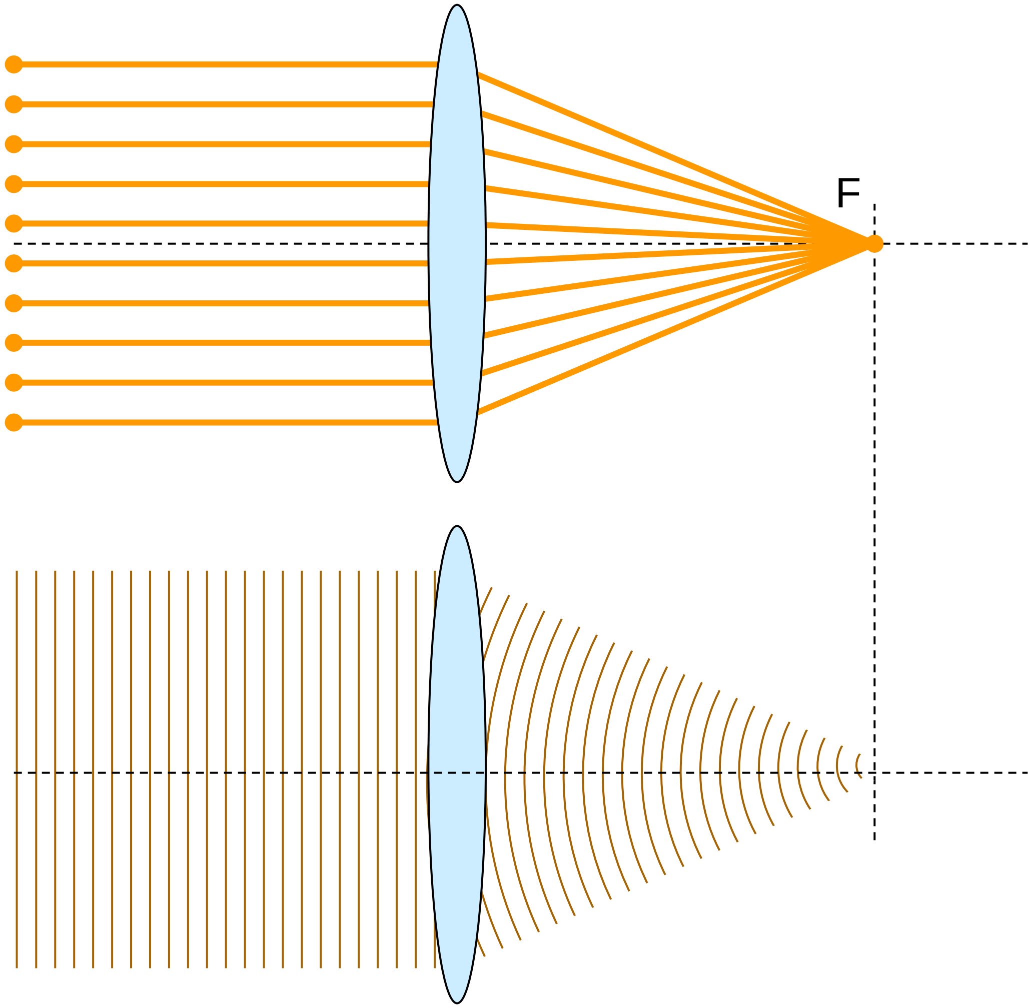
Waves and rays

- Example: Lens
  - Rays || to the axis will be focused in the focus F

    OR

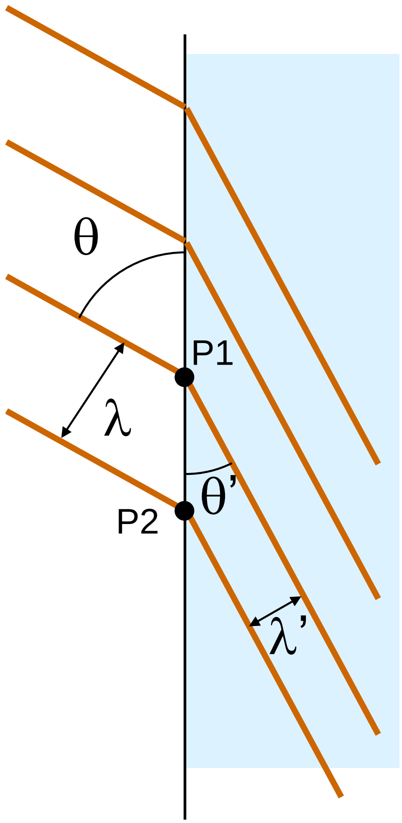
  - A plane wave is converted into a spherical wave front around F:

    waves are 'delayed' more where the lens is thicker



Reflection and refraction (Snell's law)

- Plane wave incident on the interface
- On the surface ($z = 0$)
  - phases of the incident, refracted and reflected waves must be the same



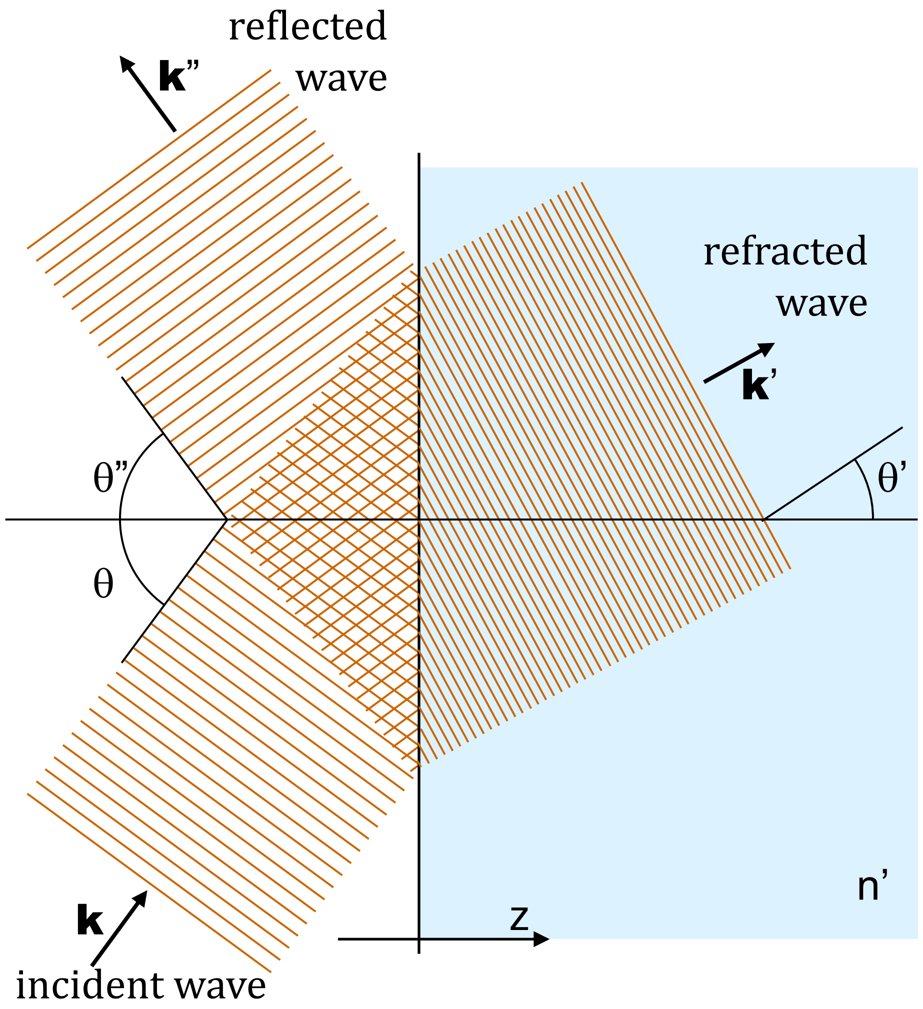
$$\mathbf{k}\cdot\mathbf{r} = \mathbf{k}'\cdot\mathbf{r}' = \mathbf{k}''\cdot\mathbf{r}''$$

$$n'\sin\theta' = n\sin\theta$$

(Snell's law)

$$\theta = \theta''$$

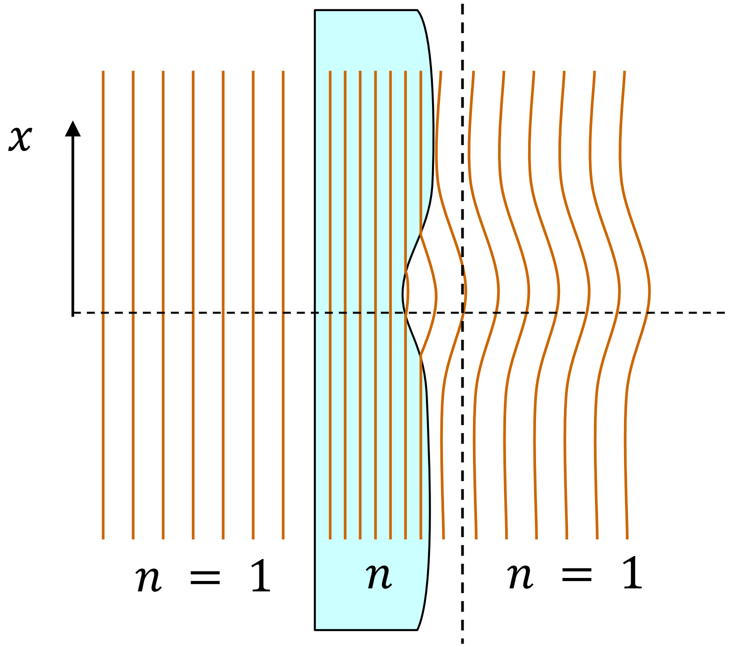
reflection

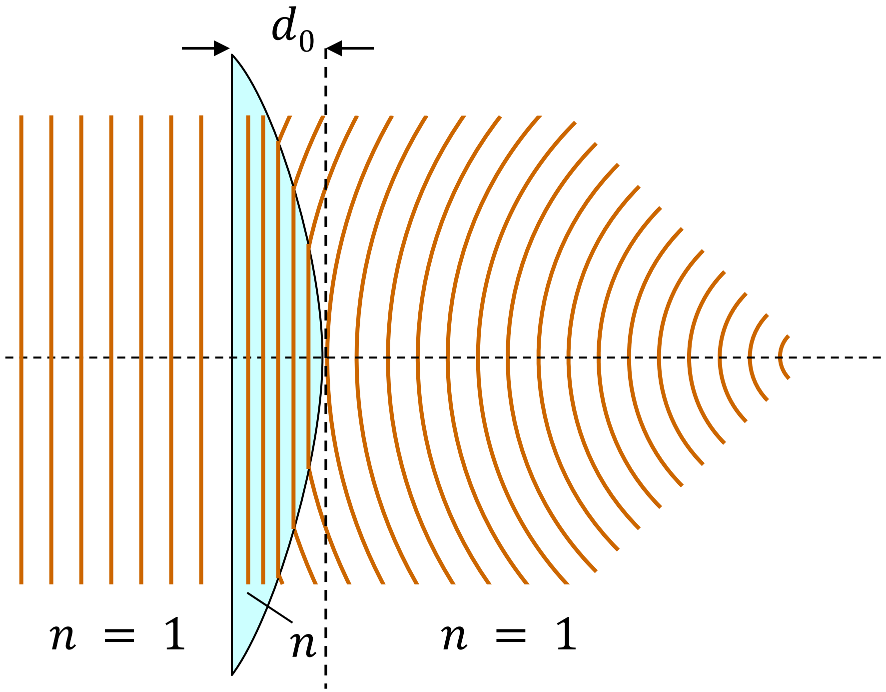


A curved plate

- Paraxial waves incident on a curved plate
  - wave propagates slower in material
  - phase delay is dependent on the local thickness: local optical path length $nd(x)$
  - Phase fronts are continuous → curved phase fronts

- E.g. lenses





Interference of two waves

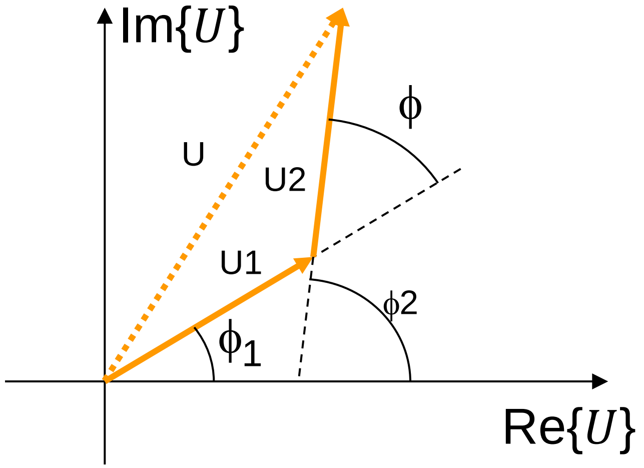
$$U = U_1 + U_2$$

$$\Downarrow$$

$$I = |U|^2 = |U_1|^2 + |U_2|^2 + U_1^* U_2 + U_1 U_2^*$$

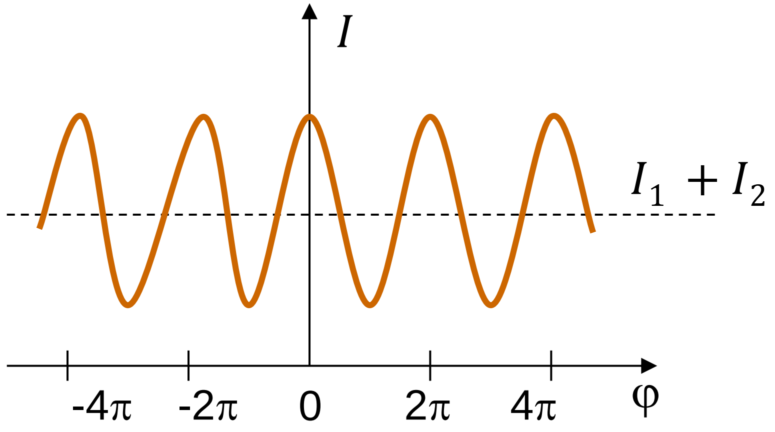
$$\Downarrow$$

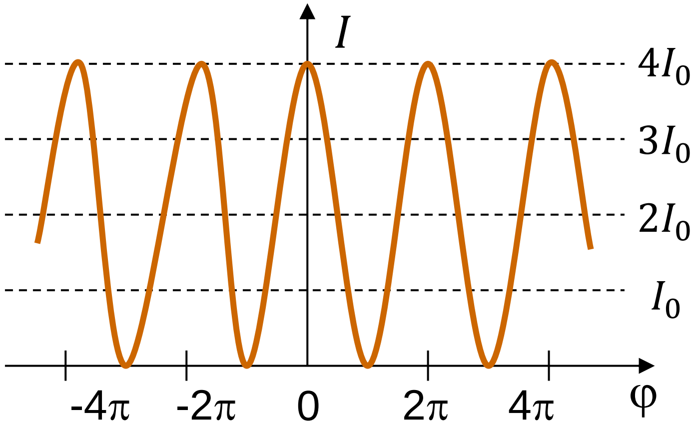
$$\phi = \phi_2 - \phi_1$$



$$\boxed{I_1 \neq I_2} \quad \boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi}$$

$$\boxed{I_1 = I_2 \triangleq I_0} \quad \boxed{I = 2I_0(1 + \cos\phi) = 4I_0\cos^2(\phi/2)}$$





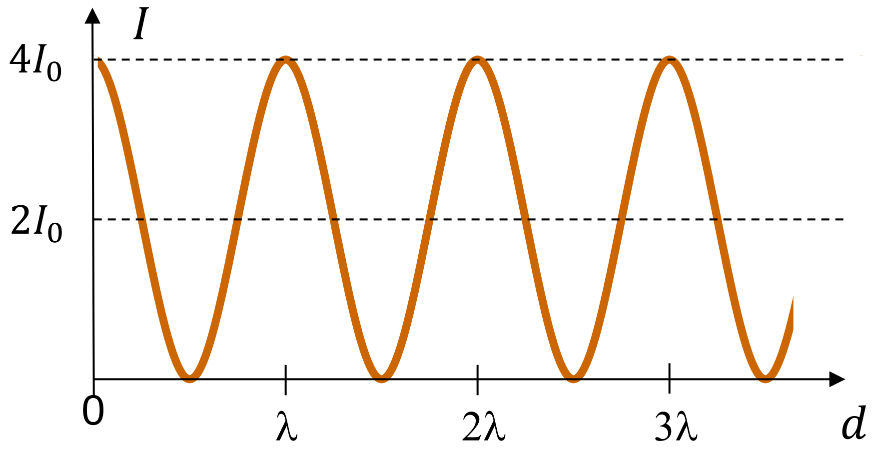
Interferometers

- Wave 1 ⟶ $U_1 = \sqrt{I_0}\mathrm{e}^{-jkz}$
- Wave 2 ⟶ $U_2 = \sqrt{I_0}\mathrm{e}^{-jk(z-d)}$

  retarded over a distance d

- Interference

$$\boxed{I = 2I_0\left[1 + \cos\left(2\pi\frac{d}{\lambda}\right)\right]}$$



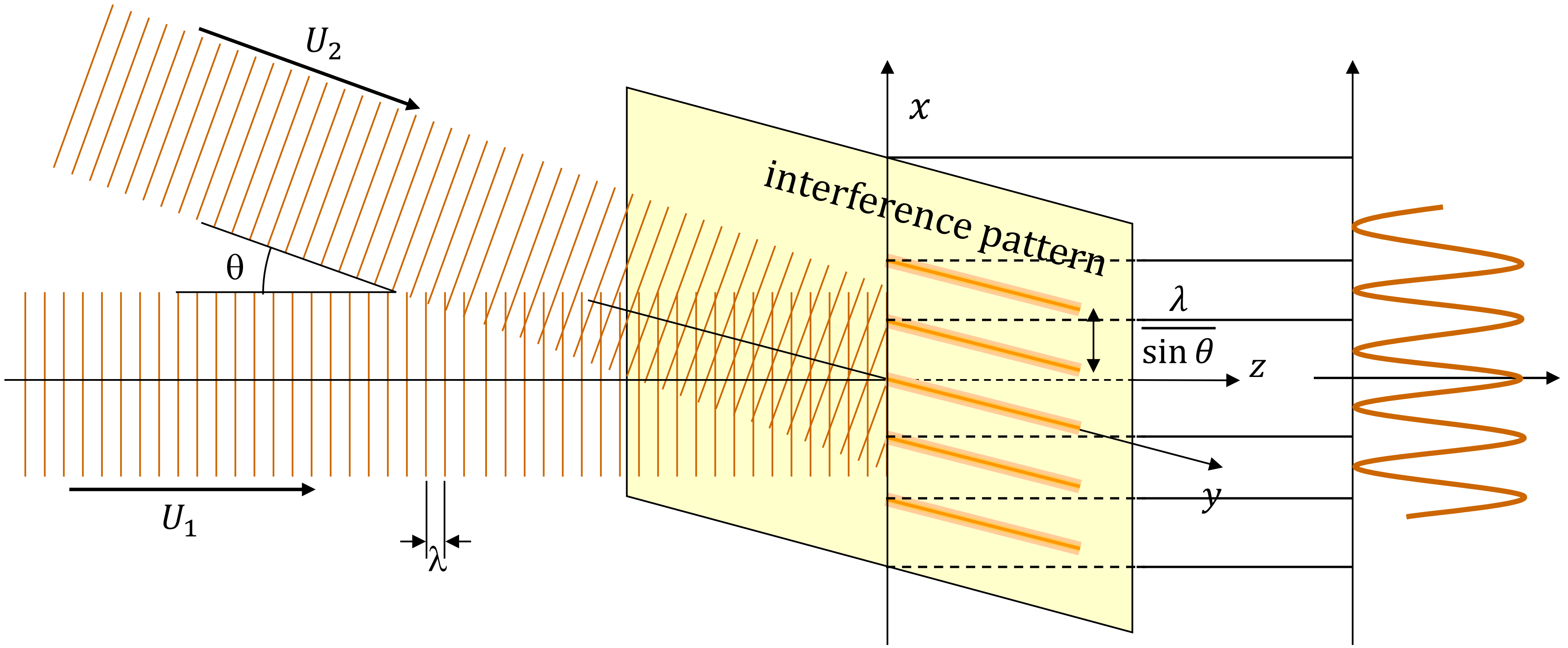
Double slit experiment

Canvas video

Interference of two plane waves

$$U_1 = \sqrt{I_0}e^{-jkz}$$
$$U_2 = \sqrt{I_0}e^{-j(k\cos(\theta)z + k\sin(\theta)x)}$$

$$\boxed{I = 2I_0[1 + \cos(k\sin(\theta)x)]}$$



Interference between multiple waves (1)

$$U_m = \sqrt{I_0} \mathrm{e}^{jm\phi}, \qquad m = 0,1,2,\dots,M-1$$

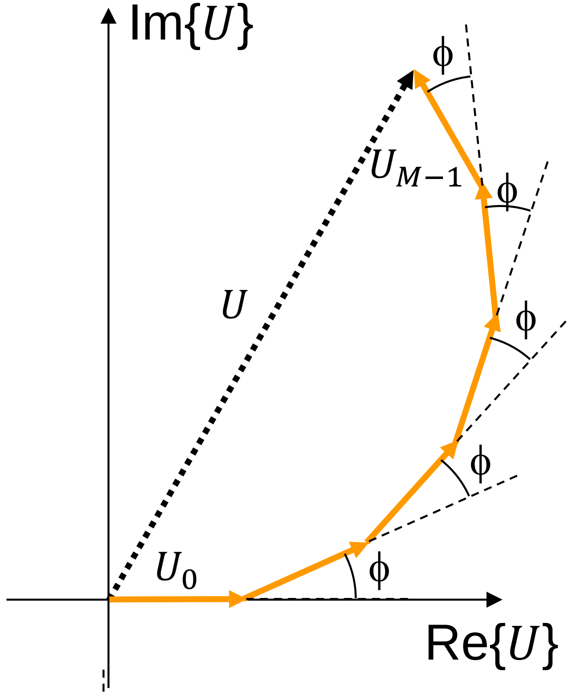
$$h = \mathrm{e}^{j\phi}$$

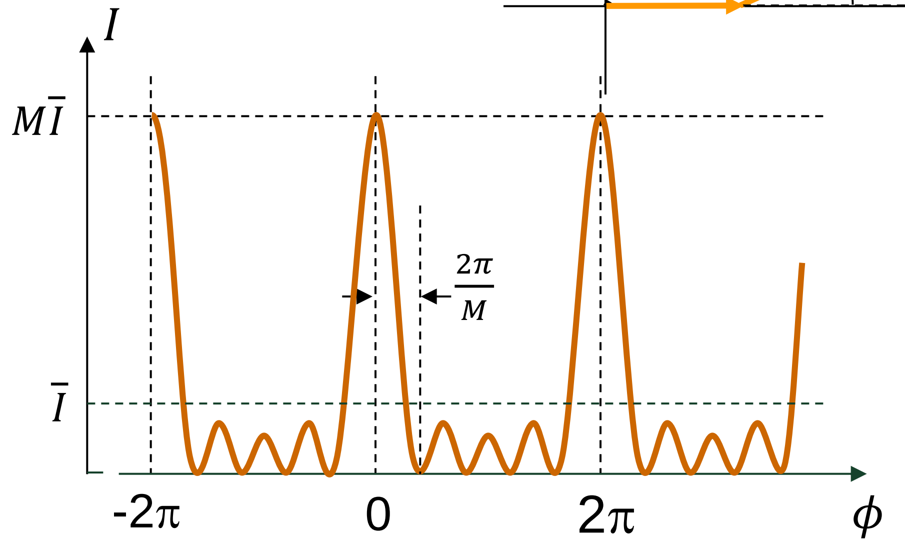
$$U = \sqrt{I_0}(1 + h + h^2 + \cdots + h^{M-1}) = \sqrt{I_0}\frac{1-h^M}{1-h}$$

$$\boxed{I = I_0 \frac{\sin^2(M\phi/2)}{\sin^2(\phi/2)}}$$

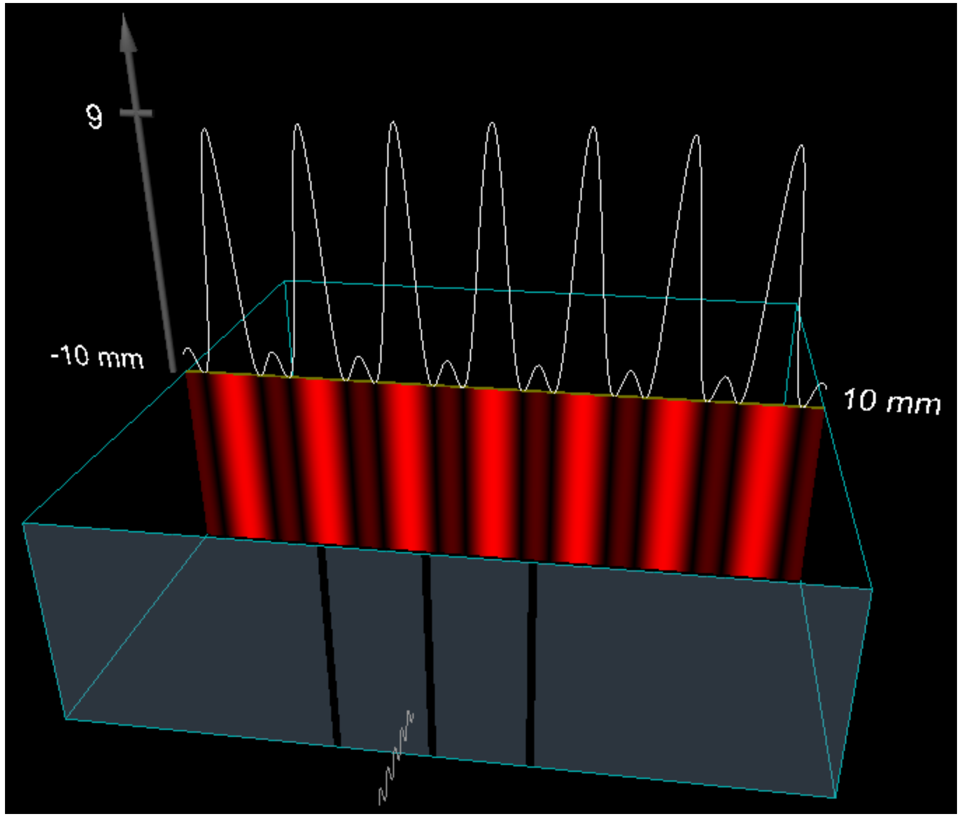
Geometric sum

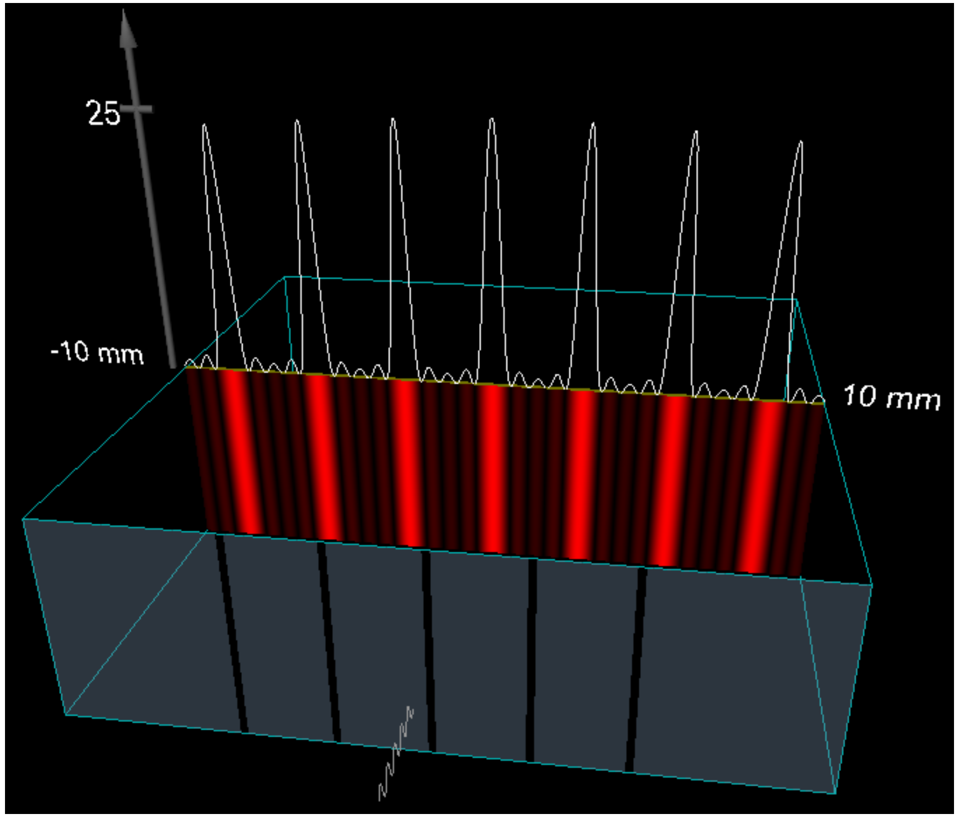
$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$$





Plane wave on multiple slits





Interference between multiple waves (2)

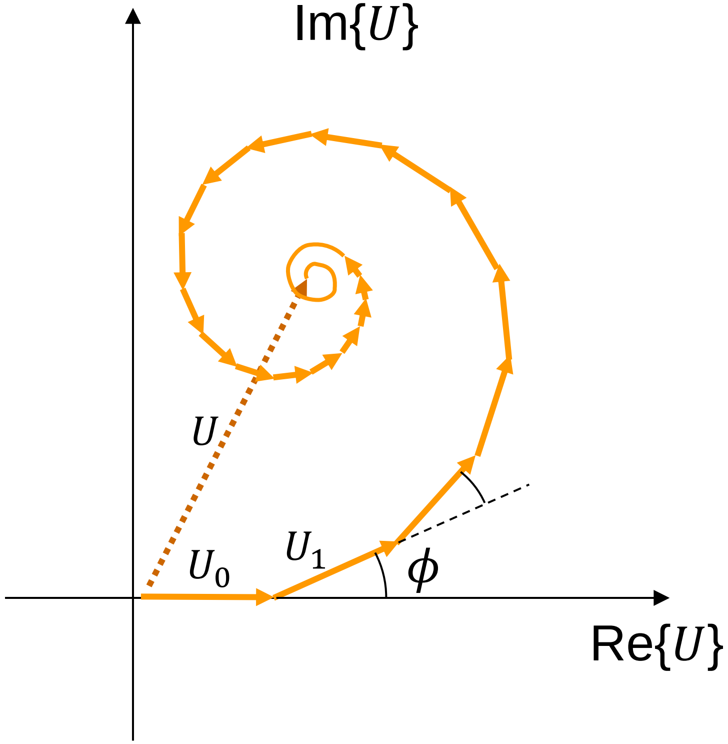
$$\begin{cases} U_0 = \sqrt{I_0}, \quad U_1 = hU_0, \quad U_2 = hU_1 = h^2 U_0, \quad \dots \\ h = |h|\mathrm{e}^{j\phi}, \quad |h| < 1 \end{cases}$$

$$U = \sqrt{I_0}(1 + h + h^2 + \cdots)$$

$$\boxed{U = \frac{\sqrt{I_0}}{1 - |h|\mathrm{e}^{j\phi}}}$$

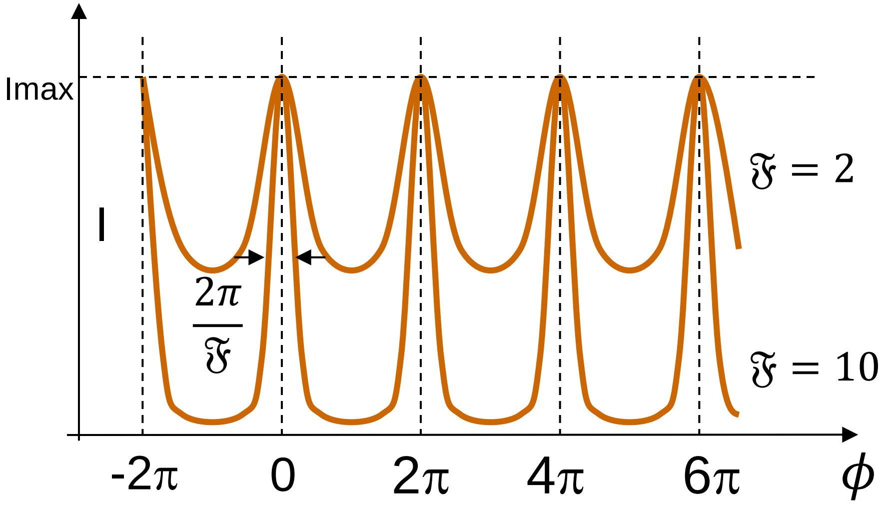
Geometric sum:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

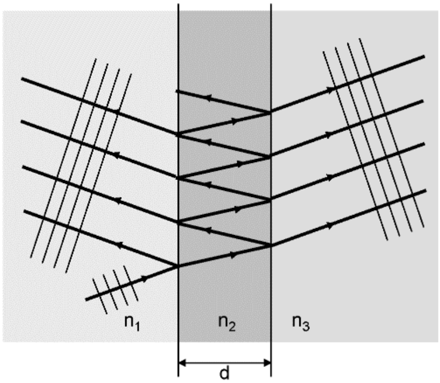


Interference between multiple waves (3)

$$\boxed{I = \frac{I_{\max}}{1 + (2\mathfrak{F}/\pi)^2 \sin^2(\phi/2)}}$$

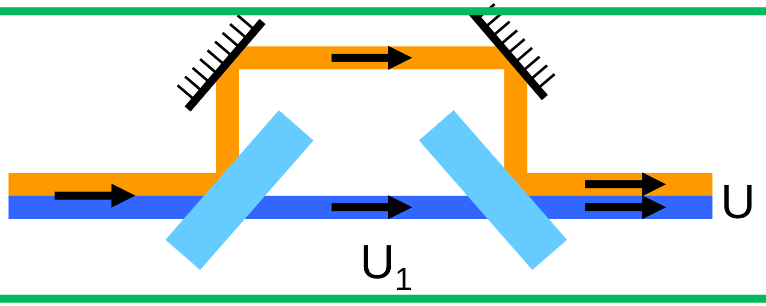


$$\begin{cases} I_{\max} = \dfrac{I_0}{(1-|h|)^2} \\ \\ \mathfrak{F} = \dfrac{\pi\sqrt{|h|}}{1-|h|} \quad \text{finesse} \end{cases}$$

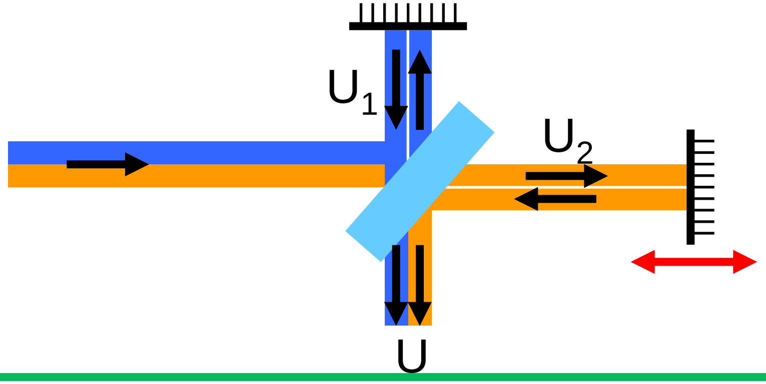


Interferometers: examples

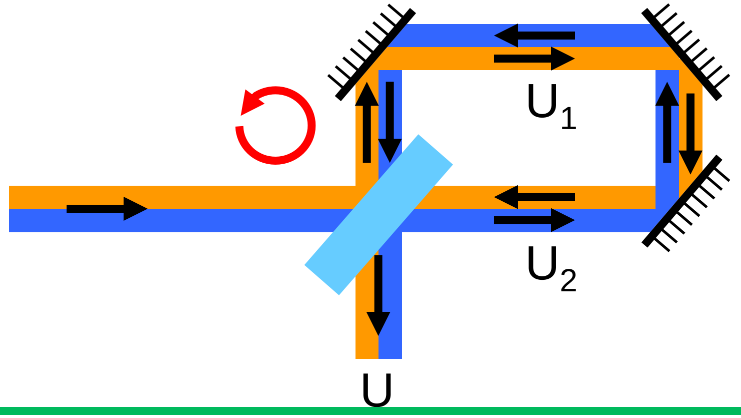
- Mach-Zehnder



- Michelson



- Sagnac

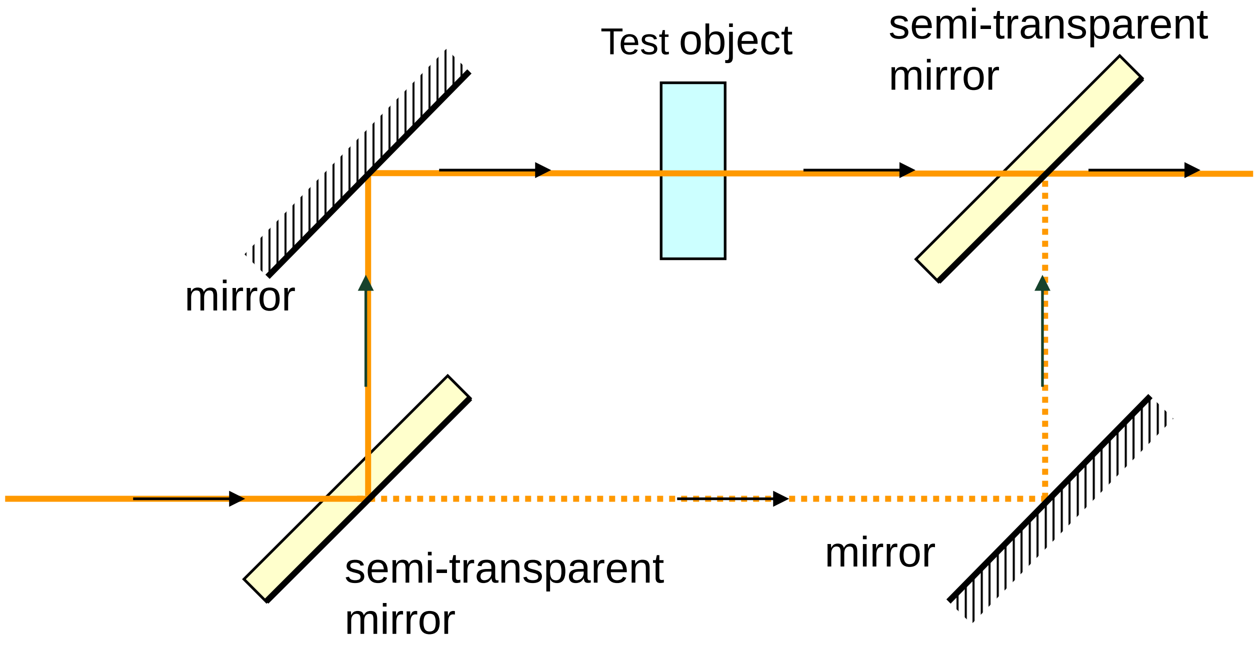


Mach-Zehnder interferometer

- Test object causes phase change in one arm

  $\rightarrow$ intensity variation

- Application: study of gases



Michelson Interferometer

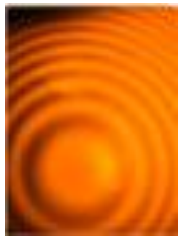
- Interference between waves reflected from <u>two mirrors</u>

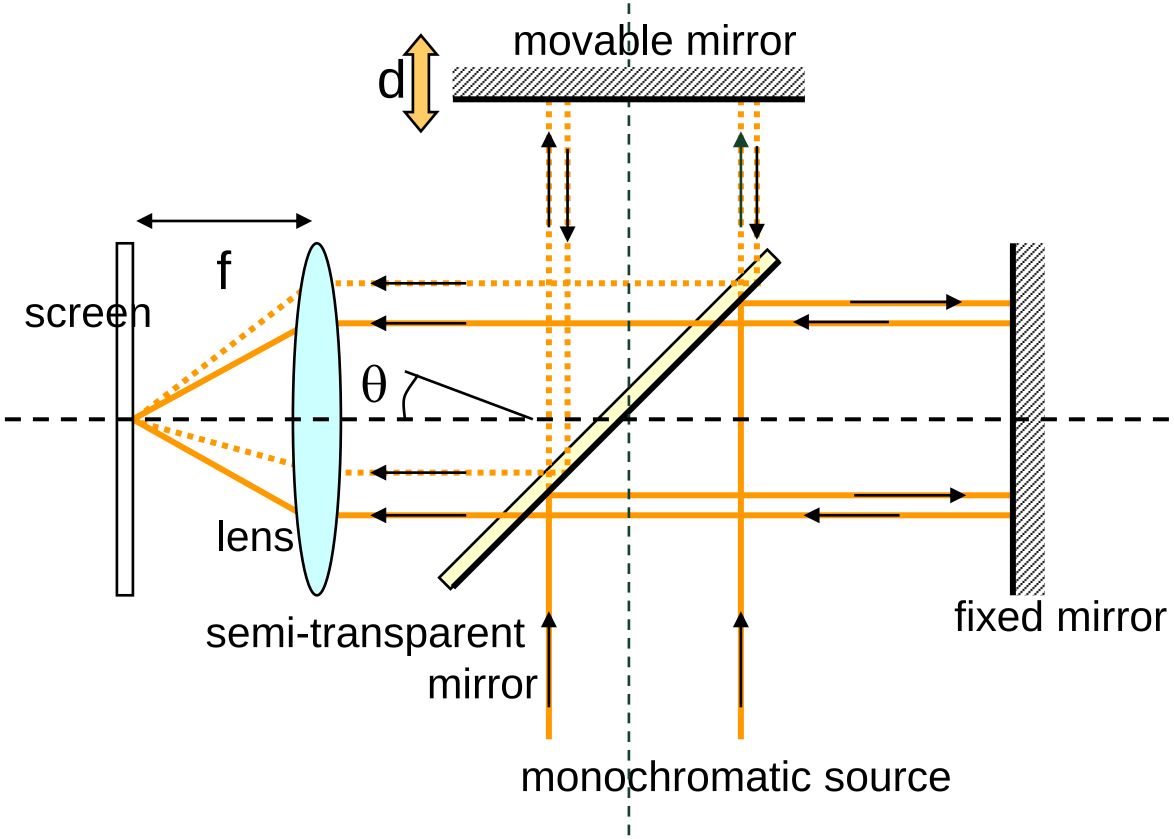
- Phase difference: $\Delta\phi = \frac{4\pi d}{\lambda}\cos\theta$

- Intensity in the focal plane:

$$I(\Delta\phi) = 4I_1\cos^2\frac{\Delta\phi}{2}$$

$\rightarrow$ concentric rings





Sagnac interferometer

- Light propagates in both directions
- Rotation: difference of the path length between two directions
- Phase difference: $\Delta\phi = 2\pi\nu\Delta t$
- Time difference Δt:

$$\Delta t = \frac{2\pi R}{c+\omega R} - \frac{2\pi R}{c-\omega R}$$
$$= \frac{4\pi\omega R^2}{c^2} = 4\frac{A\omega}{c^2}$$

with A: area of the ring

- Application: measurements of rotations (optical gyroscope)