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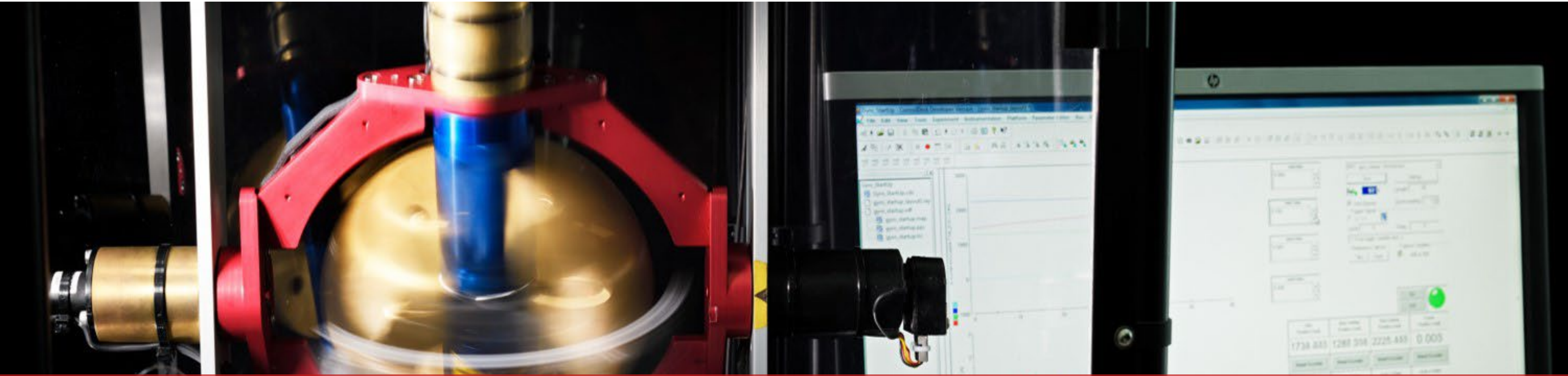


Control Systems (5ESD0)
Lecture 2: Root Locus

Tijs Donkers

Today’s Menu

Learning objective: Understand how closed-loop system properties van be modified using feedback

- Part 1: Root Locus Plots
  - What are root locus plots? (Summary of pencasts/book)
  - Sketching Rules
  - Examples of sketching

- Part 2: Quiz

Part 1: Root Locus Plots (Section 5.1, 5.2 and 5.6.1)

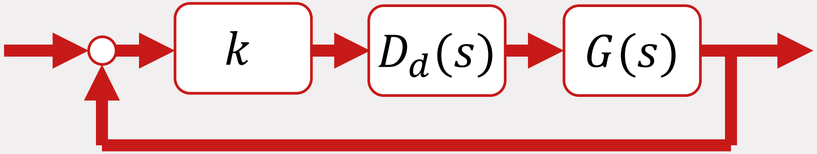
Root Locus Plots

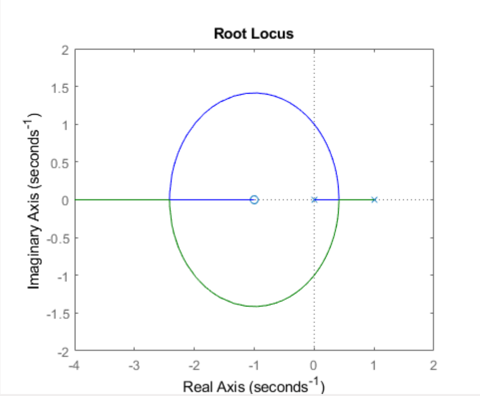
Learning objective: Understand how closed-loop system properties van be modified using feedback

Section 5.1 of the book:

- Closed-loop poles of transfer function $T(s) = \frac{kL(s)}{1+kL(s)}$ change when gain k changes
- Computed using characteristic equation $1 + kL(s) = 0$
- Important for stability and dynamic response
- Take transfer function $L(s) = \frac{b(s)}{a(s)}$

$\mathrm{D}(s) = k \cdot D_d(s)$ $L(s) = D_d(s)G(s)$





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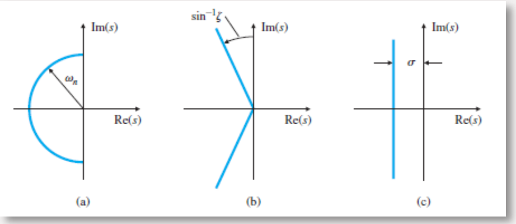
Root Locus Plots

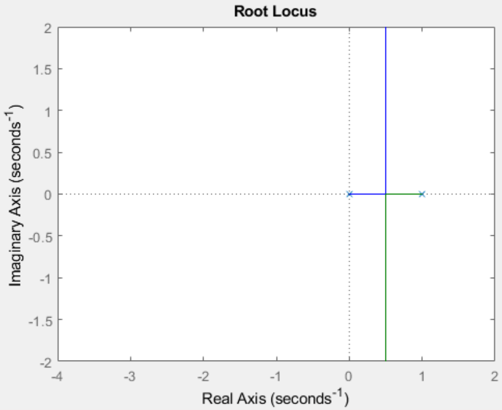
Learning objective: Understand how closed-loop system properties van be modified using feedback

Why learn plotting root loci:

- Not to design compensators (in this course)
- Time domain maps to s-domain (revisit Section 3.4)
- Understand what compensator is needed
  to satisfy performance requirements
- Understand fundamental limitations (Module 6)



Rise time overhoot setting time



Root Locus Plots

Root Locus is the set of values for s for which $1 + kL(s) = 0$ holds as function of $k \in [0, \infty)$.

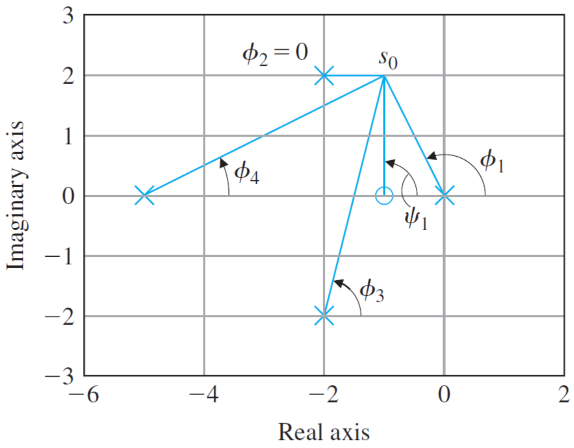
$$1 + kL(s) = 0 \Leftrightarrow L(s) = \frac{b(s)}{a(s)} = \frac{-1}{k}$$

Consequence: The root locus of $L(s)$ is the set of points in the s-plane where $\angle L(s) = -180°$.

- Angle from a zero to test point s_0 is denoted by ψ_i
- Angle from a pole to test point s_0 is denoted ϕ_i

$$L(s_0) = \frac{(s_0 - z_1)(s_0 - z_2) \ldots (s_0 - z_m)}{(s_0 - p_1)(s_0 - p_2) \ldots (s_0 - p_n)}$$

$\angle L(s_0) = \sum \psi_i - \sum \phi_i$ = multiple of -180°



Sketching Root Locus Plots

Summary of the rules of Section 5.2:

- Rule 1: n branches start at the poles and m branches end at the zeros
  - If there are more poles than zeros, $n - m$ loci go to infinity
  - If there are more zeros than poles, $m - n$ loci come from infinity
- Rule 2: The loci are on the real axis to the left of an odd number of poles and zeros
- Rule 3: Asymptotes have angle $\frac{180^o+360^o(l-1)}{n-m}$ with $l = 1, \dots, n - m$ center point $\frac{\sum p_i - \sum z_i}{n-m}$
- Rule 4: For simple poles and zeros (multiplicity 1, for higher multiplicity, see book)
  - Departure angles: $\phi_{dep} = \sum_i \psi_i - \sum_{i \neq dep} \phi_i - 180^o$
  - Arrival angles: $\psi_{arr} = \sum_i \phi_i - \sum_{i \neq arr} \psi_i + 180^o$
- Rule 5: Branches approach a point of q roots with angles $\frac{180^o+360^o(l-1)}{q}$ for $l = 1, \dots, q$

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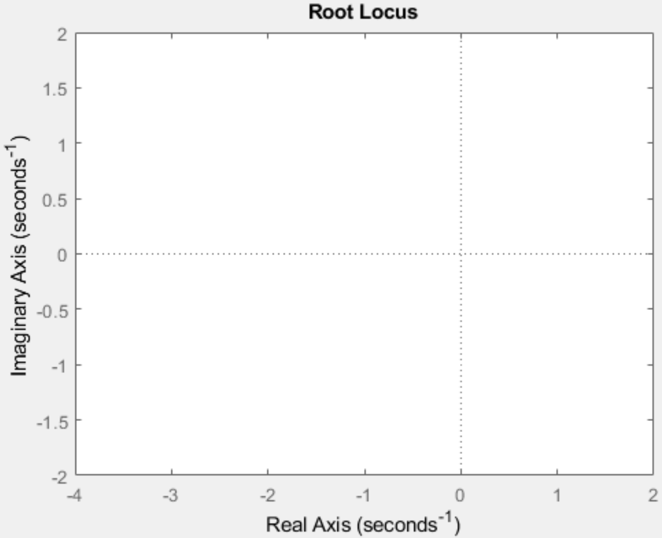
Examples on Sketching

Summary of the rules:

- Rule 1: n branches start at the poles and m branches end at the zeros
- Rule 2: The loci are on the real axis to the left of an odd number of poles and zeros
- Rule 3: Asymptotes have angle $\frac{180^o+360^o(l-1)}{n-m}$ with $l = 1, \dots, n-m$ center point $\frac{\sum p_i - \sum z_i}{n-m}$
- Rule 4: For simple poles and zeros
  - Departure angles: $\phi_{dep} = \sum_i \psi_i - \sum_{i \neq dep} \phi_i - 180^o$
  - Arrival angles: $\psi_{arr} = \sum_i \phi_i - \sum_{i \neq arr} \psi_i + 180^o$
- Rule 5: Branches approach a point of q roots with angles $\frac{180^o+360^o(l-1)}{q}$

$$L(s) = \frac{1}{s(s-1)}$$

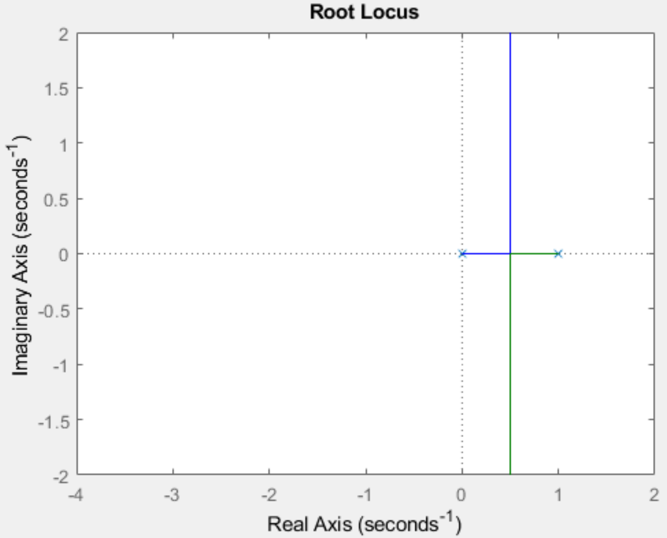


Examples on Sketching

$$L(s) = \frac{1}{s(s-1)}$$

Summary of the rules:

- Rule 1: 2 branches start at the poles, and they go to infinity (there are no zeros)
- Rule 2: Real segment $0 \leq s \leq 1$is part of the locus
- Rule 3: Asymptotes have angle $\pm 90^o$ and center point $\frac{1}{2}$
- Rule 4: Departure angles (no arrival angles, because no zeros)
  - Around $p_1 = 0$, we have $\phi_2 = 180^o$, so departure angle $\phi_{dep} = -360^o = 0$
  - Around $p_2 = 1$, we have $\phi_1 = 0^o$, so departure angle $\phi_{dep} = -180^o = 180^o$
  - Also follows from rule 3
- Rule 5: breakout angles $\pm 90^o$ (from rule 3)

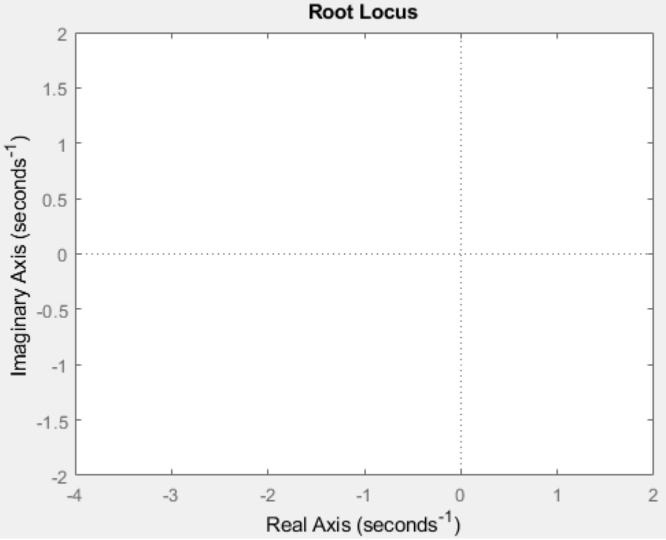


Examples on Sketching

Summary of the rules:

- Rule 1: n branches start at the poles and m branches end at the zeros
- Rule 2: The loci are on the real axis to the left of an odd number of poles and zeros
- Rule 3: Asymptotes have angle $\frac{180^o+360^o(l-1)}{n-m}$ with $l = 1, \dots, n-m$ center point $\frac{\sum p_i - \sum z_i}{n-m}$
- Rule 4: For simple poles and zeros
  - Departure angles: $\phi_{dep} = \sum_i \psi_i - \sum_{i \neq dep} \phi_i - 180^o$
  - Arrival angles: $\psi_{arr} = \sum_i \phi_i - \sum_{i \neq arr} \psi_i + 180^o$
- Rule 5: Branches approach a point of q roots with angles $\frac{180^o+360^o(l-1)}{q}$

$$L(s) = \frac{s+1}{s(s-1)}$$



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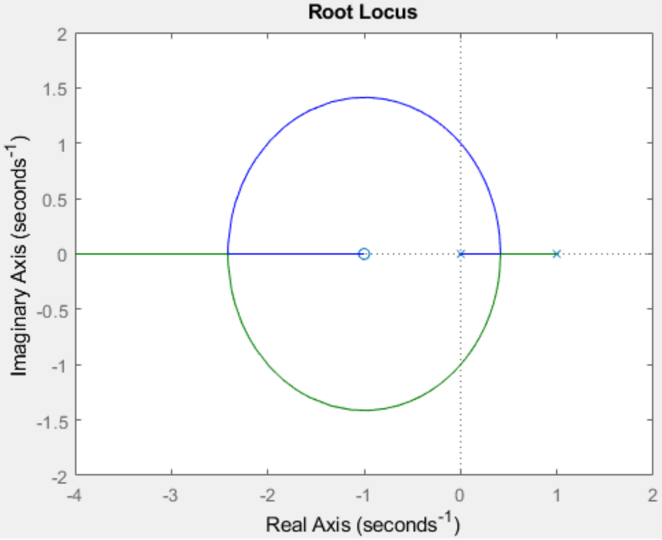
Examples on Sketching

$$L(s) = \frac{s+1}{s(s-1)}$$

Summary of the rules:

- Rule 1: 2 branches start at the poles and 1 branches ends at the zero
- Rule 2: Real segments $0 \leq s \leq 1$ and $s \leq -1$ are part of the locus
- Rule 3: Asymptote has angle 180^o
- Rule 4: Departure and arrival angles
    - Around $p_1 = 0$, $\phi_{dep} = 0^o - 180^0 - 180^o = -360^o = 0^o$
    - Around $p_2 = 1$, $\phi_{dep} = 0^o - 0^0 - 180^o = -180^o = 180^o$
    - Around $z_1 = -1$, $\psi_{arr} = 2 \cdot 180^o + 180^o = 540^o = 180^o$
- Rule 5: breakout angles $\pm 90^o$

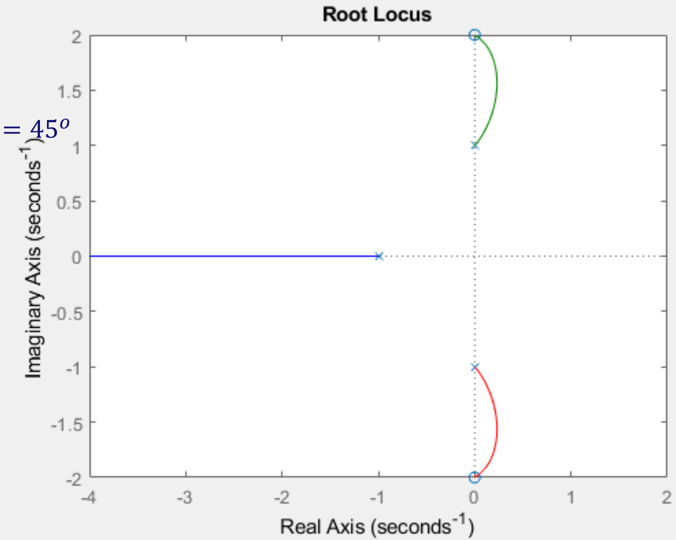


Examples on Sketching

Summary of the rules:

- Rule 1: 2 branches start at the poles and end at zeros, one branch goes to infinity
- Rule 2: Real segments $s \leq -1$ are part of the locus
- Rule 3: Asymptote have angle -180^o and center point $\frac{1}{2}$
- Rule 4: Departure angles (no arrival angles, because no zeros)
  - Around $p_1 = j$, we have $\phi_2 = 90^o$ and $\phi_3 = 45^o$ and $\psi_{1,2} = \pm 90^o$, so $\phi_{dep} = -315^o = 45^o$
  - Around $z_1 = 2j$, we have $\phi_{1,2} = 90^o$ and $\phi_3 \approx 60^o$ and $\psi_2 = 90^o$, so $\psi_{arr} = -30^o$
- Rule 5: no multiple roots on locus

$$L(s) = \frac{s^2+4}{(s+1)(s^2+1)}$$



Negative Root Locus (Section 5.6.1)

$$L(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

	$K > 0$ and $a_0 = \boldsymbol{b_0} = \mathbf{1}$	$K < 0$ and $a_0 = \boldsymbol{b_0} = \mathbf{1}$
Rule 1	n branches start at the poles and m branches end at the zeros	
Rule 2	The loci are on the real axis to the left of an **odd** number of poles and zeros	The loci are on the real axis to the left of an **even** number of poles and zeros
Rule 3	Asymptotes have angle $\frac{180^o+360^o(l-1)}{n-m}$ with $l = 1, \ldots, n-m$ Center point $\frac{\Sigma p_i - \Sigma z_i}{n-m}$	Asymptotes have angle $\frac{360^o(l-1)}{n-m}$ with $l = 1, \ldots, n-m$ Center point $\frac{\Sigma p_i - \Sigma z_i}{n-m}$
Rule 4	Departure angles: $\phi_{dep} = \sum_i \psi_i - \sum_{i \neq dep} \phi_i - 180^o$ Arrival angles: $\psi_{arr} = \sum_i \phi_i - \sum_{i \neq arr} \psi_i + 180^o$	Departure angles: $\phi_{dep} = \sum_i \psi_i - \sum_{i \neq dep} \phi_i$ Arrival angles: $\psi_{arr} = \sum_i \phi_i - \sum_{i \neq arr} \psi_i$
Rule 5	Branches approach a point of q roots with angles $\frac{180^o+360^o(l-1)}{q}$ for $l = 1, \ldots, q$	

Note: if $b_0 = -1$, multiply $L(s)$ with -1 and take $K < 0$

Part 2: Quiz

Quiz on Root Locus – Question 1/4

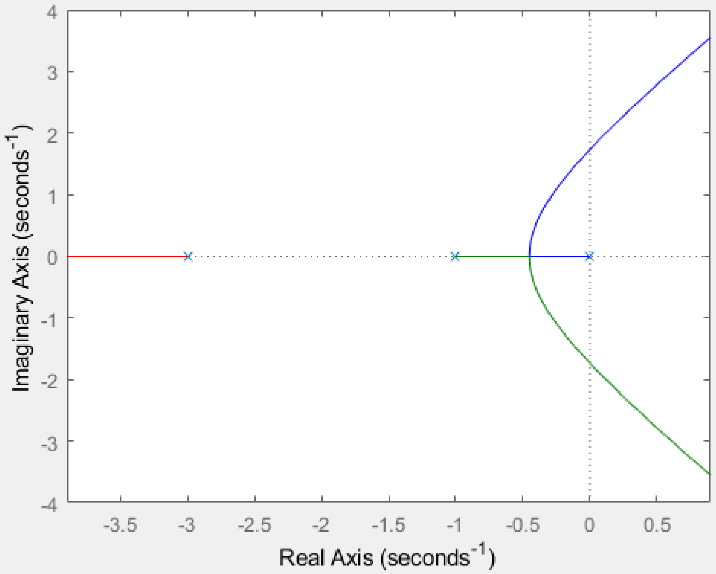
Given $L(s) = \frac{1}{s(s+1)(s+3)}$. What is the point of intersection of the asymptotes and the real axis?

a) -0.67

b) -1

c) -1.33

d) -1.67



Quiz on Root Locus – Question 2/4

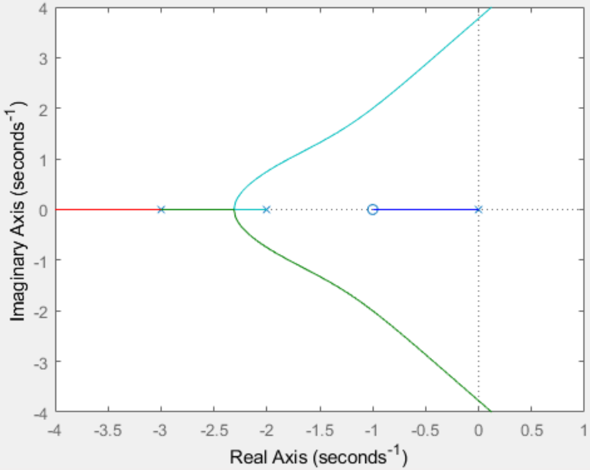
What is the corresponding plant transfer function $L(s)$ for the given root locus?

a) $L(s) = \dfrac{(s+1)}{s(s+2)^2(s+3)}$

b) $L(s) = \dfrac{(s+1)}{s(s+2)(s+3)^2}$

c) $L(s) = \dfrac{1}{s(s-1)(s+2)(s+3)}$

d) $L(s) = \dfrac{(s+1)}{s(s+2)(s+3)}$



Quiz on Root Locus – Question 3/4

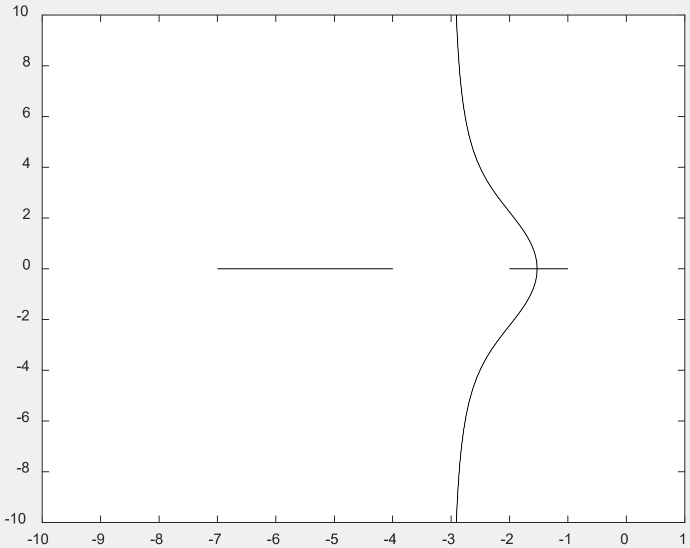
What is the corresponding plant transfer function $L(s)$ for the given root locus sketch?

a) $L(s) = \frac{(s+1)}{(s+2)(s+4)(s+7)}$

b) $L(s) = \frac{(s+4)}{(s+1)(s+2)(s+7)}$

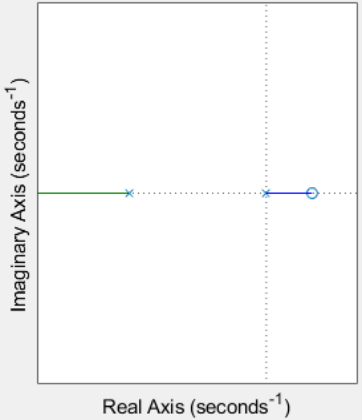
c) $L(s) = \frac{(s+7)}{(s+1)(s+2)(s+4)}$

d) $L(s) = \frac{(s+1)(s+2)}{(s+7)(s+4)}$

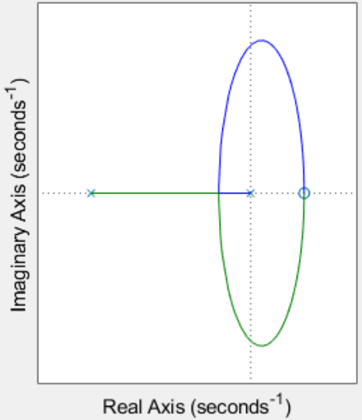


Quiz on Root Locus – Question 4/4

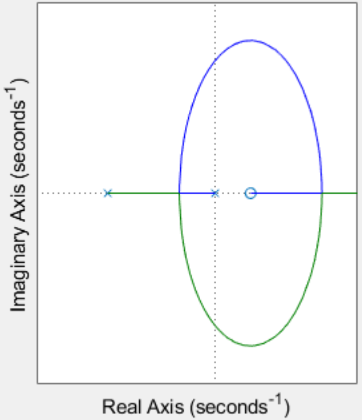
Given $L(s) = \frac{1-s}{s(s+3)}$ and $k > 0$. Find the corresponding root locus.



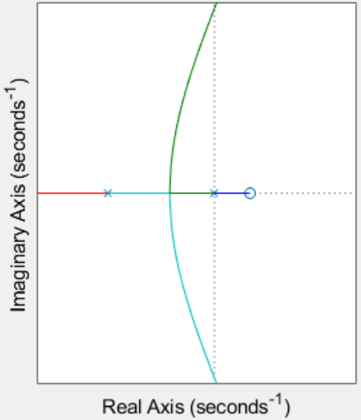
a) $L(s) = \frac{(s-1)}{s(s+3)}$



b)



c) $L(s) = \frac{(1-s)}{s(s+3)}$



d)

Wrap up

Learning objective: Understand how closed-loop system properties van be modified using feedback

- Part 1: Main Takeaways
  - Plotting root loci links open-loop behaviour to closed-loop behaviour
    - Understand why some systems become closed-loop unstable
    - Understand fundamental limitations (lecture 6)
    - Assess how adding poles and zeros in the compensator will change the closed-loop behaviour
  - Root loci can be sketched using simple rules
  - Designing controller using root loci is not a course objective (Exam: Section 5.1 – 5.3)
  - Matlab commands: rlocus or sisotool
- Part 2: Quiz
  - Sketching root loci

Suggested Reading, Exercises and Preview

- Practice with Exercise Set 2
- Watch video material / read book of Module 3 before Tuesday
  - Frequency Response / Bode diagrams