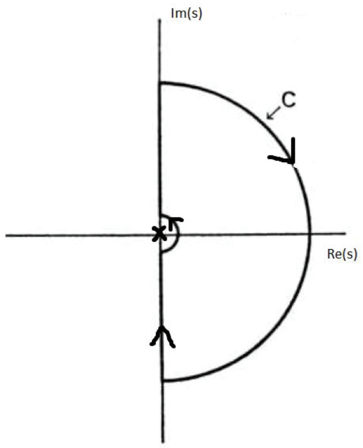
1. **Nyquist plot and Nyquist stability criterion:**
Draw the Nyquist plots for the following transfer functions and analyze the closed-loop stability according to the Nyquist criterion. Estimate the range of K for which the closed-loop systems are stable. Verify your results with MATLAB.

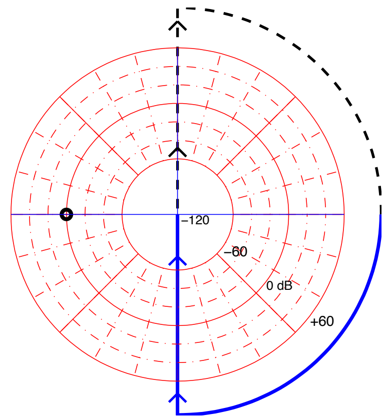
   (a) $KG(s) = \frac{1}{s}$,
   (b) $KG(s) = \frac{s+2}{s+10}$,
   (c) $KG(s) = \frac{1}{(s+10)(s+2)^2}$,
   (d) $KG(s) = \frac{(s+10)(s+1)}{(s+100)(s+2)^3}$.

Solution:

(a) The logarithmically scaled Nyquist plot is shown in Figure 1b. The plot can never encircle the point $-\frac{1}{K}$, hence the closed-loop system is stable for all $K > 0$. The contour C is given in Figure 1a. Note the half-circle with infinitely-small radius at the origin, that is, where the pole is located.



(a) The contour C.



(b) The (logarithmically scaled) Nyquist plot.

Figure 1: Nyquist for Problem 1a

(b) From Figure 2 we have that the number of clockwise encirclements is $N = 0$, and the number of unstable open-loop poles is $P = 0$, thus we derive that the number of unstable closed-loop poles is $Z = N + P = 0$. Thus, the closed-loop system is stable for $K = 1$. Moreover, it will be stable for any positive value of $K > 0$ since the Nyquist plot will never include the point $-\frac{1}{K}$ for positive K.

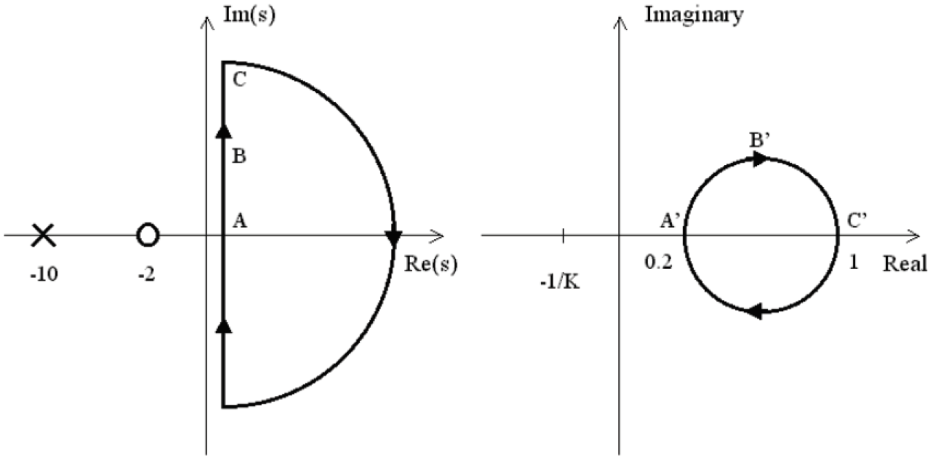


Figure 2: Problem 1b: The contour C is shown on the left, the Nyquist plot is shown on the right.

(c) The Nyquist plot is shown in Figure 3. To determine the point where the Nyquist plot crosses the negative real axis by hand, we will first look at the transfer function for $K = 1$. This transfer function is given as

$$
\begin{aligned}
G(j\omega) &= \frac{1}{(-14\omega^2 + 40) + j(-\omega^3 + 44\omega)} \frac{(-14\omega^2 + 40) - j(-\omega^3 + 44\omega)}{(-14\omega^2 + 40) - j(-\omega^3 + 44\omega)} \\
&= \frac{(-14\omega^2 + 40) + j(\omega^3 - 44\omega)}{\omega^6 + 108\omega^4 + 816\omega^2 + 1600}.
\end{aligned}
$$

The Nyquist plot crosses the real axis when $G(j\omega)$ becomes real valued, which happens when $\omega^3 - 44\omega = 0$. This is the case for $\omega = 0$ and $\omega = \sqrt{44}$.

Inserting these values into $G(j\omega)$ gives us a value of 0.025 for $\omega = 0$ and -0.0017 for $\omega = \sqrt{44}$. Since the Nyquist plot crosses the negative real axis with magnitude equal to 0.00174, it will not encircle the $-\frac{1}{K}$ point until $K = \frac{1}{0.00174} = 576$. Indeed, for $K \in (0, 576)$, we have $N = 0$, $P = 0$ and $Z = N + P = 0$, hence the closed-loop system is stable. On the other hand, when $K > 576$ we have $N = 2$ (2 clockwise encirclements), $P = 0$ and $Z = 2 + 0 = 2$, i.e., the closed-loop system has two unstable poles.

(d) The Nyquist plot is shown in Figure 4. The plot will never encircle the $-\frac{1}{K}$ point, hence the closed-loop system is stable for all $K > 0$.

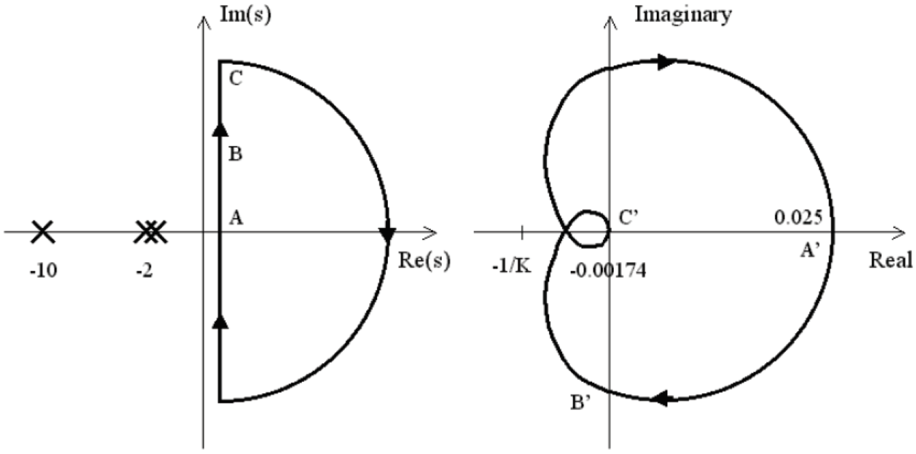


Figure 3: Problem 1c: The contour C is shown on the left, the Nyquist plot is shown on the right.

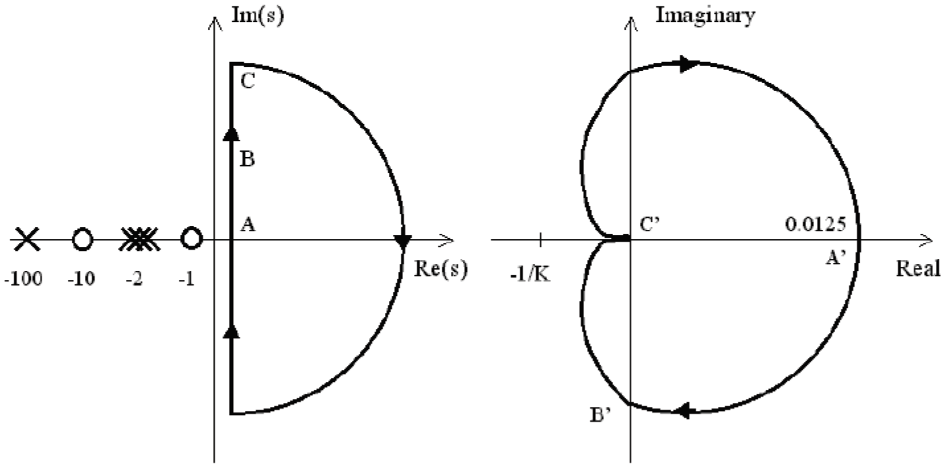
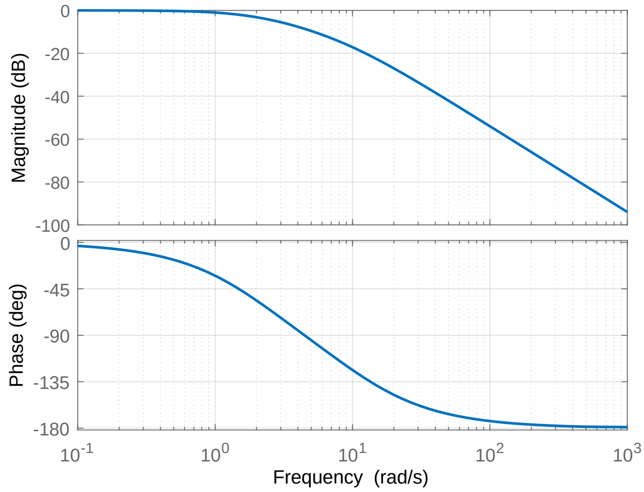


Figure 4: Problem 1d: The contour C is shown on the left, the Nyquist plot is shown on the right.

2. **Nyquist from bode plot (Exam Level Question):**
   Draw the Nyquist plots for the following bode diagrams and analyze the closed-loop stability according to the Nyquist criterion. Also, compute the gain and phase margins.

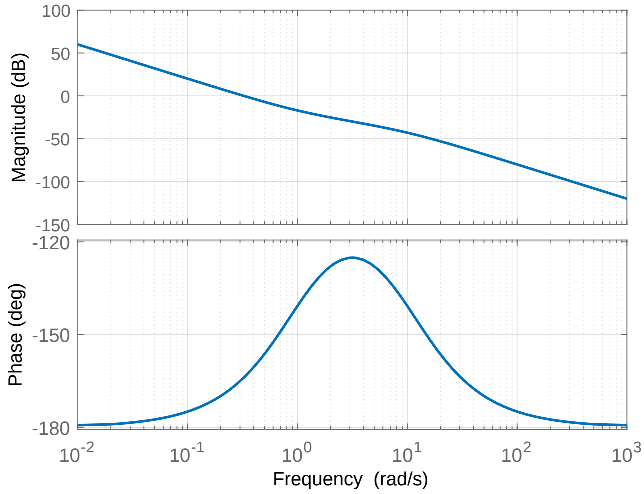
$$G(s) = \frac{20s}{s(s+2)(s+10)}$$

(a)



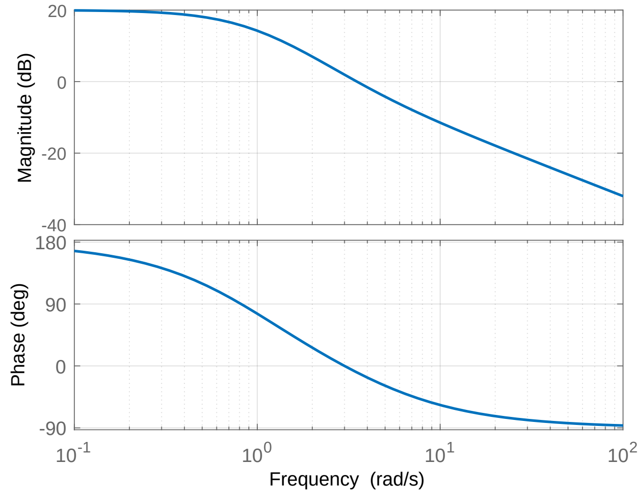
$$G(s) = \frac{s+1}{s^2(s+10)}$$

(b)



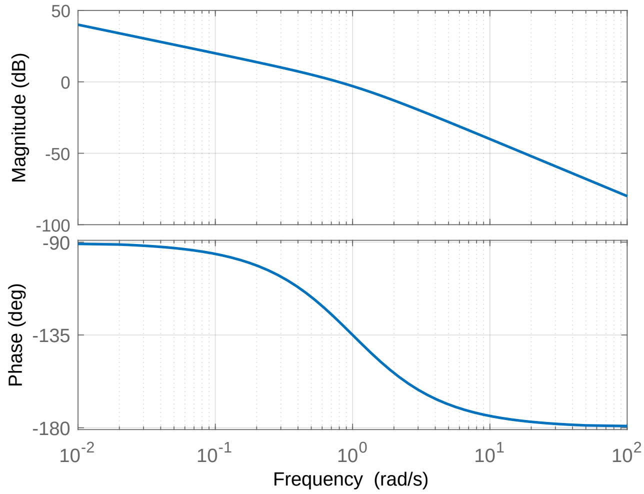
$$G(s) = \frac{2.5(s-4)}{(s+1)^2}$$

(c)



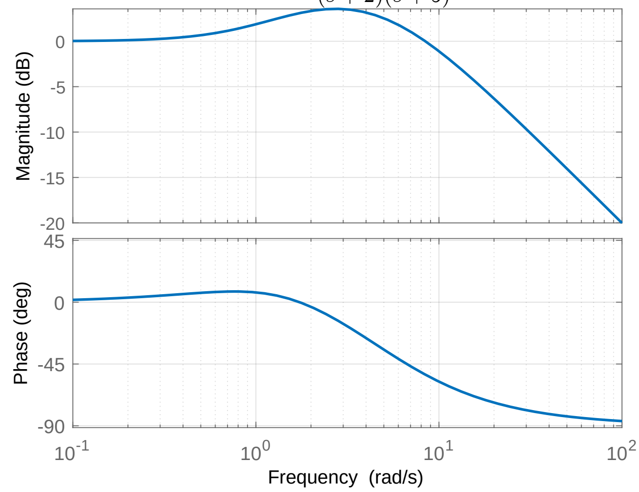
(d)

$$G(s) = \frac{s}{s^2(s+1)}$$



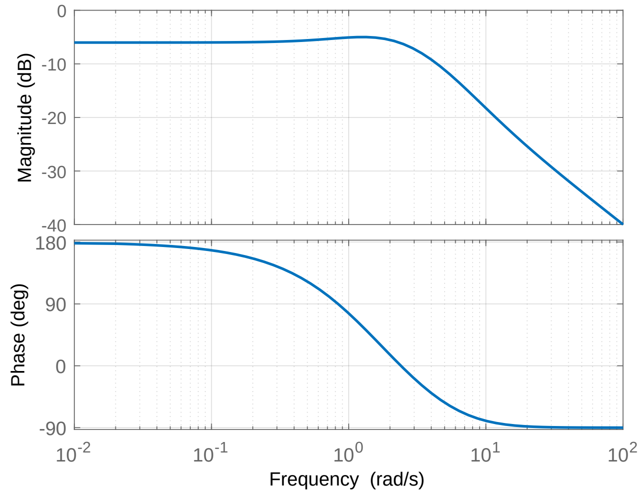
(e)

$$G(s) = \frac{10(s+1)}{(s+2)(s+5)}$$



(f)

$$G(s) = \frac{(s-1)(s+10)}{(s+5)(s+2)^2}$$



Solution:

(a) To find the magnitude and phase, we substitute $s = j\omega$ into the transfer function. This will result in,

$$G(j\omega) = \frac{20j\omega}{j\omega(j\omega+2)(j\omega+10)} = \frac{20}{(j\omega+2)(j\omega+10)},$$

which is equivalent to,

$$G(j\omega) = \frac{20}{-\omega^2 + 12j\omega + 20}$$

For the gain margin, we need to find the frequency ω where the phase $\angle G(j\omega)$ is -180 degrees. By separating the real and imaginary part of $G(j\omega)$, we get,

$$|G(j\omega)| = \frac{20}{\sqrt{(-\omega^2+20)^2 + (12\omega)^2}}, \quad \angle G(j\omega) = \arctan\left(\frac{12\omega}{-\omega^2+20}\right).$$

Since

$$\angle G(j\omega) = \arctan\left(\frac{12\omega}{-\omega^2+20}\right) = -180,$$

has no solution, i.e., the -180°phase is never reached, we conclude the gain margin is infinite. The phase margin is determined by finding the frequency ω where the magnitude $|G(j\omega)| = 1$.

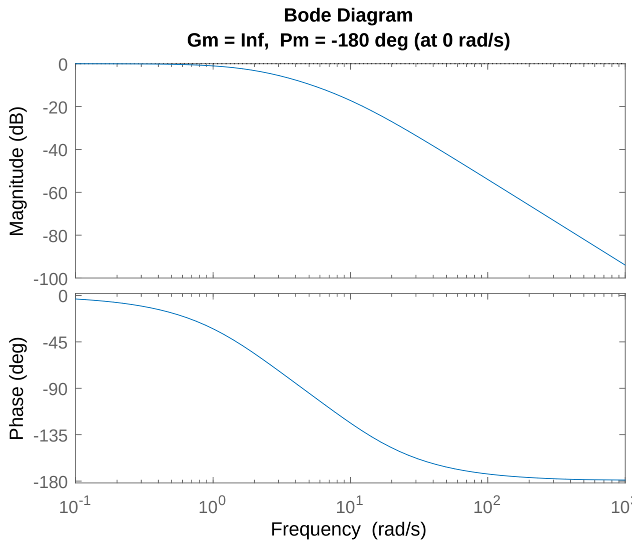
$$\begin{aligned} |G(j\omega)| = \frac{20}{\sqrt{(-\omega^2+20)^2 + (12\omega)^2}} &= 1, \\ 20 &= \sqrt{(-\omega^2+20)^2 + (12\omega)^2} \\ 400 &= \omega^4 + 104\omega^2 + 400, \\ \omega &= 0. \end{aligned}$$

$\omega = 0\text{rad/s}$ gives,

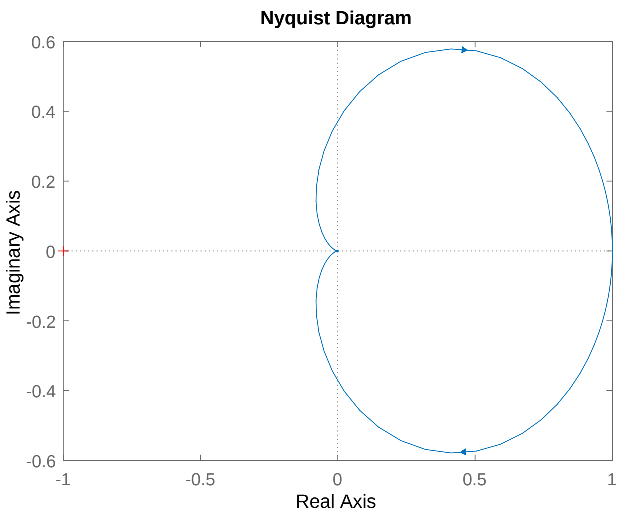
$$\angle G(j\,0) = \arctan\left(\frac{12\omega}{-\omega^2+20}\right) = 0.$$

Hence the phase margin is 180°.

Regarding the nyquist stability criterion, $Z = N + P = 0 + 0$, see the nyquist diagram below.



(a) Bode diagram (a)



(b) Nyquist diagram (a)

(b) To find the magnitude and phase, again we substitute $s = j\omega$ into the transfer function. This will result in,

$$G(j\omega) = \frac{j\omega + 1}{-\omega^2(j\omega + 10)} = \frac{j\omega + 1}{-10\omega^2 - j\omega^3}. \tag{1}$$

Separating the real and complex parts as,

$$G(j\omega) = \frac{j\omega + 1}{-10\omega^2 - j\omega^3}\frac{-10\omega^2 + j\omega^3}{-10\omega^2 + j\omega^3} = \frac{-9j\omega^3 - \omega^4 - 10\omega^2}{100\omega^4 + \omega^6}$$

For the gain margin, we need to find the frequency ω where the phase $\angle G(j\omega)$ is -180 degrees.

$$\angle G(j\omega) = \arctan\left(\frac{-9\omega^3}{-\omega^4 - 10\omega^2}\right) = -180.$$

Since there is no ω for which the above equation holds, i.e., the -180°phase is never reached for the transfer function, we conclude the gain margin is infinite.
The phase margin is determined by finding the frequency ω where the magnitude $|G(j\omega)| = 1$.

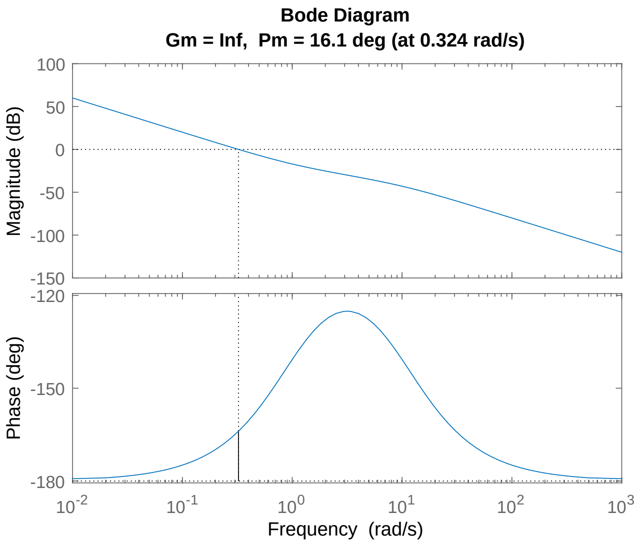
$$\begin{aligned}|G(j\omega)| = \frac{\sqrt{\omega^2+1}}{\sqrt{(-\omega^3)^2 + (-10\omega^2)^2}} &= 1,\\ \omega^2 + 1 &= \omega^6 + 100\omega^4,\\ \omega &\approx 0.324.\end{aligned}$$

$\omega = 0.324$rad/s gives,

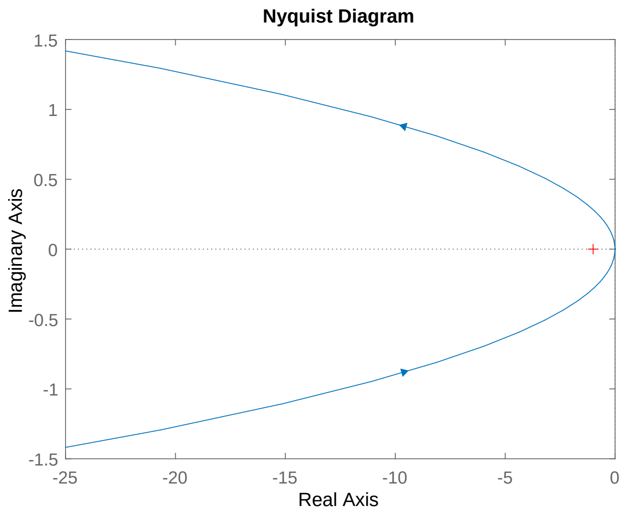
$$\angle G(0.324) = \arctan\left(\frac{-9\omega^3}{-\omega^4 - 10\omega^2}\right) \approx -164.$$

Note that the arctan above will give 16°. However, when using the arctan with complex numbers, one should not forget to check the quadrant the complex number is as the arctan only gives an angle between -90 and 90°. So in this case, $16 - 180 = -164$ is the angle of the complex number. The phase margin of the transfer function is $180 - 164 = 16$°.

Regarding the nyquist stability criterion, $Z = N + P = 0 + 0$, hence the system is stable.



(a) Bode diagram (b)



(b) Nyquist diagram (b)

(c) To find the magnitude and phase margin, again we substitute $s = j\omega$ into the transfer function,

$$G(j\omega) = \frac{2.5(j\omega - 4)}{(j\omega + 1)^2}.$$

Separating the real and complex parts gives,

$$G(j\omega)\frac{-10+15\omega^2}{1+\omega^4+2\omega^2}+i\frac{-2.5\omega^3+22.5\omega}{1+\omega^4+2\omega^2}.$$

For the gain margin, we need to find the frequency ω where the phase $\angle G(j\omega)$ is -180 degrees.

$$\angle G(j\omega)=\arctan\left(\frac{-2.5\omega^3+22.5\omega}{-10+15\omega^2}\right)=-180.$$

The above equation doesn't seem to have a solution. However, for $\omega=0$, the arctan is equal to zero and looking at $G(j\omega)$ for $\omega=0$ shows that the complex value is on the real axis at -180°. The gain margin is then calculated as,

$$GM=\frac{1}{|G(j\,0)|}=\frac{1}{10}=-20dB.$$

The phase margin is determined by finding the frequency ω where the magnitude $|G(j\omega)|=1$.

$$|G(j\omega)|=\sqrt{\frac{6.25\omega^6+112.5\omega^4+206.25\omega^2+100}{(1+\omega^4+2\omega^2)^2}}=1,$$
$$\omega=\frac{\sqrt{17+5\sqrt{265}}}{2\sqrt{2}}\vee\omega=-\frac{\sqrt{17+5\sqrt{265}}}{2\sqrt{2}}.$$

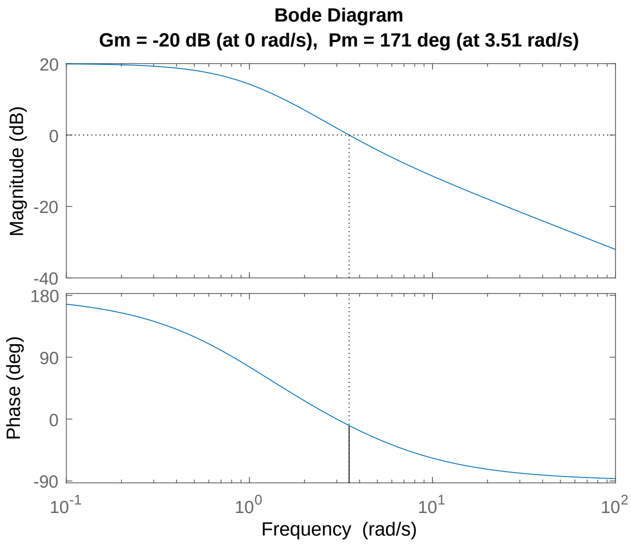
$\omega=3.507\text{rad/s}$ gives,

$$\angle G(3.507)=\arctan\left(\frac{-2.5\omega^3+22.5\omega}{-10+15\omega^2}\right)\approx-9.4123.$$

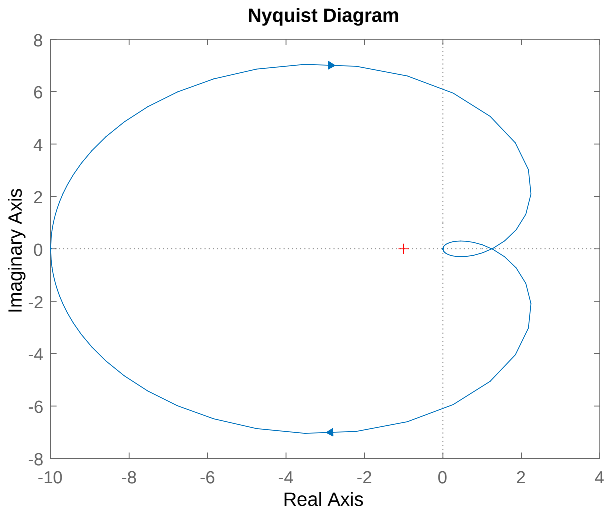
The phase margin is thus,

$$PM=180-9.4123\approx171°.$$

Regarding the nyquist stability criterion, $Z=N+P=1+0$, i.e., one clockwise encirclement of the -1 point, as can be seen below. The closed-loop system has one unstable pool.

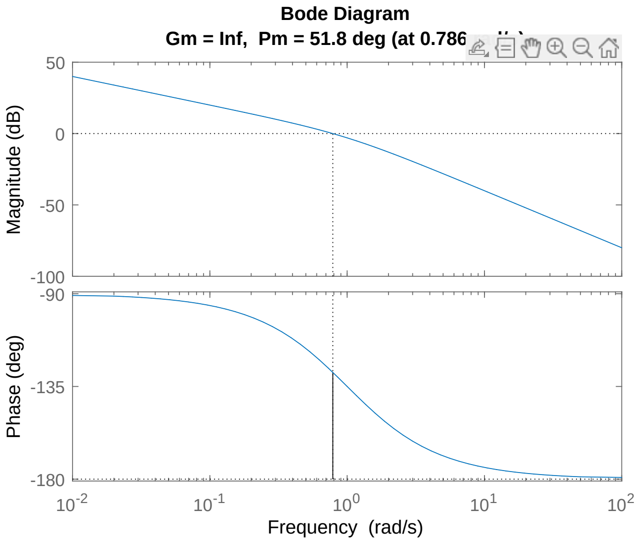


(a) Bode diagram (c)

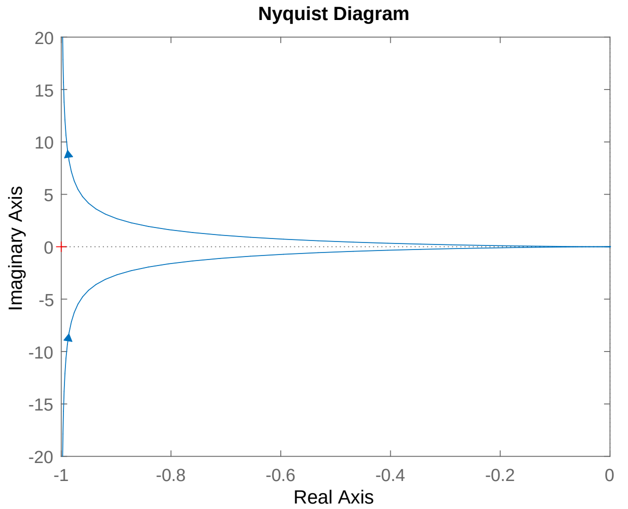


(b) Nyquist diagram (c)

(d)

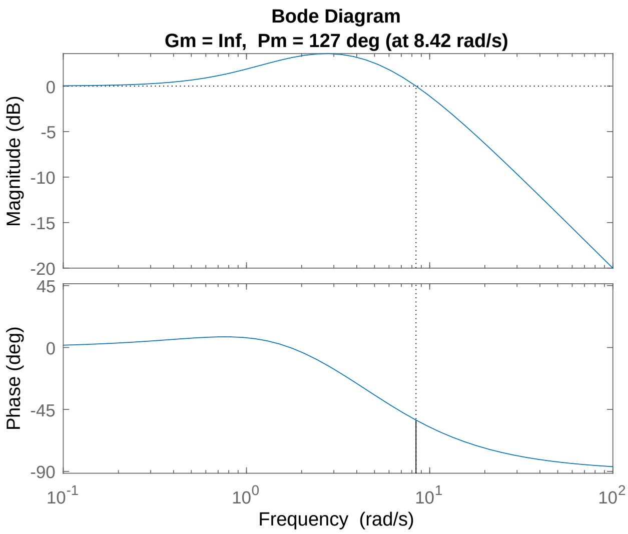


(a) Bode diagram (d)

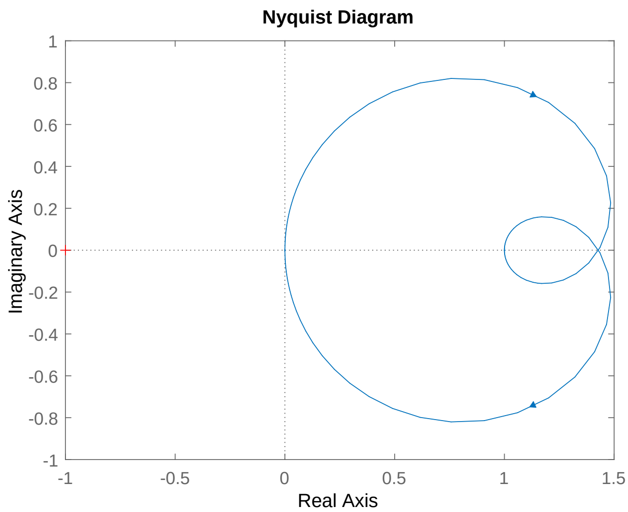


(b) Nyquist diagram (d)

(e)

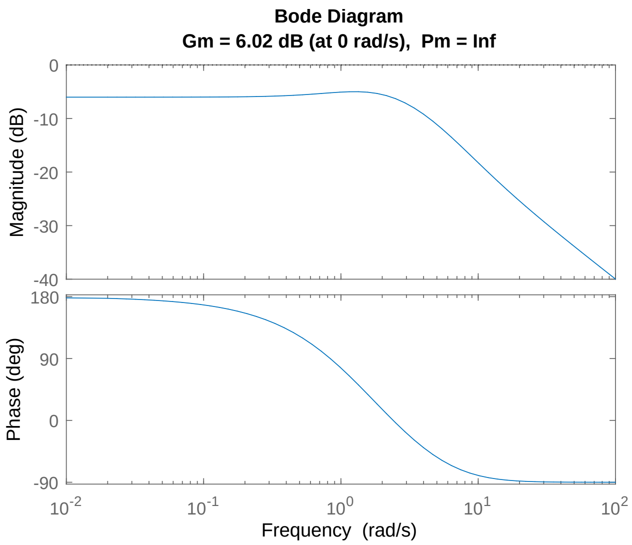


(a) Bode diagram (e)

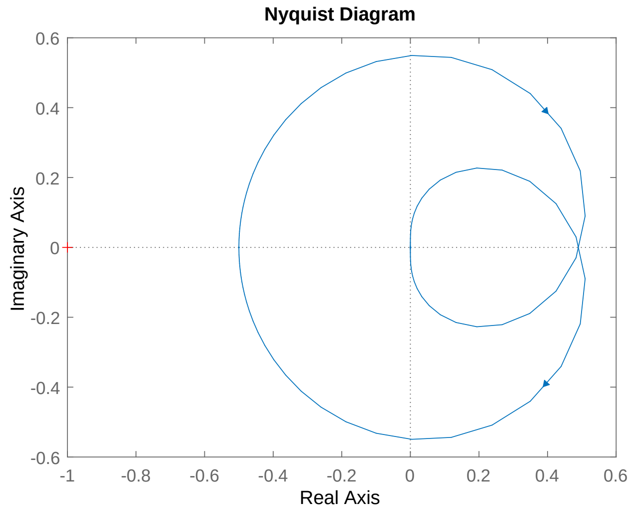


(b) Nyquist diagram (e)

(f)



(a) Bode diagram (f)



(b) Nyquist diagram (f)

3. **Stability margins:**
Consider the Nyquist plot for an open-loop stable system, shown in Figure 11.

   (a) Determine which are the gain and phase margins for the system, given that $\alpha = 0.4$, $\beta = 1.3$, and $\phi = 40^o$.

   (b) Describe what happens to the stability of the closed-loop system as the gain K goes from 0 to a very large value.

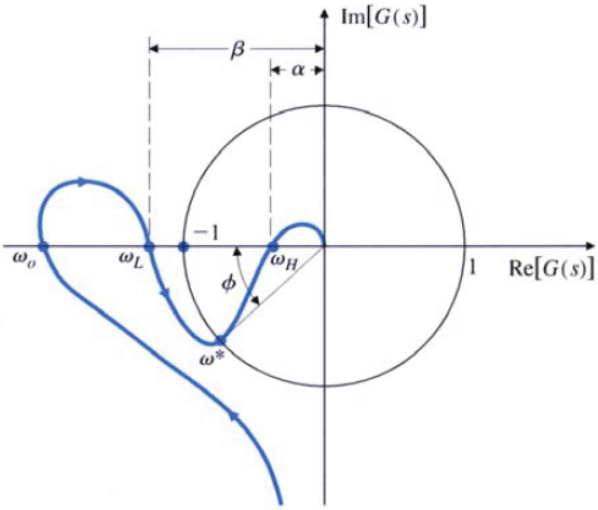


Figure 11: Problem 2: Nyquist plot.

Solution:

(a) By definition, the phase margin (PM) is the amount by which the phase can be decreased before reaching instability. In this case, it is straightforward to see that PM $= \phi = 40°$. For the gain margin (GM), there are two possibilities for which the system becomes unstable, namely, the gain is increased by a factor of $\frac{1}{\alpha}$ or decreased by a factor of β. This is a conditionally stable system (see Figure 6.40, Franklin book for a typical root locus of a conditionally stable system). Thus, we have $GM_1 = -20\log(\alpha)$ dB for $\omega = \omega_H$ and $GM_2 = 20\log(\beta)$ dB for $\omega = \omega_L$.

(b) For very low values of the gain K, the entire Nyquist plot shrinks and the -1 point occurs to the left of the negative real axis crossing at ω_0, so there would be no encirclements and the system would be stable (another way to see that is leaving the plot as it is and observing that the point $-\frac{1}{K}$ moves to the left of the ω_0 point). As the gain K increases, the -1 point occurs between ω_0 and ω_L, thus, there is an encirclement and the system becomes unstable. Further increase on the gain brings point -1 between ω_L and ω_H, thus, no encirclement happens and the system is stable. If the gain increases even more, it moves between ω_H and 0 and the system becomes unstable. A rough sketch of the Bode plot of such a system is shown in Figure 12.

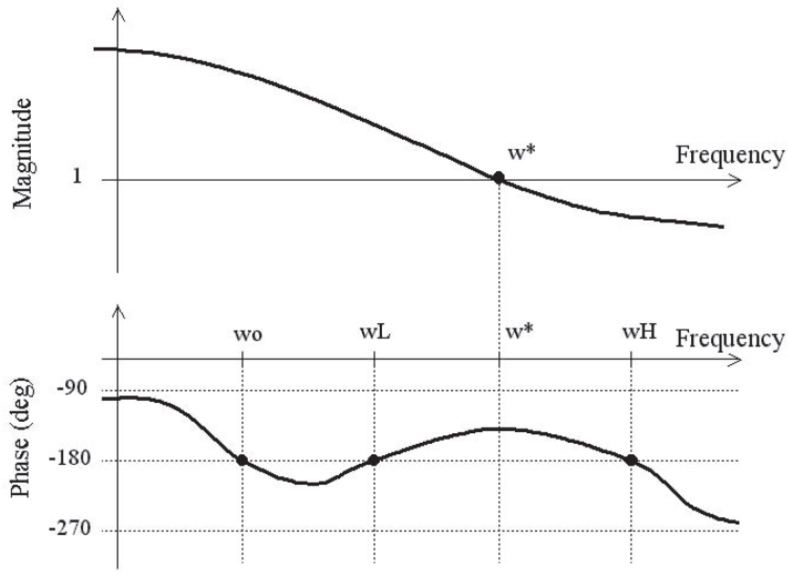


Figure 12: A rough sketch of the Bode plot for the system.

4. **Stability margins continued:**
Consider the closed-loop system in Figure 13.

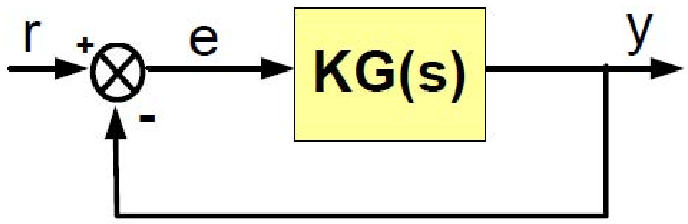


Figure 13: Block diagram of the system.

The steering dynamics of a ship are represented by the transfer function

$$\frac{V(s)}{\delta_{\mathrm{r}}(s)} = KG(s) = \frac{K\,(1 - s/0.142)}{s\,(1 + s/0.325)\,(1 + s/0.0362)},$$

where V is the ship lateral velocity (in meters per second) and δ_{r} is the rudder angle (in radians).

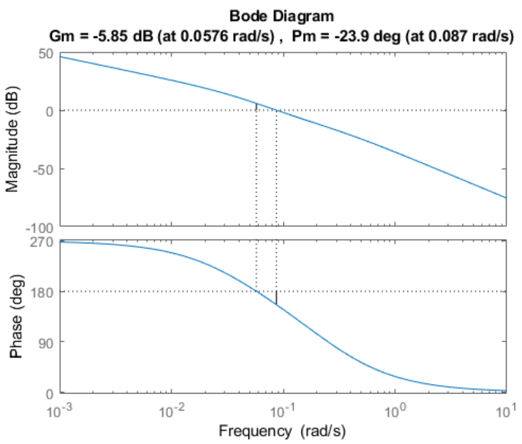
(a) Use `Matlab`'s `margin` command to evaluate the phase and gain margin of $KG(s)$ for $K = 0.2$ and asses whether the closed-loop system is stable or not. Verify your answer by also plotting the Nyquist plot.

(b) In order to stabilize the system, should you increase or decrease the gain and by how much. Verify your thinking by changing the gain.

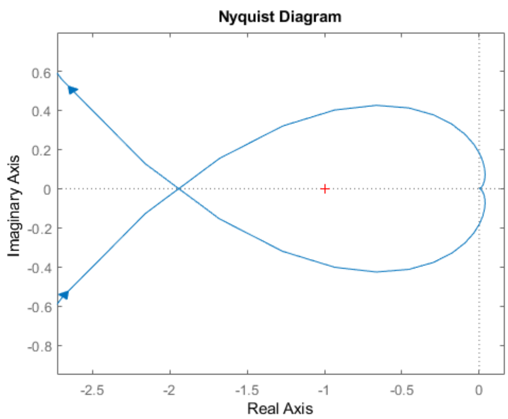
(c) Is the stabilized system well-damped? If not, how could you achieve a damping coefficient, ζ, of 0.3 by changing the gain. Verify your thinking by changing the gain.

Solution:

(a) Evaluating the gain and phase we see that both are negative which signal that the system is closed-loop unstable for $K = 0.2$. This can also be verified from the Nyquist plot shown in Figure 14b since we have an encirclement of the point -1. Hence $Z = N + P \neq 0$ and thus the system is unstable.



(a) Bode plot of the system.



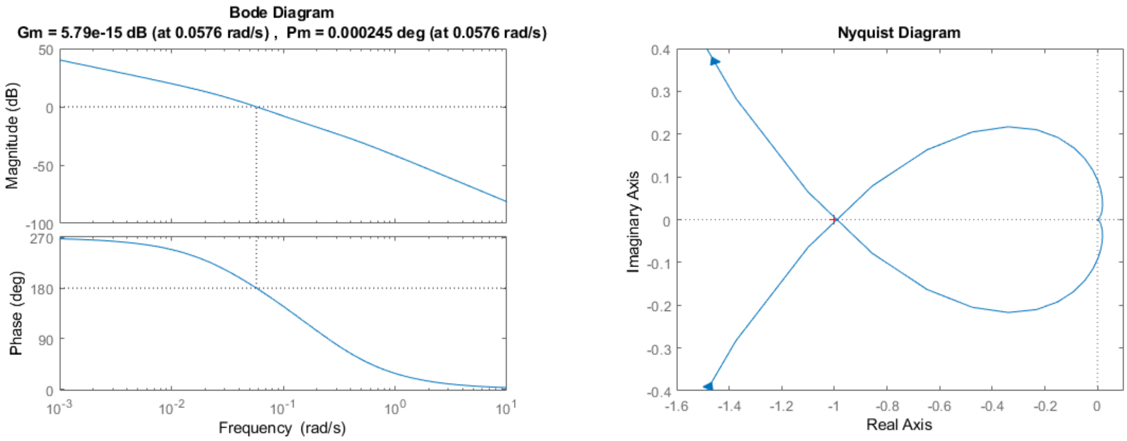
(b) Nyquist plot of the system

Figure 14: Bode and Nyquist plot for the system depicted in Figure ??

(b) We note that the gain margin is -5.85 dB. Decreasing the gain K by a factor of 5.85 dB would place the crossover frequency at a phase of 180 degrees and the system will be on the boundary of stability. Hence our new gain will be

$$K_{stab} = 0.5099K = 0.102$$

Plotting the Bode and Nyquist plot now indeed shows the system to be stable, Figure 15.



(a) Bode plot of the system. (b) Nyquist plot of the system

Figure 15: Bode and Nyquist plot for the system depicted in Figure **??** with $K_{stab} = 0.102$

(c) As an approximation we can say that the damping coefficient is

$$\zeta = \frac{PM}{100}$$

Since our phase margin is close to zero we can say that this system is sincerely under-damped. Using the same approximation for zeta, we see that for a damping coefficient of 0.3 we need a phase margin of approximately 30 degrees. From the Bode plot in Figure 15a we note that this would be achieved if the crossover frequency was located at 0.03 rad/s. The magnitude at this frequency is 8.42 dB. Hence, to place the crossover frequency at 0.03 rad/s, we need to decrease the gain by a factor of 8.42 dB resulting in

$$K_{damp} = 0.3793K_{stab}$$

Evaluating the damping of the closed-loop system with K_{damp} now shows a damping coefficient of approximately 0.3.

5. **Closed-loop bandwidth:**

Consider the open-loop system $K\,G(s) = K\,\frac{s+1}{s^2(s+10)^2}$.

(a) Determine the value of K at the stability boundary and the value(s) for which PM $= 30°$.

(b) Determine the range of K for which PM $\geq 30°$ and the closed-loop bandwidth is maximized.

Solution:

(a) The Bode plot of the system with $K = 1$ is shown in Figure 16. First let's rewrite $G(j\omega)$ as a complex number

$$\begin{aligned} G(j\omega) &= \frac{j\omega + 1}{(\omega^4 - 100\omega^2) + j(-20\omega^3)} \frac{(\omega^4 - 100\omega^2) - j(-20\omega^3)}{(\omega^4 - 100\omega^2) - j(-20\omega^3)} \\ &= \frac{(-19\omega^4 - 100\omega^2) + j(\omega^5 - 80\omega^3)}{\omega^8 + 200\omega^6 + 10000\omega^4} \end{aligned}$$

The gain margin is GM= $\frac{1}{G(j\omega_{180})}$, where the phase of $G(j\omega)$ crosses $-180°$ in ω_{180}. We know that the phase is $-180°$ when $G(j\omega)$ is real valued, which is true when $\omega^3(\omega^2 - 80) = 0$. This holds for $\omega = 0$ and $\omega = \sqrt{80}$.

Note that the real part of the transfer function goes to 0 in $\omega = 0$ and therefore we do not necessarily have an angle of $-180°$ at $\omega = 0$. From our sketch we know that the phase goes in the limit to $-180°$ as ω goes to 0, but it will actually never cross it.

Therefore we can conclude that $\omega_{180} = \sqrt{80}$. Substituting this value into $G(j\omega)$ gives GM = 64.1 dB = 1600, hence the range of K for stability is $(0, 1600)$.

The phase margin is PM=30° if the phase is $-150°$ in ω_c, where ω_c is chosen such that $|G(j\omega_c)| = 1$. First we will determine the values of ω for which the phase is $-150°$. Observing that $G(j\omega)$ is a complex number, we get that

$$\tan(-150°) = \frac{\omega_{150}^5 - 80\omega_{150}^3}{-19\omega_{150}^4 - 100\omega_{150}^2}$$

where we have a phase of $-150°$ in ω_{150}. This equation is very hard to solve by hand, but from our Bode sketch we expect 2 values for ω_{150} around the zero of the system. Iteratively substituting some values for ω into the equation gives us approximate values for ω_{150}.

The magnitude of our transfer function is 0.0188 (-34.5 dB) at $\omega_{150} = 0.8282$ rad/sec, and the magnitude is equal to 0.00198 (-54.1 dB) at $\omega_{150} = 4.44$ rad/sec. Therefore, the values of K at the points where PM= 30° are $K = \frac{1}{0.0188} = 53.2$ and $K = \frac{1}{0.00198} = 505$.

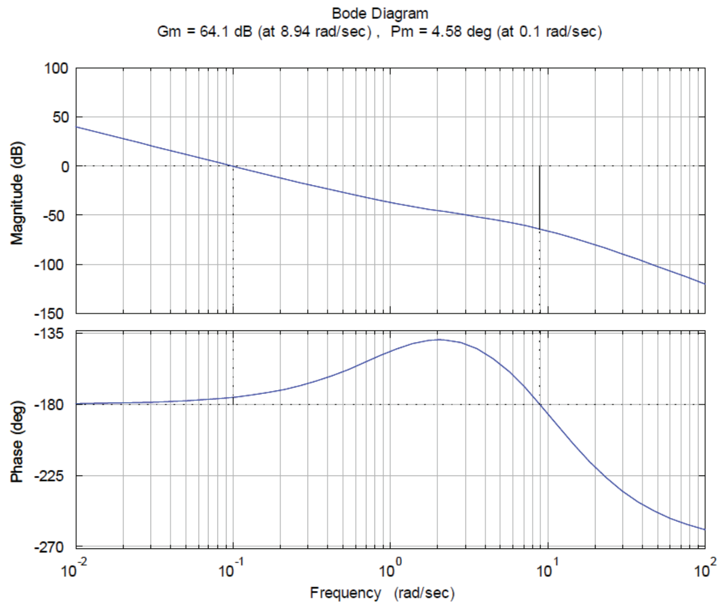


Figure 16: Bode plot with $K = 1$.

(b) From (a), the range of K for which PM $\geq$ 30° is $[53.2, 505]$. The maximum closed-loop bandwidth will occur with the maximum gain K within the allowed region. Therefore, the maximum bandwidth will occur when $K = 505$.

The Bode plot for $K = 505$ is shown in Figure 17. Looking at the point with magnitude 0.707 (-3 dB), the maximum possible closed-loop bandwidth is $\omega_{\max} = 7.7$ rad/sec.

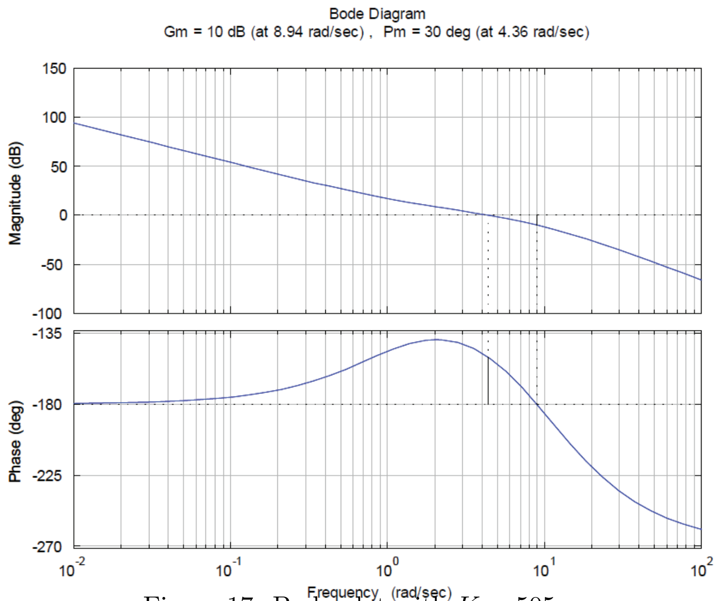


Figure 17: Bode plot with $K = 505$.