Digital Signal Processing Fundamentals (5ESC0)

The DFT

Sveta Zinger, Piet Sommen, Elisabetta Peri

(s.zinger@tue.nl, e.peri@tue.nl)

Learning Outcomes

* Definition of Discrete Fourier Transform (DFT)
* Choice of the length N of the DFT
* Properties of the DFT
* Relationship between the DFT and the other Fourier Transforms

The DFT

* The Discrete Fourier Transform

* A practical way to calculate (an approximation of) the FTD

* When transforms are used in your computer, in most cases the DFT is used because it is discrete

DFT for Periodic Signals

* We know the (I)FTD pair:

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n} \circ\!\!-\!\!\circ\; x[n] = \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} X(e^{j\theta})e^{jn\theta}\,d\theta$$

* For periodic signals, this relation gives convergence problems, as the summation becomes infinite

* We have a periodic signal (subscript p denotes 'periodic') with period N:

$$x_p[n] = x_p[n + l \cdot N] \qquad \text{for} \qquad l = 0, \pm 1, \pm 2, \ldots$$

* We look at another approach

DFT for Periodic Signals

* $x_p[n] = x_p[n + l \cdot N]$ for $l = 0, \pm 1, \pm 2, \ldots$

* We use the following characteristics of periodic signals:
  - It can only have frequencies that are multiplicities of $\theta_0 = \frac{2\pi}{N}$, besides $\theta = 0$ (DC)
  - All the frequency content will be in one period (one sequence of N samples)

* With this approach we find the (I)DFT pair:

$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \circ\!\!-\!\!\circ X[k] = \sum_{n=0}^{N-1} x_p[n] e^{-j\frac{2\pi}{N}kn}$$

* We only consider one period and in the exponent, θ is replaced by $\frac{2\pi}{N}k$
* $X[k]$ is now discrete with index k and both $x_p[n]$ and $X[k]$ are periodic

DFT for Periodic Signals

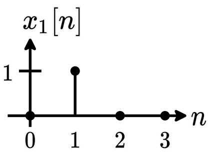
* (I)DFT pair:

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} \circ\!\!-\!\!\circ\; X[k] = \sum_{n=0}^{N-1} x_p[n]e^{-j\frac{2\pi}{N}kn}$$

* As we have seen before, **sampling in one domain means repetition in the other domain**
* $X[k]$ is a sampled function, which means we expect a repetition in the other domain
* Since $x_p[n]$ is periodic, there is repetition
* Samples $x_p[n] \equiv$ Periodic Extended (PE) finite length N sequence $x[n]$
* If no confusion, we skip subscript p

DFT for Periodic Signals: example 1 DFT

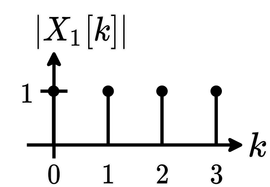
* $x_1[n] = \begin{cases} 1, & \text{for } n = 4 \cdot l + 1 \\ 0, & \text{otherwise} \end{cases}$, where l is an integer

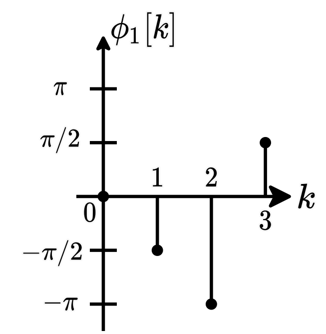


* This sequence has a period of $N = 4$, so we find a 1 at $n = 1, 5, 9, \ldots$
* Now we want to find the DFT of this sequence, so we fill in the equation:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{3} x[n] e^{-j\frac{\pi}{2}k} = e^{-j\frac{\pi}{2}\cdot \mathrm{k}}$$

* We find the magnitude and phase:
  $|X[k]| = 1$ and $\angle\{X[k]\} = -\frac{\pi}{2} \cdot k$
* Whereas θ is continuous, now we evaluate $N = 4$ points on the unit circle





DFT for Periodic Signals: example 2 IDFT

* $X_2[k] = \begin{cases} 1, & k = 2 + 16 \cdot l \\ 1, & k = 14 + 16 \cdot l \\ 0, & \text{otherwise} \end{cases}$, where l is an integer
* Now we want to find the IDFT $x_2[k]$ of $X_2[k]$:

$$x_2[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} = \frac{1}{16}\sum_{k=0}^{15} X[k]e^{j\frac{\pi}{8}kn} = \frac{1}{16}(e^{j\frac{2\pi}{8}n} + e^{j\frac{14\pi}{8}n})$$

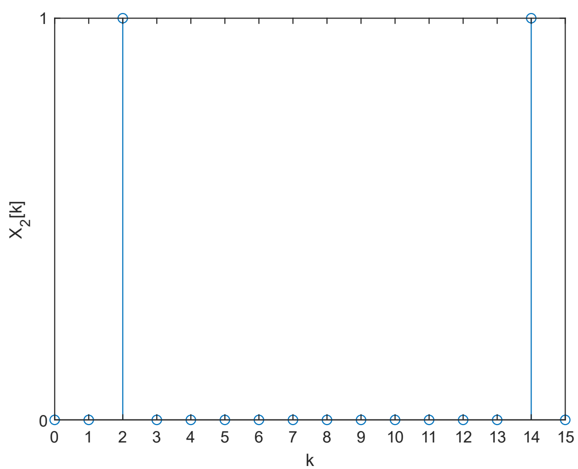
* We recognize something that looks like an Euler expression

DFT for Periodic Signals: example 2 IDFT

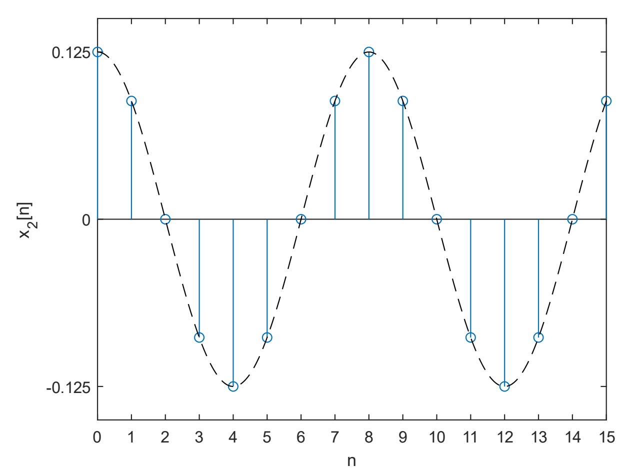
* $x_2[n] = \frac{1}{16}\left(e^{j\frac{2\pi}{8}n} + e^{j\frac{14\pi}{8}n}\right)$
* Because it is periodic, we find the same value after shifting with the period $N = 16$, therefore:

$$e^{j\frac{14\pi}{8}n} = e^{j\frac{(14-16)\pi}{8}n} = e^{j\frac{-2\pi}{8}n}$$

* Using Euler, we find: $x_2[n] = \frac{1}{8}\cos(\frac{\pi}{4}n)$



$\circ\!\!-\!\!\circ$



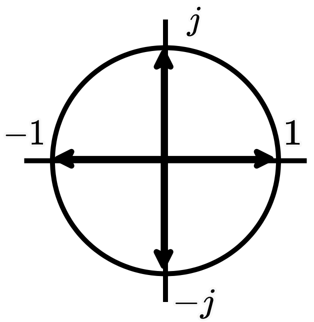
Orthogonality property

* When evaluating the DFT, often the so called twiddle factor is used:

$$W_N = e^{j\frac{2\pi}{N}}$$

* Property:

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} N & \text{for } k = l \cdot N \text{ with } l = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$



* Say we take $N = 4$, then $\sum_{n=0}^{3} e^{\frac{j\pi}{2}kn} = A_k$
* $A_0 = e^{j0} + e^{j0} + e^{j0} + e^{j0} = 1 + 1 + 1 + 1 = 4$
* $A_1 = e^{j0} + e^{j\frac{\pi}{2}} + e^{j\pi} + e^{j\frac{3\pi}{2}} = 1 + j - 1 - j = 0$
* $A_2 = A_3 = 0,\ A_4 = 1$
* Only for $k = l \cdot \mathrm{N}$ we will find a value N, otherwise we add complex exponentials each for which there is an exponential with a π phase shift, which sums to 0

Orthogonality property

* Example 3: Calculate DFT length $N = 6$ of $x_3[n] = \cos(2\pi n/6)$
* We know: $X_3[k] = \sum_{n=0}^{N-1} x_3[n] e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{5} \cos\left(\frac{2\pi n}{6}\right) e^{-j\frac{2\pi}{6}nk}$
* We use Euler: $X_3[k] = \frac{1}{2}\sum_{n=0}^{5}\left(e^{j\frac{2\pi}{6}n} + e^{-j\frac{2\pi}{6}n}\right)e^{-j\frac{2\pi}{6}nk}$
* We can split this: $= \frac{1}{2}\sum_{n=0}^{5} e^{-j\frac{2\pi}{6}n(k-1)} + \frac{1}{2}\sum_{n=0}^{5} e^{-j\frac{2\pi}{6}n(k+1)}$
* Using the twiddle factor, we know that the first sum will equal 6 for $k - 1 = 6 \cdot l$ and 0 otherwise, where l is an integer
* The second term will, likewise, equal 6 for $k + 1 = 6 \cdot l$ and 0 otherwise
* It follows that for integer l:

$$X_3[k] = \begin{cases} 3, & k = 6 \cdot l + 1 \\ 3, & k = 6 \cdot l - 1 \\ 0, & \text{otherwise} \end{cases}$$

How to choose DFT length N for periodic signals

- Now let us have a look at different signals to see how the DFT works
- We will consider periodic signals
- We will look at what effect the length N has on the DFT of a signal

How to choose DFT length N for periodic signals

* We consider the same example, but now of FTD length $N = 16$:
  $x_4[n] = x_3[n] = \cos(\frac{2\pi n}{6})$
* We use the same approach: $X_4[k] = \sum_{n=0}^{16-1} \cos(2\pi n/6)\, e^{-j\frac{2\pi}{16}nk}$

$$X_4[k] = \frac{1}{2}\sum_{n=0}^{15}\left(e^{j\frac{2\pi}{6}n} + e^{-j\frac{2\pi}{6}n}\right)e^{-j\frac{2\pi}{16}nk}$$

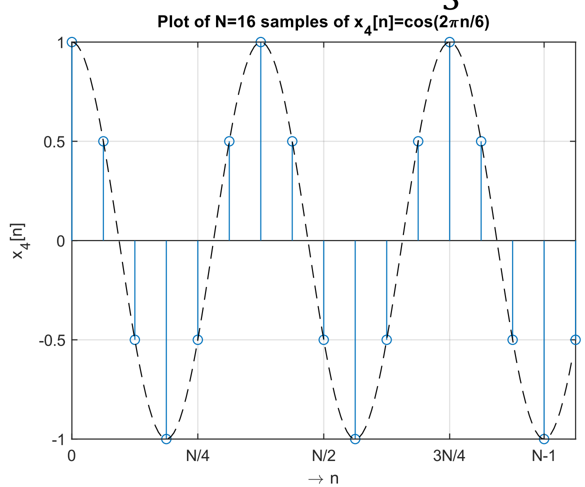
$$= \frac{1}{2}\sum_{n=0}^{15}\left(e^{j\frac{2\pi}{16}\left(\frac{16}{6}-k\right)n} + e^{-j\frac{2\pi}{16}\left(\frac{16}{6}+k\right)n}\right)$$

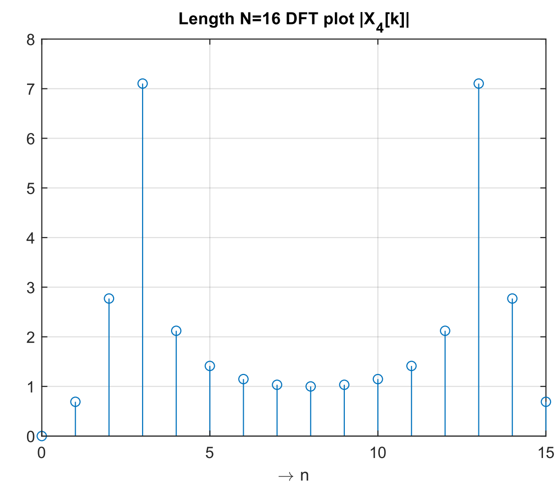
* Now if we plot the magnitude of this FTD, what will it look like?
* Note that we know what the spectrum of $x_4[n]$ will look like because of the Euler expression: delta pulses at $\theta = \pm\frac{\pi}{3}$

How to choose DFT length N for periodic signals

$$X_4[k] = \frac{1}{2}\sum_{n=0}^{15}\left(e^{j\frac{2\pi}{16}\left(\frac{16}{6}-k\right)n} + e^{-j\frac{2\pi}{16}\left(\frac{16}{6}+k\right)n}\right)$$

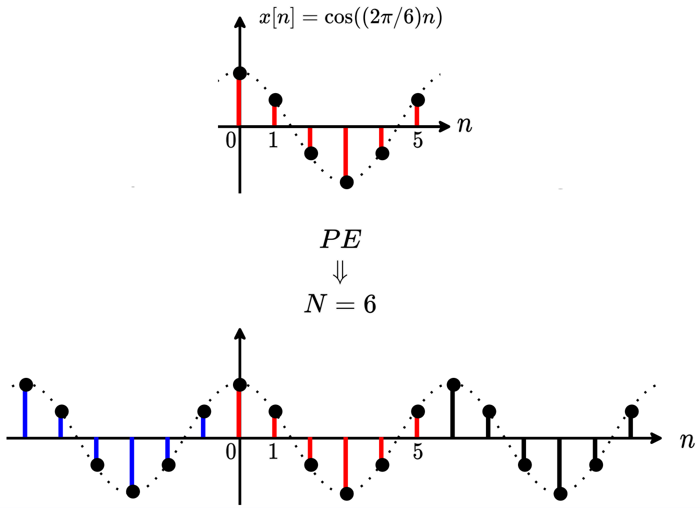
- We know the delta pulses are located at $\theta = \pm\frac{\pi}{3}$, but we used DFT length $N = 16$, so our samples k are $\frac{2\pi}{16} = \frac{\pi}{8}$ apart
- We will not reach $\theta = \pm\frac{\pi}{3}$, because it would require a value $k = \frac{16}{6}$





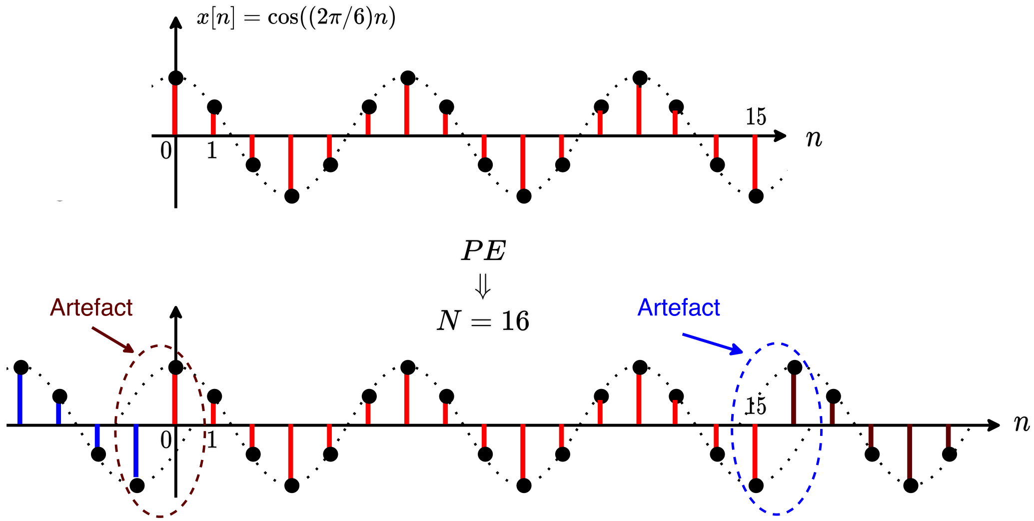
Why is the DFT plot of a sinusoidal signal not two peaks?

- <u>Explanation 1:</u> **Artefacts during Periodic Extension** (PE)
- When using the DFT, we choose N, which means that we assume our signal has a period of N samples
- In the DFT calculation, it is assumed this sequence of N samples is repeated
- If we choose N equal to the period or a multiple of the period, then the Periodic Extension will equal the input samples



Why is the DFT plot of a sinusoidal signal not two peaks?

- Explanation 1: Artefacts during Periodic Extension (PE)
- If we take the same signal but now choose N to resemble a non-integer amount of periods
- For $N = 16$ we find between 2 and 3 full periods
- By using the DFT now, we assume that this sequence is periodically extended
- The plot that we saw is the spectrum of the signal in the figure on the right
- This is not the same signal
- We find artefacts at the edges; it does not fit anymore

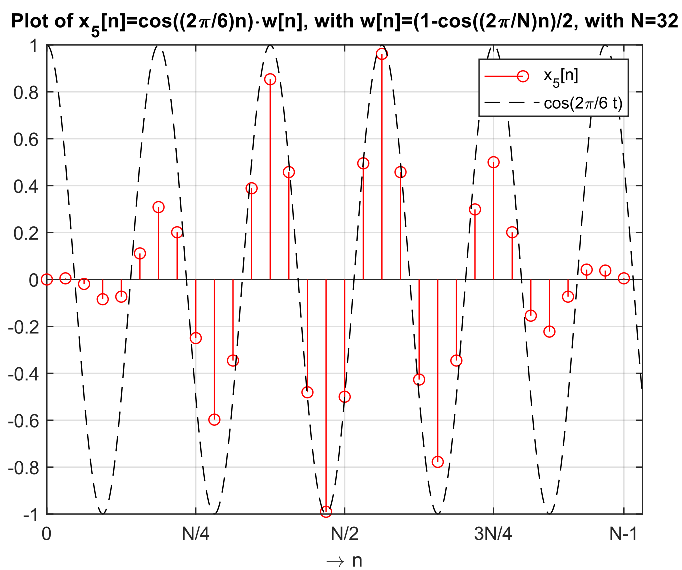


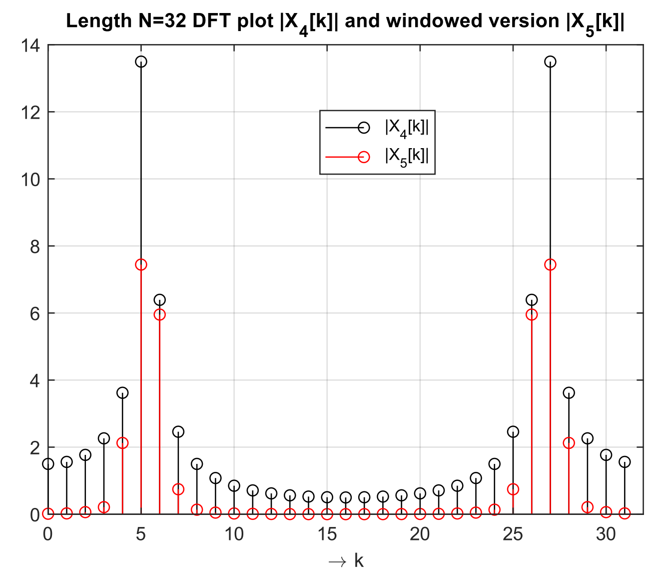
Why is the DFT plot of a sinusoidal signal not two peaks?

- If you know the period, choose the DFT length such that it fits the period
- In practice, however, we usually want to find the period, which we need the DFT for, so we have to compensate in a way
- We can **smoothen the transition at the edges of the periodic extension by windowing**
- In between the peaks we have nonzero values, which are attenuated by windowing
- Example 5: Length $N = 32$ DFT of $x_5[n] = \cos\left(\frac{2\pi n}{6}\right) \cdot w[n]$,
  where $w[n]$ is e.g. a Hanning window: $w[n] = \frac{1-\cos(2\pi n/N)}{2}$

Why is the DFT plot of a sinusoidal signal not two peaks?

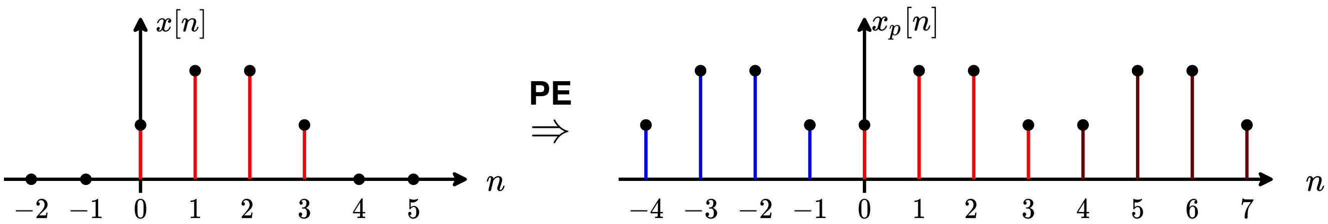
* Example 5: Length $N = 32$ DFT of $x_5[n] = \cos\left(\frac{2\pi n}{6}\right) \cdot w[n]$, where $w[n]$ is e.g. a Hanning window: $w[n] = \frac{1-\cos(2\pi n/N)}{2}$





DFT length N for finite length signals

- Finite length signal $x[n]$ and its periodic extension $x_p[n]$:
  $x[n] = 0$ for $n < 0$ and $n \geq N \quad \Rightarrow \qquad x_p[n] = x[n \bmod(N)]$

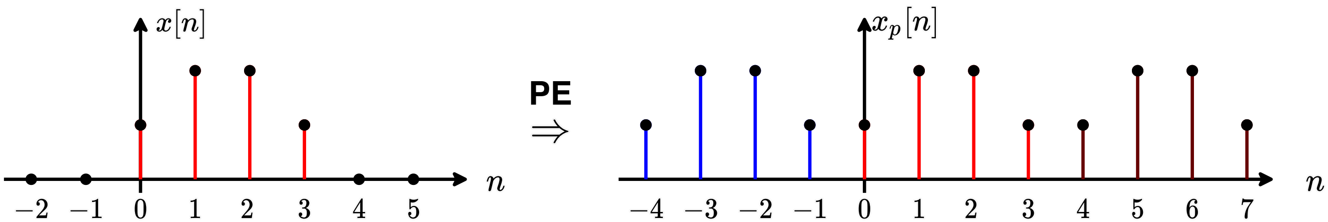


- Now if we take the FTD of $x[n]$, the bounds of the summation are limited to the nonzero values, because the FTD does not assume periodicity

- FTD: $X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n} = \sum_{n=0}^{N-1} x[n]e^{-jn\theta}$

DFT length N for finite length signals

- Now we compare the FTD to what we use in our computers: the DFT

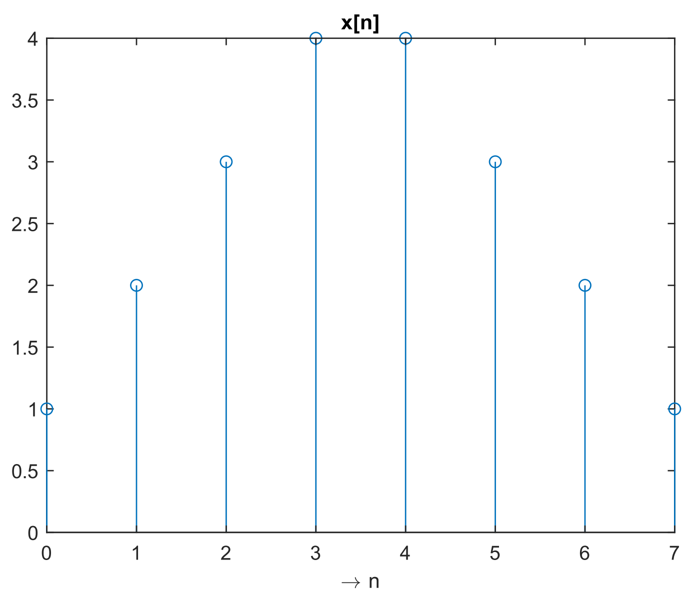


- FTD: $X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n} = \sum_{n=0}^{N-1} x[n]e^{-jn\theta}$
- DFT: $X[k] = \sum_{n=0}^{N-1} x_p[n]e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n]e^{-jnk\frac{2\pi}{N}}$
- $\Rightarrow$ If we have a finite length sequence $x[n]$ and its periodic extension $x_p[n]$, then the DFT is an exact sampled version of the FTD:

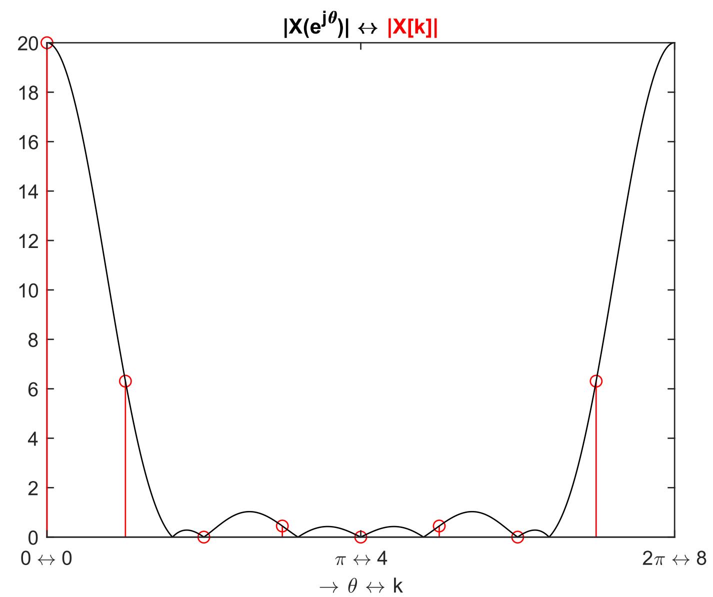
DFT coefficients ≡ samples of FTD: $X[k] = X\left(e^{j\theta}\right)\Big|_{\theta=k\cdot\frac{2\pi}{N}}$

Zero padding

* Assume we have a **signal $\boldsymbol{x}[\boldsymbol{n}]$ with 8 samples**
* Now we compute the FTD and the DFT and we plot the magnitude for both
* FTD: $X(e^{j\theta})$ ⊶ DFT($N = 8$): $X[k]$

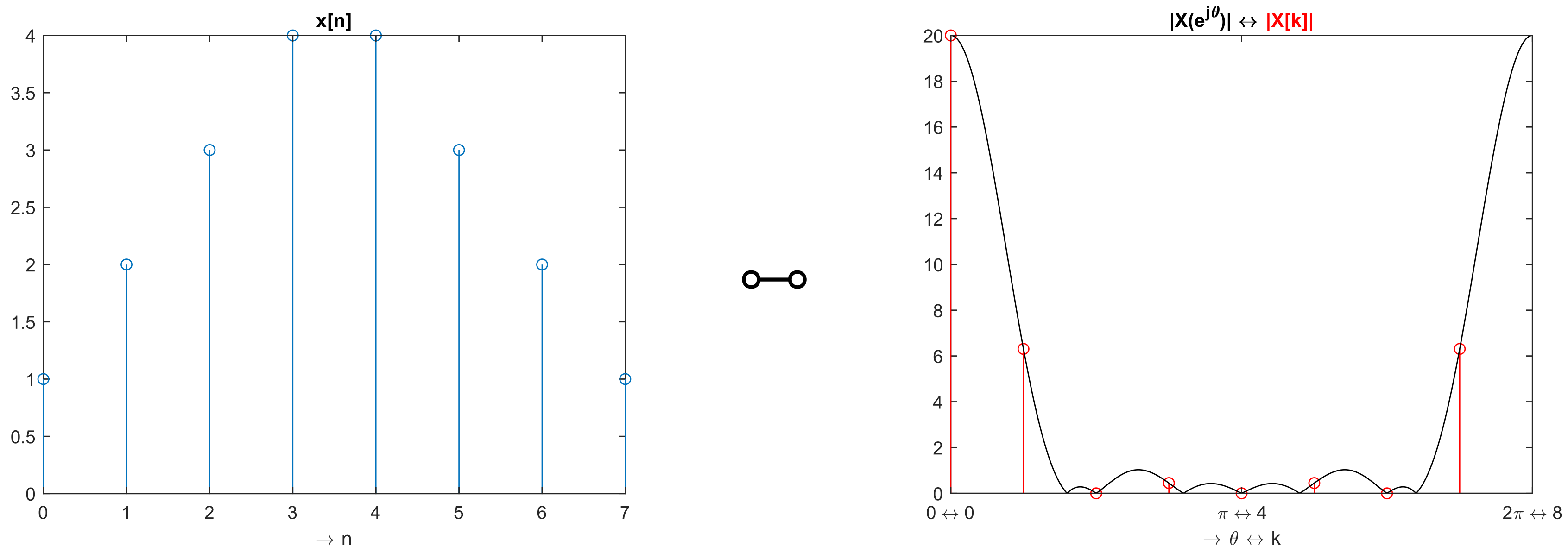


⊶



Zero padding

- In the magnitude plot, we see that for the DFT we have 7 samples that describe a period
- If we want to improve the stem plot, we need more samples, but we cannot simply change the DFT length as that will represent in a different signal



Zero padding

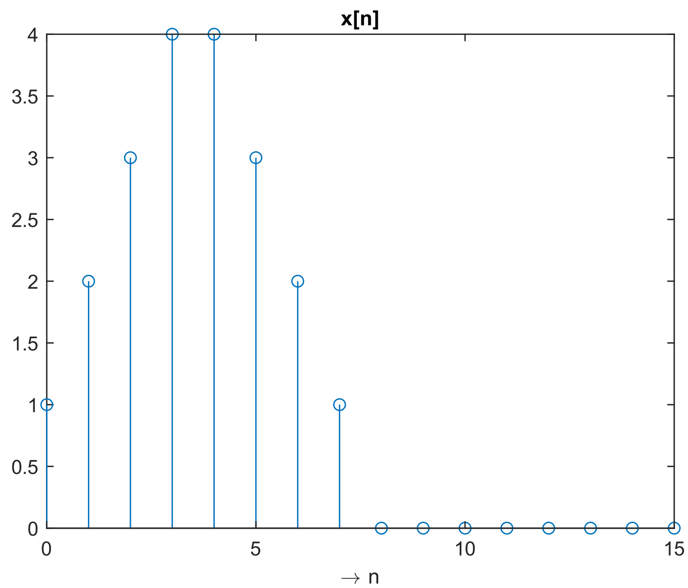
* We use **zero padding**
* We can take the same set of samples and add new samples with a value 0 to it, so we do not add new information
* In this way, we can increase the DFT length without changing what in the DFT is assumed to be our signal

* If we increase N, we decrease the space between the samples of the DFT
* $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\theta\frac{2\pi}{N}kn}$

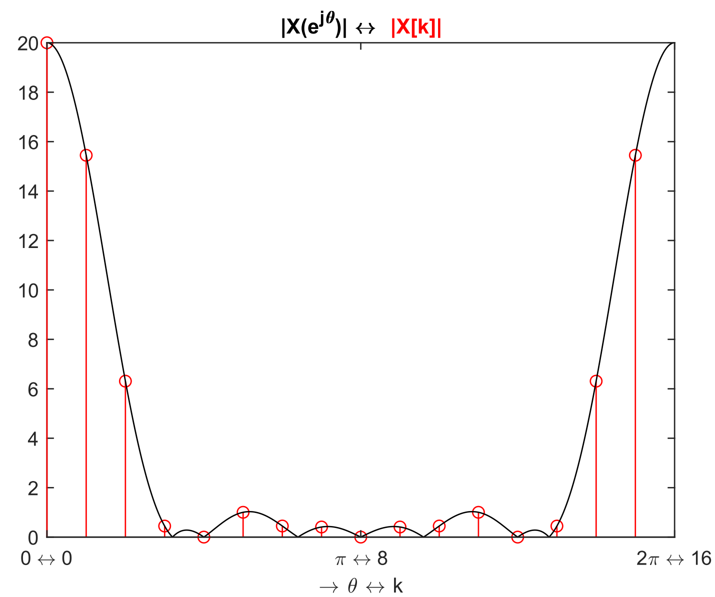
* For $N = 8$, the samples are $\frac{\pi}{4}$ apart and for $N = 16$, the samples are $\frac{\pi}{8}$ apart
* On the next slide we zero pad with 8 samples

Zero padding

- FTD: $X\left(e^{j\theta}\right) \multimap\!\circ$ DFT$(N = 16)$: $\color{red}{X[k]}$
- We can see that the same magnitude plot of the FTD is just sampled twice as frequent, because we made the DFT length twice as long without changing the information of our signal



$\circ\!\!-\!\!\circ$

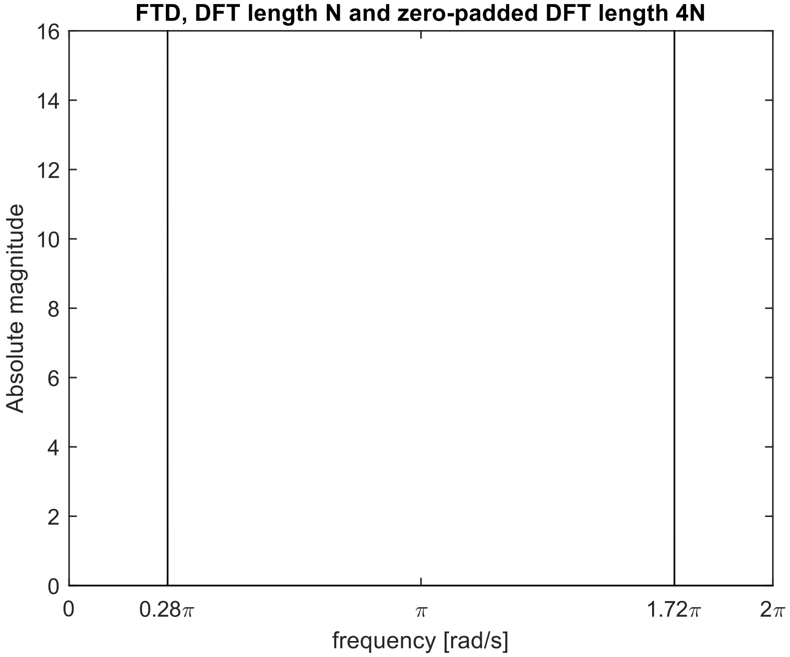


Zero padding

* Summarizing notes on zero padding:
* Zero padding = increasing DFT length by padding zeros
* **Not an increase in spectral information**
* **Only an increase in spectral resolution**

Why is the DFT plot of a sinusoidal signal not two peaks?

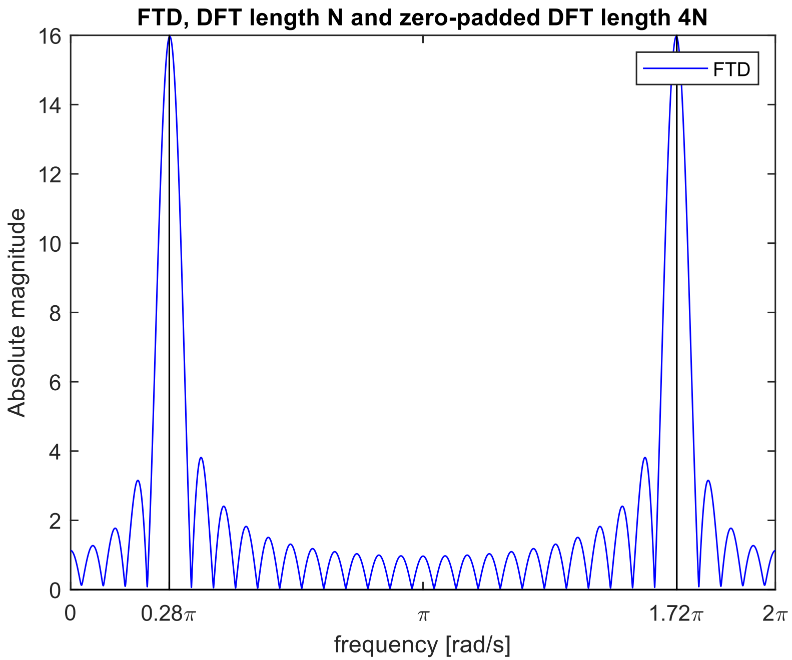
- Explanation 2: Sampling the FTD of a windowed sinusoidal signal
- The example we have seen before: $x[n] = \cos(0.28\pi n)$
- If n would span from $-\infty$ to ∞, the spectrum will consist of two delta pulses



FTD$\{x[n]\}$

Why is the DFT plot of a sinusoidal signal not two peaks?

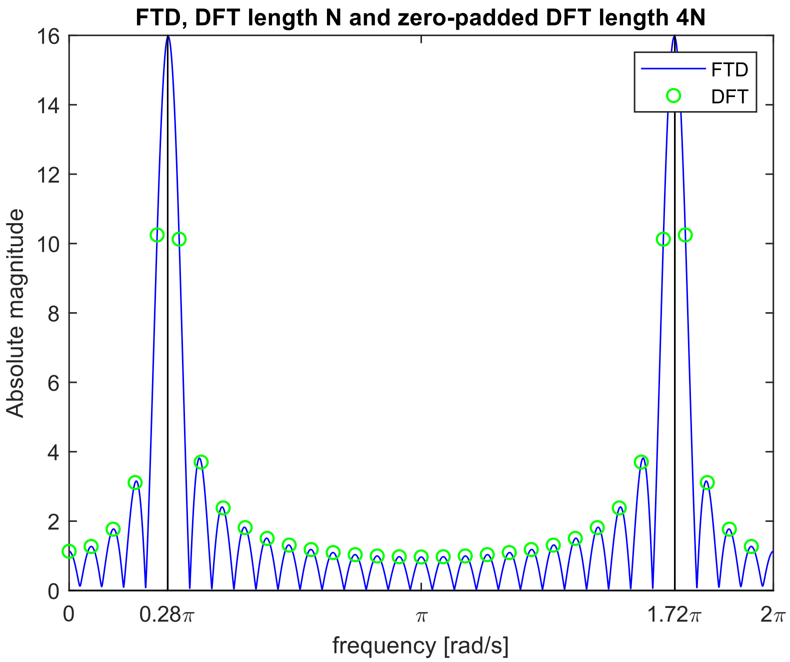
* $x[n] = \cos(0.28\pi n), \tilde{x}[n] = x[n] \cdot w_R[n],$ where $w_R[n]$ is an averaging filter
* In practice we cannot make an infinitely long signal, so we use a window
* A convolution of delta pulses with the Dirichlet function causes the blue line



FTD$\{x[n]\}$
FTD$\{\tilde{x}[n]\}$

Why is the DFT plot of a sinusoidal signal not two peaks?

* $x[n] = \cos(0.28\pi n), \tilde{x}[n] = x[n] \cdot w_R[n],$ where $w_R[n]$ is an averaging filter
* The DFT (green dots) takes N samples on the blue line of the FTD
* We can see that the peaks are in between two samples

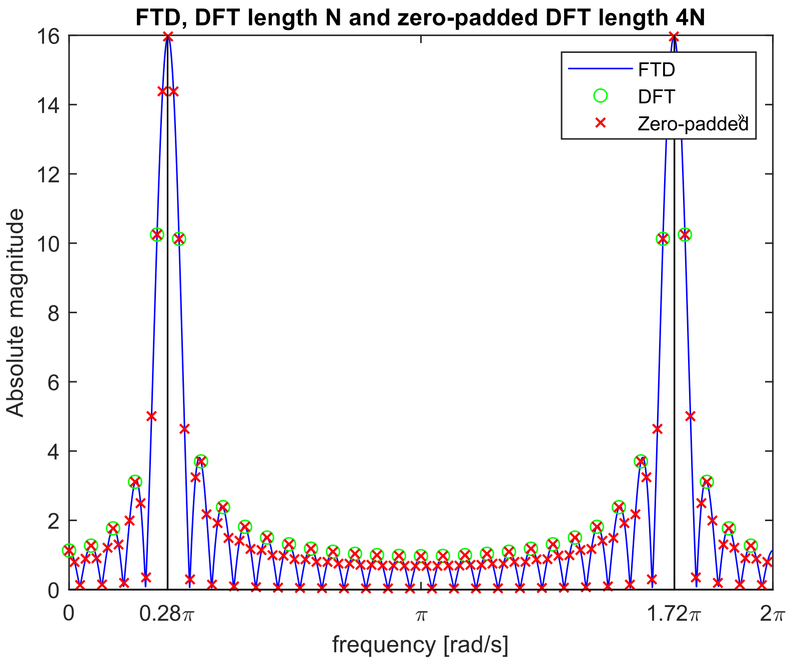


FTD$\{x[n]\}$
FTD$\{\tilde{x}[n]\}$

With $N = 32$;
$\mathrm{DFT}_N\{\tilde{x}[n]\}$

Why is the DFT plot of a sinusoidal signal not two peaks?

- $x[n] = \cos(0.28\pi n), \tilde{x}[n] = x[n] \cdot w_R[n]$, where $w_R[n]$ is an averaging filter
- If we want to improve how the DFT represents the spectrum in theory, we use zero padding to improve the spectral resolution



FTD$\{x[n]\}$
FTD$\{\tilde{x}[n]\}$

With $N = 32$;
$\mathrm{DFT}_N\{\tilde{x}[n]\}$

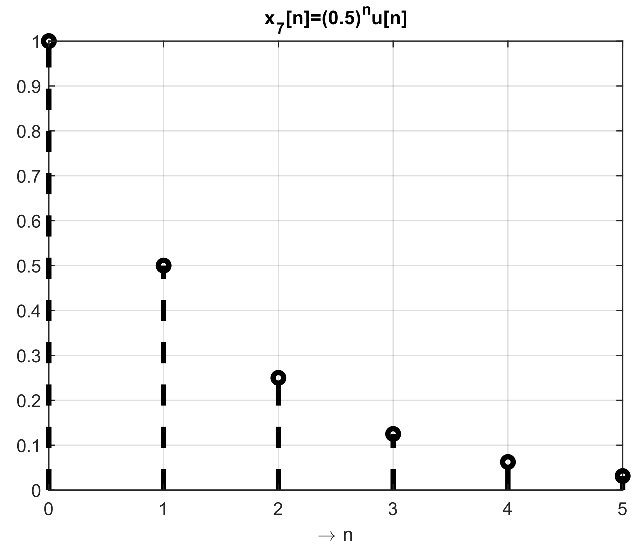
With $L = 4 \cdot N$;
$\mathrm{DFT}_L\{\tilde{x}[n]\}$

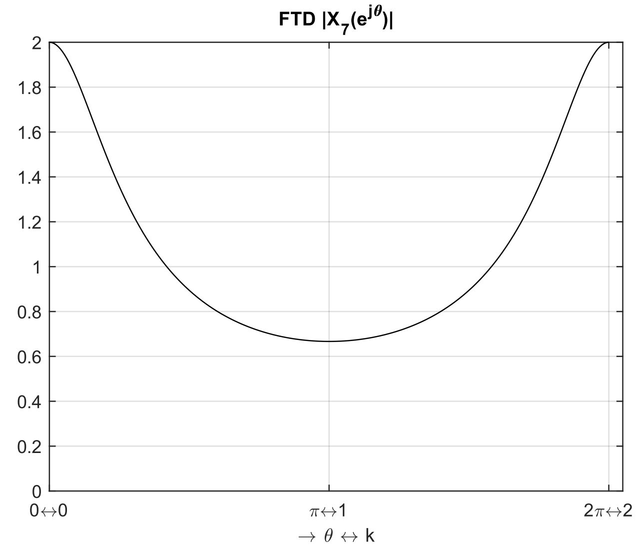
N for non periodic infinite length signals

* What do we do when our signal is infinite or the signal is much larger than our DFT length?
* The exact frequency domain description is only possible through the FTD
* Length N DFT approximation ⇒ choose N large enough
* We can only approximate the FTD through the DFT, the larger N the better the approximation

N for non periodic infinite length signals

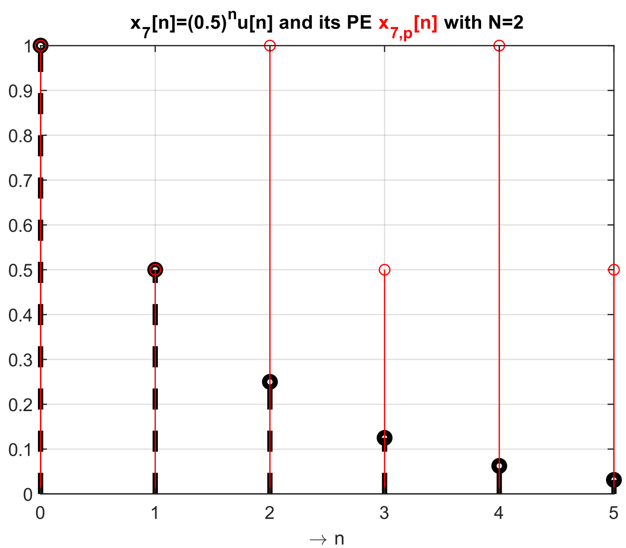
- Example 7: $x_7[n] = (0.5)^n u[n]$, choose for DFT first N samples
- This sequence is plotted in the bottom left plot
- We know the expression for the FTD: $X_7\left(e^{j\theta}\right) = \frac{1}{1-0.5e^{-j\theta}}$
- The frequency spectrum is plotted in the bottom right figure

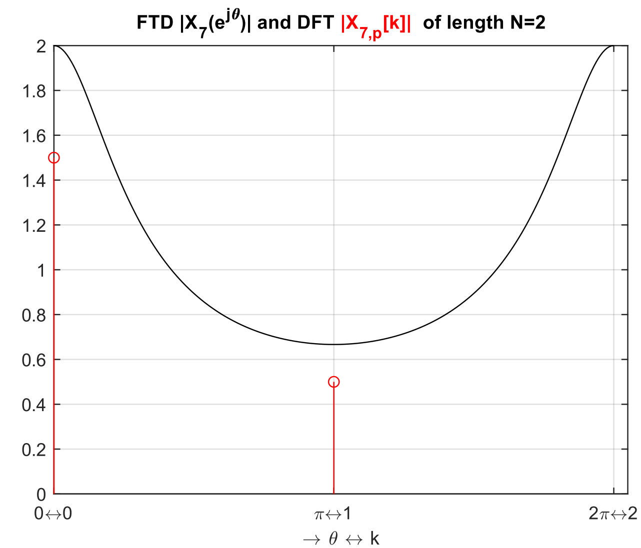




N for non periodic infinite length signals

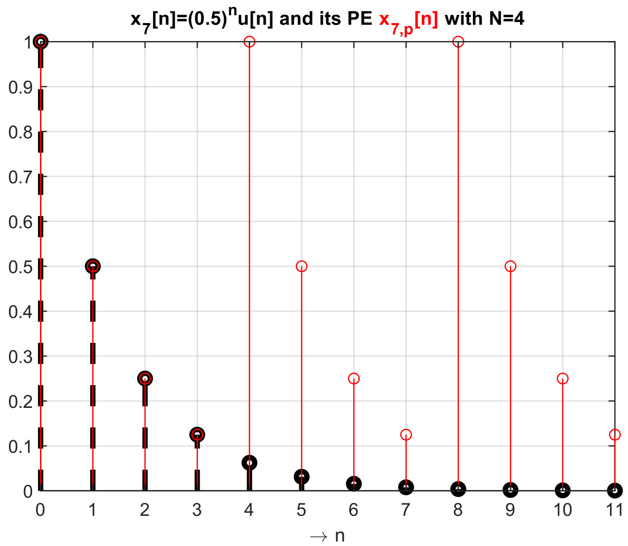
- If we apply a **DFT of length $N = 2$**, then we take the first two samples of $x_7[n]$ and intrinsically these are periodically extended because of the DFT
- Because the periodically extended $x_{7,p}[n]$ version clearly deviates from $x_7[n]$, we expect the DFT not to be a good approximation of the FTD
- In the bottom right plot we see the approximation

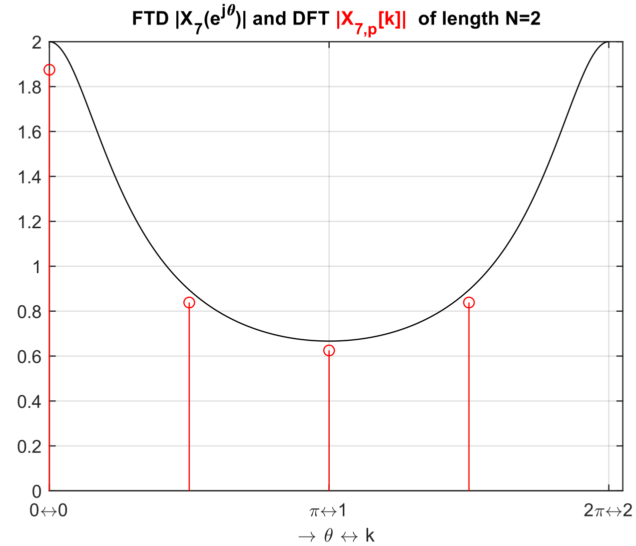




N for non periodic infinite length signals

* Now we increase the DFT **length to $N = 4$**, we take 4 samples for the PE and calculate the spectral result on 4 positions instead of 2
* Here this gives a quite good approximation, because after $n = 3$ the values for $x_7[n]$ become rather small. In general, a **higher N yields a better approximation, but it depends on the signal**





DFT properties

* The DFT has properties that are mostly similar to the transforms we have seen before

* Linearity: With DFT paris $x_1[n] \circ\!\!-\!\!\circ X_1[k]$ and $x_2[n] \circ\!\!-\!\!\circ X_2[k]$
$$ax_1[n] + bx_2[n] = aX_1[k] + bX_2[k]$$

* Symmetry: With DFT pair $x[n] \circ\!\!-\!\!\circ X[k]$ we have for real $x[n]$
$$X[k] = X^*[(-k)\mathrm{mod}(N)] = X^*[(N-K)\mathrm{mod}(N)] = X^*[N-k]$$
The magnitude is even symmetric and the phase is odd symmetric

* Note: For real $x[n]$ the DFT values $X[k]$ are known completely by $0 \leq k \leq N/2$ for even N and $0 \leq k \leq (N-1)/2$ for odd N
Because of symmetry you only need to compute half

DFT properties

- Example 8: Calculate the DFT of length $N = 6$ of $x_8[n] = \sin(2\pi n/6)$
- We use an Euler expression and the twiddle factor

$$X_8[k] = \sum_{n=0}^{6-1} x_8[n] e^{-j\frac{2\pi}{6}kn} = \sum_{n=0}^{5} \frac{1}{2j}\left(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}\right)e^{-j\frac{\pi}{3}kn}$$

$$= \frac{1}{2j}\sum_{n=0}^{5}\left(e^{-j\frac{\pi}{3}n(k-1)} - e^{-j\frac{\pi}{3}n(k+1)}\right)$$

- From the twiddle factor follows:

$$X_8[k] = \begin{cases} -3j, & \text{for } k = l\cdot 6 + 1 \text{ with } l = 0, \pm 1, \pm 2, \ldots \\ 3j, & \text{for } k = l\cdot 6 - 1 \text{ with } l = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- $\Rightarrow X_8[1] = X_8^*[-1] = X_8^*[N-1] = X_8^*[5] \Rightarrow$ symmetry

DFT properties

* Circular shift: With DFT pair $x_p[n] \circ\!\!-\!\!\circ X_p[k]$

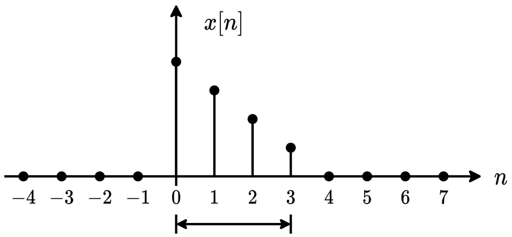
  Time shift: $x_p[n-i] \circ\!\!-\!\!\circ e^{-j\frac{2\pi}{N}ki} \cdot X_p[k]$

  Frequency shift: $e^{j\frac{2\pi}{N}ni} \cdot x_p[n] \circ\!\!-\!\!\circ X_p[k-i]$

* The shifting property works similarly, but there is a difference between circular shifting and regular shifting

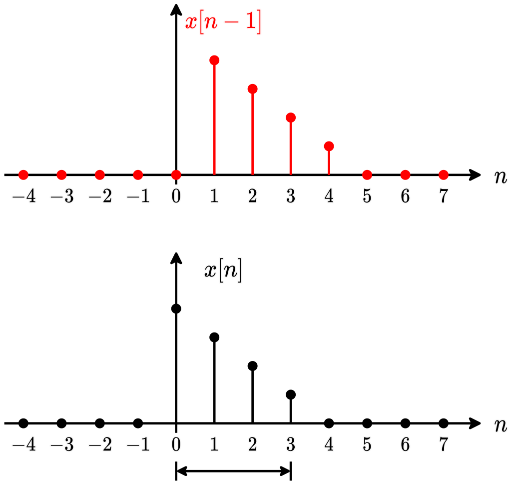
* Circular shift $\neq$ regular shift

Circular shift



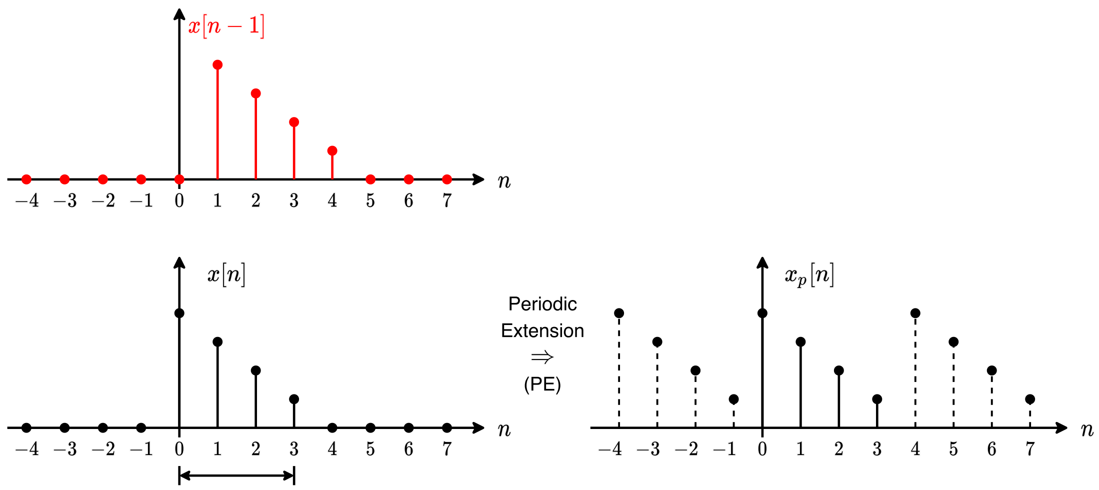
- Say we have these 4 samples
- Now we perform a regular shift of 1 sample

Circular shift



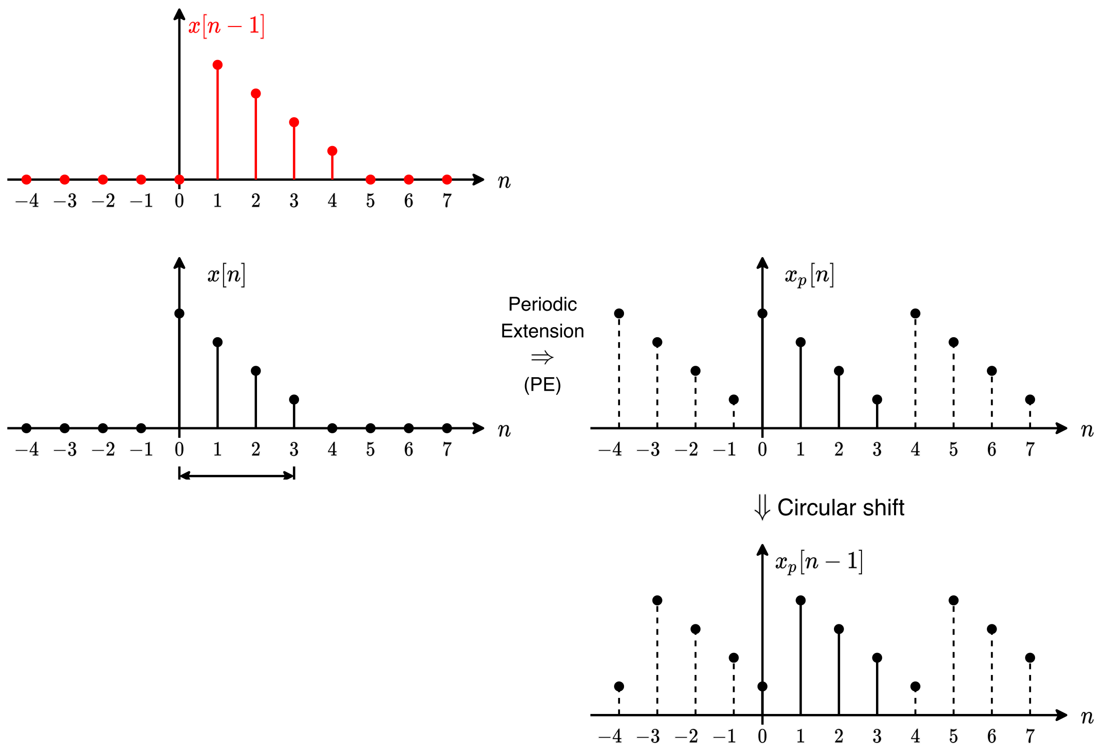
- The shifted samples are in red
- When the DFT is used, this signal becomes periodically extended automatically

Circular shift



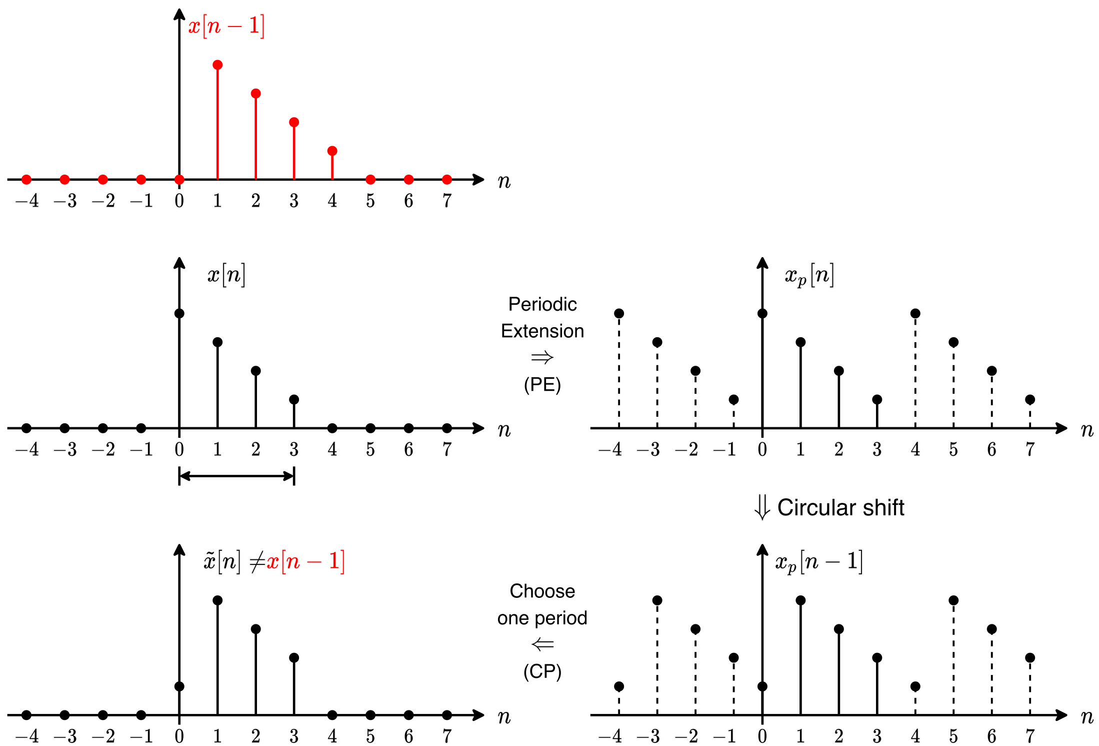
- When we are working with the DFT, we work in a periodically extended domain
- A shift in this domain (a circular shift) results in the same shift in the PE

Circular shift



* The circular shift looks like this: the samples and all repetitions are shifted one to the right
* In the DFT, we only look at one period, so we choose one period

Circular shift



- ∗ We can see that a circular shift will give a different output than a regular shift
- ∗ Shifting in the DFT domain is not equal to shifting in the FTD domain

DFT properties

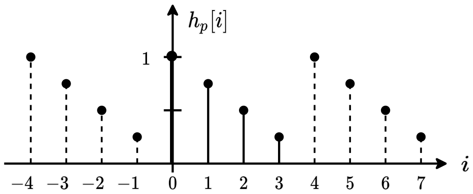
* Circular convolution($\circledast$): (note the difference with regular convolution)

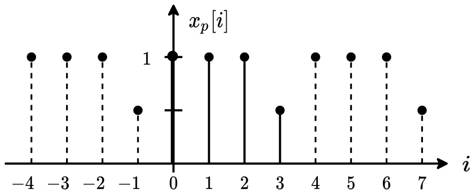
$$y_p[n] = x_p[n] \circledast h_p[n-i] = \sum_{i=0}^{N-1} x_p[i] h_p[n-i] = \sum_{i=0}^{N-1} x_p[n-i] h_p[i]$$

* The circled asterisk $\circledast$ indicates circular convolution
* In the procedure circular shifts are used instead of regular shifts
* We only look at the range from 0 to $N-1$
* Let us look at an example

DFT properties

* Example 9: Circular convolution 2 periodic signals $N = 4$



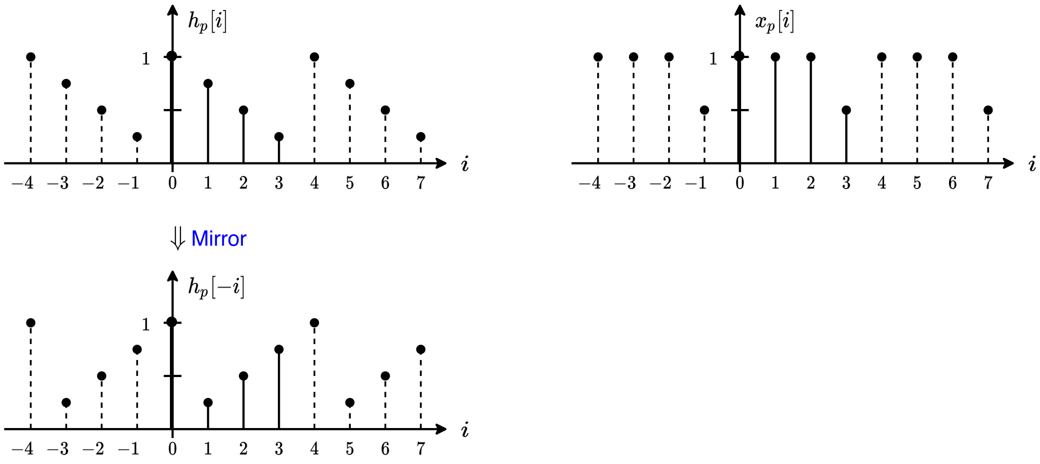


* The sequences themselves only have 4 samples, but because we deal with the DFT, we have to account for the periodic extension
* Now we have to mirror one of the two sequences

* NOTE: both sequences have the same length N=4. If not, use zero padding to equal their lengths!

DFT properties

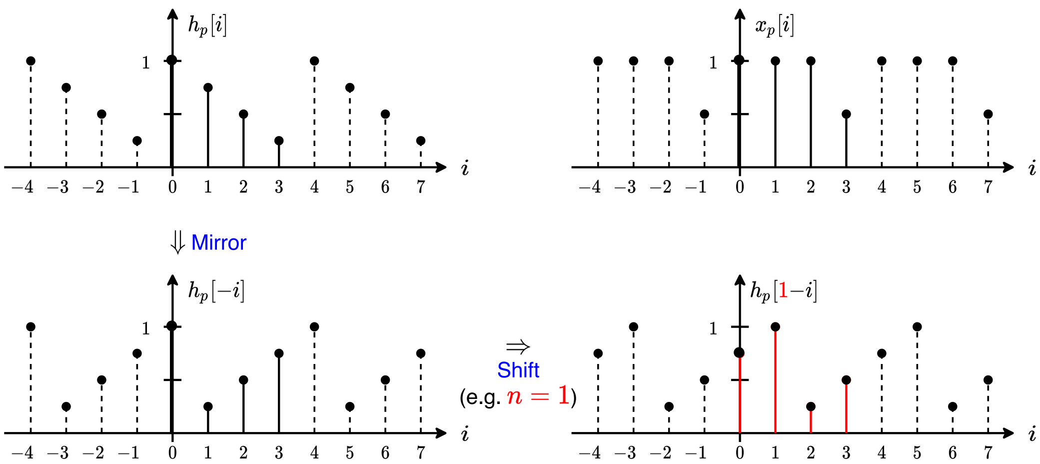
* Example 9: Circular convolution 2 periodic signals $N = 4$



* We mirror $h_p[i]$ around 0
* Notice that the periodic extensions are also mirrored
* The circular mirrored version is clearly different from mirroring only the 4 samples

DFT properties

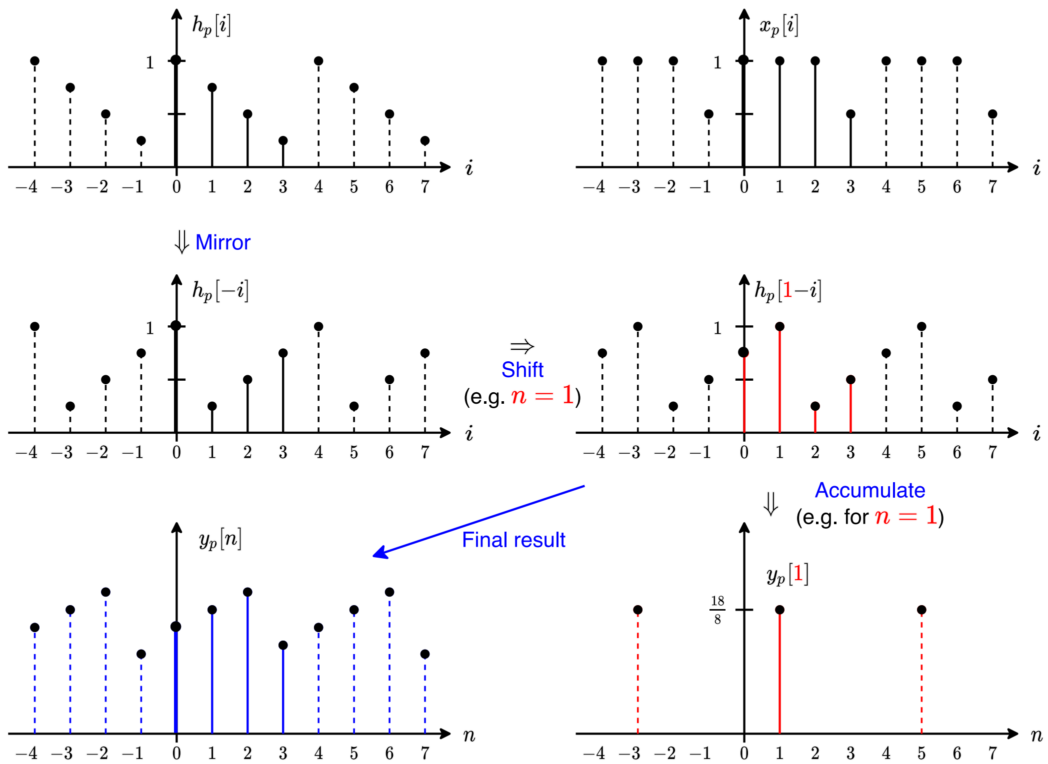
- Example 9: Circular convolution 2 periodic signals $N = 4$



- Now we shift the circularly mirrored version over one period, so $n = 0$ to $n = N - 1$
- We look at the overlap it has with $x_p[i]$ per shift n and accumulate to find $y[n]$

DFT properties

* Example 9: Circular convolution 2 periodic signals $N = 4$



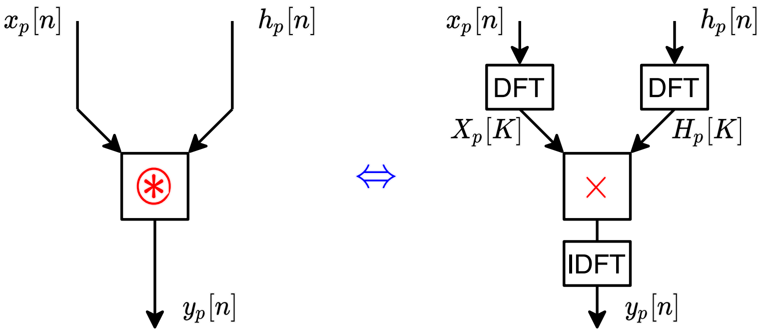
DFT properties

* Circular convolution in DFT domain:
* Convolution in one domain is multiplication in the other domain, however for the DFT it is circular convolution

$$y_p[n] = x_p[n] \circledast h_p[n]$$

$$\Updownarrow$$

$$Y_p[k] = X_p[k] \cdot H_p[k]$$



* When we apply a filter operation in Matlab, behind the screens the procedure in the figure on the right is followed

DFT properties

* Circular convolution in DFT domain:
* Circular convolution ↔ multiplication: proof

$$Y_p[k] = \sum_{n=0}^{N-1} y_p[n] e^{-j(2\pi/N)kn} = \sum_{n=0}^{N-1} \sum_{i=0}^{N-1} x_p[i] h_p[n-i]\, e^{-j(2\pi/N)kn}$$

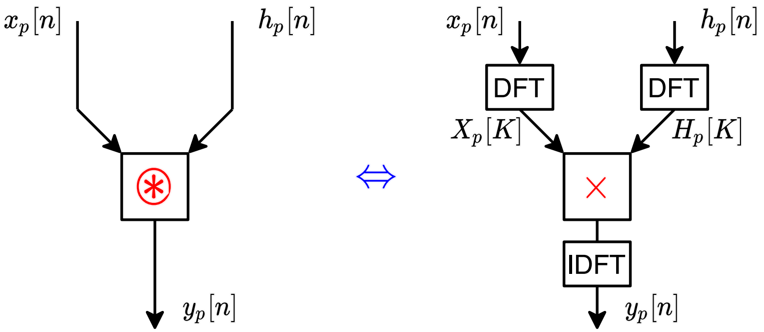
$$= \sum_{i=0}^{N-1} x_p[i] e^{-j(2\pi/N)ki} \sum_{n=0}^{N-1} h_p[n-i] e^{-j(2\pi/N)k(n-i)}$$

$$= \sum_{i=0}^{N-1} x_p[i] e^{-j(2\pi/N)ki} \sum_{l=0}^{N-1} h_p[l] e^{-j(2\pi/N)kl} = X_p[k] \cdot H_p[k]$$

DFT properties

- Circular convolution in DFT domain:

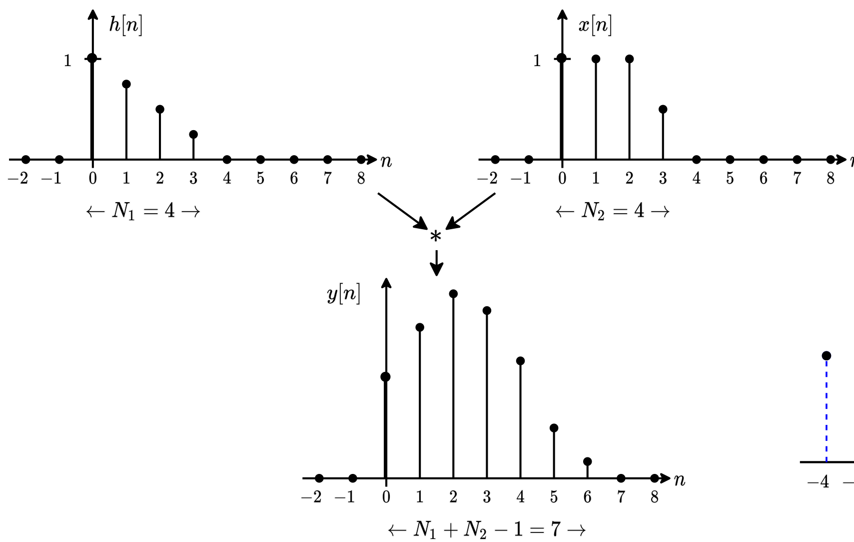
$$y_p[n] = x_p[n] \circledast h_p[n]$$
$$\Updownarrow$$
$$Y_p[k] = X_p[k] \cdot H_p[k]$$



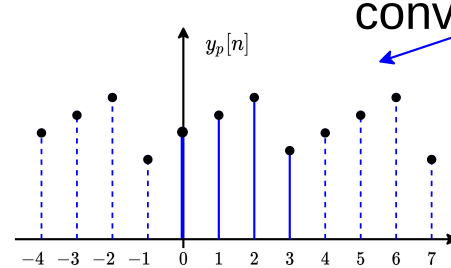
- Computers use the DFT (in Matlab the FFT, which is an efficient implementation of the DFT), but that means that if we multiply signals in frequency domain, it will always be in the DFT domain
- The counterpart is circular convolution in the time domain, whereas filters work with linear convolution
- How do we obtain the linear convolution from the circular convolution?

Convolution finite $x[n]$ with finite $h[n]$

- We have two finite length sequences $x[n]$ and $h[n]$
  Finite length $h[n]$: $N_1 = 4$
  Finite length $x[n]$: $N_2 = 4$
  We know the finite length of linear convolution will be $N_1 + N_2 - 1 = 7$



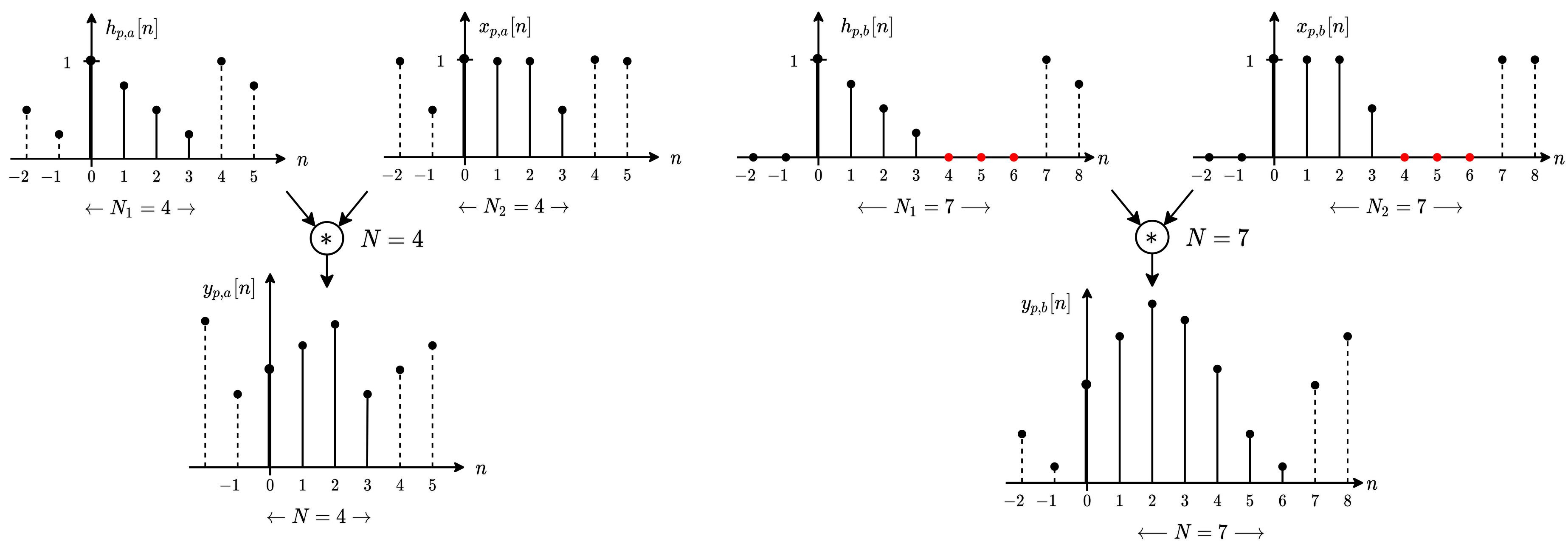
Circular convolution



- How do we change the sequences such that circular convolution will give the same outcome as linear convolution?

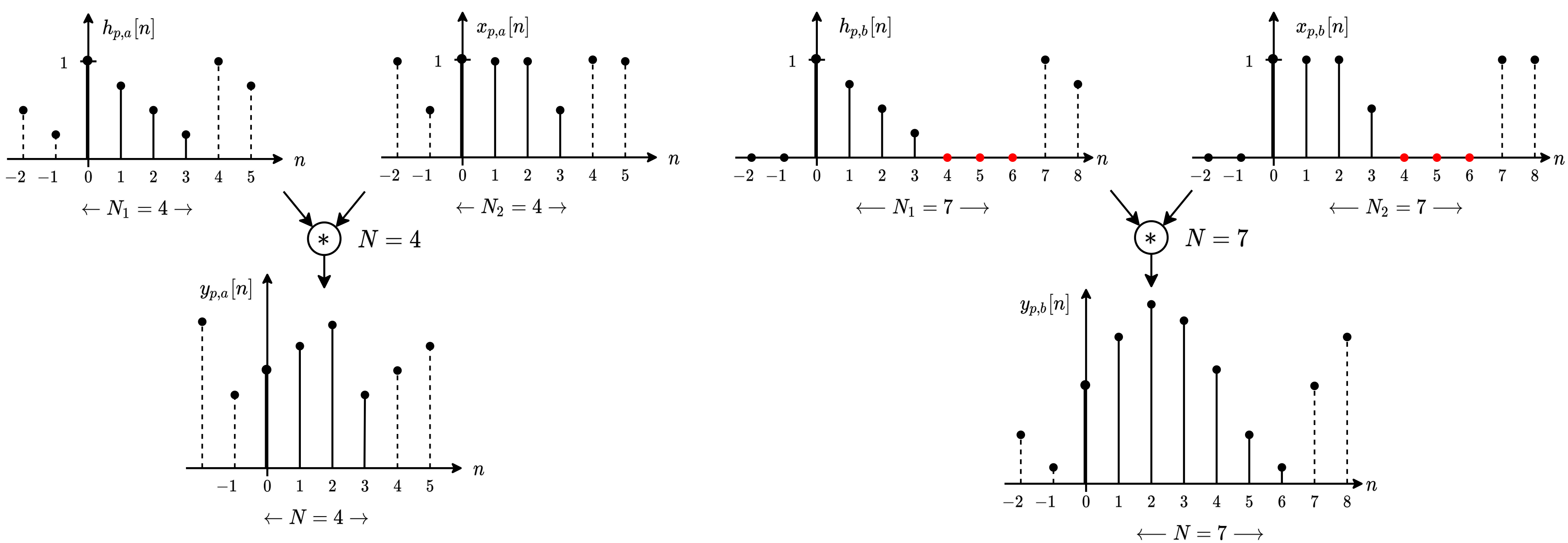
Convolution finite $x[n]$ with finite $h[n]$

* We use zero padding
* We zero-pad both sequences to at least the length of the output of the linear convolution

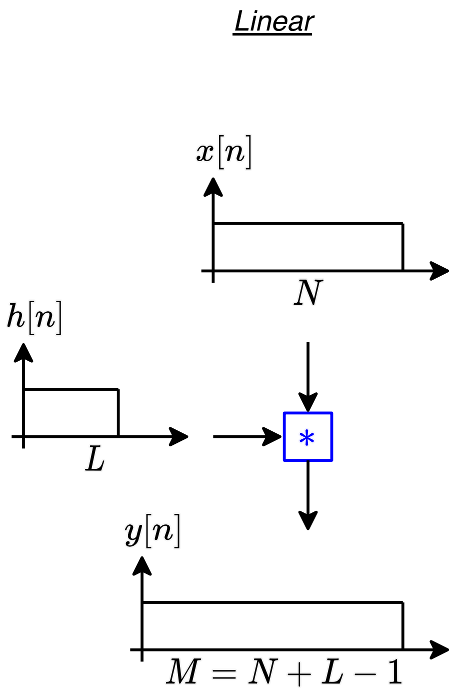


Convolution finite $x[n]$ with finite $h[n]$

* We can see that the convolved zero-padded sequences give the same result as linear convolution, however with PE, so we choose one period
* $\Rightarrow$ Circular $\equiv$ linear iff zero-padded until length $N \geq N_1 + N_2 - 1$

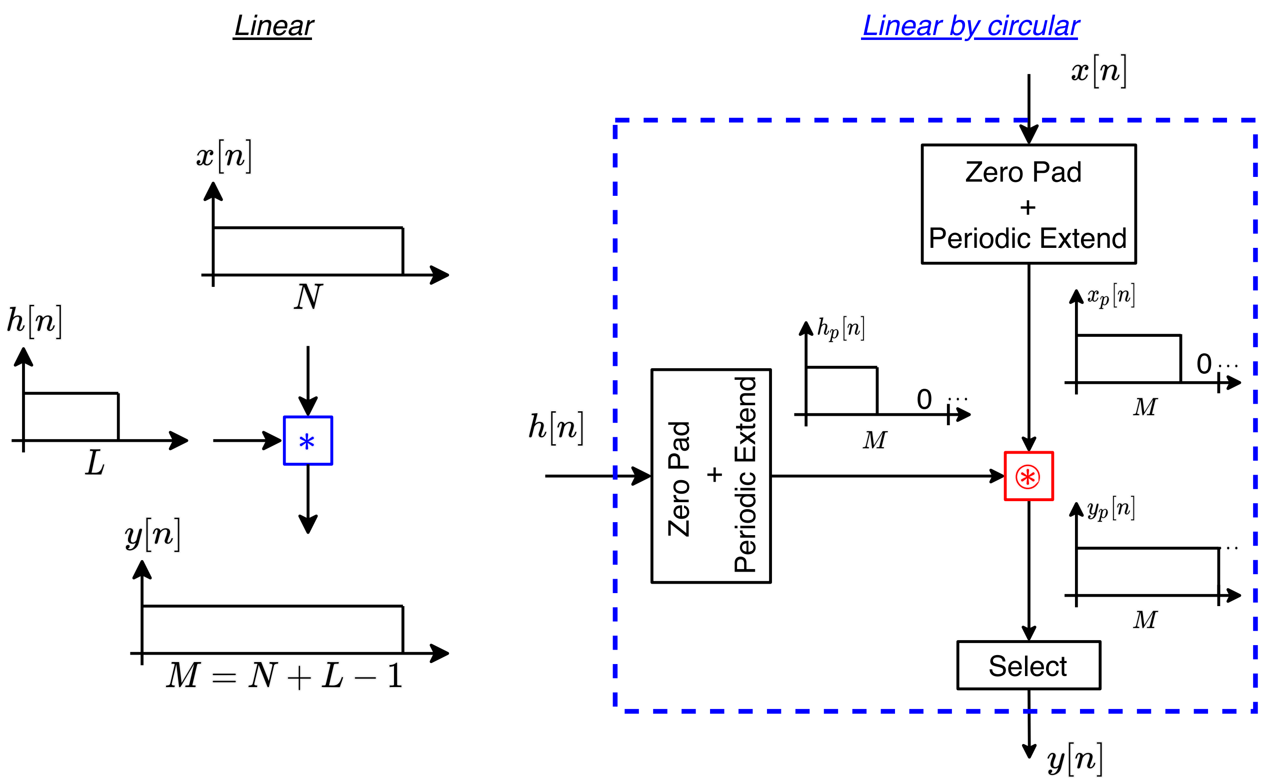


Linear by circular convolution in DFT domain



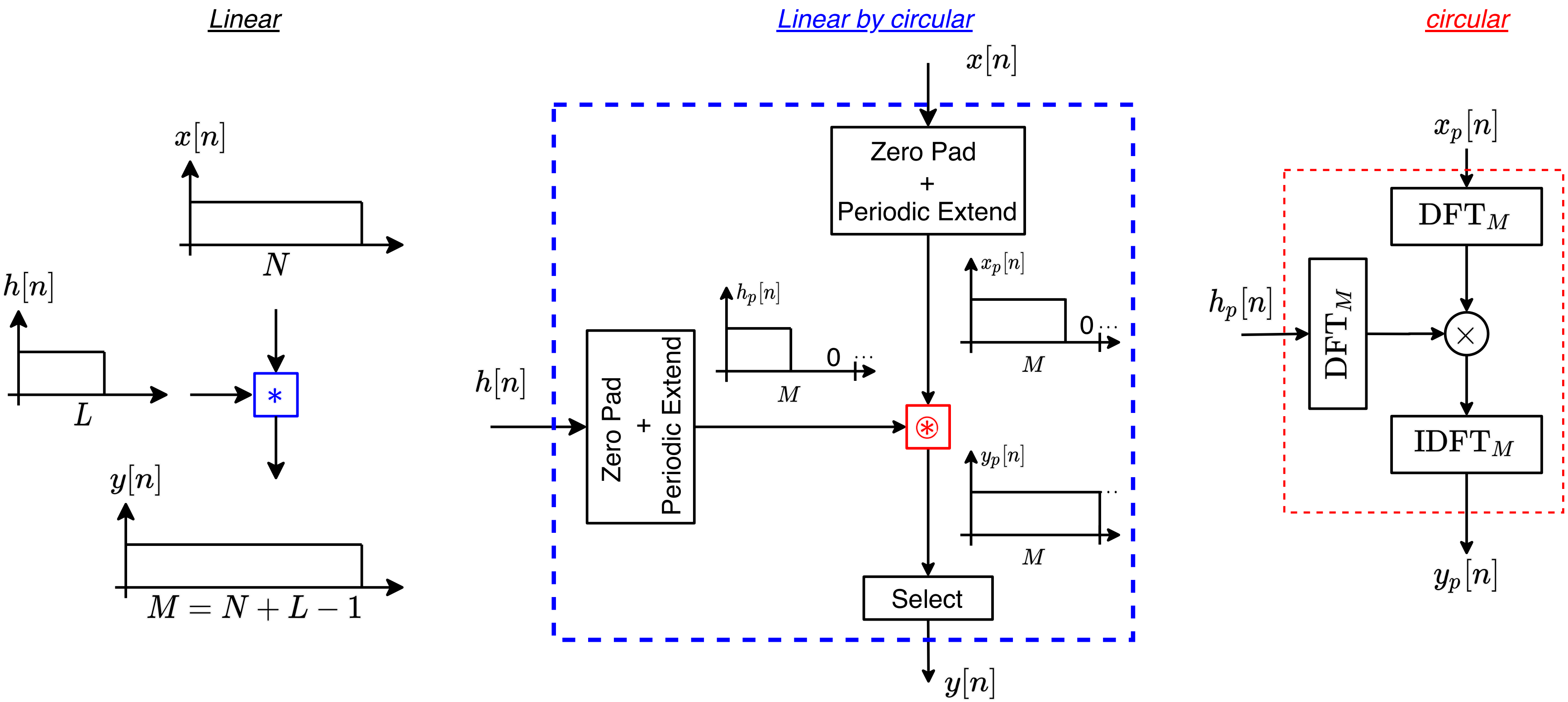
- We want to use a computer to calculate a linear convolution
- We know the length after convolution will be $M = N + L - 1$

Linear by circular convolution in DFT domain



* Usually DFTs are used, so we deal with periodic extensions
* Zero padding ensures circular convolution is equivalent to linear convolution

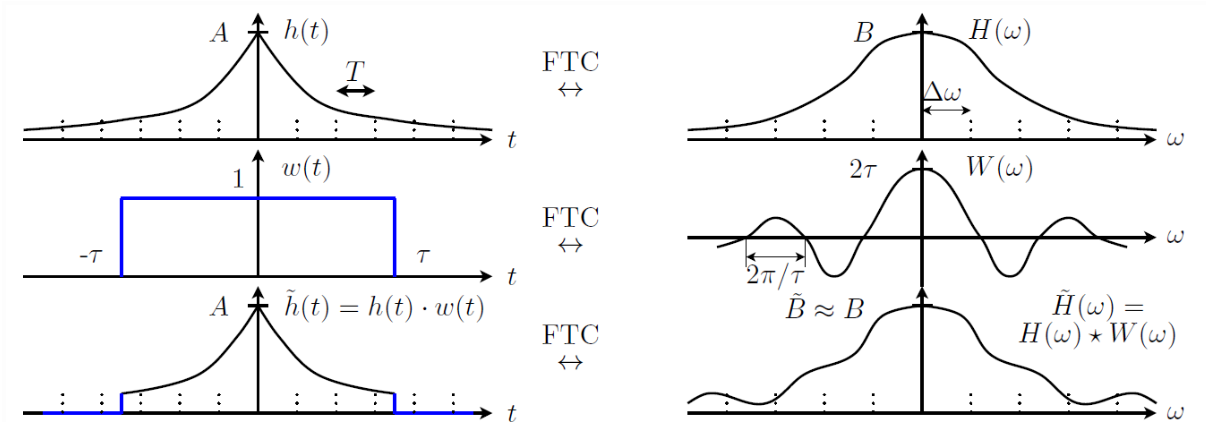
Linear by circular convolution in DFT domain



* In practice, the DFT of length M is taken (in Matlab zero padding to length M is done automatically). One period of $y_p[n]$ will equal $y[n]$

Apply DFT on continuous signal

∗ <u>Example 10</u>: Different error causes

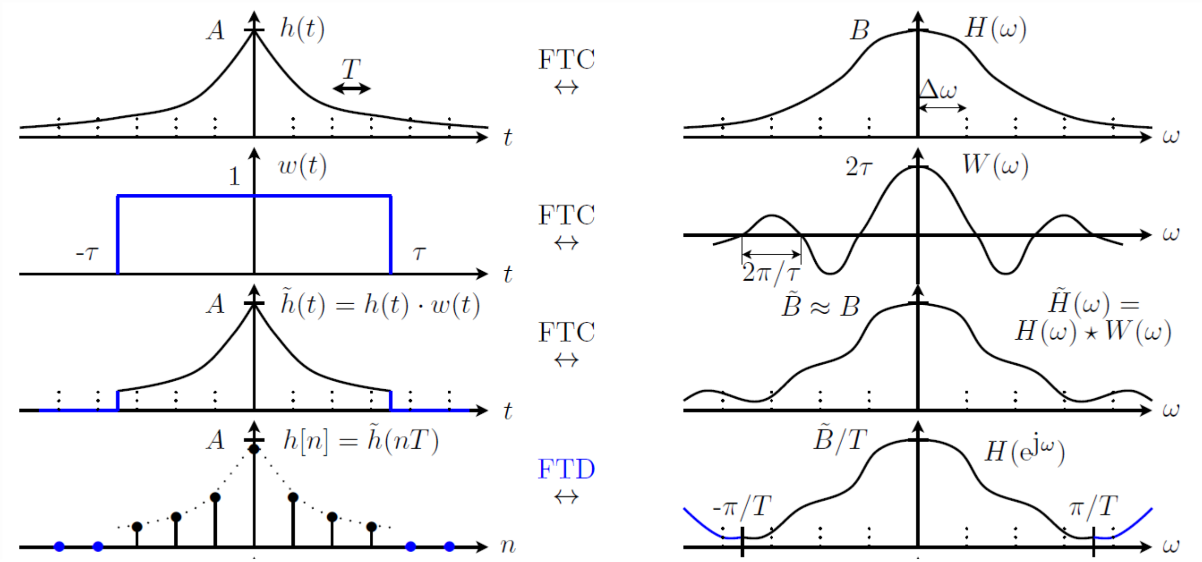


∗ If we have a time domain signal that we want to measure, we cannot measure it from minus to plus infinity, so effectively by measuring it, we **window it** (multiply with a block)

∗ In frequency domain this equals a convolution with a sinc function

Apply DFT on continuous signal

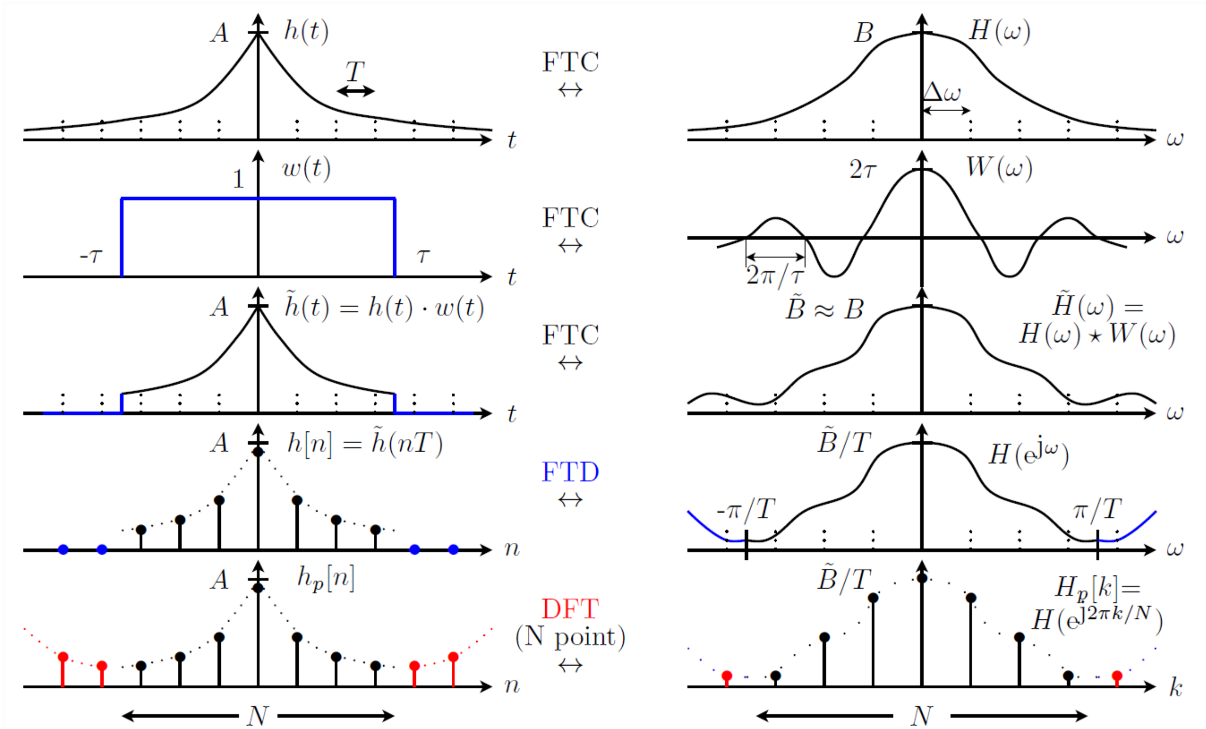
* Example 10: Different error causes



* When we sample the signal, only samples within the window are considered
* But because the DFT is used, there will be periodic extension
* The PE causes repetition where the measurements stopped → errors

Apply DFT on continuous signal

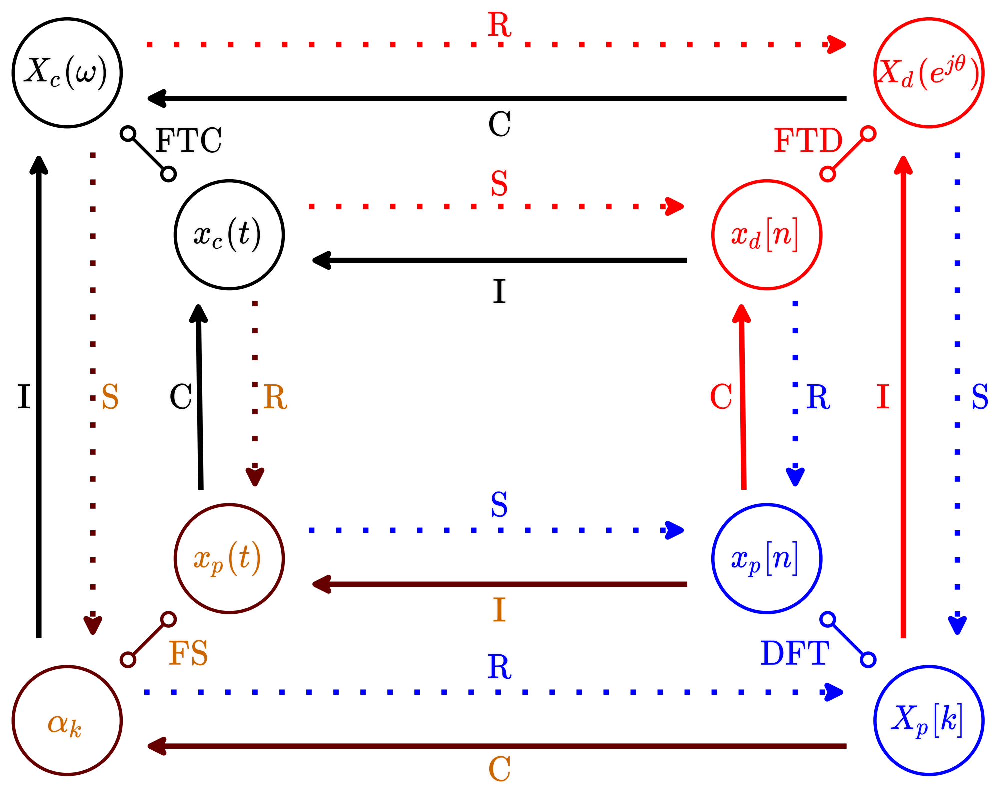
* Example 10: Different error causes



Relations between different transforms

- R- repeat
- S- sampling
- I- interpolate
- C- choose

- R/S are reversible with C/I iff the signal you repeat or sample is finite



Summary

- We discussed the DFT- the Discrete Fourier Transform:
  - The definition and equation
  - The DFT for periodic signals
  - The orthogonality property → the twiddle factor

- We looked at the effect of the DFT length N

- We explained why the DFT plot of a sinusoidal signal is not two peaks
  - Artefacts during periodic extension
  - Sampling the FTD of a windowed sinusoidal signal
- We discussed zero padding
- We looked at DFT properties
  - Circular shifts and circular convolution
  - How zero padding is used to relate circular convolution to linear convolution