1. Let C and D be nonempty sets. Prove that $C \times D = D \times C$ if and only if $C = D$. Why do we need the condition that C and D are nonempty?

   Proof. Let C and D be nonempty sets.
   ($\Rightarrow$) Suppose $C \times D = D \times C$. Let (x, y) to be an arbitrary ordered pair from set $C \times D$. Hence, $x \in C$ and $y \in D$. Since $C \times D = D \times C$ and $(x, y) \in C \times D$, $(x, y) \in D \times C$. It implies that for all x in set C is also in set D, and for all y in set D is also in set C. Therefore, $C \subseteq D$ and $D \subseteq C$. $C = D$ is the only possible result.

   ($\Leftarrow$) Suppose $C = D$. We can rewrite $C \times D$ as $D \times D$ and rewrite $D \times C$ as $D \times D$ as well. Therefore, $C \times D = D \times D = D \times C$.

   If either C or D is empty and the another one is nonempty , $C \times D = D \times C = \emptyset$. However, empty set does not equal to another nonempty set, $C \neq D$. The conclusion will not hold anymore without the condition C and D are nonempty. ■

2. Find a condition for the sets A and B such that you can create a theorem of the form "Let A and B be sets. We have $A \backslash B = B \backslash A$ if and only if (your condition on A and B). That is, you're looking to state and prove necessary and sufficient conditions for $A \backslash B = B \backslash A$.

   Solution. Let A and B be sets. We have $A \backslash B = B \backslash A$ if and only if $A = B$.
   To prove the statement above, we need to prove in both directions:
   ($\Rightarrow$) Suppose $A \backslash B = B \backslash A$. By defination, $A \backslash B$ is the set of all elements in A but not B and $B \backslash A$ is the set of all elements in B but not A. Hence, $A \backslash B$ and $B \backslash A$ disjoint. For disjoint set to be equal, they must both be empty. In this case, $A = B = \emptyset$.
   ($\Leftarrow$) Suppose $A = B$, and then $A \backslash B$ is equivalent to $A \backslash A$, which is the set of all elements in A but not A. Hence, empty set. Similarly., $B \backslash A$ is empty as well. Therefore, $A \backslash B = B \backslash A = \emptyset$. □

3. Let A be a set. The *complement* of A denoted $\overline{A}$ or A^C is defined as the set of all objects that are not in A. This can be problematic, as this can include literally anything at all: the "complement" of the set {1, 2} could include the number -3 as easily as it could include your cell phone. We then must specify the set U =*universe* in which the set exists if we are to discuss the complement of the set within that universe. So if we are talking about $U = \mathbb{Z}$ for our previous example, then the complement of {1,2} would be $\mathbb{Z} \backslash$ {1,2}; however if we were using $\mathbb{R}$ as our universe, the complement would be $\mathbb{R} \backslash \{1, 2\}$.
   Prove the following about set complements, assuming $A, B, C \subseteq U$.

   (a) $A = B$ if and only if $\overline{A} = \overline{B}$

Proof. Let A and B be two sets.
($\Rightarrow$) Suppose $A = B$. We can write $\overline{A}$ as $U - A$ and $\overline{B}$ as $U - B$. Given $A = B$, we can conclude that $U - A = U - B$ and hence $\overline{A} = \overline{B}$.
($\Leftarrow$)Suppose $\overline{A} = \overline{B}$. We also know that $\overline{A} = U - A$ and $\overline{B} = U - B$. Therefore, $U - A = U - B$ and $A = B$. ∎

(b) $\overline{\overline{A}} = A$

Proof. Let A be a set. We know that $\overline{A} = U - A$ and $\overline{\overline{A}} = U - \overline{A} = U - (U - A) = A$ by the definition of the complement set. ∎

(c) $\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$

Proof. Let A, B, and C be sets.
Let $x \in \overline{A \cup B \cup C}$. We know that $x \notin A \cup B \cup C$. It implies that $x \notin A$, $x \notin B$, and $x \notin C$ by the definition of the complement set. Hence, $x \in \overline{A}$, $x \in \overline{B}$, and $x \in \overline{C}$, implies that $x \in \overline{A} \cap \overline{B} \cap \overline{C}$. Therefore, every elements in $\overline{A \cup B \cup C}$ is also in $\overline{A} \cap \overline{B}$, $n\overline{A \cup B \cup C} \subseteq \overline{A} \cap \overline{B}$.

Let $x \in \overline{A} \cap \overline{B} \cap \overline{C}$. It implies from the definition of intersects that x is in $\overline{A}$,$\overline{B}$ and $\overline{C}$. Hence, x is $\notin A$, $x \notin B$ and $x \notin C$, or we can write it out as x is not in A, x is not in B, and x is not in C. According to the De Morgan's Law, we can conclude that x is not in A, B or C. Hence, $x \in \overline{A \cup B \cup C}$. Therefore, every elements in $\overline{A} \cap \overline{B} \cap \overline{C}$ is also in $\overline{A \cup B \cup C}$, $\overline{A} \cap \overline{B} \cap \overline{C} \subseteq \overline{A \cup B \cup C}$.

$\overline{A \cup B \cup C}$ and $\overline{A} \cap \overline{B} \cap \overline{C}$ are mutually subset of each other, which implies that $\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$. ∎

4. Let A, B, and C denote sets. Prove the following:

   (a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

   Proof.

   $A \times (B \cup C)$
   By definition of Cartesian product:
   $= \{(x, y) | (x \in A) \wedge (y \in (B \cup C)\}$
   Then, by definition of Union:
   $= \{(x, y) | (x \in A) \wedge ((y \in B) \vee (y \in C))\}$
   Distruibute the and operator using the distributive property of boolean algebra:
   $= \{((x, y) | (x \in A) \wedge (y \in B)) \vee ((x \in A) \wedge (y \in C)))\}$
   We can rewrite it into two sets:
   $= \{(x, y) | (x \in A) \wedge (y \in B)\} \cup \{(x, y) | (x \in A) \wedge (y \in C)\}$
   $= (A \times B) \cup (A \times C)$

   ■

   (b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

   Proof. Let (x, y) to be an arbitrary ordered pair from set $A \times (B \cap C)$, $(x, y) \in A \times (B \cap C)$. Then, we can state that $x \in A$ and $y \in (B \cap C)$. By the definition of intersect, since $y \in (B \cap C)$, $y \in B$ and $y \in C$ is true. By the distributive property, ($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$) has to be true as well. Putting x and y in ordered pair, we have $(x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$ from Cartesian Product. Therefore, $(x, y) \in (A \times B) \cap (A \times C)$ from the definition of intersection. Therefore, every elements in $A \times (B \cap C)$ is also in $(A \times B) \cap (A \times C)$, $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

   Let (x, y) to be an arbitrary ordered pair from set $(A \times B) \cap (A \times C)$, $(x, y) \in (A \times B) \cap (A \times C)$. By definition of intersection, we can conclude that $(x, y) \in A \times B$ and $(x, y) \in A \times C$. x, in this case, is in A, and y is in B and y is in C, $y \in B$ and $y \in C$. By definition of union, we can say that $y \in B \cup C$. Putting (x, y) into ordered pair, we can conclude $(x, y) \in A \times (B \cup C)$ from Cartesian Product. Therefore, every elements in $(A \times B) \cap (A \times C)$ is also in $A \times (B \cap C)$, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

   $A \times (B \cap C)$ and $(A \times B) \cap (A \times C)$ are mutually subset of each other, which implies that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. ■

(c) $A \times (B - C) = (A \times B) - (A \times C)$

Proof. Let (x, y) to be an arbitrary ordered pair from set $A \times (B - C)$. Then, we can state that $x \in A$ and $y \in (B - C)$. The set $B - C$ includes all all elements in set B but not in set C, or we can write as $y \in B$ and $y \notin C$. From $x \in A$ and $y \in B$, we can conclude that $(x, y) \in A \times B$ by Cartesian Product, and from $x \in A$ and $y \notin C$, we can conclude that $(x, y) \notin A \times C$ by Cartesian Product. Hence, to satisfy both statements, it implies that $(x, y) \in (A \times B) - (A \times C)$. Therefore, every elements in $A \times (B - C)$ is also in $(A \times B) - (A \times C)$, $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

Let (x, y) to be an arbitrary ordered pair from set $(A \times B) - (A \times C)$. Then, we can state that $(x, y) \in (A \times B)$ and $(x, y) \notin (A \times C)$. Hence, ($x \in A$ and $y \in B$) and not ($x \in A$ and $y \in C$). By the De Morgan's Law, we can conclude that ($x \in A$ and $y \in B$) and ($x \notin A$ or $y \notin C$). Since $x \in A$ must be true, $y \notin C$. Therefore, $x \in A$, $y \in B$ and $y \notin C$. It implies that $(x, y) \in A \times (B - C)$ from Cartesian Product. Therefore, every elements in $(A \times B) - (A \times C)$ is also in $A \times (B - C)$, $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

$A \times (B - C)$ and $(A \times B) - (A \times C)$ are mutually subset of each other, which implies that $A \times (B - C) = (A \times B) - (A \times C)$.

■

(d) $A \times (B \Delta C) = (A \times B) \Delta (A \times C)$

Proof. Let (x, y) to be an arbitrary ordered pair from set $A \times (B \Delta C)$. Then, we can state that $x \in A$ and $y \in (B \Delta C)$. The elements in set $B \Delta C$ includes everything in B and in C but not in both B and C. We can represent using symbol $y \in B \cup C$ and $y \notin B \cap C$. Hence, (x, y) is either in $A \times B$ or $A \times C$ but not in both. By definition of symmetric difference, we will have $(x, y) \in (A \times B) \Delta (A \times C)$. Therefore, every elements in $A \times (B \Delta C)$ is also in $(A \times B) \Delta (A \times C)$, $A \times (B \Delta C) \subseteq (A \times B) \Delta (A \times C)$.

Let (x, y) to be an arbitrary ordered pair from set $(A \times B) \Delta (A \times C)$. Then, we can state that $x \in A$ and $y \in B$ or $y \in C$ and $y \notin (B \cap C)$. By definition of symmetric difference, we can conclude that $y \in (B \Delta C)$. Hence, $(x, y) \in A \times (B \Delta C)$ by Cartesian Product. Therefore, every elements in $(A \times B) \Delta (A \times C)$ is also in $A \times (B \Delta C)$, $(A \times B) \Delta (A \times C) \subseteq A \times (B \Delta C)$.

$A \times (B \Delta C)$ and $(A \times B) \Delta (A \times C)$ are mutually subset of each other, which implies that $A \times (B \Delta C) = (A \times B) \Delta (A \times C)$. ■