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1. If $\phi : G \to H$ is a homomorphism with kernel $ker(\phi)$, then ϕ is injective iff $ker(\phi) = \{e\}$

   Proof. $(=>)$ Given that ϕ is injective, let $g \in G$ such that $\phi(g) = e$. From the homomorphism, we know that $\phi(e) = e$. Hence, $\phi(g) = \phi(e)$. By the injectivity, $g = e$. Therefore, $ker(\phi) = \{e\}$.

   $(<=)$ Given that $ker(\phi) = \{e\}$, let $g_1, g_2 \in G$ such that $\phi(g1) = \phi(g2)$.

   $$\begin{aligned} \phi(g_1) &= \phi(g_2) \\ \phi(g_1)\phi(g_2)^{-1} &= e \\ \phi(g_1)\phi(g_2^{-1}) &= e \\ \phi(g_1 g_2^{-1}) &= e \end{aligned}$$

   Since e is the only element in K, $g_1 g_2^{-1} = e$. It implies that $g_1 = ((g_2)^{-1})^{-1} = g_2$. Therefore, ϕ is injective.

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2. Let C be the center of G. Prove $C \trianglelefteq G$.

   Proof. We have proved that the center of G is a subgroup of G in write up 7. Now we need to show it is a normal subgroup. Let $c \in C$, $g \in G$. $gcg^{-1} = cgg^{-1} = c$. Therefore, $C \trianglelefteq G$.

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3. Prove that any intersection of normal subgroups of G is itself a normal subgroup of G.

   Proof. Let H and K be two arbitrary normal subgroups of G. We need to first prove that $H \cap K$ is a subgroup of G.

   First $H \cap K$ must contains the identity because both H and K contain it. Second, let $a, b \in H \cap K$. It implies that $a, b \in H$ and $a, b \in K$. That is $ab \in H$ and $ab \in G$, and hence $ab \in H \cap K$. Third, let $a^{-1} \in H \cap K$. It implies that $a^{-1} \in H$ and $a^{-1} inK$. Hence, $a^{-1} \in H \cap K$.

   Then, let $g \in G$ and $a \in H \cap K$. We know that $gag^{-1} \in H$ because H is a normal subgroup. Similarly, $gag^{-1} \in K$. Hence, $gag^{-1} \in H \cap K$.

   Therefore, we can conclude that any intersection of normal subgroups of G is itself a normal subgroup of G.

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4. Let $\phi : G \to H$, where ϕ is surjective. Prove that if G is cyclic, then H is cyclic. (assumed homomorphism)

   Proof. Since G is cylic, we have an element x such that $G = \langle x \rangle$. Let $g \in G$ and we can repreesnt g and x^m where m is an integer. Hence, $\phi(g) = \phi(x^m) = (\phi(x))^m = h \in H$ by homomorphism. Then we know that $H \in \langle \phi(x) \rangle$. We also know from the surjectivity that $\langle \phi(x) \rangle \in H$. Therefore, $H = \langle \phi(x) \rangle$, which proves that H is cyclic. ■

5. Let $\phi : G \to H$, where ϕ is surjective. Prove that if every element of G has finite order, then every element of H also has finite order.(assumed homomorphism)

   Proof. Since every element of G has a finite order, let $ord(g) = m$ and $g^m = e, g \in G, m \in \mathbb{Z}$. Let $h = \phi(g)$. We have $\phi(g^m) = \phi(e) = e$ and $\phi(g^m) = (\phi(g))^m$ from homomorphism. Hence, $\phi(g^m) = e = (\phi(g))^m = h^m$. Therefore, $h \in H$ has finite order(any factor of m). Since the choice of g is arbitrary, every other element of H also has finite order. ■