1. If $G = \langle a \rangle$ and $b \in G$, then the order of b is a factor of the order of a.

   Proof. Let the order of b be l. That is $b^l = e$. Since $b \in G$, b is some m power of a, then $b = a^m$. We can rewrite $b^l = e$ as $(a^m)^l = a^{ml} = e$. Hence, the order of b, l is a factor of the order of a, ml. ■

2. Let G have order 4. Prove that either G is cyclic, or every element of G is its own inverse. Conclude that every group of order 4 is abelian.

   Proof. Since G is a finite group, and the possible order of its can only be 1, 2 and 4. If G have an element a of order 1, $a^1 = e$, which is its own inverse. Then, if G have an element a of order 2,$a^2 = e$. We can concludue that a is its own inverse because $a * a = e$. If G have an element a of order 4. Then $\langle a \rangle$ is a subgroup of G and since $|a| = 4$, we have $|\langle a \rangle| = 4$. This means that $\langle a \rangle = G$ and so G is cyclic. Then, Let G have an element b of order 2. Therefore, either G is cyclic, or every element of G is its own inverse.

   Cyclic group is abelian because it is all generated by the same generator. If every element is its own inverse. That implies $a, b \in G$, such that, $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$. Therefore, every group of order 4 is abelian.

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3. Let G be finite, and $H, K \leq G$. Suppose H has index p and K has index q, where p and q are distinct primes. Prove that the index of $H \cap K$ is a multiple of pq.

   Proof. Let the order of G be n, order of H be l and the order of K be b. Then, we know that the index of H, $p = \frac{n}{l}$ and the index of K, $q = \frac{n}{b}$. We know that $H \cap K$ is a subgroup of H and of K. Then, by the Lagrange's theorem, we can conclude that $|H \cap K|$ is a factor of $|H|$ and of $|K|$. Let $|H \cap K| = z$, we can express $l = zi$ and $b = zj$. Hence, we can rewrite the $p = \frac{n}{zi}$ and $q = \frac{n}{zj}$. Or, $pi = qj = \frac{n}{z}$. We also know that i must be an integer, then p is a factor of qj. However, q is a prime number, p hence must be a factor of j, say $px = j$. Putting it in the equation, we will have $qj = qpx = \frac{n}{z}$. Therefore, $qp = pq$ is a factor of $\frac{n}{z}$, which is the index of $H \cap K$. Then, $H \cap K$ is a muutiple of pq.

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