For problems 1-3, let $H, K \leq G$.

1. Prove that $H \subseteq K \implies H \leq K$.

   Proof. Let $H \subseteq K$. Since K is a group and H is a nonempty subset of K. We are sure it is nonempty because H is a subgroup of G, where it cannot be empty. If we can show H is closed under the group operation and inverses, we can say that H is a subgroup of K.

   (1) Let h_1 and h_2 be elements of H. Since $H \leq G$, H is closed with the group operation and $h_1 * h_2$ is in H. Since $H, K \subseteq G$, they all share the same operation. Therefore, H is closed under the group operaton.

   (2) For each element h in H, its inverse is also in H because H is a group. Therefore, H has inverses for all its elements.

   We can conclude that H is a subgroup of K, H $\leq$K. ■

2. Show that $H \cap K \leq G$.

   Proof. Let $H \cap K \leq G$. Since G is a group and $H \cap K$ is a nonempty subset of G. We are sure the set is nonempty because they are both subgroups of G and hence have to share at least one element, the identity element. Then, if we can show $H \cap K$ is closed under the group operation and inverses, we can say that $H \cap K$ is a subgroup of G.

   (1) We know that for every element x in $H \cap K$, $x \in H$ and $x \in K$. Let x_1 and x_2 be elements of $H \cap K$. $x_1 * x_2$ must be in H as H is a group closed under the operation and $x1$ and $x2$ are both in H. Similarly, $x_1 * x_2$ must be in K. Hence, $x_1 * x_2$ is in $H \cap K$, it is closed under the operation.

   (2) Let x' be the inverse of x in $H \cap K$. Similarly, since x in both in H and K, its inverse is in both H and K. Hence, $H \cap K$ has inverses for all its elements.

   ■

3. Let G be an abelian group, and define HK as follows:

   $$HK = \{hk \mid h \in H \text{ and } k \in K\}$$

   Prove that $HK \leq G$.

   Proof. To prove that $HK \leq G$, we need to show that (1) HK is a subset of G. (2) HK is closed under the opreation and (3) inverses.

   (1) Since $H, K \subseteq G$, all elements in H and K are in G. Also that G is a group, so it is closed under the group opreation, hence, for every $h \in H$ and $k \in K$, hk must also be in G. Hence, HK is a subset of G.

(2)Let $h_1, h_2 \in H$ and $k_1, k_2 \in K$. $(h_1k_1)(h_2k_2) = (h_1h_2)(k_1k_2)$ because G is abelian group and H, K are also abelian groups. It implies the communitivity. We know that $h_1h_2 \in H$ and $k_1k_2 \in K$ because H and K are groups. Therefore, HK is closed under the group operation.

(3) Let $hk \in H$, and $(hk)^{-1}$ be its inverse. Since hk is in G its inverse $(hk)^{-1}$ must also be in G. Hence, $(hk)^{-1} = k^{-1}h^{-1} = h^{-1}k^{-1}$, by the communitivity from the abelian group. Since H and K are both groups, $h^{-1} \in H$ and $k^{-1} \in K$. $(hk)^{-1} \in HK.HK$ has inverses for all its elements. ■

4. Suppose a group G is generated by two elements a and b.
   Prove that $ab = ba \implies G$ is abelian.

Proof. Let $ab = ba$. We need to prove that group G has communitivity for it to be a abelian group.
Since $ab = ba$, we can conclude that a and b commute. We can also prove a and b^{-1} commute, $ab^{-1} = b^{-1}bab^{-1} = b^{-1}abb^{-1} = b^{-1}$; a^{-1} and b commute, $a^{-1}b = a^{-1}baa^{-1} = a^{-1}aba^{-1} = ba^{-1}$; a^{-1} and b^{-1} commute, $a^{-1}b^{-1} = a^{-1}b^{-1}aa^{-1} = a^{-1}ab^{-1}a^{-1} = b^{-1}a^{-1}$.
Since the group G is generated by a and b, every element can be written out only using a, a^{-1}, b and b^{-1}. We proved that a and b^{-1}, b and a^{-1}, and a^{-1} and b^{-1} commute. Given that a and b commute, and inherently a commute with a and b commute with b. We can conclude that all elements commute with each other in any generated elements by a and b. Therefore, we will be able to simplify any element in the group G as $a^x b^y$ (x,y $\in \mathbb{Z}$) after moving all a terms to the left side by communitivity, and combining like items. Similarly, we can moving all b terms to the left side and reduce it to $b^y a^x$. It implies that $a^x b^y$ and $b^y a^x$ are identical. Hence a^x commute with b^y.
Now, we need to prove that $a^x b^y$ commute with $a^q b^p$.

$$\begin{aligned}
(a^x b^y)(a^q b^p) &= a^x (b^y a^q) b^p && \text{associative} \\
&= a^x (a^q b^y) b^p && a^q \text{ commute with } b^y \\
&= (a^x a^q)(b^y b^p) && \text{associative} \\
&= a^{x+q} b^{y+p} \\
&= a^q a^x b^p b^y \\
&= a^q b^p a^x b^y && \text{associative} \\
&= (a^q b^p)(a^x b^y)
\end{aligned}$$

Therefore, we proved that group G has communitivity and hence it is abelian. ■

5. Define the center of a group to be

$$C = \{g \in G \mid gx = xg, \forall x \in G\},$$

that is, the set of all elements of G that commute with every element of G.
Prove $C \leq G$.

Proof. We know that C is a subset of the group G because $\forall g \in G$. Then, we need to prove (1) C is closed under the group operation and (2) inverses.

(1) Let $c, d \in C$. We know that $cx = xc$ and $dx = xd$. Hence, $(cd)x = cdx = cxd = xcd = x(cd)$. Hence, $cd \in C$, C is closed under the group operation.
(2) Let $c \in C$ and c^{-1} be its inverse. $c^{-1}x = c^{-1}xe = c^{-1}xcc^{-1} = c^{-1}cxc^{-1} = exc^{-1} = xc^{-1}$. Hence, $c^{-1} \in C$, C is closed under inverses.
Therefore, C is a subgroup of G, $C \leq G$. ∎