If G is a group, an **automorphism** of G is an isomorphism from G to G.

1. If G is any group, and a is any element of G, prove that $f(x) = axa^{-1}$ is an automorphism of G. We call this *conjugation* by b.

   Proof. We need to show that f is a bijecton function and satisfy that $f(xy) = f(x)f(y)$ such that x, y are both in G. Let $x, y \in G$ and $f(x) = f(y)$.

$$\begin{aligned} axa^{-1} &= aya^{-1} \\ a^{-1}axa^{-1} &= a^{-1}aya^{-1} \\ xa^{-1} &= ya^{-1} \\ xa^{-1}a &= ya^{-1}a \\ x &= y \end{aligned}$$

   Hence, f is injective. Then, let us prove that for every $y \in G$ there exists x in G such that $f(x) = y$. We have $x = a^{-1}ya$ satisfy the statement. $y = a(a^{-1}ya)a^{-1} = f(x)$ Hence, f is surjective and hence bijective.

   Lastly, we need to show that $f(xy) = f(x)f(y)$ for all $x, y \in G$.

$$\begin{aligned} f(xy) &= a(xy)a^{-1} \\ &= axya^{-1} \\ f(x)f(y) &= axa^{-1}aya^{-1} \\ &= axeya^{-1} \\ &= axya^{-1} \end{aligned}$$

   Hence, f is isomorphsm for all $x, y \in G$. Therefore, f is an automorphism of G. ■

2. Since each automorphism of G is a bijective function from G to G, it is a *permutation* of G. Define Aut(G) as the set of all automorphisms of G. Prove Aut(G)$\leq S_G$.

   Proof. First, we need to show that $Aut(G)$ is a nonempty subset of S_G. We can say that $\epsilon \in Aut(G)$ if G is a isomorphism of G. Consider the identity function for G: $\epsilon(x) = x$. It is injective as

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3. We'll prove some basic properties of order. Let $a, b, c \in G$. Show that

   (a) $\text{ord}(a) = \text{ord}(bab^{-1})$

   (b) $\text{ord}(a^{-1}) = \text{ord}(a)$

4. Now show
   (a) $\text{ord}(ab) = \text{ord}(ba)$
   (b) $\text{ord}(abc) = \text{ord}(cab) = \text{ord}(bca)$
5. Let $a \in G$, and of finite order. Prove that if a is the *only* element of order k in G, then a is in the center of G.