CS 4820, Spring 2017	**Homework 3, Problem 3**
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(3) *(10 points)*
Consider the following problem: You are given a connected undirected graph $G = (V, E)$ with distinct non-negative weights on the edges, $w_{ij} \geq 0$ for all $(i, j) \in E$. Let n and m denote the number of vertices and edges, respectively, in the graph. Give an algorithm with running time $O(m \log n)$ to partition the vertex set into two non-empty sets S and $V \setminus S$, such that the maximum weight of any edge with one end point in S and one end point in $V \setminus S$ is minimized. In other words, your algorithm should determine the set that minimizes

$$\max \{w_{ij} \mid (i, j) \in E, i \in S, j \notin S\}$$

among all sets S such that both S and $V \setminus S$ are non-empty.

Description of the Algorithm

- Change all edge values to its opposite signs for a new Graph G'.
- Use Kruskal's Algorithm to find the Minimum Spanning Tree (MST) T of the G'.
- Find the maximum weight edge of G': w'_{max}. Then $-w'_{max}$ is to minimize

$$\max \{w_{ij} \mid (i, j) \in E, i \in S, j \notin S\}$$

- Cut off this edge in the tree T, then we have two partitions S and $V \setminus S$.

Correction of the Algorithm

*Lemma*1 : The Kruskal's Algorithm is still applicable for negative edges.
Proof: Because the Kruskal's Algorithm is to use greedy algorithm to find the smallest weight edge combination. It hurts nothing if the edges are negative or positive as long as the newly inserted edge doesn't form a cycle. It is equals to add the currently maximum edge into the current graph.

*Lemma*2 : Find the maximum weight edge in MST will lead to the minimum of the objective function.
Proof: If we change the signs for the edge values, the requirement is going to be: find the maximum of the minimum weight edge that has one end in S and the other in $V \setminus S$. It is known that for arbitrary non-empty partition, we always use MST to find the minimum edge combination. If we find the maximum weight edge in the MST, we have this edge with a larger edge than any other in the MST, thus maximizing such edges with one end in S and the other in $V \setminus S$ in the graph.