MA203/MA283 Linear Algebra 2023/24
Exam Information and Advice

The exam has four questions. For full marks, you must answer (any) three of the four questions. If you answer all four, marks will be awarded for the best three. Each question has four parts, and all four parts have the same weight.

The questions will connect to the syllabus content as follows.

1. Systems of linear equations (Lectures 1 to 5)
2. Matrix algebra, linear transformations and matrices, the Rank-Nullity Theorem (Lectures 6 to 10 and Lecture 14)
3. Linear independence, spanning sets, eigenvectors, algebraic and geometric multiplicity. (Lectures 10 to 14 and Lecture 17)
4. Similarity, inner product spaces, orthogonal projection, least squares approximate solutions. (Lectures 15 to 20).

The sample questions below are intended to give a sense of what to expect in terms of style and format, and to provide some practice exam-type questions.

Sample Exam Paper

1. (a) What is meant by an *inconsistent* system of linear equations?
Explain how applying elementary row operations to the augmented matrix of a system of equations can reveal that the system is inconsistent.

(b) Find the general solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & -7 \\ 2x_1 & & & + & x_3 & + & 5x_4 & = & 0 \\ x_1 & + & 3x_2 & - & 2x_3 & + & 3x_4 & = & 7 \end{array}$$

(c) Show that the following system has infinitely many solutions.

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & -7 \\ 2x_1 & & & + & x_3 & + & 5x_4 & = & 0 \\ x_1 & + & 3x_2 & - & 2x_3 & + & 3x_4 & = & 7 \\ x_1 & + & x_2 & + & x_3 & + & 4x_4 & = & 5 \end{array}$$

(d) Answer TRUE or FALSE to each of the following.

i. Every system of linear equations with more variables than equations is consistent.
ii. A system of three linear equations in two variables could have a unique solution.
iii. A system of two linear equations in three variables could have a unique
iv. An inconsistent system can be made consistent by adding more equations.
v. If a system is consistent, then a reduced row echelon form obtained from its augmented matrix has a leading 1 corresponding to each variable.

2. (a) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Explain how to write the (standard) matrix of T.

(b) Suppose that $T : \mathbb{R}^4 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ are linear transformations whose (standard) matrices are respectively $\begin{bmatrix} 1 & -1 & 3 & 5 \\ 2 & 1 & 9 & -3 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 \\ 4 & 0 \\ 6 & -1 \end{bmatrix}$. What is the (standard) matrix of the linear transformation $S \circ T$? .

(c) Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel and image both include the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(d) State and prove the Rank-Nullity Theorem. Explain how it follows from the Rank-Nullity Theorem that the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ cannot *both* belong to both the image and kernel of any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

3. (a) Explain what is meant by a *spanning set* of a vector space. If a vector space has a linearly independent set with three elements, show that it cannot have a spanning set with two elements.

(b) Determine whether $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 1 & 2 & 4 \\ -6 & 2 & -1 \\ 8 & -7 & 6 \end{bmatrix}$, and determine the corresponding eigenvalue if so.

(c) Suppose that $\lambda_1, \dots, \lambda_k$ are *distinct* eigenvalues of a $n \times n$ matrix A, with corresponding eigenvectors $v_1, \dots, v_k$ respectively. Prove that $\{v_1, \dots, v_k\}$ are linearly independent, and deduce that A is diagonalizable if $k = n$. Give an example of a 2×2 matrix that does not have two distinct eigenvalues but is diagonalizable.

(d) For a matrix $A \in M_n(\mathbb{R})$ with an eigenvalue λ, explain what is meant by the *algebraic multiplicity* and *geometric multiplicity* of λ. Give an example of a matrix that has three distinct eigenvalues with algebraic multiplicities $1, 2$ and 2, and geometric multiplicities $1, 1$ and 2.

4. (a) What does it mean to say that the 2×2 matrices A and B are similar? Determine whether the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ are similar.

(b) State the definition of an *inner product* on a real vector space V, and explain how an inner product can be used to define the *distance* between two vectors in V.

(c) In $\mathbb{R}^3$, find the projection of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ on the 2-dimensional subspace that is spanned by $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(d) Find the least-squares approximate solution to the following overdetermined system of equations.

$$\begin{array}{rcrcr} x & - & 3y & = & 1 \\ x & + & 4y & = & -4 \\ 2x & - & y & = & 3 \end{array}$$

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Sample Exam Paper - Outline Solutions.

Q1(a) A system is inconsistent if it has no solution. This is revealed in elementary row operations by the appearence of a row whose entries are all zero except the last.

(b)
$$\left[\begin{array}{ccccc} 1 & -2 & 1 & 1 & -7 \\ 2 & 0 & 1 & 5 & 0 \\ 1 & 3 & -2 & 3 & 7 \end{array}\right] \xrightarrow[R3 \to R3 - R1]{R2 \to R2 - 2R1} \left[\begin{array}{ccccc} 1 & -2 & 1 & 1 & -7 \\ 0 & 4 & -1 & 3 & 14 \\ 0 & 5 & -3 & 2 & 14 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - R2} \left[\begin{array}{ccccc} 1 & -2 & 1 & 1 & -7 \\ 0 & 4 & -1 & 3 & 14 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right] \xleftrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccccc} 1 & -2 & 1 & 1 & -7 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 4 & -1 & 3 & 14 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - 4R2} \left[\begin{array}{ccccc} 1 & -2 & 1 & 1 & -7 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 7 & 7 & 14 \end{array}\right] \xrightarrow{R3 \times (1/7)} \left[\begin{array}{ccccc} 1 & -2 & 1 & 1 & -7 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R1 \to R1 + 2R2} \left[\begin{array}{ccccc} 1 & 0 & -3 & -1 & -7 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow[R2 \to R2 + 2R3]{R1 \to R1 + 3R3} \left[\begin{array}{ccccc} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

General Solution:

$$(x_1, x_2, x_3, x_4) = (-1-2t,\ 4-t,\ 2-t,\ t), \quad t \in \mathbb{R}$$

(c) The first three equations are from part (b). Putting $(x_1, x_2, x_3, x_4) = (-1-2t, 4-t, 2-t, t)$ in Equation 4 gives

$$-1-2t + 4-t + 2-t + 4t = 5 + 0t = 5$$

Eqn 4 is satisfied by all simultaneous

solutions of the first 3 equations

(d) FALSE, TRUE, FALSE, FALSE, FALSE

Q2 (a) The standard matrix of T is the $m \times n$ matrix that has $T(e_1), T(e_2), \ldots, T(e_n)$ as its n columns, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. The vector e_j has 1 as its entry in position j and 0 in all other positions.

(b) Matrix of $S \circ T$ is

$$\begin{bmatrix} 2 & -3 \\ 4 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 & 5 \\ 2 & 1 & 9 & -3 \end{bmatrix} = \begin{bmatrix} -4 & -5 & -21 & 19 \\ 4 & -4 & 12 & 20 \\ 4 & -7 & 9 & 33 \end{bmatrix}.$$

(c) The linear transformation defined by $T(v) = Av$, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix}$$

(d) Rank Nullity Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

$$\dim(\ker T) + \dim(\operatorname{Im} T) = n.$$

Proof Suppose $k = \dim(\ker T)$, and let $\{b_1, \ldots, b_k\}$ be a basis of $\ker T$. We can extend this to a basis $\{b_1, \ldots, b_k, c_{k+1}, \ldots, c_n\}$ of $\mathbb{R}^n$. Since every element of $\mathbb{R}^n$ is a linear combination of $b_1, \ldots, b_k, c_{k+1}, \ldots, c_n$, and $T(b_i) = 0$ for $i = 1, \ldots, k$, it follows that every element of the image of T is a linear combination of $T(c_{k+1}), \ldots, T(c_n)$. Moreover, $\{T(c_{k+1}), \ldots, T(c_n)\}$ is a linearly independent set in $\mathbb{R}^m$, for suppose

$$\alpha_{k+1} T(c_{k+1}) + \ldots + \alpha_n T(c_n) = 0$$

for scalars α_i. Then

$$T(\alpha_{k+1} c_{k+1} + \ldots + \alpha_n c_n) = 0$$

and $\alpha_{k+1} c_{k+1} + \cdots + \alpha_n c_n \in \ker T$. This means $\alpha_{k+1} c_{k+1} + \ldots + \alpha_n c_n$ is a linear combination of $b_1, \ldots, b_k$, which contradicts the linear independence of $\{b_1, \ldots, b_k, c_{k+1}, \ldots, c_n\}$ unless $\alpha_{k+1} c_{k+1} + \ldots + \alpha_n c_n = 0$. Since $\{c_{k+1}, \ldots, c_n\}$ is a linearly independent set, it follows that $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$. Hence $\{T(c_{k+1}), \ldots, T(c_n)\}$ is linearly independent, so it is a basis of the image of T, and $\dim(\operatorname{Im} T) = n - k$, as required.

If $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\2\\4\end{bmatrix}$ both belong to the image <u>and</u> the kernel of $T : \mathbb{R}^3 \to \mathbb{R}^3$, then $\dim(\ker T) + \dim(\operatorname{Im} T) \geq 2 + 2 = 4$, contrary to the Rank-Nullity Theorem.

Q3(a) S is a spanning set of the vector space V if every element of V is a linear combination of elements of S.

Suppose $\{v_1, v_2, v_3\}$ is a linearly independent subset of V, and let $S = \{s_1, s_2\}$ be a subset of V with two elements. We need to show that S does not span V. If $v_1 \notin \langle S \rangle$, then S does not span V. So assume $v_1 \in \langle S \rangle$ and write $v_1 = as_1 + bs_2$ for scalars a and b. Since v_1 belongs to a linearly independent set, $v_1 \neq 0$ and a and b are not both zero. Suppose $a \neq 0$. Then $s_1 = a^{-1}v_1 - a^{-1}bs_2$, and $s_1 \in \langle v_1, s_2 \rangle$, and $\langle s_1, s_2 \rangle \subseteq \langle v_1, s_2 \rangle$. Since $v_1 \in \langle s_1, s_2 \rangle$ also, it follows that $\langle v_1, s_2 \rangle \subseteq \langle s_1, s_2 \rangle$ and $\langle s_1, s_2 \rangle = \langle v_1, s_2 \rangle$.

If $v_2 \notin \langle v_1, s_2 \rangle$, then $v_2 \notin \langle s_1, s_2 \rangle$ and $\{s_1, s_2\}$ is not a spanning set. So assume $v_2 \in \langle s_1, s_2 \rangle$. Then $v_2 \in \langle v_1, s_2 \rangle$ and $v_2 = av_1 + bs_2$, where $b \neq 0$ since v_2 is not a scalar multiple of v_1. Then $s_2 = b^{-1}v_2 - b^{-1}av_1$, and $s_2 \in \langle v_1, v_2 \rangle$. Since $v_2 \in \langle v_1, s_2 \rangle$ also, it follows that $\langle v_1, v_2 \rangle = \langle v_1, s_2 \rangle = \langle s_1, s_2 \rangle$. But now $v_3 \notin \langle v_1, v_2 \rangle$ (since $\{v_1, v_2, v_3\}$ is linearly independent, so $v_3 \notin \langle S \rangle$ and S is not a spanning set.

(b) $$\begin{bmatrix} 1 & 2 & 4 \\ -6 & 2 & -1 \\ 8 & -7 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \\ 33 \end{bmatrix} = 11\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$ Yes an eigenvector. Eigenvalue is 11

(c) Suppose $\{v_1, \dots, v_k\}$ is linearly dependent, and let m be the least with $\{v_1, \dots, v_m\}$ linearly dependent. Then

$$\alpha_1 v_1 + \dots + \alpha_m v_m = 0$$

for scalars $\alpha_1, \dots, \alpha_m$ with $\alpha_m \neq 0$.
Then multiplying separately by λ_m and A gives

$$\alpha_1 \lambda_m v_1 + \dots + \alpha_m \lambda_m v_m = 0$$
$$\alpha_1 \lambda_1 v_1 + \dots + \alpha_m \lambda_m v_m = 0$$

Subtract to get

$$\alpha_1 (\lambda_m - \lambda_1) v_1 + \dots + \alpha_{m-1} (\lambda_m - \lambda_{m-1}) v_{m-1} = 0$$

The coefficients here are not all zero, since all of the $\lambda_m - \lambda_i$ are non-zero, and at least one of the α_i is not zero. So this is a nontrivial expression for 0 as a linear combination of $v_1, \dots, v_{m-1}$, contradicting the definition of m.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Repeated eigenvalue $\lambda = 1$, but diagonalizable.

(d) The algebraic multiplicity of λ is the number of times λ occurs as a root of the characteristic polynomial of A. The geometric multiplicity is the dimension of the space $\{v \in \mathbb{R}^n : Av = \lambda v\}$.

Example $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ Eigenvalues are 1, 2, 3.

Q4(a) A and B are similar if $A = PBP^{-1}$ for some invertible matrix P.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v$. The matrices are similar if there is a basis $\{b_1, b_2\}$ of $\mathbb{R}^2$ with $T(b_1) = b_1$ and $T(b_2) = 3b_1 + b_2$.

Take $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} x \\ y \end{bmatrix}$. Then $T(b_2) = \begin{bmatrix} x+y \\ y \end{bmatrix}$. If $T(b_2) = 3b_1 + b_2$ then

$$\begin{bmatrix} x+y \\ y \end{bmatrix} = \begin{bmatrix} 3+x \\ y \end{bmatrix} \Rightarrow y = 3, \quad x \text{ is free.}$$

For example take $x = 0$: $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.
Then $T(b_1) = b_1$, $T(b_2) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3b_1 + b_2$.

The matrix of T with respect to the basis $\{b_1, b_2\}$ is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ are similar.

(b) An inner product $\langle \cdot, \cdot \rangle$ on V is a function from $V \times V$ to $\mathbb{R}$ that satisfies the following:

1. $\langle u, v \rangle = \langle v, u \rangle \quad \forall \ u, v \in V$
2. $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$ and $\langle u, bv + cw \rangle = b\langle u, v \rangle + c\langle u, w \rangle$, for all $u, v, w \in V$ and $a, b, c \in \mathbb{R}$
3. $\langle v, v \rangle \geq 0 \quad \forall \ v \in V$, and $\langle v, v \rangle = 0$ only if $v = 0$

(c) Since $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, as $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$,

the projection on the subspace is just

the sum of the projections on $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\1\\-1\end{bmatrix}$

$$\text{proj}_{\begin{bmatrix}1\\1\\2\end{bmatrix}}\begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{(1\ 2\ 1)\cdot(1\ 1\ 2)}{(1\ 1\ 2)\cdot(1\ 1\ 2)}\begin{bmatrix}1\\1\\2\end{bmatrix} = \frac{5}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}$$

$$\text{proj}_{\begin{bmatrix}1\\1\\-1\end{bmatrix}}\begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{(1\ 2\ 1)\cdot(1\ 1\ -1)}{(1\ 1\ -1)\cdot(1\ 1\ -1)}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \frac{2}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Projection on the subspace is

$$\frac{5}{6}\begin{bmatrix}1\\1\\2\end{bmatrix} + \frac{2}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\1\end{bmatrix}$$

(d) $$\underbrace{\begin{bmatrix}1 & -3\\1 & 4\\2 & -1\end{bmatrix}}_{A}\begin{bmatrix}x\\y\end{bmatrix} = \underbrace{\begin{bmatrix}1\\-4\\3\end{bmatrix}}_{b}$$

Least squares solution is $(A^TA)^{-1}A^Tb$

$$A^Tb = \begin{bmatrix}1 & 1 & 2\\-3 & 4 & -1\end{bmatrix}\begin{bmatrix}1\\-4\\3\end{bmatrix} = \begin{bmatrix}3\\-22\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1 & 1 & 2\\-3 & 4 & -1\end{bmatrix}\begin{bmatrix}1 & -3\\1 & 4\\2 & -1\end{bmatrix} = \begin{bmatrix}6 & -1\\-1 & 26\end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{155}\begin{bmatrix}26 & 1\\1 & 6\end{bmatrix}$$

Solution: $$\frac{1}{155}\begin{bmatrix}26 & 1\\1 & 6\end{bmatrix}\begin{bmatrix}3\\-22\end{bmatrix} = \frac{1}{155}\begin{bmatrix}56\\-129\end{bmatrix}$$