OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Semester 2 Examinations 2022–2023

Course Instance Codes	2BA1, 2BDS1, 2BME1, 2BMU1, 1OA2, 1OA3, 2BCT1, 2BMS1, 2BPT1, 2BS1, 3BS9, 2FM1
Examinations	Second Year Science, Arts, Computer Science & Information Technology
Module Codes	MA283
Module	Mathematics
Paper No	1
External Examiner	Prof. C. Roney-Dougal
Internal Examiners	Prof. G. Pfeiffer Dr R. Quinlan*
Instructions:	Answer THREE of the four questions. Within each question, all four parts carry equal marks
Duration	Two hours
No. of Pages	Three (including this cover page)

Requirements:

Release in Exam Venue	Yes ☑ No ☐
MCQ	Yes ☐ No ☑
Handout	None
Statistical Tables/ Log Tables	None
Cambridge Tables	None
Graph paper	None
Log Graph Paper	None
Other Materials	Non-programmable calculators are permitted

1. (a) Find the general solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \end{array}$$

(b) Find the unique value of k for which the following system is consistent.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \\ x_1 & + & x_2 & - & x_3 & - & 5x_4 & = & k \end{array}$$

(c) Find the unique solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & 3x_4 & = & 6 \\ 2x_1 & - & x_2 & + & x_3 & + & 8x_4 & = & -1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 10x_4 & = & 6 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 9 \end{array}$$

(d) State, with a short explanation, whether each of the following is TRUE or FALSE.

i. A system of three linear equations in three variables can have infinitely many solutions.

ii. A system of three linear equations in four variables can have a unique solution.

2. (a) What is meant by a *spanning set* of a vector space V? Exactly one of the following sets is a spanning set of $\mathbb{R}^3$. Determine which one, with explanation.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Let V be the subset of $M_3(\mathbb{R})$ consisting of all matrices A with the property that $A^T + A$ is diagonal, where A^T denotes the transpose of A. Show that V is a subspace of $M_3(\mathbb{R})$, and determine its dimension.

(c) Let V be a vector space over $\mathbb{R}$, with a spanning set $S = \{v_1, v_2, v_3\}$. Show that every set of four elements of V is linearly dependent.

(d) Let A be a $m \times n$ matrix with real entries. Explain what is meant by the *row rank* and the *column rank* of A, and show that these are equal.

Questions 3 and 4 are on the next page

3. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(v) = \begin{bmatrix} 2 & -2 & 0 \\ 2 & -4 & -1 \\ 2 & -4 & 1 \end{bmatrix} v,$$

for a column vector $v \in \mathbb{R}^3$. What is $T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right)$?

Determine the matrix of T with respect to the basis $\mathcal{B}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

(b) Suppose that $v_1, \ldots, v_k$ are eigenvectors of a square matrix A, corresponding respectively to different eigenvalues $\lambda_1, \ldots, \lambda_k$. Show that $\{v_1, \ldots, v_k\}$ is a linearly independent set.

(c) Give an example of

i. A matrix in $M_2(\mathbb{R})$ whose characteristic polynomial is $X^2 + 1$.

ii. A non-zero matrix in $M_2(\mathbb{R})$, that has 0 occurring twice as an eigenvalue.

iii. A matrix in $M_2(\mathbb{R})$ that is similar to $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ and has 2 as its entry in the $(1,1)$-position.

(d) Let A be a square matrix with an eigenvalue λ. What is meant by the *algebraic multiplicity* and *geometric multiplicity* of λ as an eigenvalue of A? Explain why the geometric multiplicity cannot exceed the algebraic multiplicity.

4. (a) A scalar product $\star$ on $\mathbb{R}^3$ is defined by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \star \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_2 + a_2 b_3 + a_3 b_1.$$

Is $\star$ an inner product on $\mathbb{R}^3$? Why or why not?

(b) In $\mathbb{R}^3$, calculate the orthogonal projection of $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ on the subspace U spanned by $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(c) Consider the following inconsistent system of linear equations.

$$\begin{array}{rcrcr} x & + & 2y & = & 3 \\ 3x & + & 7y & = & 8 \\ 3x & + & 5y & = & 4 \end{array}$$

What is meant by the *least-squares approximate solution* to this system? By using the concept of projection on a subspace or otherwise, explain why the least-squares approximate solution is given by

$$\hat{x} = (A^T A)^{-1} A^T \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix},$$

where A is the coefficient matrix of the system.

(d) Find the least-squares approximate solution of the system of equations in part (c).