Semester 2 Examinations 2021–2022

Course Instance Codes	2BA1, 2BDS1, 2BME1, 2BMU1, 1OA2, 1OA3, 2BCT1, 2BMS1, 2BPT1, 2BS1, 3BS9, 2FM1
Examinations	Second Year Science, Arts, Computer Science & Information Technology
Module Codes	MA283
Module	Mathematics
Paper No	1
External Examiner	Prof. C. Roney-Dougal
Internal Examiners	Dr N. Madden Dr R. Quinlan*
Instructions:	Answer THREE of the four questions. Within each question, all four parts carry equal marks
Duration	Two hours
No. of Pages	Three (including this cover page)

Requirements:

Release in Exam Venue	Yes ☑ No ☐
MCQ	Yes ☐ No ☑
Handout	None
Statistical Tables/ Log Tables	None
Cambridge Tables	None
Graph paper	None
Log Graph Paper	None
Other Materials	Non-programmable calculators are permitted

1. (a) Determine whether each of the following is a vector space over the field of rational numbers. If your answer is *no*, give a reason. If your answer is *yes*, it is enough to just say so.

i. The set $\mathbb{Z}$ of integers (with the usual addition and multiplication by rational scalars).

ii. The set $\mathbb{R}$ of real numbers (with the usual addition and multiplication by rational scalars).

iii. The set $M_2(\mathbb{Q})$ of all 2×2 matrices with rational entries.

(b) Answer TRUE of FALSE to each of the following

i. A system of three linear equations in three variables can have a unique solution.

ii. A system of four linear equations in three variables can have a unique solution.

iii. A system of three linear equations in four variables can have a unique solution.

iv. A system of three linear equations in three variables can be inconsistent.

v. A system of three linear equations in three variables can have exactly two solutions.

(c) Find the general solution of the following system of linear equations.

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 10 \\ 3x_1 & + & x_2 & + & x_3 & - & 2x_4 & = & 1 \\ x_1 & + & 4x_2 & - & x_3 & + & 9x_4 & = & 13 \end{array}$$

(d) Find the unique value of k for which the following system has infinitely many solutions.

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 10 \\ 3x_1 & + & x_2 & + & x_3 & - & 2x_4 & = & 1 \\ x_1 & + & 4x_2 & - & x_3 & + & 9x_4 & = & 13 \\ x_1 & + & x_2 & + & x_3 & + & kx_4 & = & -1 \end{array}$$

2. (a) Let V be a vector space over a field $\mathbb{F}$, and let S be a subset of V. What does it mean to say that S is a *spanning set* of V?

Determine whether the set S below is a spanning set of the space $M_2(\mathbb{Q})$ of 2×2 matrices over $\mathbb{Q}$.

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -3 & 2 \\ 0 & 1 \end{bmatrix} \right\}$$

(b) Let V be a vector space over a field $\mathbb{F}$, and let L be a linearly independent subset of V. Let v be an element of V that does not belong to the linear span of L in V. Prove that $L \cup \{v\}$ is linearly independent.

(c) Answer TRUE or FALSE to each of the following statements about a finite-dimensional vector space V over a field $\mathbb{F}$.

i. If V has a spanning set with 5 elements, then the dimension of V is 5.

ii. If V has dimension 5, then V has a linearly independent subset with four elements.

iii. If V has dimension 5, then *every* spanning set of V has at least 5 elements.

iv. If V has a linearly independent set with 5 elements, then the dimension of V is at least 5.

v. If the dimension of V is 5, then every subset of V with four elements is linearly independent.

(d) One of the following sets is a basis of $\mathbb{R}^3$ and the other is not. Determine which is which.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

For the set above that *is* a basis of $\mathbb{R}^3$, determine the coordinates of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with respect to this basis (with the ordering of the basis elements written above).

3. (a) Suppose that $T : V \to V$ is a linear transformation, where V is a vector space of dimension n. If $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis of V, explain how to write the matrix of T with respect to $\mathcal{B}$.

(b) Give an example, with explanation, of a matrix in $M_2(\mathbb{R})$ that is not diagonalizable.

(c) What does it mean to say that two square matrices are *similar*? Determine, with explanation whether the following two matrices in $M_2(\mathbb{R})$ are similar.

$$\begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix} \quad \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix}$$

(d) Let $A = \begin{bmatrix} -1 & 1 & 1 \\ 4 & 0 & -2 \\ 4 & -3 & -1 \end{bmatrix}$. Find a 3×3 invertible matrix $P \in M_n(\mathbb{R})$ for which

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

(Note: you can do this without calculating the characteristic polynomial of A.)

4. (a) Let A be a $m \times n$ matrix with entries in $\mathbb{R}$, and let r be the row rank of A. Explain why A can be written as a product BC, where B is $m \times r$ and C is $r \times n$. Deduce that the row rank and column rank of A are equal.

(b) State the definition of an *inner product* on a real vector space V, and explain how an inner product can be used to define the *distance* between two vectors in V.

(c) Suppose that W is a subspace of a finite dimensional inner product space V. For $v \in V$, how is the *orthogonal projection* $P_W(v)$ of v on W defined? Show that this is the unique element of W that is closest to v.

(d) Find the least-squares approximate solution to the following overdetermined system of equations.

$$\begin{array}{rcrcr} x & + & 3y & = & 4 \\ x & - & 2y & = & -6 \\ 2x & - & 3y & = & 11 \end{array}$$