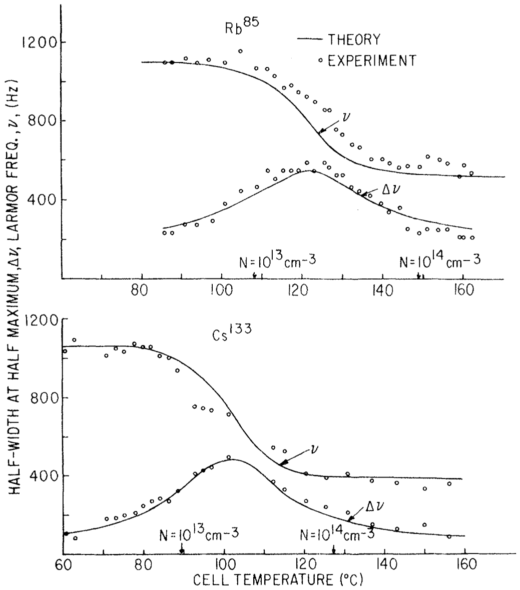

FIG. 3. Magnetic resonance frequency ν and the half width at half-maximum, $\Delta\nu$, of the magnetic resonance line in Rb^{85} and Cs^{133} versus the resonance cell temperature. Theoretical curves, calculated using the spin-exchange constant (Ref. 8), data for rubidium vapor pressure versus temperature (Ref. 9), and data for cesium vapor pressure versus temperature (Ref. 10).

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[1]E. M. Purcell and G. B. Field, Astrophys. J. 124, 542 (1956).
[2]H. W. Moos and R. H. Sands, Phys. Rev. 135, A591 (1964).
[3]N. Bloembergen, E. M. Purcell, and R. V. Pound, Phys. Rev. 73, 679 (1948).
[4]C. J. Gorter and J. H. Van Vleck, Phys. Rev. 72, 1128 (1947); J. Van Vleck, Phys. Rev. 74, 1168 (1948); P. W. Anderson and R. R. Weiss, Rev. Mod. Phys. 25, 269 (1968); J. D. Currin, Phys. Rev. 126, 1995 (1962).
[5]L. W. Anderson, F. M. Pipkin, and J. C. Baird, Phys. Rev. 120, 1279 (1960); L. W. Anderson and A. T. Ramsey, Phys. Rev. 132, 712 (1963).
[6]P. L. Bender, Phys. Rev. 132, 2154 (1963); L. C. Balling, R. J. Hanson, and F. M. Pipkin, Phys. Rev. 133, A607 (1964); L. C. Balling and F. M. Pipkin, Phys. Rev. 136, A461 (1964).
[7]F. Grossetête, J. Phys. (Paris) 25, 383 (1965), and 29, 456 (1968).
[8]H. M. Gibbs, Phys. Rev. 139, A1374 (1965); H. M. Gibbs and R. J. Hull, Phys. Rev. 153, 132 (1967).
[9]T. J. Killian, Phys. Rev. 27, 578 (1926).
[10]I. Langmuir and K. H. Kingdon, Proc. Roy. Soc., Ser. A 107, 61 (1925).

Time Evolution of a Two-Dimensional Classical Lattice System

J. Hardy and Y. Pomeau
Laboratoire de Physique des Plasmas, Faculté des Sciences, 91-Orsay, France

and

O. de Pazzis
Centre National de la Recherche Scientifique, Marseille, France
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We exhibit a dynamical model of particles in a two-dimensional lattice. We study a particular class of equilibrium states, and we investigate the simplest time correlation functions for these states. As a consequence, the propagation of sound waves with velocity $1/\sqrt{2}$ can be proved rigorously. A transport coefficient analogous to a viscosity is given by a Kubo-type integral formula. The convergence of the integral is discussed. Some preliminary computer results are reported.

We consider an infinite two-dimensional square lattice. On each lattice site there are at most four particles. Each particle may have four velocities, all equal in absolute value, but whose directions are one of the four numbers $\{1, 2, 3, 4\}$ corresponding respectively to the four unit vectors $\{(1, 0), (0, 1), (-1, 0), (0, -1)\}$. Configurations are excluded where there are at least two particles with the same velocity on the same lattice site. We denote by X an allowed configuration and by K the set of all X. It is convenient to introduce the following functions defined on K:

$$\sigma_X(p,q;i)=\begin{cases}1 \text{ if the site } (p,q) \text{ is occupied by a particle of } X \text{ of velocity } i,\\ 0 \text{ otherwise.}\end{cases}$$

Giving an X is the same as giving the values of the functions $\sigma_X(p,q;i)$.

The motion of the particles is specified as follows. During one unit of time each particle jumps one step in the direction of its velocity; then, on each lattice site where the number of particles is equal to two and where the two particles have opposite velocities, *there is a collision* which rotates the velocity of each particle through an angle of $\pi/2$. On the other lattice sites the situation is left unchanged.

If X is the initial configuration, we call $T(X)$ the new configuration after one unit of time has elapsed. Using the functions $\sigma_X(p,q;i)$, we have

$$\begin{aligned}&\sigma_{T(X)}(p,q;1)=\sigma_X(p-1,q;1)-\Psi_X(p,q), \quad \sigma_{T(X)}(p,q;2)=\sigma_X(p,q-1;2)+\Psi_X(p,q),\\ &\sigma_{T(X)}(p,q;3)=\sigma_X(p+1,q;3)-\Psi_X(p,q), \quad \sigma_{T(X)}(p,q;4)=\sigma_X(p,q+1;4)+\Psi_X(p,q),\end{aligned} \tag{1}$$

where

$$\begin{aligned}\Psi_X(p,q)=\ &\sigma_X(p-1,q;1)\sigma_X(p+1,q;3)\bar{\sigma}_X(p,q-1;2)\bar{\sigma}_X(p,q+1;4)\\ &-\bar{\sigma}_X(p-1,q;1)\bar{\sigma}_X(p+1,q;3)\sigma_X(p,q-1;2)\sigma_X(p,q+1;4), \quad \bar{\sigma}_X(p,q;i)=1-\sigma_X(p,q;i).\end{aligned}$$

In this Letter we give some preliminary results concerning the statistical mechanics of this model. The detailed proofs of the statements given below will be published elsewhere. First, we introduce further notations. Let μ denote a probability measure on K, which is homogeneous, that is, invariant under lattice translations. We can define the following quantities: $n_i(\mu)=\int\sigma_X(p,q;i)\,d\mu$. The left-hand side is independent of (p,q) by the homogeneity property.

It is easy to check that the *measures* $\mu(n_1,n_2,n_3,n_4)$, *such that the particles are uncorrelated, are invariant under T if and only if*

$$n_1n_3(1-n_2)(1-n_4)=n_2n_4(1-n_1)(1-n_3). \tag{2}$$

These measures play the role of the equilibrium states in more realistic systems. Note that (2) is satisfied in particular when $n_1=n_2=n_3=n_4=n_0$. We denote by $\mu(n_0)$, $0\leq n_0\leq 1$, the corresponding measure. The remainder of this Letter is devoted to the study of some time correlation functions of the dynamical system $[K, T, \mu(n_0)]$.[1]

First we introduce the new functions $s_X(p,q;i)=\sigma_X(p,q;i)-n_0$, whose mean values are zero. Equations (1) can be writted in terms of $s_X(p,q;i)$. The first equation of system (1), for instance, reads

$$\begin{aligned}s_{T(X)}(p,q;1)=(1-\lambda)s_X(p-1,q;1)+\lambda s_X(p,q-1;2)-\lambda s_X(p+1,q;3)\\ +\lambda s_X(p,q+1;4)+\text{“nonlinear terms,”}\end{aligned} \tag{3}$$

where $\lambda=n_0(1-n_0)$ (note that $0\leq\lambda\leq\frac{1}{4}$). The "nonlinear term" is a linear combination of products of two or three functions s_X associated to the nearest neighbors of (p,q). By iterating Eq. (3) we get for $s_{T^t(X)}(p,q;1)$ a complicated polynomial in the functions $s_X(p',q',j')$ with $|p'-p|+|q'-q|\leq t$. The coefficient of

$$\prod_{i=1}^{k} s_X(p_i',q_i';j_i')$$

in this polynomial is precisely the time correlation function

$$\langle s_{T^t(X)}(p,q;1)\prod_{i=1}^{k} s_X(p_i',q_i';j_i')\rangle,$$

where the angular brackets mean an average with respect to the probability measure $\mu(n_0)$. In the following we are only interested in the *binary correlation functions* $\langle s_{T^t(X)}(p,q;i)s_X(p',q';j)\rangle$, so that we shall only consider the coefficients of the linear part of this polynomial. In evaluating these coefficients it is tempting to neglect in the above iteration procedure the contributions to the linear terms coming from the nonlinear terms; such contributions do exist after $t\geq 3$. The approximation which consists in neglecting the correction coming from the nonlinear terms is called *the linear approximation* (LA). We make two important remarks about this approximation:

(a) Firstly, it can be shown that *the LA gives the exact values* for the time correlation functions $\langle s_{T^t(X)}(p,q;i)s_X(p',q';j)\rangle$ with $|p-p'|+|q-q'|=t$. That is, the LA gives exactly the correlation between the initial perturbation at (p,q) and the resulting perturbation at time t, on the boundary of the perturbed region. This remark can be used to prove the existence of sound waves of velocity $1/\sqrt{2}$ in the directions parallel to the bissectrix of the lattice axes. (The detailed calculations will be published elsewhere.)

(b) Secondly, for the other time correlation functions like $\langle s_{T^t(X)}(p,q;i)s_X(p,q;j)\rangle$, it gives only an estimate which should be a good approximation as long as $\lambda \simeq 0$, that is, for low collision frequencies. Actually, it can be shown that the LA corresponds to the linearized Boltzmann approximation in kinetic theory. We now select some typical results obtained with this approximation.

(I) *Mixing properties.*—We used the LA to estimate the time correlation functions at the same lattice site. The result is the following: As $t\to\infty$,

$$\langle s_{T^t(X)}(0,0;i)s_X(0,0;j)\rangle \simeq \begin{cases} C(-1)^{(i+j)/2}t^{-1/2}, & i+j \text{ even},\\ C(-1)^{(i+j)/2+1}t^{-3/2}, & i+j \text{ odd}.\end{cases} \tag{4}$$

Note that the above results are typical examples of mixing properties for the dynamical system $[K, T, \mu(n_0)]$.[1] At first glance this result is rather unexpected. In fact, one should expect an asymptotic behavior like ct^{-1} as predicted by diffusion processes in two dimensions. Nevertheless, it is possible to explain the long-time tail, like $t^{-1/2}$, if $i+j$ is even. In fact, let us set $v_X(p,q;1)=s_X(p,q;1)-s_X(p,q;3)$, $v_X(p,q;2)=s_X(p,q;2)-s_X(p,q;4)$; $v_X(p,q;1)$ $[v_X(p,q;2)]$ is the horizontal (vertical) momentum at (p,q). Then (4) implies that the horizontal (vertical) momentum relaxes like *a one-dimensional diffusion process*. This is not surprising, because the collision law is such that the horizontal (vertical) momentum integrated on each horizontal (vertical) line of lattice sites is conserved. The long-time behavior like $t^{-3/2}$ for $\langle v_X(0,0;1)v_X(0,0;2)\rangle$ seems to be more difficult to explain. Again we emphasize that (4) is not a rigorous result, but should be a good approximation as long as $\lambda\simeq 0$.

(II) *Viscosity coefficient.*—Using the same technique as Hardy and Pomeau,[2] it is possible to derive for this system the analogs of the Navier-Stokes equations after replacing finite differences by suitable partial-differentiation operators. A transport coefficient like a viscosity appears in these equations, as given by a Kubo-type formula,

$$\eta = \sum_{t=0}^{\infty}\eta(t),$$

where

$$\eta(t) = C\Big\langle \chi_X(0,0)\sum_{|p|+|q|\le t}\chi_{T^t(X)}(p,q)\Big\rangle, \quad \chi_X(p,q)\equiv\sum_{i=1}^{4}(-1)^{i+1}s_X(p,q;i). \tag{5}$$

It is possible to discuss the convergence of the series giving η in evaluating the long-time behavior of $\eta(t)$.

In order to retrieve the divergence of the Kubo formula for η as predicted by the mode-mode coupling theory[3] or the kinetic theory[4] in two-dimensional fluids, we propose a better estimate of $\eta(t)$, based on the Landau-Placzek argument. Namely, it is assumed that after a long time has elapsed, a local equilibrium is established on each lattice site. This amounts to replacing, in (5), $\chi_X(p,q)$ by the unique real solution of the following equation:

$$[n\chi - \tfrac{1}{4}(v_1^2 - v_2^2)](1-\tfrac{1}{2}n) + 2\chi[\tfrac{1}{8}n^2 + 2\chi^2 - \tfrac{1}{4}(v_1^2+v_2^2)] = 0, \tag{6}$$

where n, v_1, and v_2 respectively stand for $n_X(p,q)$, $v_X(p,q;1)$, and $v_X(p,q;2)$. Equation (6) is nothing else but Eq. (2) written in the new variables

$$n=\sum_{i=1}^{4}n_i, \quad v_1=n_1-n_3, \quad v_2=n_2-n_4, \quad \chi=\sum_{i=1}^{4}(-1)^{i+1}n_i.$$

After linearizing Eq. (6) around the equilibrium state $n=4n_0$, $v_1=v_2=0$, one gets

$$\eta(t)\simeq\begin{cases}\frac{1}{16}(1-2n_0)\Big\langle\chi_X(0,0)\sum_{|p|+|q|\le t}[v_{T^t(X)}{}^2(p,q;1)-v_{T^t(X)}{}^2(p,q;2)]\Big\rangle \text{ if } n_0\ne\frac{1}{2},\\ -\frac{1}{8}\Big\langle\chi_X(0,0)\sum_{|p|+|q|\le t}n_{T^t(X)}(p,q)[v_{T^t(X)}{}^2(p,q;1)-v_{T^t(X)}{}^2(p,q;2)]\Big\rangle, \quad n_0=\frac{1}{2}.\end{cases} \tag{7}$$

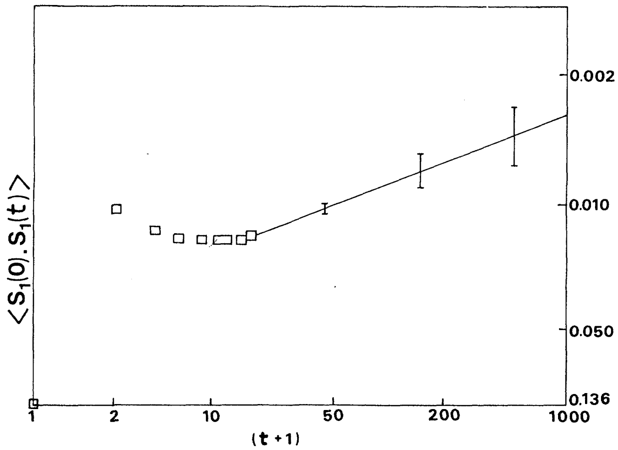

FIG. 1. $\langle s_X(0,0;1)s_{T^t(X)}(0,0;1)\rangle$ as a function of t on a log-log scale for $n_0=\frac{1}{6}$, that is, $n=\frac{2}{3}$. The slope of the straight line is near 0.6.

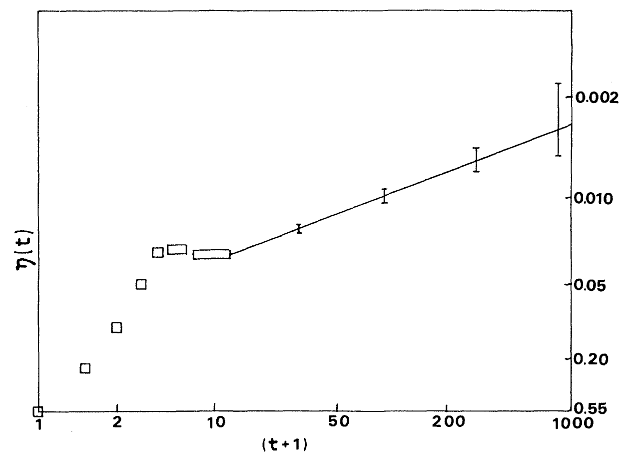

FIG. 2. $\eta(t)$ as a function of t on a log-log scale for $n_0=\frac{1}{6}$. The slope of the straight line is near 0.5.

Then we applied the LA in order to evaluate the two angular brackets in (7). The result is, as $t\rightarrow\infty$, $\eta(t)\simeq(C)t^{-1/2}$ if $n_0\neq\frac{1}{2}$, $\eta(t)\simeq(C)t^{-3/2}$ if $n_0=\frac{1}{2}$. The first estimate corresponding to the case $n_0\neq\frac{1}{2}$ can be explained by assuming that the one-dimensional diffusion process for the momentum gives the dominant contribution to $\eta(t)$; therefore, $\eta(t)\simeq(C)t^{-d/2}$ with $d=1$ instead of $d=2$ (where d is the dimensionality).

Note that for $n_0=\frac{1}{2}$ the estimate for $\eta(t)$ suggests that the series giving η is convergent. Of course, if this is true, it should be a peculiarity of the model. Nevertheless, it seems interesting to find rigorous arguments for the above estimates. This point is currently under investigation.

The model has been simulated on the Orsay computer. The number of lattice sites is 108 so that when the density is equal to 2, the particle number is 23 328. The phase-space averages are replaced by time averages; that is, we set, for instance,

$$\langle s_{T^t(X)}(0,0;1)s_X(0,0;1)\rangle\simeq\frac{1}{R}\sum_{\tau=1}^{R}s_{T^{t+\tau}(X)}(0,0;1)s_{T^\tau(X)}(0,0;1).$$

As $R\rightarrow\infty$, these time averages are supposed to give the phase-space averages. The results given in Figs. 1 and 2 correspond to $R=10^6$. We plot $\langle s_{T^t(X)}(0,0;1)s_X(0,0;1)\rangle$ and $\eta(t)$ as functions of t on log-log scales. The values 0.5 and 0.6 for the slopes are in good agreement with the theoretical predictions.

[1]A. Arnold, and A. Avez, *Problèmes Ergodiques de la Mécanique Classique* (Gauthiers-Villars, Paris, 1967).
[2]J. Hardy, and Y. Pomeau, J. Math. Phys. (N. Y.) 13, 7, 1042 (1972).
[3]Y. Pomeau, Phys. Rev. A 3, 1174 (1971).
[4]M. H. Ernst, E. G. Hauge, and J. M. J. Van Leeuwen, Phys. Rev. A 4, 2055 (1971).