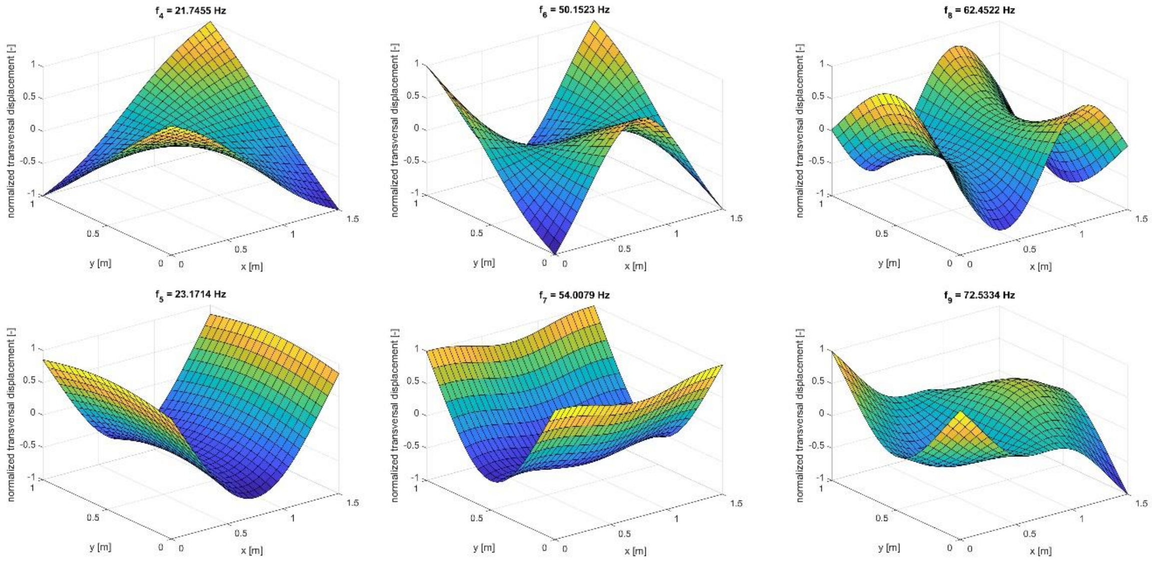
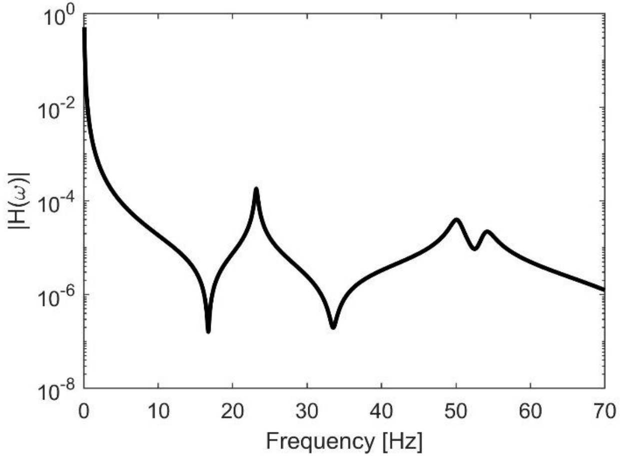




2. Numerical Modal Analysis

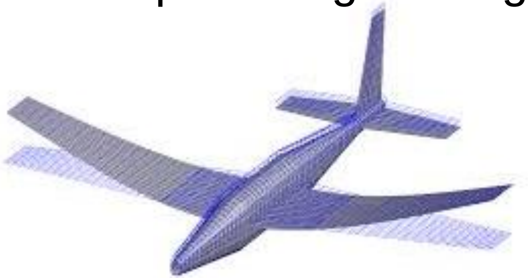
Structural Dynamics part of 4DM00

dr.ir. R.H.B. (Rob) Fey, ir. D.W.M. (Daniël) Veldman

Numerical modal analysis of linear dynamical systems

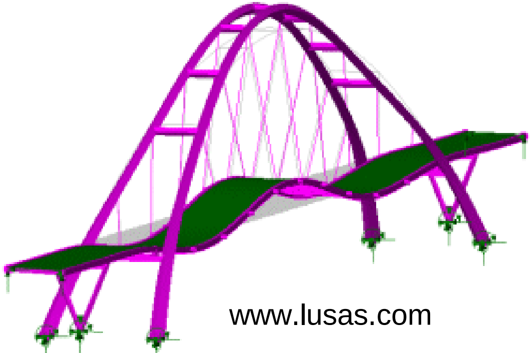
A wide field of application

Aerospace Engineering



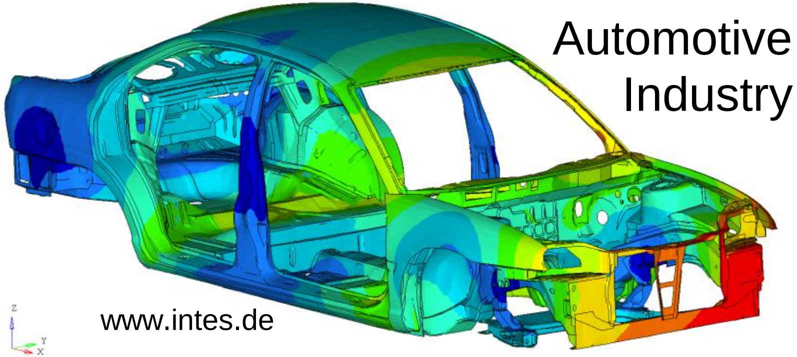
Ritter and Dillinger, IFASD-2011-143, Paris

Civil Engineering



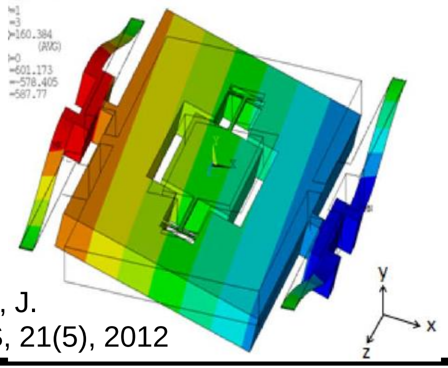
www.lusas.com

Automotive Industry



www.intes.de

MEMS



Koh and Lee, J.
Micromech S, 21(5), 2012

Mechanical systems

Modeling of a linear mechanical dynamical system:

- Newton's laws of motion, Lagrange's equation, principle of Hamilton
- For geometrically complex systems: Finite Element discretization

Equations of motion (system of 2nd order ODEs)

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t)$$

$q(t)$	$(n \times 1)$ column with dofs	displacements, rotations
$f(t)$	$(n \times 1)$ column with loads	forces, moments
M	$(n \times n)$ mass matrix	symmetric, positive definite
B	$(n \times n)$ viscous damping matrix	(semi-)positive definite
K	$(n \times n)$ stiffness matrix	(semi-)positive definite

Analysis of the response

TIME DOMAIN: Analysis of response $q(t)$ caused by $f(t)$ in time domain

- Supply initial conditions, apply standard numerical integration
- excessive computational times if n is very large (say 10^6)
  - especially in case of conditionally stable integration schemes

FREQUENCY DOMAIN: Harmonic response analysis in frequency domain:

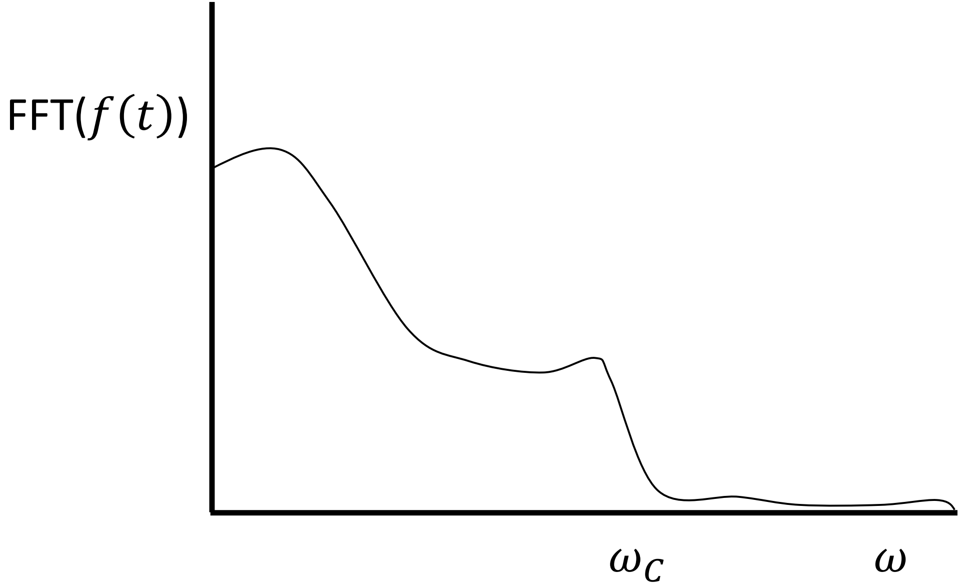
Load: $f(t) = \mathrm{Re}\left\{\hat{f}e^{j\omega t}\right\}$ Response: $q(t) = \mathrm{Re}\left\{\hat{q}e^{j\omega t}\right\}$ (NB. $\hat{f}$ and $\hat{q}$ complex)

$$\boxed{\hat{q} = (-\omega^2 M + j\omega B + K)^{-1}\hat{f}}$$

- excessive computer times if n is very large (say 10^6) and many ω's need to be evaluated
- Limited insight in contribution of modes to the response
  - Particularly problematic because high frequency modes are inaccurate (discretization)

Cut-off frequency ω_C (or f_C)

Analyse frequency content of external load $f(t)$



Mainly excitation of eigenmodes up to ω_C rad/s (or $f_C = \frac{\omega_C}{2\pi}$ Hz)

General approach in modal analysis

I. Uncouple the equations of motion
   - Use a coordinate transformation based on eigenmodes using the orthogonality property.

II. Subsequent dynamic analysis
   - Use the uncoupled equations with eigenvalues and eigenmodes in frequency range of interest.

Overview remainder of this lecture

Modal analysis in both time domain and frequency domain for:

1. Undamped systems
2. Proportionally damped systems (good approximation for weakly damped systems)
3. Generally viscously damped systems, (1. and 2. are special cases of 3.)

2b. Undamped systems

Undamped systems (time domain)

Homogeneous equations of motion: $M\,\ddot{q}(t) + Kq(t) = 0.$

Try: $q(t) = u_{Ok}e^{\lambda_k t}$

'Linear' eigenvalue problem: $(\gamma_k M + K)u_{Ok} = 0$ with $\gamma_k = \lambda_k^2 = -\omega_k^2 \leq 0.$

$$\lambda_k = \pm j\omega_{Ok} = \pm 2\pi f_{Ok} j$$

λ_k is the k-th eigenvalue (imaginary)

ω_{Ok} is the k-th angular eigenfrequency in rad/s (real),

f_{Ok} is the k-th eigenfrequency in Hz (real)

u_{Ok} is the k-th undamped eigenmode (real)

Store angular frequencies ω_{Ok} in diagonal $(n \times n)$-matrix Ω_O.

Store undamped eigenmodes u_{Ok} in (the <u>columns</u> of) the $(n \times n)$-matrix U_O.

Undamped systems (time domain)

Coordinate transformation $q(t) = U_O p(t)$, $p(t)$ are the generalized/modal DOFs.

Substitution into $M\ddot{q}(t) + Kq(t) = f(t)$ and pre-multiplication by $U_O^\top$ yields

$$U_O^\top M U_O \ddot{p}(t) + U_O^\top K U_O p(t) = U_O^\top f(t).$$

Orthogonality property: $M^* := U_O^\top M U_O$ and $K^* := U_O^\top K U_O$ are diagonal matrices (see next slide)

Nonzero entries of M^* are the **modal masses** $m_k^* = u_{Ok}^\top M u_{Ok}$.
Nonzero entries of K^* are the **modal stiffnesses** $k_k^* = u_{Ok}^\top K u_{Ok}$.

Decoupled equations of motion:

$$m_k^* \ddot{p}_k(t) + k_k^* p_k(t) = u_{Ok}^\top f(t)$$

Solve only for the $k < n_C$ for which $\omega_{Ok} < \omega_C$ and construct response $q(t) = \sum_{k=1}^{n_C} u_{Ok} p_k(t)$

Proof of orthogonality property

Orthogonality property:

$M^* := U_O^\top M U_O$ and $K^* := U_O^\top K U_O$ are diagonal matrices when $\gamma_k \neq \gamma_l$ for $l \neq k$.

Remark: the orthogonality property can still hold when there are eigenvalues with multiplicity > 1

Proof:

$$\underline{u_{Ol}^\top(\gamma_k M + K)u_{Ok} = 0.} \quad - \quad (u_{Ok} \text{ is an eigenvector})$$

$u_{Ok}^\top(\gamma_l M + K)u_{Ol} = 0$, transpose: $u_{Ol}^\top(\gamma_l M + K)u_{Ok} = 0.$ (u_{Ol} eigenvector, M and K sym.)

$$u_{Ol}^\top(\gamma_k - \gamma_l)M u_{Ok} = 0. \quad \text{(subtract)}$$

So, if $\gamma_k \neq \gamma_l$, then $u_{Ol}^\top M u_{Ok} = 0$.

But then $0 = u_{Ol}^\top(\gamma_k M + K)u_{Ok} = u_{Ol}^\top K u_{Ok}$. QED.

Modal mass and modal stiffness

Because $u_{Ok}^{\top}(\gamma_k M + K)u_{Ok} = 0$ implies that $\gamma_k m_k^* + k_k^* = 0$. Since $\gamma_k = -\omega_{Ok}^2$,

$$k_k^* = \omega_{Ok}^2 m_k^*,$$

$$\boxed{\omega_{Ok} = \sqrt{\frac{k_k^*}{m_k^*}}}$$

This can be written using in terms of diagonal matrices as

$$K^* = M^* \Omega_O^2.$$

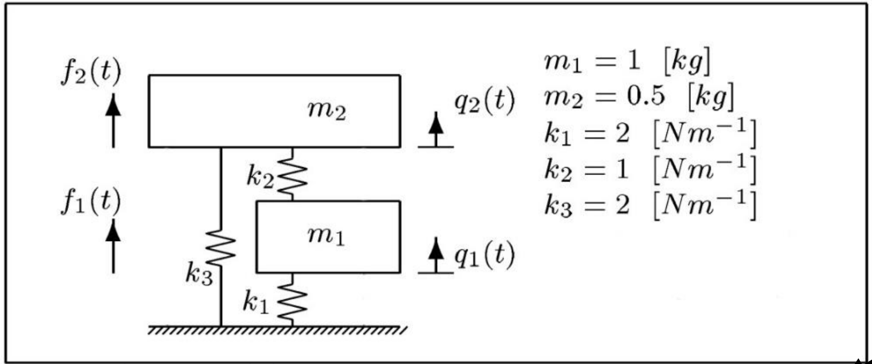
Often, eigenmodes U_O are **mass-normalized** such that

$$M^* = U_O^{\top} M U_O = I, \qquad \Rightarrow \qquad K^* = U_O^{\top} K U_O = \Omega_O^2.$$

The uncoupled equations of motion then take the form

$$I\ddot{p}(t) + \Omega_O^2 p(t) = U_O^{\top} f(t),$$
$$\ddot{p}_k(t) + \omega_{Ok}^2 p(t) = u_{Ok}^{\top} f(t).$$

Example: undamped 2-dof system



Eigenvalue problem:

Mass matrix

Stiffness matrix

$$\left(\lambda_k^2 \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\right) \begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvalues λ_k [rad/s], angular eigenfrequencies ω_{Ok} [rad/s], eigenfrequencies f_{Ok} [Hz], and mass-normalized eigenmodes u_{O1} and u_{O2}

$$\begin{bmatrix} \lambda_{1\pm} \\ \lambda_{2\pm} \end{bmatrix} = \pm j \begin{bmatrix} \omega_{O1} \\ \omega_{O2} \end{bmatrix} = \pm 2\pi j \begin{bmatrix} f_{O1} \\ f_{O2} \end{bmatrix} = \pm j \begin{bmatrix} 1.56 \\ 2.56 \end{bmatrix}, \qquad u_{O1} = \begin{bmatrix} 0.93 \\ 0.52 \end{bmatrix}, \ \ u_{O2} = \begin{bmatrix} 0.37 \\ -1.31 \end{bmatrix}.$$

Undamped systems (frequency domain)

Assume harmonic excitation and response:

$$f(t) = \mathrm{Re}\{\hat{f}e^{j\omega t}\}, \qquad p(t) = \mathrm{Re}\{\hat{p}e^{j\omega t}\}, \qquad q(t) = \mathrm{Re}\{\hat{q}e^{j\omega t}\}.$$

Substitute in $M^*\ddot{p}(t) + K^*p(t) = U_O^\top f(t)$ and use $q(t) = U_O p(t)$

$$\hat{q} = U_O(-\omega^2 M^* + K^*)^{-1}U_O^\top \hat{f} = \sum_{k=1}^{n} \frac{u_{Ok}u_{Ok}^\top}{-m_k^*\omega^2 + k_k^*}\hat{f} = \sum_{k=1}^{n} \frac{u_{Ok}u_{Ok}^\top}{m_k^*(-\omega^2 + \omega_{Ok}^2)}\hat{f}\,.$$

Matrix of Frequency Response Functions (FRFs)

$$H(\omega) = \sum_{k=1}^{n} \frac{u_{Ok}u_{Ok}^\top}{m_k^*(-\omega^2 + \omega_{Ok}^2)}$$

Remarks on undamped systems

- When the eigenmodes U_O are mass-normalized: $M^* = I, m_k^* = 1$

$$H(\omega) = \sum_{k=1}^{n} \frac{u_{Ok} u_{Ok}^{\top}}{-\omega^2 + \omega_{Ok}^2}$$

- When only interested in frequency range $\omega \in [0, \omega_C]$ rad/s, consider truncate summation upto n_C terms for which $\omega_{Ok} < \omega_C$ $(k = 1, \dots, n_C)$

$$H(\omega) \approx \sum_{k=1}^{n_C} \frac{u_{Ok} u_{Ok}^{\top}}{m_k^*(-\omega^2 + \omega_{Ok}^2)}$$

2c. Proportionally damped systems

Proportional damping

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t)$$

Definition: The system is *proportionally damped* when the damping matrix B decouples on the undamped (real) eigenmodes, i.e. when

$$B^* := U_0^\top B U_0 \text{ is diagonal.}$$

Nonzero entries of B^* are the **modal damping coefficients** b_k^* .

Write

$$b_k^* = 2m_k^* \xi_k \omega_{0k},$$

where ξ_k are the **dimensionless modal damping coefficients**.

In matrix form

$$B^* = 2M^* \Xi \Omega_0, \qquad B = 2U_0^{-\top} M^* \Xi \Omega_0 U_0^{-1},$$

where Ξ is the diagonal matrix with ξ_k on the diagonal.

Proportionally damped systems (time domain)

Homogeneous equations of motion: $M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = 0$.

Try: $q(t) = u_k e^{\lambda_k t}$

Quadratic eigenvalue problem: $\left(\lambda_k^2 M + \lambda_k B + K\right)u_k = 0$

In case of **proportional damping** it is not necessary to solve this eigenvalue problem, because the eigenmodes u_k are equal to the **undamped eigenmodes** u_{Ok}.

Coordinate transformation $q(t) = U_O p(t)$.

Substitution into $M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t)$ and pre-multiplication by $U_O^\top$

$$U_O^\top M U_O \ddot{p}(t) + U_O^\top B U_O \dot{p}(t) + U_O^\top K U_O p(t) = U_O^\top f(t)$$

$$M^* \ddot{p}(t) + B^* \dot{p}(t) + K^* p(t) = U_O^\top f(t)$$

$$m_k^* \ddot{p}_k(t) + b_k^* \dot{p}_k(t) + k_k^* p_k(t) = u_{Ok}^\top f(t)$$

$$\ddot{p}_k(t) + 2\xi_k \omega_{Ok} \dot{p}_k(t) + \omega_{Ok}^2 p_k(t) = \frac{1}{m_k^*} u_{Ok}^\top f(t)$$

Proportionally damped systems (time domain)

Eigenvalue λ_k is found by substituting $p_k(t) = \hat{p}_k e^{\lambda_k t}$ into

$$\ddot{p}_k(t) + 2\xi_k \omega_{Ok} \dot{p}_k(t) + \omega_{Ok}^2 p_k(t) = 0.$$
$$\lambda_k^2 + 2\xi_k \omega_{Ok} \lambda_k + \omega_{Ok}^2 = 0.$$

- When $0 < \xi_k < 1$, the k-th eigenmode is **under critically damped**
  Two complex conjugate eigenvalues

$$\lambda_{k\pm} = -\xi_k \omega_{Ok} \pm j\omega_{Ok}\sqrt{1-\xi_k^2}, \qquad \mu_k = -\xi_k \omega_{Ok} < 0, \qquad \omega_k = \omega_{Ok}\sqrt{1-\xi_k^2} > 0.$$

- When $\xi_k > 1$, the k-th eigenmode is **over critically damped**
  Two negative real eigenvalues

$$\lambda_{k\pm} = -\xi_k \omega_{Ok} \pm \omega_{Ok}\sqrt{\xi_k^2 - 1} = \mu_{k\pm} < 0, \qquad \omega_k = 0.$$

<u>Decoupled equations of motion:</u>
Solve only the equations for which $\omega_k < \omega_{Ok} < \omega_C$ and compute $q(t) = \sum_{k=1}^{n_C} u_{Ok} p_k(t)$

Check for proportional damping

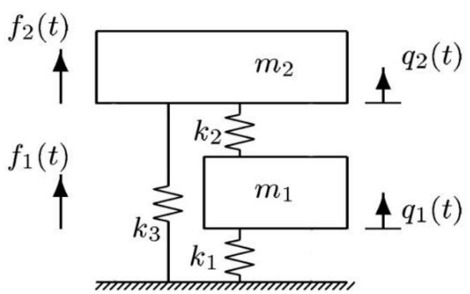
The system is proportionally damped when $\boxed{KM^{-1}B = BM^{-1}K}$ and $\gamma_k \neq \gamma_l$ for $l \neq k$.

Proof: $$BM^{-1}Ku_{Ok} = BM^{-1}\gamma_k M u_{Ok} = \gamma_k B u_{Ok}. \quad (u_{Ok} \text{ is an eigenvector})$$
$$u_{Ol}^{\top}KM^{-1}B = \gamma_l u_{Ol}^{\top}MM^{-1}B = \gamma_l u_{Ol}^{\top}B. \quad (u_{Ol} \text{ is an eigenvector})$$

Because $KM^{-1}B = BM^{-1}K$
$$0 = u_{Ol}^{\top}(BM^{-1}K - KM^{-1}B)u_{Ok} = (\gamma_k - \gamma_l)u_{Ol}^{\top}Bu_{Ok},$$
so $u_{Ol}^{\top}Bu_{Ok} = 0$ for $l \neq k$. Therefore, $U_O^{\top}BU_O$ is diagonal. QED.

Rayleigh damping: $B = \alpha M + \beta K$ is a special case of proportional damping.
Only two modes can be tuned independently.

Example 2-dof system revisited, proportional damping



$m_1 = 1 \quad [kg]$
$m_2 = 0.5 \quad [kg]$
$k_1 = 2 \quad [Nm^{-1}]$
$k_2 = 1 \quad [Nm^{-1}]$
$k_3 = 2 \quad [Nm^{-1}]$

Undamped system (previous example):

$$\Omega_O = \begin{bmatrix} \omega_{O1} & 0 \\ 0 & \omega_{O2} \end{bmatrix} = \begin{bmatrix} 1.56 & 0 \\ 0 & 2.56 \end{bmatrix},$$

$$U_O = [u_{O1} \quad u_{O2}] = \begin{bmatrix} 0.93 & 0.37 \\ 0.52 & -1.31 \end{bmatrix}.$$

'Experimentally' identified modal damping factors.

$$\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} = \begin{bmatrix} 0.156 & 0 \\ 0 & 0.256 \end{bmatrix}.$$

Using Ω_O, Ξ, and the **mass-normalized** undamped eigenmodes U_O, we can determine

$$\begin{bmatrix} \lambda_{1\pm} \\ \lambda_{2\pm} \end{bmatrix} = \begin{bmatrix} -0.24 \pm 1.54j \\ -0.66 \pm 2.48j \end{bmatrix}, \qquad B = 2(U_O^\top)^{-1}\Xi\Omega_O U_O^{-1} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}.$$

Proportionally damped systems (frequency domain)

Assume harmonic excitation and response:

$$f(t) = \mathrm{Re}\{\hat{f}e^{j\omega t}\}, \qquad p(t) = \mathrm{Re}\{\hat{p}e^{j\omega t}\}, \qquad q(t) = \mathrm{Re}\{\hat{q}e^{j\omega t}\}.$$

Substitute in $M^*\ddot{p}(t) + B^*\dot{p}(t) + K^*p(t) = U_O^\top f(t)$ and use $q(t) = U_O p(t)$

$$\hat{q} = U_O(-\omega^2 M^* + j\omega B^* + K^*)^{-1} U_O^\top \hat{f} = \sum_{k=1}^{n} \frac{u_{Ok}u_{Ok}^\top}{-m_k^*\omega^2 + b_k^*\omega j + k_k^*}\hat{f}$$

$$= \sum_{k=1}^{n} \frac{u_{Ok}u_{Ok}^\top}{m_k^*(-\omega^2 + 2\xi_k\omega_{Ok}\omega j + \omega_{Ok}^2)}\hat{f}\,.$$

Matrix of Frequency Response Functions (FRFs)

$$H(\omega) = \sum_{k=1}^{n} \frac{u_{Ok}u_{Ok}^\top}{m_k^*(-\omega^2 + 2\xi_k\omega_{Ok}\omega j + \omega_{Ok}^2)}$$

Remarks on proportionally damped systems

- When the eigenmodes U_O are mass-normalized: $M^* = I, m_k^* = 1$

$$H(\omega) = \sum_{k=1}^{n} \frac{u_{Ok}u_{Ok}^{\top}}{-\omega^2 + 2\xi_k\omega_{Ok}\omega j + \omega_{Ok}^2}$$

- When only interested in frequency range $\omega \in [0, \omega_C]$ rad/s,
  consider truncate summation upto n_C terms for which $\omega_k < \omega_{Ok} < \omega_C$ $(k = 1, \dots, n_C)$

$$H(\omega) \approx \sum_{k=1}^{n_C} \frac{u_{Ok}u_{Ok}^{\top}}{m_k^*(-\omega^2 + 2\xi_k\omega_{Ok}\omega j + \omega_{Ok}^2)}$$

- For **weakly damped** systems (i.e. $\xi_k \approx O(10^{-2})$), neglecting the off-diagonal terms in $B^* = U_O^{\top} B U_O$ still leads to good approximation of the dynamic behavior.
  Weakly damped systems can thus be approximated well by proportionally damped systems.

Example: plate

Aluminum rectangular plate

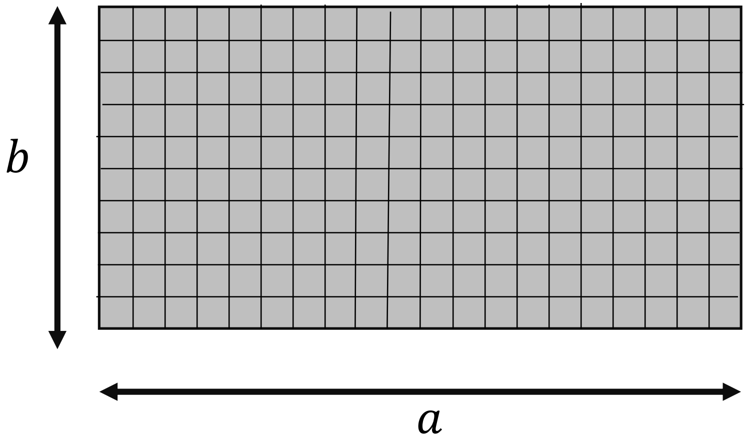
$a = 1.5$ m, $b = 1$ m,

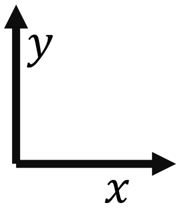
$\rho = 2700$ kg/m^3, $h = 0.01$ m,

$E = 69$ GPa, $\nu = 0.3$.

(pure bending, no in-plane deformation)

All edges are free.





FE model:

30 elements in x-direction, 20 elements in y-direction.

4 DOFs per node $\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x \partial y}\right)$.

Total number of DOFs: $n = 2604 = 4 * (30 + 1) * (20 + 1)$.

Result: mass matrix M and stiffness matrix K (both $n \times n$).

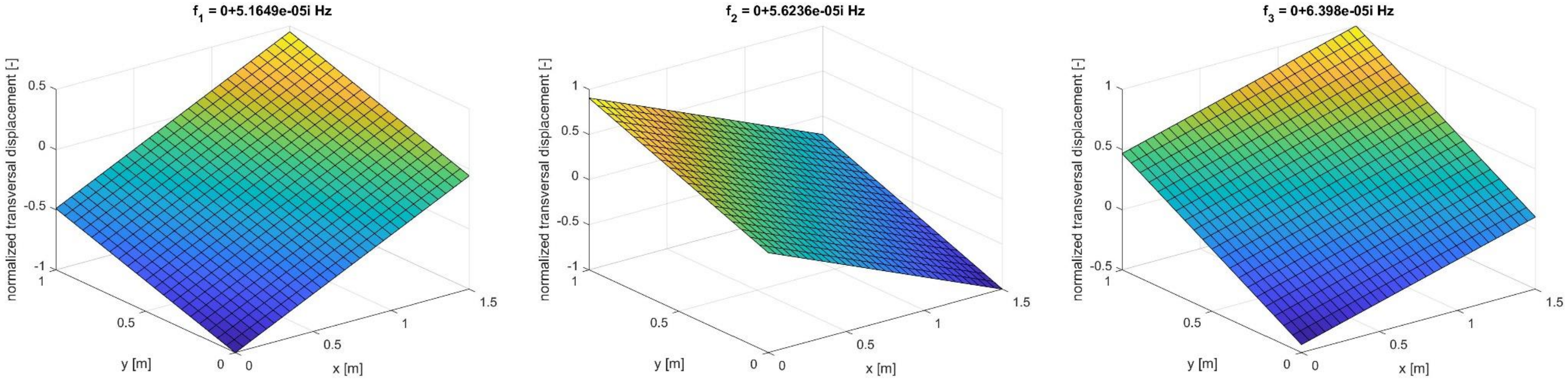
Example: plate

Undamped eigenmodes u_{Ok}

$$(\gamma_k M + K) u_{Ok} = 0.$$

First three eigenmodes are rigid body modes (1 translation + 2 rotations)

$f_{O1} \approx 0$ Hz $\quad f_{O2} \approx 0$ Hz $\quad f_{O3} \approx 0$ Hz



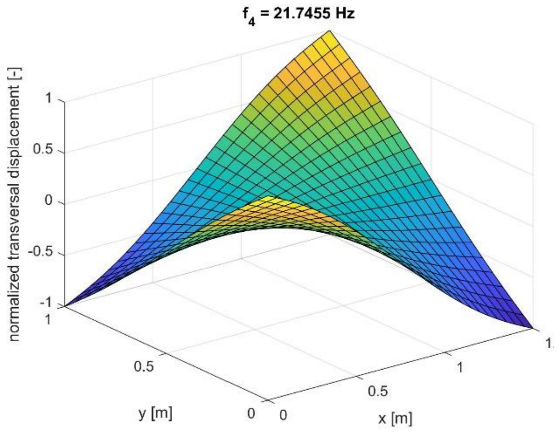
Example: plate

Undamped eigenmodes u_{Ok}

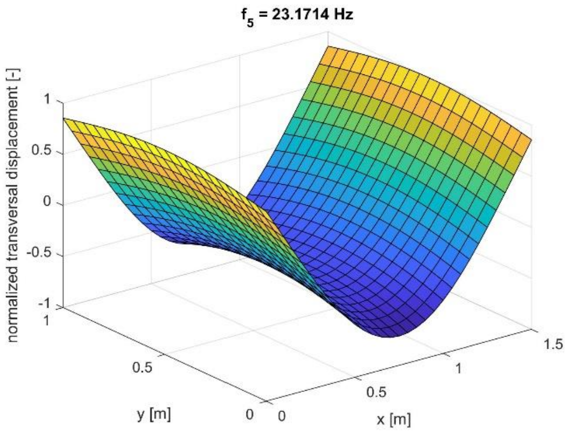
$$(\gamma_k M + K)u_{Ok} = 0.$$

Higher modes show elastic deformation.

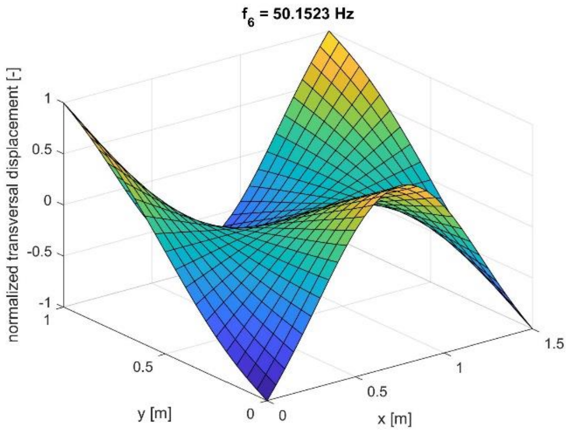
$f_{O4} = 21.7$ Hz



$f_{O5} = 23.2$ Hz



$f_{O6} \approx 50.2$ Hz



Example: plate

<u>Undamped eigenmodes u_{Ok}</u>

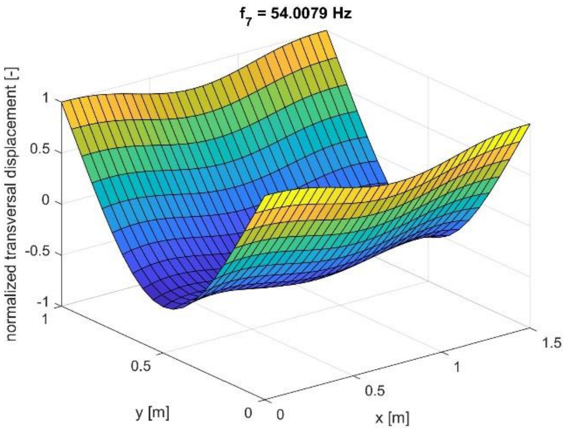
$$(\gamma_k M + K)u_{Ok} = 0.$$

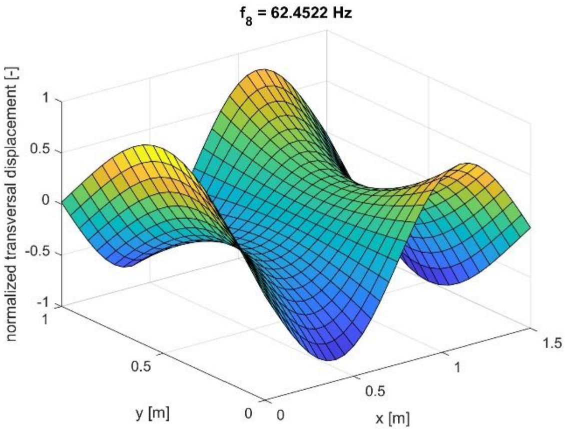
Higher modes show elastic deformation.

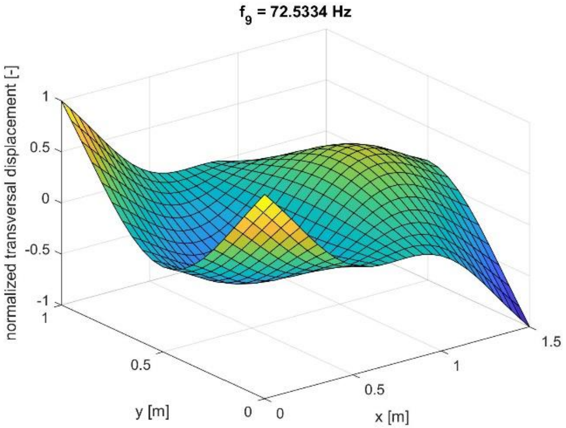
$f_{O7} = 54.0$ Hz

$f_{O8} = 62.5$ Hz

$f_{O9} \approx 72.5$ Hz



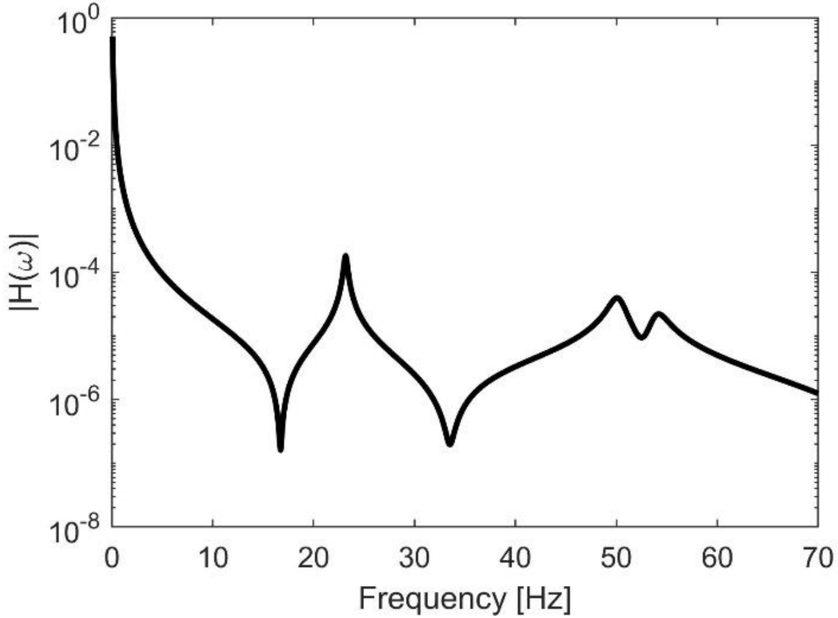




Example: plate

Use Rayleigh damping $B = 0.0001\ K$ (proportional damping: $u_k = u_{Ok}$).

Construct FRF: input is force at $(x, y) = (a/2{,}0)$, output is displacement at $(x, y) = (a/2{,}0)$



- Why does $|H(\omega)| \to \infty$ near $f = 0$ Hz?
- Why is there no resonance visible at $f_{O4} = 21.7$ Hz and at $f_{O8} = 62.5$ Hz?

2d. Generally viscously damped systems

Generally viscously damped systems (time domain)

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t)$$

<u>What are the eigenmodes?</u>

Homogeneous equations of motion: $M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = 0.$

Try: $q(t) = u_k e^{\lambda_k t}$ with $\lambda_k = \mu_k + j\omega_k.$

Quadratic eigenvalue problem:

$$\left(\lambda_k^2 M + \lambda_k B + K\right)u_k = 0.$$

Limited numerical procedures available (polyeig in Matlab, reliable?).

Note: when the matrices M, B, and K are real,
the eigenvalues λ_k are real or appear in complex conjugate pairs $\lambda_{k\pm} = \mu_k \pm j\omega_k.$

A first order system

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t)$$

Rewrite EoM as

$$\begin{bmatrix} B & M \\ M & O \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix} + \begin{bmatrix} K & O \\ O & -M \end{bmatrix} \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}. \qquad (M\dot{q}(t) - M\dot{q}(t) = 0)$$

This set of equations can be considered as a **1st order** ODE

$$C\dot{y}(t) + Dy(t) = g(t),$$

where

$$y(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \qquad g(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} B & M \\ M & O \end{bmatrix}, \qquad D = \begin{bmatrix} K & O \\ O & -M \end{bmatrix}.$$

Note: C and D are sparse and symmetric when M, B, and K are sparse and symmetric.

Intermezzo: different first-order representations

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t)$$

- Non-descriptor form: $\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t)$

$$\begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix} = \begin{bmatrix} O & I \\ -M^{-1}K & -M^{-1}B \end{bmatrix} \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}f(t) \end{bmatrix}.$$

  Not preferred: $\hat{A}$ is not sparse and symmetric.

- Non-symmetric descriptor form: $C\dot{y}(t) + Dy(t) = g(t)$

$$\begin{bmatrix} O & M \\ M & O \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix} + \begin{bmatrix} K & B \\ O & -M \end{bmatrix} \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}$$

  Not preferred: D is not symmetric.

Both forms are not preferred, especially for systems with a large number of DOFs.

Generally viscously damped systems (time domain)

$$C\dot{y}(t) + Dy(t) = g(t)$$

Homogeneous equations of motion: $C\dot{y}(t) + Dy(t) = 0.$

Try: $y(t) = v_k e^{\lambda_k t}$ with $\lambda_k = \mu_k + j\omega_k.$

Right-eigenvalue problem: $(\lambda_k C + D)v_k = 0.$

- the vectors v_k are **complex**.
- $y(t) = v_k e^{\lambda_k t}$ corresponds $q(t) = u_k e^{\lambda_k t}$. Therefore, $v_k = \begin{bmatrix} u_k \\ \lambda_k u_k \end{bmatrix}$.

In matrix form:

$$CV\Lambda + DV = O$$

$$V = [v_1 \quad v_2 \quad \cdots \quad v_{2n}], \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \lambda_{2n} \end{bmatrix}$$

The left eigenvalue problem

$$w_k^\top(\lambda_k C + D) = 0^\top \quad \Leftrightarrow \quad (\lambda_k C^\top + D^\top)w_k = 0.$$

- The eigenvalues λ_k are the same for the left and right eigenvalue problems.
- When C and D are symmetric, the left and right eigenvectors are equal, i.e. $v_k = w_k$.
- The displacement part of w_k is denoted by x_k, i.e. $w_k = \begin{bmatrix} x_k \\ \lambda_k x_k \end{bmatrix}$.

In matrix form

$$\Lambda W^\top C + W^\top D = O \quad \Leftrightarrow \quad C^\top W \Lambda + D^\top W = O$$

where $W = \begin{bmatrix} w_1 & w_2 & \cdots & w_{2n} \end{bmatrix}$.

Remark: alternative form for the right and left eigenvalue problems

$$\begin{bmatrix} O & I \\ -M^{-1}K & -M^{-1}B \end{bmatrix} v_k = \lambda_k v_k, \qquad \begin{bmatrix} O & I \\ -M^{-\top}K^\top & -M^{-\top}B^\top \end{bmatrix} w_k = \lambda_k w_k.$$

Bi-orthogonality property

The matrices $C^* := W^\top CV$ and $D^* := W^\top DV$ are diagonal matrices when $\lambda_k \neq \lambda_l$ for $l \neq k$.

Proof:

$$\begin{array}{ll}
w_l^\top(\lambda_k C + D)v_k = 0. & (v_k \text{ is a right eigenvector}) \\
\underline{w_l^\top(\lambda_l C + D)v_k = 0.} \quad - & (w_l \text{ is a left eigenvector}) \\
w_l^\top(\lambda_k - \lambda_l)Mv_k = 0. & (\text{subtract})
\end{array}$$

So, if $\lambda_k \neq \lambda_l$, then $w_l^\top C v_k = 0$. These are off-diagonal elements of C^*, so C^* is diagonal.

But then $0 = w_l^\top(\lambda_k C + D)v_k = w_l^\top D v_k$. So D^* is diagonal. QED.

Remark: the orthogonality property can still hold when there are eigenvalues with multiplicity > 1

Notation: the diagonal elements of C^* and D^* are denoted by

$$c_k^* = w_k^\top C v_k, \qquad d_k^* = w_k^\top D v_k.$$

Generally viscously damped systems (time domain)

$$C\dot{y}(t) + Dy(t) = g(t)$$

Substitute $y(t) = V\eta(t)$ and pre-multiply with $W^\top$

$$W^\top CV\dot{\eta}(t) + W^\top DV\eta(t) = W^\top g(t)$$
$$C^*\dot{\eta}(t) + D^*\eta(t) = W^\top g(t)$$
$$c_k^*\dot{\eta}_k(t) + d_k^*\eta_k(t) = x_k^\top f(t)$$
$$c_k^*\big(\dot{\eta}_k(t) - \lambda_k\eta_k(t)\big) = x_k^\top f(t)$$

These are the **uncoupled equations of motion**.

Typical approach:
solve only those equations for which $|\mathrm{Im}(\lambda_k)| < \omega_C$ $(k = 1, \dots, n_C)$ and compute

$$y(t) = \sum_{k=1}^{n_C} v_k\eta_k(t).$$

Generally viscously damped systems (frequency domain)

Assume harmonic excitation and response:

$$g(t) = \mathrm{Re}\{\hat{g}e^{j\omega t}\}, \qquad \eta(t) = \mathrm{Re}\{\hat{\eta}e^{j\omega t}\}, \qquad y(t) = \mathrm{Re}\{\hat{y}e^{j\omega t}\}.$$

Substitute in $C^*\dot{\eta}(t) + D^*\eta(t) = W^\top g(t)$ and use $y(t) = V\eta(t)$

$$\hat{y} = V(j\omega C^* + D^*)^{-1}W^\top \hat{g} = H_{2n}(\omega)\hat{g}.$$

Note that

$$\hat{y} = \begin{bmatrix} \hat{q} \\ \hat{\dot{q}} \end{bmatrix}, \qquad H_{2n}(\omega) = \begin{bmatrix} H(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}, \qquad \hat{g} = \begin{bmatrix} \hat{f} \\ 0 \end{bmatrix}.$$

All information is present in $\hat{q} = H(\omega)\hat{f}$. $\qquad (\hat{\dot{q}} = j\omega\hat{q} = j\omega H(\omega)\hat{f} = H_{21}(\omega)\hat{f})$

Because $V = \begin{bmatrix} U \\ U\Lambda \end{bmatrix}$ and $W = \begin{bmatrix} X \\ X\Lambda \end{bmatrix}$

$$\boxed{H(\omega) = U(j\omega C^* + D^*)^{-1}X^\top = \sum_{k=1}^{2n} \frac{u_k x_k^\top}{c_k^*(j\omega - \lambda_k)}.}$$

Residue representation

$$H(\omega) = \sum_{k=1}^{2n} \frac{u_k x_k^\top}{c_k^*(j\omega - \lambda_k)}$$

We may introduce the **residue matrices**

$$A_k := \frac{u_k x_k^\top}{c_k^*}.$$

With this definition:

$$H(\omega) = \sum_{k=1}^{2n} \frac{A_k}{j\omega - \lambda_k}.$$

- Rank(A_k)$=1$.
- A_k and λ_k are the **modal parameters** in Experimental Modal Analysis.
- Frequency response function for excitation at DOF e to response at node r

$$H(\omega) = \sum_{k=1}^{2n} \frac{u_k[r] x_k[e]}{c_k^*(j\omega - \lambda_k)} = \sum_{k=1}^{2n} \frac{A_k[r,e]}{j\omega - \lambda_k}.$$

Residue representation

When M, B, and K are real,
complex conjugate eigenvalues have complex conjugate residues, i.e. $\bar{A}_k = A_l$ when $\bar{\lambda}_k = \lambda_l$.

For systems without real eigenvalues (e.g. weakly damped)

$$\lambda_k = \mu_k \pm j\omega_k, \qquad A_k = A_{Rk} \pm jA_{Ik},$$

and the frequency response function can be written as

$$H(\omega) = \sum_{k=1}^{n} \frac{A_{Rk} + jA_{Ik}}{-\mu_k + j(\omega - \omega_k)} + \sum_{k=1}^{n} \frac{A_{Rk} - jA_{Ik}}{-\mu_k + j(\omega + \omega_k)}$$

$$= 2\sum_{k=1}^{n} \frac{j\omega A_{Rk} - \mu_k A_{Rk} - \omega_k A_{Ik}}{-\omega^2 + 2\mu_k j\omega + \mu_k^2 + \omega_k^2}.$$

Special case: undamped symmetric systems

- Eigenvalues: $\lambda_{k\pm} = \pm j\omega_{Ok}$ $\qquad (\mu_k = 0, \omega_k = \omega_{Ok})$

$$\Lambda = \begin{bmatrix} j\Omega_O & O \\ O & -j\Omega_O \end{bmatrix}, \qquad \Omega_O = diag([\omega_{O1} \quad \omega_{O2} \quad \cdots \quad \omega_{On}])$$

- Eigenvectors: $u_k = x_k = u_{Ok}$ (real)

$$U = [U_O \quad U_O], \qquad V = W = \begin{bmatrix} U \\ U\Lambda \end{bmatrix} = \begin{bmatrix} U_O & U_O \\ jU_O\Omega_O & U_O\Omega_O \end{bmatrix}$$

- Coefficients c_k^*

$$C^* = W^\top C V = \begin{bmatrix} 2jM^*\Omega_O & O \\ O & -2jM^*\Omega_O \end{bmatrix}, \qquad M^* = U_O^\top M U_O.$$

So $c_k^* = \pm j2m_k^*\omega_{Ok}$ (complex)

- Residues

$$A_k = \frac{u_{Ok}u_{Ok}^\top}{c_k^*} = \mp j\frac{u_{Ok}u_{Ok}^\top}{2m_k^*\omega_{Ok}}, \qquad \left(A_{Rk} = 0, A_{Ik} = \frac{-u_{Ok}u_{Ok}^\top}{2m_k^*\omega_{Ok}}\right)$$

- Frequency response function

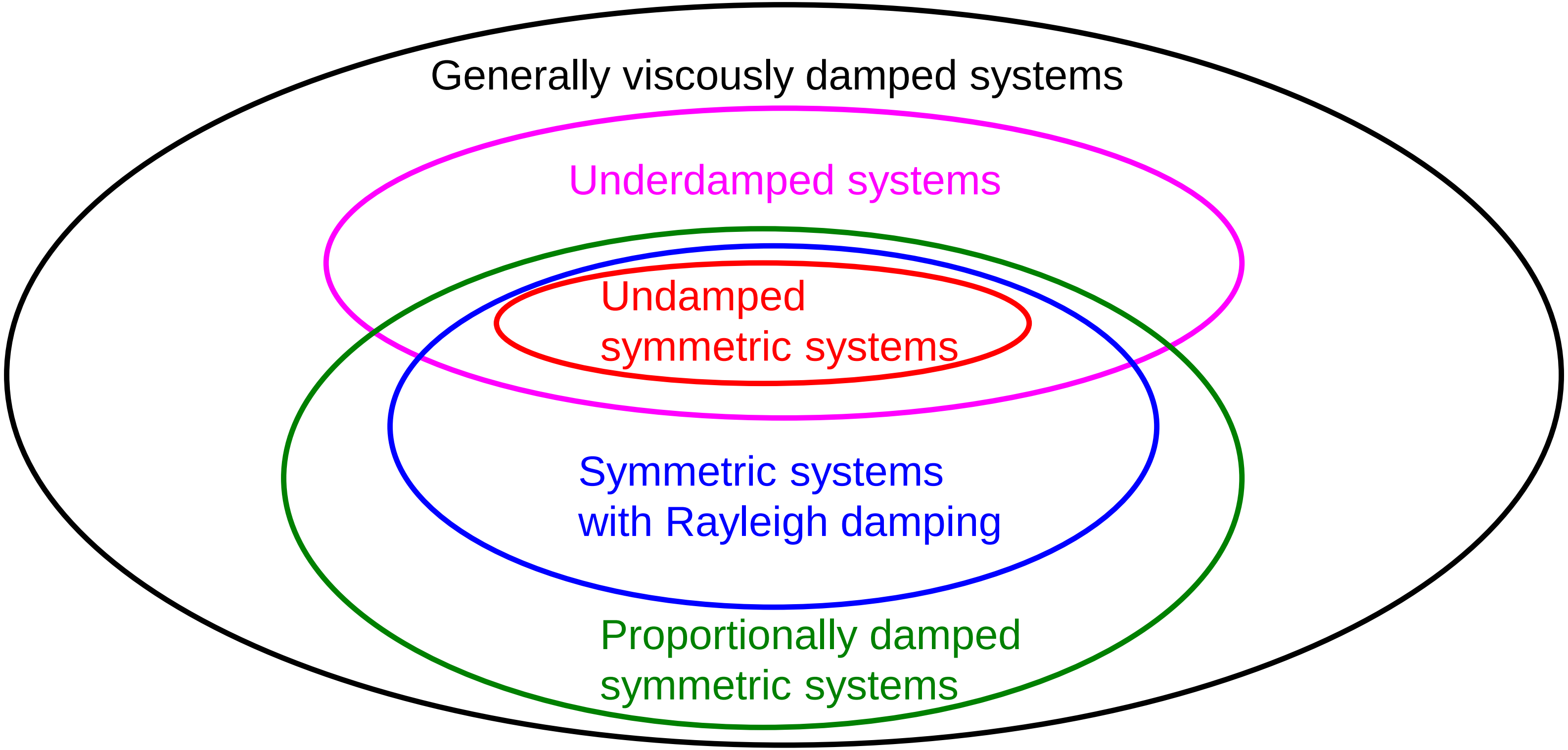
$$H(\omega) = 2\sum_{k=1}^{n} \frac{j\omega A_{Rk} - \mu_k A_{Rk} - \omega_k A_{Ik}}{-\omega^2 + 2\mu_k j\omega + \mu_k^2 + \omega_k^2} = \sum_{k=1}^{n} \frac{u_{Ok}u_{Ok}^\top}{m_k^*(-\omega^2 + \omega_{Ok}^2)}$$

Different types of damping

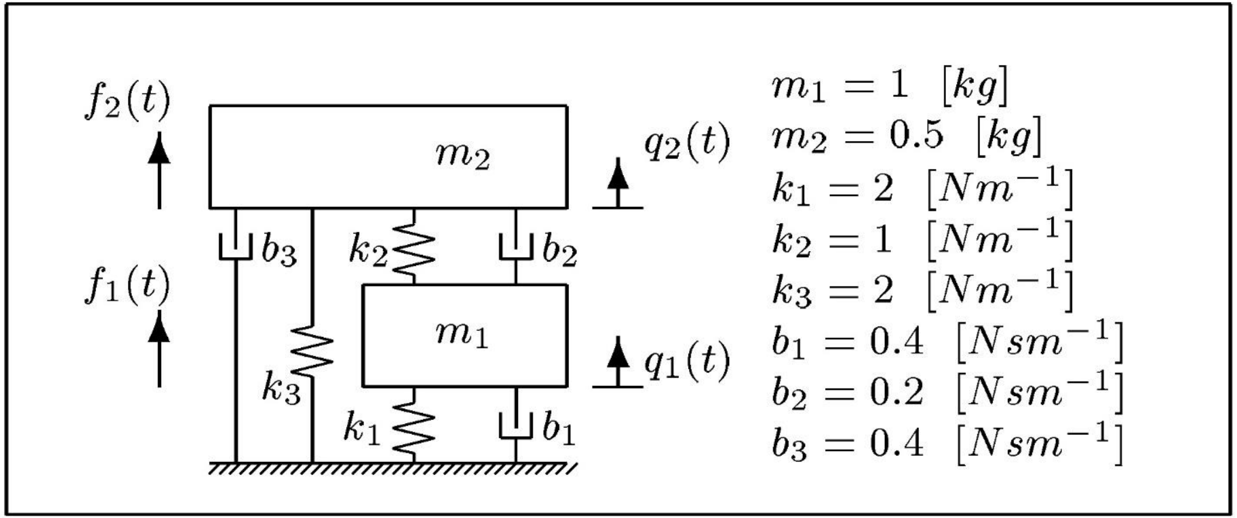
Damping:	Undamped	Proportional	General
Eigenvalues λ_k:	imaginary (complex conjugate pairs)	complex conjugate pairs and/or real	complex conjugate pairs and/or real
Eigenmodes u_k: (displacements)	real	real	complex conjugate pairs and/or real

Undercritical Damping	Proportional	General
Eigenvalues λ_k:	complex conjugate pairs	complex conjugate pairs
Eigenmodes u_k: (displacements)	real	complex conjugate pairs

Classes of systems



Example: 2-dof system



Two cases:

1. Proportional damping ($B = 0.2K$)
2. General viscous damping (take $b_3 = 0$ Ns/m)

Example: 2-dof system

Case 1. Proportional damping

$$(\lambda_k C - D)v_k = 0$$

λ_1	λ_2	λ_3	λ_4
-0.24 - 1.54j	-0.24 + 1.54j	-0.66 - 2.48j	-0.66 + 2.48j

	v_1	v_2	v_3	v_4
u_k	1	1	1	1
	0.56	0.56	-3.56	-3.56
$\lambda_k u_k$	-0.24 - 1.54j	-0.24 + 1.54j	-0.66 - 2.48j	-0.66 + 2.48j
	-0.14 - 0.87j	-0.14 + 0.87j	2.34 + 8.82j	2.34 – 8.82j

Normalize first element of eigenmode v_k to 1

- Apparently undercritically damped:
  Eigenvalues λ_k show up in complex conjugate pairs
- Displacement parts u_k of eigenmodes v_k are **all real**

Example: 2-dof system

Case 2. General viscous damping

$$(\lambda_k C - D)v_k = 0$$

λ_1	λ_2	λ_3	λ_4
-0.19 - 1.56j	-0.19 + 1.56j	-0.31 - 2.52j	-0.31 + 2.52j

	v_1	v_2	v_3	v_4
u_k	1	1	1	1
	0.55 - 0.17j	0.55 + 0.17j	-2.86 - 1.49j	-2.86 + 1.49j
$\lambda_k u_k$	-0.19 - 1.56j	-0.19 + 1.56j	-0.31 - 2.52j	-0.31 + 2.52j
	-0.38 - 0.83j	-0.38 + 0.83j	-2.87 + 7.66j	-2.87 - 7.66j

Normalize first element of eigenmode v_k to 1

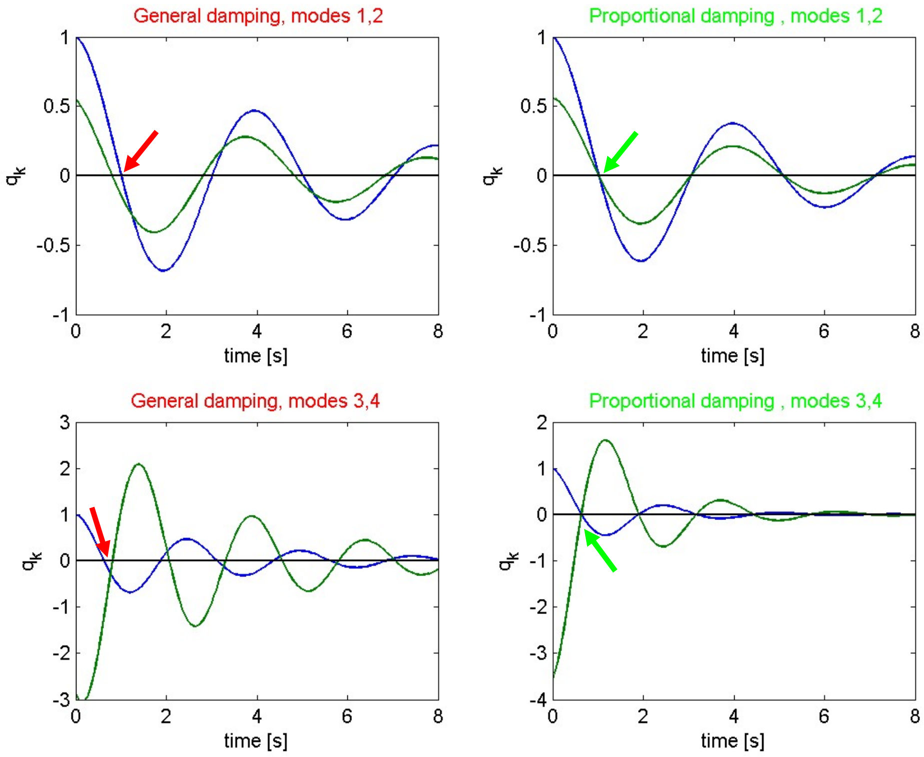
- Apparently undercritically damped:
  Eigenvalues λ_k show up in complex conjugate pairs
- Displacement parts u_k of eigenmodes v_k remain **complex** after normalization.

Example: 2-dof system

Free vibration

$$q[i](t) = \frac{1}{2}\left(u_k[i]e^{\lambda_k t} + u_{k+1}[i]e^{\lambda_{k+1} t}\right)$$

- DOFs $i = 1,2$ (blue and green lines).
- For modes 1&2 ($k = 1$) and modes 3&4 ($k = 3$)



Observe:

- For proportional damping, both DOFs become zero at the same time instant
- When the eigenmodes are complex, this is no longer the case

Example: Pinned-pinned beam

- Discrete damper (b [Ns/m]) at quarter of beam length.
- Generally viscously damped system

- Look at free-vibration of combination of 3rd-4th eigenpair:

$$q(t) = \frac{1}{2}\left(u_3 e^{\lambda_3 t} + u_4 e^{\lambda_4 t}\right), \qquad u_3 = \overline{u_4}, \qquad \lambda_3 = \overline{\lambda_4} = \mu_3 + j\omega_3.$$

- Simulations for 4 damping levels b

Example: Pinned-pinned beam

Case 1. No damping

Example: Pinned-pinned beam

Case 2. $b = 10$ Ns/m

Example: Pinned-pinned beam

Case 3. $b = 30$ Ns/m

Observe: the displacement in the middle of the beam is **nonzero** at certain time instants

⇒ Complex eigenmodes result in a time-dependent shape!

Example: Pinned-pinned beam

Case 4. $b = 100$ Ns/m

Observe: the displacement in the middle of the beam is **nonzero** at certain time instants

$\Rightarrow$ Complex eigenmodes result in a time-dependent shape!

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