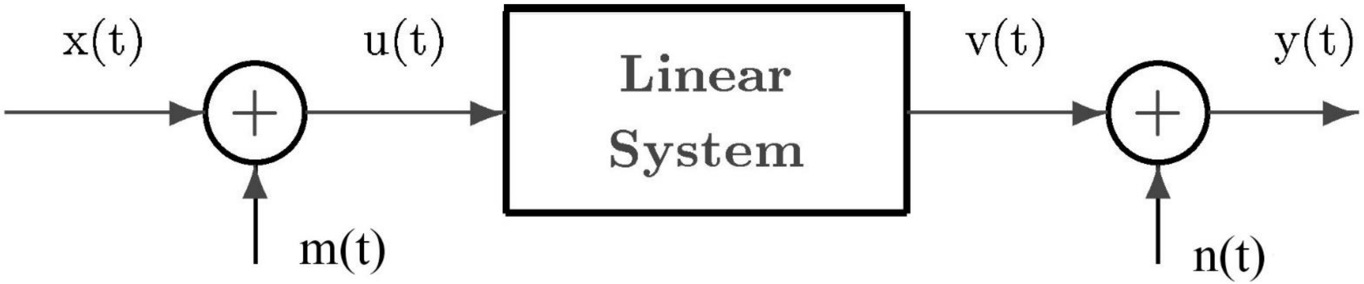

9. System identification

Structural Dynamics part of 4DM00

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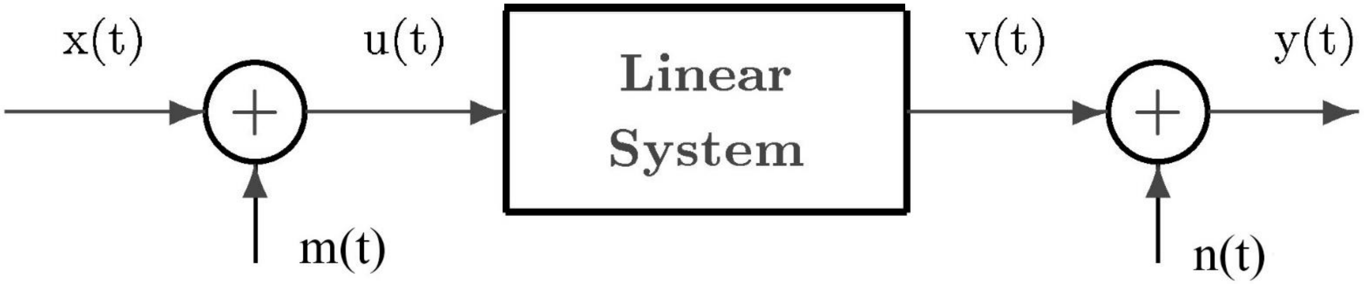
System identification

Assume a linear relation between input and output signals.
How to estimate this relation based on measurements $x(t)$ and $y(t)$?

General situation: input noise $m(t)$ and output noise $n(t)$.



First, only output noise ($m(t) \equiv 0$).

$$y(t) = h(t) \otimes x(t) + n(t)$$

Time-domain approach

Assume:

- No input noise, only output noise: $y(t) = h(t) \otimes x(t) + n(t)$.
- Input $x(t)$ and output noise $n(t)$ are independent: $R_{xn}(\tau) = \mathbb{E}[x(t)n(t+\tau)] \equiv 0$.

Then

$$R_{xy}(\tau) = \mathbb{E}[x(t)y(t+\tau)] = \mathbb{E}\left[x(t)\left[\left(\int_{-\infty}^{\infty} h(\xi)x(t+\tau-\xi)d\xi\right) + n(t+\tau)\right]\right]$$

$$= \int_{-\infty}^{\infty} h(\xi)\mathbb{E}[x(t)x(t+\tau-\xi)]d\xi + \mathbb{E}[x(t)n(t+\tau)]$$

$$= \int_{-\infty}^{\infty} h(\xi)R_{xx}(\tau-\xi)d\xi = h(\tau) \otimes R_{xx}(\tau)$$

In principle possible to estimate $h(t)$ from $R_{xx}(\tau)$ and $R_{xy}(\tau)$.
However, convolution makes time domain approach unsuitable.

Frequency domain approach

The time domain approach resulted in:

$$R_{xy}(t) = h(t) \otimes R_{xx}(t)$$

Problematic due to convolution.
⇒ transform to the frequency domain:

$$S_{xy}(f) = H_{xy}(f) S_{xx}(f)$$

$\boldsymbol{H_1}$**-estimator** for FRF $H_{xy}(f)$:

$$\boxed{\hat{H}_1(f) = \frac{\hat{S}_{xy}(f)}{\hat{S}_{xx}(f)}.}$$

Note:

- The H_1 estimator is unbiased in the absence of input noise.

TU/e

Alternative derivation in the frequency domain

Consider again a rectangular window $w_T(t)$ which is 1 for $0 \le t \le T$ and 0 otherwise.
Windowed signals $y_T(t) = y(t)w(t)$, $x_T(t) = x(t)w_T(t)$ and $n_T(t) = n(t)w_T(t)$.

Multiply $y(t) = h(t) \otimes x(t) + n(t)$ by $w_T(t)$:

$$y_T(t) = w_T(t)\big(h(t) \otimes x(t)\big) + n_T(t) = w_T(t)\big(h(t) \otimes x_T(t)\big) + n_T(t)$$

Note: future inputs do not influence past outputs $h(t) \otimes x(t) = h(t) \otimes x_T(t)$ for $t \le T$.

1. Transform to frequency domain:

$$Y_T(f) = W_T(f) \otimes \big(H_{xy}(f)X_T(f)\big) + N_T(f)$$

2. Multiply by $X_T^*(f)$

$$X_T^*(f)Y_T(f) = X_T^*(f)\Big(W_T(f) \otimes \big(H_{xy}(f)X_T(f)\big)\Big) + X_T^*(f)N_T(f)$$

3. Multiply by $1/T$ and take the limit for $T \to \infty$.

$$S_{xy}(f) = H_{xy}(f)S_{xx}(f) + S_{xn}(f) = H_{xy}(f)S_{xx}(f)$$

Note that $S_{xn}(f) = F[R_{xn}(\tau)] = 0$ when $x(t)$ and $n(t)$ are independent.

Influence of output noise on the FRF estimate

$$Y(f) = H_{xy}(f)X(f) + N(f)$$

Therefore,

$$\begin{aligned}&Y^*(f)Y(f)\\&= H_{xy}^*(f)H_{xy}(f)X^*(f)X(f) + H_{xy}^*(f)X^*(f)N(f) + H_{xy}(f)N^*(f)X(f) + N^*(f)N(f)\end{aligned}$$

Indeed, it can be shown that

$$S_{yy}(f) = \left|H_{xy}(f)\right|^2 S_{xx}(f) + H^*(f)S_{xn}(f) + H(f)S_{nx}(f) + S_{nn}(f).$$

Note that $S_{xn}(f) = \overline{S_{nx}(f)} \equiv 0$ when $x(t)$ and $n(t)$ are independent. Therefore,

$$S_{yy}(f) = \left|H_{xy}(f)\right|^2 S_{xx}(f) + S_{nn}(f)$$

Influence of output noise on the FRF estimate

$$S_{yy}(f) = \left|H_{xy}(f)\right|^2 S_{xx}(f) + S_{nn}(f), \qquad S_{nn}(f) = S_{yy}(f) - \left|H_{xy}(f)\right|^2 S_{xx}(f)$$

But $H_{xy}(f) = S_{xy}(f)/S_{xx}(f)$, therefore

$$S_{nn}(f) = S_{yy}(f) - \frac{\left|S_{xy}(f)\right|^2}{|S_{xx}(f)|^2} S_{xx}(f) = \left(1 - \frac{\left|S_{xy}(f)\right|^2}{S_{xx}(f) S_{yy}(f)}\right) S_{yy}(f).$$

We now recognize the **coherence function** $\gamma_{xy}(f)$

$$0 \leq \gamma_{xy}^2(f) = \frac{\left|S_{xy}(f)\right|^2}{S_{xx}(f) S_{yy}(f)} \leq 1.$$

We conclude:

$$\boxed{S_{nn}(f) = \left(1 - \gamma_{xy}^2(f)\right) S_{yy}(f)}$$

Influence of output noise on the FRF estimate

Recall that $y(t) = v(t) + n(t)$
$v(t)$ is the part of $y(t)$ that is not influences by the noise $n(t)$.

Because $Y(f) = V(f) + N(f)$

$$Y^*(f)Y(f) = V^*(f)V(f) + N^*(f)V(f) + V^*(f)N(f) + N^*(f)N(f)$$

Indeed, it is true that

$$S_{yy}(f) = S_{vv}(f) + S_{nv}(f) + S_{vn}(f) + S_{nn}(f).$$

It can be shown that $v(t)$ and $n(t)$ are uncorrelated when $x(t)$ and $n(t)$ are independent.

$$S_{yy}(f) = S_{vv}(f) + S_{nn}(f)$$

Influence of output noise on the FRF estimate

$$S_{nn}(f) = \left(1 - \gamma_{xy}^2(f)\right) S_{yy}(f),$$

$$S_{vv}(f) = S_{yy}(f) - S_{nn}(f) = \gamma_{xy}^2(f) S_{yy}(f)$$

The coherence function $\gamma_{xy}^2(f)$ $(0 \leq \gamma_{xy}^2(f) \leq 1)$ thus tells which part of $S_{yy}(f)$ is caused by the noise $n(t)$ and which part by $v(t)$.

$\gamma_{xy}^2(f) \to 1$: strong linear relation between input and output at frequency f

$\gamma_{xy}^2(f) \to 0$: no linear relation between input and output at frequency f, output dominated by noise and/or nonlinear effects

Variance of the estimators

- The standard deviation of the H_1-estimator $\hat{H}_1(f) = \hat{S}_{xy}(f)/\hat{S}_{xx}(f)$ is

$$\sigma_{|\hat{H}_1|} = \left(var\left[\hat{H}_1(f)\right]\right)^{1/2} \approx \frac{\left[1-\hat{\gamma}_{xy}^2(f)\right]^{\frac{1}{2}}}{\left|\hat{\gamma}_{xy}(f)\right|\sqrt{2N}}\left|\hat{H}_{xy}(f)\right|$$

- The standard deviation of the coherence estimator $\hat{\gamma}_{xy}^2(f) = \left|\hat{S}_{xy}(f)\right|^2/(\hat{S}_{xx}(f)\hat{S}_{yy}(f))$ is

$$\sigma_{\hat{\gamma}_{xy}^2} = \left(var\left[\hat{\gamma}_{xy}^2(f)\right]\right)^{1/2} \approx \frac{\sqrt{2}\left[1-\hat{\gamma}_{xy}^2(f)\right]}{\left|\hat{\gamma}_{xy}(f)\right|\sqrt{N}}\hat{\gamma}_{xy}^2(f)$$

- The standard deviation of the estimator $\hat{S}_{vv}(f) = \left|\hat{S}_{xy}(f)\right|^2/\hat{S}_{xx}(f)$ is

$$\sigma_{\hat{S}_{vv}} = \left(var\left[\hat{S}_{vv}(f)\right]\right)^{1/2} \approx \frac{\left[2-\hat{\gamma}_{xy}^2(f)\right]^{1/2}}{\left|\hat{\gamma}_{xy}(f)\right|\sqrt{N}}\hat{S}_{vv}(f)$$

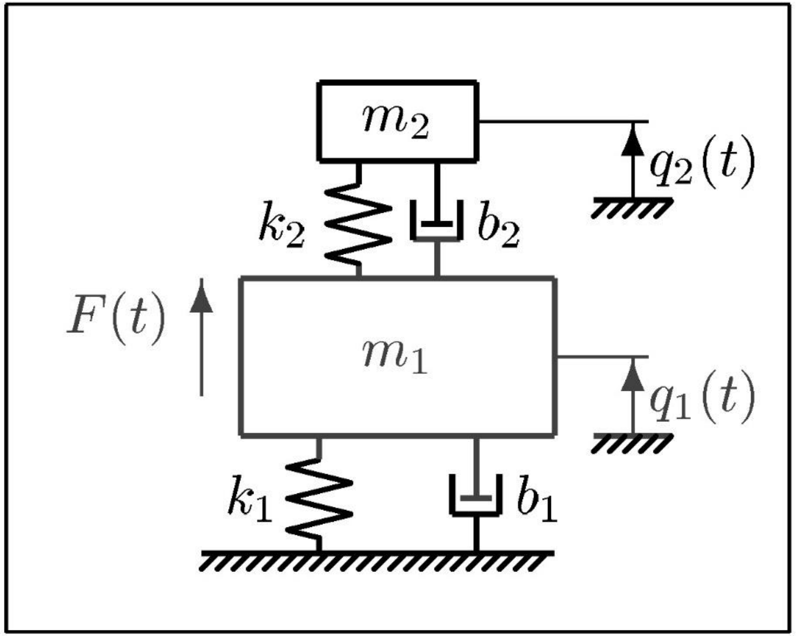
These standard deviations can be used to define 95%-confidence intervals.

Example: 2-dof system

Properties:
$m_1 = 1$ kg, $m_2 = 0.3$ kg,
$k_1 = 100$ N/m, $k_2 = 50$ N/m,
$b_1 = 1$ Ns/m, $b_2 = 0.4$ Ns/m.

Excitation: $x(t) = F(t)$
Response: $y(t) = q_1(t)$

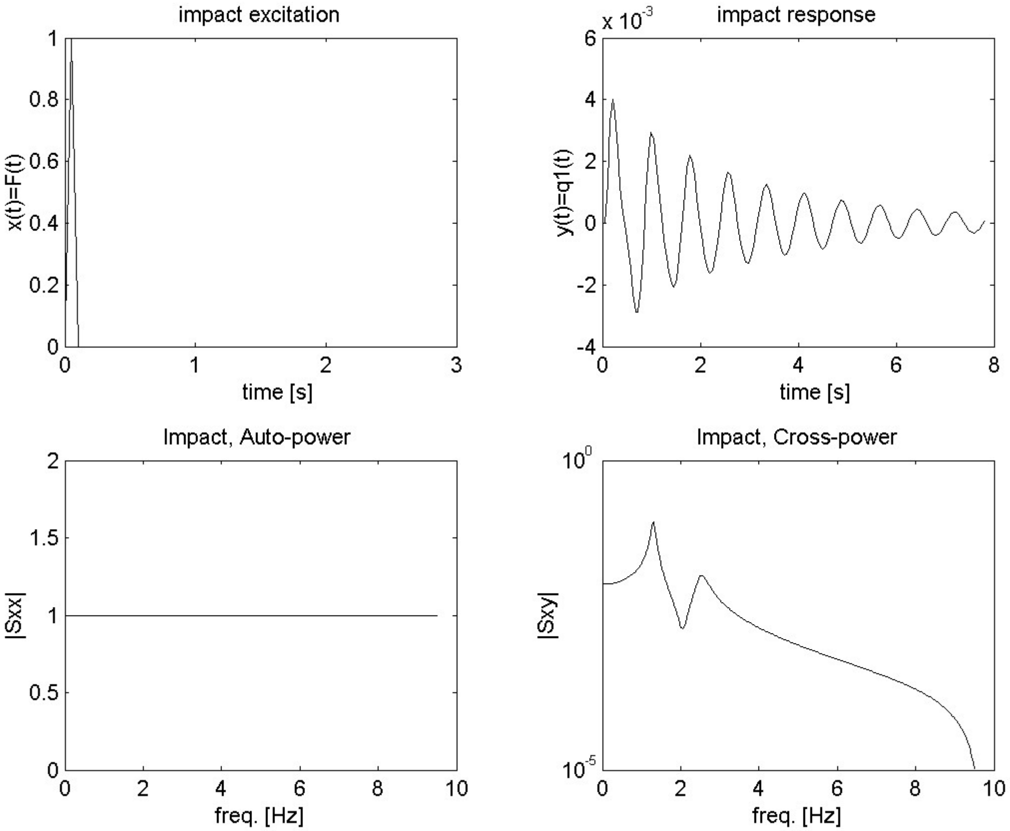
simulated experiment
no noise, so $n(t) = 0$.



Example: 2-dof system

Case 1:

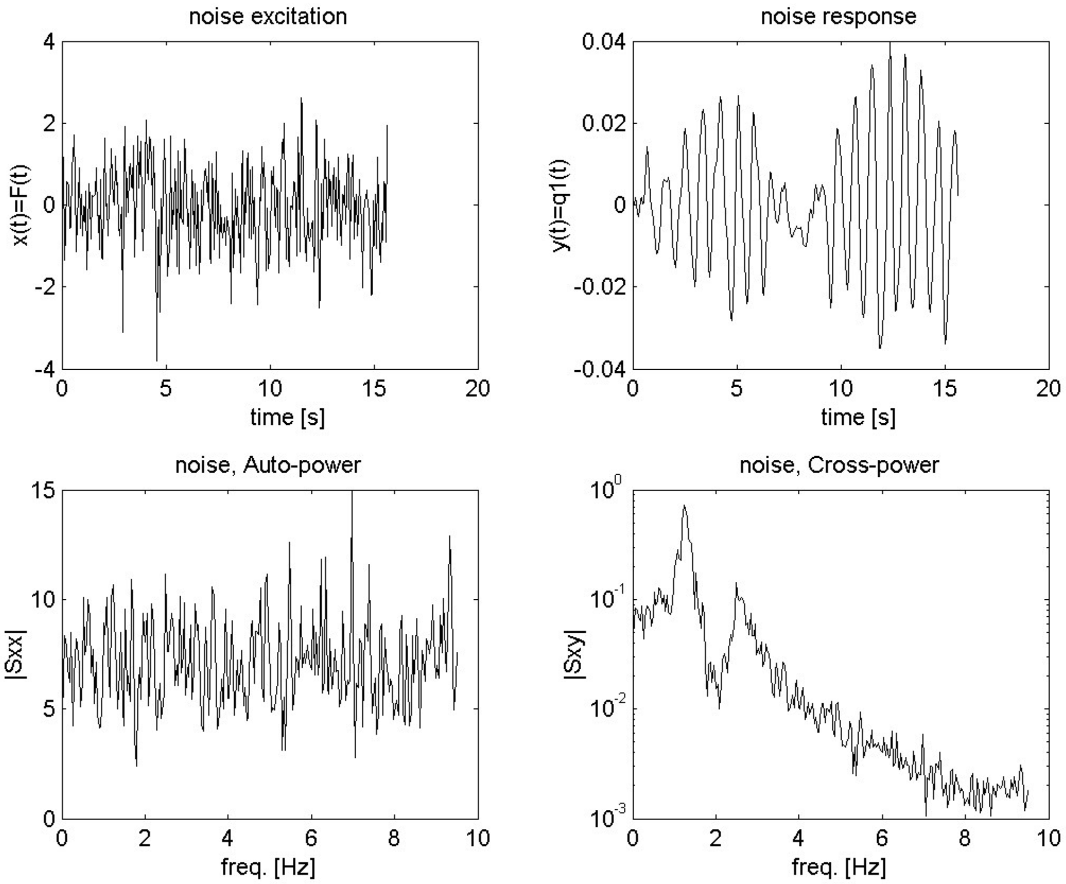
Hammer (impact) excitation
(2 records of 512 time points)



Example: 2-dof system

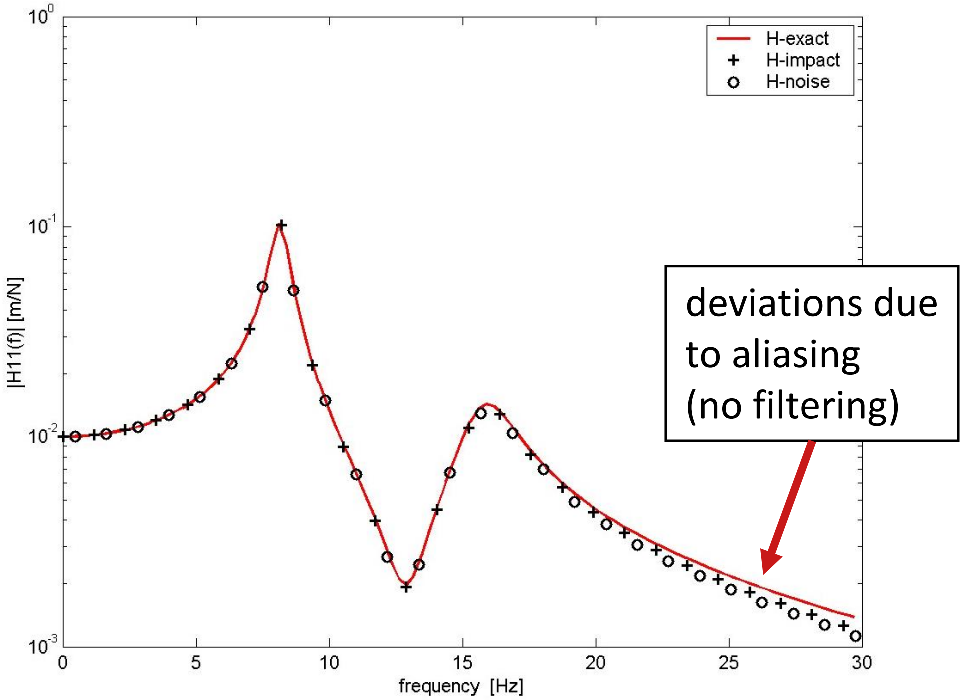
Case 2:

White noise excitation,
10 records of 512 time points,
Hanning window



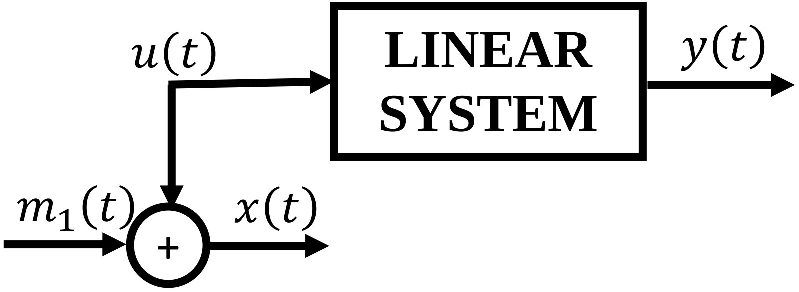
Example: 2-dof system

- Exact FRF (red)
- Estimated FRF based on **impact** excitation (case 1, +)
- Estimated FRF based on **white noise** excitation (case 2, o)



9b. Input noise

Two types of input noise

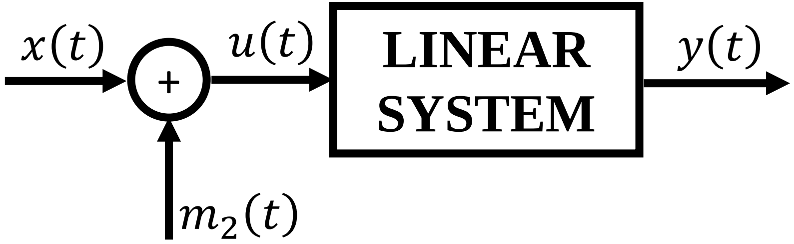


$x(t)$ and $y(t)$ are measured

$m_1(t)$: unknown noise source at input, does not go through system

$$y(t) = h(t) \otimes u(t)$$
$$= h(t) \otimes \left(x(t) - m_1(t)\right)$$

$m_1(t)$ and $u(t)$ are statistically independent
$m_1(t)$ and $x(t)$ are not statistically independent



$x(t)$ and $y(t)$ are measured

$m_2(t)$: unknown noise source at input, does go through system

$$y(t) = h(t) \otimes u(t)$$
$$= h(t) \otimes \left(x(t) + m_2(t)\right)$$

$m_2(t)$ and $x(t)$ are statistically independent
$m_2(t)$ and $u(t)$ are not statistically independent

Input noise $m_1(t)$

- $m_1(t)$ and $u(t)$ are independent
- $m_1(t)$ and $x(t)$ are not independent
- $y(t) = h(t) \otimes \left(x(t) - m_1(t)\right)$

In the frequency domain,

$$Y(f) = H_{xy}(f)\big(X(f) - M_1(f)\big)$$

Two options:

- OPTION 1: Multiply by $X^*(f)$

$$X^*(f)Y(f) = H_{xy}(f)X^*(f)\big(X(f) - M_1(f)\big)$$

$$S_{xy}(f) = H_{xy}(f)\left(S_{xx}(f) - S_{xm_1}(f)\right)$$

But $x(t)$ and $m_1(t)$ are not independent, so $S_{xm_1}(f) \neq 0$!

The H_1-estimator $\hat{S}_{xy}(f)/\hat{S}_{xx}(f)$ is biased!

Input noise $m_1(t)$

- $m_1(t)$ and $u(t)$ are independent
- $m_1(t)$ and $x(t)$ are not independent
- $y(t) = h(t) \otimes \left(x(t) - m_1(t)\right)$

In the frequency domain,

$$Y(f) = H_{xy}(f)\left(X(f) - M_1(f)\right)$$

Two options:

- OPTION 2: Multiply by $Y^*(f)$

$$Y^*(f)Y(f) = H_{xy}(f)Y^*(f)\left(X(f) - M_1(f)\right)$$

$$S_{yy}(f) = H_{xy}(f)\left(S_{yx}(f) - S_{ym_1}(f)\right)$$

$y(t)$ and $m_1(t)$ are independent
(because $u(t)$ and $m_1(t)$ are independent and $y(t)$ depends linearly on $u(t)$)
The $\boldsymbol{H_2}$**-estimator** $\hat{S}_{yy}(f)/\hat{S}_{yx}(f)$ is unbiased!

Input noise $m_2(t)$

- $m_2(t)$ and $x(t)$ are independent
- $m_2(t)$ and $u(t)$ are not independent
- $y(t) = h(t) \otimes \left(x(t) + m_2(t)\right)$

In the frequency domain,

$$Y(f) = H_{xy}(f)\left(X(f) + M_2(f)\right)$$

Two options:

- OPTION 1: Multiply by $X^*(f)$

$$X^*(f)Y(f) = H_{xy}(f)X^*(f)\left(X(f) + M_2(f)\right)$$

$$S_{xy}(f) = H_{xy}(f)\left(S_{xx}(f) + S_{xm_2}(f)\right)$$

But $x(t)$ and $m_2(t)$ are not independent, so $S_{xm_2}(f) = 0$!

The H_1-estimator $\hat{S}_{xy}(f)/\hat{S}_{xx}(f)$ is unbiased!
(it can be verified similarly that the H_2-estimator is biased).

Two estimators

So we have found two estimators:

$$\hat{H}_1(f) = \frac{\hat{S}_{xy}(f)}{\hat{S}_{xx}(f)}, \qquad \hat{H}_2(f) = \frac{\hat{S}_{yy}(f)}{\hat{S}_{yx}(f)}.$$

Note that

$$\frac{\hat{H}_1(f)}{\hat{H}_2(f)} = \frac{\hat{S}_{xy}(f)\hat{S}_{yx}(f)}{\hat{S}_{xx}(f)\hat{S}_{yy}(f)} = \frac{\left|\hat{S}_{xy}(f)\right|^2}{\hat{S}_{xx}(f)\hat{S}_{yy}(f)} = \hat{\gamma}_{xy}^2(f)$$

So the estimators will differ iff $\hat{\gamma}_{xy}^2(f) < 1$.

$\Rightarrow$ without noise $\hat{H}_1(f) = \hat{H}_2(f)$.

Example: 2-dof system

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = F(t)$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.055 & -0.0003 \\ -0.0003 & 0.03 \end{bmatrix}, \qquad K = \begin{bmatrix} 1.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix}$$

$$q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}, \qquad F(t) = \begin{bmatrix} 0 \\ F_2(t) \end{bmatrix}.$$

Simulated experiment, estimate the collocated FRF of dof 2: $H_{22}(f) = Q_2/F_2 = Y/X$

Input force $F_2(t) = x(t)$: random noise, normal pdf ($\mu_x = 0$, $\sigma_x = 1$)

After numerical integration: response $q_2(t) = y(t)$, $\sigma_y \approx 10$

Add 2 disturbance noises: $m_1(t)$, $\sigma_{m_1} = 0.2$ and $n(t)$, $\sigma_n = 2$

Discrete $x(t_k), y(t_k)$ are generated, $\Delta T = 1.2$ s, $k = 1,2, \dots, 10000$

Example: 2-dof system

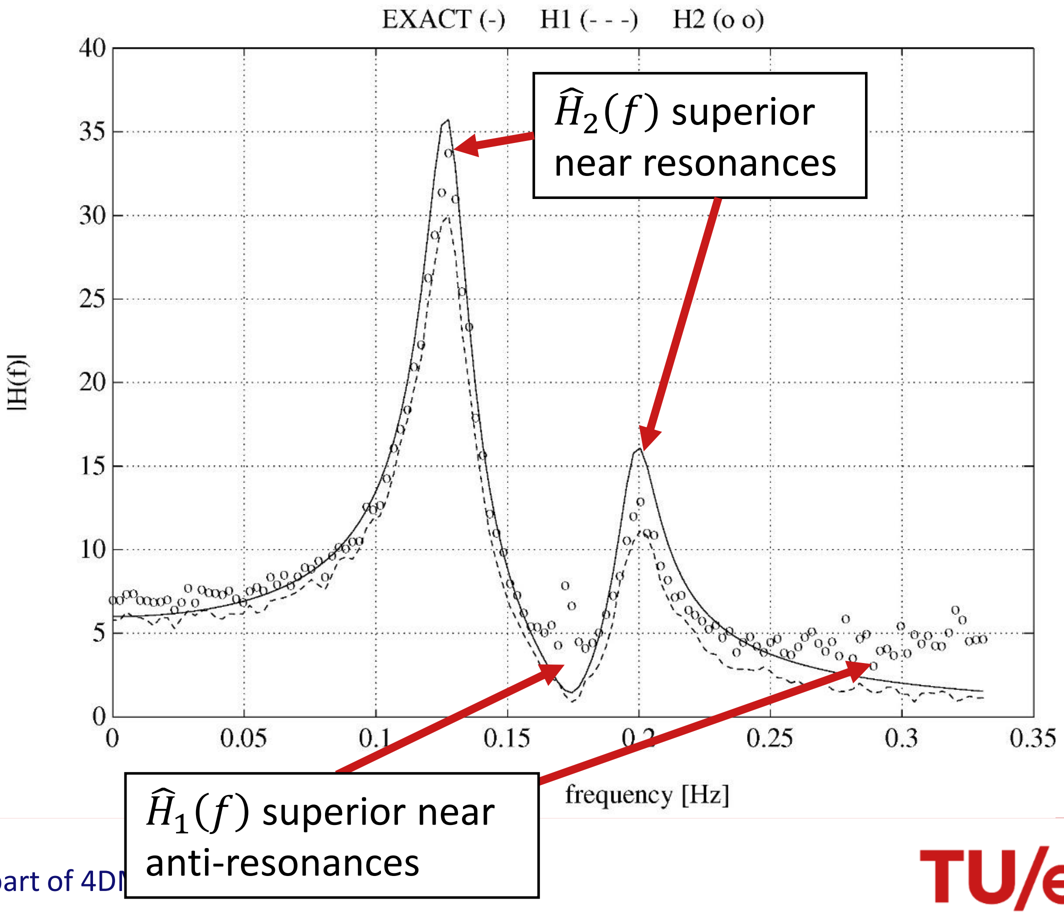
The estimators

- $\widehat{H}_1(f)$ (- - -) and
- $\widehat{H}_2(f)$ (o o)

are compared to

- the exact solution $H_{xy}(f)$ (---).

(39 records of 256 time points)



Estimation near anti-resonances and resonances

- Near anti-resonances:
  large input $u(t)$ gives small response $v(t)$
  $S_{nn}(f_{ar})$ has large influence on $S_{yy}(f_{ar}) = S_{vv}(f_{ar}) + S_{nn}(f_{ar})$
  Noise $n(t)$ strongly influences the estimator

$$\widehat{H}_2(f_{ar}) = \frac{\hat{S}_{yy}(f_{ar})}{\hat{S}_{yx}(f_{ar})}$$

- Near resonances:
  small input $u(t)$ gives large response $v(t)$
  $S_{m_1 m_1}(f_{res})$ has large influence on $S_{xx}(f_{res}) = S_{uu}(f_{res}) + S_{m_1 m_1}(f_{res})$
  Noise $m_1(t)$ strongly influences the estimator

$$\widehat{H}_1(f_{res}) = \frac{\hat{S}_{xy}(f_{res})}{\hat{S}_{xx}(f_{res})}$$

Summary

Three estimators for $H_{xy}(f)$:

- $\widehat{H}_0(f) = \frac{\hat{Y}(f)}{\hat{X}(f)}$ Rarely used, difficult to create accurate estimator

- $\widehat{H}_1(f) = \frac{\hat{S}_{xy}(f)}{\hat{S}_{xx}(f)}$ to be used when the (input or output) noise is independent of $x(t)$ superior near anti-resonances
- $\widehat{H}_2(f) = \frac{\hat{S}_{yy}(f)}{\hat{S}_{yx}(f)}$ to be used when the (input) noise is independent of $u(t)$ superior near resonances

Without disturbance noise, these 3 estimators are **equivalent**!

Warning:
For a single record $N = 1$, $\widehat{H}_1(f) = \widehat{H}_2(f)$ and $\gamma_{xy}(f) \equiv 1$! Noise cannot be seen for $N = 1$!
Choose the number of records N large enough to properly identify the influence of noise!