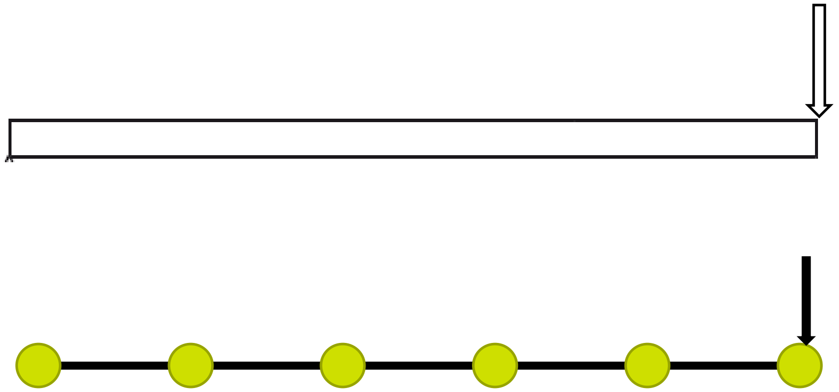

3. Finite Element Models of dynamic structures

Structural Dynamics part of 4DM00

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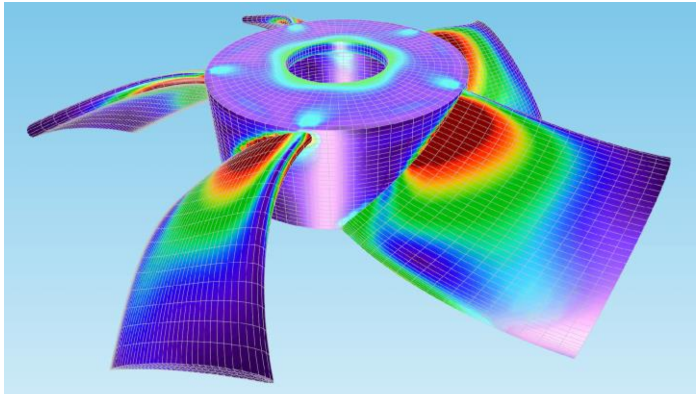
Introduction

- The Finite Element Method (FEM) is a numerical modeling and analysis procedure to approximate solutions of Partial Differential Equations (PDEs) that govern physical behavior.
- Examples
  - structural dynamics,
  - heat transfer,
  - acoustics,
  - fluid mechanics,
  - multi-physics problems.

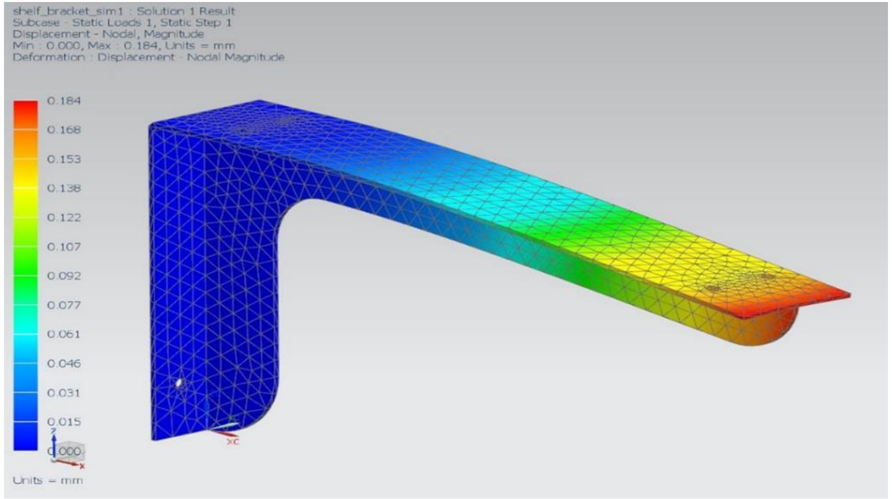


- In structural dynamics, FEM gives an approximate solution of the displacement fields.

Introduction

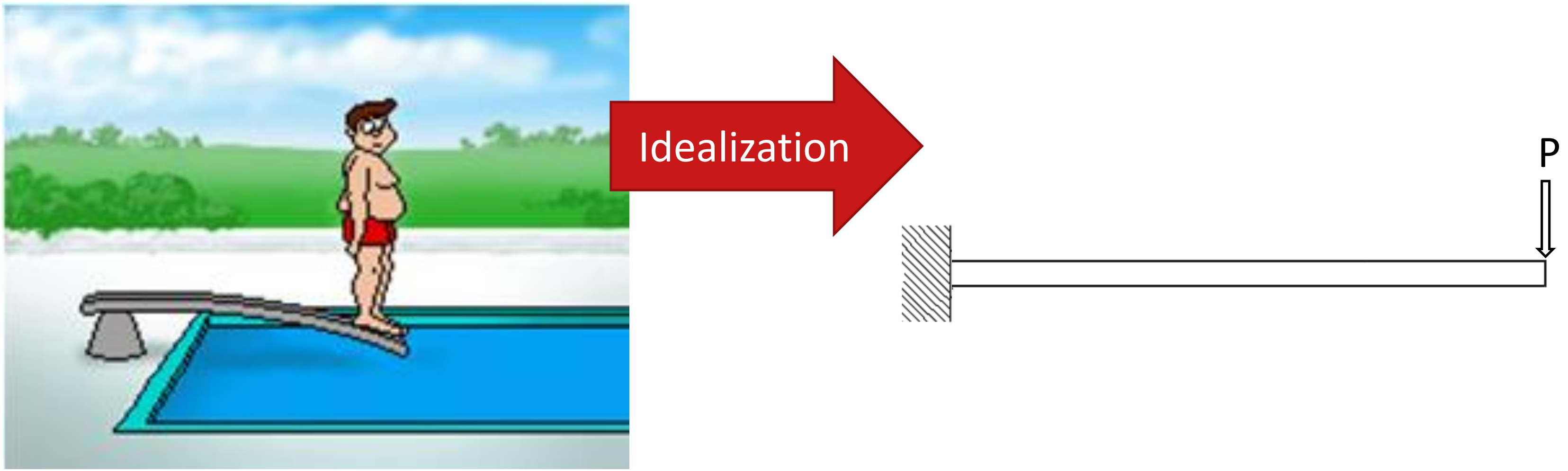
The main procedure

1. Divide a (complex) geometry into a finite number of (small) elements with relatively simple geometries.
2. Apply physical laws to each element.
3. Assemble all elements to obtain a model of the complete structure
4. Apply boundary conditions



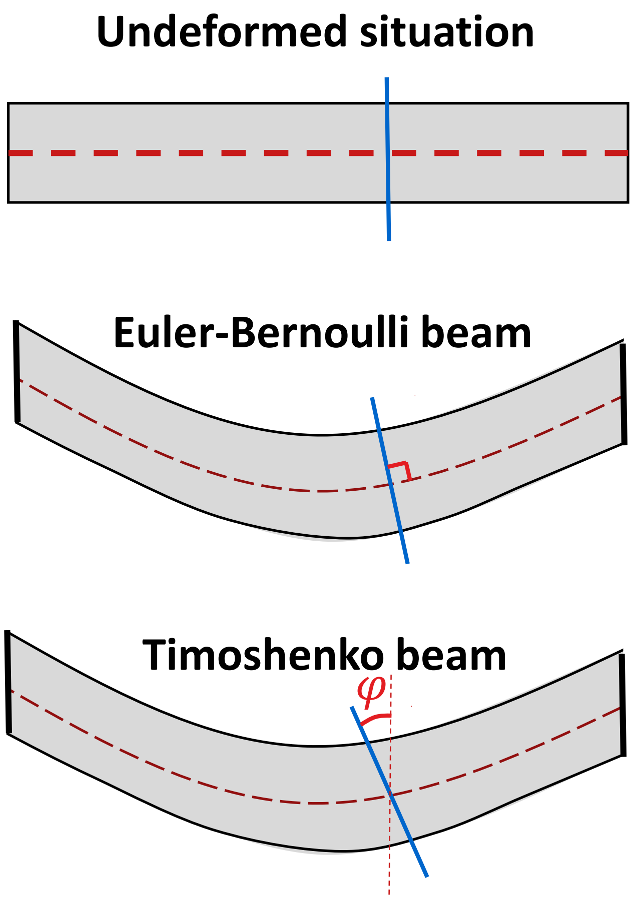
FEM Pre-processing: Idealization

STEP 0. Idealization/modelling

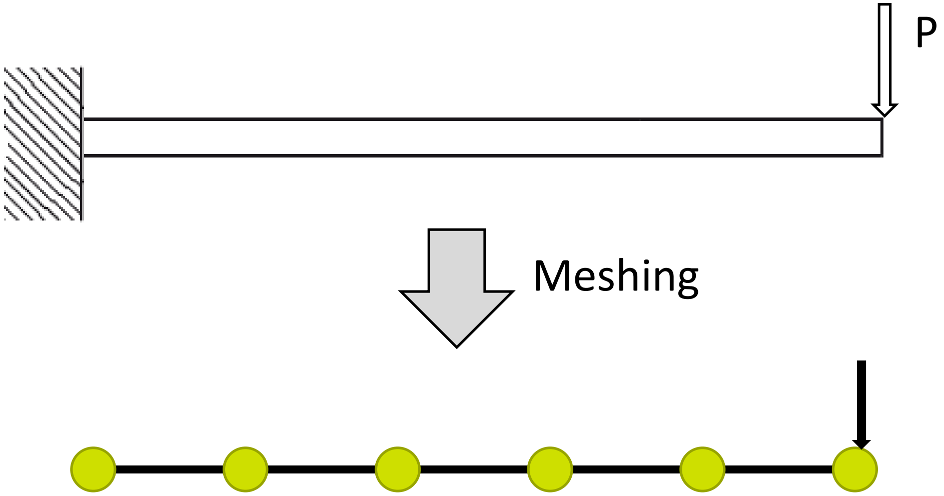


Different beam models

1. **The Euler-Bernoulli beam**
   (slender beams)
   Considers:
   - Bending
   - Translational inertia
2. **Rayleigh beam equation**
   Considers:
   - Bending
   - Rotary and translational inertia
3. **Timoshenko beam equation**
   (short, thick beams)
   Considers:
   Shear and bending
   Rotary and translational inertia



Meshing



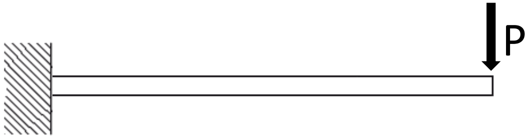
The density of the mesh depends on

- the accuracy requirement of the analysis and
- the computational resources available.

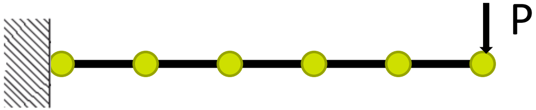
The mesh is usually not uniform. Refine the mesh where

- the displacement gradient is larger
- the accuracy is critical to the analysis.

FEM Pre-processing: To a generic element



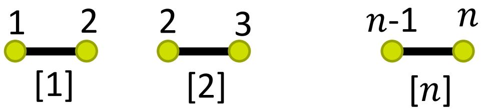
0. Idealization



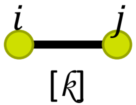
1. Discretization

2. Remove loads and supports

3. Disassemble



4. Localization



5. Generic element

The generic element

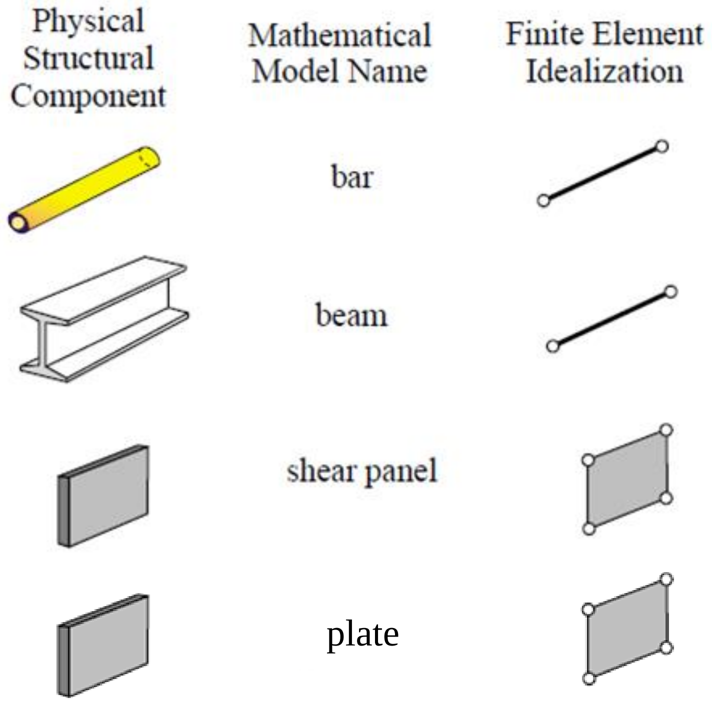
Some generic structural elements:

Bar: deforms only in axial direction.

Beam: deforms in directions perpendicular to its axis (bending, shear).

Membrane (shear panel): loading and deformation occur in-plane.

Plate: loading and deformation occur out-of-plane.

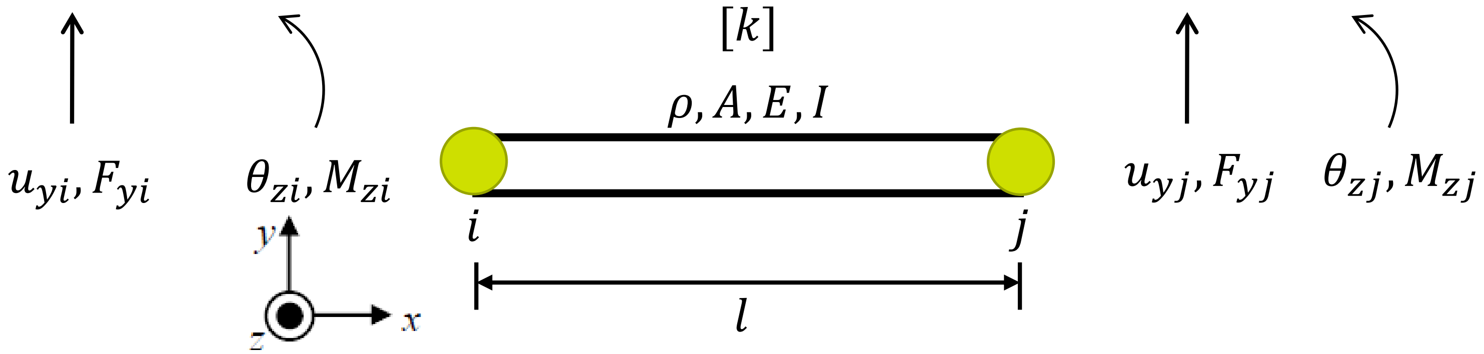


The generic element

2 DOFs per node
4 DOFs per element

SIGN CONVENTIONS

Euler-Bernoulli beam element



l	Length
A	Cross section
E	Young's modulus
ρ	Mass density
I	Second moment of area

u_{yi}	Deflection in the y-direction at node i
θ_{zi}	Rotation around the z-axis at node i
F_{yi}	Transversal force applied at node i
M_{zi}	Moment around the z-axis applied at node i

Apply physical laws to generic element

Equation of motion for the Euler-Bernoulli beam

$$M^e \ddot{q}^e(t) + K^e q^e(t) = f^e(t)$$

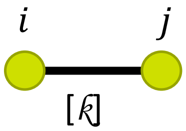
Element generalized coordinate vector and Element generalized force vector

$$q^e(t) = \begin{bmatrix} u_{yi}(t) \\ \theta_{zi}(t) \\ u_{yj}(t) \\ \theta_{zj}(t) \end{bmatrix}, \qquad f^e(t) = \begin{bmatrix} F_{yi}(t) \\ M_{zi}(t) \\ F_{yj}(t) \\ M_{zj}(t) \end{bmatrix},$$

(consistent) Element mass matrix and element stiffness matrix

$$M^e = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{sym.} & & & 4l^2 \end{bmatrix}, \qquad K^e = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym.} & & & 4l^2 \end{bmatrix}.$$

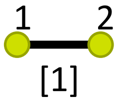
Assembly steps

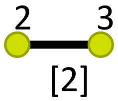


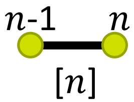
5.+6. Generic element + physical laws

$$M^e \ddot{q}^e(t) + K^e q^e(t) = f^e(t)$$

7. Physical laws for individual elements





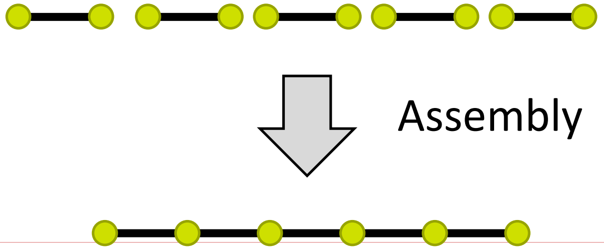


$$M^{[1]} \ddot{q}^{[1]}(t) + K^{[1]} q^{[1]}(t) = f^{[1]}(t)$$

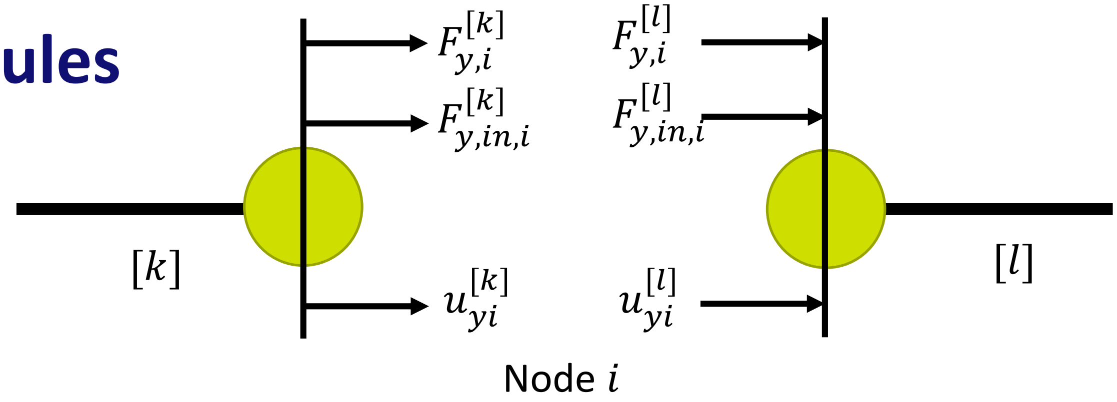
$$M^{[2]} \ddot{q}^{[2]}(t) + K^{[2]} q^{[2]}(t) = f^{[2]}(t)$$

$$M^{[n]} \ddot{q}^{[n]}(t) + K^{[n]} q^{[n]}(t) = f^{[n]}(t)$$

8. Assembly



Assembly rules

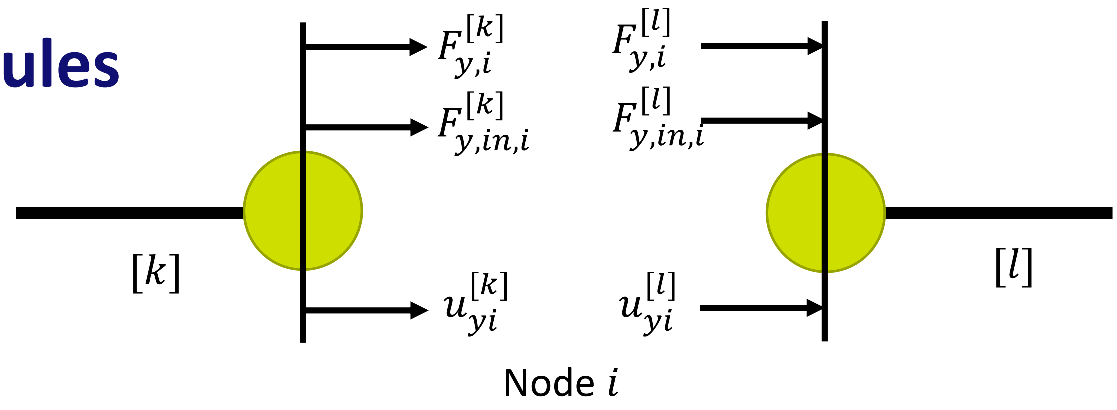


1. *Compatibility of displacements (and/or rotations)*

   The displacements and rotations of all elements meeting at a node are the same.

$$u_{yi}^{[k]}(t) = u_{yi}^{[l]}(t), \qquad \theta_{zi}^{[k]}(t) = \theta_{zi}^{[l]}(t).$$

Assembly rules



2. *Equilibrium of internal forces (and/or internal moments)*
   The sum of internal forces (and moments) exerted by all elements that meet at a node balances

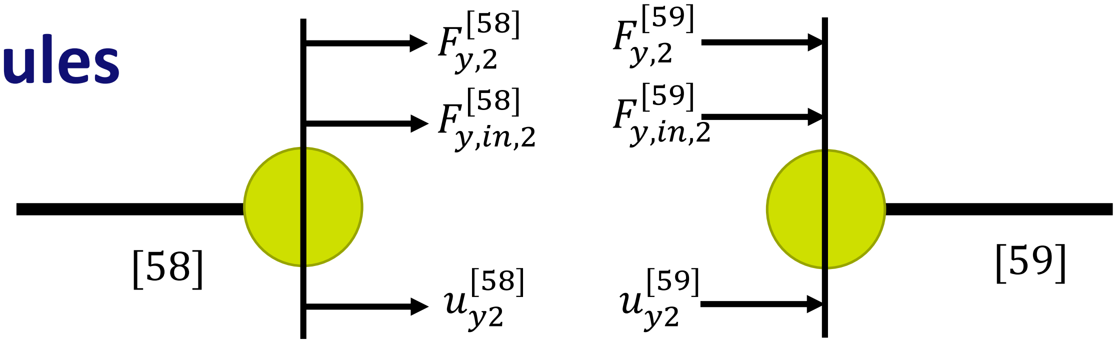
$$F_{y,in,i}^{[k]}(t) + F_{y,in,i}^{[l]}(t) = 0, \qquad M_{z,in,i}^{[k]}(t) + M_{z,in,i}^{[l]}(t) = 0$$

The net force (or moment) applied at node i is then

$$F_{y,i}(t) = F_{y,i}^{[k]}(t) + F_{y,in,i}^{[k]}(t) + F_{y,i}^{[l]}(t) + F_{y,in,i}^{[k]}(t) = F_{y,i}^{[k]}(t) + F_{y,i}^{[l]}(t)$$

$$M_{z,i}(t) = M_{z,i}^{[k]}(t) + M_{z,in,i}^{[k]}(t) + M_{z,i}^{[l]}(t) + M_{z,in,i}^{[k]}(t) = M_{z,i}^{[k]}(t) + M_{z,i}^{[l]}(t)$$

Assembly rules

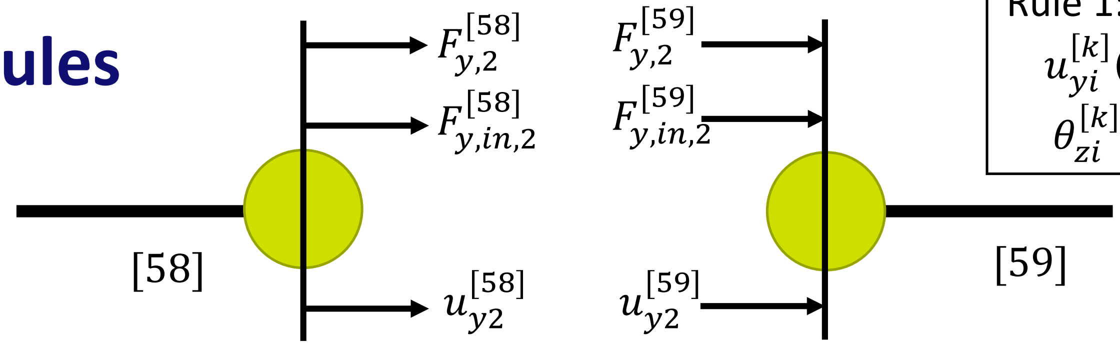


Node 2

$$\begin{bmatrix} m_{11}^{[58]} & m_{12}^{[58]} & m_{13}^{[58]} & m_{14}^{[58]} \\ m_{21}^{[58]} & m_{22}^{[58]} & m_{23}^{[58]} & m_{24}^{[58]} \\ m_{31}^{[58]} & m_{32}^{[58]} & m_{33}^{[58]} & m_{34}^{[58]} \\ m_{41}^{[58]} & m_{42}^{[58]} & m_{43}^{[58]} & m_{44}^{[58]} \end{bmatrix} \begin{bmatrix} \ddot{u}_{y1}^{[58]} \\ \ddot{\theta}_{z1}^{[58]} \\ \ddot{u}_{y2}^{[58]} \\ \ddot{\theta}_{z2}^{[58]} \end{bmatrix} + \begin{bmatrix} k_{11}^{[58]} & k_{12}^{[58]} & k_{13}^{[58]} & k_{14}^{[58]} \\ k_{21}^{[58]} & k_{22}^{[58]} & k_{23}^{[58]} & k_{24}^{[58]} \\ k_{31}^{[58]} & k_{32}^{[58]} & k_{33}^{[58]} & k_{34}^{[58]} \\ k_{41}^{[58]} & k_{42}^{[58]} & k_{43}^{[58]} & k_{44}^{[58]} \end{bmatrix} \begin{bmatrix} u_{y1}^{[58]} \\ \theta_{z1}^{[58]} \\ u_{y2}^{[58]} \\ \theta_{z2}^{[58]} \end{bmatrix} = \begin{bmatrix} F_{y1}^{[58]} \\ M_{z1}^{[58]} \\ F_{y2}^{[58]} \\ M_{z2}^{[58]} \end{bmatrix}$$

$$\begin{bmatrix} m_{11}^{[59]} & m_{12}^{[59]} & m_{13}^{[59]} & m_{14}^{[59]} \\ m_{21}^{[59]} & m_{22}^{[59]} & m_{23}^{[59]} & m_{24}^{[59]} \\ m_{31}^{[59]} & m_{32}^{[59]} & m_{33}^{[59]} & m_{34}^{[59]} \\ m_{41}^{[59]} & m_{42}^{[59]} & m_{43}^{[59]} & m_{44}^{[59]} \end{bmatrix} \begin{bmatrix} \ddot{u}_{y2}^{[59]} \\ \ddot{\theta}_{z2}^{[59]} \\ \ddot{u}_{y3}^{[59]} \\ \ddot{\theta}_{z3}^{[59]} \end{bmatrix} + \begin{bmatrix} k_{11}^{[59]} & k_{12}^{[59]} & k_{13}^{[59]} & k_{14}^{[59]} \\ k_{21}^{[59]} & k_{22}^{[59]} & k_{23}^{[59]} & k_{24}^{[59]} \\ k_{31}^{[59]} & k_{32}^{[59]} & k_{33}^{[59]} & k_{34}^{[59]} \\ k_{41}^{[59]} & k_{42}^{[59]} & k_{43}^{[59]} & k_{44}^{[59]} \end{bmatrix} \begin{bmatrix} u_{y2}^{[59]} \\ \theta_{z2}^{[59]} \\ u_{y3}^{[59]} \\ \theta_{z3}^{[59]} \end{bmatrix} = \begin{bmatrix} F_{y2}^{[59]} \\ M_{z2}^{[59]} \\ F_{y3}^{[59]} \\ M_{z3}^{[59]} \end{bmatrix}$$

Assembly rules



Rule 1: Compatability:

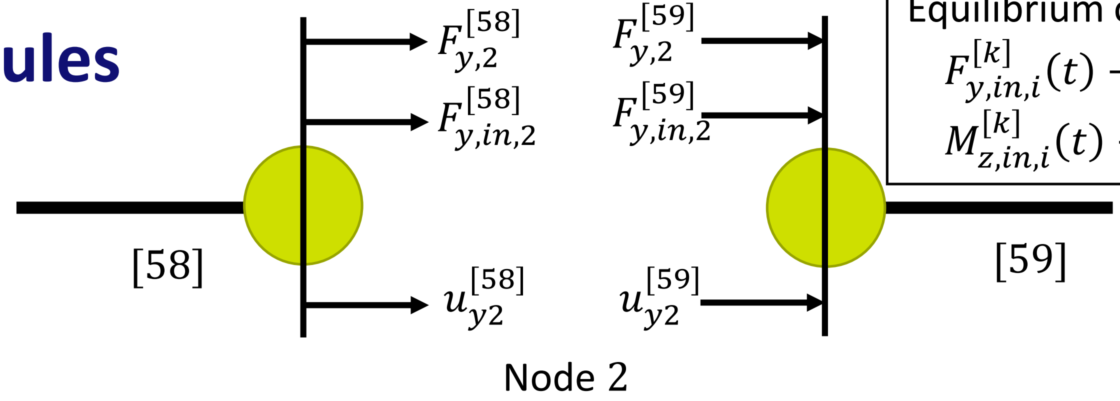
$$u_{yi}^{[k]}(t) = u_{yi}^{[l]}(t),$$
$$\theta_{zi}^{[k]}(t) = \theta_{zi}^{[l]}(t)$$

Node 2

$$\begin{bmatrix} m_{11}^{[58]} & m_{12}^{[58]} & m_{13}^{[58]} & m_{14}^{[58]} \\ m_{21}^{[58]} & m_{22}^{[58]} & m_{23}^{[58]} & m_{24}^{[58]} \\ m_{31}^{[58]} & m_{32}^{[58]} & m_{33}^{[58]} & m_{34}^{[58]} \\ m_{41}^{[58]} & m_{42}^{[58]} & m_{43}^{[58]} & m_{44}^{[58]} \end{bmatrix} \begin{bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \end{bmatrix} + \begin{bmatrix} k_{11}^{[58]} & k_{12}^{[58]} & k_{13}^{[58]} & k_{14}^{[58]} \\ k_{21}^{[58]} & k_{22}^{[58]} & k_{23}^{[58]} & k_{24}^{[58]} \\ k_{31}^{[58]} & k_{32}^{[58]} & k_{33}^{[58]} & k_{34}^{[58]} \\ k_{41}^{[58]} & k_{42}^{[58]} & k_{43}^{[58]} & k_{44}^{[58]} \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{bmatrix} = \begin{bmatrix} F_{y1}^{[58]} \\ M_{z1}^{[58]} \\ F_{y2}^{[58]} \\ M_{z2}^{[58]} \end{bmatrix}$$

$$\begin{bmatrix} m_{11}^{[59]} & m_{12}^{[59]} & m_{13}^{[59]} & m_{14}^{[59]} \\ m_{21}^{[59]} & m_{22}^{[59]} & m_{23}^{[59]} & m_{24}^{[59]} \\ m_{31}^{[59]} & m_{32}^{[59]} & m_{33}^{[59]} & m_{34}^{[59]} \\ m_{41}^{[59]} & m_{42}^{[59]} & m_{43}^{[59]} & m_{44}^{[59]} \end{bmatrix} \begin{bmatrix} \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \\ \ddot{u}_{y3} \\ \ddot{\theta}_{z3} \end{bmatrix} + \begin{bmatrix} k_{11}^{[59]} & k_{12}^{[59]} & k_{13}^{[59]} & k_{14}^{[59]} \\ k_{21}^{[59]} & k_{22}^{[59]} & k_{23}^{[59]} & k_{24}^{[59]} \\ k_{31}^{[59]} & k_{32}^{[59]} & k_{33}^{[59]} & k_{34}^{[59]} \\ k_{41}^{[59]} & k_{42}^{[59]} & k_{43}^{[59]} & k_{44}^{[59]} \end{bmatrix} \begin{bmatrix} u_{y2} \\ \theta_{z2} \\ u_{y3} \\ \theta_{z3} \end{bmatrix} = \begin{bmatrix} F_{y2}^{[59]} \\ M_{z2}^{[59]} \\ F_{y3}^{[59]} \\ M_{z3}^{[59]} \end{bmatrix}$$

Assembly rules



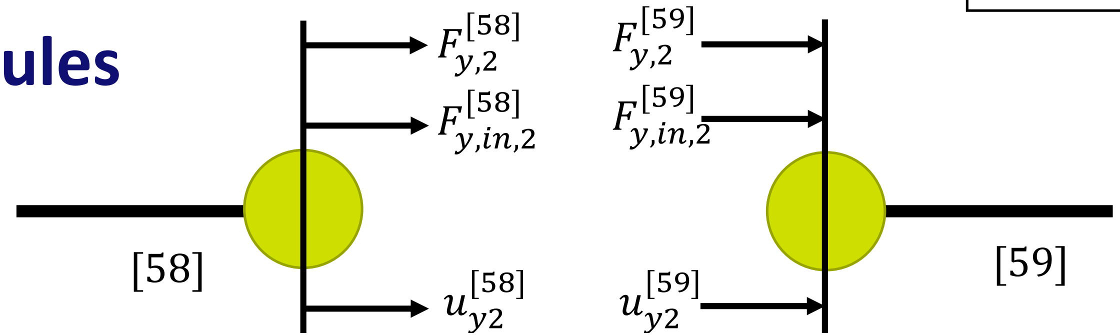
Rule 2:
Equilibrium of internal forces

$$F_{y,in,i}^{[k]}(t) + F_{y,in,i}^{[l]}(t) = 0,$$
$$M_{z,in,i}^{[k]}(t) + M_{z,in,i}^{[l]}(t) = 0$$

$$\begin{aligned}
F_{y2} &= F_{y2}^{[58]} + F_{y2}^{[59]} \\
&= m_{31}^{[58]}\ddot{u}_{y1} + m_{32}^{[58]}\ddot{\theta}_{z1} + m_{33}^{[58]}\ddot{u}_{y2} + m_{34}^{[58]}\ddot{\theta}_{z2} + k_{31}^{[58]}u_{y1} + k_{32}^{[58]}\theta_{z1} + k_{33}^{[58]}u_{y2} + k_{34}^{[58]}\theta_{z2} \\
&+ m_{11}^{[59]}\ddot{u}_{y2} + m_{12}^{[59]}\ddot{\theta}_{z2} + m_{13}^{[59]}\ddot{u}_{y3} + m_{14}^{[59]}\ddot{\theta}_{z3} + k_{11}^{[59]}u_{y2} + k_{12}^{[59]}\theta_{z2} + k_{13}^{[59]}u_{y3} + k_{14}^{[59]}\theta_{z3} \\
&= m_{31}^{[58]}\ddot{u}_{y1} + m_{32}^{[58]}\ddot{\theta}_{z1} + k_{31}^{[58]}u_{y1} + k_{32}^{[58]}\theta_{z1} \\
&+ (m_{33}^{[58]} + m_{11}^{[59]})\ddot{u}_{y2} + (m_{34}^{[58]} + m_{12}^{[59]})\ddot{\theta}_{z2} + (k_{33}^{[58]} + k_{11}^{[59]})u_{y2} + (k_{34}^{[58]} + k_{12}^{[59]})\theta_{z2} \\
&+ m_{13}^{[59]}\ddot{u}_{y3} + m_{14}^{[59]}\ddot{\theta}_{z3} + k_{13}^{[59]}u_{y3} + k_{14}^{[59]}\theta_{z3}
\end{aligned}$$

Write in vector form

Assembly rules

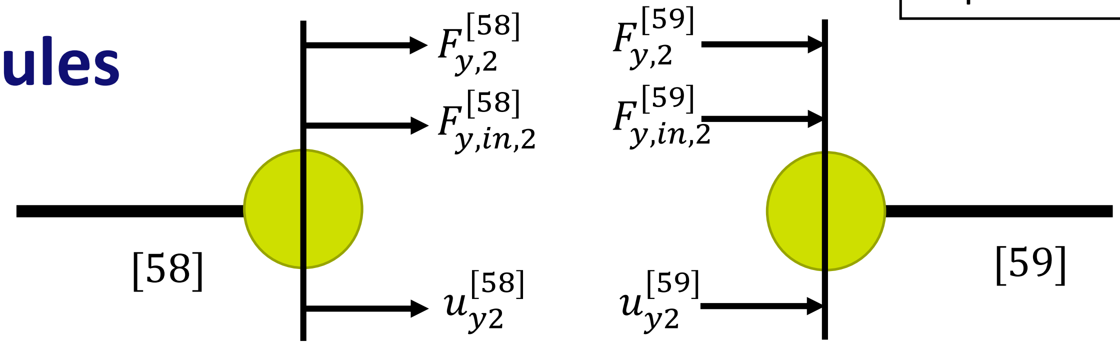


$$F_{y2} = \begin{bmatrix} m_{31}^{[58]} & m_{32}^{[58]} & m_{33}^{[58]} + m_{11}^{[59]} & m_{34}^{[58]} + m_{12}^{[59]} & m_{13}^{[59]} & m_{14}^{[59]} \end{bmatrix} \begin{bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \\ \ddot{u}_{y3} \\ \ddot{\theta}_{z3} \end{bmatrix}$$

$$+ \begin{bmatrix} k_{31}^{[58]} & k_{32}^{[58]} & k_{33}^{[58]} + k_{11}^{[59]} & k_{34}^{[58]} + k_{12}^{[59]} & k_{13}^{[59]} & k_{14}^{[59]} \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \\ u_{y3} \\ \theta_{z3} \end{bmatrix}$$

Assembly rules

Repeat for all external loads



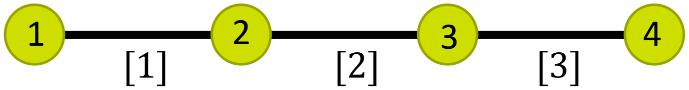
Node 2

$$\begin{bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \end{bmatrix} = \begin{bmatrix} m_{11}^{[58]} & m_{12}^{[58]} & m_{13}^{[58]} & m_{14}^{[58]} & 0 & 0 \\ m_{21}^{[58]} & m_{22}^{[58]} & m_{23}^{[58]} & m_{24}^{[58]} & 0 & 0 \\ m_{31}^{[58]} & m_{32}^{[58]} & m_{33}^{[58]} + m_{11}^{[59]} & m_{34}^{[58]} + m_{12}^{[59]} & m_{13}^{[59]} & m_{14}^{[59]} \\ m_{41}^{[58]} & m_{42}^{[58]} & m_{43}^{[58]} + m_{21}^{[59]} & m_{44}^{[58]} + m_{22}^{[59]} & m_{23}^{[59]} & m_{24}^{[59]} \\ 0 & 0 & m_{31}^{[59]} & m_{32}^{[59]} & m_{33}^{[59]} & m_{34}^{[59]} \\ 0 & 0 & m_{41}^{[59]} & m_{42}^{[59]} & m_{43}^{[59]} & m_{44}^{[59]} \end{bmatrix} \begin{bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \\ \ddot{u}_{y3} \\ \ddot{\theta}_{z3} \end{bmatrix}$$

$$+ \begin{bmatrix} k_{11}^{[58]} & k_{12}^{[58]} & k_{13}^{[58]} & k_{14}^{[58]} & 0 & 0 \\ k_{21}^{[58]} & k_{22}^{[58]} & k_{23}^{[58]} & k_{24}^{[58]} & 0 & 0 \\ k_{31}^{[58]} & k_{32}^{[58]} & k_{33}^{[58]} + k_{11}^{[59]} & k_{34}^{[58]} + k_{12}^{[59]} & k_{13}^{[59]} & k_{14}^{[59]} \\ k_{41}^{[58]} & k_{42}^{[58]} & k_{43}^{[58]} + k_{21}^{[59]} & k_{44}^{[58]} + k_{22}^{[59]} & k_{23}^{[59]} & k_{24}^{[59]} \\ 0 & 0 & k_{31}^{[59]} & k_{32}^{[59]} & k_{33}^{[59]} & k_{34}^{[59]} \\ 0 & 0 & k_{41}^{[59]} & k_{42}^{[59]} & k_{43}^{[59]} & k_{44}^{[59]} \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \\ u_{y3} \\ \theta_{z3} \end{bmatrix}$$

Algorithm: assembly of FE matrices for beams

1. Determine the number of degrees of freedom $n = 2 \times \#nodes$
2. Initialize the global stiffness matrix K and global mass matrix M as $n \times n$ null matrices
3. Initialize external force vector f as a vector with n elements
4. Assemble the global stiffness matrix K:



$$K = \begin{array}{cc} \begin{matrix} u_{y1} & \theta_{z1} & u_{y2} & \theta_{z2} & u_{y3} & \theta_{z3} & u_{y4} & \theta_{z4} \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \\ F_{y4} \\ M_{z4} \end{matrix} \end{array}$$

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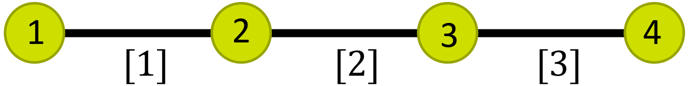
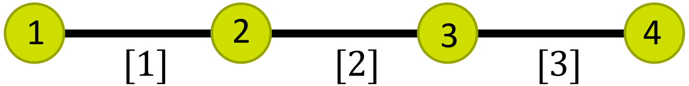


$$K = \begin{array}{c c}
\begin{matrix} \quad u_{y1} & \quad \theta_{z1} & \quad u_{y2} & \quad \theta_{z2} & \quad u_{y3} & \quad \theta_{z3} & \quad u_{y4} & \quad \theta_{z4} \end{matrix} & \\
\begin{bmatrix}
k_{11}^{[1]} & k_{12}^{[1]} & k_{13}^{[1]} & k_{14}^{[1]} & 0 & 0 & 0 & 0 \\
k_{21}^{[1]} & k_{22}^{[1]} & k_{23}^{[1]} & k_{24}^{[1]} & 0 & 0 & 0 & 0 \\
k_{31}^{[1]} & k_{32}^{[1]} & k_{33}^{[1]} & k_{34}^{[1]} & 0 & 0 & 0 & 0 \\
k_{41}^{[1]} & k_{42}^{[1]} & k_{43}^{[1]} & k_{44}^{[1]} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} &
\begin{matrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \\ F_{y4} \\ M_{z4} \end{matrix}
\end{array}$$

Algorithm: assembly of FE matrices for beams

1. Determine the number of degrees of freedom $n = 2 \times \#nodes$
2. Initialize the global stiffness matrix K and global mass matrix M as $n \times n$ null matrices
3. Initialize external force vector f as a vector with n elements
4. Assemble the global stiffness matrix K:



$$K = \begin{array}{c} \begin{array}{cccccccc} u_{y1} & \theta_{z1} & u_{y2} & \theta_{z2} & u_{y3} & \theta_{z3} & u_{y4} & \theta_{z4} \end{array} \\ \begin{bmatrix} k_{11}^{[1]} & k_{12}^{[1]} & k_{13}^{[1]} & k_{14}^{[1]} & 0 & 0 & 0 & 0 \\ k_{21}^{[1]} & k_{22}^{[1]} & k_{23}^{[1]} & k_{24}^{[1]} & 0 & 0 & 0 & 0 \\ k_{31}^{[1]} & k_{32}^{[1]} & k_{33}^{[1]} + k_{11}^{[2]} & k_{34}^{[1]} + k_{12}^{[2]} & k_{13}^{[2]} & k_{14}^{[2]} & 0 & 0 \\ k_{41}^{[1]} & k_{42}^{[1]} & k_{43}^{[1]} + k_{21}^{[2]} & k_{44}^{[1]} + k_{22}^{[2]} & k_{23}^{[2]} & k_{24}^{[2]} & 0 & 0 \\ 0 & 0 & k_{31}^{[2]} & k_{32}^{[2]} & k_{33}^{[2]} & k_{34}^{[2]} & 0 & 0 \\ 0 & 0 & k_{41}^{[2]} & k_{42}^{[2]} & k_{43}^{[2]} & k_{44}^{[2]} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \\ F_{y4} \\ M_{z4} \end{array}$$

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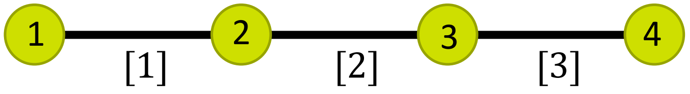
Algorithm: assembly of FE matrices for beams

1. Determine the number of degrees of freedom $n = 2 \times \#nodes$
2. Initialize the global stiffness matrix K and global mass matrix M as $n \times n$ null matrices
3. Initialize external force vector f as a vector with n elements
4. Assemble the global stiffness matrix K:



$K^{[1]}$ $K^{[2]}$ $K^{[3]}$

$$K = \begin{array}{c} \begin{array}{cccccccc} u_{y1} & \theta_{z1} & u_{y2} & \theta_{z2} & u_{y3} & \theta_{z3} & u_{y4} & \theta_{z4} \end{array} \\ \begin{bmatrix} k_{11}^{[1]} & k_{12}^{[1]} & k_{13}^{[1]} & k_{14}^{[1]} & 0 & 0 & 0 & 0 \\ k_{21}^{[1]} & k_{22}^{[1]} & k_{23}^{[1]} & k_{24}^{[1]} & 0 & 0 & 0 & 0 \\ k_{31}^{[1]} & k_{32}^{[1]} & k_{33}^{[1]} + k_{11}^{[2]} & k_{34}^{[1]} + k_{12}^{[2]} & k_{13}^{[2]} & k_{14}^{[2]} & 0 & 0 \\ k_{41}^{[1]} & k_{42}^{[1]} & k_{43}^{[1]} + k_{21}^{[2]} & k_{44}^{[1]} + k_{22}^{[2]} & k_{23}^{[2]} & k_{24}^{[2]} & 0 & 0 \\ 0 & 0 & k_{31}^{[2]} & k_{32}^{[2]} & k_{33}^{[2]} + k_{11}^{[3]} & k_{34}^{[2]} + k_{12}^{[3]} & k_{13}^{[3]} & k_{14}^{[3]} \\ 0 & 0 & k_{41}^{[2]} & k_{42}^{[2]} & k_{43}^{[2]} + k_{21}^{[3]} & k_{44}^{[2]} + k_{22}^{[3]} & k_{23}^{[3]} & k_{24}^{[3]} \\ 0 & 0 & 0 & 0 & k_{31}^{[3]} & k_{32}^{[3]} & k_{33}^{[3]} & k_{34}^{[3]} \\ 0 & 0 & 0 & 0 & k_{41}^{[3]} & k_{42}^{[3]} & k_{43}^{[3]} & k_{44}^{[3]} \end{bmatrix} \end{array} \begin{array}{c} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \\ F_{y4} \\ M_{z4} \end{array}$$

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Algorithm: assembly of FE matrices for beams

1. Determine the number of degrees of freedom $n = 2 \times \#nodes$
2. Initialize the global stiffness matrix K and global mass matrix M as $n \times n$ null matrices
3. Initialize external force vector f as a vector with n elements
4. Assemble the global stiffness matrix K.
5. Assemble the global mass matrix M:

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} u_{y1} & \theta_{z1} & u_{y2} & \theta_{z2} & u_{y3} & \theta_{z3} & u_{y4} & \theta_{z4} \end{array} \\
M = & \left[\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{c} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \\ F_{y4} \\ M_{z4} \end{array}
\end{array}
$$

Algorithm: assembly of FE matrices for beams

1. Determine the number of degrees of freedom $n = 2 \times \#nodes$
2. Initialize the global stiffness matrix K and global mass matrix M as $n \times n$ null matrices
3. Initialize external force vector f as a vector with n elements
4. Assemble the global stiffness matrix K.
5. Assemble the global mass matrix M:

$M^{[1]}$ $M^{[2]}$ $M^{[3]}$

$$M = \begin{array}{c} \\ \\ \end{array} \begin{array}{cccccccc|c} u_{y1} & \theta_{z1} & u_{y2} & \theta_{z2} & u_{y3} & \theta_{z3} & u_{y4} & \theta_{z4} & \end{array}$$

$$M = \begin{array}{c} \left[\begin{array}{cccccccc} m^{[1]}_{11} & m^{[1]}_{12} & m^{[1]}_{13} & m^{[1]}_{14} & 0 & 0 & 0 & 0 \\ m^{[1]}_{21} & m^{[1]}_{22} & m^{[1]}_{23} & m^{[1]}_{24} & 0 & 0 & 0 & 0 \\ m^{[1]}_{31} & m^{[1]}_{32} & m^{[1]}_{33} + m^{[2]}_{11} & m^{[1]}_{34} + m^{[2]}_{12} & m^{[2]}_{13} & m^{[2]}_{14} & 0 & 0 \\ m^{[1]}_{41} & m^{[1]}_{42} & m^{[1]}_{43} + m^{[2]}_{21} & m^{[1]}_{44} + m^{[2]}_{22} & m^{[2]}_{23} & m^{[2]}_{24} & 0 & 0 \\ 0 & 0 & m^{[2]}_{31} & m^{[2]}_{32} & m^{[2]}_{33} + m^{[3]}_{11} & m^{[2]}_{34} + m^{[3]}_{12} & m^{[3]}_{13} & m^{[3]}_{14} \\ 0 & 0 & m^{[2]}_{41} & m^{[2]}_{42} & m^{[2]}_{43} + m^{[3]}_{21} & m^{[2]}_{44} + m^{[3]}_{22} & m^{[3]}_{23} & m^{[3]}_{24} \\ 0 & 0 & 0 & 0 & m^{[3]}_{31} & m^{[3]}_{32} & m^{[3]}_{33} & m^{[3]}_{34} \\ 0 & 0 & 0 & 0 & m^{[3]}_{41} & m^{[3]}_{42} & m^{[3]}_{43} & m^{[3]}_{44} \end{array}\right] \end{array} \begin{array}{c} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \\ F_{y4} \\ M_{z4} \end{array}$$

Algorithm: assembly of FE matrices for beams

1. Determine the number of degrees of freedom $n = 2 \times \#nodes$
2. Initialize the global stiffness matrix K and global mass matrix M as $n \times n$ null matrices
3. Initialize external force vector f as a vector with n elements
4. Assemble the global stiffness matrix K.
5. Assemble the global mass matrix M.
6. Assemble the global force vector f:

$$f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \\ F_{y4} \\ M_{z4} \end{array}$$

Algorithm: assembly of FE matrices for beams

1. Determine the number of degrees of freedom $n = 2 \times \#nodes$
2. Initialize the global stiffness matrix K and global mass matrix M as $n \times n$ null matrices
3. Initialize external force vector f as a vector with n elements
4. Assemble the global stiffness matrix K.
5. Assemble the global mass matrix M.
6. Assemble the global force vector f:

$$f = \begin{array}{cc} \begin{bmatrix} F_{y1}^{[1]} \\ M_{z1}^{[1]} \\ F_{y2}^{[1]} + F_{y2}^{[2]} \\ M_{z2}^{[1]} + M_{z2}^{[2]} \\ F_{y3}^{[2]} + F_{y3}^{[3]} \\ F_{y3}^{[2]} + M_{z3}^{[3]} \\ F_{y4}^{[3]} \\ M_{z4}^{[3]} \end{bmatrix} & \begin{matrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \\ F_{y4} \\ M_{z4} \end{matrix} \end{array}$$

$f^{[1]}$ $f^{[2]}$ $f^{[3]}$

Remarks

- M and K are symmetric
- M and K are sparse
- K is singular (i.e. there is a rigid body mode)

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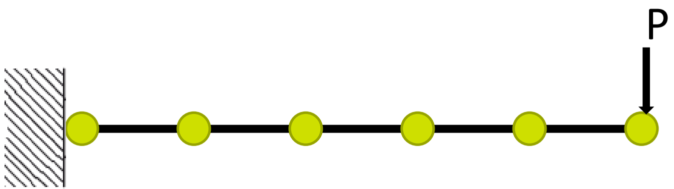
Boundary conditions

8. Merge elements

9. Apply kinematic and dynamic boundary conditions

a. Kinematic bc's: supports

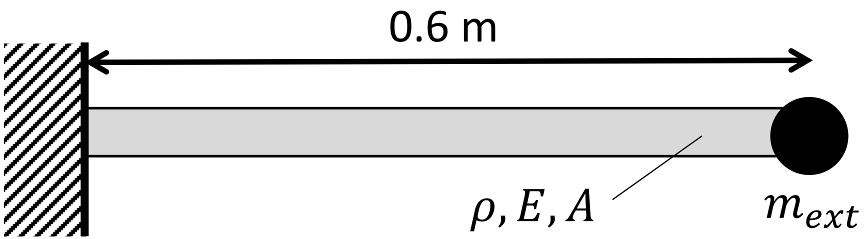
b. Dynamic bc's: external loads

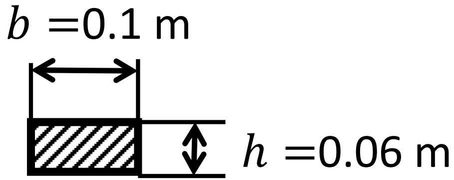


Example 1: Clamped beam with end mass

Consider the clamped aluminum beam with a point mass m_{ext} at end of the beam.

Obtain the finite element model of the system using three Euler beam elements.



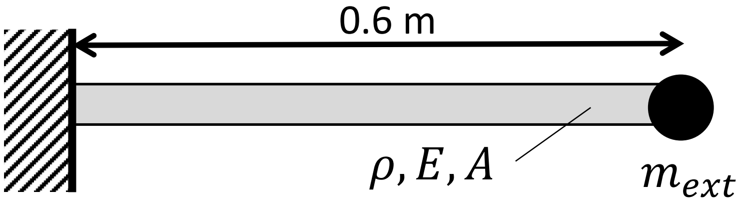


Properties of aluminum: Young's modulus E =69 GPa, Mass density $\rho = 2700$ kg/m^3

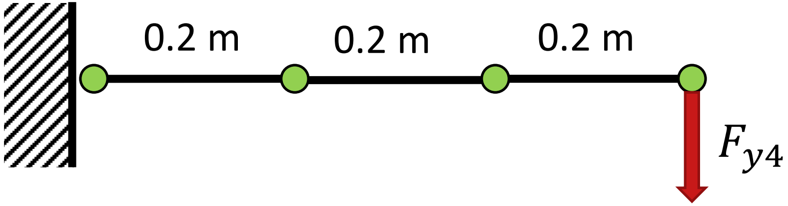
Using the given cross section, the second moment of area can be found

$$I = \frac{1}{12} b h^3 = 1.8 \times 10^{-6}\ \mathrm{m}^4$$

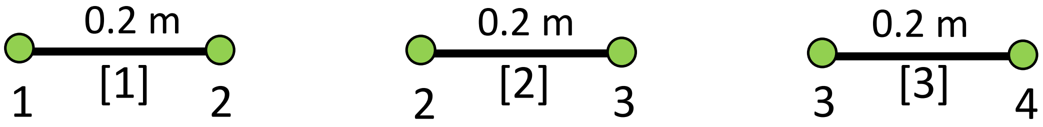
Example 1: Clamped beam with end mass



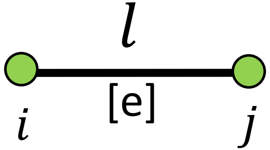
Create FE model



Disassemble



Example 1: Clamped beam with end mass

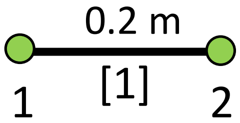


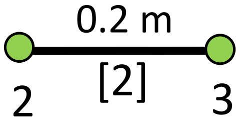
Element matrices

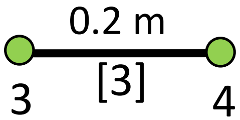
Generic element matrices

$$M^e = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{sym.} & & & 4l^2 \end{bmatrix}, \qquad K^e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym.} & & & 4l^2 \end{bmatrix}$$

Specific element matrices





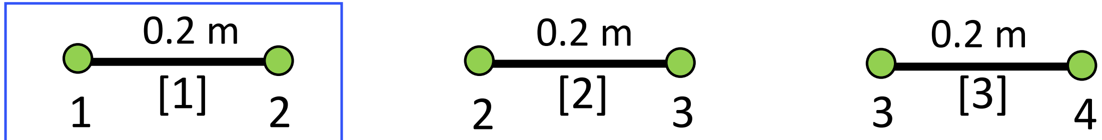


$$M^{[1]} = 7.7 \cdot 10^{-3}\begin{bmatrix} 156 & 4.4 & 54 & -2.6 \\ 4.4 & 0.16 & 2.6 & -0.12 \\ 54 & 2.6 & 156 & -4.4 \\ -2.6 & -0.12 & -4.4 & 0.16 \end{bmatrix}, \qquad K^{[1]} = 1.6 \cdot 10^{-7}\begin{bmatrix} 12 & 1.2 & -12 & 1.2 \\ 1.2 & 0.16 & -1.2 & 0.08 \\ -12 & -1.2 & 12 & -1.2 \\ 1.2 & 0.08 & -1.2 & 0.16 \end{bmatrix}$$

$$M^{[2]} = M^{[3]} = M^{[1]}, K^{[2]} = K^{[3]} = K^{[1]}.$$

Example 1: Clamped beam with end mass

Assembly

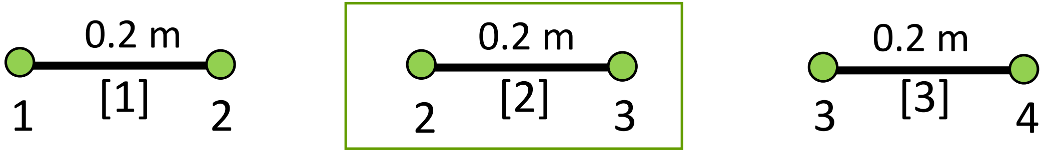


$$\begin{bmatrix} F_{y1}^{[1]} \\ M_{z1}^{[1]} \\ F_{y2}^{[1]} \\ M_{z2}^{[1]} \\ F_{y3}^{[1]} \\ M_{z3}^{[1]} \\ F_{y4}^{[1]} \\ M_{z4}^{[1]} \end{bmatrix} = 7.7143 \times 10^{-3} \begin{bmatrix} 156 & 4.4 & 54 & -2.6 & 0 & 0 & 0 & 0 \\ 4.4 & 0.16 & 2.6 & -0.12 & 0 & 0 & 0 & 0 \\ 54 & 2.6 & 156 & -4.4 & 0 & 0 & 0 & 0 \\ -2.6 & -0.12 & -4.4 & 0.16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u}_{y1}^{[1]} \\ \ddot{\theta}_{z1}^{[1]} \\ \ddot{u}_{y2}^{[1]} \\ \ddot{\theta}_{z2}^{[1]} \\ \ddot{u}_{y3}^{[1]} \\ \ddot{\theta}_{z3}^{[1]} \\ \ddot{u}_{y4}^{[1]} \\ \ddot{\theta}_{z4}^{[1]} \end{bmatrix}$$

$$+ 1.5525 \times 10^{-7} \begin{bmatrix} 12 & 1.2 & -12 & 1.2 & 0 & 0 & 0 & 0 \\ 1.2 & 0.16 & -1.2 & 0.08 & 0 & 0 & 0 & 0 \\ -12 & -1.2 & 12 & -1.2 & 0 & 0 & 0 & 0 \\ 1.2 & 0.08 & -1.2 & 0.16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{y1}^{[1]} \\ \theta_{z1}^{[1]} \\ u_{y2}^{[1]} \\ \theta_{z2}^{[1]} \\ u_{y3}^{[1]} \\ \theta_{z3}^{[1]} \\ u_{y4}^{[1]} \\ \theta_{z4}^{[1]} \end{bmatrix}$$

Example 1: Clamped beam with end mass

Assembly

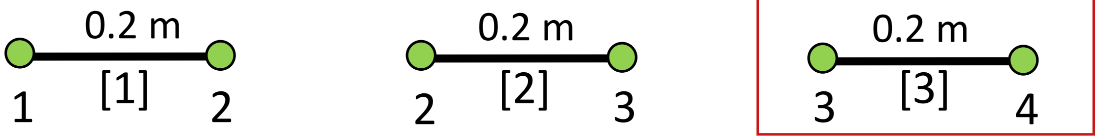


$$
\begin{bmatrix} F_{y1}^{[2]} \\ M_{z1}^{[2]} \\ F_{y2}^{[2]} \\ M_{z2}^{[2]} \\ F_{y3}^{[2]} \\ M_{z3}^{[2]} \\ F_{y4}^{[2]} \\ M_{z4}^{[2]} \end{bmatrix}
= 7.7143 \times 10^{-3}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 156 & 4.4 & 54 & -2.6 & 0 & 0 \\
0 & 0 & 4.4 & 0.16 & 2.6 & -0.12 & 0 & 0 \\
0 & 0 & 54 & 2.6 & 156 & -4.4 & 0 & 0 \\
0 & 0 & -2.6 & -0.12 & -4.4 & 0.16 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \ddot{u}_{y1}^{[2]} \\ \ddot{\theta}_{z1}^{[2]} \\ \ddot{u}_{y2}^{[2]} \\ \ddot{\theta}_{z2}^{[2]} \\ \ddot{u}_{y3}^{[2]} \\ \ddot{\theta}_{z3}^{[2]} \\ \ddot{u}_{y4}^{[2]} \\ \ddot{\theta}_{z4}^{[2]} \end{bmatrix}
$$

$$
+ 1.5525 \times 10^{-7}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 12 & 1.2 & -12 & 1.2 & 0 & 0 \\
0 & 0 & 1.2 & 0.16 & -1.2 & 0.08 & 0 & 0 \\
0 & 0 & -12 & -1.2 & 12 & -1.2 & 0 & 0 \\
0 & 0 & 1.2 & 0.08 & -1.2 & 0.16 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u_{y1}^{[2]} \\ \theta_{z1}^{[2]} \\ u_{y2}^{[2]} \\ \theta_{z2}^{[2]} \\ u_{y3}^{[2]} \\ \theta_{z3}^{[2]} \\ u_{y4}^{[2]} \\ \theta_{z4}^{[2]} \end{bmatrix}
$$

Example 1: Clamped beam with end mass

Assembly

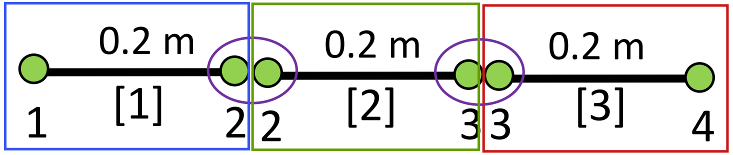


$$
\begin{bmatrix} F_{y1}^{[3]} \\ M_{z1}^{[3]} \\ F_{y2}^{[3]} \\ M_{z2}^{[3]} \\ F_{y3}^{[3]} \\ M_{z3}^{[3]} \\ F_{y4}^{[3]} \\ M_{z4}^{[3]} \end{bmatrix}
= 7.7143 \times 10^{-3}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 156 & 4.4 & 54 & -2.6 \\
0 & 0 & 0 & 0 & 4.4 & 0.16 & 2.6 & -0.12 \\
0 & 0 & 0 & 0 & 54 & 2.6 & 156 & -4.4 \\
0 & 0 & 0 & 0 & -2.6 & -0.12 & -4.4 & 0.16
\end{bmatrix}
\begin{bmatrix} \ddot{u}_{y1}^{[3]} \\ \ddot{\theta}_{z1}^{[3]} \\ \ddot{u}_{y2}^{[3]} \\ \ddot{\theta}_{z2}^{[3]} \\ \ddot{u}_{y3}^{[3]} \\ \ddot{\theta}_{z3}^{[3]} \\ \ddot{u}_{y4}^{[3]} \\ \ddot{\theta}_{z4}^{[3]} \end{bmatrix}
$$

$$
+ 1.5525 \times 10^{-7}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 12 & 1.2 & -12 & 1.2 \\
0 & 0 & 0 & 0 & 1.2 & 0.16 & -1.2 & 0.08 \\
0 & 0 & 0 & 0 & -12 & -1.2 & 12 & -1.2 \\
0 & 0 & 0 & 0 & 1.2 & 0.08 & -1.2 & 0.16
\end{bmatrix}
\begin{bmatrix} u_{y1}^{[3]} \\ \theta_{z1}^{[3]} \\ u_{y2}^{[3]} \\ \theta_{z2}^{[3]} \\ u_{y3}^{[3]} \\ \theta_{z3}^{[3]} \\ u_{y4}^{[3]} \\ \theta_{z4}^{[3]} \end{bmatrix}
$$

Example 1: Clamped beam with end mass

<u>Assembly</u>

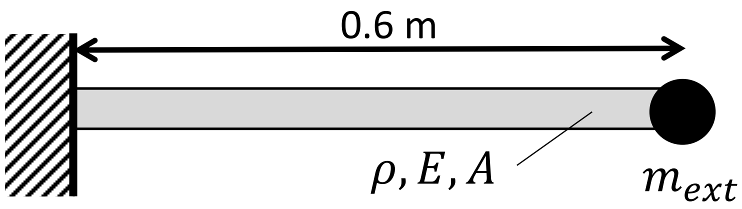


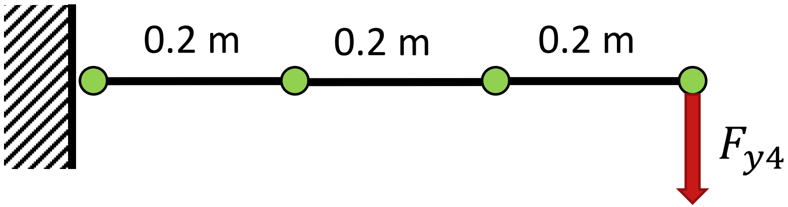
$$\begin{bmatrix} F_{y1}^{[1]} + \cancel{F_{y1}^{[2]}} + \cancel{F_{y1}^{[3]}} \\ M_{z1}^{[1]} + \cancel{M_{z1}^{[2]}} + \cancel{M_{z1}^{[3]}} \\ F_{y2}^{[1]} + F_{y2}^{[2]} + \cancel{F_{y2}^{[3]}} \\ M_{z2}^{[1]} + M_{z2}^{[2]} + \cancel{M_{z2}^{[3]}} \\ \cancel{F_{y3}^{[1]}} + F_{y3}^{[2]} + F_{y3}^{[3]} \\ \cancel{M_{z3}^{[1]}} + M_{z3}^{[2]} + M_{z3}^{[3]} \\ \cancel{F_{y4}^{[1]}} + \cancel{F_{y4}^{[2]}} + F_{y4}^{[3]} \\ \cancel{M_{z4}^{[1]}} + \cancel{M_{z4}^{[2]}} + M_{z4}^{[3]} \end{bmatrix} = 7.7143\times10^{-3} \begin{bmatrix} 156 & 4.4 & 54 & -2.6 & 0 & 0 & 0 & 0 \\ 4.4 & 0.16 & 2.6 & -0.12 & 0 & 0 & 0 & 0 \\ 54 & 2.6 & 312 & 0 & 54 & -2.6 & 0 & 0 \\ -2.6 & -0.12 & 0 & 0.32 & 2.6 & -0.12 & 0 & 0 \\ 0 & 0 & 54 & 2.6 & 312 & 0 & 54 & -2.6 \\ 0 & 0 & -2.6 & -0.12 & 0 & 0.32 & 2.6 & -0.12 \\ 0 & 0 & 0 & 0 & 54 & 2.6 & 156 & -4.4 \\ 0 & 0 & 0 & 0 & -2.6 & -0.12 & -4.4 & 0.16 \end{bmatrix} \begin{bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \\ \ddot{u}_{y3} \\ \ddot{\theta}_{z3} \\ \ddot{u}_{y4} \\ \ddot{\theta}_{z4} \end{bmatrix}$$

$$+ 1.5525 \times 10^{-7} \begin{bmatrix} 12 & 1.2 & -12 & 1.2 & 0 & 0 & 0 & 0 \\ 1.2 & 0.16 & -1.2 & 0.08 & 0 & 0 & 0 & 0 \\ -12 & -1.2 & 24 & 0 & -12 & 1.2 & 0 & 0 \\ 1.2 & 0.08 & 0 & 0.32 & -1.2 & 0.08 & 0 & 0 \\ 0 & 0 & -12 & -1.2 & 24 & 0 & -12 & 1.2 \\ 0 & 0 & 1.2 & 0.08 & 0 & 0.32 & -1.2 & 0.08 \\ 0 & 0 & 0 & 0 & -12 & -1.2 & 12 & -1.2 \\ 0 & 0 & 0 & 0 & 1.2 & 0.08 & -1.2 & 0.16 \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \\ u_{y3} \\ \theta_{z3} \\ u_{y4} \\ \theta_{z4} \end{bmatrix}$$

Example 1: Clamped beam with end mass

Boundary conditions





Kinematic boundary conditions:

$$u_{y1} = 0, \qquad \theta_{z1} = 0.$$

Dynamic boundary conditions:

$$F_{y2} = F_{y2}^{[1]} + F_{y2}^{[2]} = 0, \qquad M_{z2} = M_{z2}^{[1]} + M_{z2}^{[2]} = 0,$$
$$F_{y3} = F_{y3}^{[2]} + F_{y3}^{[3]} = 0, \qquad M_{z3} = M_{z3}^{[2]} + M_{z3}^{[3]} = 0,$$
$$F_{y4} = F_{y4}^{[3]} = -m_{ext}\,\ddot{u}_{y4}, \qquad M_{z4} = M_{z4}^{[3]} = 0.$$

Example 1: Clamped beam with end mass

The condensed matrices are obtained by deleting the rows and columns of the kinematic BCs

Condensed Mass Matrix

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 7.7143 \times 10^{-3} \begin{bmatrix} 312 & 0 & 54 & -2.6 & 0 & 0 \\ 0 & 0.32 & 2.6 & -0.12 & 0 & 0 \\ 54 & 2.6 & 312 & 0 & 54 & -2.6 \\ -2.6 & -0.12 & 0 & 0.32 & 2.6 & -0.12 \\ 0 & 0 & 54 & 2.6 & 156 + \frac{m_{ext}}{7.7143 \times 10^{-3}} & -4.4 \\ 0 & 0 & -2.6 & -0.12 & -4.4 & 0.16 \end{bmatrix} \begin{bmatrix} \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \\ \ddot{u}_{y3} \\ \ddot{\theta}_{z3} \\ \ddot{u}_{y4} \\ \ddot{\theta}_{z4} \end{bmatrix}$$

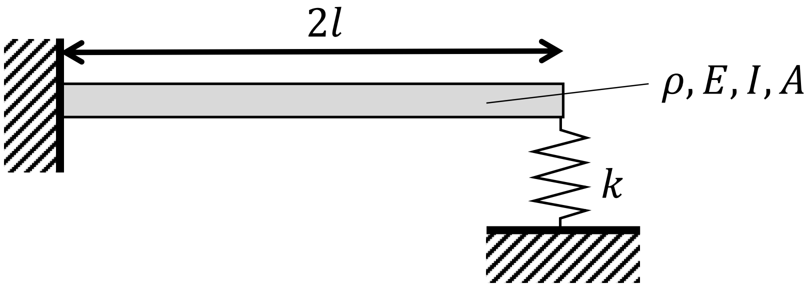
$$+ 1.5525 \times 10^{-7} \begin{bmatrix} 24 & 0 & -12 & 1.2 & 0 & 0 \\ 0 & 0.32 & -1.2 & 0.08 & 0 & 0 \\ -12 & -1.2 & 24 & 0 & -12 & 1.2 \\ 1.2 & 0.08 & 0 & 0.32 & -1.2 & 0.08 \\ 0 & 0 & -12 & -1.2 & 12 & -1.2 \\ 0 & 0 & 1.2 & 0.08 & -1.2 & 0.16 \end{bmatrix} \begin{bmatrix} u_{y2} \\ \theta_{z2} \\ u_{y3} \\ \theta_{z3} \\ u_{y4} \\ \theta_{z4} \end{bmatrix}$$

Now you can use M and K to perform a dynamic response analysis, an eigenvalue analysis, etc.

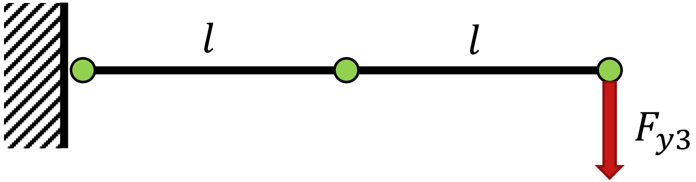
Condensed Stiffness Matrix

Example 2: Clamped beam with flexible support

Consider a clamped beam with Young's modulus E, area of cross section A, second moment of area about the axis of bending I, mass density ρ, length $2l$, and with its right end being supported vertically by a linear elastic spring with stiffness k.

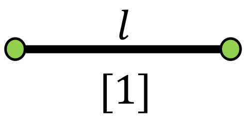


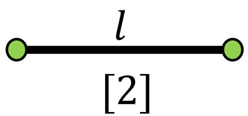
Use two beam elements to obtain the FE model.
Question: Assemble the mass and stiffness matrix of this system.



Example 2: Clamped beam with flexible support

<u>Element matrices</u>





$$M^{[1]} = M^{[2]} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{sym.} & & & 4l^2 \end{bmatrix}, \quad K^{[1]} = K^{[2]} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym.} & & & 4l^2 \end{bmatrix}$$

Example 2: Clamped beam with flexible support

Global matrices

$$M = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l & 0 & 0 \\ & 4l^2 & 13l & -3l^2 & 0 & 0 \\ & & 312 & 0 & 54 & -13l \\ & & & 8l^2 & 13l & -3l^2 \\ & & & & 156 & -22l \\ \text{sym} & & & & & 4l^2 \end{bmatrix}$$

Global mass matrix

$$q = \begin{bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \\ u_{y3} \\ \theta_{z3} \end{bmatrix}$$

Global coordinate vector

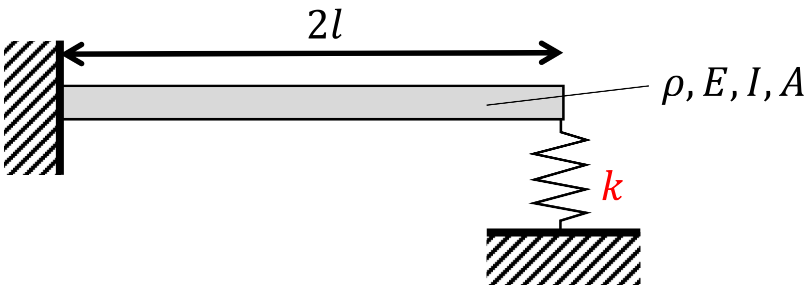
$$K = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ & 4l^2 & -6l & 2l^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6l \\ & & & 8l^2 & -6l & 2l^2 \\ & & & & 12 & -6l \\ \text{sym} & & & & & 4l^2 \end{bmatrix}$$

Global stiffness matrix

$$F = \begin{bmatrix} F_{y1}^{[1]} \\ M_{z1}^{[1]} \\ F_{y2}^{[1]} + F_{y2}^{[2]} \\ M_{z2}^{[1]} + M_{z2}^{[2]} \\ F_{y3}^{[2]} \\ M_{z3}^{[2]} \end{bmatrix}$$

Global force vector

Example 2: Clamped beam with flexible support



Boundary conditions

- $u_{y1} = 0, \theta_{z1} = 0,$
- $F_{y2}^{[1]} + F_{y2}^{[2]} = 0, M_{z2}^{[1]} + M_{z2}^{[2]} = 0,$
- $F_{y3}^{[2]} = -ku_{y3},\ M_{z3}^{[2]} = 0.$

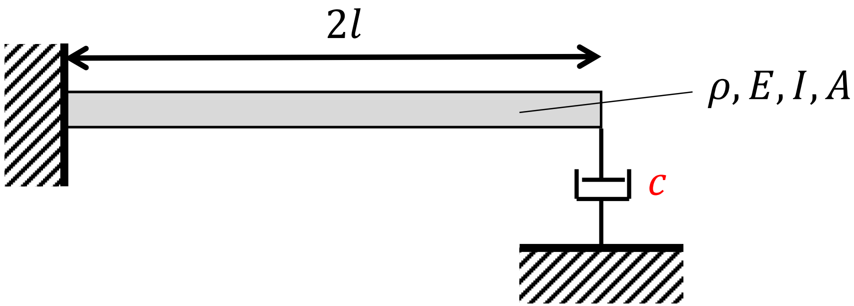
Condensed equations:

$$\frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ & 8l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{sym} & & & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \\ \ddot{u}_{y3} \\ \ddot{\theta}_{z3} \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ & 8l^2 & -6l & 2l^2 \\ & & 12+k^* & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix}\begin{bmatrix} u_{y2} \\ \theta_{z2} \\ u_{y3} \\ \theta_{z3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k^* = \frac{kl^3}{EI}$$

Example 2: Clamped beam with a damper

Boundary conditions



- $u_{y1} = 0, \theta_{z1} = 0,$
- $F_{y2}^{[1]} + F_{y2}^{[2]} = 0, M_{z2}^{[1]} + M_{z2}^{[2]} = 0,$
- $F_{y3}^{[2]} = -c\dot{u}_{y3}, M_{z3}^{[2]} = 0.$

Condensed equations:

$$\frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13l \\ & 8l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{sym} & & & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \\ \ddot{u}_{y3} \\ \ddot{\theta}_{z3} \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ & 8l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix}\begin{bmatrix} u_{y2} \\ \theta_{z2} \\ u_{y3} \\ \theta_{z3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -c\dot{u}_{y3} \\ 0 \end{bmatrix}$$

Summary

1. Breakdown the structure into elements
2. Look up mass and stiffness matrices for a generic element
3. Obtain the element mass and stiffness matrices for individual elements
4. Apply the assembly procedure to find the global mass and stiffness matrices
    - Compatibility of displacements
    - Internal force equilibrium
5. Apply boundary conditions
    - Kinematic boundary conditions
    - Dynamic boundary conditions
6. Use the assembled stiffness and mass system matrices
   e.g. to perform a dynamic response or eigenvalue analysis.