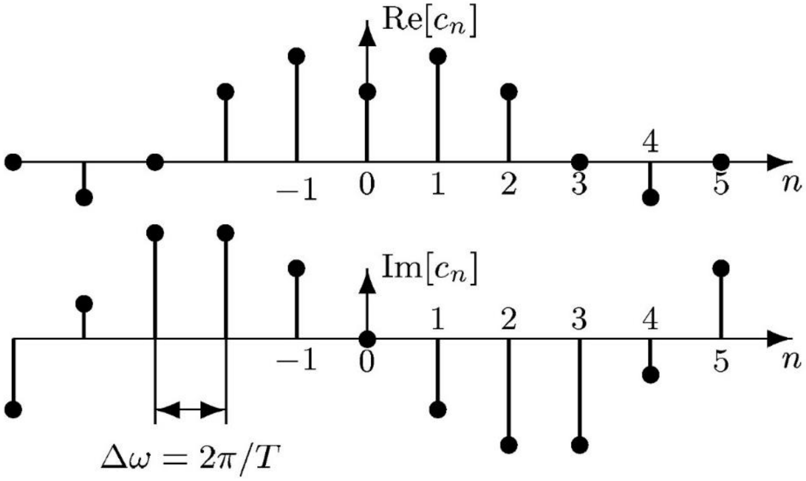

7. Fourier transformation

Structural Dynamics part of 4DM00

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Fourier transformation

The Fourier transformation is

- a unique transformation from time domain to frequency domain
- a decomposition of a signal in harmonic components (frequency, amplitude, and phase)

First: complex periodic signals $x(t) = x(t+T)$

T is the minimal period

Complex form of **Fourier-series:**

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{(2\pi f_n t)j}, \qquad c_n = \frac{1}{T}\int_0^T x(t)\, e^{-(2\pi f_n t)j} dt, \qquad n \in \mathbb{Z}$$

with $e^{(2\pi f_n t)j} = \cos(2\pi f_n t) + j\sin(2\pi f_n t)$ and $f_n = n/T$.

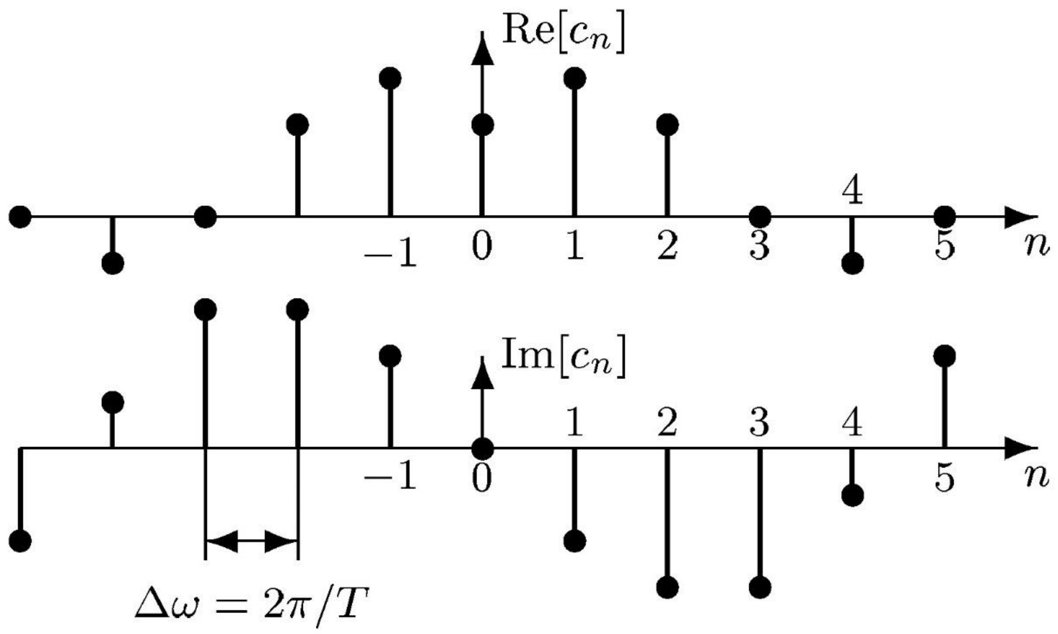
Note: if $x(t)$ real, $c_{-n} = \bar{c}_n$.

$x(t)$ and c_n have the same unit.

Frequency spectrum

periodic signal
$x(t) = x(t + T)$
$\Rightarrow$
discrete frequency spectrum ($\Delta f = 1/T$)

real signal $x(t)$
$\Rightarrow$
$c_{-n} = \overline{c_n}$



Fourier transformation

In practice non-periodic signals (e.g. impulse response, white noise)
Fourier integral needed $T \rightarrow \infty$

Compute **Fourier transform** $X(f)$ of non-periodic $x(t)$ by the **Fourier integral**

$$X(f) = F[x(t)] = \int_{t=-\infty}^{\infty} x(t)\, e^{-(2\pi f t) j} dt$$

Note: unit of $X(f)$ is $[x]$/Hz $\Rightarrow$ $X(f)$ is a spectral density.

Inverse Fourier transform:

$$x(t) = F^{-1}[X(f)] = \int_{f=-\infty}^{\infty} X(f)\, e^{(2\pi f t) j} df$$

Dirichlet condition

Fourier integral exists if $x(t)$ fulfils Dirichlet condition.

$\boxed{\underline{\textbf{Dirichlet condition:}}\ \int_{t=-\infty}^{\infty} |x(t)|\, dt \text{ exists and } x(t) \text{ piecewise smooth}}$

For signals which do not fulfil Dirichlet condition: use *generalised* Fourier integral

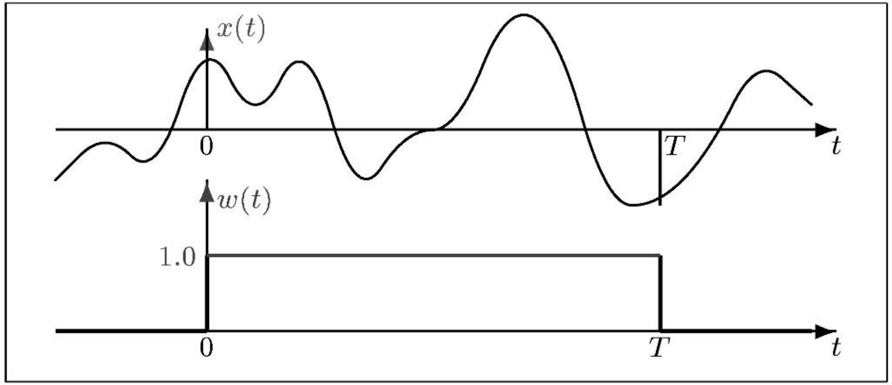
In practice $x(t)$ is only known in a time window.

Only $x_B(t) \coloneqq w(t)x(t)$ is available, with

$$w(t) = \begin{cases} 1 & \text{if} \quad t \in [0, T) \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Note: $x_B(t)$ fulfills Dirichlet condition and

$$\lim_{T\to\infty} X_B(f) = X(f)$$



Fourier transformation tables

General rules

time domain	frequency domain	Notes
$x(t) + y(t)$	$X(f) + Y(f)$	
$\alpha x(t)$	$\alpha X(f)$	
$x(\alpha t),$	$X(f/\alpha)/\lvert\alpha\rvert$	$\alpha \in \mathbb{R}\backslash 0$
$x(t - t_0)$	$X(f)e^{-2\pi j f t_0}$	$t_0 \in \mathbb{R}$
$x(t)e^{2\pi j f_0 t}$	$X(f - f_0)$	$f_0 \in \mathbb{R}$
$\dot{x}(t)$	$2\pi j f X(f)$	
$t x(t)$	$(j/2\pi)\, dX(f)/df$	
$\bar{x}(t)$	$X(-f)$	

Selected signals

time domain	frequency domain	
$\delta(t - t_0)$	$e^{-2\pi j f t_0}$	Dirac function
$e^{-2\pi j f_0 t}$	$\delta(f + f_0)$	
$\cos(2\pi f_0 t)$	$\dfrac{\delta(f - f_0) + \delta(f + f_0)}{2}$	
$\sin(2\pi f_0 t)$	$-j\dfrac{\delta(f - f_0) - \delta(f + f_0)}{2}$	
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\sum_{n=-\infty}^{\infty} \delta(f - n/T)$	pulse train

Example: Fourier transform of $w(t)$

$$w(t) = \begin{cases} 1 & \text{if} \quad t \in [0, T) \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Compute $W(f) = F[w(t)]$

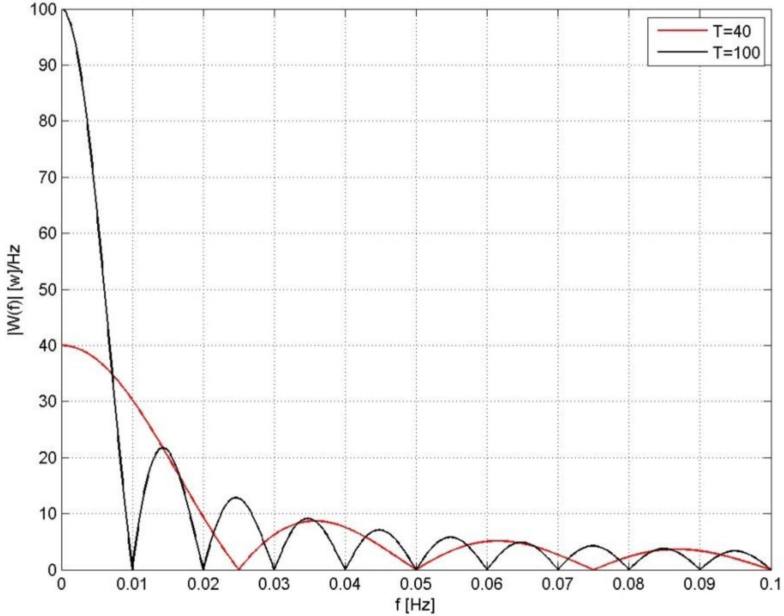
$$W(f) = \int_{-\infty}^{\infty} w(t) e^{-2\pi j f t} dt$$

$$= \int_0^T (\cos(2\pi f t) - j\sin(2\pi f t))dt$$

$$= \frac{1}{2\pi f}(\sin(2\pi f T) + j\cos(2\pi f T) - j)$$

$$= T\frac{\sin(\pi f T)}{\pi f T} e^{-\pi j f T}$$

Red line $T = 40$, Black line $T = 100$



$W(f) \rightarrow \delta(f)$ **for** $T \rightarrow \infty$.

Convolution

Convolution in time domain

$$x(t) = x_1(t) \otimes x_2(t) = \int_{\tau=-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

becomes in the frequency domain

$$X(f) = \int_{t=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \right] e^{-(2\pi jft)}dt$$

Introduce $\xi = t - \tau$ (note that $d\xi = dt$) and note that$e^{-2\pi jft} = e^{2\pi jf\tau}e^{2\pi jf\xi}$

$$X(f) = \int_{\tau=-\infty}^{\infty} x_1(\tau)e^{-(2\pi jf\tau)}d\tau \int_{\xi=-\infty}^{\infty} x_2(\xi)e^{-(2\pi jf\xi)}d\xi = X_1(f)X_2(f)$$

Convolution in time domain is equivalent to **multiplication** in frequency domain.

Convolution

time domain	frequency domain
$x_1(t)x_2(t)$	$X_1(f) \otimes X_2(f)$
$x_1(t) \otimes x_2(t)$	$X_1(f)X_2(f)$

Do not calculate $x(t)$ via convolution in the time domain!

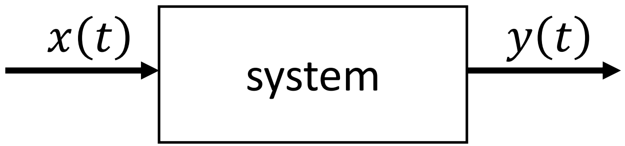
1. transform $x_1(t), x_2(t)$ to frequency domain: $X_1(f), X_2(f)$
2. apply simple multiplication $X(f) = X_1(f)X_2(f)$
3. transform $X(f)$ back to time domain: $x(t) = F^{-1}[X(f)]$

Availability of Fast Fourier Transformation algorithm essential (will be treated later on)

Impulse response and FRF

Consider linear time-invariant (LTI) system with

- input $x(t)$
- output $y(t)$



For example, a 1-dof mass-spring-damper system.

- The **impulse response function** $h(t)$ is the output $y(t)$ resulting from the input $x(t) = \delta(t)$. In other words, if $x(t) = \delta(t)$ (Dirac) then $y(t) = h(t)$.
- For general inputs $x(t)$, the output $y(t)$ is the convolution of $h(t)$ with $x(t)$

$$y(t) = h(t) \otimes x(t).$$

  Note that special case $x(t) = \delta(t)$ indeed results in $y(\mathrm{t}) = h(t) \otimes \delta(t) = h(t)$.
- Transform to the frequency domain:

$$Y(f) = H(f)X(f)$$

$\boxed{\text{The Fourier transform of the impulse response } h(t) \text{ is the FRF } H(f)}$

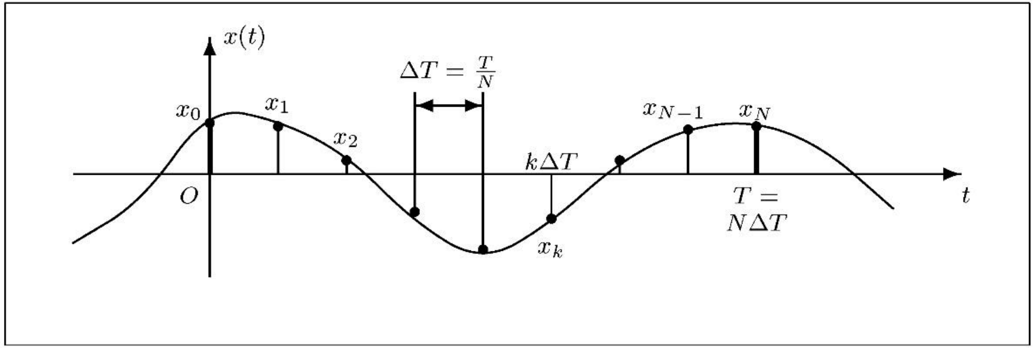
TU/e

7b. Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT)

Recall notation:

- $x(t)$ is the continuous time signal for $-\infty < t < \infty$
- $x_B(t)$ is the windowed signal that is equal to $x(t)$ for $0 \leq t \leq T$ and zero otherwise



Digitization of signal with N (equidistant) **samples**:

$$x_B(t_k) = x_B\ (k\Delta T), \qquad \Delta T = \frac{T}{N}, \qquad k = 0,1,\dots,N-1.$$

Discrete Fourier Transform (DFT)

Discrete approximation of Fourier integral (unit: $[x]$/Hz)

$$X_B(f) = \int_{t=0}^{T} x_B(t) e^{-(2\pi f t)j} dt \approx \sum_{k=0}^{N-1} x_B(k\Delta T) e^{-2\pi j f k \Delta T} \Delta T$$

Observe: the windowed part $x_B(t)$ can be viewed as 1 period of **periodic** signal

$$x_p(t) = \sum_{l=-\infty}^{\infty} x_B(t - lT)$$

Fundamental frequency of $x_B(t)$ is $f_1 = 1/T$.

Evaluate $X_B(f)$ therefore only at discrete values $f_n = n f_1 = \frac{n}{T},\ n \in \mathbb{Z}$.

Discrete Fourier Transform (DFT, unit: $[x]$/Hz):

$$X_B(f_n) = \Delta T \sum_{k=0}^{N-1} x_B(k\Delta T) e^{-2\pi j k n \frac{\Delta T}{T}} = \Delta T \sum_{k=0}^{N-1} x_B(k\Delta T) e^{-2\pi j \frac{kn}{N}}, \qquad n \in \mathbb{Z}$$

Two forms of DFT

In practice two forms of DFT are used (difference: factor $1/T$):

1. Spectral density (approximation of Fourier integral, unit: $[x]$/Hz):

$$X(f_n) = X(n) = \Delta T \sum_{k=0}^{N-1} x(k\Delta T) e^{-2\pi j \frac{kn}{N}}, \qquad n = 0,1,\dots,N-1.$$

2. Frequency spectrum ($X(f_n)$ is equivalent with c_n, unit: $[x]$):

$$X(f_n) = X(n) = \frac{\Delta T}{T} \sum_{k=0}^{N-1} x(k\Delta T) e^{-2\pi j \frac{kn}{N}}, \qquad n = 0,1,\dots,N-1.$$

Note: fft in matlab computes

$$X(f_n) = X(n) = \sum_{k=0}^{N-1} x(k\Delta T) e^{-2\pi j \frac{kn}{N}}, \qquad n = 0,1,\dots,N-1.$$

For spectral density, multiply with $T/N = \Delta T$. For frequency spectrum, multiply with $1/N$.

Periodicity

DFT is periodic with sample frequency

$$f_N = \frac{1}{\Delta T} = \frac{N}{T}$$

Proof: for each integer l

$$X(f_{n+lN}) = \Delta T \sum_{k=0}^{N-1} x(k\Delta T) e^{-2\pi j \frac{k(n+lN)}{N}} = \Delta T \sum_{k=0}^{N-1} e^{-2\pi jkl} x(k\Delta T) e^{-2\pi j \frac{kn}{N}} = X(f_n).$$

QED.

Therefore, evaluate $X(f_n)$ only for N discrete frequencies

$$X(f_n) = \Delta T \sum_{k=0}^{N-1} x(k\Delta T) e^{-2\pi j \frac{kn}{N}}, \qquad n = 0,1,\ldots,N-1.$$

The essential information of a **complex** signal $x(k\Delta T)$, $k = 0,1,\ldots,N-1$ is thus contained in the **complex** DFT $X(f_n)$, $n = 0,1,\ldots,N-1$.

Folding property

For **real** $x(k\Delta T)$, the DFT $X(f_n)$ is still **complex**, but

$$X\left(f_{(N/2)+n}\right) = \bar{X}\left(f_{(N/2)-n}\right).$$

The DFT is thus folded in a special way around **folding frequency**

$$f_{fold} = f_{N/2} = \frac{f_N}{2} = \frac{N}{2T}.$$

Indeed, in both time and frequency domain, a real signal $x(k\Delta T)$ is defined by N numbers:

- $X(f_0)$ and $X\left(f_{N/2}\right)$ are real: 2 numbers
- $X(f_n), n = 1,2,\ldots,(N/2) - 1$ are complex: $2\left(\left(\frac{N}{2}\right) - 1\right) = N - 2$ numbers.

The essential information of a **real** signal $x(k\Delta T), k = 0,1,\ldots,N-1$ is thus contained in the **complex** DFT $X(f_n), n = 0,1,\ldots,N/2$.

Inverse DFT

Again two forms:

1. Inverse of the Spectral density
$$x(t_k) = x(k) = \frac{1}{T}\sum_{n=0}^{N-1} X(f_n)e^{2\pi j\frac{kn}{N}}, \qquad k = 0,1,\dots,N-1.$$
2. Inverse of the Frequency spectrum
$$x(t_k) = x(k) = \sum_{n=0}^{N-1} X(f_n)e^{2\pi j\frac{kn}{N}}, \qquad k = 0,1,\dots,N-1.$$

Note: ifft in matlab computes
$$x(k\Delta T) = x(k) = \frac{1}{N}\sum_{n=0}^{N-1} X(f_n)e^{2\pi j\frac{kn}{N}}, \qquad k = 0,1,\dots,N-1.$$

For inverse of spectral density, multiply with $1/\Delta T$.
For inverse of frequency spectrum, multiply with N

Periodicity of inverse DFT

The inverse DFT $x(t_k)$is periodic with period $T = N\Delta T$ (for both forms).

<u>Proof:</u>

$$x(t_{k+lN}) = \frac{1}{T}\sum_{n=0}^{N-1} X(f_n)e^{2\pi j\frac{(k+lN)n}{N}} = \frac{1}{T}\sum_{n=0}^{N-1} e^{2\pi jnl} X(f_n)e^{2\pi j\frac{kn}{N}} = x(t_k).$$

Example: DFT of $w(t)$

$$w(t) = \begin{cases} 1 & \text{if} \quad t \in [0,10) \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Measurement time: $T = 200$ s.

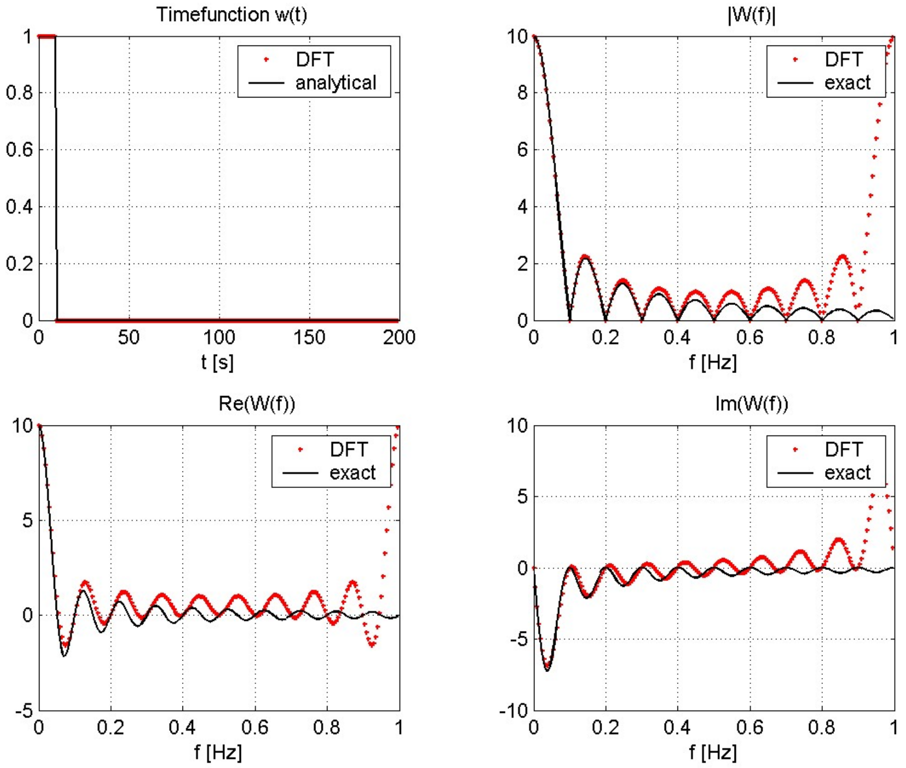
Number of time points: $N = 200$.

- Fundamental frequency (resolution)

$$f_1 = \Delta f = \frac{1}{T} = 0.005 \text{ Hz.}$$

- Sample time (increment): $\Delta T = \frac{T}{N} = 1$ s.
- Sample frequency: $f_N = \frac{1}{\Delta T} = 1$ Hz.
- Folding frequency: $f_{fold} = \frac{f_N}{2} = 0.5$ Hz.

Compute the spectral density.



Example: DFT of $w(t)$

$$w(t) = \begin{cases} 1 & \text{if} \quad t \in [0,10) \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Measurement time: $T = 200$ s.

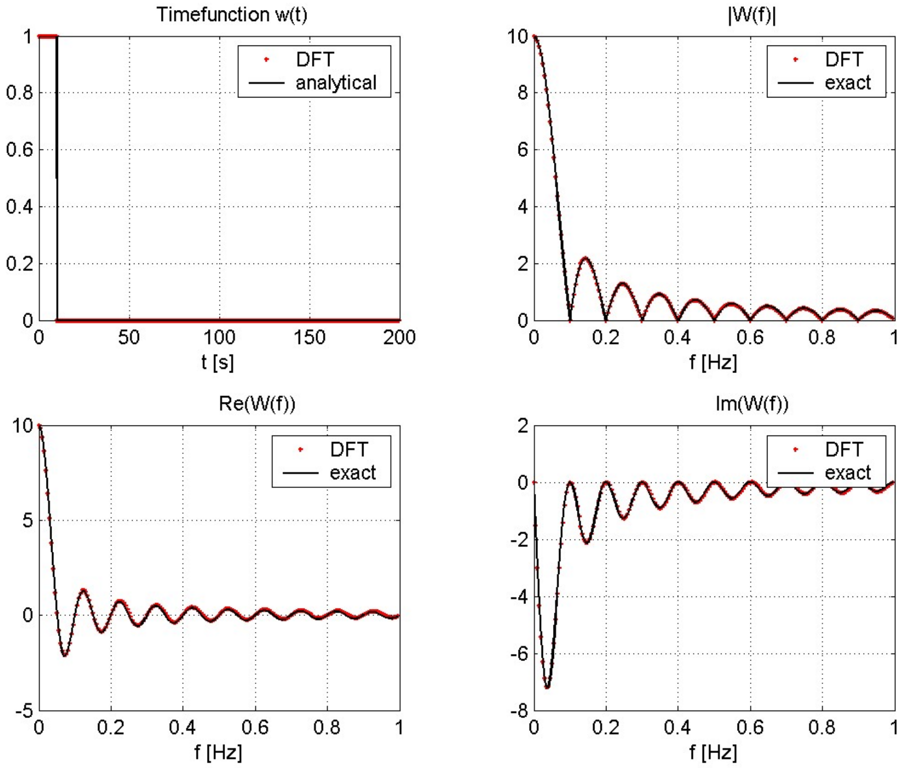
Number of time points: $\boldsymbol{N = 2000}$.

- Fundamental frequency (resolution)

$$f_1 = \Delta f = \frac{1}{T} = 0.005 \text{ Hz}.$$

- Sample time (increment): $\Delta T = \frac{T}{N} = \mathbf{0.1}$ **s.**
- Sample frequency: $f_N = \frac{1}{\Delta T} = \mathbf{10}$ **Hz.**
- Folding frequency: $f_{fold} = \frac{f_N}{2} = \mathbf{5}$ **Hz.**

Compute the spectral density.



Fast Fourier Transformation (FFT)

$$X(f_n) = X(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k\Delta T)e^{-2\pi j\frac{kn}{N}}, \qquad n = 0,1,\ldots,N-1.$$

DFT: N^2 complex multiplications needed, expensive for large N!
However, a lot of identical multiplications…

FFT: algorithm, see book
When $N = 2^m$ ($m \in \mathbb{N}$) only $N\log_2 N$ complex multiplications.

For $N = 2048 = 2^{11}$, FFT reduces the cpu-time by a factor 186!

7c. Error sources

Error sources in DFT/FFT

1. **Signal leakage**
   Basic problem: $x(t)$ is defined for $-\infty < t < \infty$,
   but only the interval $0 \leq t < T$ is taken into account.
   In other words, we consider $x_B(t)$ instead of $x(t)$.

2. **Aliasing**
   Basic problem: the continuous-time signal $x(t)$ (or in fact $x_B(t)$)
   is only sampled in a limited number of discrete values.

Main questions:

a) How do these error sources influence the DFT/FFT?

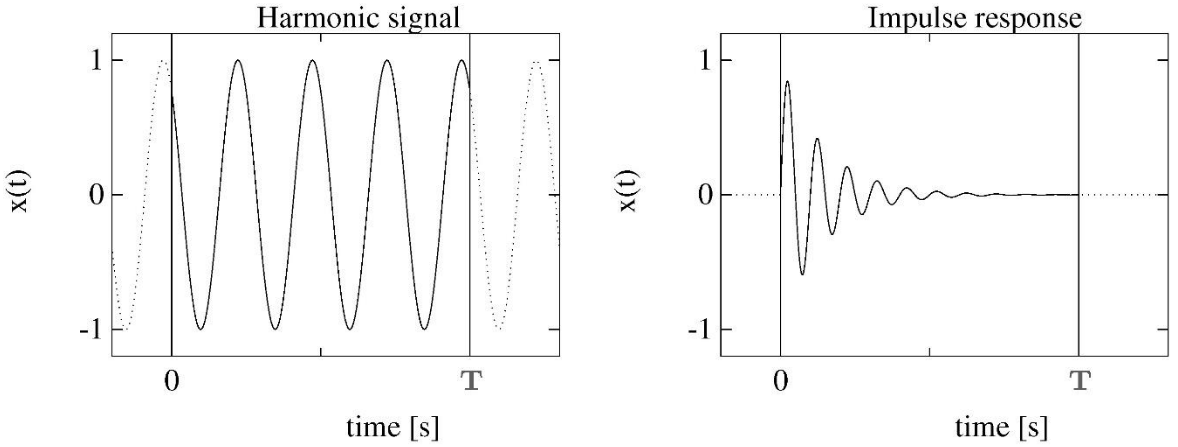
b) How can the effect of these error sources be reduced?

Signal leakage

The application of a rectangular window generally leads to signal leakage!

Two exceptions:

- Window length is a multiple of period of a periodic signal.
- Transient signal (e.g. impulse response) which is (practically) damped out before the end of the window



Signal leakage

Signal leakage is caused by the windowing function $w(t)$

$$x_B(t) = w(t)x(t),$$

In the frequency domain:

$$X_B(f) = F[x_B(t)] = W(f) \otimes X(f).$$

Because $\delta(f) \otimes X(f) = X(f)$:
the better $W(f)$ resembles $\delta(f)$, the better $X_B(f)$ resembles $X(f)$ and
the better $w(t)$ resembles 1 (for all $-\infty < t < \infty$), the better $X_B(f)$ resembles $X(f)$.

Signal leakage thus results in:

- Energy in a certain frequency will be spread out over neighbouring dummy frequencies
- Resonance peaks become wider
  $\Rightarrow$ very close peaks cannot be distinguished.

Example: Signal leakage

Signal: $x(t) = \cos(2\pi f_x t)$
Rectangular window $w(t)$

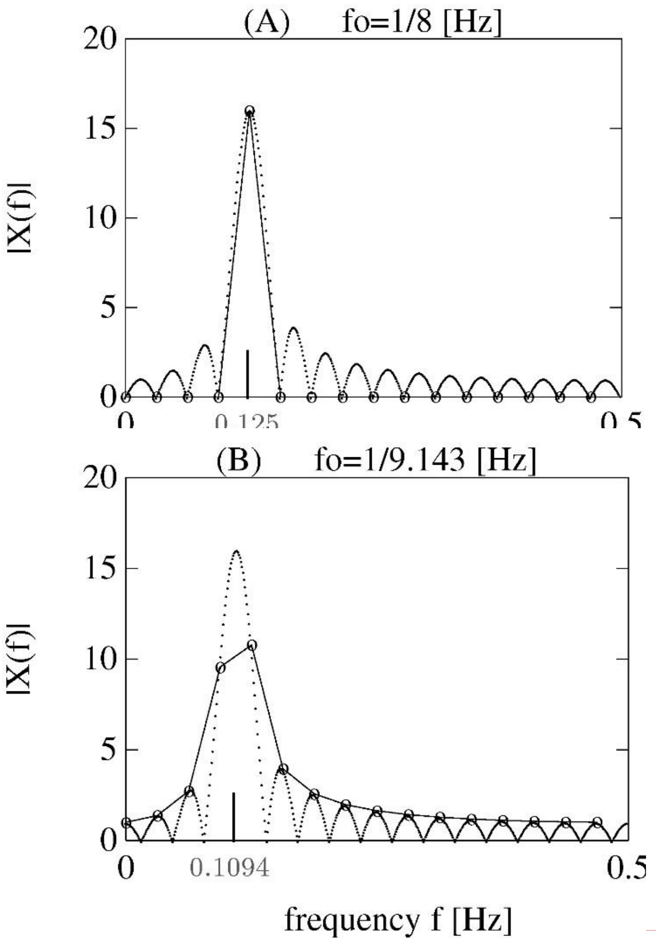
Theoretical FT: $X(f) = \frac{1}{2}[\delta(f - f_x) + \delta(f + f_x)]$
Windowed FT: $X_B(f) = \frac{1}{2}[W(f - f_x) + W(f + f_x)]$

Case A: $f_x = 1/8$ Hz, $T = 4/f_x = 32$ s, $N = 32.$

Case B: $f_x = 1/9.143$ Hz, $T = 3.5/f_x = 32$ s, $N = 32.$

Look at the open dots at the discrete frequencies f_n.

Case A: No leakage
Case B: Some energy has leaked to frequencies around f_x.



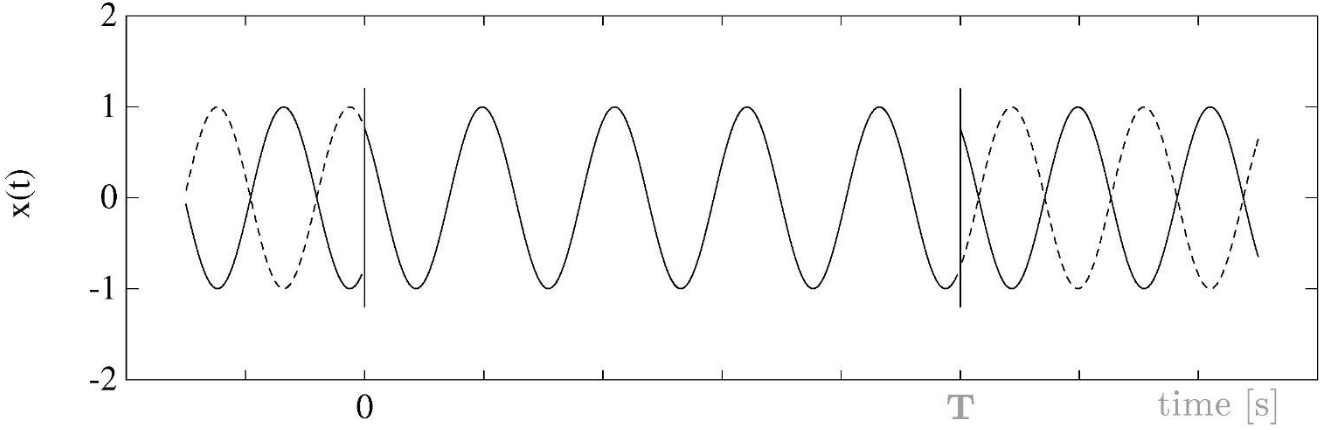
Example: Signal leakage

Signal: $x(t) = \cos(2\pi f_x t)$
Rectangular window $w(t)$

Case A: $f_x = 1/8$ Hz, $T = 4/f_x = 32$ s, $N = 32.$

Case B: $f_x = 1/9.143$ Hz, $T = 3.5/f_x = 32$ s, $N = 32.$

For Case B, we see a discontinuous signal in the time domain!

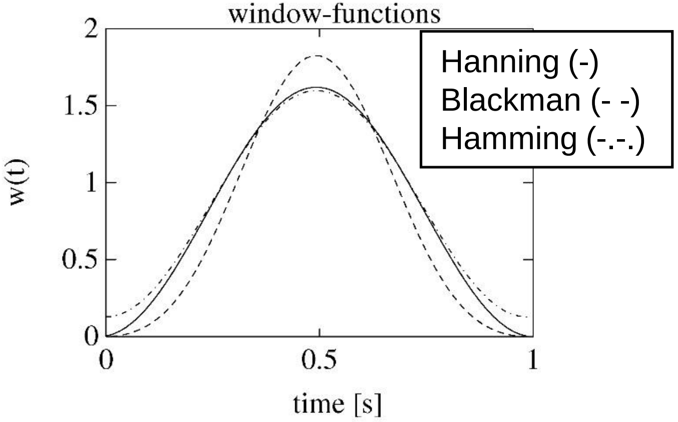


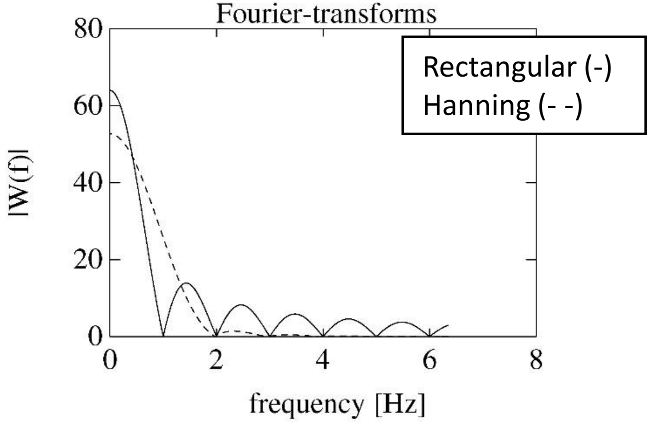
TU/e

Solution to signal leakage

- in the time domain:
  edge discontinuities should be reduced.
- in the frequency domain:
  use a window function $w(t)$ for which $W(f) = F[w(t)]$ resembles a Dirac delta $\delta(f)$.

Windowing functions





For unaffected mean power of $w(t)x(t)$

$$MSV = \frac{1}{T}\int_0^T w^2(t)\, x^2(t) dt,$$

require:

$$\frac{1}{T}\int_0^T w^2(t)\, dt = 1$$

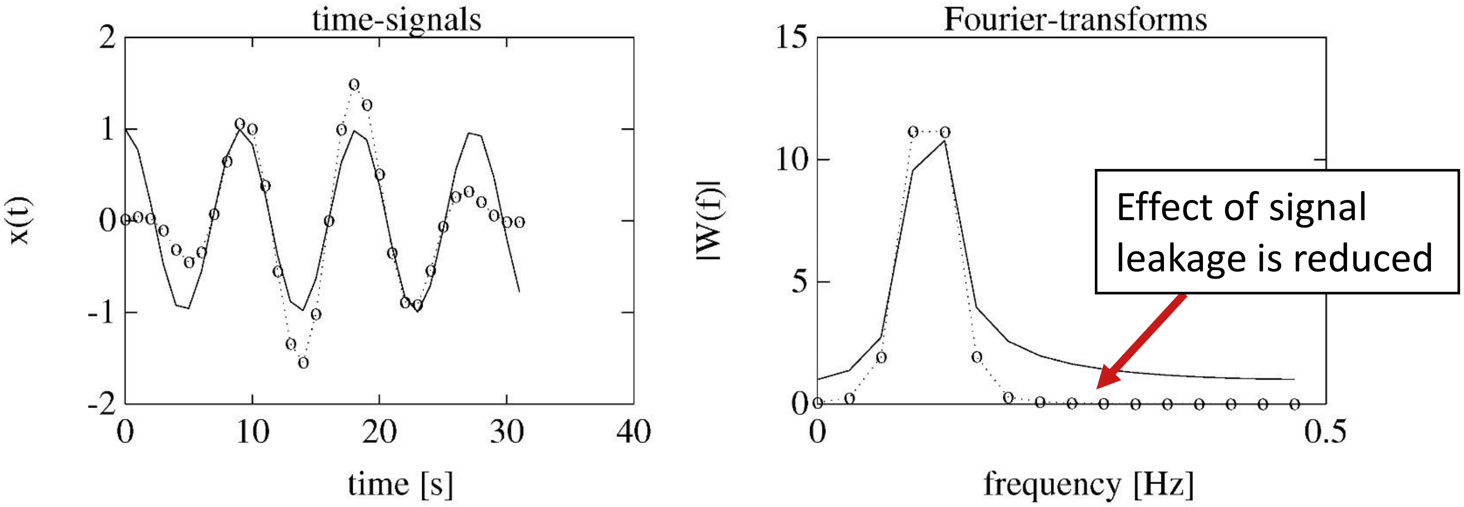
Hanning window compared to Rectangular window:

- side lobes are almost missing (++)
- central peak a bit wider (-)

Example: Signal leakage

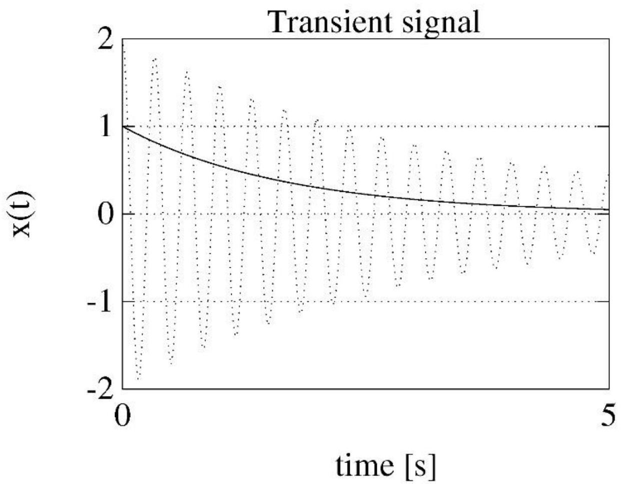
Signal: $x(t) = \cos(2\pi f_x t)$
Again, consider Case B: $f_x = 1/9.143$ Hz, $T = 3.5/f_x = 32$ s, $N = 32$.

Rectangular window $w(t) = 1$ for $0 \leq t < T$ (solid lines) and
Hanning window $w_H(t) = \sqrt{2/3}(1 - \cos(2\pi t/T))$ (dotted lines with open dots).



Windowing in practice

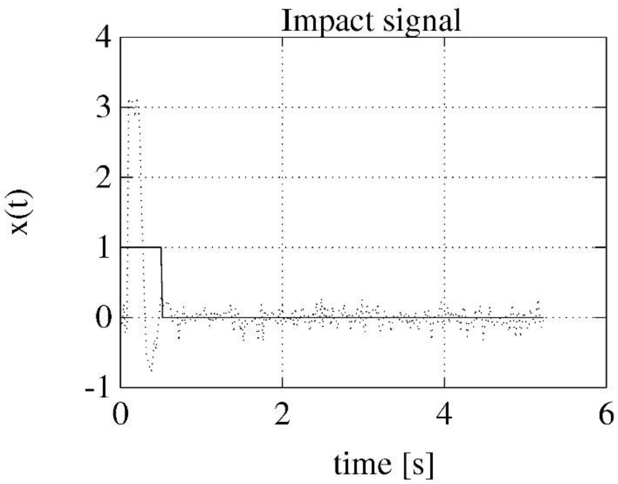
Lightly damped structures



Transient not yet damped out at the end of measurement interval

Natural to use **exponential window.**
This introduces numerical damping!

Hammer (impact) excitation



After impact: noise, swinging of hammer, putting hammer on table. All not related to hammer impact on the structure.

Use **rectangular window.**

Aliasing

Related to **sampling** of $x(t)$ in discrete time points $t_n = n\Delta T = \frac{nT}{N}$.
Sampling frequency $f_N = 1/\Delta T$.

Approximate Fourier transform $X(f) = F[x(t)]$ using **only** the samples $x(t_n)$.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-(2\pi f t) j} dt \approx \sum_{n=-\infty}^{\infty} x(n\Delta T) e^{-2\pi j f n \Delta T} \Delta T.$$

The RHS can be considered as the Fourier transform of the pulse train

$$x_B(t) = x(t)\, \Delta T \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)\,.$$

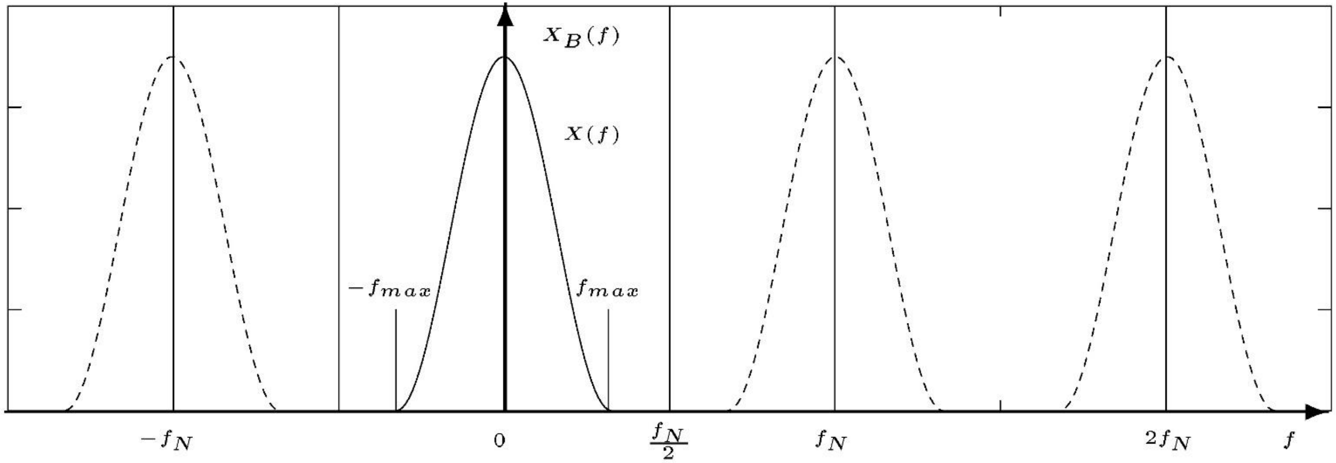
Taking the Fourier transform of $x_B(t)$ yields

$$X_B(f) = X(f) \otimes \sum_{n=-\infty}^{\infty} \delta(f - n f_N) = \sum_{n=-\infty}^{\infty} X(f + n f_N)\,.$$

Aliasing

$$X_B(f) = \sum_{n=-\infty}^{\infty} X(f + nf_N)$$

Situation I: sufficiently high sampling frequency f_N, so $|X(f)| = 0$ for $f > \frac{f_N}{2} = f_{N/2}$.

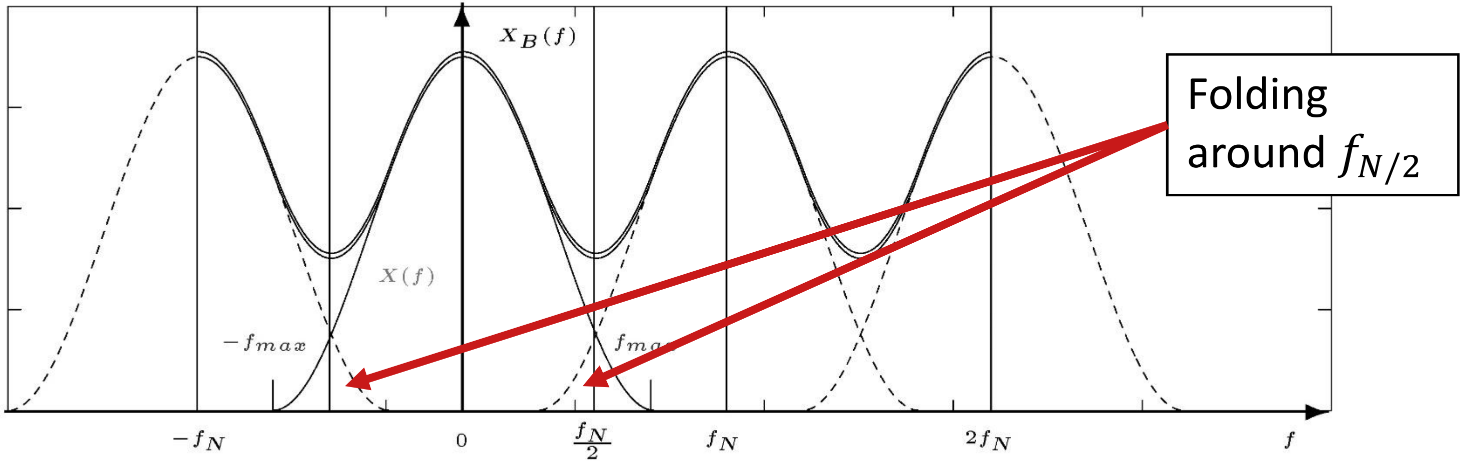


- $X_B(f)$ is periodic with period f_N, $X(f)$ is not periodic!
- $X_B(f)$ is **exactly equal** to $X(f)$ in $-f_{N/2} \leq f \leq f_{N/2}$

Aliasing

$$X_B(f) = \sum_{n=-\infty}^{\infty} X(f + nf_N)$$

Situation II: sampling frequency f_N is too low, so $|X(f)| > 0$ for (some) $f > \frac{f_N}{2} = f_{N/2}$.



- $X_B(f)$ is periodic with period f_N, $X(f)$ is not periodic!
- $X_B(f)$ is **not equal** to $X(f)$ in $-f_{N/2} \leq f \leq f_{N/2}$, **aliasing** occurs!

Aliasing

Avoid aliasing by choosing the sampling time ΔT such that the **Nyquist (folding)** frequency satisfies:

$$f_{fold} = f_{Nyquist} = f_{N/2} = \frac{1}{2\Delta T} > f_{max}.$$

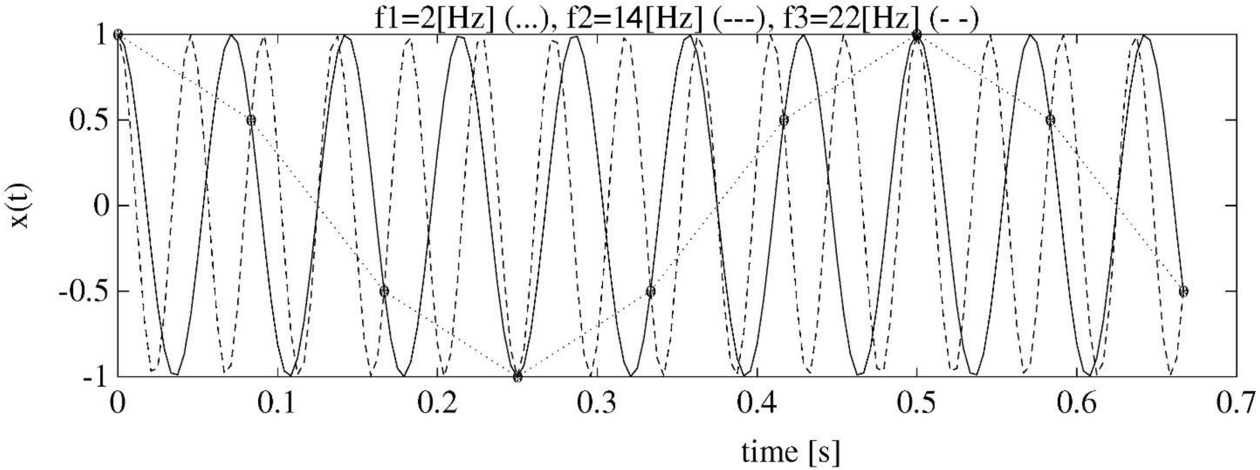
- f_{max} may be very high, then **analogue low-pass filtering** should be applied.

Aliasing is close related to **Shannon's sampling theorem:**

If $X(f) = 0$ for all $|f| > f_{max}$ then the continuous function $x(t)$ is fully determined by samples at the discrete time points $t_n = n\Delta T, n \in \mathbb{Z}$ with $\Delta T = 1/(2f_{max})$.

Example: aliasing

Consider signals $x_i(t) = \cos(2\pi f_i t)$ with $f_2 = 14$ Hz (solid line) and $f_3 = 22$ Hz (dashed line).



Sample time: $\Delta T = 0.0833$ s, sampling frequency $f_N = 12$ Hz.
Samples of $x_2(t)$ and $x_3(t)$ coincide with samples of $x_1(t)$ with $f_1 = 2$ Hz.

Problems could be expected: $f_{fold} = 1/(2\Delta T) = 6$ Hz

TU/e

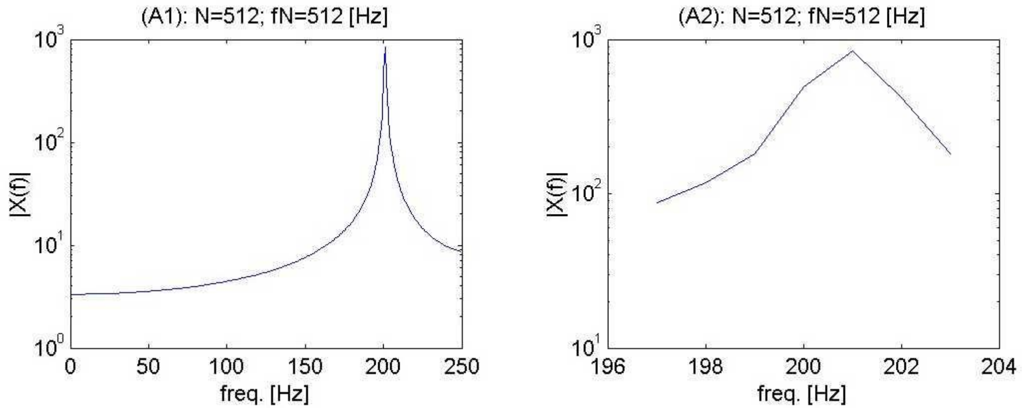
Zoom-FFT

Method to obtain detailed information in small frequency band.

Motivation: consider the signal

$$x(t) = 2\sin(2\pi f_1 t) + 4\cos(2\pi f_2 t)\,, \qquad f_1 = 200.4\ \text{Hz}, f_2 = 201.3\ \text{Hz}.$$

Compute FFT $N = 512$, $T = 1$ s, so $f_{fold} = 256$ Hz.



Not clear that we are dealing with two frequency components. Resolution $\Delta f = 1$ Hz is too low!

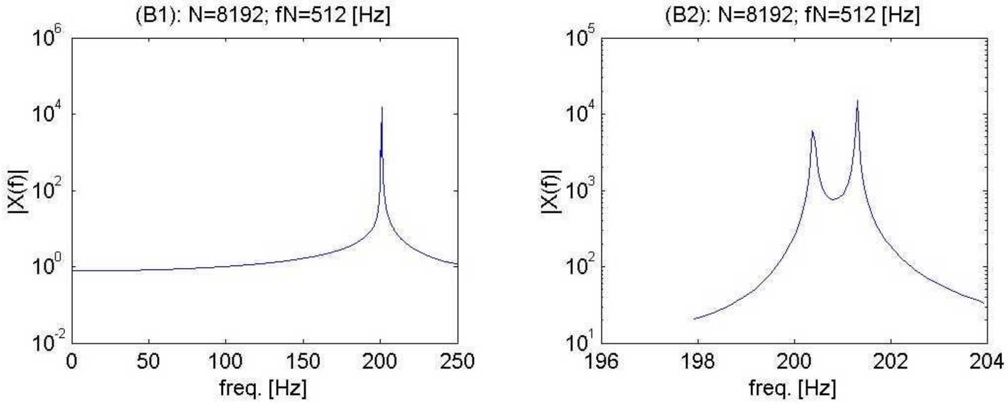
Zoom-FFT

Method to obtain detailed information in small frequency band.

Motivation: consider the signal

$$x(t) = 2\sin(2\pi f_1 t) + 4\cos(2\pi f_2 t)\,, \qquad f_1 = 200.4\text{ Hz}, f_2 = 201.3\text{ Hz}.$$

Possible solution: compute FFT $N = 8192$, $T = 16$ s, so $f_{fold} = 256$ Hz and $\Delta f = 0.0625$ Hz



Disadvantage: N is large, FFT will take much time. Only information around 200 Hz is interesting.

Zoom-FFT

Another solution: Zoom-FFT! Now N can be kept small.

Frequency band of interest $f_{min} < f < f_{max}$.

Introduce $g := f - f_{min}$, so we are interested in $0 \leq g \leq f_b := f_{max} - f_{min}$.

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi jft}dt = \int_{-\infty}^{\infty} x(t)e^{-2\pi jf_{min}t}e^{-2\pi jgt}dt = \int_{-\infty}^{\infty} y(t)e^{-2\pi jgt}dt = Y(g)$$

where

$$y(t) := x(t)e^{-2\pi jf_{min}t}.$$

Observe:

- Complex $y(t)$ is obtained by **frequency modulation** of real $x(t)$.
- $Y(g)$ can be computed using a standard FFT with $g_N = f_b = 1/\Delta T$.

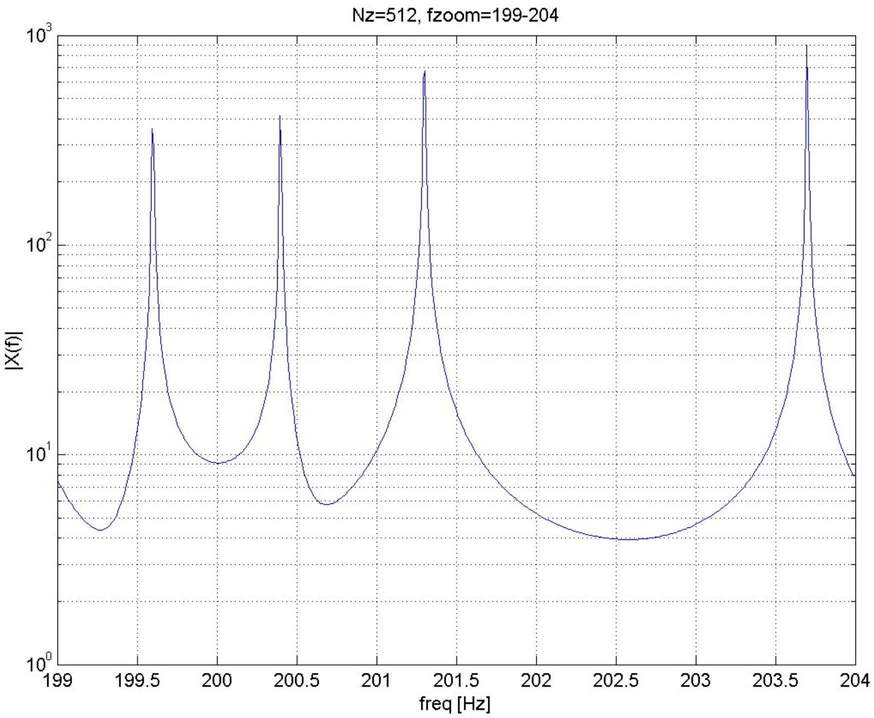
Example: Zoom-FFT

We choose $f_{min} = 199$ Hz, $f_{max} = 204$ Hz
$\Rightarrow f_b = g_N = f_{max} - f_{min} = 5$ Hz
$\Rightarrow \Delta T = 1/f_b = 0.2$ s.

We also choose $N = 512$ (small)
$\Rightarrow T = N\Delta T = 102.4$ s, (large)
$\Rightarrow \Delta f = \frac{1}{T} \approx 0.01$ Hz (small)

Compute $Y(g)$ and shift result to

$$f = g + f_{min}.$$



Observe:
Not only peaks at 200.4 and 201.3 Hz, b but also at 199.6 and 203.7 Hz (dummy frequencies)!

Remarks on Zoom-FFT

- There is no folding frequency at $g_{N/2} = (f_{max} - f_{min})/2$ Hz because $y(t)$ is complex! Only if f_{min} is a multiple of f_b a folding frequency will arise.
- Dummy frequencies can be understood as follows:

$$y(t) = \sin(2\pi f_1 t)\, e^{-2\pi j f_{min}} = \frac{j}{2}\left[-e^{-2\pi j (f_1 - f_{min})} + -e^{-2\pi j(-f_1 - f_{min})}\right],$$

so an harmonic at $f = f_1$ introduces two peaks at $g_{1a} = f_1 - f_{min}$ and $g_{1b} = -f_1 - f_{min}$.
Furthermore, the FFT $Y(g)$ is periodic with $g_{N/2} = (f_{max} - f_{min})/2$ Hz.
The peaks thus also appear at $g = g_{1a} + n g_{N/2}$ and $g = g_{1b} + n g_{N/2}, n \in \mathbb{Z}$.

With $f_1 = 200.4$ Hz, $f_{min} = 199$ Hz, and $f_{max} = 204$ Hz,
we thus get $g_{1a} = 1.4$ Hz and $g_{1b} = -399.4$ Hz.
Due to the periodicity of $Y(g)$ with 5 Hz,
g_{1b} leads to a peak at $g = 0.6$ Hz,
which corresponds to $f_b = g_b + f_{min} = 199.6$ Hz!

For $f_2 = 201.3$ Hz,
$g_{1a} = 2.3$ Hz and $g_{1b} = -400.3$ Hz.
g_{1b} leads to a peak at $g = 4.7$ Hz,
which corresponds to $f = 203.7$ Hz!