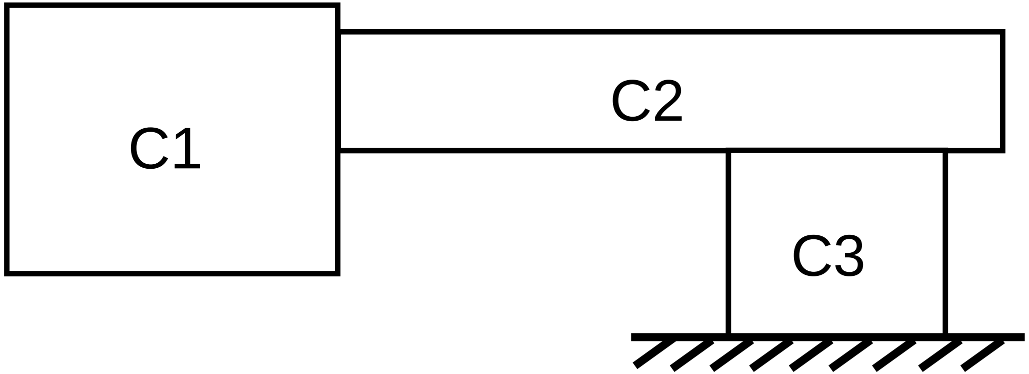

5. Substructuring, reduction, and coupling

Structural Dynamics part of 4DM00

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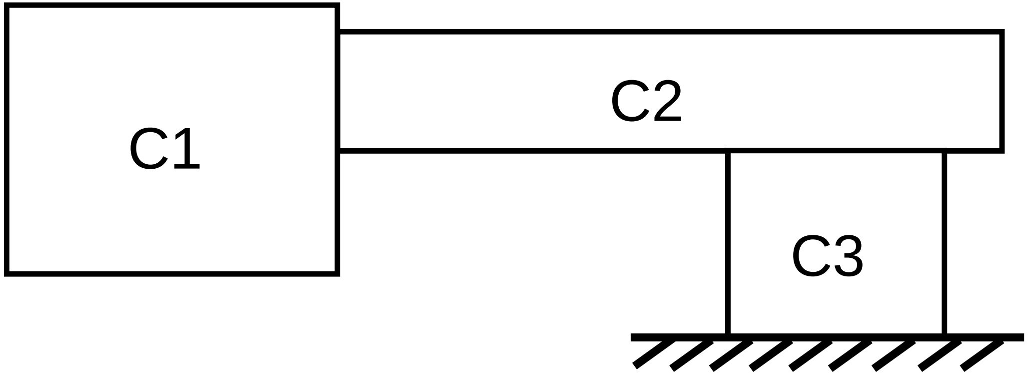
Substructuring, reduction, and coupling

Complex mechanical system: 3D Finite Element model with 10^6 dofs

An accurate, reduced dynamic model of the system in a frequency range of interest can be derived by:

1. reducing the system model itself, but also by….
2. coupling of **reduced** substructure/component/superelement models

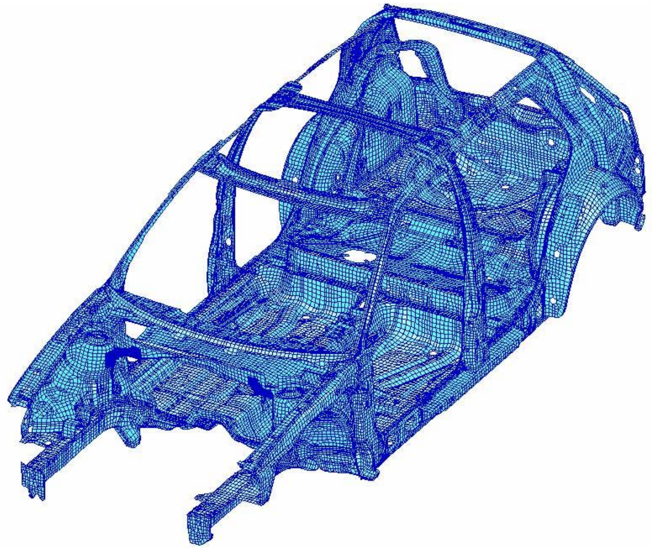


Motivation for distinguishing components

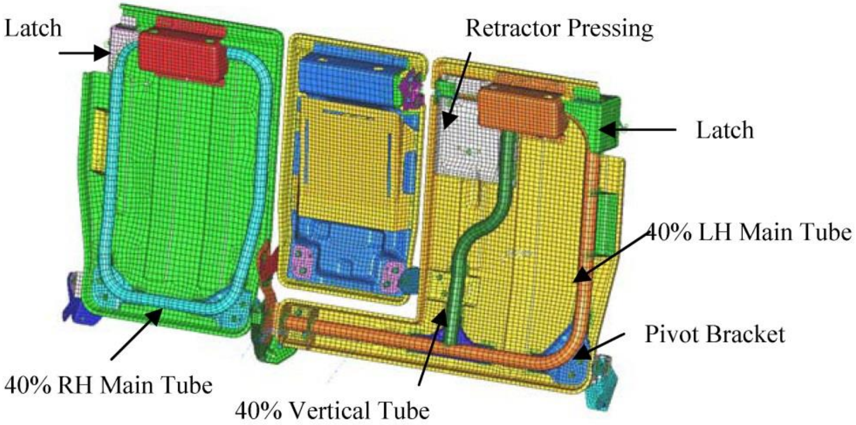
- Main contractor responsible for dynamic behaviour of total system, subcontractors responsible for individual components. Dynamic characteristics of components are required.
- Dynamic properties of optional designs of some components still need to be evaluated to determine the optimal design, whereas designs of other components are already fixed.
- Some components can be modelled theoretically, others need to be identified experimentally (unknown damping characteristics)
- Identical components, e.g. blades of windturbine.

Example

Car can be divided in substructures (car body, back seat, etc)



FE model of car body



FE model of a part of a back seat

Motivation for reducing the number of DOFs

Reduce calculation times for dynamic analyses while retaining accuracy

1. Frequency range of interest: limited number of modes active
2. Discretization errors increase in higher frequency modes
3. Need to reduce computational times increases in case of:
   a) multiple load cases
   b) parameter studies
   c) nonlinear substructure(s)

Substructuring, reduction & coupling can be applied in the:

1) time domain
2) frequency domain

General reduction procedure (time domain)

Equations of motion of unreduced/original component/system:

$$M\ddot{q}(t) + B\dot{q}(t) + Kq(t) = f(t)$$

$q(t)$: column with the n original DOFs.

Approximate $q(t)$ by linear combination of n_p Ritz columns t_i:

$$q(t) \approx \sum_{i=1}^{n_p} t_i p_i(t) = Tp(t)$$

$p(t)$: column with n_p generalized dof's, $n_p \ll n$.
T: Ritz reduction matrix, dimension (n, n_p).

General reduction procedure (time domain)

Approximate kinetic and elastic energy:

- Kinetic energy: $T_k = \frac{1}{2}\dot{q}^\top M\dot{q} \approx \frac{1}{2}\dot{p}^\top T^\top MT\dot{p} =: \frac{1}{2}\dot{p}^\top M^{red}\dot{p}$
- Elastic energy: $V_e = \frac{1}{2}q^\top Kq \approx \frac{1}{2}p^\top T^\top KTp =: \frac{1}{2}p^\top K^{red}p$
- Virtual work of non-conservative loads:

$$\delta A = \delta q^\top(f - B\dot{q}) \approx \delta p^\top(T^\top f - T^\top BT\dot{p}) =: \delta p^\top\left(f^{red} - B^{red}\dot{p}\right)$$

Reduced equations of motion of component/system (Lagrange):

$$M^{red}\ddot{p}(t) + B^{red}\dot{p}(t) + K^{red}p(t) = f^{red}(t)$$

- $M^{red} = T^\top MT$
- $B^{red} = T^\top BT$
- $K^{red} = T^\top KT$
- $f^{red}(t) = T^\top f(t)$

Reduction methods in the time domain

1. Dynamic reduction using eigenmodes (see 2. Numerical Modal Analysis)

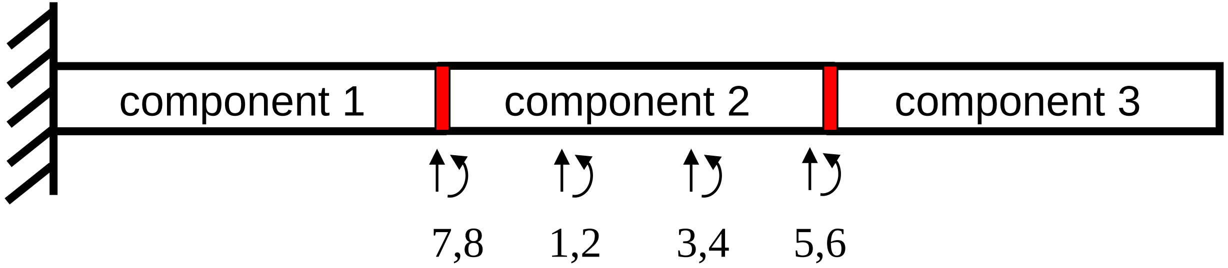
2. Guyan reduction or static reduction

Component Mode Synthesis (CMS), dynamic reduction (usually on component level):

3. Craig-Bampton

4. Rubin

Component 2 of a cantilever beam system will be used in examples:



5b. Guyan reduction or static reduction

Guyan reduction or static reduction

Partition DOFs $q(t)$ as

$$q(t) = \begin{bmatrix} q_m(t) \\ q_l(t) \end{bmatrix}$$

- $q_m(t)$: master DOFs, length n_m
- $q_l(t)$: local DOFs, length n_l

Usually $n_m \ll n_l$

Partition equation of motion accordingly

$$\begin{bmatrix} M_{mm} & M_{ml} \\ M_{lm} & M_{ll} \end{bmatrix} \begin{bmatrix} \ddot{q}_m(t) \\ \ddot{q}_l(t) \end{bmatrix} + \begin{bmatrix} K_{mm} & K_{ml} \\ K_{lm} & K_{ll} \end{bmatrix} \begin{bmatrix} q_m(t) \\ q_l(t) \end{bmatrix} = \begin{bmatrix} f_m(t) \\ f_l(t) \end{bmatrix}$$

Guyan reduction or static reduction

Partitioned equations of motion:

$$\begin{bmatrix} M_{mm} & M_{ml} \\ M_{lm} & M_{ll} \end{bmatrix} \begin{bmatrix} \ddot{q}_m(t) \\ \ddot{q}_l(t) \end{bmatrix} + \begin{bmatrix} K_{mm} & K_{ml} \\ K_{lm} & K_{ll} \end{bmatrix} \begin{bmatrix} q_m(t) \\ q_l(t) \end{bmatrix} = \begin{bmatrix} f_m(t) \\ f_l(t) \end{bmatrix}$$

Assume $|K_{lm} q_m(t) + K_{ll} q_l(t)|$ is much larger than $|f_l(t)|$ and $|M_{lm} \ddot{q}_m(t) + M_{ll} \ddot{q}_l(t)|$.

Therefore,

$$K_{lm} q_m(t) + K_{ll} q_l(t) \approx 0, \qquad \Rightarrow \quad \boxed{q_l(t) \approx -K_{ll}^{-1} K_{lm} q_m(t)}$$

$q(t)$ is static response for prescribed $q_m(t)$
(external forces at local DOFs are neglected)

Guyan reduction or static reduction

$$q_l(t) \approx -K_{ll}^{-1} K_{lm} q_m(t)$$

Transformation / Ritz matrix:

$$q(t) = \begin{bmatrix} q_m(t) \\ q_l(t) \end{bmatrix} \approx \begin{bmatrix} I \\ -K_{ll}^{-1} K_{lm} \end{bmatrix} q_m(t) =: T_{nm}^G q_m(t)$$

Reduced equations of motion:

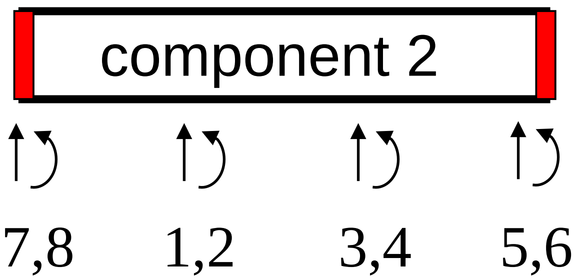
$$M_{mm}^G \ddot{q}_m(t) + K_{mm}^G q_m(t) = f_m^G(t)$$

where

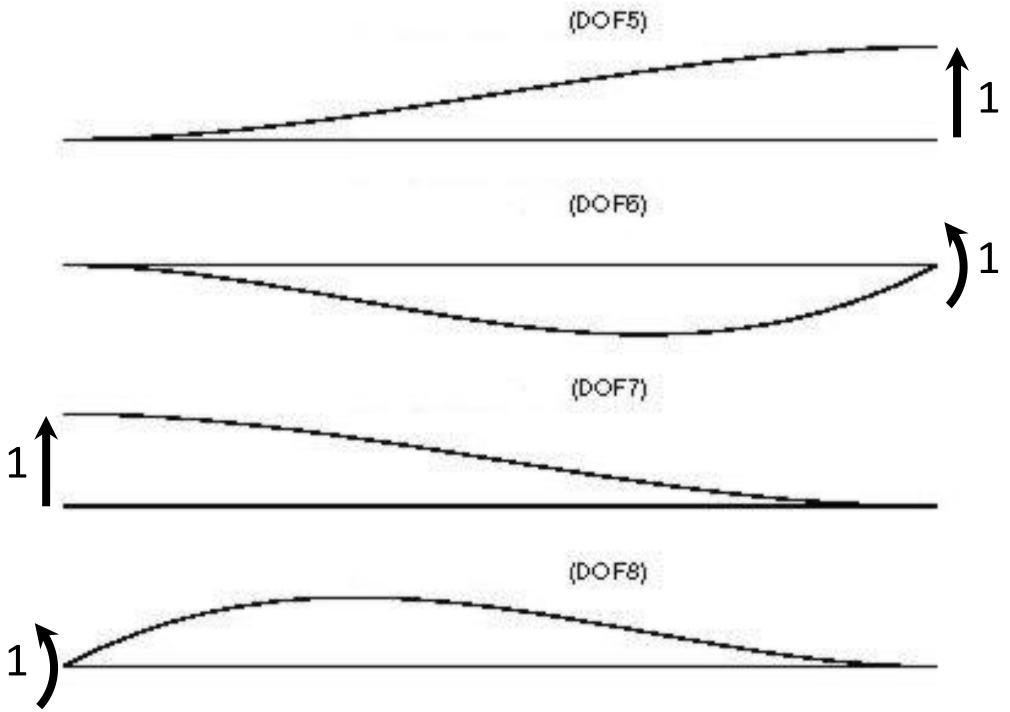
$$M_{mm}^G = (T_{nm}^G)^\top M T_{nm}^G, \qquad K_{mm}^G = (T_{nm}^G)^\top K T_{nm}^G, \qquad f_m^G(t) = (T_{nm}^G)^\top f(t)$$

Example: Guyan reduction

Component 2 of cantilever beam model



Select DOF 5-8 as master DOFs in component 2.



Remarks on Guyan reduction

- Easy to implement, present in many FE packages
- Problem: how to select master dof's?
  What is the influence of a particular selection $q_m(t)$ on model accuracy???
- If $f_l(t) \equiv 0$: reduced model statically equivalent to unreduced model

5c. The CMS-method of Craig-Bampton

CMS-method of Craig-Bampton

Partition

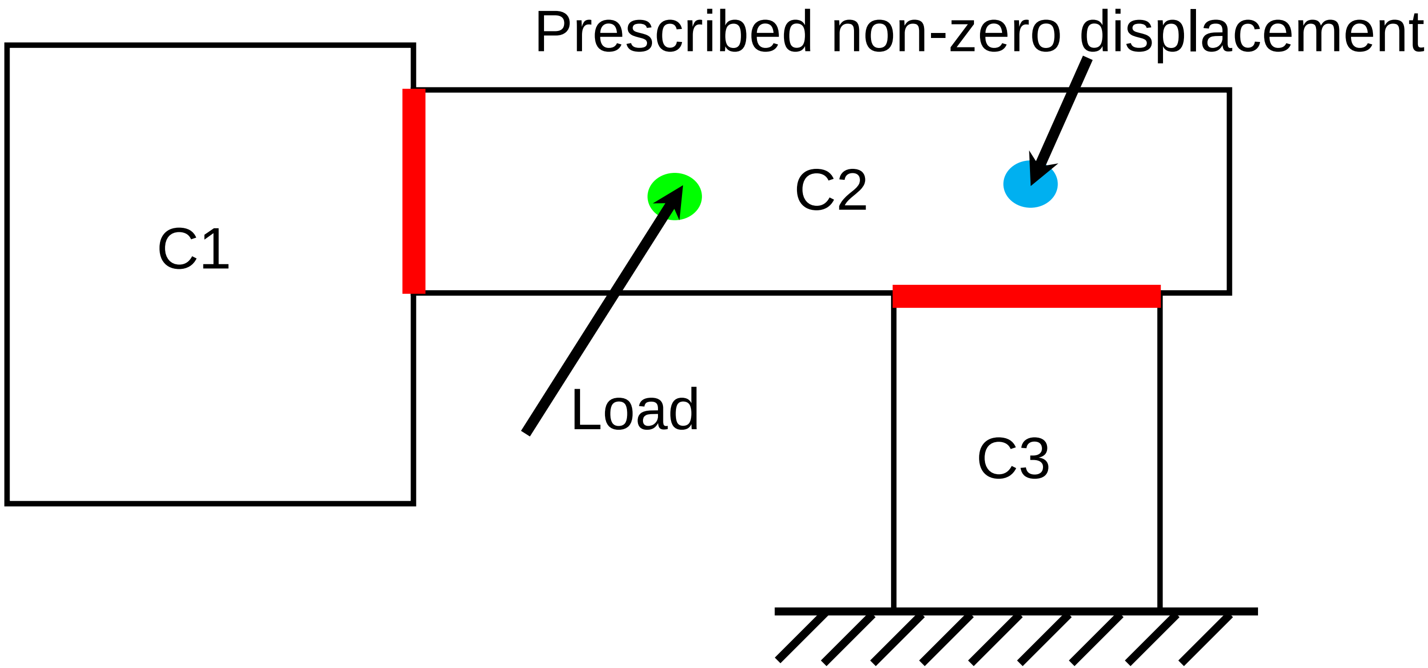
$$q(t) = \begin{bmatrix} q_B(t) \\ q_I(t) \end{bmatrix}$$

- $q(t)$: DOFs of considered component
- $q_B(t)$: boundary DOFs (length n_B) that consist of
  - DOFs at which the reduced component is coupled to other (reduced) components
  - externally loaded DOFs
  - DOFs with prescribed non-zero displacements
- $q_I(t)$: internal DOFs (length n_I, remaining DOFs, non-loaded DOFs)

Partition equation of motion accordingly

$$\begin{bmatrix} M_{BB} & M_{BI} \\ M_{IB} & M_{II} \end{bmatrix} \begin{bmatrix} \ddot{q}_B(t) \\ \ddot{q}_I(t) \end{bmatrix} + \begin{bmatrix} K_{BB} & K_{BI} \\ K_{IB} & K_{II} \end{bmatrix} \begin{bmatrix} q_B(t) \\ q_I(t) \end{bmatrix} = \begin{bmatrix} f_B(t) \\ f_I(t) \end{bmatrix}$$

Example: boundary DOFs of component 2



CMS-method of Craig-Bampton

Approximation:

$$q(t) = \begin{bmatrix} q_B(t) \\ q_I(t) \end{bmatrix} \approx q^S(t) + q^D(t)$$

- $q^S(t)$: dynamic response due to linear combination of n_B static modes
- $q^D(t)$: dynamic response due to linear combination of limited set of dynamic modes

Static modes:

$q^S(t)$ is the static response resulting from prescribed $q_B(t)$.

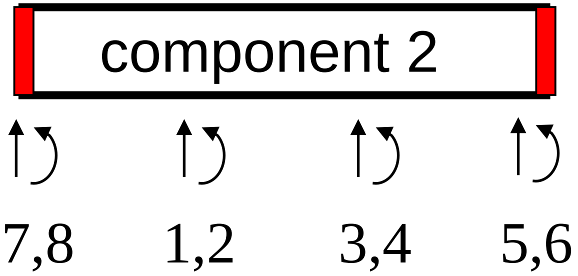
$$q^S(t) = \begin{bmatrix} q_B^S(t) \\ q_I^S(t) \end{bmatrix} = \begin{bmatrix} I_{BB} \\ -K_{II}^{-1} K_{IB} \end{bmatrix} q_B(t) = \begin{bmatrix} I_{BB} \\ T_{IB} \end{bmatrix} q_B(t) =: T_{nb} q_B(t)$$

Columns of T_{nb} (size (n, n_B)) contain all constraint modes.

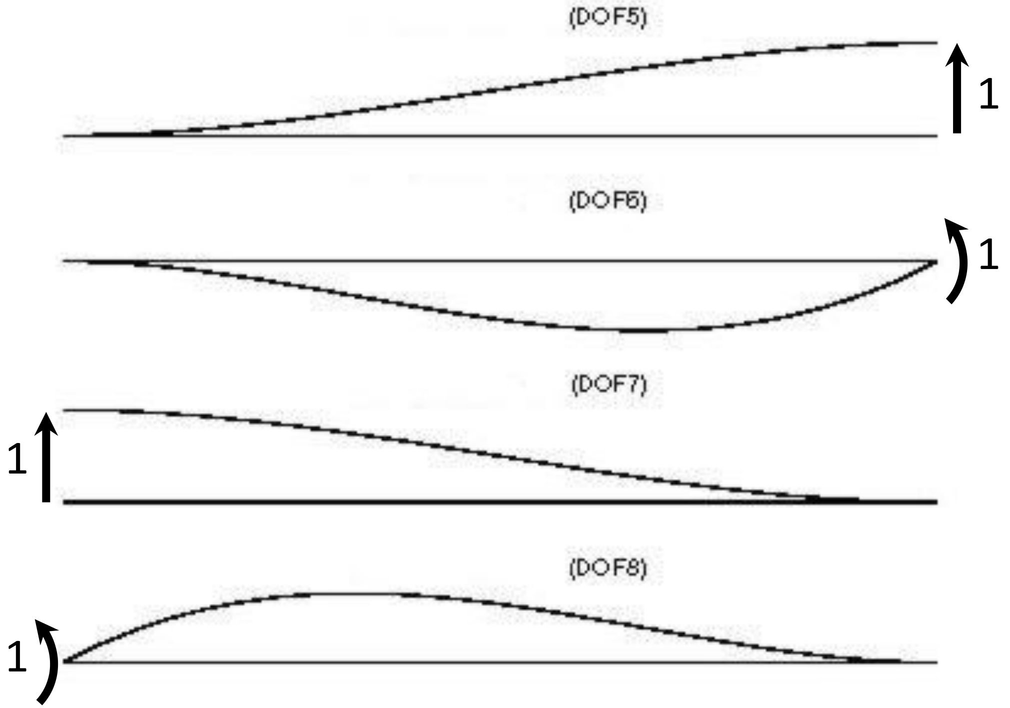
This transformation is analogous to Guyan reduction!

Example: boundary DOFs and static modes

Component 2 of cantilever beam model



For component 2, the constraint modes are defined by boundary dof’s 5-8



CMS-method of Craig-Bampton

Approximation:

$$q(t) = \begin{bmatrix} q_B(t) \\ q_I(t) \end{bmatrix} \approx q^S(t) + q^D(t)$$

- $q^S(t)$: dynamic response due to linear combination of n_B static modes
- $q^D(t)$: dynamic response due to linear combination of limited set of dynamic modes

Fixed-interface eigenmodes (dynamic modes)

$$q^D(t) = \begin{bmatrix} q_B^D(t) \\ q_I^D(t) \end{bmatrix} = \begin{bmatrix} O_{BK} \\ \Phi_{IK} \end{bmatrix} p_K(t)$$

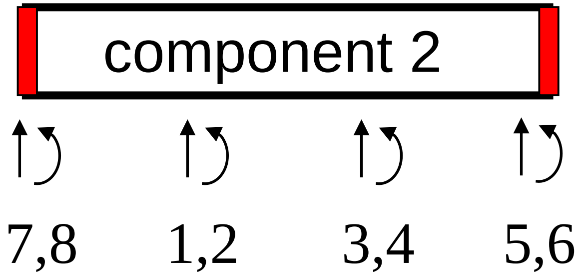
- $q_B^D(t) \equiv 0$: Boundary/interface dofs are fixed.
- Φ_{IK}: kept fixed-interface eigenmodes. Found solving the eigenvalue problem:

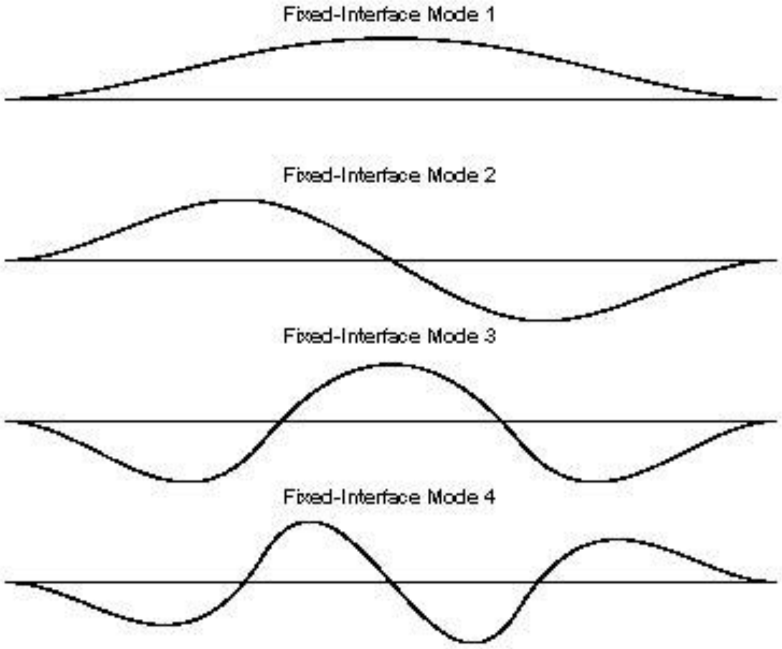
  $$\left(-\omega_j^2 M_{II} + K_{II}\right)\varphi_{Ij} = 0, \qquad \Phi_{IK} = [\varphi_{I1} \quad \varphi_{I2} \quad \cdots \quad \varphi_{In_K}].$$

  Φ_{IK} contains n_K modes φ_{Ij} for which $\omega_j < \omega_C$ (user defined).
  Kept angular eigenfrequencies are stored in $\Omega_{KK} = \mathrm{diag}([\omega_1 \quad \omega_2 \quad \cdots \quad \omega_{n_K}])$.

Example: fixed-interface eigenmodes (dynamic modes)

Component 2 of cantilever beam model





For the fixed-interface eigenmodes, the boundary DOFs 5-8 are set to zero

CMS-method of Craig-Bampton

Transformation

$$q(t) \approx q^S(t) + q^D(t) = \begin{bmatrix} I_{BB} \\ T_{IB} \end{bmatrix} q_B(t) + \begin{bmatrix} O_{BK} \\ \Phi_{IK} \end{bmatrix} p_K(t) = \begin{bmatrix} I_{BB} & O_{BK} \\ T_{IB} & \Phi_{IK} \end{bmatrix} \begin{bmatrix} q_B(t) \\ p_K(t) \end{bmatrix} = T^{CB} p(t)$$

Transformation matrix T^{CB} is $n \times (n_B + n_K)$. Generally, $n_B + n_K \ll n$.

Reduced equations of motion:

$$M^{CB} \ddot{p}(t) + K^{CB} p(t) = f^{CB}(t)$$

where

$$M^{CB} = (T^{CB})^\top M T^{CB} = \begin{bmatrix} M_{BB}^{CB} & M_{BK}^{CB} \\ \text{sym.} & I_{KK} \end{bmatrix}$$

$$K^{CB} = (T^{CB})^\top K T^{CB} = \begin{bmatrix} K_{BB}^{CB} & O_{BK} \\ \text{sym.} & \Omega_{KK}^2 \end{bmatrix}$$

$$f^{CB}(t) = (T^{CB})^\top f(t) = \begin{bmatrix} f_B(t) \\ \Phi_{IK}^\top f_I(t) \end{bmatrix}$$

Remarks on the method of Craig-Bampton

- If $n_K = 0$ (only constraint modes) we have Guyan's method
- If $n_K = n_I$ no reduction takes place, only coordinate transformation.
- If $f_I(t) = 0$ the reduced model is statically exact
- Reduced component models can be coupled easily (direct stiffness method, see later)
- It is unclear up to which frequency the reduced component model is accurate
- Experimental derivation of reduced model is very difficult

5d. The CMS-method of Rubin

CMS-method of Rubin

Partition:

$$q(t) = \begin{bmatrix} q_B(t) \\ q_W(t) \\ q_R(t) \end{bmatrix}$$

where

- $q(t)$: DOFs of considered component
- $q_B(t)$: boundary DOFs (length n_B) that consist of
  - DOFs at which the reduced component is coupled to other (reduced) components
  - externally loaded DOFs
  - DOFs with prescribed non-zero displacements
- $q_R(t)$: **minimal** set of dof's capable of suppressing rigid body modes
- $q_W(t)$: remaining DOFs.

Notation that will be used as well:

$$q(t) = \begin{bmatrix} q_B(t) \\ q_I(t) \end{bmatrix} = \begin{bmatrix} q_E(t) \\ q_R(t) \end{bmatrix}, \qquad \Rightarrow \qquad q_I(t) = \begin{bmatrix} q_W(t) \\ q_R(t) \end{bmatrix}, q_E(t) = \begin{bmatrix} q_B(t) \\ q_W(t) \end{bmatrix}.$$

M, K, and $f(t)$ are partitioned accordingly.

CMS-method of Rubin

Approximation:

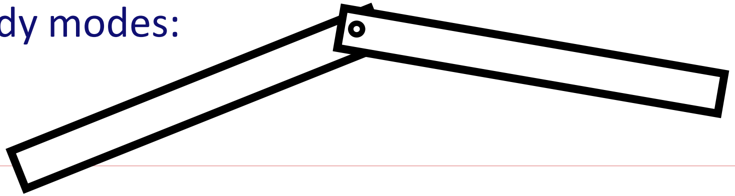
$$q(t) = \Phi_B p_B(t) + \Phi_R p_R(t) + \Phi_K p_K(t) = \begin{bmatrix} \Phi_B & \Phi_R & \Phi_K \end{bmatrix} \begin{bmatrix} p_B(t) \\ p_R(t) \\ p_K(t) \end{bmatrix} = T_1^R p_1(t)$$

- Φ_B: residual flexibility modes (length n_B static correction modes)
- Φ_R: rigid body modes (length n_R)
- Φ_K: kept elastic **free**-interface modes (length n_K)
- $p_1(t)$: generalized coordinates (length n_P)

Note: the Craig-Bampton method uses the **fixed**-interface modes Φ_{IK}. $\Phi_{IK} \neq \Phi_K$!

For significant reduction: $n_P = n_B + n_R + n_K \ll n$

Recall: A body in 3D has maximal 6 rigid body modes (3 translations & 3 rotations).
Mechanisms result in extra rigid body modes:



CMS-method of Rubin

Φ_B, Φ_R, and Φ_K are related to the (free-interface) eigenvalue problem.

$$\left(-\omega_j^2 M + K\right)\varphi_j = 0.$$

Sort all solutions of the eigenvalue problem such that $\omega_j \leq \omega_{j+1}$.
Store all eigenvalues and eigenvectors φ_j in

$$\Omega = \mathrm{diag}([\omega_1 \quad \cdots \quad \omega_n]), \qquad \Phi = [\varphi_1 \quad \dots \quad \varphi_n].$$

Use mass-normalization $\Phi^\top M \Phi = I$.

Rigid body modes $\boldsymbol{\Phi}_R$ and kept elastic free-interface modes $\boldsymbol{\Phi}_K$

$$\Phi = [\Phi_R \quad \Phi_K \quad \Phi_D].$$

- Φ_R are the rigid body modes
- Φ_K are the **kept** free-interface eigenmodes
- Φ_D are the **deleted** free-interface eigenmodes

n_K is chosen based on cut-off frequency ω_C such that $\omega_{n_K} \leq \omega_C < \omega_{n_K+1}$.

Alternative computation of Φ_R

Recall: $q(t) = \begin{bmatrix} q_E(t) \\ q_R(t) \end{bmatrix}$

Due to proper choice of $q_R(t)$ it is possible to compute

$$K\widetilde{\Phi}_R = O_R\,, \qquad \begin{bmatrix} K_{EE} & K_{ER} \\ K_{RE} & K_{RR} \end{bmatrix} \begin{bmatrix} \widetilde{\Phi}_{ER} \\ I_{RR} \end{bmatrix} = \begin{bmatrix} O_{ER} \\ O_{RR} \end{bmatrix}, \qquad \widetilde{\Phi}_R = \begin{bmatrix} \widetilde{\Phi}_{ER} \\ I_{RR} \end{bmatrix} = \begin{bmatrix} -K_{EE}^{-1} K_{ER} \\ I_{RR} \end{bmatrix}.$$

However, $\widetilde{\Phi}_R$ is typically not mass normalized, i.e. $\widetilde{\Phi}_R^\top M \widetilde{\Phi}_R \neq I_{RR}$.

Note that $K\widetilde{\Phi}_R = O_R$ implied that $K\widetilde{\Phi}_R Z = O_R$. So $\widetilde{\Phi}_R Z$ are also rigid body modes.

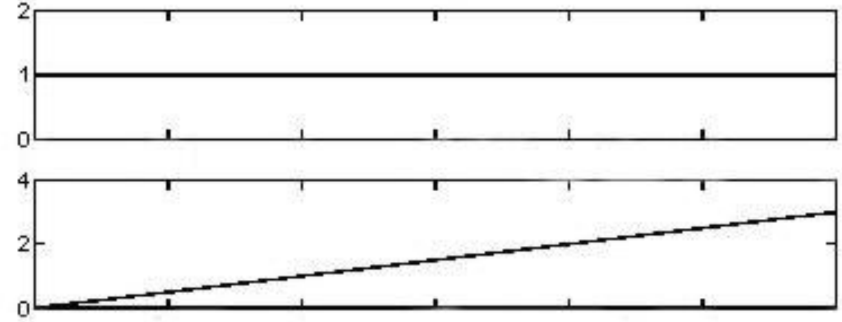
Find **transformation Z** such that $\Phi_R = \widetilde{\Phi}_R Z$ satisfies $\Phi_R^\top M \Phi_R = I_{RR}$.

1. Compute $\widetilde{M} := \widetilde{\Phi}_R^\top M \widetilde{\Phi}_R$
2. Solve eigenvalue problem $\widetilde{M}\tilde{z}_j = \gamma_j \tilde{z}_j$.
   These eigenvectors satisfy $\tilde{z}_j^\top \widetilde{M} \tilde{z}_k = 0$ for $j \neq k$.
3. Define $\widetilde{m}_j := \tilde{z}_j^\top \widetilde{M} \tilde{z}_j$ and $z_j := \tilde{z}_j / \sqrt{\widetilde{m}_j}$
   Then $z_j^\top \widetilde{M} z_k = 0$ for $j \neq k$ and $z_j^\top \widetilde{M} z_j = 1$.
4. Define $Z := \begin{bmatrix} z_1 & z_2 & \cdots & z_{n_R} \end{bmatrix}$.
   Then $Z^\top \widetilde{M} Z = I_{RR}$. It follows that $\Phi_R^\top M \Phi_R = Z^\top \widetilde{\Phi}_R^\top M \widetilde{\Phi}_R Z = Z^\top \widetilde{M} Z = I_{RR}$.

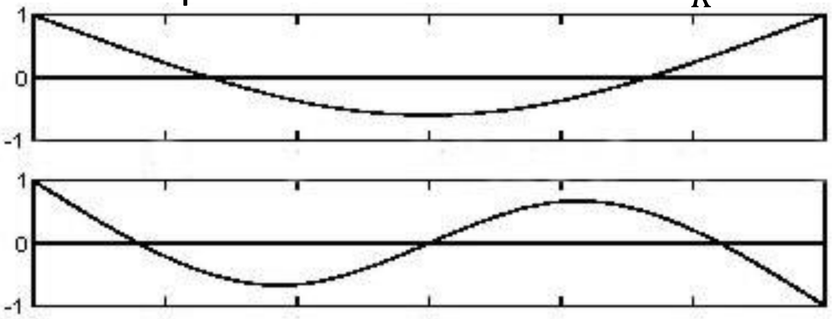
Example: Φ_R and Φ_K

Component 2 of cantilever beam model

Rigid body modes $\tilde{\Phi}_R = \begin{bmatrix} -K_{EE}^{-1}K_{ER} \\ I_{RR} \end{bmatrix}$



Kept free-interface modes Φ_K



There are two rigid body modes
(1 translation + 1 rotation)

We consider $n_K = 2$ kept free-interface modes.

CMS-method of Rubin

Φ_B, Φ_R, and Φ_K are related to the (free-interface) eigenvalue problem.

$$\left(-\omega_j^2 M + K\right)\varphi_j = 0.$$

Residual flexibility modes $\boldsymbol{\Phi_B}$

$$H(\omega) = \sum_{j=1}^{n_R+n_K} \frac{\varphi_j \varphi_j^\top}{-\omega^2 + \omega_j^2} + \sum_{j=n_R+n_K+1}^{n} \frac{\varphi_j \varphi_j^\top}{-\omega^2 + \omega_j^2}$$

For $\omega \ll \omega_C$,

$$H(\omega) = \sum_{j=1}^{n_R+n_K} \frac{\varphi_j \varphi_j^\top}{-\omega^2 + \omega_j^2} + \sum_{j=n_R+n_K+1}^{n} \frac{\varphi_j \varphi_j^\top}{\omega_j^2}$$

Residual flexibility $= \Phi_D \Omega_{DD}^{-2} \Phi_D^\top$

Partition $\Omega = \begin{bmatrix} \Omega_{RR} & O & O \\ O & \Omega_{KK} & O \\ O & O & \Omega_{DD} \end{bmatrix}$ according to $\Phi = [\Phi_R \quad \Phi_K \quad \Phi_D]$.

Residual flexibility modes are $\Phi_B = \Phi_D \Omega_{DD}^{-2} \Phi_D^\top B$ where $B = \begin{bmatrix} I_{BB} \\ O_{IB} \end{bmatrix}$

Unit loads are applied at boundary DOFs

Computation of $\boldsymbol{\Phi_B}$

$$\Phi_B = \Phi_D \Omega_{DD}^{-2} \Phi_D^\top B$$

Problem: in practice Φ_D is often not available!

Compute Φ_B therefore as follows:

- In case rigid body modes are absent:

$$\boxed{\Phi_B = (K^{-1} - \Phi_K \Omega_{KK}^{-2} \Phi_K^\top) B.}$$

  Note that $\Phi^\top K \Phi = \Omega^2$ implies that $K^{-1} = \Phi \Omega^{-2} \Phi^\top = \Phi_K \Omega_{KK}^{-2} \Phi_K^\top + \Phi_D \Omega_{DD}^{-2} \Phi_D^\top$.

- In case of rigid body modes:

$$\Phi_B = (P^\top G P - \Phi_K \Omega_{KK}^{-2} \Phi_K^\top) B,$$

  with

$$P = I - M \Phi_R \Phi_R^\top, \qquad G = \begin{bmatrix} K_{EE}^{-1} & O_{ER} \\ O_{RE} & O_{RR} \end{bmatrix}$$

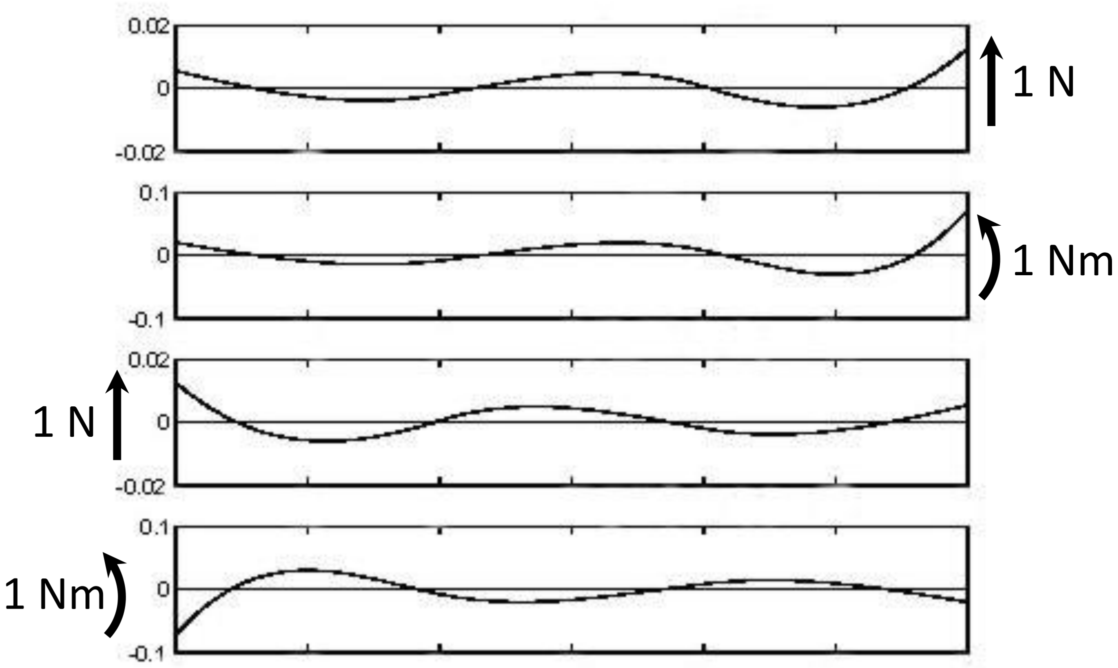
  Proof: see book

Example: Φ_B

Component 2 of cantilever beam

There are 4 boundary dof's, therefore also 4 residual flexibility modes Φ_B

Residual flexibility modes Φ_B



CMS-method of Rubin

$$q(t) = \begin{bmatrix} q_B(t) \\ q_I(t) \end{bmatrix} \approx \begin{bmatrix} \Phi_{BB} & \Phi_{BR} & \Phi_{BK} \\ \Phi_{IB} & \Phi_{IR} & \Phi_{IK} \end{bmatrix} \begin{bmatrix} p_B(t) \\ p_R(t) \\ p_K(t) \end{bmatrix} = T_1^R p_1(t)$$

Remaining problem: we need $q_B(t)$ instead of $p_B(t)$ for coupling to other components.

$$p_1(t) = \begin{bmatrix} p_B(t) \\ p_R(t) \\ p_K(t) \end{bmatrix} = \begin{bmatrix} \Phi_{BB}^{-1} & -\Phi_{BB}^{-1}\Phi_{BR} & -\Phi_{BB}^{-1}\Phi_{\mathrm{BK}} \\ O_{RB} & I_{RR} & O_{RK} \\ O_{KB} & O_{KR} & I_{KK} \end{bmatrix} \begin{bmatrix} q_B(t) \\ p_R(t) \\ p_K(t) \end{bmatrix} = T_2^R p(t).$$

Combine the two transformation matrices:

$$q(t) = T_1^R p_1(t) = T_1^R T_2^R p(t) = T^R p(t), \qquad T^R \coloneqq T_1^R T_2^R.$$

Reduced equations of motion for the Rubin method:

$$M^R \ddot{p}(t) + K^R p(t) = f^R(t),$$

with

$$M^R = (T^R)^\top M T^R, \qquad K^R = (T^R)^\top K T^R, \qquad f^R(t) = (T^R)^\top f(t).$$

Remarks on the method of Rubin

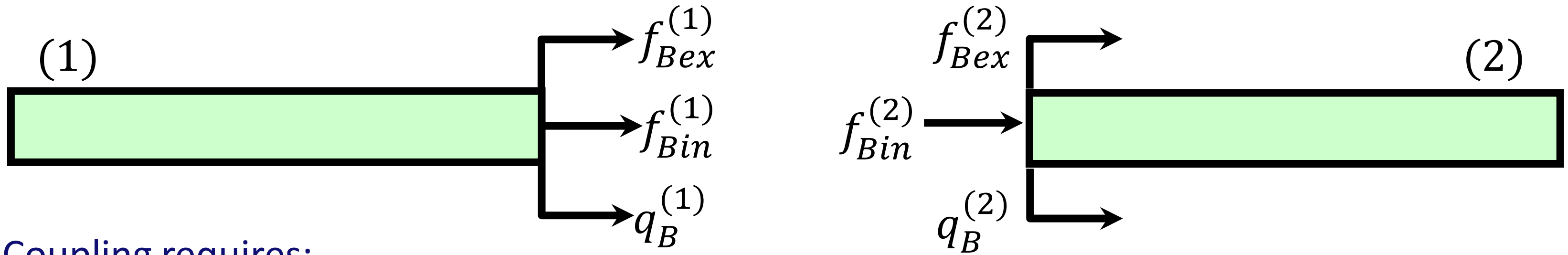
- To get substantial reduction: $n_B \ll n_D$
- If $f_I(t) = 0$ the reduced model is statically exact
- Numerical coupling of reduced component models requires T_2^R
- Reduced component model accurate up to cut-off frequency ω_C
- Experimental modal analysis can be used for experimental derivation of reduced model:
    - free-interface eigenmodes (component on soft springs)
    - residual flexibility modes via residual terms FRF

5e. Coupling and impedance coupling

Coupling of reduced substructure/component models

2 Substructures with their equations of motion ($j = 1,2$):

$$\begin{bmatrix} M_{BB}^{(j)} & M_{BP}^{(j)} \\ sym & M_{PP}^{(j)} \end{bmatrix} \begin{bmatrix} \ddot{q}_B^{(j)}(t) \\ \ddot{p}^{(j)}(t) \end{bmatrix} + \begin{bmatrix} K_{BB}^{(j)} & K_{BP}^{(j)} \\ sym & K_{PP}^{(j)} \end{bmatrix} \begin{bmatrix} q_B^{(j)}(t) \\ p^{(j)}(t) \end{bmatrix} = \begin{bmatrix} f_{Bex}^{(j)}(t) + f_{Bin}^{(j)}(t) \\ f_P^{(j)}(t) \end{bmatrix}$$



Coupling requires:

- equilibrium of internal forces: $f_{Bin}^{(1)}(t) = -f_{Bin}^{(2)}(t)$
- connectivity of interface DOFs: $q_B(t) = q_B^{(1)}(t) = q_B^{(2)}(t)$

Coupling of reduced substructure/component models

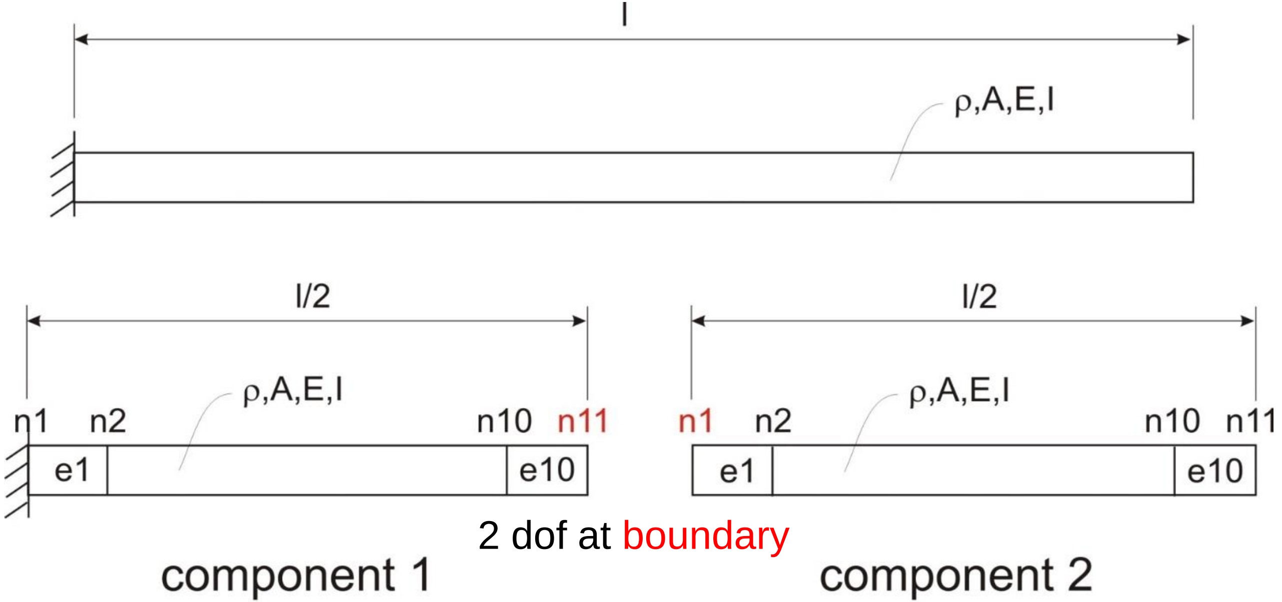
Reduced system equations (direct stiffness method):

$$\begin{bmatrix} M_{BB}^{(1)} + M_{BB}^{(2)} & M_{BP}^{(1)} & M_{BP}^{(2)} \\ sym & M_{PP}^{(1)} & O \\ sym & sym & M_{PP}^{(2)} \end{bmatrix} \begin{bmatrix} \ddot{q}_B(t) \\ \ddot{p}^{(1)}(t) \\ \ddot{p}^{(2)}(t) \end{bmatrix} + \begin{bmatrix} K_{BB}^{(1)} + K_{BB}^{(2)} & K_{BP}^{(1)} & K_{BP}^{(2)} \\ sym & K_{PP}^{(1)} & O \\ sym & sym & K_{PP}^{(2)} \end{bmatrix} \begin{bmatrix} q_B(t) \\ p^{(1)}(t) \\ p^{(2)}(t) \end{bmatrix} = \begin{bmatrix} f_{Bex}^{(1)}(t) + f_{Bex}^{(2)}(t) \\ f_P^{(1)}(t) \\ f_P^{(2)}(t) \end{bmatrix}$$

IMPORTANT: up to which frequency is this model accurate?
Is in general unknown.
For the Rubin method: approximately up to ω_C

Example: CMS methods

Clamped-free beam, pure bending,
2 DOF per node (transversal displacement + rotation)



$E = 2.1 \cdot 10^{11}$ Pa, $I = 10^{-4}/12$ m^4, $A = 0.01$ m^2, $l = 4$ m, $\rho = 7850$ kg/m^3

Example: CMS methods

Fair comparison between Craig-Bampton and Rubin method:
number of DOF of reduced system must be equal!

Component 1 (reduced to 4 dof):

C-B: 2 constraint modes + 2 fixed-interface eigenmodes
R: 2 residual flexibility modes + 2 elastic free-interface eigenmodes

Constraint modes and residual flexibility modes are defined for the 2 boundary DOF of node 11 in component 1

Component 2 (reduced to 5 dof):

C-B: 2 constraint modes + 3 fixed-interface eigenmodes
R: 2 residual flex modes + 2 rigid body modes +1 elastic free-interface eigenmode

Constraint modes and residual flexibility modes are defined for the 2 DOF of node 1 in component 2. In this case: constraint modes are rigid body modes!

Example: Results of Craig-Bampton method

Eigenfreq no	Comp 1 fixed-interface eigenmodes	Comp 2 fixed-interface eigenmodes
1	132.92	20.888
2	366.48	130.91
3		366.62

Eigenfreq no	Comp 1, red.	Comp 2, red.	System, red.	System, exact	Difference [%]
1	20.889	0	5.2220	5.2220	0
2	131.05	0	32.729	32.726	0.009
3	367.96	133.07	91.642	91.633	0.010
4	1491.7	367.80	180.10	179.56	0.301
5		1491.7	297.84	296.83	0.340
6			457.74	443.41	3.232
7			967.06	619.31	56.15

Example: Results of Rubin method

Eigenfreq no	Comp 1 free-interface eigenmodes	Comp 2 free-interface eigenmodes
1	20.888	0
2	130.91	0
3		132.92

Eigenfreq no	Comp 1, red.	Comp 2, red.	System, red.	System, exact	Difference [%]
1	20.888	0	5.2220	5.2220	0
2	130.91	0	32.726	32.726	0
3	403.97	132.92	91.635	91.633	0.002
4	2294.0	403.83	179.70	179.56	0.134
5		2298.1	305.60	296.83	2.955
6			550.76	443.41	24.21
7			1429.7	619.31	130.9

Reduction and coupling in the frequency domain

Reduction and coupling procedures can also be carried out in the frequency domain.

- Reduction method: **modal truncation (+ residual flexibility)**

- Coupling method: **Impedance coupling**.

Reduction in the frequency domain

Take not all n modes into account, but only a limited number n_C based on the frequency range of interest (see 2. Numerical Modal Analysis)

$$H(\omega) = \sum_{k=1}^{n} \frac{u_{Ok} u_{Ok}^T}{m_k^* (-\omega^2 + \omega_{Ok}^2)} \approx \sum_{k=1}^{n_C} \frac{u_{Ok} u_{Ok}^T}{m_k^* (-\omega^2 + \omega_{Ok}^2)}$$

Improve accuracy by taking **residual flexibilities** related to deleted modes into account

$$H_B(\omega) = \sum_{k=1}^{n} \frac{u_{Ok} u_{BOk}^T}{m_k^* (-\omega^2 + \omega_{Ok}^2)} \approx \sum_{k=1}^{n_C} \frac{u_{Ok} u_{BOk}^T}{m_k^* (-\omega^2 + \omega_{Ok}^2)} + \sum_{k=n_C+1}^{n} \frac{u_{Ok} u_{BOk}^T}{m_k^* \omega_{Ok}^2}$$

The residual flexibility term (last term) plays the same role as:

- Constraint modes in the Craig-Bampton method
- Residual flexibility modes in the Rubin method

The method can be applied on system or component level.

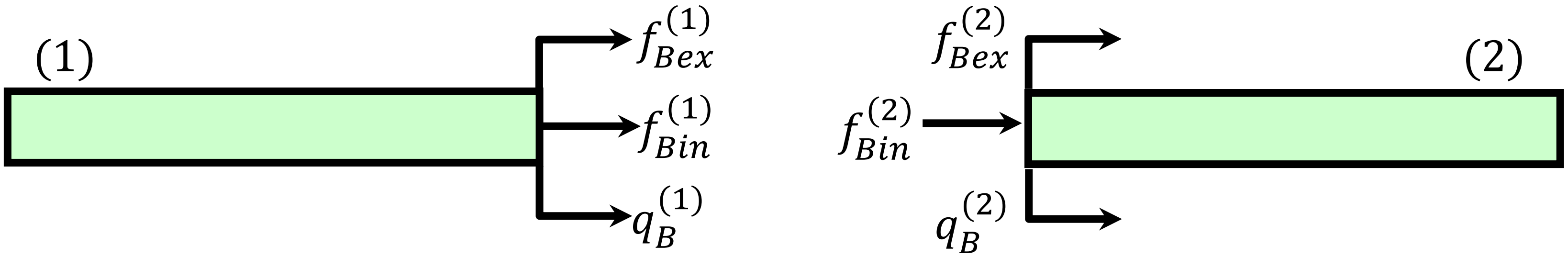
TU/e

Impedance coupling

Component coupling by means of component FRF matrices.

Given: FRF matrix for each component $j = 1,2$:

$$\begin{bmatrix} \hat{q}_B^{(j)} \\ \hat{q}_I^{(j)} \end{bmatrix} = \begin{bmatrix} H_{BB}^{(j)}(\omega) & H_{BI}^{(j)}(\omega) \\ H_{IB}^{(j)}(\omega) & H_{II}^{(j)}(\omega) \end{bmatrix} \begin{bmatrix} \hat{f}_{Bin}^{(j)} + \hat{f}_{Bex}^{(j)} \\ \hat{f}_I^{(j)} \end{bmatrix}.$$



Coupling requires:

- equilibrium of internal forces: $\hat{f}_{Bin}^{(1)} = -\hat{f}_{Bin}^{(2)}$
- connectivity of interface DOFs: $\hat{q}_B = \hat{q}_B^{(1)} = \hat{q}_B^{(2)}$

Impedance coupling

Leads to the following FRF matrix for the coupled system (proof: see book section 1.7)

$$\begin{bmatrix} \hat{q}_B \\ \hat{q}_I^{(1)} \\ \hat{q}_I^{(2)} \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) & H_{13}(\omega) \\ \text{sym.} & H_{22}(\omega) & H_{23}(\omega) \\ \text{sym.} & \text{sym.} & H_{33}(\omega) \end{bmatrix} \begin{bmatrix} \hat{f}_{Bex}^{(1)} + \hat{f}_{Bex}^{(2)} \\ \hat{f}_I^{(1)} \\ \hat{f}_I^{(2)} \end{bmatrix}$$

where

- $H_{11} = H_\kappa = \left[\left[H_{BB}^{(1)} \right]^{-1} + \left[H_{BB}^{(2)} \right]^{-1} \right]^{-1}$
- $H_{12} = H_\kappa \left[H_{BB}^{(1)} \right]^{-1} H_{BI}^{(1)}$, $\qquad H_{13} = H_\kappa \left[H_{BB}^{(2)} \right]^{-1} H_{BI}^{(2)}$
- $H_{22} = H_{II}^{(1)} - H_{IB}^{(1)} \left[H_{BB}^{(1)} \right]^{-1} H_{BI}^{(1)} + H_{IB}^{(1)} \left[H_{BB}^{(1)} \right]^{-1} H_\kappa \left[H_{BB}^{(1)} \right]^{-1} H_{BI}^{(1)}$
- $H_{23} = H_{IB}^{(1)} \left[H_{BB}^{(1)} \right]^{-1} H_\kappa \left[H_{BB}^{(2)} \right]^{-1} H_{BI}^{(2)}$
- $H_{33} = H_{II}^{(2)} - H_{IB}^{(2)} \left[H_{BB}^{(2)} \right]^{-1} H_{BI}^{(2)} + H_{IB}^{(2)} \left[H_{BB}^{(2)} \right]^{-1} H_\kappa \left[H_{BB}^{(2)} \right]^{-1} H_{BI}^{(2)}$

3 inversions needed!

Impedance coupling

Alternative expression (see Gordis et al., J. of Sound & Vibration, Vol. 150, 1991, pp. 139-158)

$$\begin{bmatrix} \hat{q}_B \\ \hat{q}_I^{(1)} \\ \hat{q}_I^{(2)} \end{bmatrix} = \left\{ \begin{bmatrix} H_{BB}^{(1)} & H_{BI}^{(1)} & O \\ H_{IB}^{(1)} & H_{II}^{(1)} & O \\ O & O & H_{II}^{(2)} \end{bmatrix} - \begin{bmatrix} H_{BB}^{(1)} \\ H_{IB}^{(1)} \\ -H_{IB}^{(2)} \end{bmatrix} \left[H_{BB}^{(1)} + H_{BB}^{(2)} \right]^{-1} \begin{bmatrix} H_{BB}^{(1)} \\ H_{IB}^{(1)} \\ -H_{IB}^{(2)} \end{bmatrix}^{\mathsf{T}} \right\} \begin{bmatrix} \hat{f}_{Bex}^{(1)} + \hat{f}_{Bex}^{(2)} \\ \hat{f}_I^{(1)} \\ \hat{f}_I^{(2)} \end{bmatrix}$$

Observe: factor between curly brackets is system FRF.

Only one inversion needed!