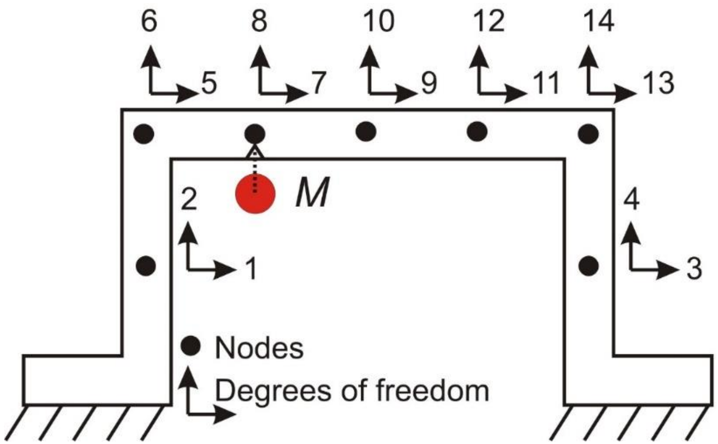


4. Sensitivity analysis

Structural Dynamics part of 4DM00

dr.ir. R.H.B. (Rob) Fey, ir. D.W.M. (Daniël) Veldman

Sensitivity analysis

General problem:

How does the dynamic behavior change when a specific design parameter value changes?

Possible design parameters:
point mass, spring stiffness, plate thickness, Youngs' modulus, radius of cylinder.

Sensitivity analysis

Problem: How do the dynamic characteristics (eigenvalues λ_k and eigenmodes v_k) change when a design parameter value changes?

We consider a general viscous damping model:

$$(\lambda_k C + D)v_k = 0$$

Design parameters are denoted by: $p_\alpha, p_\beta, \dots$
Matrices C and D depend on design parameters,

$$C = C(p_\alpha, p_\beta, \dots), \qquad D = D(p_\alpha, p_\beta, \dots)$$

Partial derivative of a matrix $A(p_\alpha, p_\beta, \dots)$ w.r.t. design parameter p_α is denoted by

$$A_{,\alpha} = \frac{\partial A}{\partial p_\alpha}$$

Problem: find $\lambda_{k,\alpha}$ and $v_{k,\alpha}$ (and $w_{k,\alpha}$)

Adjoint method (eigenvalue sensitivity)

- Eigenvalue problem

$$(\lambda_k C + D)v_k = 0$$

- Differentiate w.r.t. p_α

$$\left(\lambda_{k,\alpha} C + \lambda_k C_{,\alpha} + D_{,\alpha}\right)v_k + (\lambda_k C + D)v_{k,\alpha} = 0$$

- Premultiply with $w_k^\top$ (which satisfies $w_k^\top(\lambda_k C + D) = 0^\top$)

$$w_k^\top\left(\lambda_{k,\alpha} C + \lambda_k C_{,\alpha} + D_{,\alpha}\right)v_k = 0.$$

$$\boxed{\lambda_{k,\alpha} = -\frac{w_k^\top\left(\lambda_k C_{,\alpha} + D_{,\alpha}\right)v_k}{w_k^\top C v_k}}$$

Adjoint method (right eigenvector sensitivity)

$$\left(\lambda_{k,\alpha} C + \lambda_k C_{,\alpha} + D_{,\alpha}\right) v_k + (\lambda_k C + D) v_{k,\alpha} = 0$$

- Write $v_{k,\alpha}$ in terms of eigenvectors v_k

$$v_{k,\alpha} = \sum_{p=1}^{2n} a_{kp} v_p$$

- Insert and premultiply with w_l^{T}

$$w_l^{\mathsf{T}} \left(\lambda_{k,\alpha} C + \lambda_k C_{,\alpha} + D_{,\alpha}\right) v_k + \sum_{p=1}^{2n} w_l^{\mathsf{T}} (\lambda_k C + D) a_{kp} v_p = 0$$

- For $l \neq k$, the bi-orthogonality property shows that $w_l^{\mathsf{T}} C v_k = w_l^{\mathsf{T}} D v_k = 0$

$$w_l^{\mathsf{T}} \left(\lambda_k C_{,\alpha} + D_{,\alpha}\right) v_k + a_{kl} w_l^{\mathsf{T}} (\lambda_k C + D) v_l = 0$$

- Finally, as $w_l^{\mathsf{T}} D v_l = -\lambda_l w_l^{\mathsf{T}} C v_l$, we can rewrite $w_l^{\mathsf{T}} (\lambda_k C + D) v_l = (\lambda_k - \lambda_l) w_l^{\mathsf{T}} C v_l$

$$\boxed{a_{kl} = -\frac{w_l^{\mathsf{T}} \left(\lambda_k C_{,\alpha} + D_{,\alpha}\right) v_k}{(\lambda_k - \lambda_l) w_l^{\mathsf{T}} C v_l} \qquad (k \neq l)}$$

Adjoint method (left eigenvector sensitivity)

$$w_k^\top\left(\lambda_{k,\alpha}C + \lambda_k C_{,\alpha} + D_{,\alpha}\right) + w_{k,\alpha}^\top(\lambda_k C + D) = 0$$

- Write $w_{k,\alpha}$ in terms of eigenvectors w_k

$$w_{k,\alpha} = \sum_{p=1}^{2n} b_{kp} w_p$$

- Insert and postmultiply with v_l

$$w_k^\top\left(\lambda_{k,\alpha}C + \lambda_k C_{,\alpha} + D_{,\alpha}\right)v_l + \sum_{p=1}^{2n} b_{kp} w_p^\top(\lambda_k C + D)v_l \ = 0$$

- For $l \neq k$, the bi-orthogonality property shows that $w_l^\top C v_k = w_l^\top D v_k = 0$

$$w_k^\top\left(\lambda_k C_{,\alpha} + D_{,\alpha}\right)v_l + b_{kl} w_l^\top(\lambda_k C + D)v_l = 0$$

- Finally, as $w_l^\top D v_l = -\lambda_l w_l^\top C v_l$

$$\boxed{b_{kl} = -\frac{w_k^\top\left(\lambda_k C_{,\alpha} + D_{,\alpha}\right)v_l}{(\lambda_k - \lambda_l) w_l^\top C v_l} \qquad (k \neq l)}$$

What are a_{kk} and b_{kk}?

Formulas for a_{kl} and b_{kl} on previous slides are not well-defined for $k = l$.

Two options to choose a_{kk} and b_{kk}.

OPTION 1: $a_{kk} = b_{kk} = 0$.

Motivation: If v_k is an eigenmode, then zv_k is also an eigenmode for any $z \in \mathbb{C}$.
When $|a_{kk}|$ compared to the other coefficients a_{kl},
$v_{k,\alpha} \approx a_{kk}v_k$ only shows a rescaling of v_k (not a true change in the eigenmode).
Setting $a_{kk} = 0$ gives a better idea about the change in shape.

Easy to implement.

Is equivalent to requiring that $w_k^\top C v_{k,\alpha} = 0$ and $v_k^\top C w_{k,\alpha} = 0$.

Normalization rule

OPTION 2: Choose a_{kk} and b_{kk} according to normalization rule.

Depends on the **normalization rule** for the eigenvectors v_k and w_k. We normalize

- v_k such that $v_k\left[i_{max,k}\right] = 1$ where $i_{max,k} = \max_i |v_k[i]|$ and
- w_k such that $w_k\left[l_{max,k}\right] = 1$ where $l_{max,k} = \max_l |w_k[l]|$.

MATLAB:
[~,imaxk]=max(vk)

(Element of largest magnitude in v_k and w_k is scaled to 1)

Warning: other normalization rules such as

- $w_k^\top C v_k = 1$ and $v_k^H v_k = w_k^H w_k$ or
- $v_k^H v_k = 1$ and $w_k^H w_k = 1$

do not allow to determine a_{kk} and b_{kk} uniquely (see slides at the end).

Adjoint method (coefficients a_{kk} and b_{kk})

The coefficient a_{kk} now follows from

$$v_k\left[i_{max,k}\right] = 1, \qquad v_{k,\alpha}\left[i_{max,k}\right] = \sum_{p=1}^{2n} a_{kp} v_p\left[i_{max,k}\right] = 0,$$

so that

$$\boxed{a_{kk} = -\sum_{\substack{p=1,\\ p\neq k}}^{2n} a_{kp} v_p\left[i_{max,k}\right].}$$

Similarly,

$$\boxed{b_{kk} = -\sum_{\substack{p=1,\\ p\neq k}}^{2n} b_{kp} w_p\left[l_{max,k}\right].}$$

Remark: adjoint method can be used to find higher-order derivatives, but formulas are complicated.

Direct method (eigenvalue+right eigenvector)

- Differentiate $(\lambda_k C + D)v_k = 0$ w.r.t. p_α:
$$(\lambda_{k,\alpha} C + \lambda_k C_{,\alpha} + D_{,\alpha})v_k + (\lambda_k C + D)v_{k,\alpha} = 0$$
- Collects unknowns $v_{k,\alpha}$ and $\lambda_{k,\alpha}$ on the left hand side
$$[\lambda_k C + D \quad C v_k] \begin{bmatrix} v_{k,\alpha} \\ \lambda_{k,\alpha} \end{bmatrix} = -(\lambda_k C_{,\alpha} + D_{,\alpha})v_k$$
$2n$ equations, $2n+1$ unknowns.
- Remove one unknown with normalization rule $v_k[i_{max,k}] = 1 \quad \Rightarrow \quad v_{k,\alpha}[i_{max,k}] = 0$.
Solve $2n$ equations
$$\boxed{B y_{,\alpha} = r_\alpha}$$
for the $2n$ unknowns $y_{,\alpha}$ where
  - B is equal to $[\lambda_k C + D \quad C v_k]$ with the $i_{max,k}$-th column removed
  - $y_{,\alpha}$ is equal to $\begin{bmatrix} v_{k,\alpha} \\ \lambda_{k,\alpha} \end{bmatrix}$ with the $i_{max,k}$-th element removed
  - $r_\alpha = -(\lambda_k C_{,\alpha} + D_{,\alpha})v_k$

TU/e

Remarks on the direct method

- Similar derivation possible for eigenvalue λ_k and left eigenvectors w_k

- Higher-order derivatives such as $\lambda_{k,\alpha\beta}$, $v_{k,\alpha\beta}$, and $w_{k,\alpha\beta}$ can be obtained similarly. Example for the right-eigenvector approach:
$$By_{,\alpha\beta} = r_{\alpha\beta}$$
where
  - B is equal to $[\lambda_k C + D \quad Cv_k]$ with the i_{mk}-th column removed (same as before!)
  - $y_{,\alpha\beta}$ is equal to $\begin{bmatrix} v_{k,\alpha\beta} \\ \lambda_{k,\alpha\beta} \end{bmatrix}$ with the i_{mk}-th element removed
  - $r_{\alpha\beta} = -\left(\lambda_{k,\alpha} C_{,\beta} + \lambda_{k,\beta} C_{,\alpha} + \lambda_k C_{,\alpha\beta} + D_{\alpha\beta}\right) v_k$
    $-\left(\lambda_{k,\alpha} C + \lambda_k C_{,\alpha} + D_{,\alpha}\right) v_{k,\beta} - \left(\lambda_{k,\beta} C + \lambda_k C_{,\beta} + D_{,\beta}\right) v_{k,\alpha}$

Comparison

Adjoint Method	Direct Method
$C_{,\alpha}$ and $D_{,\alpha}$ needed	$C_{,\alpha}$ and $D_{,\alpha}$ needed
Calculation of $\lambda_{k,\alpha}$ for a single k is cheap, only the corresponding λ_k, v_k, and w_k are needed	For each single k: Calculation of $\lambda_{k,\alpha}$, $v_{k,\alpha}$, and $w_{k,\alpha}$ requires λ_k, v_k, and w_k and the solution of a set of linear algebraic equations
Calculation of $v_{k,\alpha}$ and $w_{k,\alpha}$ is expensive, solution of complete eigenvalue problem is needed	
More than 1 design parameter does not need much additional effort	More than 1 design parameter has only consequences for right hand side
	Second-order derivatives are easily obtained

System matrix derivatives

Note that:

$$C_{,\alpha} = \begin{bmatrix} B_{,\alpha} & M_{,\alpha} \\ M_{,\alpha} & O \end{bmatrix}, \qquad D_{,\alpha} = \begin{bmatrix} K_{,\alpha} & O \\ O & M_{,\alpha} \end{bmatrix}$$

So we need to determine $M_{,\alpha}$, $B_{,\alpha}$ and $K_{,\alpha}$.

Two situations:

1. Analytical expressions:

$$M_{,\alpha} = \frac{\partial M}{\partial p_\alpha}, \quad \text{etc.}$$

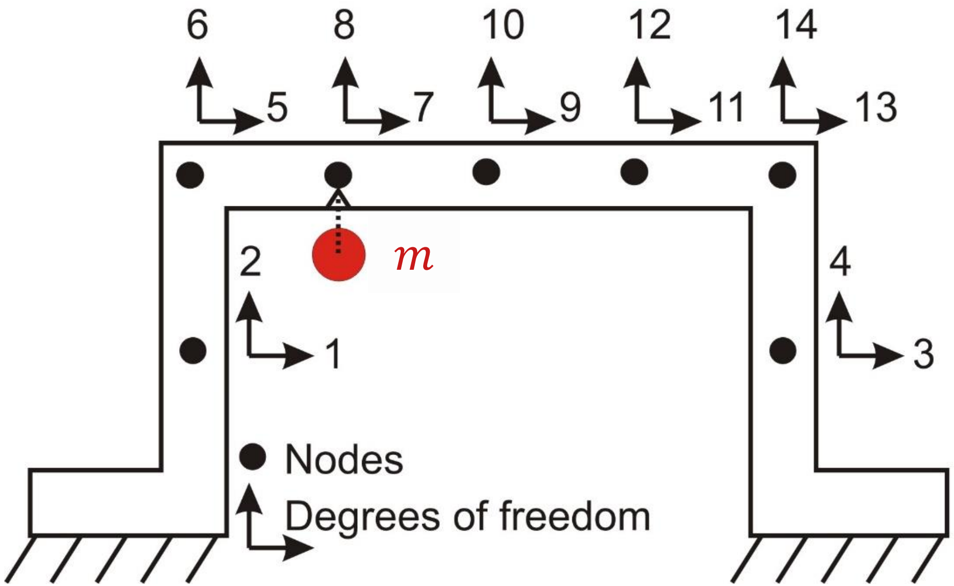
Examples: discrete mass, discrete spring stiffness, discrete viscous damping constant, Young's modulus, mass density.

2. Numerical approximations:

$$M_{,\alpha} \approx \frac{\Delta M}{\Delta p_\alpha} = \frac{M(p_\alpha + \Delta p_\alpha) - M(p_\alpha)}{\Delta p_\alpha}, \quad \text{etc.}$$

Examples: radius of cylinder, distance between plate stiffeners

Example 1: Portal frame with point mass

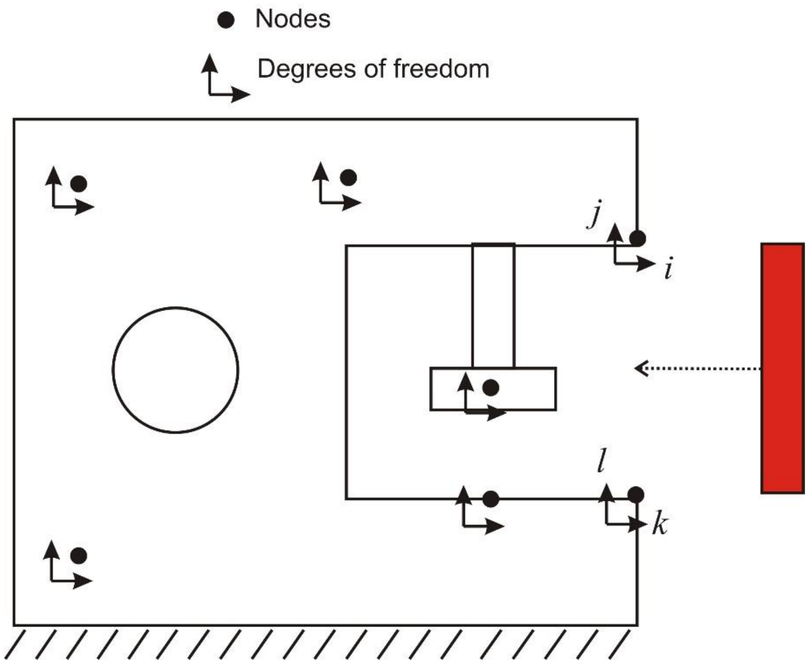


$$M_{,\alpha} = \begin{array}{cc} & \begin{array}{cccc} & 7 & 8 & \end{array} \\ \begin{array}{c} \\ 7 \\ 8 \\ \end{array} & \begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix} \end{array}$$

$$K_{,\alpha} = B_{,\alpha} = O$$

Design parameter is discrete mass m.

Example 2: Milling machine

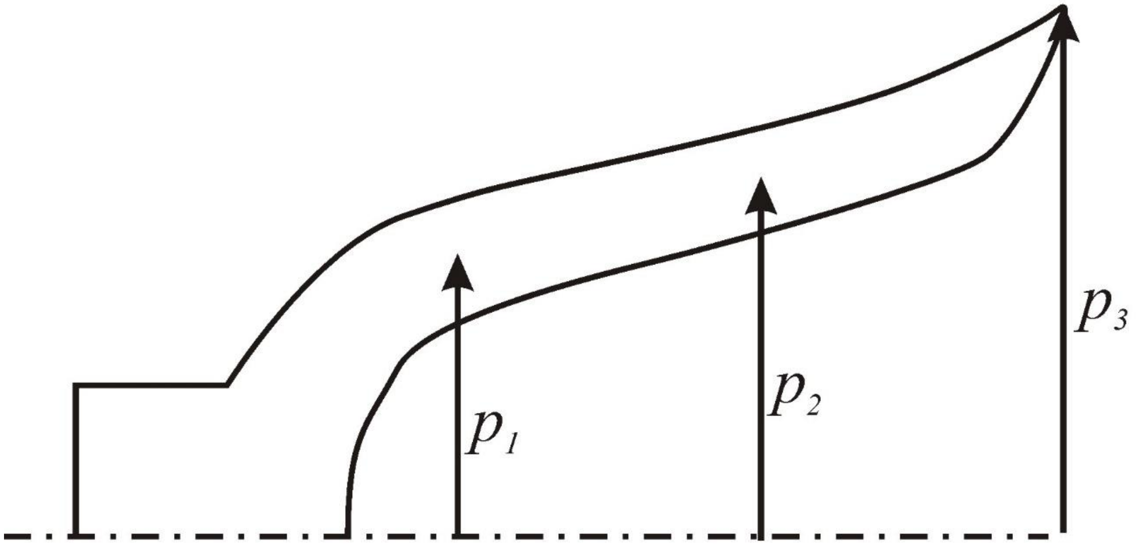


$$K_{,\alpha} = \begin{array}{c c} & \begin{array}{cc} j & l \end{array} \\ \begin{array}{c} j \\ l \end{array} & \begin{bmatrix} \ddots & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & \ddots \end{bmatrix} \end{array}$$

$$M_{,\alpha} = B_{,\alpha} = O$$

Design parameter is axial stiffness k of (massless) spring between nodes j and l.

Example 3: Bell



Only numerical approximations are possible:

$$M_{,\alpha} \approx \frac{\Delta M}{\Delta p_\alpha}$$

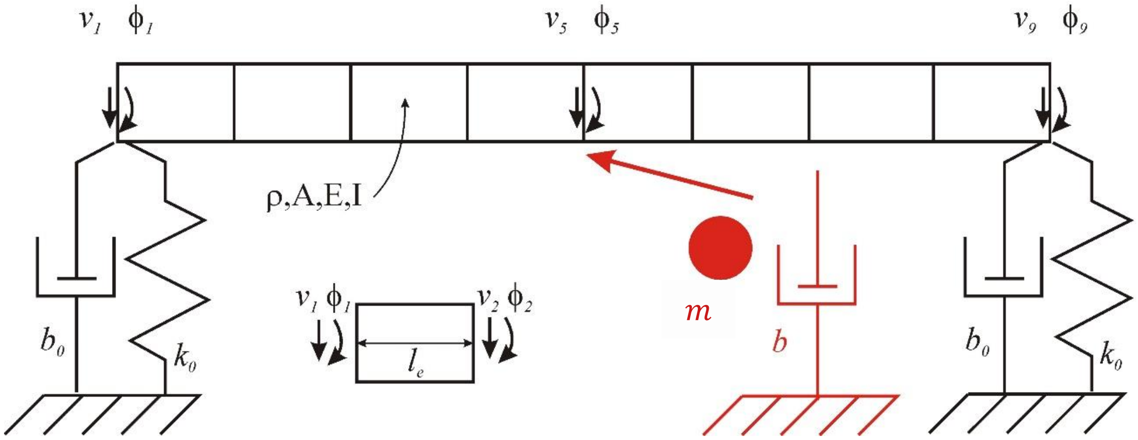
$$K_{,\alpha} \approx \frac{\Delta K}{\Delta p_\alpha}$$

The 3 design parameters are the radii at three axial positions

Example: sensitivity analysis

Beam model

- 8 elements, 18 DOFs
- $l_{tot} = 1$ m,
  $\rho A = 7$ kg/m,
  $EI = 1.5 \cdot 10^4$ Nm^2,
  $k_o = 2 \cdot 10^6$ N/m,
  $b_o = 2 \cdot 10^3$ Ns/m.

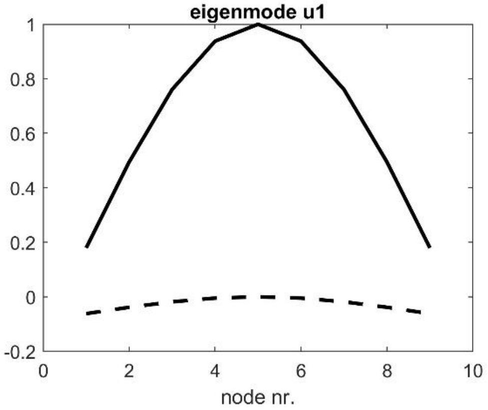


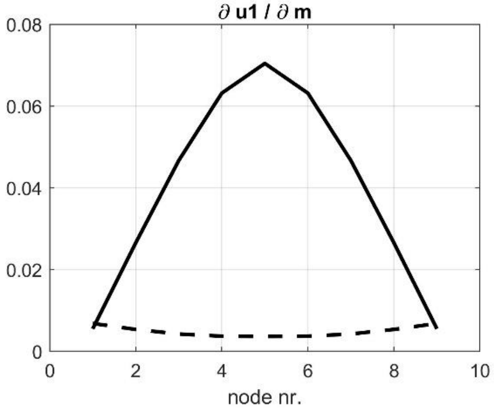
$\lambda_1 = -18.43 + 404.3j$ (first bending mode, weakly damped)

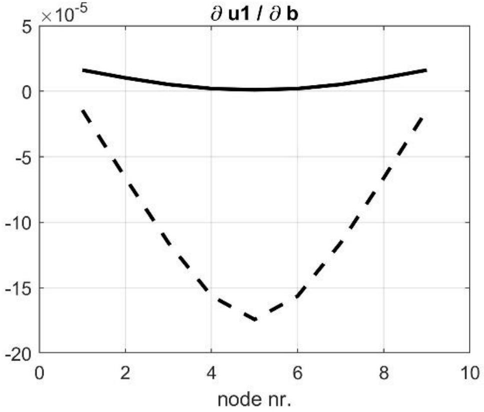
$\partial\lambda_1/\partial m = +6.5 - 53j$ $\partial\lambda_1/\partial b = -0.13 - 0.01j$

Sensitivity of first eigenmode

$a_{kk} = 0$ according to OPTION 1





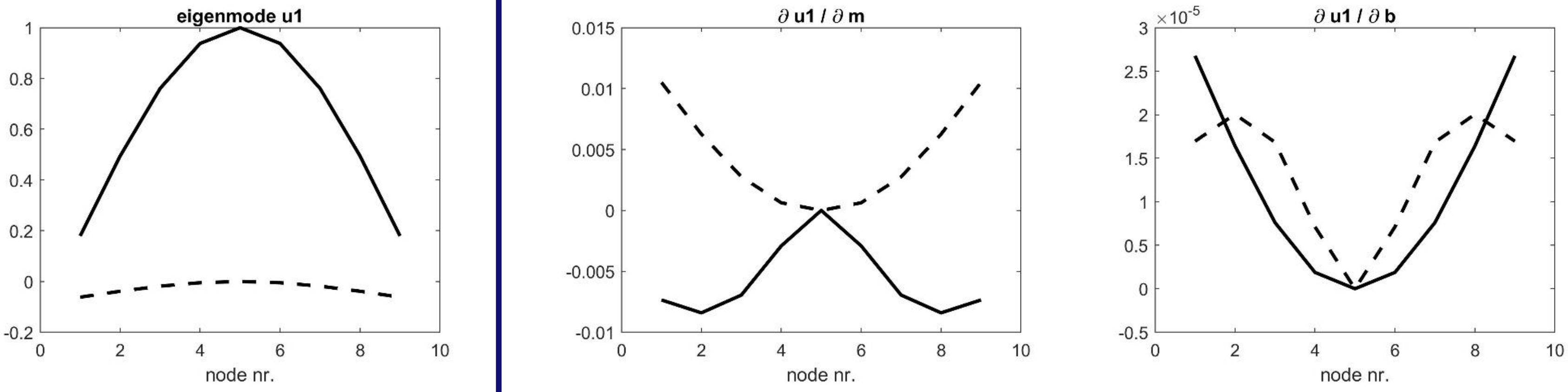


Figures show the real parts (solid lines) and imaginary parts (dashed lines)

Adding mass m increases bending, adding damping b decreases bending.

Sensitivity of first eigenmode

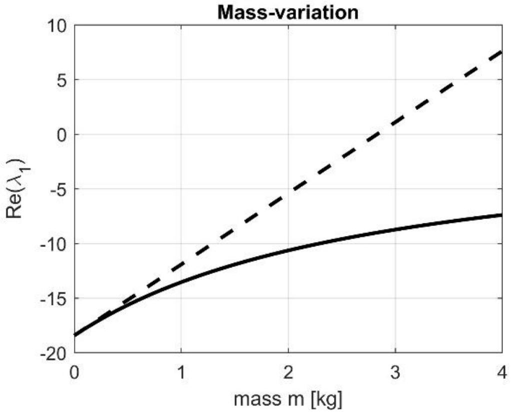
a_{kk} according to OPTION 2

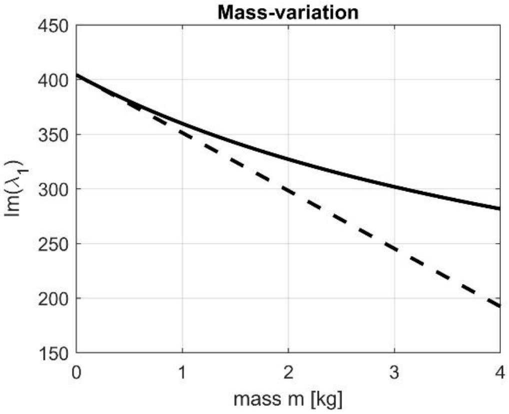


Figures show the real parts (solid lines) and imaginary parts (dashed lines)

Adding mass m increases bending, adding damping b decreases bending.

Linear approximations





Solid lines: exact eigenvalue: $\lambda_1(m)$
Dashed lines: linear approximation: $\lambda_1(m=0) + \frac{\partial \lambda_1}{\partial m}(m=0)m$

Accurate approximation for eigenvalue λ_1 if $m < 0.5$ kg
For comparison $\rho A l_{tot} = 7$ kg

4b. Model updating

Model updating using eigenvalue sensitivity

Given: λ_e, a column of e experimentally measured eigenvalues.

Given: a numerical model $(\lambda_k C(p) + D(p))v_k = 0$
depending on a vector of q parameters p.
Note: typically, $q < 2e$.

Problem: find/adapt the parameter values p such that $\lambda_n = \lambda_n(p)$, a vector containing e eigenvalues of the numerical model, match with λ_e.
Note: Corresponding eigenmodes should be similar.

Model updating using eigenvalue sensitivity

Iterative approach: $p_{(i)}$ denotes the parameter values at iteration i

For small parameter variations $\Delta p_{(i)}$ around $p_{(i)}$

$$\lambda_n\left(p_{(i)} + \Delta p_{(i)}\right) \approx \lambda_n\left(p_{(i)}\right) + \left.\frac{\partial \lambda_n}{\partial p}\right|_{p=p_{(i)}} \Delta p_{(i)} =: \lambda_{n(i)} + S_{(i)} \Delta p_{(i)},$$

where

$$\lambda_{n(i)} \coloneqq \lambda_n\left(p_{(i)}\right), \qquad S_{(i)} = \left.\frac{\partial \lambda_n}{\partial p}\right|_{p=p_{(i)}}.$$

Note: $S_{(i)}$ is computed using sensitivity analysis.

Introduce

$$\Delta\lambda_{(i)} \coloneqq \lambda_e - \lambda_{n(i)} - S_{(i)} \Delta p_{(i)}$$

Note: $\Delta\lambda_{(i)} \neq \lambda_e - \lambda_n\left(p_{(i)} + \Delta p_{(i)}\right)$, but $\Delta\lambda_{(i)} \approx \lambda_e - \lambda_n\left(p_{(i)} + \Delta p_{(i)}\right)$ for small $\Delta p_{(i)}$.

Model updating using eigenvalue sensitivity

The number of parameters q is typically small (i.e. $q < 2e$).

$\Rightarrow$ it is typically impossible to achieve $\lambda_n(p) = \lambda_e$.

Least squares approach: Minimize

$$\varepsilon_{(i)} = \Delta\lambda_{(i)}^H W \, \Delta\lambda_{(i)}$$

- $\varepsilon_{(i)}$ is a scalar-valued cost function
- $\Delta\lambda_{(i)} = \lambda_e - \lambda_{n(i)} - S_{(i)}\Delta p_{(i)}$
- W is a real, positive definite, symmetric (often diagonal) weighting matrix.

Optimal $\boldsymbol{\Delta p_{(i)}}$: $\dfrac{\partial\varepsilon}{\partial\Delta p_{(i)}} = 0, \qquad \mathrm{Re}\big(S_{(i)}^H W S_{(i)}\big)\,\Delta p_{(i)} = \mathrm{Re}\Big(S_{(i)}^H W\big(\lambda_e - \lambda_{n(i)}\big)\Big)$

Update: Parameter update $p_{(i+1)} = p_{(i)} + \Delta p_{(i)}$ and model update $\lambda_{n(i+1)}$ and $S_{(i+1)}$.

Stop when $\varepsilon_{(i)}$ and/or the relative changes $\Delta p_{\alpha(i)}/p_{\alpha(i)}$ $(\alpha = 1, \dots, q)$ are small enough.

Problems with normalization rule 3

$$v_k^H v_k = 1 \text{ and } w_k^H w_k = 1$$

The normalization rule does not uniquely define the eigenvectors:

- Set $\tilde{v}_k = \mu v_k$ for some $\mu \in \mathbb{C}$.
- $\tilde{v}_k^H \tilde{v}_k = \bar{\mu} v_k^H \mu v_k = |\mu|^2 v_k^H v_k$, so only absolute value of μ is fixed by normalization rule.
- Still leaves the possibility to set $\mu = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

The rule is also not sufficient to determine a_{kk} and b_{kk}.
Illustration for a_{kk}:

- Differentiate $v_k^H v_k = 1$ w.r.t. p_α

$$v_{k,\alpha}^H v_k + v_k^H v_{k,\alpha} = 2\mathrm{Re}\left(v_k^H v_{k,\alpha}\right) = 0$$

- Expand $v_{k,\alpha}$ in the eigenmodes v_p: $v_{k,\alpha} = \sum_p a_{kp} v_p$

$$2\mathrm{Re}(a_{kk}) v_k^H v_k = -2 \sum_{\substack{p=1 \\ p \neq k}}^{2n} \mathrm{Re}\left(a_{kp} v_k^H v_p\right).$$

- So only the real part of a_{kk} is determined. Imaginary part is still free!

Problems with normalization rule 1

$$w_k^\top C v_k = 1 \text{ and } v_k^H v_k = w_k^H w_k$$

The normalization rule does not uniquely define the eigenvectors:

- Assume v_k and w_k satisfy the normalization rule, i.e. $w_k^\top C v_k = 1$ and $v_k^H v_k = w_k^H w_k$.
- Set $\tilde{v}_k = \mu_1 v_k$ and $\tilde{w}_k = \mu_2 w_k$ for some $\mu_1, \mu_2 \in \mathbb{C}$.
- Requiring that $\tilde{v}_k^H \tilde{v}_k = \tilde{w}_k^H \tilde{w}_k$ implies that $|\mu_1|^2 v_k^H v_k = |\mu_2|^2 w_k^H w_k$.
- Because $v_k^H v_k = w_k^H w_k$, it follows that $|\mu_1| = |\mu_2|$.

- So we can write $\mu_1 = re^{i\theta_1}$ and $\mu_2 = re^{i\theta_2}$ for some $r \geq 0$ and $\theta_1, \theta_2 \in \mathbb{R}$.
- Requiring $1 = \tilde{w}_k^\top C \tilde{v}_k = r^2 e^{i(\theta_1+\theta_2)} w_k^\top C v_k = r^2 e^{i(\theta_1+\theta_2)}$ shows that $r = 1$ and $\theta_2 = -\theta_1$.
- Still leaves the possibility to choose any $\theta_1 \in \mathbb{R}$.

Problems with normalization rule 1

$$w_k^\top C v_k = 1 \text{ and } v_k^H v_k = w_k^H w_k$$

The rule is also not sufficient to determine a_{kk} and b_{kk}.

- From the book we find the conditions

$$a_{kk} + b_{kk} = -w_k^\top C_{,\alpha} v_k, \qquad \mathrm{Re}\left(v_{k,\alpha}^H v_k\right) = \mathrm{Re}\left(w_{k,\alpha}^H w_k\right).$$

- Writing $v_{k,\alpha} = \sum_p a_{kp} v_p$ and $w_{k,\alpha} = \sum_p b_{kp} w_p$, the latter condition can be rewritten as

$$2\mathrm{Re}(a_{kk}) v_k^H v_k - 2\mathrm{Re}(b_{kk}) w_k^H w_k = -2 \sum_{\substack{p=1 \\ p \neq k}}^{2n} \mathrm{Re}\left(a_{kp} v_k^H v_p\right) + 2 \sum_{\substack{p=1 \\ p \neq k}}^{2n} \mathrm{Re}\left(b_{kp} w_k^H w_p\right).$$

- There are 4 unknowns ($\mathrm{Re}(a_{kk})$, $\mathrm{Im}(a_{kk})$, $\mathrm{Re}(b_{kk})$, $\mathrm{Im}(b_{kk})$) but only 3 equations.
- Imaginary parts of a_{kk} and b_{kk} are not uniquely determined. Solution set is of the form:

$$a_{kk} = a_{kk,0} + jC, \qquad b_{kk} = b_{kk,0} - jC, \qquad C \in \mathbb{R}.$$

FRF derivatives

Sensitivity of FRF due to change in design parameter p_α.

$$H(\omega, p_\alpha) = \sum_{k=1}^{2n} \frac{A_k(p_\alpha)}{(j\omega - \lambda_k(p_\alpha))}, \qquad A_k(p_\alpha) = \frac{u_k(p_\alpha) x_k^\top(p_\alpha)}{w_k^\top(p_\alpha) C v_k(p_\alpha)}.$$

Differentiate to p_α:

$$H_{,\alpha}(\omega) = \sum_{k=1}^{2n} \left[\frac{A_{k,\alpha}}{(j\omega - \lambda_k)} + \frac{A_k \lambda_{k,\alpha}}{(j\omega - \lambda_k)^2} \right].$$

Estimation based on first order Taylor series expansion:

$$H(\omega, p_\alpha + \Delta p_\alpha) = H(\omega, p_\alpha) + H_{,\alpha}(\omega, p_\alpha)\Delta p_\alpha + O\{\Delta p_\alpha^2\}.$$

Only meaningful if for relatively small Δp_α.
Not meaningful for weakly damped systems: Higher order terms $\frac{Q(\Delta p_\alpha)^i}{(j\omega - \lambda_k)^{i+1}}$ will not be small!

Alternative 1: finite difference approximation

1. Determine $H(\omega, p_\alpha)$ using $\lambda_k(p_\alpha), v_k(p_\alpha)$, and $w_k(p_\alpha)$.
2. Choose a (sufficiently small) Δp_α and
   determine $H(\omega, p_\alpha + \Delta p_\alpha)$ using $\lambda_k(p_\alpha + \Delta p_\alpha), v_k(p_\alpha + \Delta p_\alpha)$, and $w_k(p_\alpha + \Delta p_\alpha)$.
3. Form the differential quotient:

$$H_{,\alpha} \approx \frac{H(\omega, p_\alpha + \Delta p_\alpha) - H(\omega, p_\alpha)}{\Delta p_\alpha} = \frac{\Delta H}{\Delta p_\alpha}.$$

Alternative 2: implicit differentiation

$$H(\omega, p_\alpha) = H = Z^{-1} = [-\omega^2 M + j\omega B + K]^{-1}$$

Differentiate the relation $HZ = I$:

$$H_{,\alpha} Z + H Z_{,\alpha} = O.$$

It follows that

$$H_{,\alpha} = -H Z_{,\alpha} Z^{-1} = -H Z_{,\alpha} H.$$

It is now only necessary to determine:

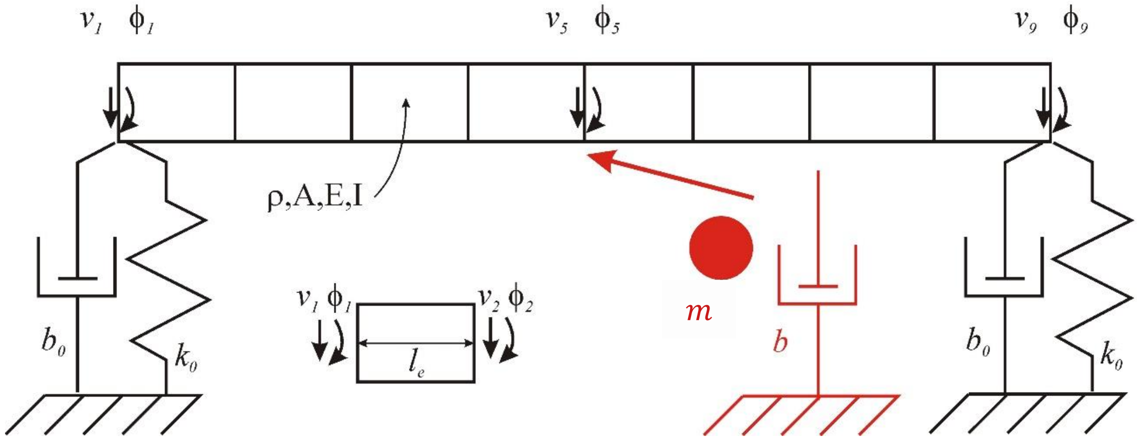
$$Z_{,\alpha} = -\omega^2 M_{,\alpha} + j\omega B_{,\alpha} + K_{,\alpha}.$$

This alternative is useful in case of simple discrete design parameters such as masses, springs, dampers between structure and ground.
In that case, $Z_{,\alpha}$ has only 1 non-zero element and only 1 column/row of H needed.

Example: FRF sensitivity analysis

Beam model

- 8 elements, 18 DOFs
- $l_{tot} = 1$ m,
  $\rho A = 7$ kg/m,
  $EI = 1.5 \cdot 10^4$ Nm^2,
  $k_o = 2 \cdot 10^6$ N/m,
  $b_o = 2 \cdot 10^3$ Ns/m.



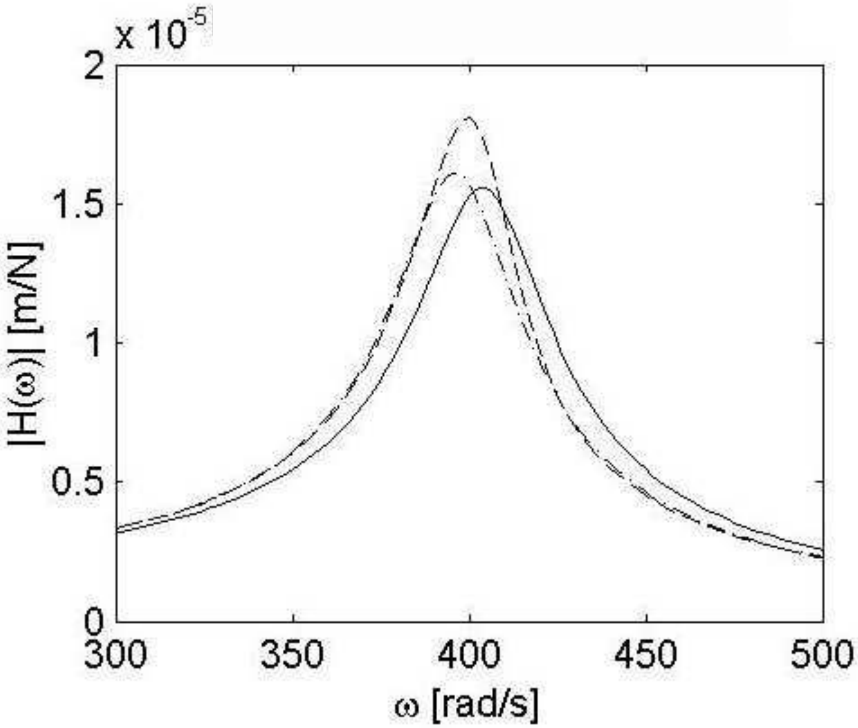
Consider the FRF

$$H(\omega) = v_4 / F_4 \, .$$

Example: FRF sensitivity analysis

- Solid line: FRF $H_0(\omega)$ of system without m
- Dashed-dotted line: FRF $H_{0.15}(\omega)$ with design parameter $m = 0.15$ kg (2% of total beam mass)
- Dashed line: Estimation $H_0(\omega) + H_{,m}(\omega)m$

Estimation and $H_{0.15}(\omega)$ differ near resonance



Example: FRF sensitivity analysis

- Solid line: FRF $H_0(\omega)$ of system without b
- Dashed-dotted line: FRF $H_{50}(\omega)$ with design parameter $b = 50$ Ns/m (2.5% of end dampers)
- Dashed line: Estimation $H_0(\omega) + H_{,b}(\omega)b$

Estimation and $\boldsymbol{H_{50}(\omega)}$ differ near resonance