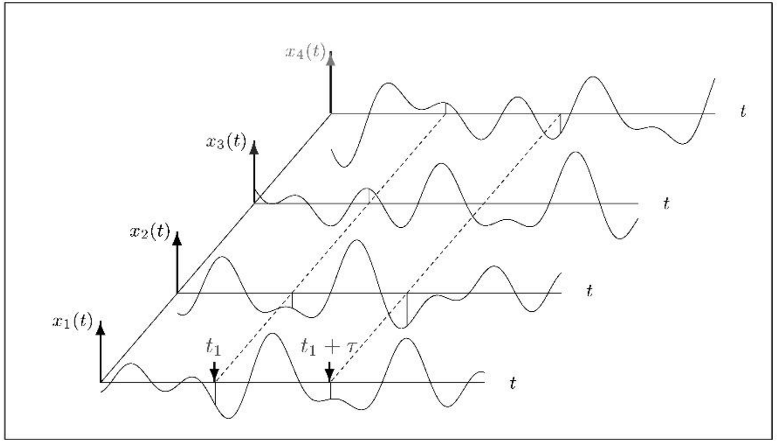

8. Random processes

Structural Dynamics part of 4DM00

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Random processes

So far, we considered deterministic signals.

In experiments, we have **non-deterministic** signals resulting from **random processes** (e.g. signals polluted by measurement noise)

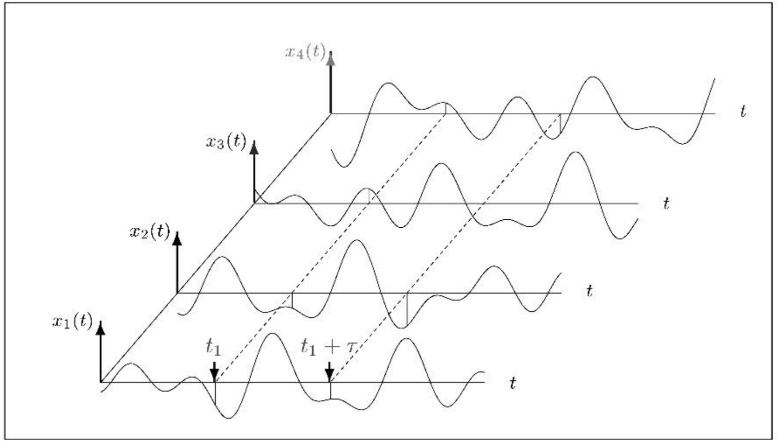
Goal: determine **statistical properties** of such random processes

Typically, this can be done using techniques like **averaging**

Records and ensembles

Each registration of a random process is called a **record**
A collection of records is called an **ensemble**

Example:
4 records of response measurements by driving 4 laps on a test circuit form an ensemble



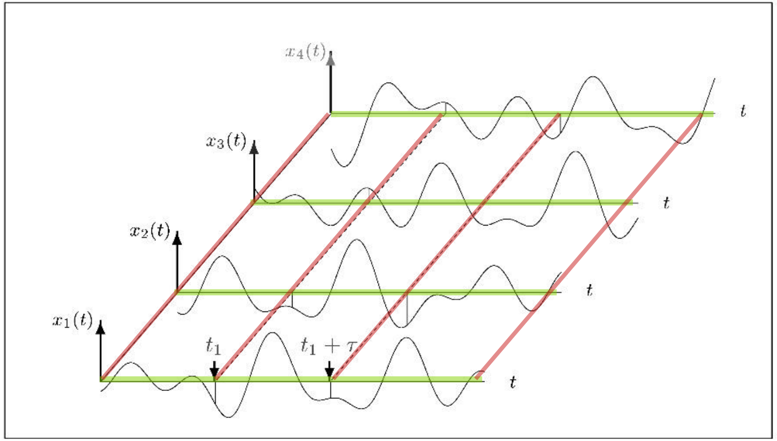
DAF test circuit Sint Oedenrode

Stationary and ergodic processes

Assumption: random processes are **stationary** and **ergodic**

Implication: statistical properties do not change with time and they can be deduced from a single, sufficiently long realization of the process

Two types of averages



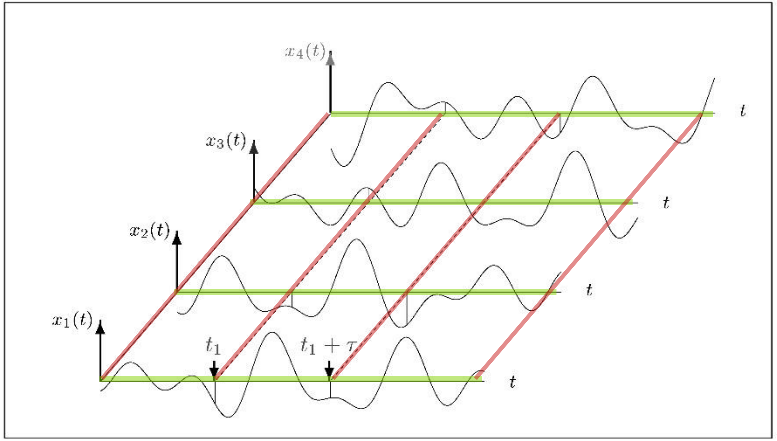
Time averaging

$$\mu_x(k) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x_k(t)dt$$

Record averaging

$$\mu_x(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t)$$

Two types of correlation functions



Auto correlation (time averaged)

$$R_{xx}(k,\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x_k(t)x_k(t+\tau)dt$$

Auto correlation (record averaged)

$$R_{xx}(t,\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x_k(t)x_k(t+\tau)$$

Stationary and ergodic processes

A random process is called **stationary** when

$$\mu_x(t) = \mu_x, \qquad R_{xx}(t,\tau) = R_{xx}(\tau),$$

In other words, the record-average $\mu_x(t)$ and the record-averaged correlation function $R_{xx}(t,\tau)$ do not depend on time t.

A random process is called **ergodic** when

$$\mu_x(k) = \mu_x, \qquad R_{xx}(k,\tau) = R_{xx}(\tau),$$

In other words, the time-average $\mu_x(k)$ and the time-averaged correlation function $R_{xx}(k,\tau)$ do not depend on the record k.

Assumption: random processes are **stationary** and **ergodic**

Implication: statistical properties do not change with time and they can be deduced from a single, sufficiently long realization of the process

TU/e

Stationary and ergodic processes

Assumption: random processes are **stationary** and **ergodic**

In practice:

- difficult to prove stationarity and ergodicity, however…
- in many situations we may assume stationarity and ergodicity (elaboration of coming theory becomes more straightforward)

To meet this assumption:

- collect record(s) under **equal experimental conditions** (e.g. constant temperature, avoid external disturbances, etc.)

Probability distribution and probability density

Consider a fixed time instant t. What is the probability that $x(t) \leq x_1$?

Answer is given by the **probability distribution function** $P(x)$ of the (stationary) process:

$$Prob[x(t) \leq x_1] = P(x_1, t) = P(x_1).$$

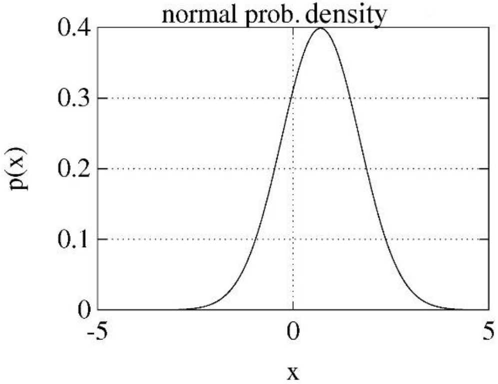
Using $P(x)$, we can define the **probability density function** $p(x)$:

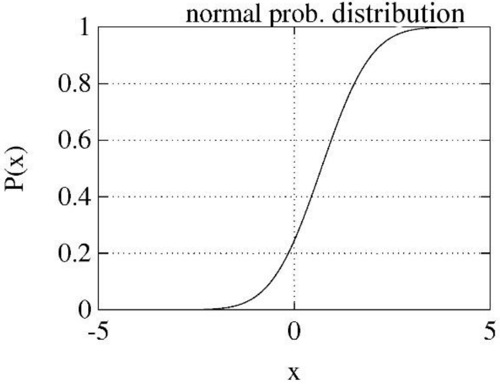
$$p(x) = \frac{dP}{dx}(x), \qquad \Rightarrow \qquad P(x_1) = \int_{-\infty}^{x_1} p(x)\, dx\,.$$

Example: Gaussian process

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

μ: mean
σ: standard deviation
In the figure: $\mu = 0.7$, $\sigma = 1.0$.





Probability distribution and probability density

Process is stationary and ergodic $\Rightarrow$ it suffices to consider only one sample $x(t)$ $(0 \leq t < \infty)$

Stochastic properties of $x(t_1)$ can be determined from the sample $x(t)$:

- **Expected value** is equal to the time-average of the signal:

$$\mathbb{E}[x(t = t_1)] = \int_{-\infty}^{\infty} xp(x)\,dx = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt = \mu_x.$$

- **Mean square value**

$$MSV = \mathbb{E}[x^2(t_1)] = \int_{-\infty}^{\infty} x^2 p(x)\,dx = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt = R_{xx}(0) =: \psi_x^2.$$

- **Variance** σ_x^2 and the **standard deviation** σ_x

$$\boxed{\sigma_x^2 = \psi_x^2 - \mu_x^2}$$

$$\sigma_x^2 = \mathbb{E}[(x(t_1) - \mu_x)^2] = \int_{-\infty}^{\infty} (x - \mu_x)^2 p(x)\,dx = \lim_{T\to\infty} \frac{1}{T}\int_0^T (x(t) - \mu_x)^2\,dt$$

Auto correlation

Auto correlation function (of a stationary ergodic process):

$$R_{xx}(\tau) = \mathbb{E}[x(t)x(t+\tau)] = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)x(t+\tau)dt.$$

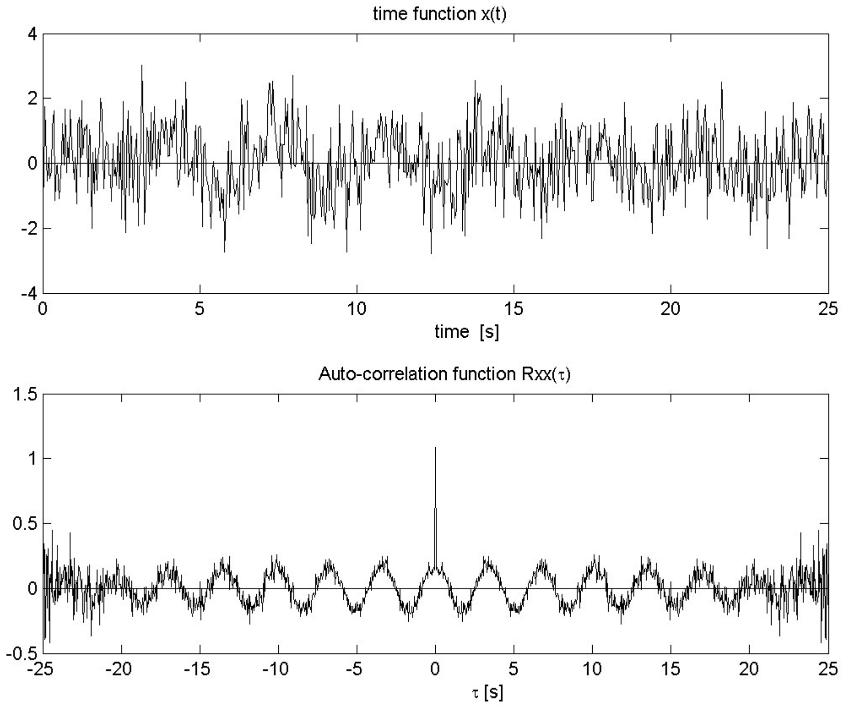
Measure for internal (periodic) structure in a (random) signal

- for $\tau = 0$ holds: $R_{xx}(0) = \psi_x^2$
- $R_{xx}(\tau) \leq R_{xx}(0)$
- symmetric: $R_{xx}(\tau) = R_{xx}(-\tau)$.

Example: Auto correlation

$x(t) = x_1(t) + x_2(t)$

- $x_1(t) = 0.5\sin(2\pi f t)$,
  period time$= 1/f = 3.333$ s
- $x_2(t)$ random signal,
  normal pdf ($\mu = 0, \sigma = 1$)



Cross correlation

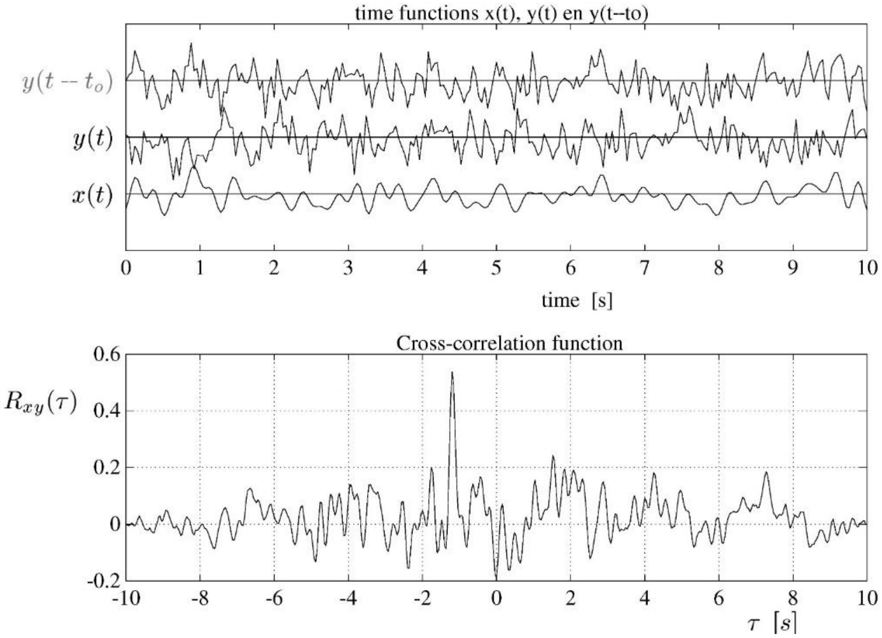
Cross correlation function for two stationary ergodic random processes $x(t)$ and $y(t)$:

$$R_{xy}(\tau) = \mathbb{E}[x(t)y(t+\tau)] = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)y(t+\tau)dt$$

$R_{xy}(\tau)$ gives information about coherence between $x(t)$ and $y(t+\tau)$

- $R_{xy}(\tau) = R_{yx}(-\tau)$,
- but $R_{xy}(\tau)$ in general not symmetric, i.e. generally $R_{xy}(\tau) \neq R_{xy}(-\tau)$.

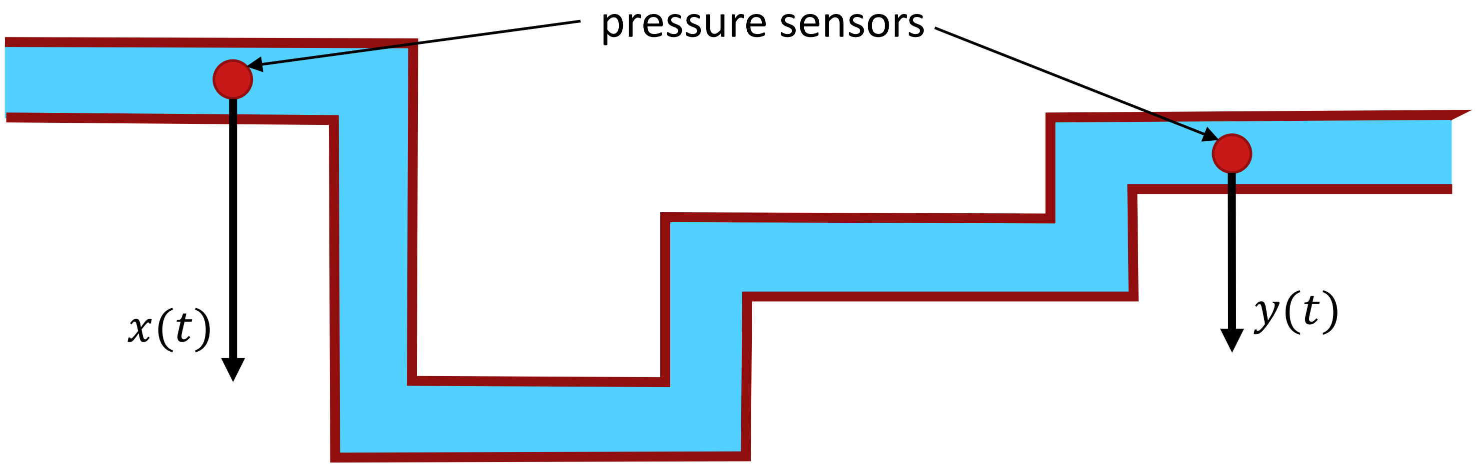
Example: cross correlation



$x(t)$ and $y(t + \tau)$ have something in common for $\tau = -t_0 = -1.2$ s!

Application of cross correlation

Determine the length of a pipe network:



Determine the value of τ at which a peak occurs in $R_{xy}(\tau)$
Length of the network can be estimated based on the propagation speed of pressure waves

8b. Random processes in the frequency domain

Auto Power Spectral Density $S_{xx}(f)$

Process is stationary and ergodic ⇒ it suffices to consider only one sample $x(t)$ ($0 \leq t < \infty$)

Introduce rectangular window $w_T(t)$ (which is 1 when $1 \leq t < T$ and zero otherwise).
Define:

$$X_T(f) = \int_0^T x(t)w_T(t)e^{-2\pi jft}dt\,, \qquad x(t)w_T(t) = \int_{-\infty}^{\infty} X_T(f)e^{2\pi jft}df\,,$$

The MSV of $x(t)w_T(t)$ is then

$$MSV = \frac{1}{T}\int_0^T x^2(t)w_T^2(t)dt = \frac{1}{T}\int_0^T x(t)w_T(t)\int_{-\infty}^{\infty} X_T(f)e^{2\pi jft}df\,dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} X_T(f)\int_0^T x(t)w_T(t)e^{2\pi jft}dt\,df = \frac{1}{T}\int_{-\infty}^{\infty} X_T(f)\,\overline{X_T(f)}\,df = \frac{1}{T}\int_{-\infty}^{\infty} |X_T(f)|^2\,df$$

Auto power spectral density $S_{xx}(f)$

$$MSV = \frac{1}{T}\int_{-\infty}^{\infty} |X_T(f)|^2 \, df$$

In electrical systems

$x(t) \sim V(t), P = \frac{V^2}{R}$

Note that the MSV represents power.

The MSV resulting from frequencies $f \in [f_1, f_1 + \Delta f]$ is given by

$$\frac{1}{T}\int_{f_1}^{f_1+\Delta f} |X_T(f)|^2 \, df \, .$$

So $\frac{1}{T}|X_T(f)|^2$ is the power density of the frequency spectrum of $x(t)w_T(t)$.

The auto power spectral density of the signal $x(t)$ (without window) is thus

$$S_{xx}(f) \coloneqq \lim_{T\to\infty} \frac{1}{T}|X_T(f)|^2 \, .$$

Properties of auto power spectral density $S_{xx}(f)$

- $S_{xx}(f)$ is related to $R_{xx}(\tau)$ by the Fourier transform (Wiener-Khintchine relation)

$$S_{xx}(f) = \int_{\tau=-\infty}^{\infty} R_{xx}(\tau)\, e^{-2\pi j f \tau} d\tau, \qquad R_{xx}(\tau) = \int_{f=-\infty}^{\infty} S_{xx}(f)\, e^{2\pi j f \tau} df.$$

- $S_{xx}(f)$ is real

- $S_{xx}(f)$ is symmetric: $S_{xx}(f) = S_{xx}(-f)$

- $MSV = \psi_x^2 = \int_{-\infty}^{\infty} S_{xx}(f) df = R_{xx}(0)$

- Nonzero average of μ_x (DC-component) leads to a Dirac function at $f = 0$ in $S_{xx}(f)$

Estimator $\hat{S}_{xx}(f)$

Problem: the computation of $S_{xx}(f)$ requires the infinite sample $x(t)$ for $0 \leq t < \infty$.

- Based on N records $x_k(t)$ on $0 \leq t < T$, we can compute the **estimator** $\hat{S}_{xx}(f)$

$$\hat{S}_{xx}(f) = \frac{1}{NT}\sum_{k=1}^{N}\left[\overline{X_{T,k}(f)}X_{T,k}(f)\right], \qquad X_{T,k}(f) = F[x_k(t)].$$

- The variance of the estimator $\hat{S}_{xx}(f)$ is given by

$$\sigma^2_{S_{xx}}(f) = var\big(S_{xx}(f)\big) = \frac{S^2_{xx}(f)}{N} \approx \frac{\hat{S}^2_{xx}(f)}{N}$$

  Variance decreases by increasing the number of records N!

- 95% confidence interval:

$$\hat{S}_{xx}(f)[1-2V] \leq S_{xx}(f) \leq \hat{S}_{xx}(f)[1+2V]$$

  Variation coefficient $V = \sigma_{S_{xx}}(f)/\hat{S}_{xx}(f)$.

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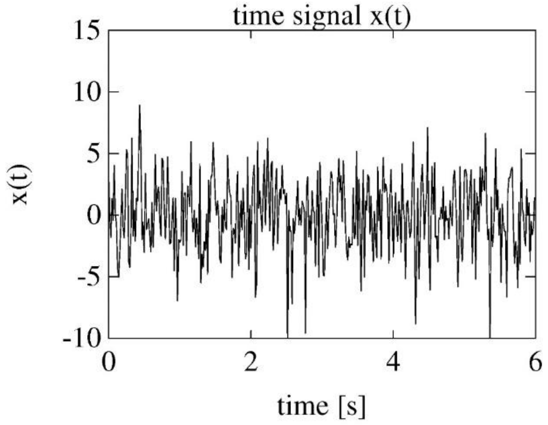
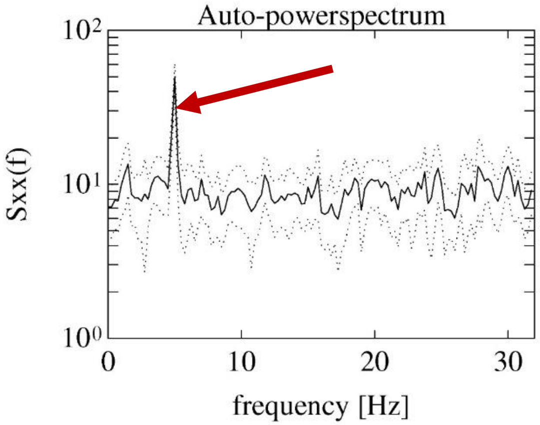
Example: Estimator $\widehat{S}_{xx}(f)$

$$x(t) = \sin(2\pi f_0 t) + n(t)$$

$f_0 = 5$ Hz, $n(t)$: Gaussian white noise $\mu_n = 0$, $\sigma_n = 3$, $\Delta T = 0.015625$ s
$N = 25$ records $X_{B,k}(f)$ of 256 points, Hanning window.



part of time signal



estimator $\hat{S}_{xx}(f)$ and
95% confidence interval

Warning: averaging of $X(f)$

Averaging of $|X_{T,k}(f)|^2$ makes sense.

Averaging of the $X_{T,k}(f)$ is tricky, and often not a good idea.

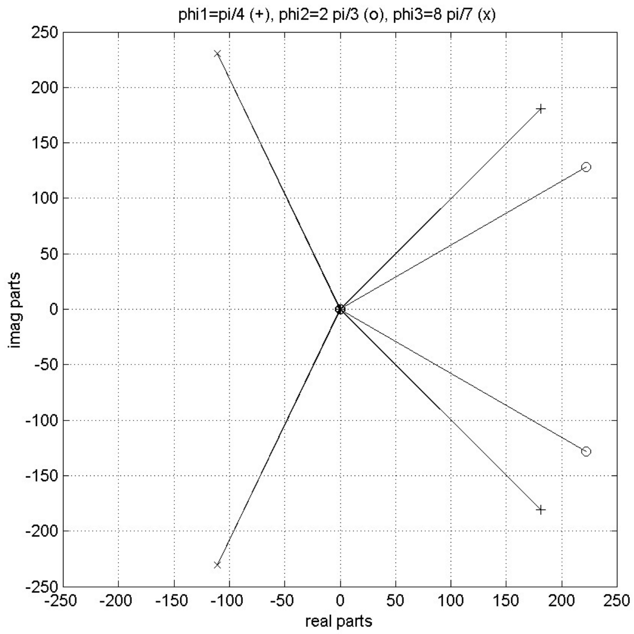
Example: consider 6 records

$$x_k(t) = \sin(2\pi f_o(t - t_k)).$$

Only the starting times t_k differ.

$$X_k(f) = X_o(f)e^{2\pi j f \phi_k}$$

with $\phi_k := -2\pi f_o t_k, X_o(f) = F[\sin(2\pi f_o t)]$.



- Phase angles directly related to starting times t_k
- Averaging (to get rid of measurement noise) converges to zero!
- Solution: signal triggering, make t_k equal (not always possible)

Cross power spectral density $S_{xy}(f)$

Introduce rectangular window $w_T(t)$ (which is 1 when $1 \leq t < T$ and zero otherwise).

Consider Fourier transforms of windowed signals

$$X_T(f) = F[x(t)w_T(t)], \qquad Y_T(f) = F[y(t)w_T(t)].$$

Definition:

$$S_{xy}(f) = \lim_{T \to \infty} \frac{1}{T} \overline{X_T(f)} Y_T(f)$$

Properties of $S_{xy}(f)$

- $S_{xy}(f)$ is related to the cross correlation $R_{xy}(\tau)$ by

$$S_{xy}(f) = \int_{\tau=-\infty}^{\infty} R_{xy}(\tau)\, e^{-2\pi j f \tau} d\tau, \qquad R_{xy}(\tau) = \int_{f=-\infty}^{\infty} S_{xy}(f)\, e^{2\pi j f \tau} df$$

- $S_{xy}(f)$ (complex) contains phase information (in contrast to $S_{xx}(f)$ (real))
- $S_{xy}(f) = \overline{S_{yx}(f)}$

- The **coherence function** $\gamma_{xy}(f)$ is defined by

$$\gamma_{xy}^2(f) := \frac{\left|S_{xy}(f)\right|^2}{S_{xx}(f) S_{yy}(f)}$$

- $\gamma_{xy}(f)$ is real and $0 \leq \gamma_{xy}(f) \leq 1$.

Estimator $\widehat{S}_{xy}(f)$

Estimator:

$$\hat{S}_{xy}(f) = \frac{1}{NT}\sum_{k=1}^{N}\left[\overline{X_{T,k}(f)}Y_{T,k}(f)\right]$$

Variance of the estimator:

$$\sigma^2_{S_{xy}}(f) = var\left(S_{xy}(f)\right) = \frac{\left|S_{xy}(f)\right|^2}{N\gamma^2_{xy}(f)} \approx \frac{\left|\hat{S}_{xy}(f)\right|^2}{N\hat{\gamma}^2_{xy}(f)},$$

95% confidence interval:

$$\hat{S}_{xy}(f)[1-2V] \leq S_{xy}(f) \leq \hat{S}_{xy}(f)[1+2V], \qquad V = \sigma_{S_{xy}}(f)/\hat{S}_{xy}(f).$$

Example: cross power spectral density $\widehat{S}_{xy}(f)$

$x(t) = \sum_{k=1}^{3}[sin(2\pi f_k t + \varphi_k)]$

$f_1 = 0.93, f_2 = 1.57, f_3 = 3.52$ Hz,
$\varphi_1 = 0.85, \varphi_2 = 2.87, \varphi_3 = 1.83$ rad

$y(t) = x(t) + n(t),$

$n(t)$: normal distributed pdf, $\mu_n = 0$, $\sigma_n = 3$

25 records of 256 time points.

$\Delta T = 500/(25 * 256) \approx 0.078$ s.

- Peaks at $f_k = 0.93,1.57,3.52$ Hz as expected
- Phase is random except at $f_k = 0.93,1.57,3.52$ Hz where it is approx. 0.

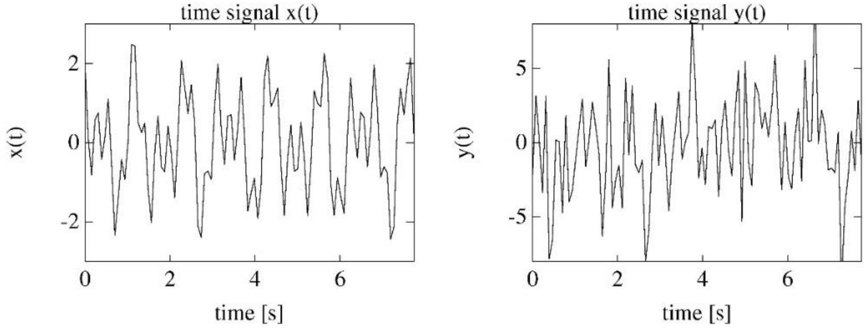


Fig. 2.29 3-harmonic signal (left) and with additional noise (right)

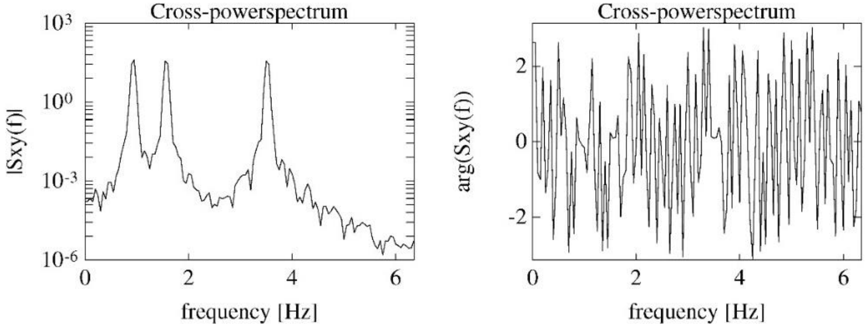


Fig. 2.30 Cross power spectrum of the two signals $x(t)$ and $y(t)$