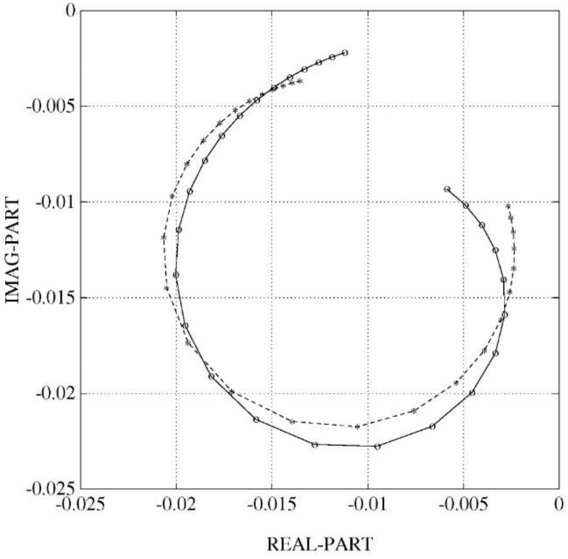

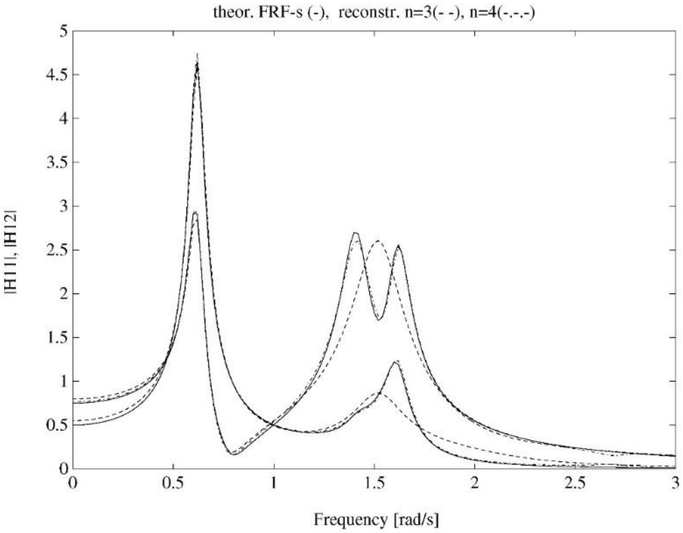

11. Modal parameter fit procedures

Structural Dynamics part of 4DM00

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Introduction

Frequency Response Function (FRF) for excitation DOF l to response DOF i

$$Y_i(\omega) = H[i,l](\omega)X_l(\omega).$$

Note: $H[i,l](\omega)$ can be measured.

Objective: determine eigenvalues λ_k and eigenmodes u_k (via A_k)

For a model of an **underdamped** system with n modes:

$$H[i,l](\omega) = \sum_{k=1}^{n}\left[\frac{A_{kR}[i,l] + jA_{kI}[i,l]}{-\mu_k + j(\omega - \nu_k)} + \frac{A_{kR}[i,l] - jA_{kI}[i,l]}{-\mu_k + j(\omega + \nu_k)}\right]$$

unit: [m/N]

Complex conjugate pairs of:

- Eigenvalues $\lambda_k = \mu_k + j\nu_k$
- Residue matrices $A_k = A_{kR} + jA_{kI}$

Introduction

Recall:

- eigenvalues and eigenmodes follow from right and left eigenvalue problems

$$(\lambda_k C + D)v_k = 0, \qquad w_k^\top(\lambda_k C + D) = 0.$$

  Assume C, D symmetric, then:

$$v_k = \begin{bmatrix} u_k \\ \lambda_k u_k \end{bmatrix} = w_k = \begin{bmatrix} x_k \\ \lambda_k x_k \end{bmatrix}$$

- **Residues** A_k

$$A_k = A_{kR} + jA_{kI} = \frac{u_k u_k^\top}{c_k^*}, \qquad c_k^* = v_k^\top C v_k.$$

Scaling freedom in eigenmodes: c_k^* cannot be determined from experiments, choose $c_k^* = 1$.

Note that $\mathrm{rank}(A_k) = 1$. So when we know

- the i -th row of A_k: $u_k[i]u_k^\top / c_k^*$, or
- the l -th column of A_k: $u_k u_k[l] / c_k^*$,

we know A_k completely. $\Rightarrow$ $\boxed{\text{it suffices to measure only one row or one column of } H(\omega).}$

Introduction

Assume l -th column is measured using s response sensors:

$$H[i,l](\omega) = \sum_{k=1}^{n} \left[\frac{A_{kR}[i,l] + jA_{kI}[i,l]}{-\mu_k + j(\omega - \nu_k)} + \frac{A_{kR}[i,l] - jA_{kI}[i,l]}{-\mu_k + j(\omega + \nu_k)} \right], \qquad i = 1, \dots, s.$$

We have $2n(s+1)$ unknowns (modal parameters):

- μ_k and ν_k, for $k = 1, \dots, n$
- $A_{kR}[i,l]$ and $A_{kI}[i,l]$, for $k = 1, \dots, n$,$i = 1, \dots, s$, (l is fixed)

We have $s * N/2$ complex or $s * N$ real equations:

- s measured FRF's $H[i,l]$, ($i = 1, \dots, s$ and l is fixed)
- for each FRF $H[i,l](\omega_m)$: values at $N/2$ discrete frequencies ω_m, $m = 1,2, \dots, N/2$

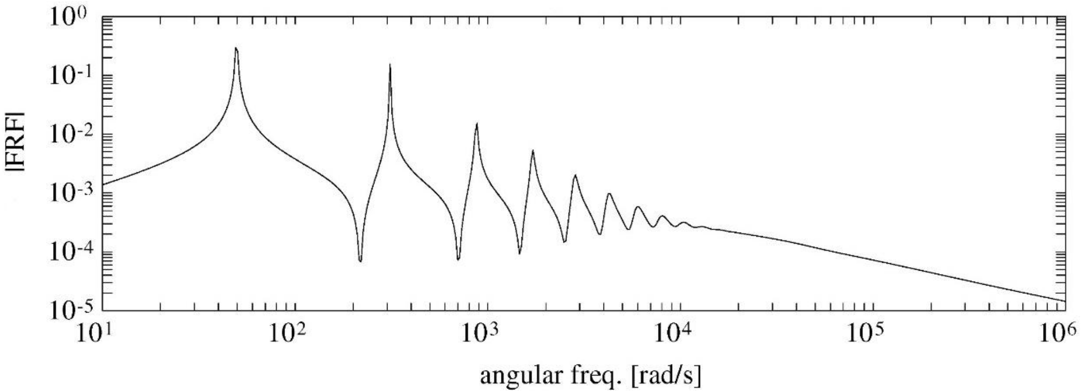
Solvable (e.g. using least squares approach) if: $sN \geq 2n(s+1)$
(as N is typically large, this condition often easily satisfied)

Problem

Choosing n (the number of modes in frequency range of interest)

Example: cantilever beam with 18 elements, 36 DOFs.
Parameters: $l = 1.8$ m, $A = 9.0 \cdot 10^{-4}$ m^2, $I = 7 \cdot 10^{-8}$ m^4, $\rho = 7800$ kg/m^3.
Rayleigh damping: $B = 0.1M + 2 \cdot 10^{-5}K$
Collocated FRF (velocity/force) at the (free) end of the beam.

Only 8 to 9 modes can clearly be distinguished.



Problem

Choosing n (the number of modes in frequency range of interest)

- Highly/overcritically damped modes: counting of peaks leads to incorrect value of n
- Moreover, closely spaced modes may occur:
  e.g. two modes with (almost) equal damped eigenfreq's appear in the FRF as one peak

Conclusion:

- A good choice for n may be difficult
- Estimation of modal parameters is far from trivial

Introduction

Problem: Find approximation of modal parameters based on a set of measured FRF’s.

A variety of methods exist in literature and commercial software
(e.g. LMS, ME’scopeVES, STAR modal)

General idea:

1. Extract complex eigenvalues λ_k from one or more well-chosen FRF’s
   (λ_k independent of chosen excitation and response dof)
   The λ_k’s of interest have imaginary parts ν_k lying in the frequency range of interest…
2. Next: use complete row or column of $H(\omega)$ to calculate
   the row or column of the residue matrix A_k.
   Then, by choosing some normalization c_k^*, the eigenmode u_k is also known.

Outline

Three main groups of modal parameter fit techniques:

1. simple single mode techniques:
   e.g. circle fit method

2. multi-mode techniques in the frequency domain

3. multi-mode techniques in the time domain:
   e.g. least squares complex exponential (lsce) technique

11b. Circle fit

Circle fit

Assumptions:

- Only 1 mode, say mode k, is relevant (no interaction with other modes). Contribution of other modes is approximated by complex constant $R_k + jI_k$
- Mode is weakly damped ($-\mu_k/\nu_k$ is small). For $\omega \approx \nu_k$:

$$\frac{\bar{A}_k[i,l]}{-\mu_k+j(\omega+\nu_k)} \text{ is negligible compared to } \frac{A_k[i,l]}{-\mu_k+j(\omega-\nu_k)}$$

We thus have for $\omega \approx \nu_k$:

$$H[i,l](\omega) \approx \frac{A_{kR}[i,l] + jA_{kI}[i,l]}{-\mu_k + j(\omega - \nu_k)} + R_k[i,l] + jI_k[i,l]$$

Circle fit

$$H[i,l](\omega) \approx \frac{A_{kR}[i,l] + jA_{kI}[i,l]}{-\mu_k + j(\omega - \nu_k)} + R_k[i,l] + jI_k[i,l]$$

Note that: $$\frac{1}{-\mu_k + j(\omega - \nu_k)} = \frac{-1}{2\mu_k} - \frac{1}{2\mu_k}\frac{[\mu_k + (\omega - \nu_k)j]^2}{\mu_k^2 + (\omega - \nu_k)^2} = -\frac{1}{2\mu_k}\left(1 + e^{j\varphi_k(\omega)}\right).$$

with $\varphi_k(\omega) = 2\arctan\left(\frac{\omega - \nu_k}{\mu_k}\right)$ (so $\varphi_k(\omega = \nu_k) = 0$)

This is a **circle** in the complex plane: centre $\left(\frac{-1}{2\mu_k}, 0\right)$, radius $\left|\frac{1}{2\mu_k}\right|$

$$H[i,l](\omega) \approx \left[A_{kR}[i,l] + jA_{kI}[i,l]\right]\left[-\frac{1}{2\mu_k}\left(1 + e^{j\varphi_k(\omega)}\right)\right] + R_k[i,l] + jI_k[i,l]$$

Circle fit

Define: $A_{kR} + jA_{kI} := A_{kM}e^{j\psi_k}$ (so $A_{kM} = \sqrt{A_{kR}^2 + A_{kI}^2}$, $\psi_k = \angle(A_{kR} + jA_{kI})$)

$$H[i,l](\omega) \approx \left[R_k - \frac{A_{kR}}{2\mu_k}\right] + j\left[I_k - \frac{A_{kI}}{2\mu_k}\right] - \frac{A_{kM}}{2\mu_k}e^{j(\varphi_k(\omega)+\psi_k)}$$

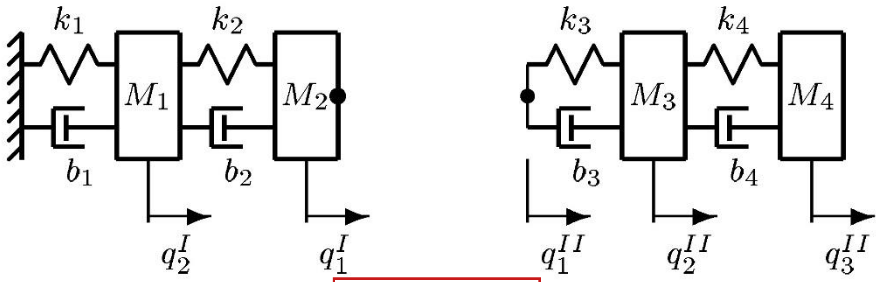
Recall that $\varphi_k(\nu_k) = 0$, so that ψ_k=angle/argument of the last term at $\omega = \nu_k$.

This is again a circle in the complex plane with:

- centre $R_0 + jI_0 = \left[R_k - \frac{A_{kR}}{2\mu_k}\right] + j\left[I_k - \frac{A_{kI}}{2\mu_k}\right]$
- radius $r_k = \frac{A_{kM}}{|2\mu_k|} = \frac{\sqrt{A_{kR}^2+A_{kI}^2}}{|2\mu_k|}$

6 unknown parameters: $\nu_k, \mu_k, A_{kR}, A_{kI}, R_k, I_k$,
but determining R_0, I_0, and r_k gives only three equations…

Example: coupled 4-dof system



Subsystem (I)
$(q^I)^T = [q_1^I, q_2^I]$

Coupling:
$q_1^I = q_1^{II}$

Subsystem (II)
$(q^{II})^T = [q_1^{II}, q_2^{II}, q_3^{II}]$

Coupled: $q^T = [q_2^I, q_1^I, q_2^{II}, q_3^{II}]$

DATA:

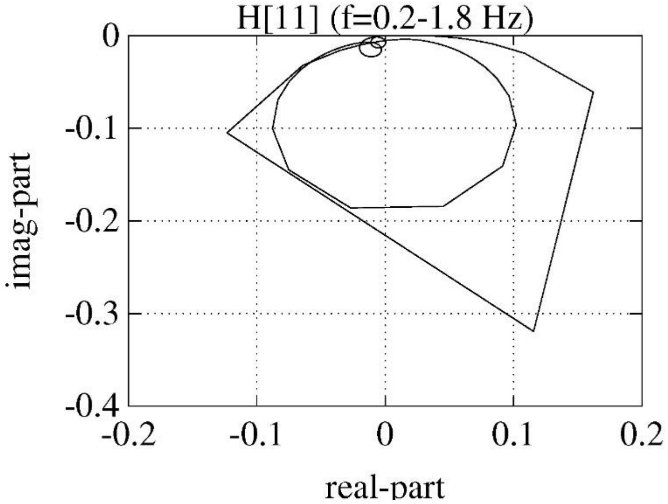
$m_1 = 2; m_2 = 2; m_3 = 1; m_4 = 1; [kg]$

$k_1 = 100; k_2 = 50; k_3 = 20; k_4 = 10; [N/m]$

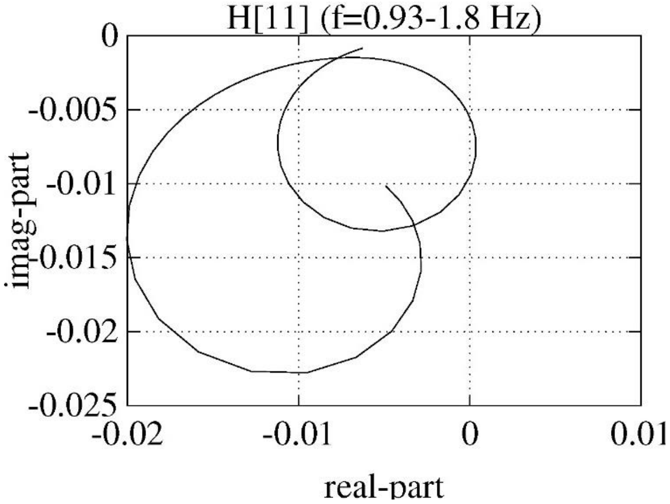
$b_1 = 0.6; b_2 = 0.6; b_3 = 0.3; b_4 = 0.3; [Ns/m]$

Consider the FRF from $F_1^I + F_1^{II}$ to $q_1^I = q_1^{II}$.

Example: coupled 4-dof system



Nyquist plot for $f \in [0.2, 1.8]$ Hz



Nyquist plot for $f \in [0.93, 1.8]$ Hz

Approximately circles ⇒ little interaction, weakly damped.

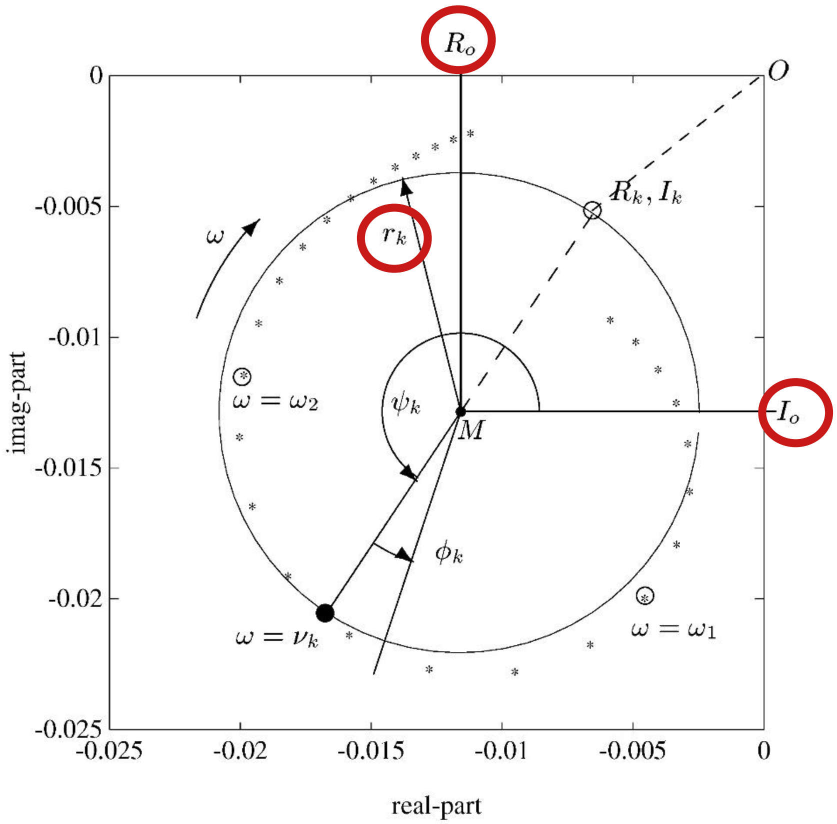
Problem: estimate parameters for mode 3 using circle fit

Example: coupled 4-dof system

Assume that $H[i, l](\omega_i)$ (*) come from experiment

Use least-squares technique to fit circle:
centre R_0, I_0, radius r_k

How to determine the modal parameters?



Circle fit (ν_k)

$$H[i,l](\omega) \approx \overbrace{\left[R_k - \frac{A_{kR}}{2\mu_k}\right]}^{R_0} + j\overbrace{\left[I_k - \frac{A_{kI}}{2\mu_k}\right]}^{I_0} - \frac{A_{kM}}{2\mu_k} e^{j(\varphi_k(\omega)+\psi_k)}$$

Define

$$H_0(\omega) \coloneqq H[i,l](\omega) - R_0 - jI_0 \approx -\frac{A_{kM}}{2\mu_k} e^{j(\varphi_k(\omega)+\psi_k)} = -\frac{A_{kM}}{2\mu_k} e^{j\theta_k(\omega)}$$

where $\theta_k(\omega) \coloneqq \varphi_k(\omega) + \psi_k$.
Note that $\boxed{\theta_k(\omega_i) \approx \angle H_0(\omega_i)}$ and that $\angle H_0(\omega_i)$ can be computed.

Now recall that $\varphi_k(\omega) = 2\arctan((\omega - \nu_k)/\mu_k)$,

$$\frac{d\theta_k(\omega)}{d\omega} = \frac{\partial\varphi_k(\omega)}{\partial\omega} = \frac{2\mu_k}{\mu_k^2 + (\omega-\nu_k)^2}, \qquad \frac{d^2\theta_k(\omega)}{d\omega^2} = \frac{-4\mu_k^2(\omega-\nu_k)}{[\mu_k^2 + (\omega-\nu_k)^2]^2}.$$

$\frac{d^2\theta_k(\omega)}{d\omega^2} = 0$ for $\omega = \nu_k \Rightarrow \frac{d\theta_k(\omega)}{d\omega}$ has a local extremum.

Determine ν_k as the point ω for which $\frac{d\theta_k(\omega)}{d\omega}$ has a local extremum.

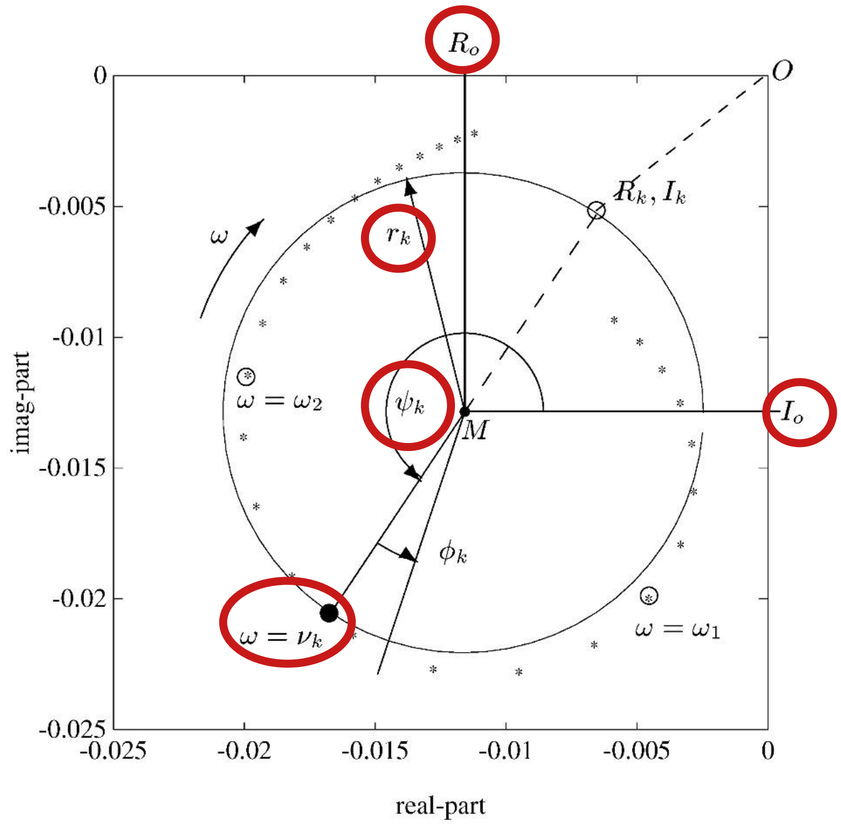
Example: coupled 4-dof system

Assume that $H[i,l](\omega_i)$ (*) come from experiment

Use least-squares technique to fit circle:
centre R_0, I_0, radius r_k

When $\Delta\omega$ is constant between two subsequent *, ν_k is found where $\Delta\theta_k$ is maximal.

Note that $\theta_k(\omega) = \varphi_k(\omega) + \psi_k$ and $\varphi_k(\nu_k) = 0$ imply that we also find $\psi_k = \angle H_0(\nu_k)$



Circle fit (μ_k)

Select ω_1 and ω_2 where $\omega_1 < \nu_k < \omega_2$.

$$\varphi_k(\omega) = 2\arctan\left(\frac{\omega - \nu_k}{\mu_k}\right), \qquad \Rightarrow \qquad \frac{\omega - \nu_k}{\mu_k} = \tan(\varphi_k(\omega)/2)$$

Two values $\omega = \omega_1$ and $\omega = \omega_2$, so two equations

$$\frac{\omega_1 - \nu_k}{\mu_k} = \tan(\varphi_k(\omega_1)/2), \qquad \frac{\omega_2 - \nu_k}{\mu_k} = \tan(\varphi_k(\omega_2)/2).$$

Note that $\varphi_k(\omega_i)$ is now available as $\varphi_k(\omega_i) = \theta_k(\omega_i) - \psi_k \approx \angle H_0(\omega_i) - \psi_k$.

Eliminate ν_k and solve for μ_k

$$\mu_k = \frac{\omega_1 - \omega_2}{\tan(\varphi_k(\omega_1)/2) - \tan(\varphi_k(\omega_2)/2)} < 0$$

Example: coupled 4-dof system

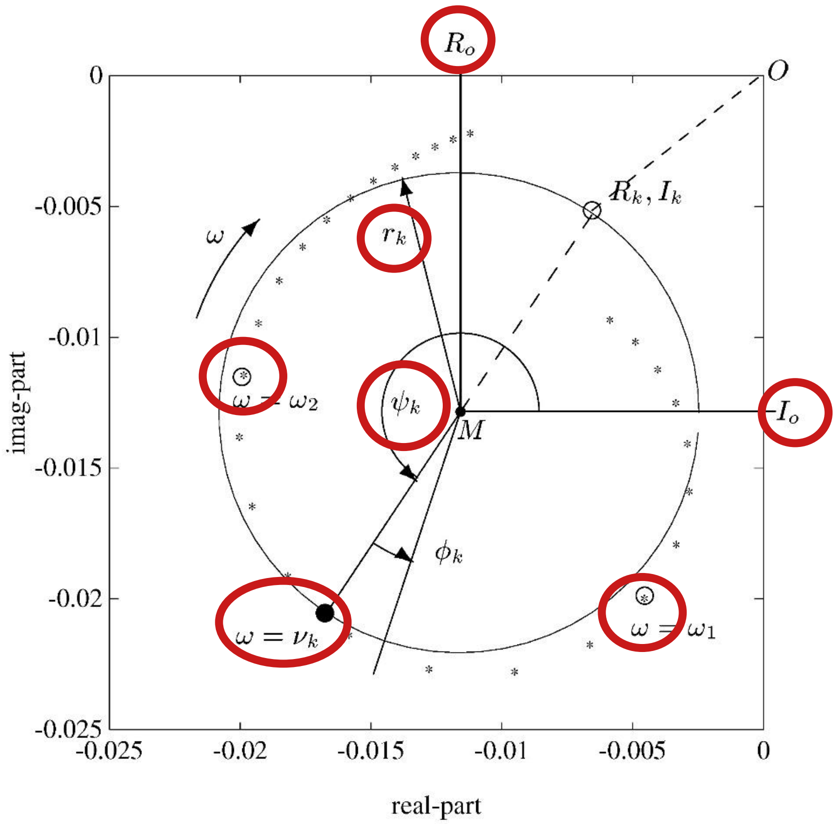
Assume that $H[i,l](\omega_i)$ (*) come from experiment

Use least-squares technique to fit circle:
centre R_0, I_0, radius r_k

When $\Delta\omega$ is constant between two subsequent *, ν_k is found where $\Delta\varphi_k$ is maximal.

Note that $\theta_k(\omega) = \varphi_k(\omega) + \psi_k$ and $\varphi_k(\nu_k) = 0$ imply that we also find $\psi_k = \angle H_0(\nu_k)$

Determine μ_k using the selected frequencies $\omega_1 < \nu_k < \omega_2$.



Circle fit (remaining parameters)

$$r_k = \frac{A_{kM}}{|2\mu_k|}, \qquad \psi_k = \angle(A_{kR} + jA_{kI}).$$

Magnitude and A_{kM} and phase ψ_k of the complex number $A_{kR} + jA_{kI}$ are known.
$\Rightarrow$ easy to determine A_{kR} and A_{kI}.

Finally, R_k and I_k are determined from

$$R_k - \frac{A_{kR}}{2\mu_k} = R_0, \qquad I_k - \frac{A_{kI}}{2\mu_k} = I_0$$

Example: coupled 4-dof system

Assume that $H[i,l](\omega_i)$ (*) come from experiment

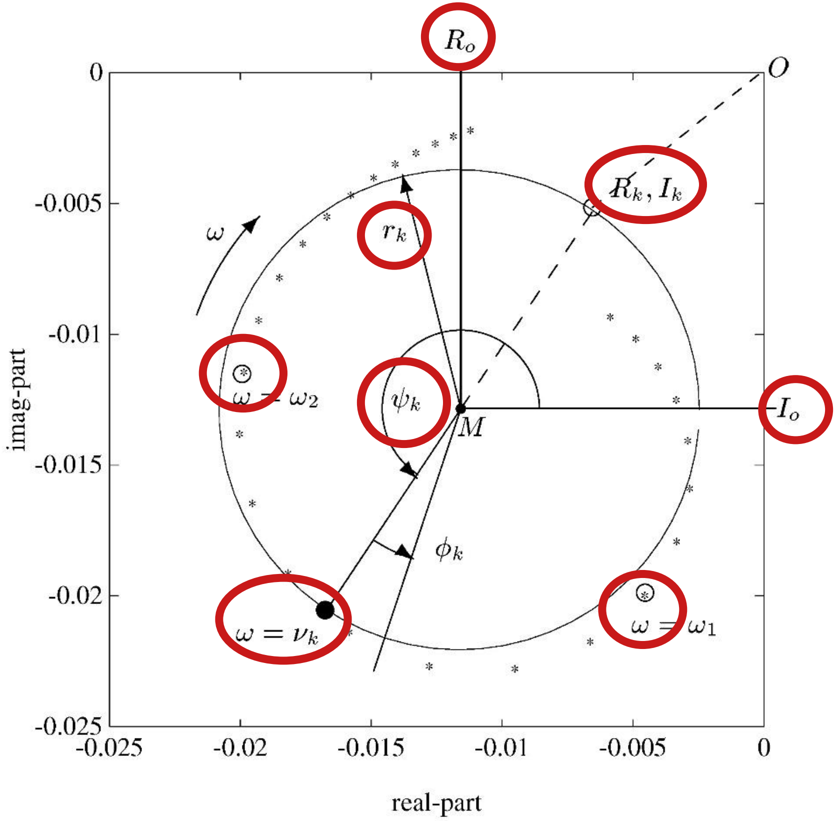
Use least-squares technique to fit circle:
centre R_0, I_0, radius r_k

When $\Delta\omega$ is constant between two subsequent *, ν_k is found where $\Delta\varphi_k$ is maximal.

Note that $\theta_k(\omega) = \varphi_k(\omega) + \psi_k$ and $\varphi_k(\nu_k) = 0$ imply that we also find $\psi_k = \angle H_0(\nu_k)$

Determine μ_k using the selected frequencies $\omega_1 < \nu_k < \omega_2$.

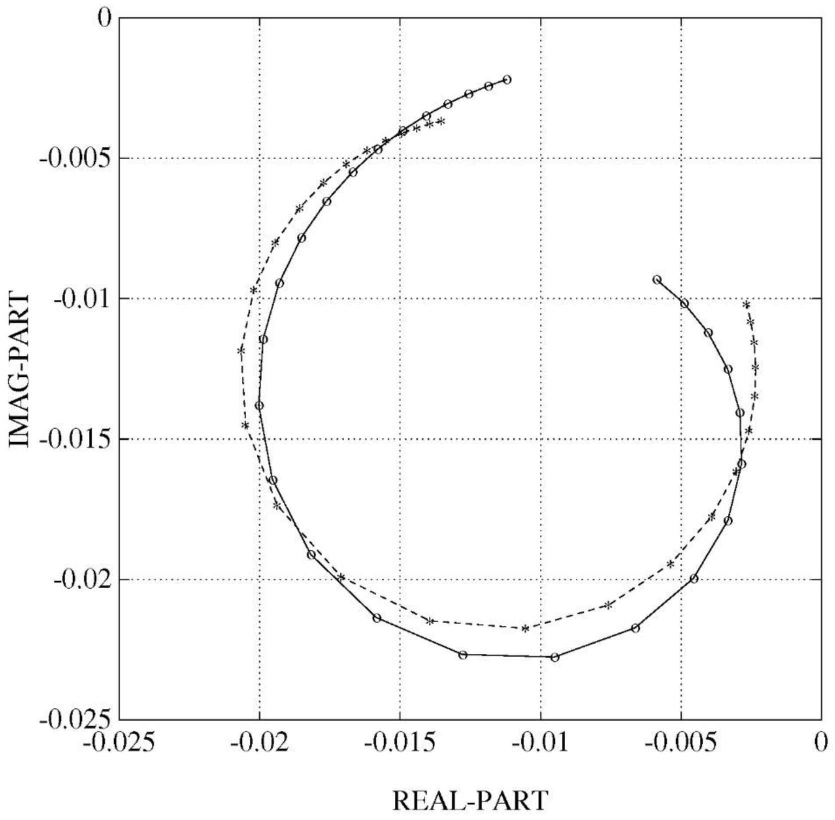
The point (R_k, I_k) lies on the circle opposite to the point corresponding to $\omega = \nu_k$



Example: coupled 4-dof system

Found data for mode 3:

- $\lambda_k = \mu_k + j\nu_k = -0.3231 + 6.5031j$
  weak damping?
- $A_{kR} + jA_{kI} = -0.0026 - 0.0053j$
  OK, ψ_k in 3th quadrant!
- $R_k + jI_k = -0.0074 - 0.0044j$
  some interaction other modes



Comparison of experimental data (solid line) and the obtained approximation (dashed line)

11c. A multi-mode fit in the frequency domain

A multi-mode-fit in the frequency domain

Situation:
In a frequency band $\omega \in [\omega_{min}, \omega_{max}]$, multiple modes $k = m_1, \dots, m_2$ are important.

$$H[i,l](\omega) = \sum_{k=1}^{n} \left[\frac{A_{kR}[i,l] + jA_{kI}[i,l]}{-\mu_k + j(\omega - \nu_k)} + \frac{A_{kR}[i,l] - jA_{kI}[i,l]}{-\mu_k + j(\omega + \nu_k)} \right]$$

Split the summation into three groups:

$$H[i,l](\omega) = \underbrace{\sum_{k=1}^{m_1-1} [\quad]}_{\boxed{\nu_k \ll \omega_{min}}} + \sum_{k=m_1}^{m_2} \left[\frac{A_{kR}[i,l] + jA_{kI}[i,l]}{-\mu_k + j(\omega - \nu_k)} + \frac{A_{kR}[i,l] - jA_{kI}[i,l]}{-\mu_k + j(\omega + \nu_k)} \right] + \underbrace{\sum_{k=m_2+1}^{n} [\quad]}_{\boxed{\nu_k \gg \omega_{max}}}.$$

Motivation of approximations

Assume proportional damping ($A_{kR} = 0$, $A_{kI} \leq 0$) and weak damping:

- Term with $\nu_k \ll \omega_{min}$.
  Note that $\omega > \omega_{min} \gg \nu_k \approx |\mu_k + j\nu_k|$ (assuming weak damping):
  $$\sum_{k=1}^{m_1-1} \frac{-2A_{kI}[i,l]\nu_k}{-\omega^2 - 2j\omega\mu_k + (\mu_k^2 + \nu_k^2)} \approx \sum_{k=1}^{m_1-1} \frac{2A_{kI}[i,l]\nu_k}{\omega^2} = \frac{-1}{\widehat{m}_{il}\omega^2}$$

- Term with $\nu_k \gg \omega_{max}$.
  Note that $\omega < \omega_{max} \ll \nu_k \approx |\mu_k + j\nu_k|$(assuming weak damping):
  $$\sum_{k=m_2+1}^{n} \frac{-2A_{kI}[i,l]\nu_k}{-\omega^2 - 2j\omega\mu_k + (\mu_k^2 + \nu_k^2)} \approx \sum_{k=m_2+1}^{n} \frac{-2A_{kI}[i,l]\nu_k}{\mu_k^2 + \nu_k^2} = \hat{s}_{il}$$

Residual mass and residual flexibility

Formulae on the previous slide are **not used** in practice,
but make the following approximation plausible:

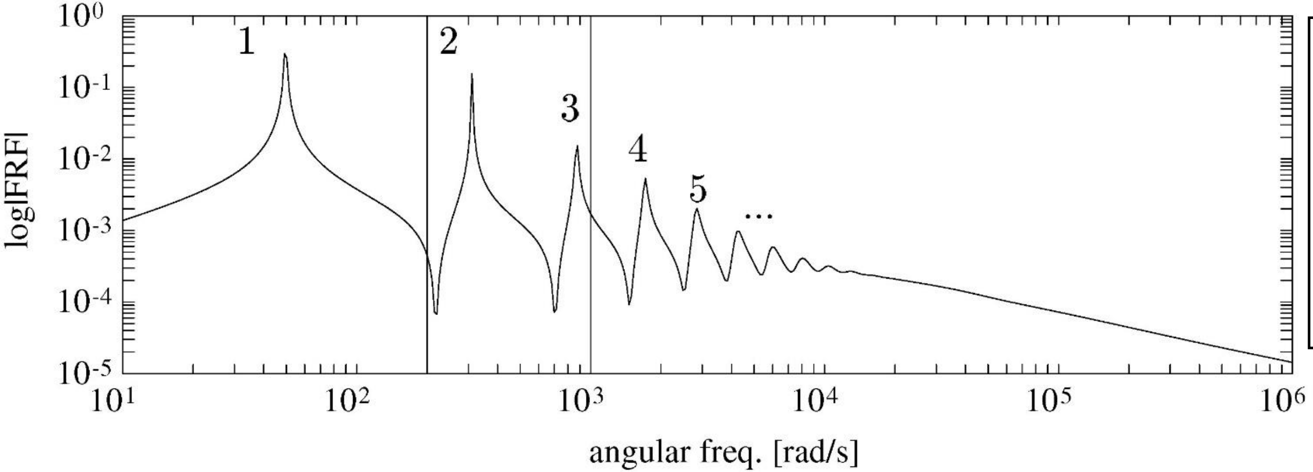
$$H[i,l](\omega) \approx \frac{-1}{m_{il}^*\omega^2} + \sum_{k=m_1}^{m_2}\left[\frac{A_{kR}[i,l] + jA_{kI}[i,l]}{-\mu_k + j(\omega - \nu_k)} + \frac{A_{kR}[i,l] - jA_{kI}[i,l]}{-\mu_k + j(\omega + \nu_k)}\right] + s_{il}^*$$

- s_{il}^*: residual flexibility
- m_{il}^*: residual mass

Modes $k = m_1, \dots m_2$ are exactly represented,
remaining modes are approximated in $\omega \in [\omega_{min}, \omega_{max}]$.

Correct selection of m_1 and m_2 is important first step in procedure.

Example: Cantilever beam with Rayleigh damping



36 beam elements
Rayleigh damping
$B = 0.1M + 2 \cdot 10^{-5} K$

input: force on free end
output: vel. of free end

Select frequency band: $\omega_{min} = 200 \leq \omega \leq 1000 = \omega_{max}$ rad/s

Modes 2 and 3 dominant

Approximation of modes 1 and 4 questionable
Approximation of modes 5,6 etc probably correct

A multi-mode-fit in the frequency domain

$$H[i,l](\omega) \approx \frac{-1}{m_{il}^{*}\omega^{2}} + \sum_{k=m_1}^{m_2}\left[\frac{A_{kR}[i,l] + jA_{kI}[i,l]}{-\mu_k + j(\omega - \nu_k)} + \frac{A_{kR}[i,l] - jA_{kI}[i,l]}{-\mu_k + j(\omega + \nu_k)}\right] + s_{il}^{*}$$

Number of selected modes $m = m_2 - m_1 + 1$.

Number of unknowns:

- m_{il}^{*}: 1 unknown
- $\mu_k, \nu_k, A_{kR}[i,l], A_{kI}[i,l]$ for $k = m_1, \ldots, m_2$: $4m$ unknowns
- s_{il}^{*}: 1 unknown

Total: $4m + 2$ unknowns.

Observe: approximation is linear in the parameters $1/m_{il}^{*}$, $A_{kR}[i,l]$, $A_{kI}[i,l]$, and s_{il}^{*}.

Algorithm (part 1/2)

For each measured FRF $H[i,l](\omega_p)$:

1) create initial estimates $\left(\mu_k^{(0)}, \nu_k^{(0)}\right)$ for $k = m_1, \ldots, m_2$

   e.g. by using circle fit or another single-mode fit technique

2) Determine parameters $a_0 := s_{il}^*$, $a_1 := -1/m_{il}^*$, $a_{k3} := A_{kR}[i,l]$, and $a_{k4} = A_{kI}[i,l]$ by minimizing the *least-squares* error in

$$H[i,l](\omega_p) = a_0 + a_1 f_1(\omega_p) + \sum_{k=m_1}^{m_2} \left[(a_{k3} + j a_{k4}) f_{k2}(\omega_p) + (a_{k3} - j a_{k4}) f_{k3}(\omega_p)\right]$$

   with

$f_1(\omega) = 1/\omega^2$, $f_{k2}(\omega) = 1/(-\mu_k^{(0)} + j(\omega - \nu_k^{(0)}))$, and $f_{k3}(\omega) = \frac{1}{-\mu_k^{(0)} + j\left(\omega + \nu_k^{(0)}\right)}$

   results in initial estimates $a_0^{(0)}$, $a_1^{(0)}$, $a_{k3}^{(0)}$, and $a_{k4}^{(0)}$.

3) Start an iterative **nonlinear** least-squares algorithm for all parameters $\mu_k^{(i)}, \nu_k^{(i)}, a_0^{(i)}, a_1^{(i)}, a_{2k}^{(i)}, a_{3k}^{(i)}$ using $\mu_k^{(0)}, \nu_k^{(0)}, a_0^{(0)}, a_1^{(0)}, a_{2k}^{(0)}, a_{3k}^{(0)}$ as first estimates.

Algorithm (part 2/2)

4) Calculate the eigenvalue averages $\hat{\lambda}_k = \hat{\mu}_k + j\hat{\nu}_k$ by averaging μ_k and ν_k estimated for each measured FRF

5) Finally, using the eigenvalue averages $\hat{\lambda}_k = \hat{\mu}_k + j\hat{\nu}_k$, apply the **linear** least-squares algorithm again (step 2) to obtain for each measured FRF: $\hat{a}_0, \hat{a}_1, \hat{a}_{k3}, \hat{a}_{k4}$

11d. A multi-mode fit in the time domain

A multi-mode-fit in the time domain

Least Squares Complex Exponential (LSCE) technique.
Advantage: does not rely on initial estimates.

Frequency domain	**Time domain**
One column or one row of measured Frequency Response Functions (FRFs)	One column or one row of measured Impulse Response Functions (IRFs)
$H[i,l](\omega_k) = \sum_{r=1}^{2n} \frac{A_r[i,l]}{j\omega_k - \lambda_r}$	$h[i,l](t_k) = \sum_{r=1}^{2n} A_r[i,l] e^{\lambda_r t_k}$

$\xrightarrow{\text{IFFT}}$

Approach: first find λ_r, then one row or one column of $A_r[i,l]$

An LSCE technique in the time domain

Assume equidistant time points: $t_k = k\Delta T$ ($k = 0,1,\dots,N-1$)
Define: $U_r = e^{\lambda_r \Delta T}$ (for $r = 1,2,\dots,2n$)

$$\boxed{h[i,l](t_k) = \sum_{r=1}^{2n} A_r[i,l]e^{\lambda_r k\Delta T} = \sum_{r=1}^{2n} A_r[i,l]U_r^k}$$

Define polynomial with roots U_r ($r = 1,2,\dots,2$)

$$p(U) = \prod_{r=1}^{2n}(U - U_r) = \sum_{k=0}^{2n} \alpha_{2n-k}U^k = \alpha_0 U^{2n} + \alpha_1 U^{2n-1} + \cdots \alpha_{2n-1}U + \alpha_{2n}$$

Note that $\alpha_0 = 1$ and that

$$\boxed{p(U_r) = \sum_{k=0}^{2n} \alpha_{2n-k}U_r^k = 0.}$$

Also note that λ_r unknown $\Rightarrow$ U_r unknown $\Rightarrow$ α_k unknown.

An LSCE technique in the time domain

$$h[i,l](t_k) = \sum_{r=1}^{2n} A_r[i,l] U_r^k, \qquad \sum_{k=0}^{2n} \alpha_{2n-k} U_r^k = 0.$$

Strategy to determine λ_r $(r = 1, \ldots, 2n)$ using one single $h[i,l]$:

1. determine coefficients α_r $(r = 1, \ldots, 2n)$, take $\alpha_0 = 1$
2. determine U_r $(r = 1, \ldots, 2n)$ using polynomial roots solver
3. determine λ_r $(r = 1, \ldots, 2n)$ using $U_r = e^{\lambda_r \Delta T}$

As we consider only one FRF in this procedure, we drop the index $[i,l]$.

Determining α_r

$$h(t_k) = \sum_{r=1}^{2n} A_r U_r^k$$

1. Write out for $k = 0,1, \dots, 2n$

$$\begin{array}{ccccccccc}
h(t_0) & = & A_1 & + & A_2 & + & \cdots & + & A_{2n} \\
h(t_1) & = & A_1 U_1 & + & A_2 U_2 & + & \cdots & + & A_{2n} U_{2n} \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
h(t_{2n}) & = & A_1 U_1^{2n} & + & A_2 U_2^{2n} & + & \cdots & + & A_{2n} U_{2n}^{2n}
\end{array}$$

Determining α_r

$$h(t_k) = \sum_{r=1}^{2n} A_r U_r^k$$

1. Write relations out for $k = 0,1, \dots, 2n$
2. Multiply the first equation by α_{2n}, the second by $\alpha_{2n} - 1$, etc.

$$\begin{array}{ccccccccc}
\alpha_{2n} h(t_0) & = & \alpha_{2n} A_1 & + & \alpha_{2n} A_2 & + & \cdots & + & \alpha_{2n} A_{2n} \\
\alpha_{2n-1} h(t_1) & = & \alpha_{2n-1} A_1 U_1 & + & \alpha_{2n-1} A_2 U_2 & + & \cdots & + & \alpha_{2n-1} A_{2n} U_{2n} \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\alpha_0 h(t_{2n}) & = & \alpha_0 A_1 U_1^{2n} & + & \alpha_0 A_2 U_2^{2n} & + & \cdots & + & \alpha_0 A_{2n} U_{2n}^{2n}
\end{array}$$

Determining α_r

Recall: $p(U_r) = \alpha_0 U_r^{2n} + \alpha_1 U_r^{2n-1} + \cdots \alpha_{2n-1} U_r + \alpha_{2n} = 0$

$$h(t_k) = \sum_{r=1}^{2n} A_r U_r^k$$

1. Write relations out for $k = 0,1,\ldots,2n$
2. Multiply the first equation by α_{2n}, the second by $\alpha_{2n} - 1$, etc.
3. Add all equations:

$$\begin{array}{ccccccccc}
\alpha_{2n} h(t_0) & = & \alpha_{2n} A_1 & + & \alpha_{2n} A_2 & + & \cdots & + & \alpha_{2n} A_{2n} \\
\alpha_{2n-1} h(t_1) & = & \alpha_{2n-1} A_1 U_1 & + & \alpha_{2n-1} A_2 U_2 & + & \cdots & + & \alpha_{2n-1} A_{2n} U_{2n} \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\alpha_0 h(t_{2n}) & = & \alpha_0 A_1 U_1^{2n} & + & \alpha_0 A_2 U_2^{2n} & + & \cdots & + & \alpha_0 A_{2n} U_{2n}^{2n}
\end{array}$$

$$\sum_{k=0}^{2n} \alpha_{2n-k} h(t_k) = A_1 p(U_1) + A_2 p(U_2) + \cdots + A_{2n} p(U_{2n}) = 0$$

We have found one linear equation for the coefficients α_k!

Determining α_r

Recall: $p(U_r) = \alpha_0 U_r^{2n} + \alpha_1 U_r^{2n-1} + \cdots \alpha_{2n-1} U_r + \alpha_{2n} = 0$

For a second equation:

$$h(t_k) = \sum_{r=1}^{2n} A_r U_r^k$$

1. Write relations out for $k = 1,2,\ldots,2n+1$
2. Multiply the first equation by α_{2n}, the second by $\alpha_{2n} - 1$, etc.
3. Add all equations:

$$\begin{array}{ccccccccc}
\alpha_{2n} h(t_1) & = & \alpha_{2n} A_1 U_1 & + & \alpha_{2n} A_2 U_2 & + & \cdots & + & \alpha_{2n} A_{2n} U_{2n} \\
\alpha_{2n-1} h(t_2) & = & \alpha_{2n-1} A_1 U_1^2 & + & \alpha_{2n-1} A_2 U_2^2 & + & \cdots & + & \alpha_{2n-1} A_{2n} U_{2n}^2 \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\alpha_0 h(t_{2n+1}) & = & \alpha_0 A_1 U_1^{2n+1} & + & \alpha_0 A_2 U_2^{2n+1} & + & \cdots & + & \alpha_0 A_{2n} U_{2n}^{2n+1}
\end{array}$$

$$\sum_{k=0}^{2n} \alpha_{2n-k} h(t_{k+1}) = A_1 U_1 p(U_1) + A_2 U_2 p(U_2) + \cdots + A_{2n} U_{2n} p(U_{2n}) = 0$$

Determining α_r

Recall: $p(U_r) = \alpha_0 U_r^{2n} + \alpha_1 U_r^{2n-1} + \cdots \alpha_{2n-1} U_r + \alpha_{2n} = 0$

For the $(m+1)$-th equation:

$$h(t_k) = \sum_{r=1}^{2n} A_r U_r^k$$

1. Write relations out for $k = m, 2, \dots, 2n + m$
2. Multiply the first equation by α_{2n}, the second by $\alpha_{2n} - 1$, etc.
3. Add all equations:

$$\begin{array}{ccccccccc}
\alpha_{2n} h(t_m) & = & \alpha_{2n} A_1 U_1^m & + & \alpha_{2n} A_2 U_2^m & + & \cdots & + & \alpha_{2n} A_{2n} U_{2n}^m \\
\alpha_{2n-1} h(t_{m+1}) & = & \alpha_{2n-1} A_1 U_1^{m+1} & + & \alpha_{2n-1} A_2 U_2^{m+1} & + & \cdots & + & \alpha_{2n-1} A_{2n} U_{2n}^{m+1} \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\alpha_0 h(t_{2n+m}) & = & \alpha_0 A_1 U_1^{2n+m} & + & \alpha_0 A_2 U_2^{2n+m} & + & \cdots & + & \alpha_0 A_{2n} U_{2n}^{2n+m}
\end{array}$$

$$\sum_{k=0}^{2n} \alpha_{2n-k} h(t_{k+m}) = A_1 U_1^m p(U_1) + A_2 U_2^m p(U_2) + \cdots + A_{2n} U_{2n}^m p(U_{2n}) = 0$$

Determining α_r

In this way, we find a total of M linear equations for the coefficients α_k.

$$B\alpha = R$$

with

$\alpha = [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_{2n}]^\top$,

$R = -[h(t_{2n}) \quad h(t_{2n+1}) \quad \dots \quad h(t_{2n+M-1})]^\top$

,

$$B = \begin{bmatrix} h(t_{2n-1}) & h(t_{2n-2}) & \cdots & h(t_0) \\ h(t_{2n}) & h(t_{2n-1}) & \cdots & h(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ h(t_{2n+M-2}) & h(t_{2n+M-3}) & \cdots & h(t_{M-1}) \end{bmatrix}$$

$\boxed{\text{Recall: } \alpha_0 = 1}$

B is rectangular of size $(M, 2n)$, we take: $M \geq 2n$.

Determining α_r

$$B\alpha = R$$

When $M > 2n$, the system is overdetermined. ⇒ use least squares approach.

We define the scalar least squares error ε:

$$\varepsilon = [B\alpha - R]^T [B\alpha - R]$$

A minimum is obtained for $d\varepsilon/d\alpha = 0$ leading to:

$$\alpha = [B^T B]^{-1} B^T R$$

Determining λ_r

$$p(U) = \alpha_0 U^{2n} + \alpha_1 U^{2n-1} + \cdots \alpha_{2n-1} U + \alpha_{2n}$$

Coefficients α_r of the polynomial are now available.

Use standard routine to find roots U_r $(r = 1, \ldots, 2n)$ of $p(U)$

The eigenvalues $\lambda_r = \mu_r + j\nu_r$ now follow from:

$$U_r = e^{\lambda_r \Delta T}, \qquad \Rightarrow \qquad \mu_r = \frac{ln|U_r|}{\Delta T}, \qquad \nu_r = \frac{arg\, U_r}{\Delta T}.$$

TU/e

Determining $A_r (= A_r[i, l])$

$$h(t_k) = \sum_{r=1}^{2n} A_r U_r^k$$

Write out for $k = 0,1, \dots, P-1$.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ U_1 & U_2 & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ U_1^{P-1} & U_2^{P-1} & \cdots & U_{2n}^{P-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2n} \end{bmatrix} = \begin{bmatrix} h(t_0) \\ h(t_1) \\ \vdots \\ h(t_{P-1}) \end{bmatrix}$$

As $U_r (r = 1,2, \dots, 2n)$ are now available,
P linear equations in the $2n$ unknowns $A_r (r = 1,2, \dots, 2n)$.

For $P \geq 2n$, A_r can be determined using a least squares approach.

Multiple IRFs

Until now: only a single impulse response function $h(t) = h[i,l](t)$ was used!

Note: eigenvalues λ_r are the same for each IRF
$\Rightarrow U_r$ are the same for each IRF $\Rightarrow \alpha_1, \alpha_2, \dots, \alpha_{2n}$ are independent of the specific IRF!

In case of multiple IRFs:

OPTION 1: repeat above procedure for each IRF and take average of estimated μ_r and average of estimated ν_r

OPTION 2: better: take into account all, say s, measured IRFs at the same time.

Multiple IRFs

Can thus be determined from one large system considering the s measured IRFs

$$B_{tot}\alpha = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{bmatrix} \alpha = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_s \end{bmatrix} = R_{tot}$$

$s * M$ equations for $2n$ parameters α

Again solution via least squares approach: $\alpha = [B_{tot}^\top B_{tot}]^{-1} B_{tot}^\top R_{tot}$

Elements of $B_{tot}^\top B_{tot}$ and $B_{tot}^\top R_{tot}$ are related to correlation functions of the IRFs
$\Rightarrow$ can be calculated efficiently by IFFT of measured auto- and cross power spectral densities

Then:

1. determine $A_r[i,l]$ for each measured IRF $h[i,l](t)$ ($i = 1, \ldots, s$)
2. combine $A_r[i,l]$ ($i = 1, \ldots, s$) to form column $A_r[l] \sim u_r$

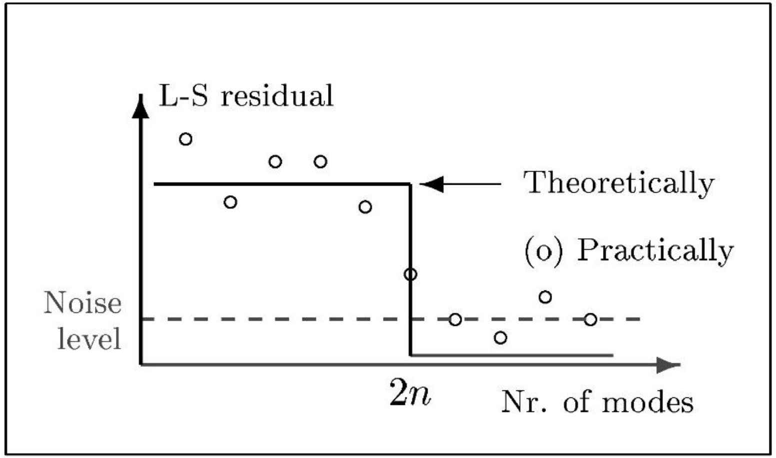
LSCE method in the time domain

Main problem: choose number of modes n

Counting of peaks may be meaningless in case of:

- closely spaced peaks
- multiple modes at same frequency
- highly damped modes

Possible solution:
evaluate least squares error $\varepsilon(\alpha)$ for several n

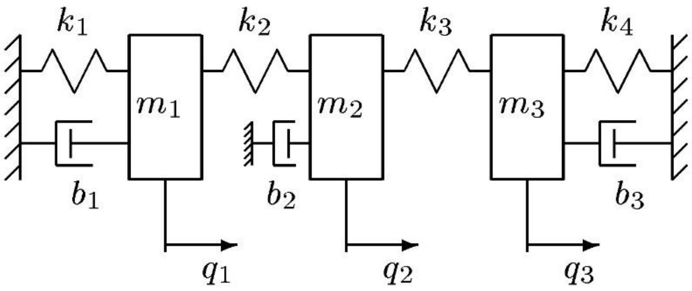


Fit with **too little modes** will lead to large error
because not all resonances can be represented effectively

Fit with **too many modes** will lead to **computational modes** and will still show a (small) error due to measurement noise, etc.

LS error ε will stabilize on certain level despite increase of n
Recognition of computational modes (non-physical modes) needs practical experience!

Example: 3-dof model with 2 closely spaced modes



Dof's: $\underline{q}^T = [q_1, q_2, q_3]$

Data:

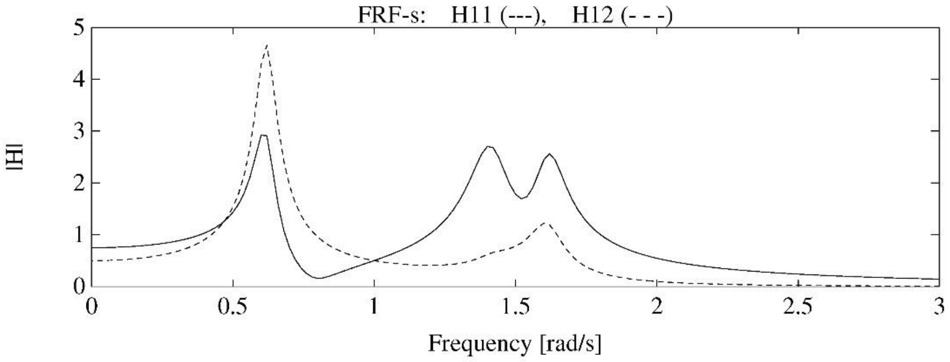
$$m_1 = 1; m_2 = 2; m_3 = 1 \ [kg]$$
$$k_1 = k_2 = k_3 = k_4 = 1 \ [N/m]$$
$$b_1 = b_2 = 0.1; b_3 = 0.2 \ [Ns/m]$$

Eigenvalues:

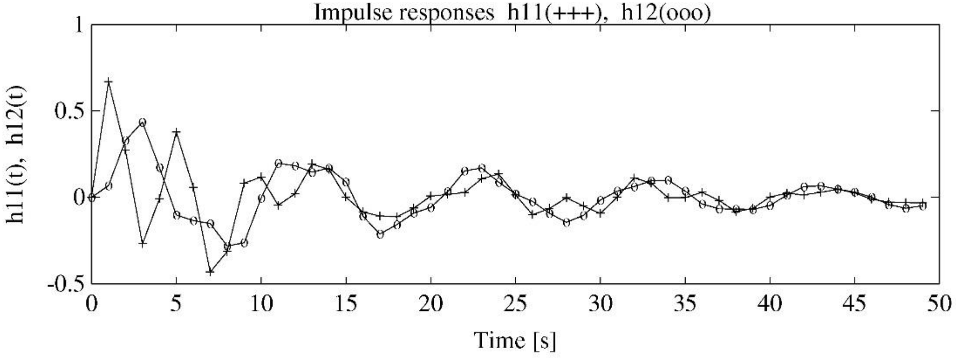
$$\lambda_{1,2} = -0.0389 \pm 0.6172j$$
$$\lambda_{3,4} = -0.0754 \pm 1.4140j$$
$$\lambda_{5,6} = -0.0607 \pm 1.6138j$$

Example: 3-dof model with 2 closely spaced modes

FRF's H_{11} and H_{12}:
in H_{12}, the last two peaks merge



Identification:
IRFs $h_{11}(t)$ and $h_{12}(t)$,
$\Delta T = 1$ s, $0 \leq t < 100$ s
5% G. w. noise



Example: 3-dof model with 2 closely spaced modes

We use: two estimates for number of modes n: $n = 3$ and $n = 4$
number of repetitions $M = 40$, number of residue equations $P = 40$
$h_{11}(t)$ and $h_{12}(t)$ are simultaneously used in estimating the eigenvalues.

Results estimation:

Eigenvalue	$n=3$	$n=4$
$\lambda^e_{1,2}$	$-0.0425 \pm 0.6179\ j$	$-0.0401 \pm 0.6173\ j$
$\lambda^e_{3,4}$	$-0.1216 \pm 1.5218\ j$	$-0.0898 \pm 1.4102\ j$
$\lambda^e_{5,6}$	$-0.4526 \pm 2.0542\ j$	$-0.0760 \pm 1.6198\ j$
$\lambda^e_{7,8}$	- - - - - - -	$-0.2224 \pm 2.8124\ j$

For comparison:
True eigenvalues
$\lambda_{1,2} = -0.0389 \pm 0.6172j$
$\lambda_{3,4} = -0.0754 \pm 1.4140j$
$\lambda_{5,6} = -0.0607 \pm 1.6138j$

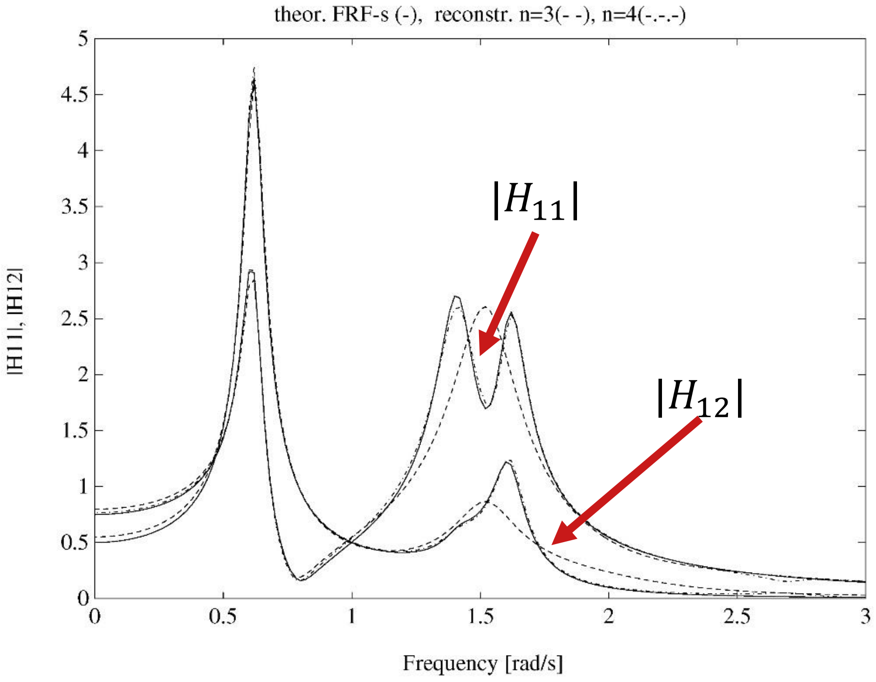
$n = 3$: moderately accurate, real part of $\lambda^e_{5,6}$ looks suspicious
$n = 4$: accurate, effect noise handled better, two computational modes $\lambda^e_{7,8}$

Example: 3-dof model with 2 closely spaced modes

For both $|H_{11}|$ and $|H_{12}|$:

- one exact curve (solid)
- two reconstructions ($n = 3{,}4$, dashed)

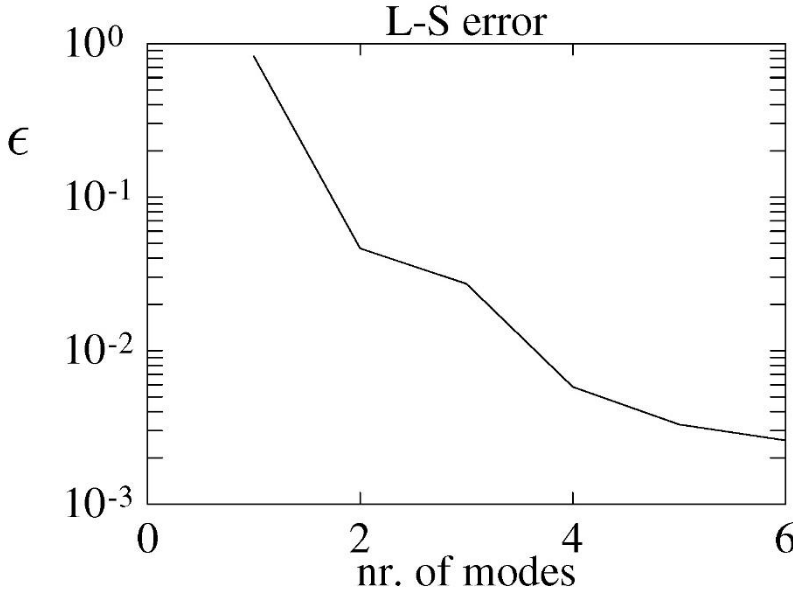
Computational modes $\lambda^e_{7,8}$ do not lead to relevant distortion



Example: 3-dof model with 2 closely spaced modes

General decrease of least squares error ε when n increases

$n = 4$ sufficient



Concluding remarks LSCE

Some trial and error will be needed for optimal results

In commercial packages:

some additional techniques to improve robustness

Review Experimental Modal Analysis

EMA can not be used as a black box!

Many parameters influence the final results!

Stage/Chapter/Lecture	parameters
Experiments	type of excitation signal, sensors, anti-aliasing filter
FFT	T, N, window type
System identification	input/output noise, number of records (averaging), FRF estimator used (H_1, H_2)
Modal parameter fit procedures	Number of FRF's (s) and location of sensors For LSCE technique also n, M, and P.