Group 5 - Project 2: Dynamic Allocation and VaR of a Portfolio

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1. Static allocation

Q1.1 : Give the expression for the optimal portfolio weights.

We consider an investor allocating wealth among two risky assets (stocks and bonds) and a risk-free asset. Denote

$$\tilde{\alpha} = \begin{pmatrix} \alpha_s \\ \alpha_b \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_s \\ \mu_b \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{ss} & \Sigma_{sb} \\ \Sigma_{sb} & \Sigma_{bb} \end{pmatrix},$$

where:

- α_s, α_b are the (unconstrained) portfolio weights in stocks and bonds;
- the weight in the risk-free asset is $1 - e'\tilde{\alpha}$;
- μ_s, μ_b are the expected returns of stocks and bonds;
- Σ is the covariance matrix of stock and bond returns;
- R_f is the (constant) risk-free rate;
- $\lambda > 0$ is the investor's coefficient of risk-aversion.

The mean–variance objective is

$$\max_{\tilde{\alpha}} \quad \underbrace{\mu_p}_{\text{portfolio mean}} - \frac{\lambda}{2} \underbrace{\sigma_p^2}_{\text{portfolio variance}} = \underbrace{\left[\tilde{\alpha}' \mu \; + \; (1 - e'\tilde{\alpha}) \, R_f \right]}_{\mu_p} - \frac{\lambda}{2} \underbrace{\left(\tilde{\alpha}' \, \Sigma \, \tilde{\alpha} \right)}_{\sigma_p^2}.$$

We compute the first order derivative of the mean-variance criterion with respect to $\tilde{\alpha}$:

$$\nabla_{\tilde{\alpha}} f = \mu - R_f e - \lambda \Sigma \, \tilde{\alpha}.$$

Taking the gradient and setting to zero:

$$\nabla_{\tilde{\alpha}} f = \mu - R_f e - \lambda \Sigma \, \tilde{\alpha} \; = \; 0 \quad \Longrightarrow \quad \boxed{\tilde{\alpha}^* = \frac{1}{\lambda} \, \Sigma^{-1} \big(\mu - R_f e \big)}.$$

The weight invested in the risk-free asset is $1 - e'\tilde{\alpha}^*$.

Q1.2 : Assume that expected returns are given by sample means and that the covariance matrix Σ is given by the sample covariance matrix. Compute the optimal weight vector, denoted by $\tilde{\alpha}^*$, for $\lambda = 2$ and 10.

Using the sample estimates from weekly return data (2001–2024), we compute optimal portfolio weights for two levels of risk aversion, $\lambda = 2$ and $\lambda = 10$, based on the mean-variance framework. The expected return vector, average risk-free rate, and sample covariance matrix are:

$$\hat{\mu} = \begin{pmatrix} 0.001137 \\ 0.000568 \end{pmatrix}, \quad \overline{R}_f = 0.000061, \quad \hat{\Sigma} = \begin{pmatrix} 6.39 \times 10^{-4} & -3.31 \times 10^{-5} \\ -3.31 \times 10^{-5} & 5.20 \times 10^{-5} \end{pmatrix}.$$

The optimal portfolio weights $\widehat{\alpha}^*(\lambda) = \frac{1}{\lambda}\,\widehat{\Sigma}^{-1}(\widehat{\mu} - \overline{R}_f\, e)$ for each level of risk aversion are:

λ	α_s (SMI)	α_b (SGBI)	$\alpha_{\text{risk-free}}$
2	1.13	5.60	-5.73
10	0.23	1.12	-0.352

The sample mean excess returns ($\widehat{\mu}_s - \overline{R}_f$ and $\widehat{\mu}_b - \overline{R}_f$) are approximately 0.001076 for the SMI and 0.000507 for the Swiss Govt. Bonds Index (SGBI). While equities offer the higher risk premium, they are also substantially more volatile, as evidenced by the covariance matrix. Bonds, in contrast, exhibit both lower variance and a slightly negative correlation with equities, enhancing their attractiveness from a diversification standpoint. With $\lambda = 2$, a relatively risk-tolerant investor allocates heavily into risky assets, particularly bonds, and finances this exposure by borrowing through a short position in cash. This leads to a highly leveraged portfolio, with over 670% gross exposure to the risky component. As risk aversion rises to $\lambda = 10$, the allocations become more moderate, with lower positions in both risky assets and a reduced reliance on leverage. This outcome is typical of unconstrained mean-variance optimization: capital flows to assets with the most favorable risk-adjusted returns, and when the estimated sharpe ratio of one asset dominates—here, the bond index—allocation tilts heavily in that direction.

2. Estimation of a GARCH model

Q2.1: For both stocks and bonds, provide evidence on the non-normality (KolmogorovSmirnov test) and the auto-correlation (Ljung-Box test, with 4 lags) of the excess returns and squared excess returns.

This analysis specifically investigates the non-normality and autocorrelation of both the excess returns and squared excess returns of stocks and bonds using the Kolmogorov-Smirnov (K-S) and Ljung-Box tests. The results are presented in the following table.

Test / Lag	Stocks (ex_s)	Bonds (ex_b)
K-S p-value		
Excess return	≈ 0.0000	0.0010
Squared Excess Return	≈ 0	≈ 0
Ljung–Box p-value (Excess returns)		
Lag 1	≈ 0	0.3605
Lag 2	≈ 0	0.4387
Lag 3	≈ 0	0.5101
Lag 4	≈ 0	0.6545
Ljung–Box p-value (Squared Excess Returns)		
Lag 1	≈ 0	≈ 0
Lag 2	≈ 0	≈ 0
Lag 3	≈ 0	≈ 0
Lag 4	≈ 0	≈ 0

Table 1: K-S and Ljung-Box tests results summary (p-values)

The **Kolmogorov-Smirnov test** provides strong evidence against the null hypothesis of normality for both stock and bond **excess returns** (p-values = 0.0000 and 0.0010, respectively). These findings indicate a significant deviation from normality in both series. The **Ljung-Box test** reveals significant evidence of autocorrelation in the **stock** excess returns, with p-values for the first four lags effectively zero. For **bonds**, **no** significant **autocorrelation** is detected at the same lags, indicating a lack of linear dependence. For **squared excess returns**, both series display **Ljung-Box test**'s p-values near zero (p-value < 0.01) across all lags considered. This suggests **strong autocorrelation in the second moments** and provides robust evidence of volatility clustering, a feature often associated with nonlinear dependence in financial return dynamics.

In addition to the Kolmogorov-Smirnov (K-S) test, the Lilliefors test is applied to assess the normality of both the excess returns and squared excess returns for stocks and bonds. Unlike the K-S test, which assumes that the mean and variance are known, the Lilliefors test adjusts for the fact that these parameters are estimated from the sample itself. This adjustment makes the Lilliefors test a more appropriate and conservative approach when dealing with financial time series data.

Table 2: Lilliefors Test results for excess returns and squared excess returns

Test / Lag	Lilliefors Stat	p-value
Excess Returns (ex_s)		
Returns	0.0868	0.0010
Squared Returns	0.3540	0.0010
Excess Returns (ex_b)		
Returns	0.0571	0.0010
Squared Returns	0.3567	0.0010

The **Lilliefors test** results, reported in Table 2, confirm substantial departures from normality in both the excess returns and their squared transformations. For the excess returns, the test statistics are 0.0868 for stocks and 0.0571 for bonds, with corresponding p-values of 0.0010 in both cases. These results reinforce the earlier rejection of normality by the Kolmogorov–Smirnov test, albeit with a slightly more conservative inference. For the squared excess returns, the Lilliefors statistics rise to 0.3540 and 0.3567 for stocks and bonds respectively—values that would be difficult to reconcile with any Gaussian distribution. This outcome is entirely expected, as squaring induces non-negativity and skews the distribution, thereby violating key properties of normality.

Table 3: ARCH-LM Test Results for Excess Returns and Squared Returns

Test / Lag	**ARCH-LM p-value (ex_s)**	**ARCH-LM p-value (ex_b)**
Returns	≈ 0.0000	≈ 0.0000
Squared Returns	≈ 0.0000	≈ 0.0000

To conclude, we assess the presence of autoregressive conditional heteroskedasticity using the **ARCH-LM test**, summarized in Table 3. For both stocks and bonds, the test decisively rejects the null of constant conditional variance when applied to both the raw excess returns and their squared counterparts. These results confirm the **presence of time-varying volatility**, aligning closely with the results observed in the Ljung–Box test for squared returns. The convergence of evidence across these tests suggests that while linear autocorrelation is present only in the stock time series, nonlinear dependence in volatility dynamics is a feature of both asset classes.

Q2.2: Estimate an AR(1) model on stock and bond returns, to filter out autocorrelation. We now denote by $\hat{\varepsilon}_{i,t+1}$ the residuals of the AR(1) model. Comment the regression estimation (goodness-of-fit, parameter estimates).

To investigate and filter out potential linear serial dependence in asset returns, we estimate AR(1) models on the weekly returns of the Swiss government bond index (R_b) and the Swiss Market Index (R_s).

$$R_{i,t} = \alpha_i + \rho_i R_{i,t-1} + \varepsilon_{i,t}, \quad i \in \{s, b\}$$

	Bond Returns (R_b)	**Stock Returns (R_s)**
const	0.0006***	0.0013
	(0.0002)	(0.001)
ρ	-0.0242	-0.1165***
	(0.0283)	(0.0281)
R-squared	0.001	0.014
Adj. R-squared	0.000	0.013
Prob (F-stat)	0.392	3.6×10^{-5}
Durbin-Watson	2.000	2.009
Skew	-0.406	-0.692
Kurtosis	8.484	8.154
Prob(JB)	0.00	0.00
No. Observations	1251	1251
ARCH-LM (χ^2) p-value	0.0000***	0.0000***
ARCH-LM (F-test) p-value	0.0000***	0.0000***

Standard errors in parentheses. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

Table 4: OLS Regression Results for AR(1) Models on Bond and Stock Returns

In the AR(1) model estimated on **bond yields (R_b)**, the auto-regressive coefficient is -0.0242 and is not statistically significant, with a p-value of 0.392. This suggests no significant serial correlation in weekly bond yields. The constant is positive and significant at the 1% level, but very close to zero. The R^2 value is extremely low (0.001), suggesting that the model has almost no explanatory capacity.

Turning to the **residuals**, the Durbin–Watson statistic is exactly 2.0, consistent with the absence of serial correlation in residuals. The ARCH-LM test show a p-value of 0.0000, which leads to reject the null hypothesis of homoskedasticity. This implies that the residuals of the AR(1) model have heteroskedasticity, i.e., non-constant variance over time, and suggests the need for a GARCH model to correctly capture the volatility dynamics.

In the case of **equity returns (R_s)**, the estimated AR(1) coefficient is -0.1165 and is highly significant, with a p-value below 0.0001. This indicates the presence of negative autocorrelation in returns. The constant, unlike the bond case, is not significant at the 5% level. Again, however, the R^2 remains very low (0.014), indicating that the model captures only a small part of the variability in the data.

Turning to the **residuals**, the Durbin–Watson statistic of 2.009 supports the absence of residual autocorrelation at lag 1. Again, the ARCH-LM tests return zero p-values, showing a strong heteroskedasticity in the residues. This confirms that a GARCH model for dynamically modelling volatility is also appropriate for equity returns.

We also examine the empirical distribution of the AR(1) residuals $\hat{\varepsilon}_{i,t}$ for $i \in \{s, b\}$. These residuals reveal significant deviations from normality, with similar leptokurtosis (Kurtosis ≈ 8.3) and negative skewness (Skew ≈ -0.5) in both series. The Jarque–Bera test strongly rejects normality in both cases (p-values $<$ 0.001), indicating the presence of fat tails and asymmetry.

Overall, the AR(1) specification appears sufficient to capture the linear dynamics in the conditional mean of both bond and equity returns, as indicated by the absence of significant autocorrelation in the residuals (D–W = 2). However, strong evidence of time-varying volatility emerges from the ARCH-LM tests on the residuals. This, combined with the pronounced skewness, excess kurtosis, and rejection of normality from the Jarque–Bera test, underscores the presence of non-linear dependence in the volatility process. These findings justify the use of GARCH-type models to capture conditional heteroskedasticity, with non-Gaussian innovation distributions (such as the Student's t or skewed t) likely offering a better empirical fit.

Q2.3: Estimate the GARCH(1,1) model for residuals using the conditional ML technique. Comment the parameter estimates (in particular, the sum $\alpha_i + \beta_i$). Test the null hypothesis that $\alpha_i + \beta_i = 1$ against the alternative that $\alpha_i + \beta_i < 1$, at the 5% significance level.

Table 5: GARCH(1,1) Estimation Results for Bond and Stock Returns

	Bond Returns (r_b)	**Stock Returns (r_s)**
ω	0.0258***	0.5147***
	(0.0072)	(0.1030)
$\omega_{\textbf{(un-scaled)}}$	0.000003***	0.000051***
	(0.0072)	(0.1030)
α	0.1312***	0.2223***
	(0.0276)	(0.0309)
β	0.8244***	0.6979***
	(0.0324)	(0.0362)
$\alpha + \beta$	0.9556	0.9202
Wald test $H_0 : \alpha + \beta = 1$	$z = -2.71$, $p = 0.0034$***	$z = -3.40$, $p = 0.0003$***

Standard errors in parentheses. * p $<$ 0.1, ** p $<$ 0.05, *** p $<$ 0.01.

We estimate GARCH(1,1) models on the AR(1) residuals for both bond and stock returns using conditional maximum likelihood. The estimated parameters, reported in Table 5, are all statistically significant at the 1% level. For bonds, the volatility dynamics are captured by $\hat{\omega}_{(un-scaled)} = 0.000003$, $\hat{\alpha} = 0.1312$, and $\hat{\beta} = 0.8244$, with a persistence measure $\hat{\alpha} + \hat{\beta} = 0.9556$. For stocks, we find $\hat{\omega}_{(un-scaled)} = 0.00005$, $\hat{\alpha} = 0.2223$, and $\hat{\beta} = 0.6979$, leading to $\hat{\alpha} + \hat{\beta} = 0.9202$. These values indicate a high degree of volatility persistence in both series, consistent with empirical findings in financial markets.

The composition of the GARCH parameters also reflects differences in market behavior. Bond return volatility is driven more heavily by its past values, as suggested by a relatively high β, while its response to new information is moderate ($\alpha = 0.1312$). In contrast, stock return volatility reacts more sharply to recent shocks ($\alpha = 0.2223$), but exhibits less persistence, with a lower $\beta = 0.6979$. This suggests that equity markets are more reactive to new information, whereas bond markets adjust more gradually.

To assess whether these volatility processes are covariance stationary, we perform Wald tests for the null hypothesis $H_0 : \alpha + \beta = 1$ against the alternative $H_1 : \alpha + \beta < 1$. The resulting test statistics are $z = -2.71$ (p = 0.0034) for bonds and $z = -3.40$ (p = 0.0003) for stocks. In both cases, the null is rejected at the 5%

significance level, confirming that the conditional variance is mean-reverting and that the unconditional variance is finite.

These findings support the use of the GARCH(1,1) specification for modeling the conditional volatility of both bond and stock returns. The significance of both α and β underscores the necessity of accounting for both innovation-driven and persistence-based dynamics in volatility modeling. Moreover, the rejection of the unit root in variance ensures the models remain dynamically stable and consistent with a stationary volatility structure.

The graphical illustration below depicts the 52-week-ahead volatility forecasts generated by the estimated GARCH(1,1) models, along with the corresponding unconditional volatilities. For both stocks and bonds, the conditional forecasted volatility gradually converges to the unconditional level, confirming the mean-reverting nature of the conditional variance process. The stock volatility, while higher, converges more rapidly due to the lower persistence ($\alpha + \beta$) relative to bonds. These trajectories visually reinforce the earlier statistical conclusions: both processes are stationary, and shocks to volatility dissipate over time at a speed determined by the persistence in the GARCH parameters.

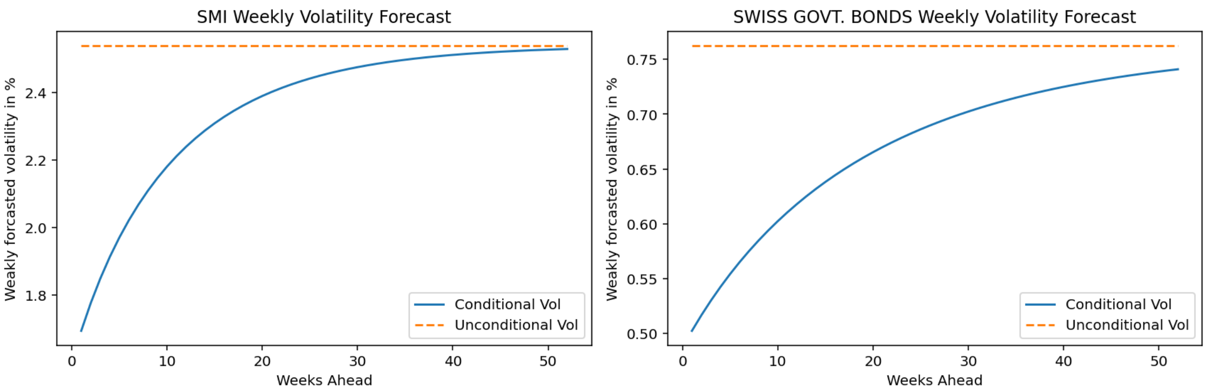

Figure 1: Weekly volatility forecasts for each asset class

3 Dynamic allocation

Q3.1: Plot the time series of optimal weights for stocks and bonds for the two approaches: with constant expected returns and volatility on the one hand ($\tilde{\alpha}^*$ of point 1) and for time-varying volatility on the other hand ($\tilde{\alpha}_t$). Comment your results for $\lambda = 2$ and 10.

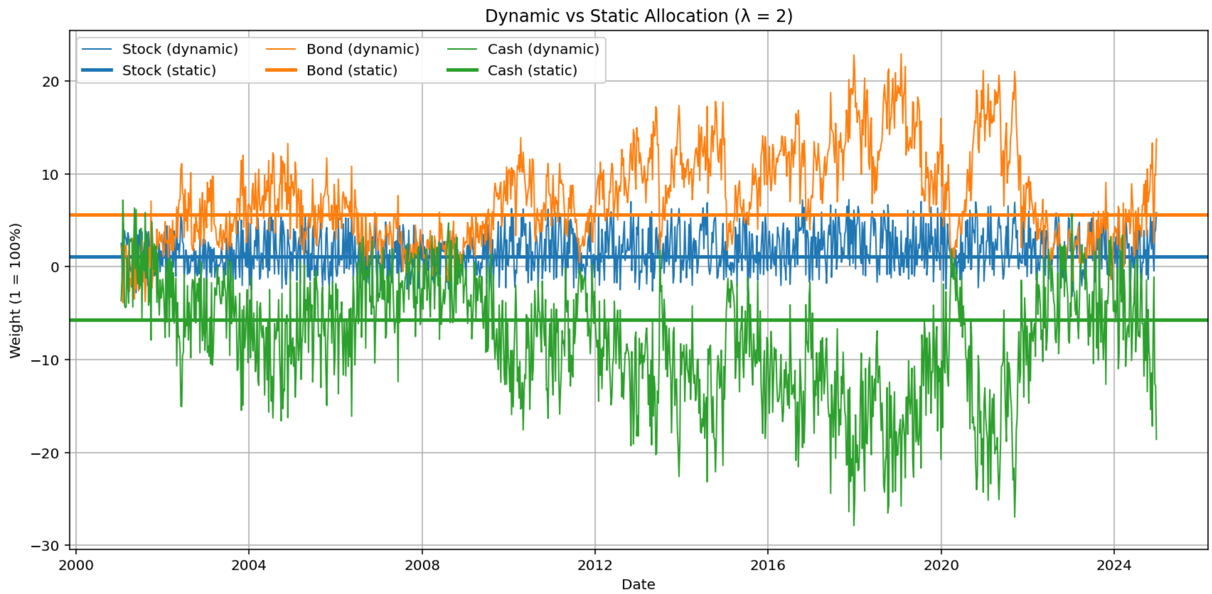

Figure 2: Dynamic vs Static Portfolio Weights ($\lambda = 2$)

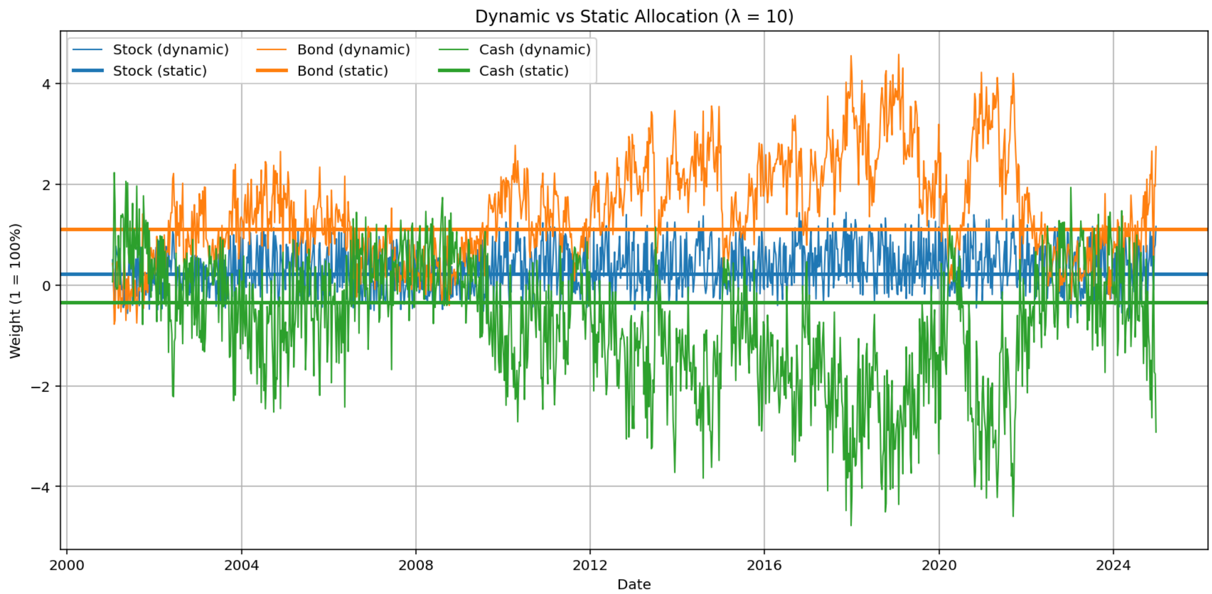

Figure 3: Dynamic vs Static Portfolio Weights ($\lambda = 10$)

The figures illustrate the time evolution of optimal portfolio weights for equities (SMI), bonds (SGBI), and cash under both static and dynamic allocation strategies, for two levels of risk aversion: $\lambda = 2$ and $\lambda = 10$.

Under the **static strategy**, weights are computed using constant sample estimates of expected returns and the variance-covariance matrix, as computed in Q1.2. These weights are constant over time and appear as horizontal lines in the plots. For $\lambda = 2$, the optimal static allocation is: 1.13 in equities, 5.60 in bonds, and –5.73 in cash. This aggressive position reflects leveraged exposure to risky assets, particularly bonds, due to their favorable risk-return trade-off. For $\lambda = 10$, the static allocation is more conservative: 0.23 in equities, 1.12 in bonds, and –0.35 in cash. The investor still borrows to invest in risky assets, but the leverage is much lower, reflecting higher risk aversion.

The **dynamic allocation strategy** responds to time-varying forecasts of expected returns and volatilities obtained through AR(1)-GARCH(1,1) models.

For the more **risk-tolerant investor** ($\lambda = 2$), this results in highly volatile and aggressive rebalancing behavior. The **equity** weight fluctuates substantially, ranging from highly negative to strongly positive, indicating frequent short and long positions. These swings reflect responsiveness to expected returns and conditional volatility updates. **Bond** allocations exhibit greater variability over the sample and tend to decrease alongside equities during crisis periods such as the 2008 financial crisis and the 2020 COVID-19 shock, reflecting a broader reduction in exposure to risky assets. Notably, in periods of heightened market uncertainty, the model shifts the portfolio heavily toward cash. These defensive allocations, particularly visible in 2008 and 2020, suggest a flight-to-safety mechanism embedded in the optimization process: when volatility spikes and expected returns deteriorate, the model reduces exposure to risky assets. **Cash** weights often take on large negative values, indicating leveraged positions when market conditions are perceived as favorable. This aggressiveness can boost returns, but it also leads to high turnover and greater sensitivity to estimation error and transaction costs.

The behavior for the more **risk-averse investor** ($\lambda = 10$), while exhibiting reduced leverage, remains qualitatively similar to the $\lambda = 2$ case. Dynamic **equity** and **bond** weights continue to fluctuate substantially over time, though within tighter bounds. The primary difference lies in the degree of aggressiveness: the $\lambda = 10$ strategy avoids the extreme overexposures and deep **cash** shortfalls observed under lower risk aversion. This indicates the continued use of leverage, albeit at a more moderate scale. During episodes of heightened volatility—such as the 2008 global financial crisis and the COVID-19—market shocks the portfolio reduces exposure to both equities and bonds while simultaneously increasing allocation to cash, demonstrating a clear and economically sound reaction to risk surges.

Overall, the level of risk aversion plays a central role in shaping allocation behavior across both static and dynamic strategies. For $\lambda = 2$, the investor adopts an aggressive stance regardless of the approach. In the static case, this manifests through highly leveraged positions in both stocks and bonds. Under the dynamic strategy, the portfolio becomes even more reactive, frequently adjusting exposures in response to changing forecasts, with substantial swings in weights and a persistent reliance on leverage.

Q3.2: Compute the cumulative returns of the optimal portfolio for the two approaches (you use the optimal portfolio weights and the realized returns). For instance, for the dynamic approach, you have $CR_t = \prod_{j=2}^{t}(1+R_{p,j})$ where $R_{p,t+1} = \tilde{\alpha}_{s,t}R_{s,t+1} + \tilde{\alpha}_{b,t}R_{b,t+1} + (1-\tilde{\alpha}_{s,t}-\tilde{\alpha}_{b,t})R_{f,t}$ is the ex-post portfolio return. (If you get something explosive, take the log of the portfolio returns by $r_{p,t+1} = \log(1+R_{p,t+1})$ and calculate the cumulative returns as the cumulative sum of $r_{p,t+1}$.) Plot the two time-series of cumulative returns. Which allocation strategy performs the best? What factor(s) could change your opinion?

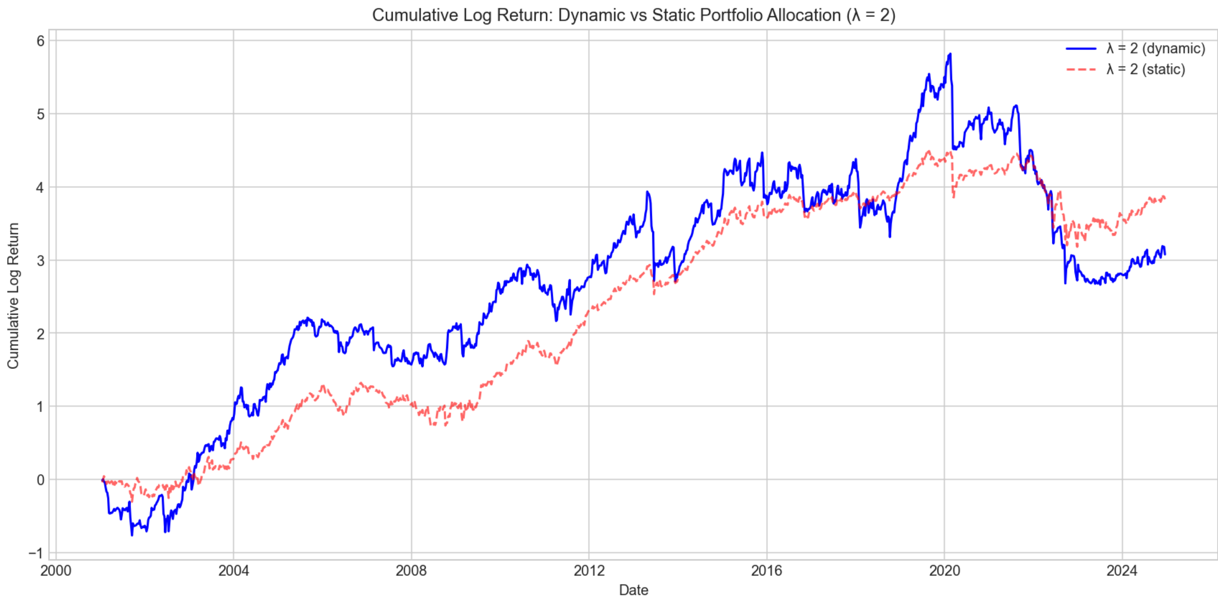

Figure 4: Dynamic vs Static Portfolio Cumulative Log Returns for $\lambda = 2$
Final Cumulative Log Returns (in%) are 307.58% (dynamic) and 382.96% (static)

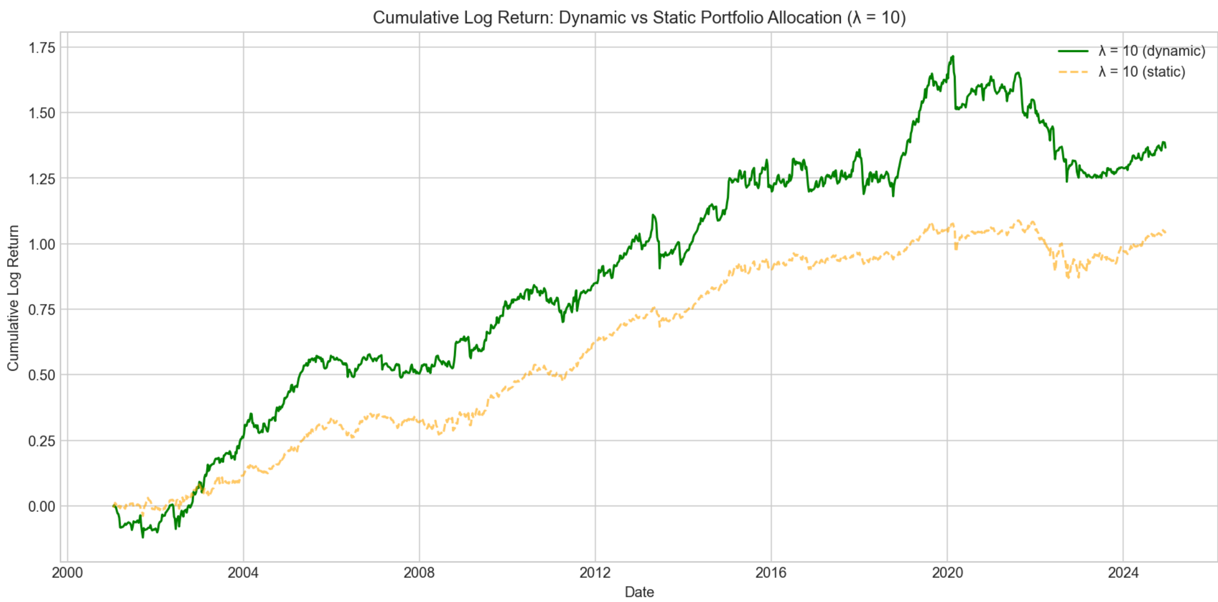

Figure 5: Dynamic vs Static Portfolio Cumulative Log Returns for $\lambda = 10$
Final Cumulative Log Returns (in%) are 136.63% (dynamic) and 103.99% (static)

We compare the cumulative returns of optimal portfolios constructed using static and dynamic allocation strategies, across two different risk aversion levels: $\lambda = 2$ and $\lambda = 10$. The cumulative returns are computed by combining realized returns with portfolio weights estimated from either the static or the dynamic optimization framework. We also consider cumulative log returns as a more robust performance metric. The following table summarizes the final cumulative returns and cumulative log returns achieved by each strategy (note: the dynamic portfolios do not include transaction costs):

Portfolio	**Annualized Cum. Log Return**	**Annualized Cum. Return**
Dynamic Allocation, $\lambda = 2$	12.82%	13.65%
Static Allocation, $\lambda = 2$	**15.96%**	**17.27%**
Dynamic Allocation, $\lambda = 10$	**5.69%**	**5.84%**
Static Allocation, $\lambda = 10$	4.33%	4.42%

Table 6: Annualized cumulative Log returns and cumulative simple returns (in %) by portfolio strategy and λ

From the graphs and the table, we observe a clear distinction between the two risk aversion regimes:

- In case of **low risk aversion** ($\lambda = 2$), the static allocation outperforms the dynamic strategy in terms of both cumulative log return and cumulative return. The static portfolio achieves a final cumulative log return of 15.96% compared to 12.82% for the dynamic approach. This superior performance may be attributed to the stability of fixed weights in the presence of estimation errors that dynamically changing weights exacerbate (especially under highly leveraged positions).
- In contrast, in case of **high risk aversion** ($\lambda = 10$), the dynamic allocation slightly outperforms the static one, with a higher cumulative log return (5.69% vs. 4.33%) and cumulative return (5.84% vs. 4.42%). This result is consistent with the idea that for highly risk-averse investors, the ability to adaptively adjust exposure to changing risk estimates may yield better results, particularly under volatile conditions where dynamic risk control is essential.

While dynamic allocation strategies are theoretically appealing due to their ability to adapt portfolio weights in response to time-varying forecasts of risk and return, their realized performance is sensitive to a number of practical and model-driven factors.

One major concern is **estimation uncertainty**: the weights in the dynamic strategy depend directly on forecasts from AR(1)-GARCH models, and even moderate forecast errors in the conditional mean or variance can result in suboptimal allocations or excessive portfolio turnover. This sensitivity may lead to large swings in weights that degrade performance. Additionally, the presence of **structural breaks or regime shifts** in financial markets can render model-based forecasts less reliable, favoring static allocations that avoid over-reacting to transient patterns.

Transaction costs are another critical consideration. Although they are not incorporated in the raw return calculations, they become especially relevant for dynamic strategies, which naturally induce higher turnover. In practice, even modest trading frictions can materially erode the performance gains from dynamic rebalancing.

The **timing of rebalancing** also plays a role: static portfolios benefit from smoother, more stable exposures, whereas dynamically adjusted weights may be more exposed to extreme realizations, particularly if model signals respond sharply to recent volatility spikes.

In conclusion, while dynamic allocation offers the advantage of conditional risk control and model responsiveness, its success depends on forecast accuracy and the cost-efficiency of implementation. The empirical results suggest that the **static allocation strategy with low risk aversion** ($\lambda = 2$) delivered the **highest overall cumulative performance**, achieving a cumulative log return of 15.96%. However, **under**

high risk aversion ($\lambda = 10$), the **dynamic allocation strategy** performed better than the static one, achieving a cumulative log return of 5.69%, benefiting from its ability to adjust to time-varying volatility and expected returns.

Q3.3: Now we introduce the transaction cost you must pay for the dynamic portfolio as a percentage f of a simplified portfolio turnover proxy given by the following formula: $TC_t = (|\tilde{\alpha}_{s,t} - \tilde{\alpha}_{s,t-1}| + |\tilde{\alpha}_{b,t} - \tilde{\alpha}_{b,t-1}|) \times f$ (There are no transaction costs for the risk-free asset.) At each period, the actual return of the dynamic is therefore the realized portfolio return minus the transaction cost TC_t. We assume that the transaction cost of the static portfolio is negligible (in fact, it should be rebalanced to maintain constant weights). For what value of f are the static (question 1.2) and the dynamic (question 3.2) portfolio allocations equally performing?

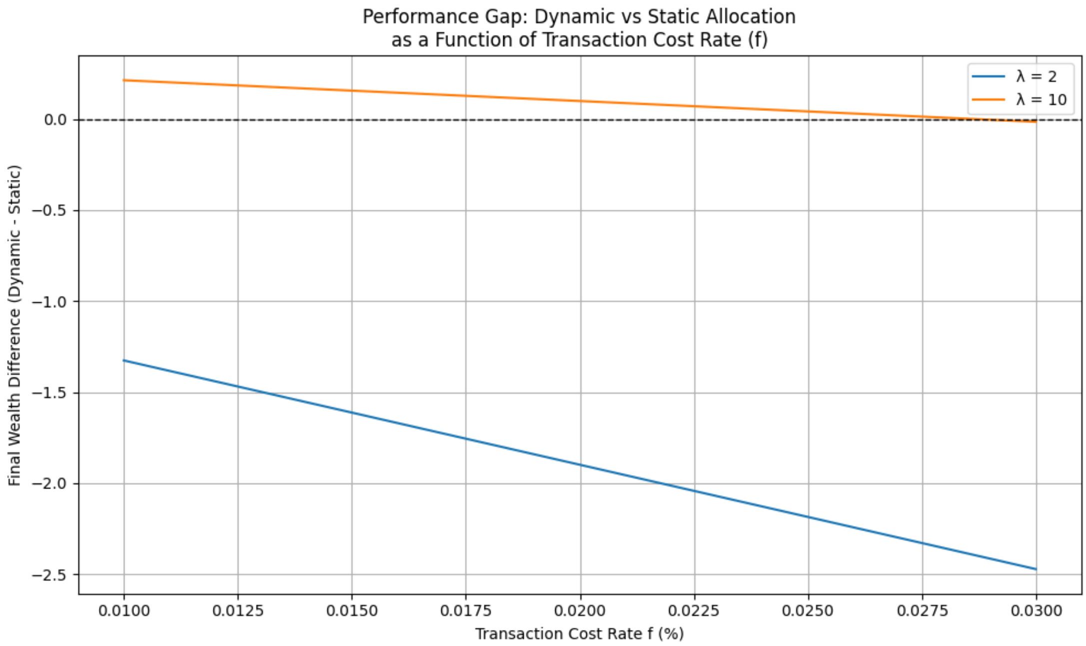

Figure 6: Performance Gap between Dynamic and Static Allocation as a Function of the Transaction Cost Rate (in %). The y-axis reports the final wealth difference (dynamic portfolio cumulative log return minus static portfolio cumulative log return) for different values of the transaction cost rate f, and for two levels of risk aversion ($\lambda = 2$ in blue and $\lambda = 10$ in orange).

The plot illustrates the final wealth difference between dynamic and static portfolio allocations as a function of the transaction cost rate f, for two levels of risk aversion, $\lambda = 2$ and $\lambda = 10$. The vertical axis represents the performance gap, defined as the difference in cumulative log returns between the dynamic and static portfolios after transaction costs have been deducted from the dynamic portfolio:

$$\Delta CR_t^{\text{net}}(f) = (CR_t^{\text{dynamic}} - TC_t) - CR_t^{\text{static}}.$$

Transaction costs for the dynamic portfolio are modeled as a function of turnover, with the following specification:

$$TC_t = \underbrace{(|\tilde{\alpha}_{s,t} - \tilde{\alpha}_{s,t-1}| + |\tilde{\alpha}_{b,t} - \tilde{\alpha}_{b,t-1}|)}_{turnover} \cdot f,$$

where f is the unit transaction cost rate applied to the absolute change in weights of risky assets between consecutive rebalancing dates.

For the **risk-tolerant investor** ($\lambda = 2$), the dynamic portfolio underperforms the static one across the entire range of transaction costs. This shows that even in the hypothetical case of zero frictions, the dynamic strategy does not dominate.
Conversely, for the **risk-averse investor** ($\lambda = 10$), the dynamic strategy initially outperforms the static allocation in the absence of transaction costs, but the performance advantage diminishes with increasing f. The intersection of the dynamic and static strategies occurs at:

$$f^* \approx \mathbf{0.0286\%} \quad \text{(unit transaction cost rate)},$$

which represents the *break-even transaction cost rate*, i.e., the level of f at which the cumulative returns under both strategies is equal.

To contextualize the economic significance of this threshold, we also report its annualized equivalent. Since transaction costs accumulate weekly as a function of portfolio turnover, we compute the annualized cost impact implied by the break-even transaction cost rate f^* using the compounding formula:

$$\text{Annualized Cost Impact} = (1 + f^* \cdot \overline{\tau}_{\text{weekly}})^{52} - 1,$$

where $f^* = 0.0286\%$ is the break-even transaction cost rate (in unit terms), and $\overline{\tau}_{\text{weekly}}$ is the average weekly turnover observed in the dynamic strategy with $\lambda = 10$. The total and average turnover values are reported below:

	Avg. weekly Turnover ($\overline{\tau}_{\text{weekly}}$)	Avg. annual Turnover ($\overline{\tau}_{\text{annual}}$)	Tot. Turnover
Turnover	0.914	47.6259	1143.02

Table 7: Average and Total Portfolio Turnover for $\lambda = 10$

Using these values, we obtain:

$$\text{Annualized Cost Impact} = (1 + 0.000286 \times 0.914)^{52} - 1 \approx 1.3694\%.$$

This result can be interpreted as the compounded yearly reduction in cumulative return due to transaction costs under the break-even scenario. For completeness, we also report the total and average transaction costs derived from observed turnover and f^*:

	Avg. weekly TC	Avg. annual TC	Tot. TC
Transaction Cost	0.00026	0.0136	0.32

Table 8: Total (over the sample), Average weekly and annual Transaction Costs (TC) at Break-even f^* for $\lambda = 10$

These values confirm that even modest frictions can compound to economically meaningful losses over time.

Conclusion:

- For $\lambda = 2$: the dynamic strategy never outperforms the static one for any feasible value of f, indicating the dominance of the static approach regardless of transaction costs (break-even f does not exist).
- For $\lambda = 10$: the break-even transaction cost rate threshold is $f^* \approx 0.0286\%$, equivalent to an annualized cost impact of $\approx 1.3694\%$.

This analysis highlights the sensitivity of dynamic strategies to the implementation of frictions, and suggests that their theoretical benefits require very low transaction costs and low turnover to be realized in practice.

4 Computing the VaR of a portfolio

Q4.1: Compute the unconditional mean $\bar{L}_p = \mathbb{E}[L_p]$ and variance $\sigma_p^2 = \mathbb{V}[L_p]$ of the portfolio loss. Assuming that the loss process is iid $\mathcal{N}(\bar{L}_p, \sigma_p^2)$, compute the expected 1-day loss and the variance forecast for the 1-day loss. Deduce the (unconditional) quantile for probability $\theta = 99\%$ and the (unconditional) VaR for the portfolio, which we denote by $VaR_p^{\textbf{(Uncond)}}$.

Portfolio	$\bar{L}_p$ **(Mean)**	$\hat{\sigma}_p^2$ **(Variance)**	$\textbf{VaR}_p^{\textbf{(Uncond)}}$
Dynamic ($\lambda = 2$)	-0.0013	0.0011	0.0771
Static ($\lambda = 2$)	-0.0008	0.0003	0.0434
Dynamic ($\lambda = 10$)	-0.0002	≈ 0	0.0154
Static ($\lambda = 10$)	-0.0001	≈ 0	0.0086

Table 9: Unconditional Mean, Variance, and VaR (99%) of Portfolio Losses

Under the assumption that the portfolio loss process is i.i.d. and normally distributed with constant unconditional mean $\bar{L}_p$ and variance σ_p^2, we compute the expected 1-day loss, the variance, and the unconditional Value-at-Risk, computed using the 99% quantile of the Gaussian distribution $z_{0.99} \approx 2.326$. The unconditional Value-at-Risk is given by the expression:

$$\text{VaR}_p^{(\text{Uncond})} = \bar{L}_p + z_{0.99} \cdot \sigma_p$$

For the **lower level of risk aversion ($\lambda = 2$)**, both static and dynamic portfolios exhibit elevated levels of (unconditional) risk. The **dynamic** strategy, in particular, displays the highest unconditional mean loss, variance, and consequently the largest estimated unconditional VaR. This outcome reflects its aggressive rebalancing behavior, which responds acutely to return and volatility forecasts, leading to greater exposure to adverse market outcomes. Even the **static** allocation for $\lambda = 2$, while more stable, shows comparatively large risk metrics due to the strong leverage implied by the optimal weights.

At the **higher level of risk aversion ($\lambda = 10$)**, we observe a substantial attenuation of risk. The **static** portfolio exhibits the lowest unconditional variance and mean loss, resulting in the smallest unconditional VaR across all configurations—remaining well below the 1% threshold. The **dynamic** strategy with $\lambda = 10$, while still responsive to time-varying market conditions, maintains more conservative weight profiles. Its associated unconditional risk metrics remain closer to the static case than to the dynamic $\lambda = 2$ portfolio, highlighting the dampening effect of stronger risk aversion.

These findings underscore that lower risk aversion amplifies the risk-return trade-off inherent in dynamic allocation, increasing the potential for extreme losses under an i.i.d. Gaussian assumption. Conversely, greater risk aversion enforces discipline in the allocation process, constraining portfolio variance and resulting in lower unconditional VaR.

Q4.2: Estimate an AR(1)-GARCH(1,1) model to describe the dynamics of portfolio losses $L_{p,t+1}$: $L_{p,t+1} = a + \rho L_{p,t} + \varepsilon_{p,t+1}$, $\varepsilon_{p,t+1} = \sigma_{p,t+1} z_{p,t+1}$, $\sigma^2_{p,t+1} = \omega + \alpha\varepsilon^2_{p,t} + \beta\sigma^2_{p,t}$. Obtain forecasts of the 1-day portfolio loss $\mu_{p,t+1} = a + \rho L_{p,t}$ and the 1-day variance forecast $\sigma^2_{p,t+1}$. Define the temporal evolution of the (conditional) quantile for probability $\theta = 99\%$ of the 1-day loss and compute the temporal evolution of the (conditional) VaR of the portfolio, which we denote by $VaR^{(\mathbf{GARCH})}_{p,t+1}$.

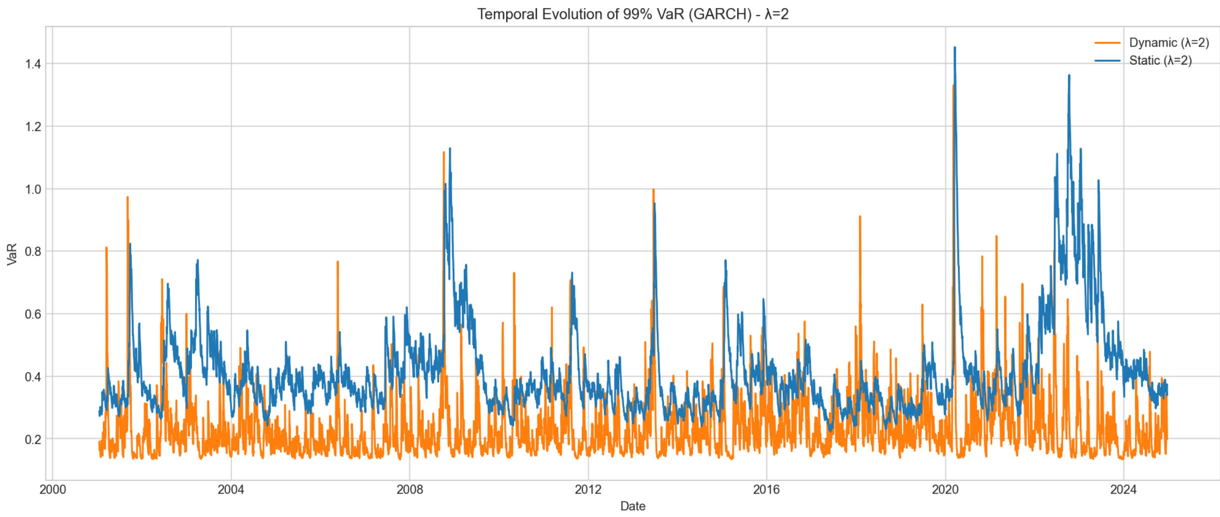

Figure 7: Temporal Evolution of 99% conditional $VaR^{(\mathrm{GARCH})}_{p,t+1}$ for $\lambda = 2$

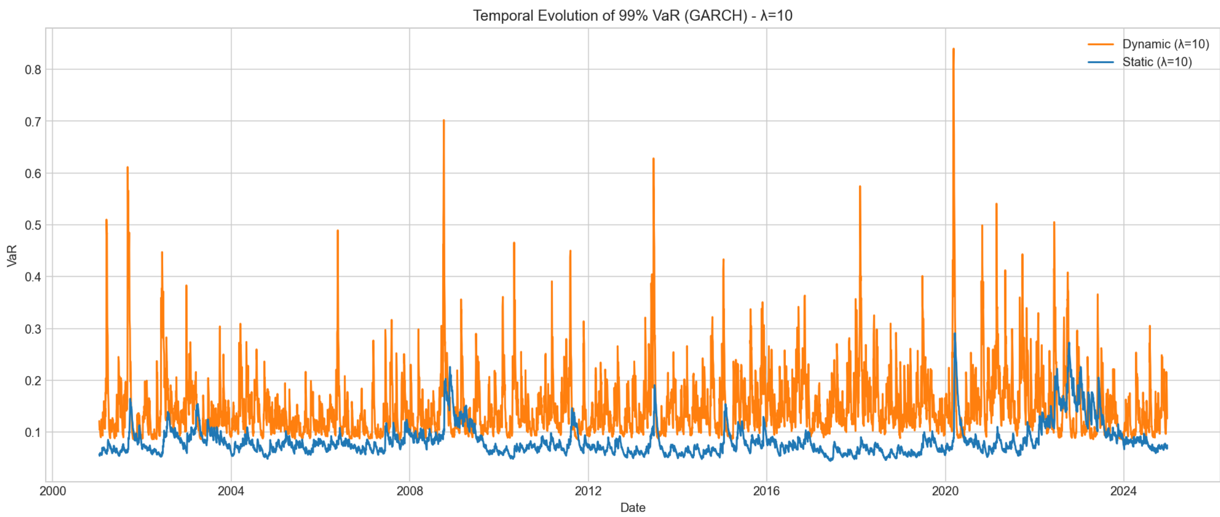

Figure 8: Temporal Evolution of 99% conditional $VaR^{(\mathrm{GARCH})}_{p,t+1}$ for $\lambda = 10$

Portfolio	**Mean**	**Std. Dev.**	**Min**	**Max**
Dynamic ($\lambda = 2$)	0.233	0.102	0.134	1.331
Static ($\lambda = 2$)	0.414	0.149	0.223	1.453
Dynamic ($\lambda = 10$)	0.148	0.064	0.085	0.84
Static ($\lambda = 10$)	0.083	0.029	0.044	0.290

Table 10: Descriptive statistics of 99% conditional $VaR^{(\mathrm{GARCH})}_{p,t+1}$

To capture the time-varying risk in portfolio losses, we first estimate an AR(1)-GARCH(1,1) model of the form:

$$L_{p,t+1} = a + \rho L_{p,t} + \varepsilon_{p,t+1}, \quad \varepsilon_{p,t+1} = \sigma_{p,t+1} z_{p,t+1}, \quad \sigma^2_{p,t+1} = \omega + \alpha \varepsilon^2_{p,t} + \beta \sigma^2_{p,t}$$

Using this specification, we generate 1-day ahead forecasts of the conditional mean $\mu_{p,t+1} = a + \rho L_{p,t}$ and conditional variance $\sigma^2_{p,t+1}$. Under the assumption of normal residuals, we define the conditional VaR of the portfolio loss as:

$$VaR^{(\text{GARCH})}_{p,t+1} = \mu_{p,t+1} + z_{0.99} \cdot \sigma_{p,t+1}$$

where $z_{0.99} \approx 2.326$ is the 99% quantile of a standard normal distribution. Unlike the unconditional VaR, this risk measure evolves over time with the forecasted mean and volatility, which react to new market information.

From the plots, we observe that for $\lambda = 2$, both **dynamic** and **static** portfolios exhibit substantial volatility in their risk estimates, with prominent spikes during periods of market turbulence such as the 2008 global financial crisis, the COVID-19 shock in 2020, and the early 2000s downturn. Although the dynamic portfolio occasionally reacts more acutely to volatility surges, the **static** portfolio under $\lambda = 2$ consistently exhibits a higher average VaR (0.414), a greater standard deviation, and a larger maximum (1.453). This may initially seem counterintuitive, as one might expect dynamic rebalancing to carry more risk. However, the elevated VaR in the static case arises from its unchanging, leveraged exposure to risky assets, which are held constantly through both calm and crisis periods. This results in persistently high conditional variance estimates from the GARCH model. In contrast, the dynamic portfolio with $\lambda = 2$ actively adjusts its composition in response to recent volatility, allowing it to de-risk during adverse conditions, which helps mitigate its loss exposure.

For $\lambda = 10$, we see a substantial drop in VaR magnitudes. The **static** strategy has the lowest overall risk exposure, with a mean VaR of only 0.083 and very limited standard deviation. The **dynamic** portfolio still reflects sensitivity to volatility shocks, reaching a max VaR of 0.840, but its mean remains moderate at 0.148, consistent with a more conservative investment strategy.

The numerical results support these interpretations. The dynamic portfolio with $\lambda = 2$ has a lower average VaR (0.233) than its static counterpart (0.414), benefiting from active volatility-aware rebalancing. Meanwhile, the static portfolio's unwavering leverage in a low risk aversion context amplifies its exposure to periods of sustained high volatility, inflating its average and peak risk levels. For $\lambda = 10$, both strategies display dampened risk, with static allocation delivering the lowest and most stable VaR path across the entire sample.

These findings highlight a key trade-off: while dynamic strategies provide adaptability, under low risk aversion they can still exhibit significant short-term risk swings. Conversely, static strategies may deliver deceptively high conditional VaR when allocations are heavily skewed toward riskier assets, lacking the flexibility to reduce exposure as market conditions evolve.

Q4.3: We model the distributions of the minima of the standardized residuals ($\hat{z}_{p,t+1} = (L_{p,t+1} - \mu_{p,t+1})/\sigma_{p,t+1}$), which are assumed to be iid. For each quarterly subsample τ (use 60 days per quarter), compute the maximum of $\hat{z}_t$ and store them in the time series m_τ, for $\tau = 1, \ldots, T/60$. Estimate the parameters $(\xi, \varpi, \psi)'$ of the generalized extreme value distribution (GEV), which is defined as follows: $H_{\xi,\varpi,\psi}(m_\tau) = H_\xi\left(\frac{m_\tau - \varpi}{\psi}\right) = \exp\left(-\left[1 + \xi\left(\frac{m_\tau - \varpi}{\psi}\right)\right]^{-1/\xi}\right)$ for $\tau = 1, \ldots, T/60$. Comment the values of the estimated parameters.

Strategy	**Shape (ξ)**	**Location (ω)**	**Scale (ψ)**
Dynamic ($\lambda = 2$)	0.0128	0.7743	0.2760
Dynamic ($\lambda = 10$)	0.0105	0.2452	0.0874
Static ($\lambda = 2$)	-0.0227	0.2375	0.0586
Static ($\lambda = 10$)	-0.0231	0.2375	0.0586

Table 11: Estimated GEV parameters for each strategy and λ

The estimation of the generalized extreme value (GEV) distribution parameters for the quarterly maxima of standardized residuals from the four portfolio strategies reveals distinct tail behavior and risk characteristics across static and dynamic allocations.

The **shape parameter** ξ, which determines the tail behavior of the generalized extreme value distribution, is slightly positive for both **dynamic** strategies ($\xi = 0.0128$ for $\lambda = 2$, $\xi = 0.0105$ for $\lambda = 10$). These values are close to zero, indicating a tail behavior that is approximately Gumbel-like but with a slight tendency toward heavier (Fréchet-type) tails. Importantly, such small positive values suggest that while extreme events may occur, their probability does not decay as slowly as in a clearly heavy-tailed distribution. Rather, these estimates reflect a moderate tail risk—consistent with the idea that dynamic allocations react to evolving volatility and market signals but are also subject to model constraints that limit unbounded exposure. The proximity of ξ to zero also raises the possibility that the actual tail behavior lies near the boundary between the Gumbel and Fréchet domains. In practice, this implies that small sample uncertainty or model misspecification could influence whether the estimated distribution is interpreted as light- or heavy-tailed. Still, the slightly positive sign under dynamic strategies may reflect their higher responsiveness and potential to engage in riskier positions during turbulent episodes. In contrast, the **static** strategies exhibit shape parameters close to zero ($\xi = -0.0227$ for $\lambda = 2$, and $\xi = -0.0231$ for $\lambda = 10$), approximating the Gumbel case. This suggests thinner tails and an exponential decay in the tail distribution, implying lower probabilities of extreme outcomes. The stability of the static allocation, which does not respond to market updates, naturally limits the emergence of tail risk in these portfolios.

The **location parameter** ω, representing the central tendency of the maxima, is substantially larger for the **dynamic** strategy with $\lambda = 2$ ($\omega = 0.7743$), highlighting the tendency of this strategy to experience larger block maxima in standardized losses. For the dynamic strategy with higher risk aversion ($\lambda = 10$), ω drops to 0.2452, indicating milder extremes. The **static** strategies, which both have $\omega \approx 0.2375$, demonstrate uniform and more contained tail behavior across different levels of risk aversion.

Finally, the **scale parameter** ψ, measuring the dispersion of the extreme values, aligns with this narrative. **Dynamic** allocations exhibit higher dispersion ($\psi = 0.2760$ for $\lambda = 2$, $\psi = 0.0874$ for $\lambda = 10$), indicating more variable and unpredictable tail behavior. In contrast, the **static** allocations display much tighter spreads ($\psi = 0.0586$), confirming the reduced volatility and more predictable loss profile inherent to these strategies.

Overall, these findings emphasize that dynamic portfolio strategies—especially under low risk aversion—are more exposed to extreme losses due to their higher responsiveness and leverage. The static allocations, while less adaptive, offer greater protection against tail risk, regardless of the investor's risk tolerance. The GEV parameter estimates thus underscore a fundamental trade-off: dynamic flexibility versus extreme risk containment.

Q4.4: Compute the 99% quantile for the distribution of the maximum m_τ. Deduce the 99% quantile of the distribution of the standardized residuals $\hat{z}_{p,t+1}$.

The 99% quantile of the GEV distribution fitted to the maxima m_τ is computed as:

$$q_{0.99}(m_\tau) = \varpi + \frac{\psi}{\xi}\left((-\log(0.99))^{-\xi} - 1\right),$$

where ϖ is the location parameter, ψ the scale, and ξ the shape parameter of the fitted GEV distribution.

The 99% quantile of the standardized residuals $\hat{z}_{p,t+1}$ is then deduced using the transformation that links the block maxima distribution to the underlying residual distribution, under the assumption that maxima are computed over N-sized blocks. Specifically, it is given by:

$$q_{0.99}(\hat{z}) = \varpi + \frac{\psi}{\xi}\left((-N \cdot \log(0.99))^{-\xi} - 1\right),$$

where $N = 60$ is the block size. This formula follows from extreme value theory and captures the rare-event quantile for the one-day standardized loss distribution implied by the tail behavior of the block maxima.

Strategy	**99% quantile of maxima m_τ**	**99% quantile of $\hat{z}$**
Dynamic ($\lambda = 2$)	2.0820	0.9144
Dynamic ($\lambda = 10$)	0.6574	0.2895
Static ($\lambda = 2$)	0.4933	0.2670
Static ($\lambda = 10$)	0.4935	0.2670

Table 12: 99% Quantiles of Maxima m_τ and Standardized Residuals $\hat{z}$

The table above reports the estimated 99% quantiles for the distribution of the block maxima m_τ and the deduced 99% quantiles of the standardized residuals $\hat{z}_{p,t+1}$ for each portfolio strategy. These values summarize the tail risk of the loss distributions based on the GEV framework under the block maxima approach.

Starting with the 99% quantile of maxima, the highest value is observed for the **dynamic** portfolio with low risk aversion ($\lambda = 2$), with $q_{0.99}(m_\tau) = 2.0820$. This reflects the increased likelihood of large maximum losses in aggressive, high-turnover strategies that are more sensitive to new information. The second-highest value corresponds to the dynamic portfolio with $\lambda = 10$, whose more conservative risk preferences reduce the exposure to extreme events, yielding $q_{0.99}(m_\tau) = 0.6574$. The **static** portfolios, which rely on fixed allocations and do not respond to changing market conditions, exhibit substantially lower maxima quantiles (approximately 0.493), confirming their reduced susceptibility to tail risk.

These findings are further reinforced by the deduced quantiles of the standardized residuals. For $\lambda = 2$, the **dynamic** strategy yields $q_{0.99}(\hat{z}) = 0.9144$, again the highest among all configurations. The corresponding quantile for the dynamic $\lambda = 10$ portfolio is considerably lower at 0.2895, reflecting both the risk-averse nature of the allocation and the lower scale parameter estimated in Q4.3. Notably, the **static** portfolios yield nearly identical and modest standardized quantiles (≈ 0.2670), underscoring the stability and predictability of their loss profiles.

These results are consistent with theoretical expectations: **dynamic** more aggressive portfolios (low λ) generate heavier tails and greater potential for extreme outcomes, as evidenced by higher quantiles for both m_τ and $\hat{z}$. In contrast, higher risk aversion or **static** allocation strategies mitigate such risks, showing thinner tails and lower tail quantiles, implying more stable and less extreme loss behavior.

Q4.5: Use the 1-day loss forecast $\mu_{p,t+1}$ and the 1-day variance forecast $\sigma^2_{p,t+1}$ to compute the temporal evolution of the 99% quantile of the loss distribution. Compute the temporal evolution of the 99% VaR for a 1-day horizon, which we denote by $VaR^{(\mathbf{GEV})}_{p,t+1}$.

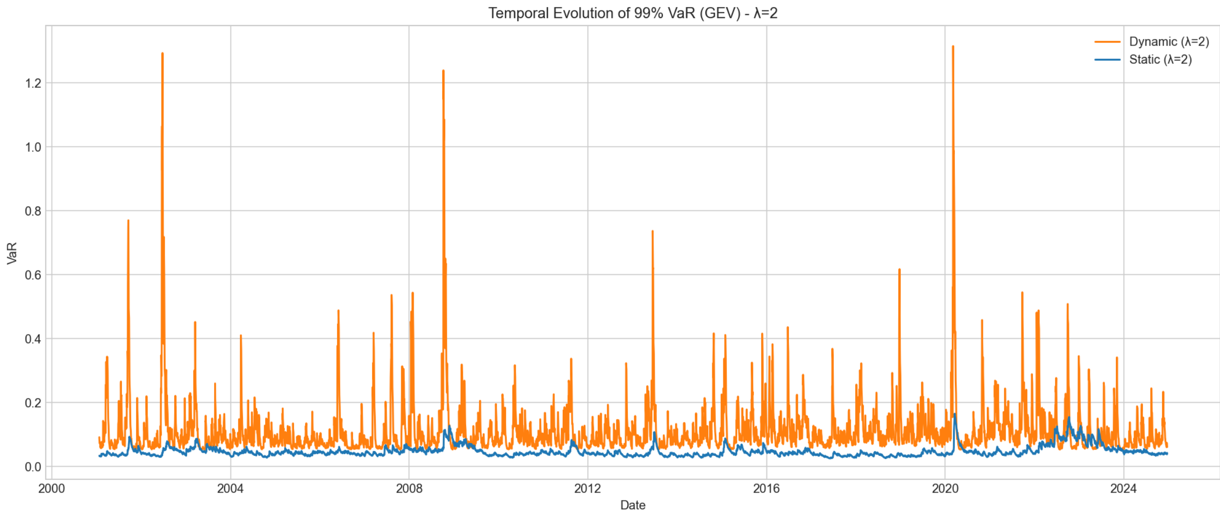

Figure 9: Temporal Evolution of 99% $VaR^{(\mathrm{GEV})}_{p,t+1}$ for $\lambda = 2$

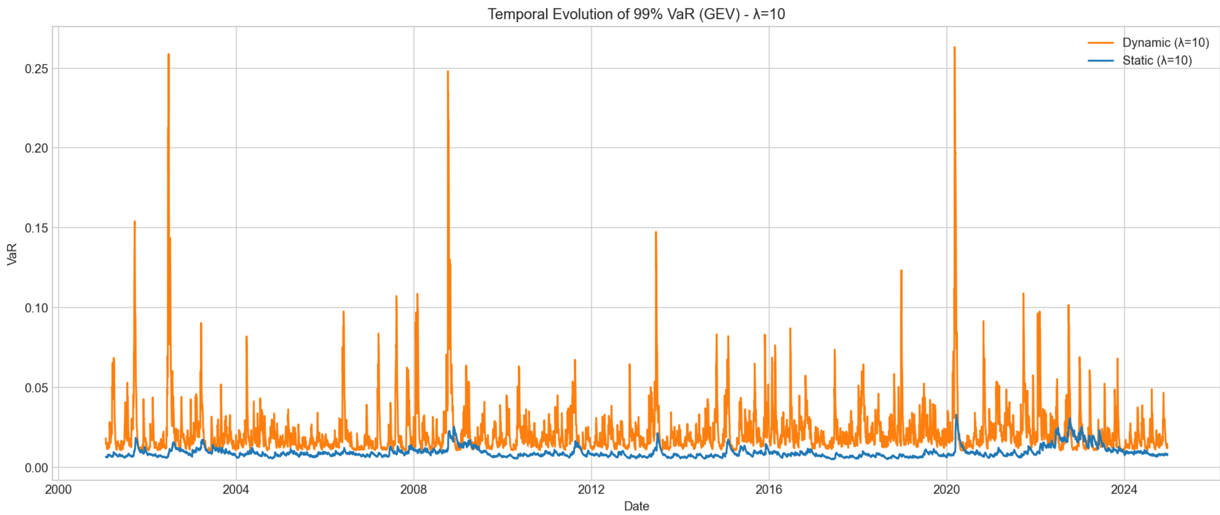

Figure 10: Temporal Evolution of 99% $VaR^{(\mathrm{GEV})}_{p,t+1}$ for $\lambda = 10$

We now examine the dynamic evolution of portfolio risk through the lens of Value-at-Risk estimated using generalized extreme value (GEV) theory. Specifically, we compute:

$$VaR_{p,t+1}^{(\mathrm{GEV})} = \mu_{p,t+1} + q_{0.99}(\hat{z}) \cdot \sigma_{p,t+1},$$

where $q_{0.99}(\hat{z})$ is the 99% quantile of the standardized residual distribution, obtained through the GEV-based transformation of block maxima m_τ. This approach should allows us to capture extreme losses and tail risk more accurately than conventional Gaussian-based methods.

Strategy	**Mean**	**Std. Dev.**	**Min**	**Max**
Static ($\lambda = 2$)	0.0465	0.0172	0.0245	0.1664
Static ($\lambda = 10$)	0.0092	0.0034	0.0048	0.0333
Dynamic ($\lambda = 2$)	0.0905	0.0400	0.0514	0.0527
Dynamic ($\lambda = 10$)	0.0181	0.0080	0.0102	0.1057

Table 13: Summary Statistics of Time-Varying 99% $VaR_{p,t+1}^{(\mathrm{GEV})}$

From the summary statistics and time-series plots, several insights emerge. For both levels of risk aversion ($\lambda = 2$ and $\lambda = 10$), **dynamic** strategies exhibit much higher volatility in $VaR_{p,t+1}^{(\mathrm{GEV})}$ compared to their static counterparts. This is particularly pronounced for the risk-tolerant dynamic strategy ($\lambda = 2$), which shows extreme spikes in the 99% $VaR_{p,t+1}^{(\mathrm{GEV})}$ during crisis periods such as 2001–2002, the 2008 global financial crisis, and the COVID-19 market shock in 2020. Its maximum value exceeds 0.09, almost twice the magnitude of the corresponding static VaR (0.046), highlighting the exposure of dynamic strategies to tail risk.

The **static** strategies, in contrast, exhibit smoother and lower VaR levels across time. The static portfolio with $\lambda = 10$ is the most conservative, with the lowest mean and maximum $VaR_{p,t+1}^{(\mathrm{GEV})}$ values. This is consistent with its limited weight variability and avoidance of leveraged positions. The static VaRs are more robust and predictable, though potentially underestimating risk during turbulent periods due to their inability to respond to volatility shifts.

The **dynamic** portfolio with $\lambda = 10$ represents a middle ground: it adapts to evolving market conditions and shows moderate variation in VaR. Although less extreme than the dynamic $\lambda = 2$ case, it still reacts visibly to periods of stress. This confirms that time-varying allocations can be beneficial for capturing shifts in tail risk, but their effectiveness is contingent on the investor's risk aversion.

Overall, these results emphasize the trade-off between reactivity and risk exposure. GEV-based VaR, by incorporating extreme loss behavior, offers a powerful diagnostic for portfolio tail risk. Dynamic strategies provide flexibility but also expose investors to heightened extremes, especially under low risk aversion. In contrast, static strategies offer predictability and stability but may miss rapid shifts in market conditions. The magnitude and dispersion of $VaR_{p,t+1}^{(\mathrm{GEV})}$ across strategies provide a concrete illustration of this fundamental balance.

Q4.6: Compare the VaR obtained with the three approaches ($VaR_p^{(\mathbf{Uncond})}$, $VaR_{p,t+1}^{(\mathbf{GARCH})}$ and $VaR_{p,t+1}^{(\mathbf{GEV})}$). Why is the last approach more relevant for computing the VaR of portfolio losses?

To evaluate portfolio risk under different modeling assumptions, we compare three Value-at-Risk (VaR) methodologies: (1) unconditional Gaussian VaR, (2) conditional GARCH-based VaR, and (3) tail-focused GEV-based VaR. Each of these methods rests on different assumptions about the distributional and temporal properties of financial returns.

1. Unconditional VaR ($VaR_p^{(\mathbf{Uncond})}$):
This approach assumes that portfolio losses $L_{p,t}$ are i.i.d. and normally distributed with constant mean $\bar{L}_p$ and variance σ_p^2. The VaR is computed as:

$$VaR_p^{(\text{Uncond})} = \bar{L}_p + z_{0.99} \cdot \hat{\sigma}_p$$

The primary appeal lies in its simplicity and closed-form computation. However, this model completely neglects time-varying volatility, autocorrelation, and tail thickness—all empirically evident features of financial return series. Consequently, the estimated VaR is often too smooth and understates the true magnitude of tail risk, particularly during crisis periods. This weakness is especially visible for the dynamic $\lambda = 2$ strategy, where the unconditional VaR fails to capture elevated volatility and downside exposure.

2. Conditional GARCH-Based VaR ($VaR_{p,t+1}^{(\mathbf{GARCH})}$):
This method models both the conditional mean and volatility of returns using an AR(1)-GARCH(1,1) framework:

$$L_{p,t+1} = a + \rho L_{p,t} + \varepsilon_{p,t+1}, \quad \varepsilon_{p,t+1} = \sigma_{p,t+1} z_{p,t+1}$$
$$VaR_{p,t+1}^{(\text{GARCH})} = \mu_{p,t+1} + z_{0.99} \cdot \sigma_{p,t+1}$$

This specification introduces time variation and persistence in volatility, capturing clustering and shocks. It is particularly effective in identifying periods of elevated risk, as seen by the conditional VaR spikes for static $\lambda = 2$ and dynamic $\lambda = 10$ portfolios. However, the method still assumes conditionally Gaussian innovations $z_{p,t+1}$, and therefore the quantile $z_{0.99}$ remains constant. This leads to underestimation of tail risk in fat-tailed data, and becomes problematic if the i.i.d. assumption for residuals or standardized residuals is violated.

3. Extreme Value Theory-Based VaR ($VaR_{p,t+1}^{(\mathbf{GEV})}$):
To overcome the weaknesses of Gaussian-based methods, we apply extreme value theory. We model the maxima of standardized residuals over 60-day blocks using the Generalized Extreme Value (GEV) distribution. The resulting VaR is computed as:

$$VaR_{p,t+1}^{(\text{GEV})} = \mu_{p,t+1} + \hat{q}_{0.99}^{(\hat{z})} \cdot \sigma_{p,t+1}$$

Here, $\hat{q}_{0.99}^{(\hat{z})}$ is the 99% quantile derived from the GEV distribution fit to the empirical maxima series m_τ. This approach explicitly accounts for heavy-tailed behavior, allowing for better modeling of rare but severe losses. In our empirical findings, the GEV-based VaR yields intermediate risk estimates between the GARCH and unconditional methods, but critically improves the representation of tail risk.

Each approach carries trade-offs:

- *Unconditional VaR* ignores all temporal and tail dynamics. It is not suitable for strategies with volatility dynamics or exposure to extremes.
- *GARCH-based VaR* captures second-moment time dependence but still underrepresents the extreme left tail. It assumes conditionally normal innovations, which empirical evidence (e.g., high kurtosis in residuals) often contradicts.

- *GEV-based VaR* models the tail explicitly and provides a more refined risk estimate. However, it relies on strong assumptions: (i) that standardized residuals are i.i.d., and (ii) that quarterly maxima are extreme enough to follow the asymptotic GEV law. Estimation may suffer if block sizes are too small or if dependence across blocks is strong.

In particular, the assumption of i.i.d. standardized residuals is critical. If residual autocorrelation or volatility clustering remains (due to mis-specification of AR-GARCH models), the GEV fit may be inconsistent. While block maxima over 60 days help reduce time dependence, the choice of block length must balance between sample size and independence. Moreover, cross-asset correlation and structural breaks may also bias the estimation of extremes, especially in dynamic portfolios.

Despite its sensitivity to modeling assumptions, the GEV-based VaR is the most theoretically appropriate for tail-focused risk management. It accommodates fat tails, incorporates conditional volatility, and models the risk of large losses directly from the tail behavior. While GARCH offers good volatility modeling, and unconditional VaR provides a benchmark, only the GEV approach captures the risk of rare but impactful losses, making it most relevant for stress testing and capital adequacy assessment in dynamic portfolios.