Empirical Methods in Finance

Project #2: Dynamic Allocation and VaR of a Portfolio

Due Friday May 26, 2025, at the beginning of the class

The objective of this assignment is to highlight the so-called "volatility timing", i.e., the additional return an investor can expect when she is able to forecast the dynamics of expected returns and volatility correctly. We also compute the Value-at-Risk of this portfolio when its return is described as a GARCH model or using extreme value theory.

The file "Codes_and_Groups.xlsx" indicates which dataset you should use. If the "Currency to use" is USD, you should collect stock and bond indexes in USD (with the mnemonic x(RI) U$) and the U.S. short-term rate. If the "Currency to use" is not the USD, you should collect stock and bond indexes in local currency (with the mnemonic x(RI)) and the local short-term rate.

Your first task is to collect the dataset that includes the three assets that are assigned to your group, from January 2001 to December 2024 at weekly frequency. The one-week interest rate is annualized and expressed in %. In the following, I assume that the second asset is a bond index to simplify the exposition.

Compute simple returns for stocks and bonds ($R_{s,t}$ and $R_{b,t}$) and compute the weekly risk-free asset ($R_{f,t}$) by dividing the annualized rate by 52. Be careful, do not multiply returns by 100.

Please write a report to answer questions below, i.e., do not comment your code and do not use screenshots from your code. For each question, answer as **concisely and precisely** as possible. Your results must be clearly presented and commented.

Please follow the question numbering.

Personal advice: Do not wait until the week before the deadline to start working on your project. As you can see, there is quite a lot of work to be done. If you wait until the deadline, you will not have time to hand in a proper assignment.

1 Static allocation

An investor would like to invest her wealth in a portfolio composed of stocks, bonds, and cash. The allocation is based on the mean-variance criterion, defined as

$$\max_{\{\tilde{\alpha}\}} \quad \mu_p - \frac{\lambda}{2}\sigma_p^2$$

where $\mu_p = \tilde{\alpha}'\mu + (1 - e'\tilde{\alpha})R_f$ and $\sigma_p^2 = \tilde{\alpha}'\Sigma\tilde{\alpha}$ are the expected portfolio return and variance of the portfolio return. We denote $\mu = (\mu_s, \mu_b)'$ the (2,1) vector of expected returns for stocks and bonds, e the (2,1) vector of ones, Σ the (2,2) covariance matrix of returns, and λ is the degree of risk aversion. Weights $\tilde{\alpha} = (\tilde{\alpha}_s, \tilde{\alpha}_b)'$ are unconstrained, and the weight on the risk-free asset is given by $1 - e'\tilde{\alpha}$. R_f is the average risk-free rate over the sample.

Q1.1 Give the expression for the optimal portfolio weights. (Hint: compute the first-order derivative of the mean-variance criterion with respect to $\tilde{\alpha}$.)

Q1.2 Assume that expected returns are given by sample means and that the covariance matrix Σ is given by the sample covariance matrix. Compute the optimal weight vector, denoted by $\tilde{\alpha}^*$, for $\lambda = 2$ and 10.

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2 Estimation of a GARCH model

We consider the following model, for $i = s, b$ (stocks and bonds):

$$\begin{aligned} R_{i,t+1} &= a_i + \rho_i R_{i,t} + \varepsilon_{i,t+1} \\ \varepsilon_{i,t+1} &= \sigma_{i,t+1} z_{i,t+1} \\ \sigma^2_{i,t+1} &= \omega_i + \alpha_i \varepsilon^2_{i,t} + \beta_i \sigma^2_{i,t} \end{aligned}$$

where the usual assumptions apply.

Q2.1 For both stocks and bonds, provide evidence on the non-normality (Kolmogorov-Smirnov test) and the auto-correlation (Ljung-Box test, with 4 lags) of the excess returns and squared excess returns.

Q2.2 Estimate an AR(1) model on stock and bond returns, to filter out autocorrelation. We now denote by $\hat{\varepsilon}_{i,t+1}$ the residuals of the AR(1) model. Comment the regression estimation (goodness-of-fit, parameter estimates).

Q2.3 Estimate the GARCH(1,1) model for residuals using the conditional ML technique. Comment the parameter estimates (in particular, the sum $\alpha_i + \beta_i$). Test the null hypothesis that $\alpha_i + \beta_i = 1$ against the alternative that $\alpha_i + \beta_i < 1$, at the 5% significance level.

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3 Dynamic allocation

For each date t and for both assets, $i = s, b$, we compute the expected returns $\mu_{i,t+1} = a_i + \rho_i R_{i,t}$ using the estimated AR(1) processes and the expected variances $\sigma^2_{i,t+1}$ using the estimated GARCH(1,1) processes for stocks and bonds. We assume that the cross-correlation between residuals $\rho_{sb} = Corr[\hat{\varepsilon}_{s,t+1}, \hat{\varepsilon}_{b,t+1}]$ is constant over time. Then, the expected covariance between residuals is given by $\sigma_{sb,t+1} = \rho_{sb} \times \sigma_{s,t+1} \times \sigma_{b,t+1}$. The covariance matrix at time $t+1$ is therefore

$$\Sigma_{t+1} = \begin{pmatrix} \sigma^2_{s,t+1} & \sigma_{sb,t+1} \\ \sigma_{sb,t+1} & \sigma^2_{b,t+1} \end{pmatrix}.$$

Then, using the same procedure as in point 2, we compute the optimal weight vector $\tilde{\alpha}_t$ for stocks and bonds at the end of each date t, that maximizes the mean-variance

$$\max_{\{\tilde{\alpha}_t\}} \quad \mu_{p,t+1} - \frac{\lambda}{2} \sigma^2_{p,t+1}$$

where $\mu_{p,t+1} = \tilde{\alpha}_t'\mu_{t+1} + (1 - e'\tilde{\alpha}_t)R_{f,t}$ and $\sigma^2_{p,t+1} = \tilde{\alpha}_t'\Sigma_{t+1}\tilde{\alpha}_t$ are the expected portfolio return and variance of the portfolio return, with $\mu_{t+1} = (\mu_{s,t+1}, \mu_{b,t+1})'$.

Q3.1 Plot the time series of optimal weights for stocks and bonds for the two approaches: with constant expected returns and volatility on the one hand ($\tilde{\alpha}^*$ of point 1) and for time-varying volatility on the other hand ($\tilde{\alpha}_t$). Comment your results for $\lambda = 2$ and 10.

Q3.2 Compute the cumulative returns of the optimal portfolio for the two approaches (you use the optimal portfolio weights and the realized returns). For instance, for the dynamic approach, you have

$$CR_t = \prod_{j=2}^{t}(1 + R_{p,j})$$

where $R_{p,t+1} = \tilde{\alpha}_{s,t}R_{s,t+1} + \tilde{\alpha}_{b,t}R_{b,t+1} + (1 - \tilde{\alpha}_{s,t} - \tilde{\alpha}_{b,t})R_{f,t}$ is the ex-post portfolio return. (If you get something explosive, take the log of the portfolio returns by $r_{p,t+1} = \log(1 + R_{p,t+1})$ and calculate the cumulative returns as the cumulative sum of $r_{p,t+1}$.) Plot the two time-series of cumulative returns. Which allocation strategy performs the best? What factor(s) could change your opinion?

Q3.3 Now we introduce the transaction cost you must pay for the dynamic portfolio as a percentage f of a simplified portfolio turnover proxy given by the following formula:

$$TC_t = (|\tilde{\alpha}_{s,t} - \tilde{\alpha}_{s,t-1}| + |\tilde{\alpha}_{b,t} - \tilde{\alpha}_{b,t-1}|) \times f$$

(There are no transaction costs for the risk-free asset.) At each period, the actual return of the dynamic is therefore the realized portfolio return minus the transaction cost TC_t. We assume that the transaction cost of the static portfolio is negligible (in fact, it should be rebalanced to maintain constant weights). For what value of f are the static (question 1.2) and the dynamic (question 3.2) portfolio allocations equally performing?

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4 Computing the VaR of a portfolio

We now use the daily data to compute the daily return of the portfolio. For the static allocation, just use the optimal weight vector $\tilde{\alpha}^*$ to compute the portfolio returns for all days. For the dynamic allocation, use the weekly optimal weights $\tilde{\alpha}_t$ for the 5 coming days until you rebalance your portfolio. We define $\tilde{R}_{p,t+1}$ as the negative return (or loss), i.e., $L_{p,t+1} = -R_{p,t+1}$.

Q4.1 Compute the unconditional mean $\bar{L}_p = E[L_p]$ and variance $\sigma^2_p = V[L_p]$ of the portfolio loss. Assuming that the loss process is iid $N(\bar{L}_p, \sigma^2_p)$, compute the expected 1-day loss and the variance forecast for the 1-day loss. Deduce the (unconditional) quantile for probability $\theta = 99\%$ and the (unconditional) VaR for the portfolio, which we denote by $VaR_p^{(Uncond)}$.

Q4.2 Estimate a AR(1)-GARCH(1,1) model to describe the dynamics of portfolio losses $L_{p,t+1}$

$$\begin{aligned} L_{p,t+1} &= a + \rho L_{p,t} + \varepsilon_{p,t+1} \\ \varepsilon_{p,t+1} &= \sigma_{p,t+1} z_{p,t+1} \\ \sigma^2_{p,t+1} &= \omega + \alpha \varepsilon^2_{p,t} + \beta \sigma^2_{p,t} \end{aligned}$$

and obtain forecasts of the 1-day portfolio loss $\mu_{p,t+1} = a + \rho L_{p,t}$ and the 1-day variance forecast $\sigma^2_{p,t+1}$. Define the temporal evolution of the (conditional) quantile for probability $\theta = 99\%$ of the 1-day loss and compute the temporal evolution of the (conditional) VaR of the portfolio, which we denote by $VaR^{(GARCH)}_{p,t+1}$.

Q4.3 We model the distributions of the minima of the standardized residuals ($\hat{z}_{p,t+1} = (L_{p,t+1} - \mu_{p,t+1})/\sigma_{p,t+1}$), which are assumed to be iid. For each quarterly subsample τ (use 60 days per quarter), compute the maximum of $\hat{z}_t$ and store them in the time series m_τ, for $\tau = 1, ..., T/60$. Estimate the parameters (ξ, ϖ, ψ)' of the generalized extreme value distribution (GEV), which is defined as follows:

$$H_{\xi,\varpi,\psi}(m_\tau) = H_\xi\left(\frac{m_\tau - \varpi}{\psi}\right) = \exp\left(-\left(1 + \xi\frac{m_\tau - \varpi}{\psi}\right)^{-1/\xi}\right)$$

for $\tau = 1, ..., T/60$. Comment the values of the estimated parameters.

Q4.4 Compute the 99% quantile for the distribution of the maximum m_τ. Deduce the 99% quantile of the distribution of the standardized residuals $\hat{z}_{p,t+1}$.

Q4.5 Use the 1-day loss forecast $\mu_{p,t+1}$ and the 1-day variance forecast $\sigma^2_{p,t+1}$ to compute the temporal evolution of the 99% quantile of the loss distribution. Compute the temporal evolution of the 99% VaR for a 1-day horizon, which we denote by $VaR^{(GEV)}_{p,t+1}$.

Q4.6 Compare the VaR obtained with the three approaches ($VaR^{(Uncond)}_p$, $VaR^{(GARCH)}_{p,t+1}$ and $VaR^{(GEV)}_{p,t+1}$). Why is the last approach more relevant for computing the VaR of portfolio losses?