Cartan formalism

Based on Narcos Alpha Playlist PSI 18/19 - Gravitational Physics Review

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(%i2) `info:build_info()$info@version;`

(%o2)

5.38.1

(%i2) `reset()$kill(all)$`

(%i1) `derivabbrev:true$`

(%i2) `ratprint:false$`

(%i3) `fpprintprec:5$`

(%i4) `if get('vect,'version)=false then load(vect)$`

(%i5) `if get('cartan,'version)=false then load(cartan)$`

(%i6) `if get('format,'version)=false then load(format)$`

(%i7) `norm(u):=block(ratsimp(radcan(√(u.u))))$`

(%i8) `normalize(v):=block(v/norm(v))$`

(%i9) `angle(u,v):=block([junk:radcan(√((u.u)*(v.v)))],acos(u.v/junk))$`

(%i10) `mycross(va,vb):=[va[2]*vb[3]-va[3]*vb[2],va[3]*vb[1]-va[1]*vb[3],va[1]*vb[2]-va[2]*vb[1]]$`

(%i11) `declare(trigsimp,evfun)$`

1 Polar coordinates

(%i12) `kill(labels,Tr,ξ,r,θ)$`

(%i2) `assume(0≤r)$`
`assume(0≤θ,θ≤2*π)$`

(%i3) `ξ:[r,θ]$`

(%i4) `dim:length(ξ)$`

Transformation formula

(%i5) `Tr:[r*cos(θ),r*sin(θ)]$`

Initialize vect package

(%i6) `scalefactors(append([Tr],ξ))$`

(%i7) `sf:reverse(rest(reverse(sf)));`

$$[1, r] \tag{sf}$$

(%i8) `sfprod;`

$$r \tag{\%o8}$$

(%i9) `dimension;`

$$2 \tag{\%o9}$$

Jacobian

(%i10) `J:jacobian(Tr,ξ);`

$$\begin{pmatrix} \cos(\theta) & -r\,\sin(\theta) \\ \sin(\theta) & r\,\cos(\theta) \end{pmatrix} \tag{J}$$

Covariant metric tensor

(%i11) `lg:trigsimp(transpose(J).J);`

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{lg}$$

(%i12) `Jdet:trigsimp(determinant(J));`

$$r \tag{Jdet}$$

Contravariant metric tensor

(%i13) `ug:invert(lg);`

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \tag{ug}$$

Line element

(%i14) `ldisplay(ds`^{2}`=diff(`ξ`).lg.transpose(diff(`ξ`)))$`

$$ds^2 = r^2\,\mathrm{del}\,(\theta)^2 + \mathrm{del}(r)^2 \qquad (\%t14)$$

Define the frame

(%i16) `e[r]:`$\sqrt{}$`(ug)[1]$`
`e[`θ`]:`$\sqrt{}$`(ug)[2]$`

(%i17) `ldisplay(e:apply('matrix,[e[r],e[`θ`]]))$`

$$e = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \qquad (\%t17)$$

Initialize cartan package

(%i18) `init_cartan(`ξ`)$`

(%i19) `cartan_basis;`

$$[dr, d\theta] \qquad (\%o19)$$

(%i20) `cartan_coords;`

$$[r, \theta] \qquad (\%o20)$$

(%i21) `cartan_dim;`

$$2 \qquad (\%o21)$$

(%i22) `extdim;`

$$2 \qquad (\%o22)$$

Define the coframe ω

(%i25) ω`[r]:dr$`
ω`[`θ`]:r*d`θ`$`
`ldisplay(`ω`:[`ω`[r],`ω`[`θ`]])$`

$$\omega = [dr, r\,d\theta] \qquad (\%t25)$$

Verify $\langle \underline{\omega}^a \mid \underline{e}_b \rangle = \delta^a{}_b$

(%i26) `genmatrix(lambda([i,j],e[`ξ`[i]]|`ω`[`ξ`[j]]),cartan_dim,cartan_dim);`

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\%o26)$$

Calculate the external derivative of the coframe

(%i27) `ldisplay(dω:ext_diff(ω))$`

$$d\omega = [0, dr\, d\theta] \tag{\%t27}$$

Generic Connection 1-form Θ

(%i28) `A:[a_1,a_2]$`

(%i32)
```
kill(Θ)$
Θ:zeromatrix(dim,dim)$
Θ[1,2]:-Θ[2,1]:A.cartan_basis$
ldisplay(Θ)$
```

$$\Theta = \begin{pmatrix} 0 & -a_2\, d\theta - a_1\, dr \\ a_2\, d\theta + a_1\, dr & 0 \end{pmatrix} \tag{\%t32}$$

Change matrix multiplication operator

(%i33) `matrix_element_mult:"~"$`

(%i34) `ldisplay(λ:list_matrix_entries(expand(Θ.ω)))$`

$$\lambda = [-a_1 r\, dr\, d\theta, -a_2 dr\, d\theta] \tag{\%t34}$$

Restore matrix multiplication operator

(%i35) `matrix_element_mult:"*"$`

Cartan's First structural equation $\mathrm{d}\omega^i = \Theta_j{}^i \wedge \omega^j$

(%i36) `Eq:zeromatrix(dim,dim)$`

(%i37)
```
for i thru dim do for j thru dim do
Eq[i,j]:format(coeff(coeff(dω,cartan_basis[i]),cartan_basis[j])=
coeff(coeff(-λ,cartan_basis[i]),cartan_basis[j]),%list)$
```

(%i38) `Eqs:apply('append,list_matrix_entries(Eq))$`

(%i39) `linsol:linsolve(Eqs,A);`

solve: dependent equations eliminated: (1 8 7 2 4 3)

$$[a_1 = 0, a_2 = 1] \tag{linsol}$$

(%i40) `ldisplay(λ:at(λ,linsol))$`

$$\lambda = [0, -dr\, d\theta] \tag{\%t40}$$

(%i41) `is(dω=-λ);`

$$\text{true} \tag{\%o41}$$

Update Connection 1-form Θ

(%i42) `ldisplay(Θ:at(Θ,linsol))$`

$$\Theta = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix} \quad (\%t42)$$

Update Connection 2-form dΘ

(%i43) `ldisplay(dΘ:trigsimp(matrixmap(edit,ext_diff(Θ))))$`

$$d\Theta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (\%t43)$$

Update coefficients

(%i44) `ldisplay(A:at(A,linsol))$`

$$A = [0, 1] \quad (\%t44)$$

Change matrix multiplication operator

(%i45) `matrix_element_mult:"~"$`

Cartan's Second structural equation: $\Omega_j{}^i = \mathrm{d}\Theta_j{}^i + \Theta_k{}^i \wedge \Theta_j{}^k$

Curvature 2-form Ω

(%i46) `ldisplay(Ω:matrixmap(edit,dΘ+Θ.Θ))$`

$$\Omega = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (\%t46)$$

Restore matrix multiplication operator

(%i47) `matrix_element_mult:"*"$`

Forms in terms of the coframe σ

(%i48) `Eqs:makelist(σ[ξ[i]]=ω[ξ[i]],i,1,cartan_dim);`

$$[\sigma_r = dr, \sigma_\theta = r\, d\theta] \quad (\mathrm{Eqs})$$

(%i49) `linsol:linsolve(Eqs,cartan_basis);`

$$\left[dr = \sigma_r, d\theta = \frac{\sigma_\theta}{r}\right] \quad (\mathrm{linsol})$$

Connection 1-form Θ

(%i50) `ldisplay(Θ:ev(Θ,linsol,fullratsimp))$`

$$\Theta = \begin{pmatrix} 0 & -\frac{\sigma_\theta}{r} \\ \frac{\sigma_\theta}{r} & 0 \end{pmatrix} \quad (\%t50)$$

Curvature 2-form Ω

(%i51)
```
ldisplay(Ω:ev(Ω,linsol,fullratsimp))$
```

$$\Omega = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{\%t51}$$

Clean up

(%i53)
```
forget(0≤r)$
forget(0≤θ,θ≤2*π)$
```

2 2-Sphere

(%i54) `kill(labels,Tr,ξ,θ,φ)$`

(%i3)
```
assume(0≤θ,θ≤π)$
assume(0≤sin(θ))$
assume(0≤φ,φ≤2*π)$
```

(%i4) `ξ:[θ,φ]$`

(%i5) `dim:length(ξ)$`

Transformation formula

(%i6) `Tr:[sin(θ)*cos(φ),sin(θ)*sin(φ),cos(θ)]$`

Initialize vect package

(%i7) `scalefactors(append([Tr],ξ))$`

(%i8) `sf;`

$$[1, \sin(\theta)] \tag{\%o8}$$

(%i9) `sfprod;`

$$\sin(\theta) \tag{\%o9}$$

(%i10) `dimension;`

$$2 \tag{\%o10}$$

Jacobian

(%i11) `J:jacobian(Tr,ξ);`

$$\begin{pmatrix} \cos(\theta)\ \cos(\phi) & -\sin(\theta)\ \sin(\phi) \\ \cos(\theta)\ \sin(\phi) & \sin(\theta)\ \cos(\phi) \\ -\sin(\theta) & 0 \end{pmatrix} \tag{J}$$

Covariant metric tensor

(%i12) `lg:trigsimp(transpose(J).J);`

$$\begin{pmatrix} 1 & 0 \\ 0 & \sin(\theta)^2 \end{pmatrix} \tag{lg}$$

Contravariant metric tensor

(%i13) `ug:invert(lg);`

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin(\theta)^2} \end{pmatrix} \tag{ug}$$

Line element

(%i14) `ldisplay(ds`^{2}`=diff(`ξ`).lg.transpose(diff(`ξ`)))$`

$$ds^2 = \sin(\theta)^2 \operatorname{del}(\phi)^2 + \operatorname{del}(\theta)^2 \tag{\%t14}$$

Define the frame

(%i16) `e[`θ`]:`$\sqrt{}$`(ug)[1]$`
`e[`ϕ`]:`$\sqrt{}$`(ug)[2]$`

(%i17) `ldisplay(e:apply('matrix,[e[`θ`],e[`ϕ`]]))$`

$$e = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin(\theta)} \end{pmatrix} \tag{\%t17}$$

Initialize cartan package

(%i18) `init_cartan(`ξ`)$`

(%i19) `cartan_basis;`

$$[d\theta, d\phi] \tag{\%o19}$$

(%i20) `cartan_coords;`

$$[\theta, \phi] \tag{\%o20}$$

(%i21) `cartan_dim;`

$$2 \tag{\%o21}$$

(%i22) `extdim;`

$$2 \tag{\%o22}$$

Define the coframe ω

(%i25) ω`[`θ`]:d`θ`$`
ω`[`ϕ`]:sin(`θ`)*d`ϕ`$`
`ldisplay(`ω`:[`ω`[`θ`],`ω`[`ϕ`]])$`

$$\omega = [d\theta, d\phi \sin(\theta)] \tag{\%t25}$$

Verify $\langle \underline{\omega}^a \mid \underline{e}_b \rangle = \delta^a{}_b$

(%i26) `genmatrix(lambda([i,j],e[`ξ`[i]]|`ω`[`ξ`[j]]),cartan_dim,cartan_dim);`

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{\%o26}$$

Calculate the external derivative of the coframe

(%i27) `ldisplay(dω:ext_diff(ω))$`

$$d\omega = [0, d\theta\, d\phi \cos(\theta)] \quad (\%t27)$$

Generic Connection 1-form Θ

(%i28) `A:[a_1,a_2]$`

(%i32)
```
kill(Θ)$
Θ:zeromatrix(dim,dim)$
Θ[1,2]:-Θ[2,1]:A.cartan_basis$
ldisplay(Θ)$
```

$$\Theta = \begin{pmatrix} 0 & -a_2\, d\phi - a_1\, d\theta \\ a_2\, d\phi + a_1\, d\theta & 0 \end{pmatrix} \quad (\%t32)$$

Change matrix multiplication operator

(%i33) `matrix_element_mult:"~"$`

(%i34) `ldisplay(λ:list_matrix_entries(expand(Θ.ω)))$`

$$\lambda = [-a_1\, d\theta\, d\phi \sin(\theta), -a_2\, d\theta\, d\phi] \quad (\%t34)$$

Restore matrix multiplication operator

(%i35) `matrix_element_mult:"*"$`

Cartan's First structural equation

(%i36) `Eq:zeromatrix(dim,dim)$`

(%i37)
```
for i thru dim do for j thru dim do
Eq[i,j]:format(coeff(coeff(dω,cartan_basis[i]),cartan_basis[j])=
coeff(coeff(-λ,cartan_basis[i]),cartan_basis[j]),%list)$
```

(%i38) `Eqs:apply('append,list_matrix_entries(Eq))$`

(%i39) `linsol:linsolve(Eqs,A);`

solve: dependent equations eliminated: (1 8 7 2 3 4)

$$[a_1 = 0, a_2 = \cos(\theta)] \quad \text{(linsol)}$$

(%i40) `ldisplay(λ:at(λ,linsol))$`

$$\lambda = [0, -d\theta\, d\phi \cos(\theta)] \quad (\%t40)$$

(%i41) `is(dω=-λ);`

true (%o41)

Update Connection 1-form Θ

(%i42) `ldisplay(Θ:at(Θ,linsol))$`

$$\Theta = \begin{pmatrix} 0 & -d\phi\,\cos(\theta) \\ d\phi\,\cos(\theta) & 0 \end{pmatrix} \tag{\%t42}$$

Update Connection 2-form dΘ

(%i43) `ldisplay(dΘ:trigsimp(matrixmap(edit,ext_diff(Θ))))$`

$$d\Theta = \begin{pmatrix} 0 & d\theta\,d\phi\,\sin(\theta) \\ -d\theta\,d\phi\,\sin(\theta) & 0 \end{pmatrix} \tag{\%t43}$$

Update coefficients

(%i44) `ldisplay(A:at(A,linsol))$`

$$A = [0, \cos(\theta)] \tag{\%t44}$$

Change matrix multiplication operator

(%i45) `matrix_element_mult:"~"$`

Cartan's Second structural equation: $\Omega_j{}^i = \mathrm{d}\Theta_j{}^i + \Theta_k{}^i \wedge \Theta_j{}^k$

Curvature 2-form Ω

(%i46) `ldisplay(Ω:matrixmap(edit,dΘ+Θ.Θ))$`

$$\Omega = \begin{pmatrix} 0 & d\theta\,d\phi\,\sin(\theta) \\ -d\theta\,d\phi\,\sin(\theta) & 0 \end{pmatrix} \tag{\%t46}$$

Restore matrix multiplication operator

(%i47) `matrix_element_mult:"*"$`

Forms in terms of the coframe σ

(%i48) `Eqs:makelist(σ[ξ[i]]=ω[ξ[i]],i,1,cartan_dim);`

$$[\sigma_\theta = d\theta, \sigma_\phi = d\phi\,\sin(\theta)] \tag{Eqs}$$

(%i49) `linsol:linsolve(Eqs,cartan_basis);`

$$\left[d\theta = \sigma_\theta, d\phi = \frac{\sigma_\phi}{\sin(\theta)}\right] \tag{linsol}$$

Connection 1-form Θ

(%i50) `ldisplay(Θ:ev(Θ,linsol,fullratsimp))$`

$$\Theta = \begin{pmatrix} 0 & -\frac{\cos(\theta)\,\sigma_\phi}{\sin(\theta)} \\ \frac{\cos(\theta)\,\sigma_\phi}{\sin(\theta)} & 0 \end{pmatrix} \tag{\%t50}$$

Curvature 2-form Ω

(%i51)
```
ldisplay(Ω:ev(Ω,linsol,fullratsimp))$
```

$$\Omega = \begin{pmatrix} 0 & \sigma_\theta\, \sigma_\phi \\ -\sigma_\theta\, \sigma_\phi & 0 \end{pmatrix} \tag{\%t51}$$

Clean up

(%i54)
```
forget(0≤θ,θ≤π)$
forget(0≤sin(θ))$
forget(0≤ϕ,ϕ≤2*π)$
```

3 Spherical coordinates

(%i12) `kill(labels,Tr,ξ,r,θ,φ)$`

(%i4)
```
assume(0≤r)$
assume(0≤θ,θ≤π)$
assume(0≤sin(θ))$
assume(0≤φ,φ≤2*π)$
```

(%i5) `ξ:[r,θ,φ]$`

(%i6) `dim:length(ξ)$`

Transformation formula

(%i7) `Tr:[r*sin(θ)*cos(φ),r*sin(θ)*sin(φ),r*cos(θ)]$`

Initialize vect package

(%i8) `scalefactors(append([Tr],ξ))$`

(%i9) `sf;`

$$[1, r, r\,\sin(\theta)] \tag{\%o9}$$

(%i10) `sfprod;`

$$r^2\,\sin(\theta) \tag{\%o10}$$

(%i11) `dimension;`

$$3 \tag{\%o11}$$

Jacobian

(%i12) `ldisplay(J:jacobian(Tr,ξ))$`

$$J = \begin{pmatrix} \sin(\theta)\,\cos(\phi) & r\,\cos(\theta)\,\cos(\phi) & -r\,\sin(\theta)\,\sin(\phi) \\ \sin(\theta)\,\sin(\phi) & r\,\cos(\theta)\,\sin(\phi) & r\,\sin(\theta)\,\cos(\phi) \\ \cos(\theta) & -r\,\sin(\theta) & 0 \end{pmatrix} \tag{\%t12}$$

Covariant metric tensor

(%i13) `lg:trigsimp(transpose(J).J);`

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin(\theta)^2 \end{pmatrix} \tag{lg}$$

(%i14) `Jdet:trigsimp(determinant(J));`

$$r^2\,\sin(\theta) \tag{Jdet}$$

Line element

(%i15) `ldisplay(ds^2=diff(ξ).lg.transpose(diff(ξ)))$`

$$ds^2 = r^2 \sin(\theta)^2 \operatorname{del}(\phi)^2 + r^2 \operatorname{del}(\theta)^2 + \operatorname{del}(r)^2 \tag{\%t15}$$

Define the frame

(%i18)
```
e[r]:[1,0,0]$
e[θ]:[0,1/r,0]$
e[φ]:[0,0,1/r/sin(θ)]$
```

(%i19) `ldisplay(e:apply('matrix,[e[r],e[θ],e[φ]]))$`

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r\,\sin(\theta)} \end{pmatrix} \tag{\%t19}$$

Initialize cartan package

(%i20) `init_cartan(ξ)$`

(%i21) `cartan_basis;`

$$[dr, d\theta, d\phi] \tag{\%o21}$$

(%i22) `cartan_coords;`

$$[r, \theta, \phi] \tag{\%o22}$$

(%i23) `cartan_dim;`

$$3 \tag{\%o23}$$

(%i24) `extdim;`

$$3 \tag{\%o24}$$

Define the coframe ω

(%i28)
```
ω[r]:dr$
ω[θ]:r*dθ$
ω[φ]:r*sin(θ)*dφ$
ldisplay(ω:[ω[r],ω[θ],ω[φ]])$
```

$$\omega = [dr, r\;d\theta, r\;d\phi\;\sin(\theta)] \tag{\%t28}$$

Verify

(%i29) `genmatrix(lambda([i,j],e[ξ[i]]|ω[ξ[j]]),cartan_dim,cartan_dim);`

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{\%o29}$$

Calculate the external derivative of the coframe

(%i30) `ldisplay(dω:ext_diff(ω))$`

$$d\omega = [0, dr\ d\theta, dr\ d\phi\ \sin(\theta) + r\ d\theta\ d\phi\ \cos(\theta)] \tag{\%t30}$$

Generic Connection 1-form Θ

(%i33)
```
A:[a_1,a_2,a_3]$
B:[b_1,b_2,b_3]$
C:[c_1,c_2,c_3]$
```

(%i39)
```
kill(Θ)$
Θ:zeromatrix(dim,dim)$
Θ[1,2]:-Θ[2,1]:A.cartan_basis$
Θ[1,3]:-Θ[3,1]:B.cartan_basis$
Θ[2,3]:-Θ[3,2]:C.cartan_basis$
ldisplay(Θ)$
```

$$\Theta = \begin{pmatrix} 0 & -a_3\,d\phi - a_2\,d\theta - a_1\,dr & -b_3\,d\phi - b_2\,d\theta - b_1\,dr \\ a_3\,d\phi + a_2\,d\theta + a_1\,dr & 0 & -c_3\,d\phi - c_2\,d\theta - c_1\,dr \\ b_3\,d\phi + b_2\,d\theta + b_1\,dr & c_3\,d\phi + c_2\,d\theta + c_1\,dr & 0 \end{pmatrix} \tag{\%t39}$$

Change matrix multiplication operator

(%i40) `matrix_element_mult:"~"$`

(%i41) `λ:list_matrix_entries(expand(Θ.ω))$`

(%i42) `map(ldisp,λ)$`

$$-b_2 r\ d\theta\ d\phi\ \sin(\theta) - b_1 r\ dr\ d\phi\ \sin(\theta) + a_3 r\ d\theta\ d\phi - a_1 r\ dr\ d\theta \tag{\%t42}$$

$$-c_2 r\ d\theta\ d\phi\ \sin(\theta) - c_1 r\ dr\ d\phi\ \sin(\theta) - a_3\,dr\ d\phi - a_2\,dr\ d\theta \tag{\%t43}$$

$$-c_3 r\ d\theta\ d\phi - b_3\,dr\ d\phi + c_1 r\ dr\ d\theta - b_2\,dr\ d\theta \tag{\%t44}$$

Restore matrix multiplication operator

(%i45) `matrix_element_mult:"*"$`

Cartan's First structural equation

(%i46) `Eq:zeromatrix(dim,dim)$`

(%i47)
```
for i thru dim do for j thru dim do
Eq[i,j]:format(coeff(coeff(dω,cartan_basis[i]),cartan_basis[j])=
coeff(coeff(-λ,cartan_basis[i]),cartan_basis[j]),%list)$
```

(%i48) `Eqs:apply('append,list_matrix_entries(Eq))$`

(%i49) `linsol:linsolve(Eqs,append(A,B,C));`

solve: dependent equations eliminated: (1 27 26 25 2 3 13 14 15 11 10 17 19 12 16 20 21 18)

$$[a_1 = 0, a_2 = 1, a_3 = 0, b_1 = 0, b_2 = 0, b_3 = \sin(\theta), c_1 = 0, c_2 = 0, c_3 = \cos(\theta)] \quad \text{(linsol)}$$

(%i50) `ldisplay(λ:at(λ,linsol))$`

$$\lambda = [0, -dr\, d\theta, -dr\, d\phi \sin(\theta) - r\, d\theta\, d\phi \cos(\theta)] \quad (\%t50)$$

(%i51) `is(dω=-λ);`

$$\text{true} \quad (\%o51)$$

Update Connection 1-form Θ

(%i52) `ldisplay(Θ:at(Θ,linsol))$`

$$\mathit{\Theta} = \begin{pmatrix} 0 & -d\theta & -d\phi \sin(\theta) \\ d\theta & 0 & -d\phi \cos(\theta) \\ d\phi \sin(\theta) & d\phi \cos(\theta) & 0 \end{pmatrix} \quad (\%t52)$$

Update Connection 2-form dΘ

(%i53) `ldisplay(dΘ:trigsimp(matrixmap(edit,ext_diff(Θ))))$`

$$d\mathit{\Theta} = \begin{pmatrix} 0 & 0 & -d\theta\, d\phi \cos(\theta) \\ 0 & 0 & d\theta\, d\phi \sin(\theta) \\ d\theta\, d\phi \cos(\theta) & -d\theta\, d\phi \sin(\theta) & 0 \end{pmatrix} \quad (\%t53)$$

Update coefficients

(%i56)
```
ldisplay(A:at(A,linsol))$
ldisplay(B:at(B,linsol))$
ldisplay(C:at(C,linsol))$
```

$$A = [0, 1, 0] \quad (\%t54)$$

$$B = [0, 0, \sin(\theta)] \quad (\%t55)$$

$$C = [0, 0, \cos(\theta)] \quad (\%t56)$$

Change matrix multiplication operator

(%i57) `matrix_element_mult:"~"$`

Cartan's Second structural equation: $\Omega_j{}^i = \mathrm{d}\Theta_j{}^i + \Theta_k{}^i \wedge \Theta_j{}^k$

Curvature 2-form Ω

```
(%i58) ldisplay(Ω:matrixmap(edit,dΘ+Θ.Θ))$
```

$$\Omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{\%t58}$$

Restore matrix multiplication operator

```
(%i59) matrix_element_mult:"*"$
```

Forms in terms of the coframe σ

```
(%i60) Eqs:makelist(σ[ξ[i]]=ω[ξ[i]],i,1,cartan_dim);
```

$$[\sigma_r = dr, \sigma_\theta = r\, d\theta, \sigma_\phi = r\, d\phi\, \sin(\theta)] \tag{Eqs}$$

```
(%i61) linsol:linsolve(Eqs,cartan_basis);
```

$$\left[dr = \sigma_r, d\theta = \frac{\sigma_\theta}{r}, d\phi = \frac{\sigma_\phi}{r\, \sin(\theta)}\right] \tag{linsol}$$

Connection 1-form Θ

```
(%i62) ldisplay(Θ:ev(Θ,linsol,fullratsimp))$
```

$$\Theta = \begin{pmatrix} 0 & -\frac{\sigma_\theta}{r} & -\frac{\sigma_\phi}{r} \\ \frac{\sigma_\theta}{r} & 0 & -\frac{\cos(\theta)\,\sigma_\phi}{r\,\sin(\theta)} \\ \frac{\sigma_\phi}{r} & \frac{\cos(\theta)\,\sigma_\phi}{r\,\sin(\theta)} & 0 \end{pmatrix} \tag{\%t62}$$

Curvature 2-form Ω

```
(%i63) ldisplay(Ω:ev(Ω,linsol,fullratsimp))$
```

$$\Omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{\%t63}$$

Clean up

```
(%i67) forget(0≤r)$
       forget(0≤θ,θ≤π)$
       forget(0≤sin(θ))$
       forget(0≤φ,φ≤2*π)$
```