Horse saddle

Based on Physics Unsimplified Video 2. Two Dimensional Spaces and Line Elements

Written by Daniel Volinski at danielvolinski@yahoo.es

```
(%i2) info:build_info()$info@version;
```

$$ \tag{\%o2} $$

5.38.1

```
(%i2) reset()$kill(all)$
(%i1) derivabbrev:true$
(%i2) ratprint:false$
(%i3) fpprintprec:5$
(%i4) load(linearalgebra)$
(%i5) if get('draw,'version)=false then load(draw)$
(%i6) wxplot_size:[1024,768]$
(%i7) if get('optvar,'version)=false then load(optvar)$
(%i8) if get('rkf45,'version)=false then load(rkf45)$
(%i9) declare(trigsimp,evfun)$
(%i10) declare(s,mainvar)$
```

Settings

```
(%i11) z:y^2-x^2$
(%i12) ζ:[x,y,z]$
(%i13) dζ:diff(ζ)$
```

Line element

```
(%i14) ldisplay(ds^2=ratsimp(dζ.dζ))$
```

$$ds^2 = -8xy\ \mathrm{del}(x)\ \mathrm{del}(y) + \left(4y^2+1\right)\ \mathrm{del}(y)^2 + \left(4x^2+1\right)\ \mathrm{del}(x)^2 \tag{\%t14}$$

Reduce coordinates

```
(%i15) ζ:reverse(rest(reverse(ζ)))$
(%i16) depends(ζ,s)$
```

Lagrangian

```
(%i17) L:subst([del(x)='diff(x,s),del(y)='diff(y,s)],ratsimp(dζ.dζ));
```

$$\left(4y^2+1\right)\ \left(y_s\right)^2 - 8x\ \left(x_s\right) y\ \left(y_s\right) + \left(4x^2+1\right)\ \left(x_s\right)^2 \tag{L}$$

Momentum Conjugate

```
(%i18) ldisplay(P_x:ev(diff(L,’diff(x,s))))$
```

$$P_x = 2\left(4x^2+1\right)\,(x_s) - 8xy\,(y_s) \tag{\%t18}$$

```
(%i19) linsolve(p_x=P_x,diff(x,s)),factor;
```

$$\left[x_s = \frac{8xy\,(y_s)+p_x}{2\,(4x^2+1)}\right] \tag{\%o19}$$

```
(%i20) ldisplay(P_y:ev(diff(L,’diff(y,s))))$
```

$$P_y = 2\left(4y^2+1\right)\,(y_s) - 8x\,(x_s)\,y \tag{\%t20}$$

```
(%i21) linsolve(p_y=P_y,diff(y,s)),factor;
```

$$\left[y_s = \frac{8x\,(x_s)\,y+p_y}{2\,(4y^2+1)}\right] \tag{\%o21}$$

Generalized Forces

```
(%i22) ldisplay(F_x:diff(L,x))$
```

$$F_x = 8x\,(x_s)^2 - 8\,(x_s)\,y\,(y_s) \tag{\%t22}$$

```
(%i23) ldisplay(F_y:diff(L,y))$
```

$$F_y = 8y\,(y_s)^2 - 8x\,(x_s)\,(y_s) \tag{\%t23}$$

Euler-Lagrange Equations

```
(%i24) aa:el(L,ζ,s)$
(%i27) bb:ev(aa,eval,diff)$
(%i28) bb[1]:subst([k[0]=-E],-bb[1])$
(%i31) bb[1]:rhs(bb[1])=lhs(bb[1])$
       bb[2]:lhs(bb[2])-rhs(bb[2])=0$
       bb[3]:lhs(bb[3])-rhs(bb[3])=0$
```

Conservation Laws

```
(%i32) radcan(bb[1]);
```

$$E = \left(4y^2+1\right)\,(y_s)^2 - 8x\,(x_s)\,y\,(y_s) + \left(4x^2+1\right)\,(x_s)^2 \tag{\%o32}$$

Equations of Motion

```
(%i33) map(ldisp,part(bb,[2,3]))$
```

$$-8xy\ \left(y_{ss}\right)-8x\left(y_s\right)^2+2\left(4x^2+1\right)\ \left(x_{ss}\right)+8x\left(x_s\right)^2=0 \tag{\%t33}$$

$$2\left(4y^2+1\right)\ \left(y_{ss}\right)+8y\left(y_s\right)^2-8x\ \left(x_{ss}\right)y-8\left(x_s\right)^2y=0 \tag{\%t34}$$

Solve for second derivative of coordinates

```
(%i35) geodesic:linsolve(part(bb,[2,3]),diff(ζ,s,2))$
(%i36) map(ldisp,geodesic:fullratsimp(geodesic))$
```

$$x_{ss}=\frac{4x\left(y_s\right)^2-4x\left(x_s\right)^2}{4y^2+4x^2+1} \tag{\%t36}$$

$$y_{ss}=-\frac{4y\left(y_s\right)^2-4\left(x_s\right)^2y}{4y^2+4x^2+1} \tag{\%t37}$$

Check Conservation of Energy

```
(%i38) radcan(rhs(bb[1]));
```

$$\left(4y^2+1\right)\ \left(y_s\right)^2-8x\ \left(x_s\right)y\ \left(y_s\right)+\left(4x^2+1\right)\ \left(x_s\right)^2 \tag{\%o38}$$

```
(%i39) subst(geodesic,diff(rhs(bb[1]),s)),fullratsimp;
```

$$0 \tag{\%o39}$$

Analytical solution

```
(%i40) params:[x_0=1,Vx_0=1,y_0=1,Vy_0=1]$
(%i41) declare([x_0,Vx_0,y_0,Vy_0],constant)$
(%i45) atvalue(x(s),[s=0],x_0)$
       atvalue(diff(x(s),s),[s=0],Vx_0)$
       atvalue(y(s),[s=0],y_0)$
       atvalue(diff(y(s),s),[s=0],Vy_0)$
(%i46) desolve(convert(geodesic,ζ,s),convert(ζ,ζ,s));
```

$$[\mathrm{x}(s)=\mathrm{ilt}((4\,\mathrm{laplace}\left(\frac{\mathrm{x}(s)\ \left(\mathrm{y}(s)_s\right)^2}{4\mathrm{y}(s)^2+4\mathrm{x}(s)^2+1},s,g644\right)-4\,\mathrm{laplace}\left(\frac{\mathrm{x}(s)\ \left(\mathrm{x}(s)_s\right)^2}{4\mathrm{y}(s)^2+4\mathrm{x}(s)^2+1},s,g644\right)+x_0g644+Vx_0)/g644^2,g644, \tag{\%o46}$$

Reduce Order

```
(%i48) Z:[X,Y]$
       depends(Z,s)$
(%i50) gradef(x,s,X)$
       gradef(y,s,Y)$
```

Solve for second derivative of coordinates

```
(%i51) bb:ev(aa,eval,diff)$
```

```
(%i52) bb[1]:subst([k[0]=-E],-bb[1])$
```

```
(%i55) bb[1]:rhs(bb[1])=lhs(bb[1])$
       bb[2]:lhs(bb[2])-rhs(bb[2])=0$
       bb[3]:lhs(bb[3])-rhs(bb[3])=0$
```

```
(%i56) geodesic:linsolve(part(bb,[2,3]),diff(ζ,s,2))$
```

```
(%i57) map(ldisp,geodesic:fullratsimp(geodesic))$
```

$$X_s = \frac{\left(4Y^2 - 4X^2\right) x}{4y^2 + 4x^2 + 1} \tag{\%t57}$$

$$Y_s = -\frac{\left(4Y^2 - 4X^2\right) y}{4y^2 + 4x^2 + 1} \tag{\%t58}$$

Numerical solution

(%i66)
```
funcs:append(ζ,Z)$ldisplay(funcs)$
initial:[500,-400,-3,2]$ldisplay(initial)$
odes:append(Z,map(rhs,geodesic))$ldisplay(odes)$
interval:[s,0,200]$ldisplay(interval)$
```

$$\mathit{funcs}=[x,y,X,Y] \tag{\%t60}$$

$$\mathit{initial}=[500,-400,-3,2] \tag{\%t62}$$

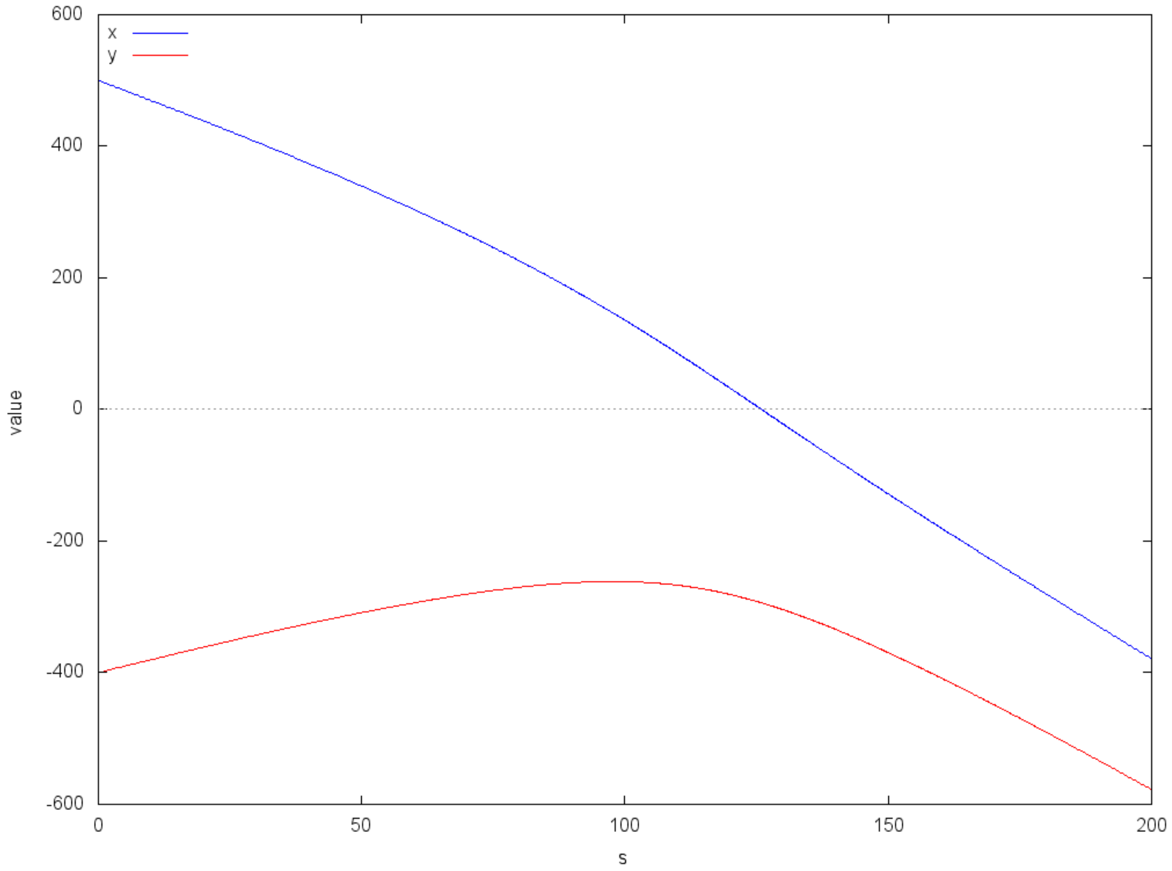
$$\mathit{odes}=\left[X,Y,\frac{\left(4Y^2-4X^2\right)x}{4y^2+4x^2+1},-\frac{\left(4Y^2-4X^2\right)y}{4y^2+4x^2+1}\right] \tag{\%t64}$$

$$\mathit{interval}=[s,0,200] \tag{\%t66}$$

(%i67)
```
rksol:rkf45(odes,funcs,initial,interval, absolute_tolerance=1E-12,report=true),params$
```

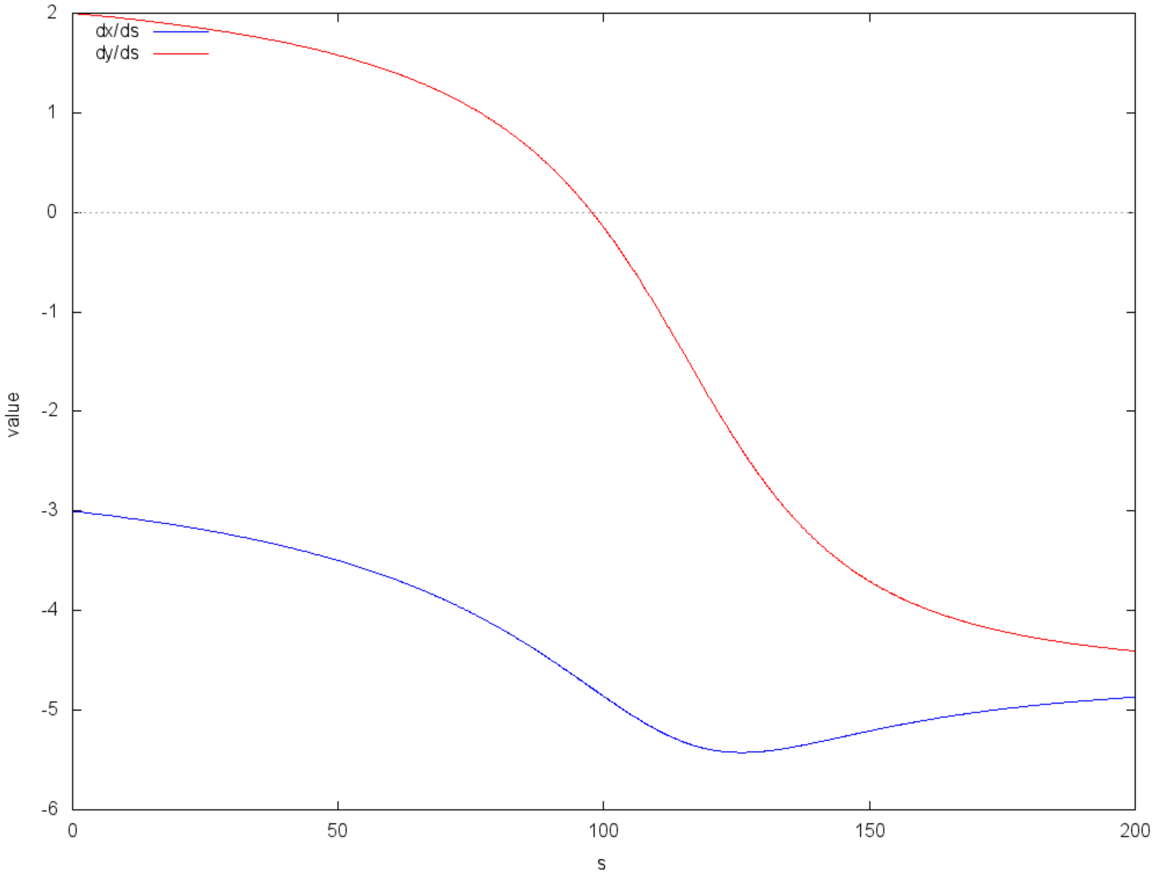
Info: rkf45:
Integration points selected:1078
Total number of iterations:1078
Bad steps corrected:1
Minimum estimated error:4.4610^{-14}
Maximum estimated error:5.334210^{-13}
Minimum integration step taken:0.10883
Maximum integration step taken:0.39993

(%i68)
```
wxplot2d([[discrete,map(lambda([u],part(u,[1,2])),rksol)], [discrete,map(lambda([u],part(u,[1,3]
[style,[lines,1]],[xlabel,"s"],[ylabel,"value"], [legend,"x","y"], [gnuplot_preamble,"set
key top left"])$
```



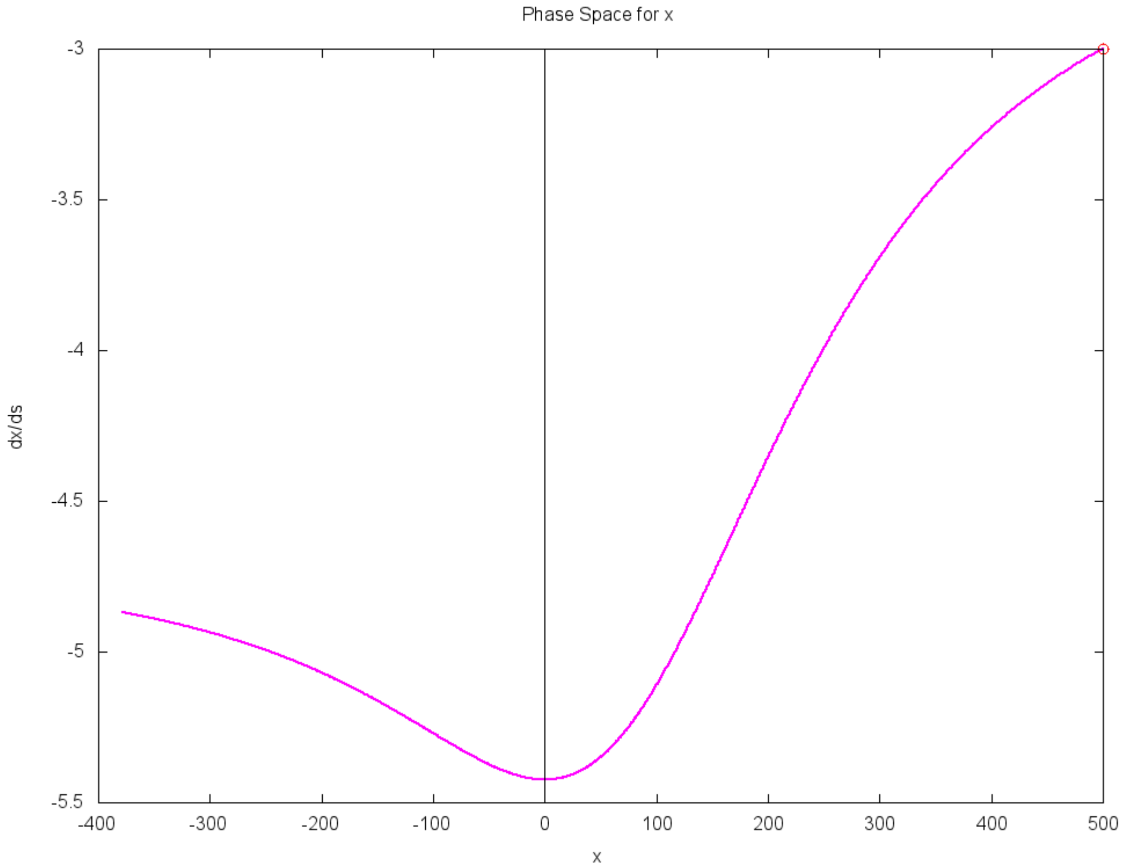
(%t68)

```
(%i69) wxplot2d([[discrete,map(lambda([u],part(u,[1,4])),rksol)], [discrete,map(lambda([u],part(u,[1,5]
       [style,[lines,1]],[xlabel,"s"],[ylabel,"value"], [legend,"dx/ds","dy/ds"],
       [gnuplot_preamble,"set key top left"])$
```



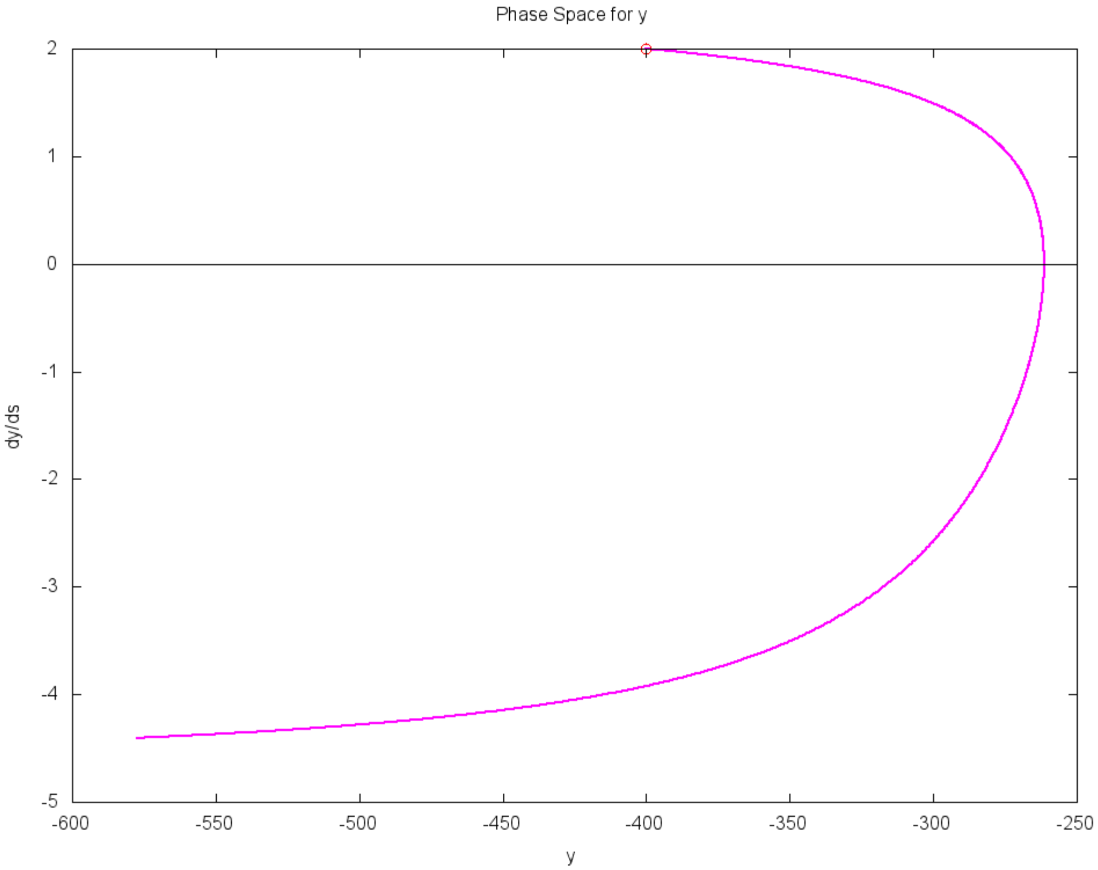
(%t69)

```
(%i70) wxplot2d([[discrete,map(lambda([u],part(u,[2,4])),rksol)], [discrete,[part(initial,[1,3])]]],[axe
       [title,"Phase Space for x"],[point_type,circle], [style,[lines,2],[points,3]],[color,magenta,red]
       [xlabel,"x"],[ylabel,"dx/ds"],[legend,false])$
```



(%t70)

```
(%i71) wxplot2d([[discrete,map(lambda([u],part(u,[3,5])),rksol)], [discrete,[part(initial,[2,4])]]],[ax
       [title,"Phase Space for y"],[point_type,circle], [style,[lines,2],[points,3]],[color,magenta,red]
       [xlabel,"y"],[ylabel,"dy/ds"],[legend,false])$
```



(%t71)

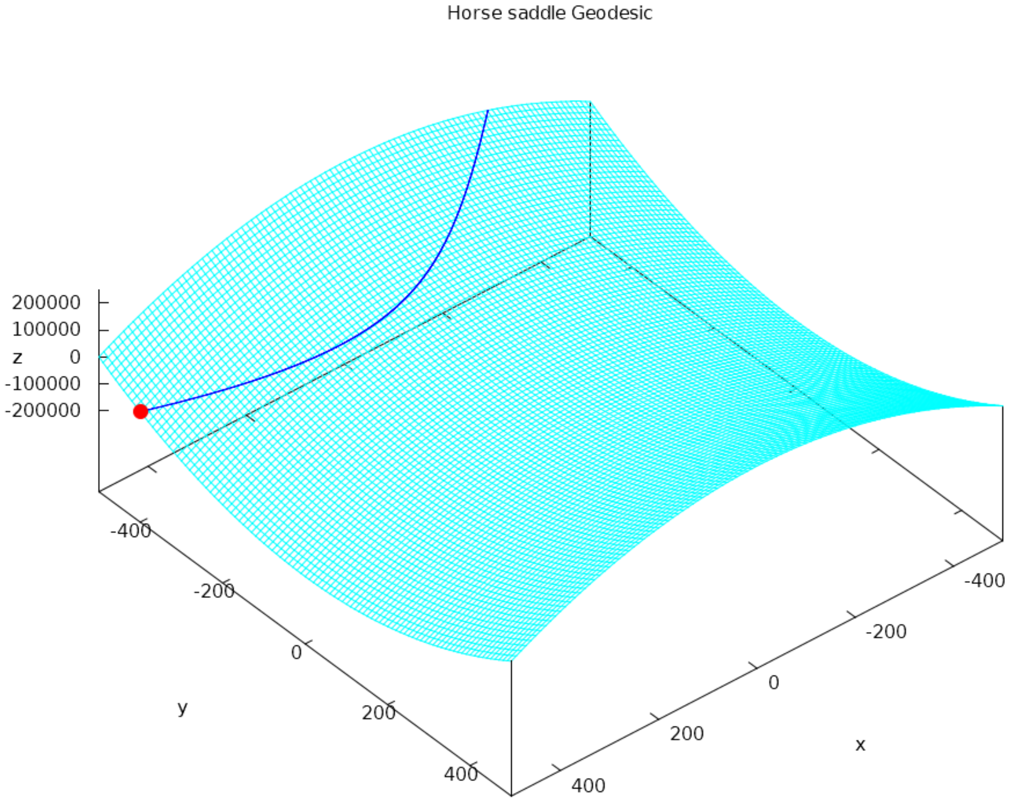
```
(%i72) draw3d(
       title = "Horse saddle Geodesic",
       file_name = "Horse_saddle_Geodesic",
       terminal = 'pngcairo,
       xu_grid = 100, yv_grid = 100,
       xrange = [-500,500], yrange = [-500,500],
       xlabel = "x", ylabel = "y", zlabel = "z",
       dimensions = wxplot_size,
       view = [27,140],

       color = cyan,
       explicit(z,x,-500,500,y,-500,500),

       color = blue,
       point_size = 1,
       point_type = -1,
       points_joined = true,
       points(makelist([k[2],k[3],k[3]^2-k[2]^2],k,rksol)),

       color = red,
       point_size = 2,
       point_type = filled_circle,
       points_joined = false,
       points([[initial[1],initial[2],initial[2]^2-initial[1]^2]]) )$
```

```
(%i73) show_image("Horse_saddle_Geodesic.png")$
```



(%t73)