Web REDUCE (6745), 2-Apr-2024 ...

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% Gravitational Physics
% Lectures by Ruth Gregory

load_package excalc$

pform {A,B,C}=0$
fdomain A=A(r),B=B(r),C=C(r)$

write "Define the Ansatz Schwarzschild coframe"$
coframe o(t)      = A               * d t,
        o(r)      = B               * d r,
        o(theta)  = C               * d theta,
        o(phi)    = C * sin(theta) * d phi
  with metric g = + o(t) * o(t) - o(r) * o(r)
                  - o(theta) * o(theta) - o(phi) * o(phi)$

frame e$
displayframe;
DETM!*;

on fancy;
on nero$
factor o,^$

write "Verify"$
e(-k) _| o(l);

clear omega$
riemannconx omega$
write "Display the connection form"$
omega(k,-l);

write "Display the connection form in Matrix"$
coords := {t, r, theta, phi}$
matrix Momega(4, 4)$
for k := 1:4 do for l := 1:4 do
Momega(k, l) := omega(part(coords, k), part(coords, l))$
Momega;

clear curv,riemann,ricci,riccisc$
pform curv(k,l)=2,{riemann(a,b,c,d),ricci(a,b),riccisc}=0$

index_symmetries curv(k,l): antisymmetric,
                 riemann(k,l,m,n): antisymmetric in {k,l},{m,n}
                                   symmetric in {{k,l},{m,n}},
                 ricci(k,l): symmetric;

write "Display the curvature form"$
curv(k,-l) := d omega(k,-l) + omega(k,-m) ^ omega(m,-l);

write "Display the curvature form in Matrix"$
linelength 200$
matrix Mcurv(4, 4)$
for k := 1:4 do for l := 1:4 do
Mcurv(k, l) := curv(part(coords, k), part(coords, l))$
Mcurv;

write "Display the Riemann Tensor all up"$
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riemann(a,b,c,d) := e(d) _| ( e(c) _| curv(a,b) );
write "Display the Riemann Tensor"$
riemann(a,-b,-c,-d);
write "Display the Riemann Tensor all down"$
riemann(-a,-b,-c,-d);

write "Display the Ricci Tensor"$
ricci(-a,-b) := riemann(c,-a,-d,-b) * g(-c,d);

write "Display the Ricci Scalar"$
riccisc := ricci(-a,-b) * g(a,b);

write "Display the Einstein Tensor"$
clear einstein$
pform einstein(a)=3$
einstein(-a) := (1/2) * curv(b,c) ^ #( o(-b) ^ o(-c) ^ o(-a) );

showtime;
end;

*** .*. redefined

*** × redefined

*** ^ redefined
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Define the Ansatz Schwarzschild coframe

$$
\begin{aligned}
o^t &= d\,t\,a \\
o^r &= d\,r\,b \\
o^\theta &= d\,\theta\,c \\
o^\phi &= d\,\phi\,\sin(\theta)\,c
\end{aligned}
$$

$$-1$$

Verify

$$\mathrm{ns}_t{}^t{:=}1$$

$$\mathrm{ns}_r{}^r{:=}1$$

$$\mathrm{ns}_\theta{}^\theta{:=}1$$

$$\mathrm{ns}_\phi{}^\phi{:=}1$$

Display the connection form

$$\omega^{r}{}_{t} := \frac{o^{t}\,\frac{\partial a}{\partial r}}{a\,b}$$

$$\omega^{t}{}_{r} := \frac{o^{t}\,\frac{\partial a}{\partial r}}{a\,b}$$

$$\omega^{\theta}{}_{r} := \frac{o^{\theta}\,\frac{\partial c}{\partial r}}{b\,c}$$

$$\omega^{\phi}{}_{r} := \frac{o^{\phi}\,\frac{\partial c}{\partial r}}{b\,c}$$

$$\omega^{r}{}_{\theta} := \frac{-o^{\theta}\,\frac{\partial c}{\partial r}}{b\,c}$$

$$\omega^{\phi}{}_{\theta} := \frac{o^{\phi}\,\cos(\theta)}{\sin(\theta)\,c}$$

$$\omega^{r}{}_{\phi} := \frac{-o^{\phi}\,\frac{\partial c}{\partial r}}{b\,c}$$

$$\omega^{\theta}{}_{\phi} := \frac{-o^{\phi}\,\cos(\theta)}{\sin(\theta)\,c}$$

Display the connection form in Matrix

$$\begin{pmatrix} 0 & \frac{-o^{t}\,\frac{\partial a}{\partial r}}{a\,b} & 0 & 0 \\ \frac{o^{t}\,\frac{\partial a}{\partial r}}{a\,b} & 0 & \frac{o^{\theta}\,\frac{\partial c}{\partial r}}{b\,c} & \frac{o^{\phi}\,\frac{\partial c}{\partial r}}{b\,c} \\ 0 & \frac{-o^{\theta}\,\frac{\partial c}{\partial r}}{b\,c} & 0 & \frac{o^{\phi}\,\cos(\theta)}{\sin(\theta)\,c} \\ 0 & \frac{-o^{\phi}\,\frac{\partial c}{\partial r}}{b\,c} & \frac{-o^{\phi}\,\cos(\theta)}{\sin(\theta)\,c} & 0 \end{pmatrix}$$

Display the curvature form

$$\mathrm{curv}^{r}{}_{t} := \frac{o^{t} \wedge o^{r} \left(-\frac{\partial^2 a}{\partial r^2}\, b + \frac{\partial a}{\partial r}\, \frac{\partial b}{\partial r}\right)}{a\, b^{3}}$$

$$\mathrm{curv}^{\theta}{}_{t} := \frac{-o^{t} \wedge o^{\theta}\, \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}}{a\, b^{2}\, c}$$

$$\mathrm{curv}^{\phi}{}_{t} := \frac{-o^{t} \wedge o^{\phi}\, \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}}{a\, b^{2}\, c}$$

$$\mathrm{curv}^{t}{}_{r} := \frac{o^{t} \wedge o^{r} \left(-\frac{\partial^2 a}{\partial r^2}\, b + \frac{\partial a}{\partial r}\, \frac{\partial b}{\partial r}\right)}{a\, b^{3}}$$

$$\mathrm{curv}^{\theta}{}_{r} := \frac{o^{r} \wedge o^{\theta} \left(-\frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\, b\right)}{b^{3}\, c}$$

$$\mathrm{curv}^{\phi}{}_{r} := \frac{o^{r} \wedge o^{\phi} \left(-\frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\, b\right)}{b^{3}\, c}$$

$$\mathrm{curv}^{t}{}_{\theta} := \frac{-o^{t} \wedge o^{\theta}\, \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}}{a\, b^{2}\, c}$$

$$\mathrm{curv}^{r}{}_{\theta} := \frac{o^{r} \wedge o^{\theta} \left(\frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r} - \frac{\partial^2 c}{\partial r^2}\, b\right)}{b^{3}\, c}$$

$$\mathrm{curv}^{\phi}{}_{\theta} := \frac{o^{\theta} \wedge o^{\phi} \left(\left(\frac{\partial c}{\partial r}\right)^{2} - b^{2}\right)}{b^{2}\, c^{2}}$$

$$\mathrm{curv}^{t}{}_{\phi} := \frac{-o^{t} \wedge o^{\phi}\, \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}}{a\, b^{2}\, c}$$

$$\mathrm{curv}^{r}{}_{\phi} := \frac{o^{r} \wedge o^{\phi} \left(\frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r} - \frac{\partial^2 c}{\partial r^2}\, b\right)}{b^{3}\, c}$$

$$\mathrm{curv}^{\theta}{}_{\phi} := \frac{o^{\theta} \wedge o^{\phi} \left(-\left(\frac{\partial c}{\partial r}\right)^{2} + b^{2}\right)}{b^{2}\, c^{2}}$$

Display the curvature form in Matrix

$$\begin{pmatrix} 0 & \frac{o^{t} \wedge o^{r} \left(\frac{\partial^2 a}{\partial r^2}\, b - \frac{\partial a}{\partial r}\, \frac{\partial b}{\partial r}\right)}{a\, b^{3}} & \frac{o^{t} \wedge o^{\theta}\, \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}}{a\, b^{2}\, c} & \frac{o^{t} \wedge o^{\phi}\, \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}}{a\, b^{2}\, c} \\ \frac{o^{t} \wedge o^{r} \left(-\frac{\partial^2 a}{\partial r^2}\, b + \frac{\partial a}{\partial r}\, \frac{\partial b}{\partial r}\right)}{a\, b^{3}} & 0 & \frac{o^{r} \wedge o^{\theta} \left(-\frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\, b\right)}{b^{3}\, c} & \frac{o^{r} \wedge o^{\phi} \left(-\frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\, b\right)}{b^{3}\, c} \\ \frac{-o^{t} \wedge o^{\theta}\, \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}}{a\, b^{2}\, c} & \frac{o^{r} \wedge o^{\theta} \left(\frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r} - \frac{\partial^2 c}{\partial r^2}\, b\right)}{b^{3}\, c} & 0 & \frac{o^{\theta} \wedge o^{\phi} \left(\left(\frac{\partial c}{\partial r}\right)^{2} - b^{2}\right)}{b^{2}\, c^{2}} \\ \frac{-o^{t} \wedge o^{\phi}\, \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}}{a\, b^{2}\, c} & \frac{o^{r} \wedge o^{\phi} \left(\frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r} - \frac{\partial^2 c}{\partial r^2}\, b\right)}{b^{3}\, c} & \frac{o^{\theta} \wedge o^{\phi} \left(-\left(\frac{\partial c}{\partial r}\right)^{2} + b^{2}\right)}{b^{2}\, c^{2}} & 0 \end{pmatrix}$$

Display the Riemann Tensor all up

$$\mathrm{riemann}^{t\,r\,t\,r} := \frac{-\frac{\partial^2 a}{\partial r^2}\,b + \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}}{a\,b^3}$$

$$\mathrm{riemann}^{t\,\theta\,t\,\theta} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\mathrm{riemann}^{r\,\theta\,r\,\theta} := \frac{-\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\mathrm{riemann}^{t\,\phi\,t\,\phi} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\mathrm{riemann}^{r\,\phi\,r\,\phi} := \frac{-\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\mathrm{riemann}^{\theta\,\phi\,\theta\,\phi} := \frac{\left(\frac{\partial c}{\partial r}\right)^2 - b^2}{b^2\,c^2}$$

Display the Riemann Tensor

$$\mathrm{riemann}^{r}{}_{t\,r\,t} := \frac{\frac{\partial^2 a}{\partial r^2}\,b - \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}}{a\,b^3}$$

$$\mathrm{riemann}^{t}{}_{r\,r\,t} := \frac{\frac{\partial^2 a}{\partial r^2}\,b - \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}}{a\,b^3}$$

$$\mathrm{riemann}^{\theta}{}_{t\,\theta\,t} := \frac{\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\mathrm{riemann}^{t}{}_{\theta\,\theta\,t} := \frac{\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\mathrm{riemann}^{\phi}{}_{t\,\phi\,t} := \frac{\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\mathrm{riemann}^{t}{}_{\phi\,\phi\,t} := \frac{\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\mathrm{riemann}^{r}{}_{t\,t\,r} := \frac{-\frac{\partial^2 a}{\partial r^2}\,b + \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}}{a\,b^3}$$

$$\mathrm{riemann}^{t}{}_{r\,t\,r} := \frac{-\frac{\partial^2 a}{\partial r^2}\,b + \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}}{a\,b^3}$$

$$\mathrm{riemann}^{\theta}{}_{r\,\theta\,r} := \frac{\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} - \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\text{riemann}^{r}{}_{\theta\,\theta\,r} := \frac{-\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\text{riemann}^{\phi}{}_{r\,\phi\,r} := \frac{\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} - \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\text{riemann}^{r}{}_{\phi\,\phi\,r} := \frac{-\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\text{riemann}^{\theta}{}_{t\,t\,\theta} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\text{riemann}^{t}{}_{\theta\,t\,\theta} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\text{riemann}^{\theta}{}_{r\,r\,\theta} := \frac{-\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\text{riemann}^{r}{}_{\theta\,r\,\theta} := \frac{\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} - \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\text{riemann}^{\phi}{}_{\theta\,\phi\,\theta} := \frac{-\left(\frac{\partial c}{\partial r}\right)^2 + b^2}{b^2\,c^2}$$

$$\text{riemann}^{\theta}{}_{\phi\,\phi\,\theta} := \frac{\left(\frac{\partial c}{\partial r}\right)^2 - b^2}{b^2\,c^2}$$

$$\text{riemann}^{\phi}{}_{t\,t\,\phi} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\text{riemann}^{t}{}_{\phi\,t\,\phi} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\text{riemann}^{\phi}{}_{r\,r\,\phi} := \frac{-\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\text{riemann}^{r}{}_{\phi\,r\,\phi} := \frac{\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} - \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\text{riemann}^{\phi}{}_{\theta\,\theta\,\phi} := \frac{\left(\frac{\partial c}{\partial r}\right)^2 - b^2}{b^2\,c^2}$$

$$\text{riemann}^{\theta}{}_{\phi\,\theta\,\phi} := \frac{-\left(\frac{\partial c}{\partial r}\right)^2 + b^2}{b^2\,c^2}$$

Display the Riemann Tensor all down

$$\text{riemann}_{t\,r\,t\,r} := \frac{-\frac{\partial^2 a}{\partial r^2}\,b + \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}}{a\,b^3}$$

$$\text{riemann}_{t\,\theta\,t\,\theta} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\mathrm{riemann}_{r\,\theta\,r\,\theta} := \frac{-\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\mathrm{riemann}_{t\,\phi\,t\,\phi} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}}{a\,b^2\,c}$$

$$\mathrm{riemann}_{r\,\phi\,r\,\phi} := \frac{-\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}\,b}{b^3\,c}$$

$$\mathrm{riemann}_{\theta\,\phi\,\theta\,\phi} := \frac{\left(\frac{\partial c}{\partial r}\right)^2 - b^2}{b^2\,c^2}$$

Display the Ricci Tensor

$$\mathrm{ricci}_{t\,t} := \frac{\frac{\partial^2 a}{\partial r^2}\,b\,c - \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}\,c + 2\,\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}\,b}{a\,b^3\,c}$$

$$\mathrm{ricci}_{r\,r} := \frac{-\frac{\partial^2 a}{\partial r^2}\,b\,c + \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}\,c + 2\,\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r}\,a - 2\,\frac{\partial^2 c}{\partial r^2}\,a\,b}{a\,b^3\,c}$$

$$\mathrm{ricci}_{\theta\,\theta} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}\,b\,c + \frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r}\,a\,c - \frac{\partial^2 c}{\partial r^2}\,a\,b\,c - \left(\frac{\partial c}{\partial r}\right)^2 a\,b + a\,b^3}{a\,b^3\,c^2}$$

$$\mathrm{ricci}_{\phi\,\phi} := \frac{-\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}\,b\,c + \frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r}\,a\,c - \frac{\partial^2 c}{\partial r^2}\,a\,b\,c - \left(\frac{\partial c}{\partial r}\right)^2 a\,b + a\,b^3}{a\,b^3\,c^2}$$

Display the Ricci Scalar

$$\mathrm{riccisc} := \frac{2\,\left(\frac{\partial^2 a}{\partial r^2}\,b\,c^2 - \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}\,c^2 + 2\,\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}\,b\,c - 2\,\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r}\,a\,c + 2\,\frac{\partial^2 c}{\partial r^2}\,a\,b\,c + \left(\frac{\partial c}{\partial r}\right)^2 a\,b - a\,b^3\right)}{a\,b^3\,c^2}$$

Display the Einstein Tensor

$$\mathrm{einstein}_{t} := \frac{o^r \wedge o^\theta \wedge o^\phi\,\left(-2\,\frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r}\,c + 2\,\frac{\partial^2 c}{\partial r^2}\,b\,c + \left(\frac{\partial c}{\partial r}\right)^2 b - b^3\right)}{b^3\,c^2}$$

$$\mathrm{einstein}_{r} := \frac{o^t \wedge o^\theta \wedge o^\phi\,\left(-2\,\frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}\,c - \left(\frac{\partial c}{\partial r}\right)^2 a + a\,b^2\right)}{a\,b^2\,c^2}$$

$$\mathrm{einstein}_{\theta} := \frac{o^t \wedge o^r \wedge o^\phi\,\left(\frac{\partial^2 a}{\partial r^2}\,b\,c - \frac{\partial a}{\partial r}\,\frac{\partial b}{\partial r}\,c + \frac{\partial a}{\partial r}\,\frac{\partial c}{\partial r}\,b - \frac{\partial b}{\partial r}\,\frac{\partial c}{\partial r}\,a + \frac{\partial^2 c}{\partial r^2}\,a\,b\right)}{a\,b^3\,c}$$

$$\mathrm{einstein}_{\phi} := \frac{o^{t} \wedge o^{r} \wedge o^{\theta} \left(-\frac{\partial^2 a}{\partial r^2}\, b\, c + \frac{\partial a}{\partial r}\, \frac{\partial b}{\partial r}\, c - \frac{\partial a}{\partial r}\, \frac{\partial c}{\partial r}\, b + \frac{\partial b}{\partial r}\, \frac{\partial c}{\partial r}\, a - \frac{\partial^2 c}{\partial r^2}\, a\, b\right)}{a\, b^3\, c}$$

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Time: 24921 ms  plus GC time: 4 ms
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