Calculus of Variations

Based on Maths For All Playlist Calculus of Variations

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(%i2)
```
info:build_info()$info@version;
```

(%o2)

5.38.1

(%i2)
```
reset()$kill(all)$
```

(%i1)
```
derivabbrev:true$
```

(%i2)
```
ratprint:false$
```

(%i3)
```
fpprintprec:5$
```

(%i4)
```
load(linearalgebra)$
```

(%i5)
```
if get('draw,'version)=false then load(draw)$
```

(%i6)
```
wxplot_size:[1024,768]$
```

(%i8)
```
load(odes)$ load(contrib_ode)$
```

(%i9)
```
if get('optvar,'version)=false then load(optvar)$
```

(%i10)
```
if get('optmiz,'version)=false then load(optmiz)$
```

1 Straight line

Lagrangian

(%i11) `depends(y,x)$`

(%i12) `L:`$\sqrt{}$`(1+'diff(y,x)`^{2}`);`

$$\sqrt{(y_x)^2+1} \tag{L}$$

Momentum Conjugate

(%i13) `ldisplay(P_y:diff(L,'diff(y,x)))$`

$$P_y = \frac{y_x}{\sqrt{(y_x)^2+1}} \tag{\%t13}$$

Generalized Forces

(%i14) `ldisplay(F_y:diff(L,y))$`

$$F_y = 0 \tag{\%t14}$$

Euler-Lagrange Equation

(%i15) `aa:el(L,y,x)$`

(%i18) `bb:ev(aa,eval,diff)$`

(%i19) `map(ldisp,bb:fullratsimp(bb))$`

$$\frac{1}{\sqrt{(y_x)^2+1}} = k_0 \tag{\%t19}$$

$$\frac{\sqrt{(y_x)^2+1}\,(y_{xx})}{(y_x)^4+2(y_x)^2+1} = 0 \tag{\%t20}$$

$$\frac{y_x}{\sqrt{(y_x)^2+1}} = k_1 \tag{\%t21}$$

(%i23)
```
bb[1]:reverse(subst([k[0]=E],bb[1]))$
bb[3]:reverse(subst([k[1]=P],bb[3]))$
```

Conservation Laws

(%i24) `map(ldisp,part(bb,[1,3]))$`

$$E = \frac{1}{\sqrt{(y_x)^2+1}} \tag{\%t24}$$

$$P = \frac{y_x}{\sqrt{(y_x)^2 + 1}} \tag{\%t25}$$

Equations of Motion

(%i26) `[sol]:solve(bb[2],'diff(y,x,2));`

$$[y_{xx} = 0] \tag{\%o26}$$

(%i27) `odesol:ode2(sol,y,x);`

$$y = \%k2x + \%k1 \tag{odesol}$$

(%i28) `method;`

$$constcoeff \tag{\%o28}$$

(%i29) `bc2(odesol,x=x_0,y=y_0,x=x_1,y=y_1);`

$$y = \frac{x\,(y_0 - y_1)}{x_0 - x_1} - \frac{x_1 y_0 - x_0 y_1}{x_0 - x_1} \tag{\%o29}$$

2 Brachistochrone problem

Lagrangian

(%i30) `depends(y,x)$`

(%i31) `L:√(1+'diff(y,x)^2)/√(2*g*y);`

$$\frac{\sqrt{(y_x)^2+1}}{\sqrt{2}\,\sqrt{g y}} \tag{L}$$

Momentum Conjugate

(%i32) `ldisplay(P_y:diff(L,'diff(y,x)))$`

$$P_y = \frac{y_x}{\sqrt{2}\,\sqrt{g y}\,\sqrt{(y_x)^2+1}} \tag{\%t32}$$

Generalized Forces

(%i33) `ldisplay(F_y:diff(L,y))$`

$$F_y = -\frac{g\,\sqrt{(y_x)^2+1}}{2^{\frac{3}{2}}\,(g y)^{\frac{3}{2}}} \tag{\%t33}$$

Euler-Lagrange Equation

(%i34) `aa:el(L,y,x)$`

(%i36) `bb:ev(aa,eval,diff)$`

(%i37) `map(ldisp,bb:fullratsimp(bb))$`

$$\frac{\sqrt{g y}}{\sqrt{2} g y\,\sqrt{(y_x)^2+1}} = k_0 \tag{\%t37}$$

$$\frac{2y\,(y_{xx}) - (y_x)^4 - (y_x)^2}{\sqrt{g y}\,\sqrt{(y_x)^2+1}\,\left(2^{\frac{3}{2}} y\,(y_x)^2 + 2^{\frac{3}{2}} y\right)} = -\frac{\sqrt{(y_x)^2+1}}{2^{\frac{3}{2}} y\,\sqrt{g y}} \tag{\%t38}$$

Conservation Laws

(%i39) `bb[1]:reverse(subst([k[0]=E],bb[1]));`

$$E = \frac{\sqrt{g y}}{\sqrt{2} g y\,\sqrt{(y_x)^2+1}} \tag{\%o39}$$

Equations of Motion

(%i40) `[sol]:solve(bb[2],'diff(y,x,2));`

$$\left[y_{xx} = -\frac{{(y_x)}^2 + 1}{2y}\right] \tag{\%o40}$$

(%i41) `odesol:ode2(sol,y,x)$`

(%i42) `method;`

$$\mathit{freeofx} \tag{\%o42}$$

(%i43) `map(ldisp,odesol)$`

$$\frac{e^{-2\%k1}\left(e^{\%k1}\sqrt{y}\,\mathrm{atan}\left(\frac{e^{-\frac{\%k1}{2}}\sqrt{1-e^{\%k1}\,y}}{\sqrt{y}}\right)+e^{\frac{3\%k1}{2}}\,y\,\sqrt{1-e^{\%k1}\,y}\right)}{\sqrt{y}} = x + \%k2 \tag{\%t43}$$

$$-\frac{e^{-2\%k1}\left(e^{\%k1}\sqrt{y}\,\mathrm{atan}\left(\frac{e^{-\frac{\%k1}{2}}\sqrt{1-e^{\%k1}\,y}}{\sqrt{y}}\right)+e^{\frac{3\%k1}{2}}\,y\,\sqrt{1-e^{\%k1}\,y}\right)}{\sqrt{y}} = x + \%k2 \tag{\%t44}$$

3 Optimal control

```
(%i45) aa:ham(['diff(v,t,1)=f,'diff(x,t,1)=v])$
(%i49) bb:ev(aa,eval)$
(%i50) bb:subst([aux[1]=aux_1,aux[2]=aux_2],bb);
```

$$[aux_2 v + aux_1 f,\, aux_{1t} = -aux_2,\, aux_{2t} = 0,\, aux_2 = c_2] \tag{bb}$$

```
(%i51) map(ldisp,bb)$
```

$$aux_2 v + aux_1 f \tag{\%t51}$$

$$aux_{1t} = -aux_2 \tag{\%t52}$$

$$aux_{2t} = 0 \tag{\%t53}$$

$$aux_2 = c_2 \tag{\%t54}$$

```
(%i56) atvalue(aux_1(t),[t=0],a)$
       atvalue(aux_2(t),[t=0],b)$
(%i57) printprops(all,atvalue)$
```

$$aux_1(0) = a,\ aux_2(0) = b$$

```
(%i58) convert(part(bb,[2,3]),[aux_1,aux_2],t);
```

$$[aux_1(t)_t = -aux_2(t),\, aux_2(t)_t = 0] \tag{\%o58}$$

```
(%i59) convert([aux_1,aux_2],[aux_1,aux_2],t);
```

$$[aux_1(t),\, aux_2(t)] \tag{\%o59}$$

```
(%i60) desol:desolve(%th(2),%th(1));
```

$$[aux_1(t) = a - bt,\, aux_2(t) = b] \tag{desol}$$

```
(%i61) convert(bb[1],[aux_1,aux_2],t);
```

$$aux_2(t)v + f\ aux_1(t) \tag{\%o61}$$

```
(%i62) ev(%,desol);
```

$$bv + f\ (a - bt) \tag{\%o62}$$

4 Calculus of Variations: Functionals

Based on NPTEL-NOC IITM Video Calculus of Variations: Functionals

(%i63) `kill(labels,x,y)$`

(%i1) `ζ:[x,y]$`

(%i2) `f(x,y):=x*y$`

(%i4) `ϕ(x,y):=x^2+y^2$ b:1$`

(%i5) `stapoints(f(x,y),[],[ϕ(x,y)-b],ζ)$`

$$stapts_1 = \left[x = \frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}, eqmult_1 = \frac{1}{2}\right] \tag{\%t5}$$

$$objsub = -\frac{1}{2} \tag{\%t6}$$

$$gradsub = [0, 0, 0] \tag{\%t7}$$

$$stapts_2 = \left[x = -\frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, eqmult_1 = \frac{1}{2}\right] \tag{\%t8}$$

$$objsub = -\frac{1}{2} \tag{\%t9}$$

$$gradsub = [0, 0, 0] \tag{\%t10}$$

$$stapts_3 = \left[x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, eqmult_1 = -\frac{1}{2}\right] \tag{\%t11}$$

$$objsub = \frac{1}{2} \tag{\%t12}$$

$$gradsub = [0, 0, 0] \tag{\%t13}$$

$$stapts_4 = \left[x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}, eqmult_1 = -\frac{1}{2}\right] \tag{\%t14}$$

$$objsub = \frac{1}{2} \tag{\%t15}$$

$$gradsub = [0, 0, 0] \tag{\%t16}$$

(%i17) `map(ldisp,stapts)$`

$$\left[x = \frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}, eqmult_1 = \frac{1}{2}\right] \tag{\%t17}$$

$$\left[x = -\frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, eqmult_1 = \frac{1}{2}\right] \tag{\%t18}$$

$$\left[x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, eqmult_1 = -\frac{1}{2}\right] \tag{\%t19}$$

$$\left[x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}, eqmult_1 = -\frac{1}{2}\right] \tag{\%t20}$$

(%i21) `makelist(at(f(x,y),stapts[i]),i,1,length(stapts));`

$$\left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \tag{\%o21}$$

(%i22) `makelist(at(`ϕ`(x,y),stapts[i]),i,1,length(stapts));`

$$[1, 1, 1, 1] \tag{\%o22}$$

(%i23) `lagrangian;`

$$eqmult_1\ \left(y^2 + x^2 - 1\right) + xy \tag{\%o23}$$

(%i24) `grad;`

$$[y + 2\,eqmult_1 x, 2\,eqmult_1 y + x, y^2 + x^2 - 1] \tag{\%o24}$$

(%i25) `decslkmults;`

$$[x, y, eqmult_1] \tag{\%o25}$$

(%i26) `gradient(decslkmults);`

$$[y + 2\,eqmult_1 x, 2\,eqmult_1 y + x, y^2 + x^2 - 1] \tag{\%o26}$$

(%i27) `G:list_matrix_entries(jacobian([f(x,y)],`ζ`));`

$$[y, x] \tag{G}$$

(%i28) `J:list_matrix_entries(jacobian([lagrangian],decslkmults));`

$$[y + 2\,eqmult_1 x, 2\,eqmult_1 y + x, y^2 + x^2 - 1] \tag{J}$$

(%i29) `sol:algsys(grad,decslkmults)$`

(%i30) `map(ldisp,sol)$`

$$\left[x = \frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}, eqmult_1 = \frac{1}{2}\right] \tag{\%t30}$$

$$\left[x = -\frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, eqmult_1 = \frac{1}{2}\right] \tag{\%t31}$$

$$\left[x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, eqmult_1 = -\frac{1}{2}\right] \tag{\%t32}$$

$$\left[x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}, eqmult_1 = -\frac{1}{2}\right] \tag{\%t33}$$

5 Examples

Based on Bsc Maths Aligarh Video Calculus of Variations

5.1

Find the extremals of the following functionals:

$$\int_{x_0}^{x_1} (x + y')\, y' \,\mathrm{d}x$$

(%i34) `kill(labels,x,y)$`

(%i1) `ζ:[x,y]$`

(%i2) `depends(y,x)$`

Lagrangian

(%i3) `L:(x+'diff(y,x))*'diff(y,x);`

$$(y_x)\ (y_x + x) \tag{L}$$

Momentum Conjugate

(%i4) `ldisplay(P:diff(L,'diff(y,x)))$`

$$P = 2\,(y_x) + x \tag{\%t4}$$

Generalized Forces

(%i5) `ldisplay(F:diff(L,x))$`

$$F = (y_x)\ (y_{xx} + 1) + (y_x + x)\ (y_{xx}) \tag{\%t5}$$

Euler-Lagrange Equation

(%i6) `aa:el(L,y,x)$`

(%i8) `bb:ev(aa,eval,diff)$`

(%i9) `map(ldisp,bb:fullratsimp(bb))$`

$$2\,(y_{xx}) + 1 = 0 \tag{\%t9}$$

$$2\,(y_x) + x = k_1 \tag{\%t10}$$

Solution

(%i11) `odesol:ode2(bb[1],y,x);`

$$y = -\frac{x^2}{4} + \%k2 x + \%k1 \tag{odesol}$$

(%i12) `method;`

$$variation of parameters \quad (\%o12)$$

(%i13) `declare([x_1,y_1,x_2,y_2],constant)$`

(%i14) `params:[x_1=0,y_1=0,x_2=10,y_2=10]$`

(%i15) `odesol:bc2(odesol,x=x_1,y=y_1,x=x_2,y=y_2);`

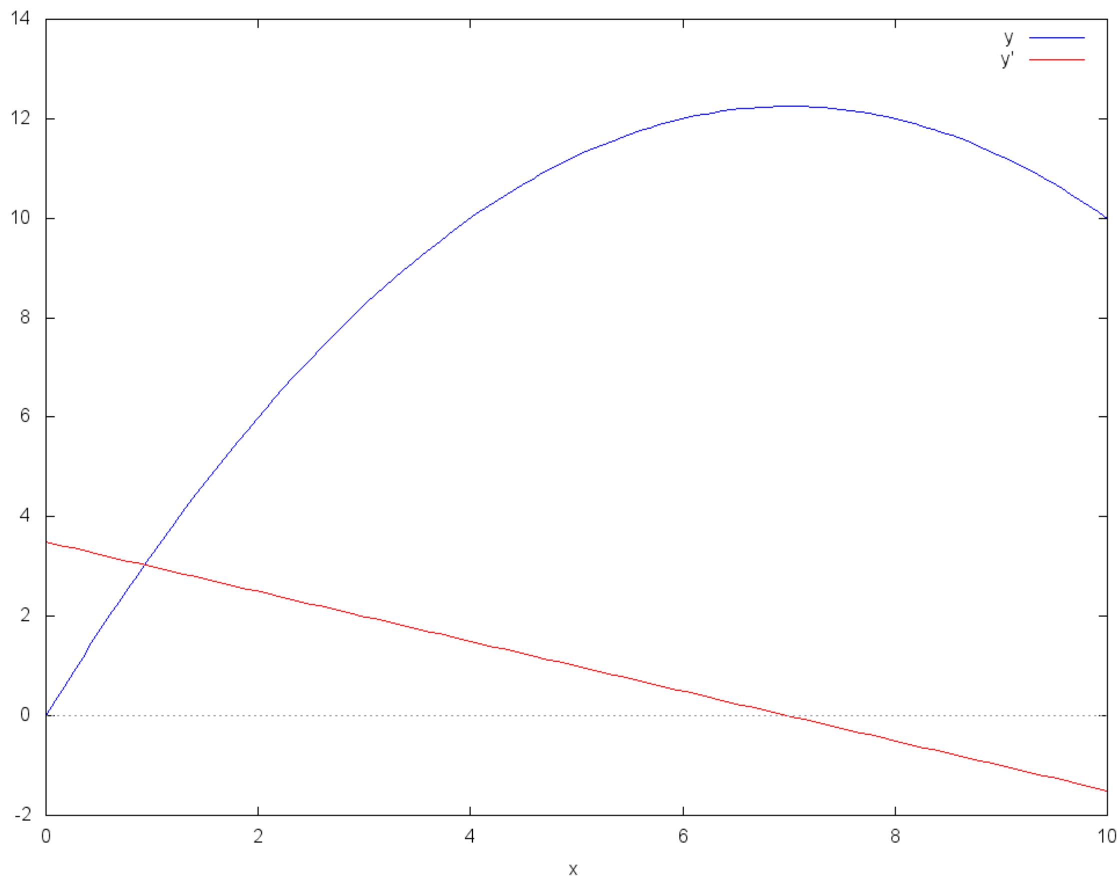
$$y = -\frac{x^2}{4} + \frac{\left(-4y_2 + 4y_1 - {x_2}^2 + {x_1}^2\right) x}{4x_1 - 4x_2} - \frac{x_1 \left(-4y_2 - {x_2}^2\right) + 4x_2 y_1 + {x_1}^2 x_2}{4x_1 - 4x_2} \quad \text{(odesol)}$$

(%i16) `bb[2],odesol,eval,diff,expand,factor;`

$$\frac{4y_2 - 4y_1 + {x_2}^2 - {x_1}^2}{2\left(x_2 - x_1\right)} = k_1 \quad (\%o16)$$

Graphics

(%i17) `wxplot2d([y,diff(y,x)],[x,x_1,x_2],[ylabel,""], [legend,"y","y'"]),odesol,params$`



(%t17)

5.2

Find the extremals of the following functionals:

$$\int_{x_0}^{x_1} \frac{{y'}^2}{x^3}\,\mathrm{d}x$$

(%i18) `kill(labels,x,y)$`

(%i1) `ζ:[x,y]$`

(%i2) `depends(y,x)$`

Lagrangian

(%i3) `L:'diff(y,x)^2/x^3;`

$$\frac{{(y_x)}^2}{x^3} \tag{L}$$

Momentum Conjugate

(%i4) `ldisplay(P:diff(L,'diff(y,x)))$`

$$P=\frac{2\,(y_x)}{x^3} \tag{\%t4}$$

Generalized Forces

(%i5) `ldisplay(F:diff(L,x))$`

$$F=\frac{2\,(y_x)\,\left(y_{xx}\right)}{x^3}-\frac{3{(y_x)}^2}{x^4} \tag{\%t5}$$

Euler-Lagrange Equation

(%i6) `aa:el(L,y,x)$`

(%i8) `bb:ev(aa,eval,diff)$`

(%i9) `map(ldisp,bb:fullratsimp(bb))$`

$$\frac{2x\,\left(y_{xx}\right)-6\,(y_x)}{x^4}=0 \tag{\%t9}$$

$$\frac{2\,(y_x)}{x^3}=k_1 \tag{\%t10}$$

Solution

(%i11) `odesol:ode2(bb[1],y,x);`

$$y=\%k2\,x^4-\frac{\%k1}{4} \tag{odesol}$$

(%i12) `method;`

$$exact \tag{\%o12}$$

(%i13) `declare([x_1,y_1,x_2,y_2],constant)$`

(%i14) `params:[x_1=0,y_1=0,x_2=10,y_2=10]$`

(%i15) `odesol:bc2(odesol,x=x_1,y=y_1,x=x_2,y=y_2);`

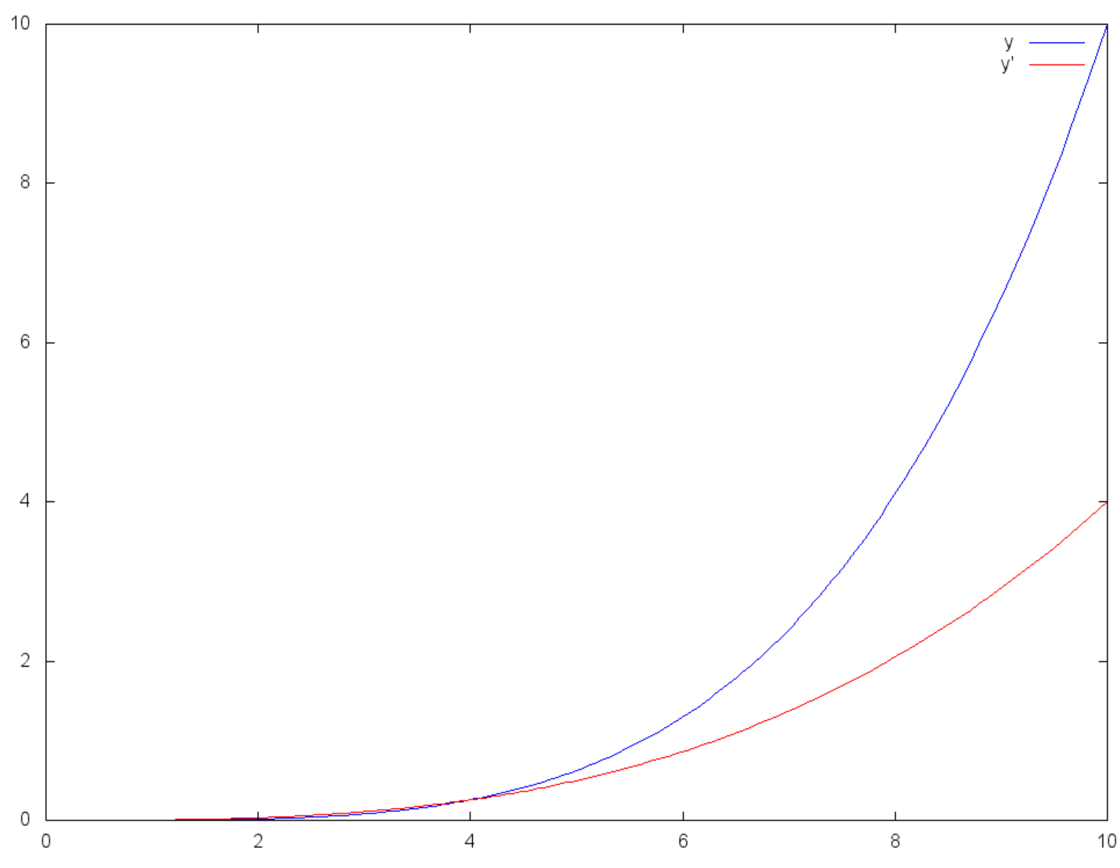
$$y=\frac{(y_1-y_2)\,x^4}{{x_1}^4-{x_2}^4}-\frac{4{x_2}^4y_1-4{x_1}^4y_2}{4\,({x_1}^4-{x_2}^4)} \tag{odesol}$$

(%i16) `bb[2],odesol,eval,diff;`

$$\frac{8\,(y_1-y_2)}{{x_1}^4-{x_2}^4}=k_1 \tag{\%o16}$$

Graphics

(%i17) `wxplot2d([y,diff(y,x)],[x,x_1,x_2],[ylabel,""], [legend,"y","y’"]),odesol,params$`



(%t17)

5.3

Find the extremals of the following functionals:

$$\int_{x_0}^{x_1} (1 + x^2\, y')y' \,\mathrm{d}x$$

(%i18) `kill(labels,x,y)$`

(%i1) `ζ:[x,y]$`

(%i2) `depends(y,x)$`

Lagrangian

(%i3) `L:(1+x^2*'diff(y,x))*'diff(y,x);`

$$(y_x)\,\left(x^2\,(y_x)+1\right) \tag{L}$$

Momentum Conjugate

(%i4) `ldisplay(P:diff(L,'diff(y,x)))$`

$$P = 2x^2\,(y_x)+1 \tag{\%t4}$$

Generalized Forces

(%i5) `ldisplay(F:fullratsimp(diff(L,x)))$`

$$F = \left(2x^2\,(y_x)+1\right)\,(y_{xx})+2x\,(y_x)^2 \tag{\%t5}$$

Euler-Lagrange Equation

(%i6) `aa:el(L,y,x)$`

(%i8) `bb:ev(aa,eval,diff)$`

(%i9) `map(ldisp,bb:fullratsimp(bb))$`

$$2x^2\,(y_{xx})+4x\,(y_x)=0 \tag{\%t9}$$

$$2x^2\,(y_x)+1=k_1 \tag{\%t10}$$

Solution

(%i11) `odesol:ode2(bb[1],y,x);`

$$y = \frac{\%k2}{x}+\%k1 \tag{odesol}$$

(%i12) `method;`

$$exact \tag{\%o12}$$

(%i13) `declare([x_1,y_1,x_2,y_2],constant)$`

(%i14) `params:[x_1=1,y_1=1,x_2=10,y_2=10]$`

(%i15) `odesol:bc2(odesol,x=x_1,y=y_1,x=x_2,y=y_2);`

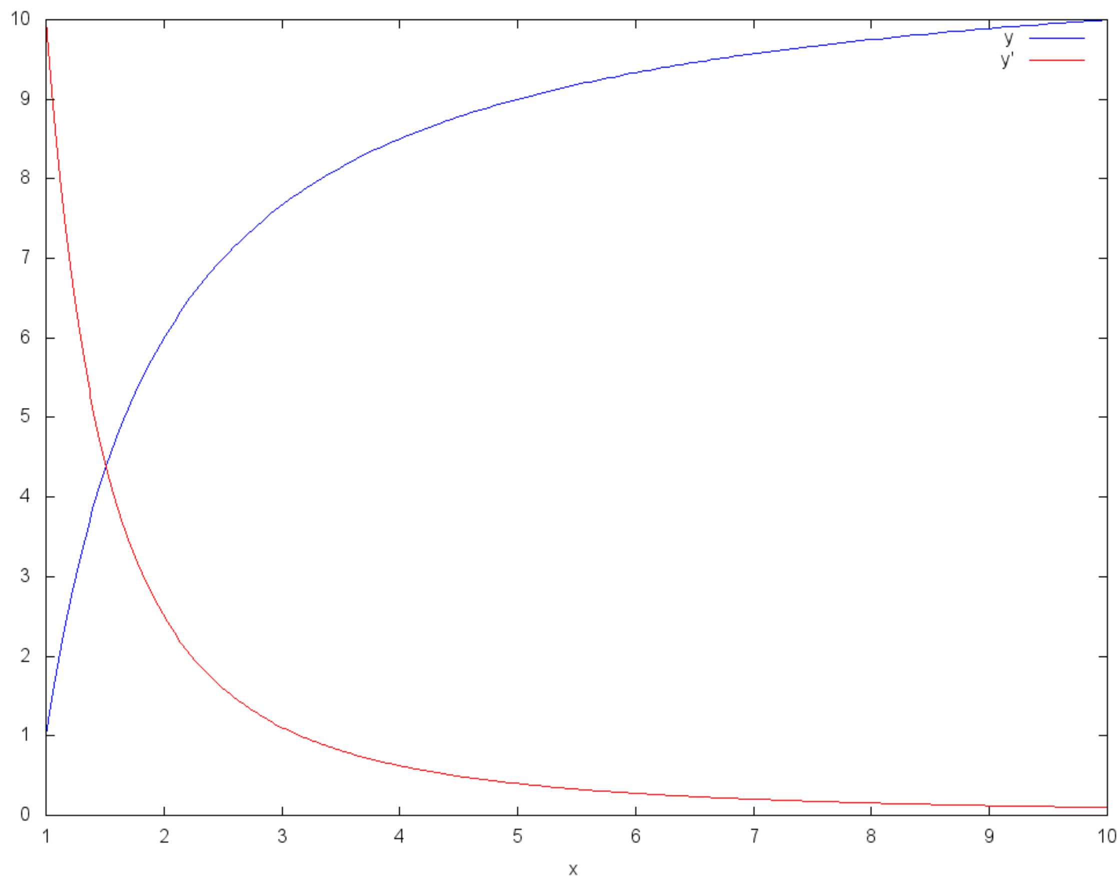
$$y = \frac{x_1 y_1 - x_2 y_2}{x_1 - x_2} - \frac{x_1 x_2 y_1 - x_1 x_2 y_2}{(x_1 - x_2)\, x} \qquad \text{(odesol)}$$

(%i16) `bb[2],odesol,eval,diff,expand,factor;`

$$\frac{2 x_1 x_2 y_2 - 2 x_1 x_2 y_1 + x_2 - x_1}{x_2 - x_1} = k_1 \qquad \text{(\%o16)}$$

Graphics

(%i17) `wxplot2d([y,diff(y,x)],[x,x_1,x_2],[ylabel,""], [legend,"y","y'"]),odesol,params$`



(%t17)

5.4

Find the extremals of the following functionals:

$$\int_{x_0}^{x_1} (y^2 + {y'}^2 - 2\,y\,\sin(x))\,\mathrm{d}x$$

(%i18) `kill(labels,x,y)$`

(%i1) `ζ:[x,y]$`

(%i2) `depends(y,x)$`

Lagrangian

(%i3) `L:y^2+'diff(y,x)^2-2*y*sin(x);`

$$(y_x)^2 + y^2 - 2\sin(x)y \qquad \text{(L)}$$

Momentum Conjugate

(%i4) `ldisplay(P:diff(L,'diff(y,x)))$`

$$P = 2\,(y_x) \qquad \text{(\%t4)}$$

Generalized Forces

(%i5) `ldisplay(F:fullratsimp(diff(L,x)))$`

$$F = 2\,(y_x)\,(y_{xx}) + (2y - 2\sin(x))\,(y_x) - 2\cos(x)y \qquad \text{(\%t5)}$$

Euler-Lagrange Equation

(%i6) `aa:el(L,y,x)$`

(%i7) `bb:ev(aa,eval,diff)$`

(%i8) `map(ldisp,bb:fullratsimp(bb))$`

$$2\,(y_{xx}) = 2y - 2\sin(x) \qquad \text{(\%t8)}$$

Solution

(%i9) `odesol:ode2(bb[1],y,x);`

$$y = \frac{\sin(x)}{2} + \%k1\,e^{x} + \%k2\,e^{-x} \qquad \text{(odesol)}$$

(%i10) `method;`

$$variationofparameters \qquad \text{(\%o10)}$$

(%i11) `declare([x_1,y_1,x_2,y_2],constant)$`

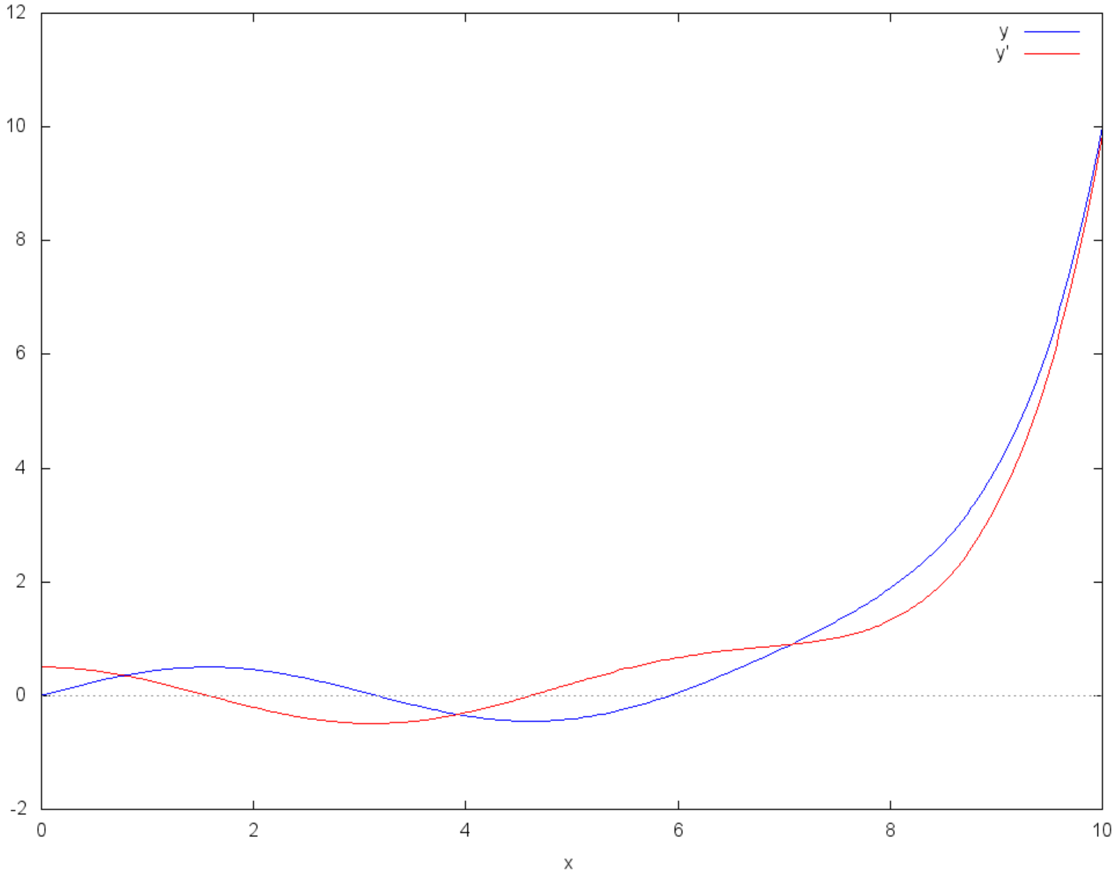
(%i12) `params:[x_1=0,y_1=0,x_2=10,y_2=10]$`

(%i13) `odesol:bc2(odesol,x=x_1,y=y_1,x=x_2,y=y_2);`

$$y = \frac{\sin(x)}{2} - \frac{\left(-e^{x_2}\sin(x_2) + e^{x_1}\sin(x_1) + 2e^{x_2}y_2 - 2e^{x_1}y_1\right)e^x}{2e^{2x_1} - 2e^{2x_2}} + \frac{\left(e^{2x_1}\left(2e^{x_2}y_2 - e^{x_2}\sin(x_2)\right) + e^{2x_2+x_1}\sin(x_1) - 2e^{2x_2+x_1}y_1\right)}{2e^{2x_1} - 2e^{2x_2}} \quad \text{(odesol)}$$

Graphics

(%i14) `wxplot2d([y,diff(y,x)],[x,x_1,x_2],[ylabel,""], [legend,"y","y'"]),odesol,params$`



(%t14)