Ecuación del movimiento de un sistema compuesto

Based on Física Hasta Que Te Aburras MECÁNICA LAGRANGIANA (EJERCICIO #4) [Ecuación del movimiento de un sistema compuesto]

Written by Daniel Volinski at danielvolinski@yahoo.es

(%i2)
```
info:build_info()$info@version;
```

(%o2)

5.38.1

(%i2)
```
reset()$kill(all)$
```

(%i1)
```
derivabbrev:true$
```

(%i2)
```
ratprint:false$
```

(%i3)
```
fpprintprec:5$
```

(%i4)
```
if get('draw,'version)=false then load(draw)$
```

(%i5)
```
wxplot_size:[1024,768]$
```

(%i6)
```
if get('optvar,'version)=false then load(optvar)$
```

(%i7)
```
if get('rkf45,'version)=false then load(rkf45)$
```

(%i8)
```
declare(trigsimp,evfun)$
```

(%i9)
```
declare(t,mainvar)$
```

1 Settings

(%i10)
```
orderless(m,g,K,l,ω)$
```

(%i11)
```
declare([m,g,K,l,l_0,ω],constant)$
```

(%i12)
```
assume(m>0,g>0,K>0,l>0,l_0,ω>0)$
```

(%i13)
```
params:[m=1.0,g=9.8,K=1.0,l=2.0,l_0=1.0,ω=0.1]$
```

(%i14)
```
τ:60$
```

Generalized coordinates

(%i15)
```
depends(θ,t)$
```

Geometry

(%i16)
```
r_1:[l*sin(θ)*cos(ω*t),l*sin(θ)*sin(ω*t),l*cos(θ)]$
```

(%i17)
```
v_1:diff(r_1,t)$
```

(%i18)
```
r_2:[l*sin(θ)*cos(ω*t+π),l*sin(θ)*sin(ω*t+π),l*cos(θ)]$
```

(%i19)
```
v_2:diff(r_2,t)$
```

2 Lagrangian Formalism

Kinetic Energy

(%i20) `ldisplay(T:trigsimp(`$\frac{1}{2}$`*m*v_1.v_1+`$\frac{1}{2}$`*m*v_2.v_2))$`

$$T = m\,l^2\,(\theta_t)^2 + m\,l^2\,\omega^2\,\sin(\theta)^2 \tag{\%t20}$$

Potential Energy

(%i21) `ldisplay(V:2*m*g*l*cos(`θ`)+`$\frac{1}{2}$`*K*(2*l*cos(`θ`)-l_0)`^{2}`)$`

$$V = \frac{K\,(2l\,\cos(\theta) - l_0)^2}{2} + 2mgl\,\cos(\theta) \tag{\%t21}$$

Lagrangian

(%i22) `ldisplay(L:expand(T-V))$`

$$L = m\,l^2\,(\theta_t)^2 + m\,l^2\,\omega^2\,\sin(\theta)^2 - 2K\,l^2\cos(\theta)^2 + 2Kl\,l_0\cos(\theta) - 2mgl\,\cos(\theta) - \frac{K\,{l_0}^2}{2} \tag{\%t22}$$

Momentum Conjugate

(%i23) `ldisplay(P:ev(diff(L,'diff(`θ`,t))))$`

$$P = 2m\,l^2\,(\theta_t) \tag{\%t23}$$

(%i24) `linsolve(p=P,diff(`θ`,t)),expand;`

$$\left[\theta_t = \frac{p}{2m\,l^2}\right] \tag{\%o24}$$

Generalized Forces

(%i25) `ldisplay(F:diff(L,`θ`))$`

$$F = 2m\,l^2\,\omega^2\,\cos(\theta)\,\sin(\theta) + 4K\,l^2\,\cos(\theta)\,\sin(\theta) - 2Kl\,l_0\sin(\theta) + 2mgl\,\sin(\theta) \tag{\%t25}$$

Euler-Lagrange Equation

(%i26) `aa:el(L,`θ`,t)$`

(%i28) `bb:ev(aa,eval,diff)$`

(%i29) `bb[1]:subst([k[0]=-E],-bb[1])$`

(%i30) `bb[2]:lhs(bb[2])-rhs(bb[2])=0$`

Conservation Laws

(%i31) `bb[1],expand,trigreduce;`

$$\frac{m\,l^2\,\omega^2\,\cos\left(2\theta\right)}{2}+K\,l^2\,\cos\left(2\theta\right)+m\,l^2\,\left(\theta_t\right)^2-2K\,l\,l_0\cos\left(\theta\right)+2mgl\,\cos\left(\theta\right)+\frac{K\,{l_0}^2}{2}-\frac{m\,l^2\,\omega^2}{2}+K\,l^2=E \quad (\%o31)$$

Equations of Motion

(%i32) `bb[2],expand,trigreduce;`

$$-m\,l^2\,\omega^2\,\sin\left(2\theta\right)-2K\,l^2\,\sin\left(2\theta\right)+2m\,l^2\,\left(\theta_{tt}\right)+2K\,l\,l_0\sin\left(\theta\right)-2mgl\,\sin\left(\theta\right)=0 \quad (\%o32)$$

Solve for second derivative of coordinates

(%i33) `linsol:linsolve(bb[2],diff(θ,t,2));`

$$\left[\theta_{tt}=\frac{\left(\left(ml\,\omega^2+2Kl\right)\,\cos\left(\theta\right)-K\,l_0+mg\right)\sin\left(\theta\right)}{ml}\right] \quad (\mathrm{linsol})$$

Check Conservation of Energy

(%i34) `lhs(bb[1]);`

$$m\,l^2\,\left(\theta_t\right)^2-m\,l^2\,\omega^2\sin\left(\theta\right)^2+2K\,l^2\cos\left(\theta\right)^2-2K\,l\,l_0\cos\left(\theta\right)+2mgl\,\cos\left(\theta\right)+\frac{K\,{l_0}^2}{2} \quad (\%o34)$$

(%i35) `subst(linsol,diff(lhs(bb[1]),t)),fullratsimp;`

$$0 \quad (\%o35)$$

Equilibrium points

(%i36) `bb[2],[diff(θ,t)=0,diff(θ,t,2)=0];`

$$-2m\,l^2\,\omega^2\,\cos\left(\theta\right)\,\sin\left(\theta\right)-4K\,l^2\,\cos\left(\theta\right)\,\sin\left(\theta\right)+2K\,l\,l_0\sin\left(\theta\right)-2mgl\,\sin\left(\theta\right)=0 \quad (\%o36)$$

(%i37) `sol:solve(%,cos(θ));`

$$\left[\cos\left(\theta\right)=\frac{K\,l_0-mg}{ml\,\omega^2+2Kl}\right] \quad (\mathrm{sol})$$

Small angles approximation

(%i38) `declare(θ_0,constant)$`

(%i39) `depends(ϵ,t)$`

(%i40) `ldisplay(Cθ:ratdisrep(taylor(cos(θ_0+ϵ),ϵ,0,1)))$`

$$C\theta=\cos\left(\theta_0\right)-\sin\left(\theta_0\right)\epsilon \quad (\%t40)$$

(%i41) `ldisplay(S`θ`:ratdisrep(taylor(sin(`θ`_0+`ϵ`),`ϵ`,0,1)))$`

$$S\theta = \cos(\theta_0)\epsilon + \sin(\theta_0) \tag{\%t41}$$

(%i42) `bb[2],[`θ`=`θ`_0+`ϵ`],diff,eval;`

$$-2m\,l^2\,\omega^2\,\cos(\epsilon+\theta_0)\sin(\epsilon+\theta_0)-4K\,l^2\,\cos(\epsilon+\theta_0)\sin(\epsilon+\theta_0)+2Kl\,l_0\sin(\epsilon+\theta_0)-2mgl\,\sin(\epsilon+\theta_0)+2m\,l^2\,(\epsilon_{tt})=0 \tag{\%o42}$$

(%i43) `%,[cos(`θ`_0+`ϵ`)=C`θ`,sin(`θ`_0+`ϵ`)=S`θ`],expand;`

$$2m\,l^2\,(\epsilon_{tt})+2m\,l^2\,\omega^2\,\cos(\theta_0)\sin(\theta_0)\epsilon^2+4K\,l^2\,\cos(\theta_0)\sin(\theta_0)\epsilon^2+2m\,l^2\,\omega^2\sin(\theta_0)^2\epsilon+4K\,l^2\sin(\theta_0)^2\epsilon-2m\,l^2\,\omega^2\cos(\theta_0)^2\epsilon-4K\,l \tag{\%o43}$$

(%i44) `%,[`ϵ^2`=0],trigsimp;`

$$l^2\,\left(2m\,(\epsilon_{tt})+K\,\left(\left(4\sin(\theta_0)^2-4\cos(\theta_0)^2\right)\epsilon-4\cos(\theta_0)\sin(\theta_0)\right)\right)+l\,\left(K\,(2l_0\cos(\theta_0)\epsilon+2l_0\sin(\theta_0))+mg\,(-2\cos(\theta_0)\epsilon-2\,\mathrm{s} \tag{\%o44}$$

3 Hamiltonian Formalism

Legendre Transformation

(%i45) `Legendre:linsolve([p=P],['diff(θ,t)]);`

$$\left[\theta_t = \frac{p}{2m\,l^2}\right] \tag{Legendre}$$

Hamiltonian

(%i46) `ldisplay(H:ev(p*'diff(θ,t)-L,Legendre))$`

$$H = -m\,l^2\,\omega^2 \sin(\theta)^2 + 2K\,l^2 \cos(\theta)^2 - 2Kl\,l_0 \cos(\theta) + 2mgl\,\cos(\theta) + \frac{p^2}{4m\,l^2} + \frac{K\,{l_0}^2}{2} \tag{\%t46}$$

Equations of Motion

(%i47) `Hq:makelist(Hq[i],i,1,2)$`

(%i49) `Hq[1]:'diff(θ,t)=diff(H,p)$ Hq[2]:'diff(p,t)=-diff(H,θ)$`

(%i50) `map(ldisp,Hq)$`

$$\theta_t = \frac{p}{2m\,l^2} \tag{\%t50}$$

$$p_t = 2m\,l^2\,\omega^2\,\cos(\theta)\,\sin(\theta) + 4K\,l^2\,\cos(\theta)\,\sin(\theta) - 2Kl\,l_0 \sin(\theta) + 2mgl\,\sin(\theta) \tag{\%t51}$$

4 Reduce Order

(%i52) `depends(Θ,t)$`

(%i53) `gradef(θ,t,Θ)$`

Euler-Lagrange Equations

(%i54) `aa:el(L,θ,t)$`

(%i56) `bb:ev(aa,eval,diff)$`

(%i57) `bb[1]:subst([k[0]=-E],-bb[1])$`

(%i58) `bb[2]:lhs(bb[2])-rhs(bb[2])=0$`

Conservation Laws

(%i59) `bb[1],expand;`

$$-m\,l^2\,\omega^2\sin\left(\theta\right)^2+2K\,l^2\cos\left(\theta\right)^2-2Kl\,l_0\cos\left(\theta\right)+2mgl\,\cos\left(\theta\right)+m\,l^2\,\Theta^2+\frac{K\,{l_0}^2}{2}=E \tag{\%o59}$$

Equations of Motion

(%i60) `bb[2],expand;`

$$-2m\,l^2\,\omega^2\,\cos\left(\theta\right)\,\sin\left(\theta\right)-4K\,l^2\,\cos\left(\theta\right)\,\sin\left(\theta\right)+2Kl\,l_0\sin\left(\theta\right)-2mgl\,\sin\left(\theta\right)+2m\,l^2\,\left(\Theta_t\right)=0 \tag{\%o60}$$

Solve for second derivative of coordinates

(%i61) `linsol:linsolve(bb[2],diff(θ,t,2));`

$$\left[\Theta_t=\frac{\left(\left(ml\,\omega^2+2Kl\right)\,\cos\left(\theta\right)-K\,l_0+mg\right)\sin\left(\theta\right)}{ml}\right] \tag{linsol}$$

Numerical solution (Lagrangian)

(%i62)
```
kill(labels)$
```

(%i8)
```
funcs:[θ,Θ]$ldisplay(funcs)$
initial:[1,0]$ldisplay(initial)$
odes:[Θ,rhs(linsol[1])]$ldisplay(odes)$
interval:[t,0,τ]$ldisplay(interval)$
```

$$funcs = [\theta, \Theta] \tag{\%t2}$$

$$initial = [1, 0] \tag{\%t4}$$

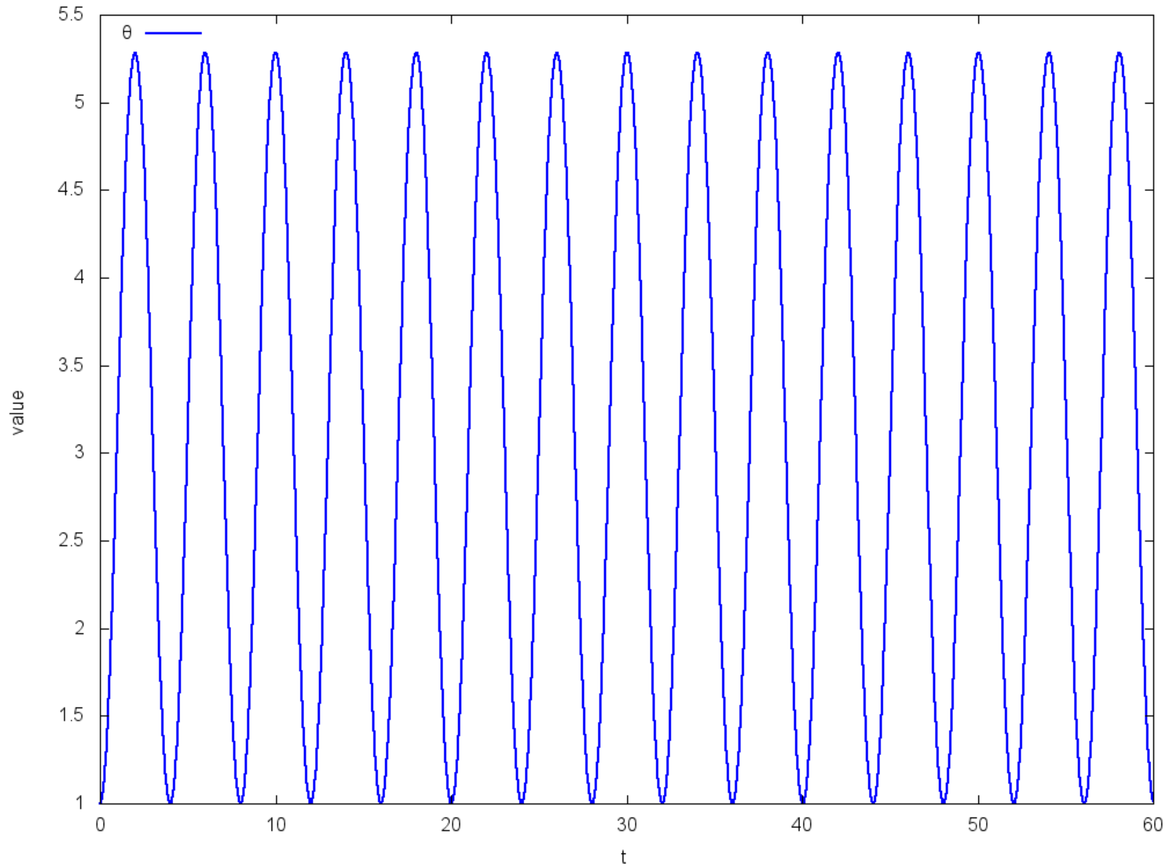
$$odes = \left[\Theta, \frac{\left(\left(ml\,\omega^2 + 2Kl\right)\, \cos\left(\theta\right) - K\, l_0 + mg\right) \sin\left(\theta\right)}{ml}\right] \tag{\%t6}$$

$$interval = [t, 0, 60] \tag{\%t8}$$

(%i9)
```
rksol:rkf45(odes,funcs,initial,interval, absolute_tolerance=1E-8,report=true),params$
```

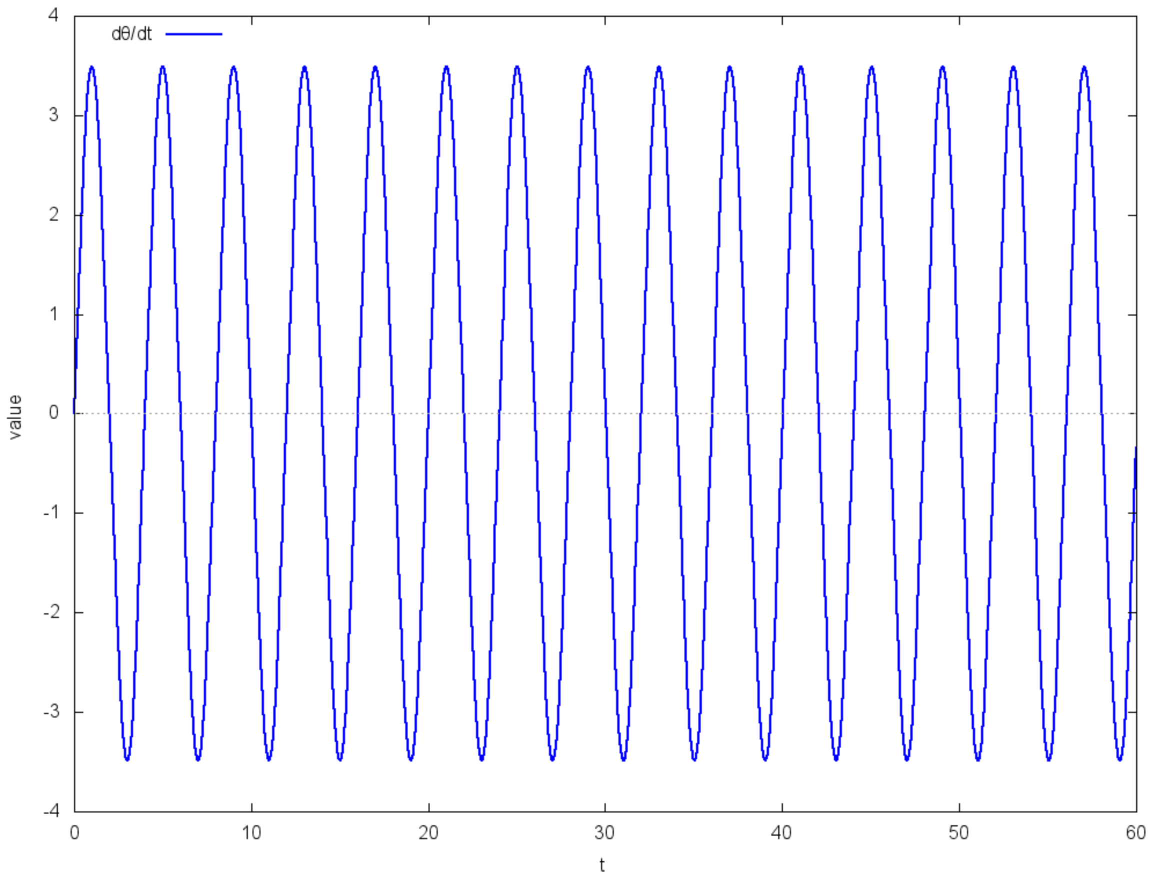
Info: rkf45:
Integration points selected:3106
Total number of iterations:3125
Bad steps corrected:20
Minimum estimated error:5.079110^{-10}
Maximum estimated error:9.917910^{-9}
Minimum integration step taken:0.012151
Maximum integration step taken:0.027203

(%i10) `wxplot2d([discrete,map(lambda([u],part(u,[1,2])),rksol)], [style,[lines,2]],[xlabel,"t"],[ylabel`
`[legend,"θ"],[gnuplot_preamble,"set key top left"])$`



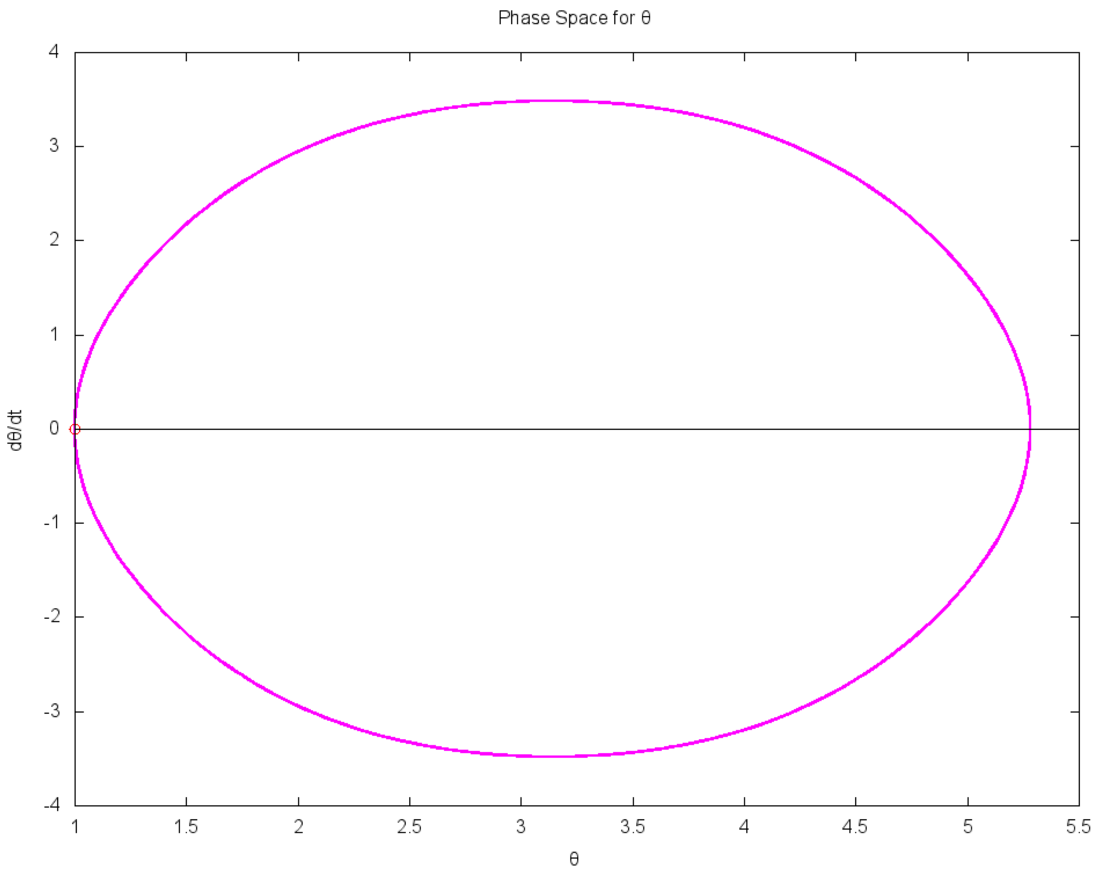
(%t10)

```
(%i11) wxplot2d([discrete,map(lambda([u],part(u,[1,3])),rksol)], [style,[lines,2]],[xlabel,"t"],[ylabel
       [legend,"dθ/dt"],[gnuplot_preamble,"set key top left"])$
```



(%t11)

(%i12) `wxplot2d([[discrete,map(lambda([u],part(u,[2,3])),rksol)], [discrete,[part(initial,[1,2])]]],[ax`
`[title,"Phase Space for θ"],[point_type,circle], [style,[lines,2],[points,3]],[color,magenta,red]`
`[xlabel,"θ"],[ylabel,"dθ/dt"],[legend,false])$`



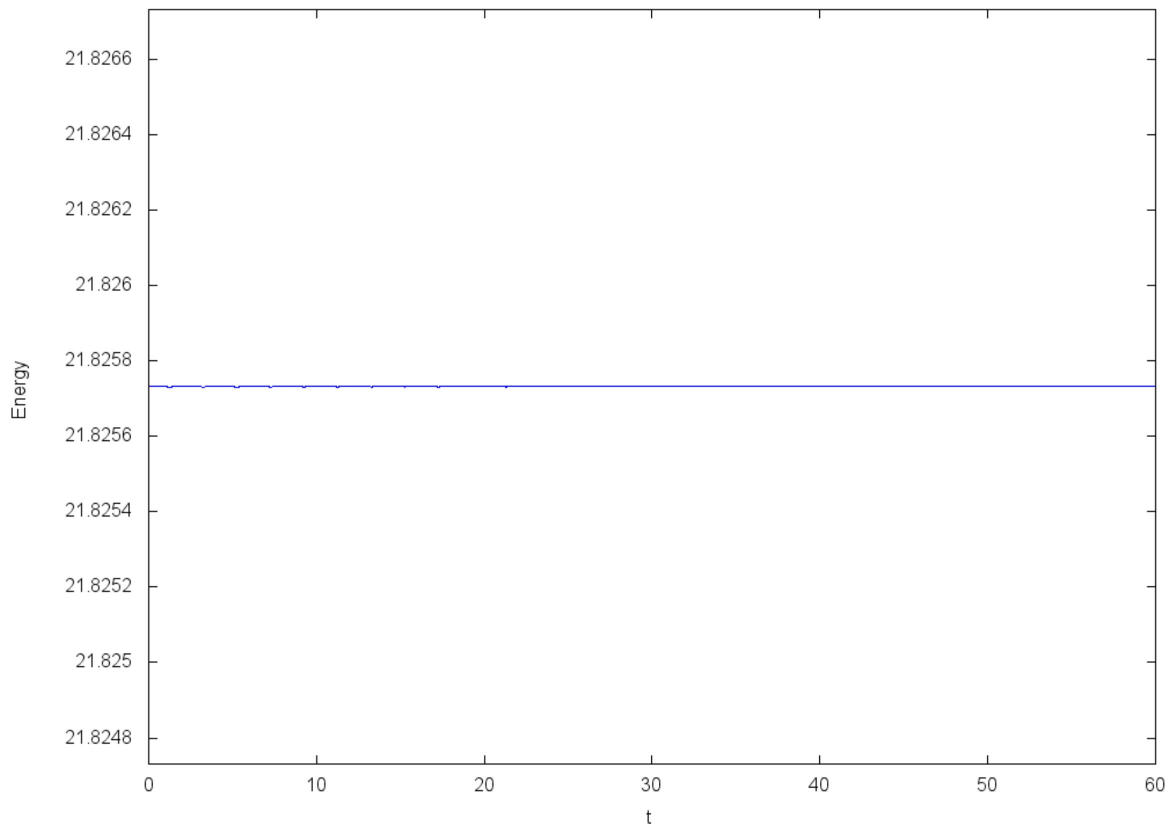
(%t12)

Check Conservation of Energy using the Numerical Data

(%i13)
```
W:bb[1],map("=",funcs,initial),params,numer,eval;
```

$$21.826 = E \tag{W}$$

(%i14)
```
wxplot2d([discrete,makelist([first(rkline), ev(lhs(bb[1]),map("=",funcs,rest(rkline)),params)],rk
[xlabel,"t"],[y,lhs(W)-0.001,lhs(W)+0.001],[ylabel,"Energy"])$
```



(%t14)

5 Graphics

(%i15) `kill(labels)$`

(%i1) `wxanimate_framerate:60$`

(%i2) `wxanimate_autoplay:false$`

(%i3) `rksol:rk(odes,funcs,initial,[t,0,τ/2,0.1]),params$`

(%i4)
```
set_draw_defaults(proportional_axes = xy, delay = 1, xtics = 1, ytics = 1, xrange =
[-1,1], yrange = [-1,1])$
```

Create animated GIF file

(%i5)
```
draw(terminal = 'animated_gif, file_name = "Sistema compuesto", makelist(gr2d(
color = red, point_type = filled_circle, point_size = 2, points_joined = true,
line_width = 2, key = sconcat("t=",float(t)/10," s"), points([[0.0,0.0],
[sin(rksol[t][2]),cos(rksol[t][2])]])), t,1,length(rksol))),params$
```

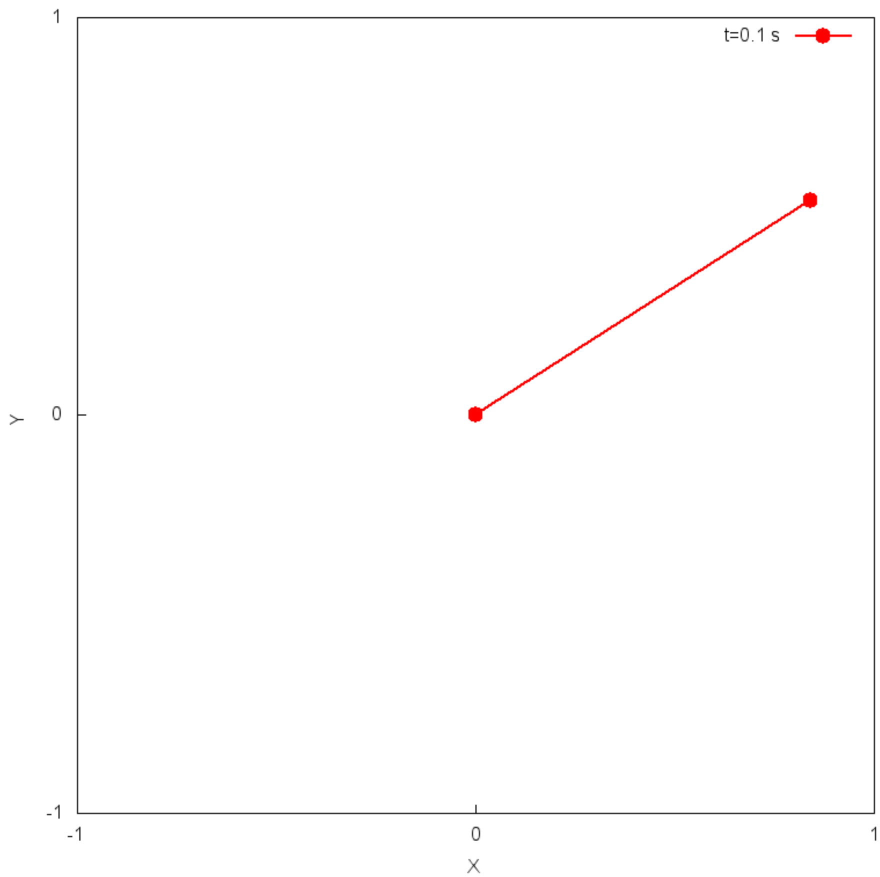
(%i6) `time(%);`

$$[0.031] \tag{\%o6}$$

(%i7) `wxanimate_framerate:30$`

(%i9)
```
print("Click the figure to start animation")$ with_slider_draw( t,makelist(i,i,1,length(rksol)),
color = red, point_type = filled_circle, point_size = 2, points_joined = true,
line_width = 2, key = sconcat("t=",float(t)/10," s"), points([[0.0,0.0],
[sin(rksol[t][2]),cos(rksol[t][2])]])),params$
```

Click the figure to start animation



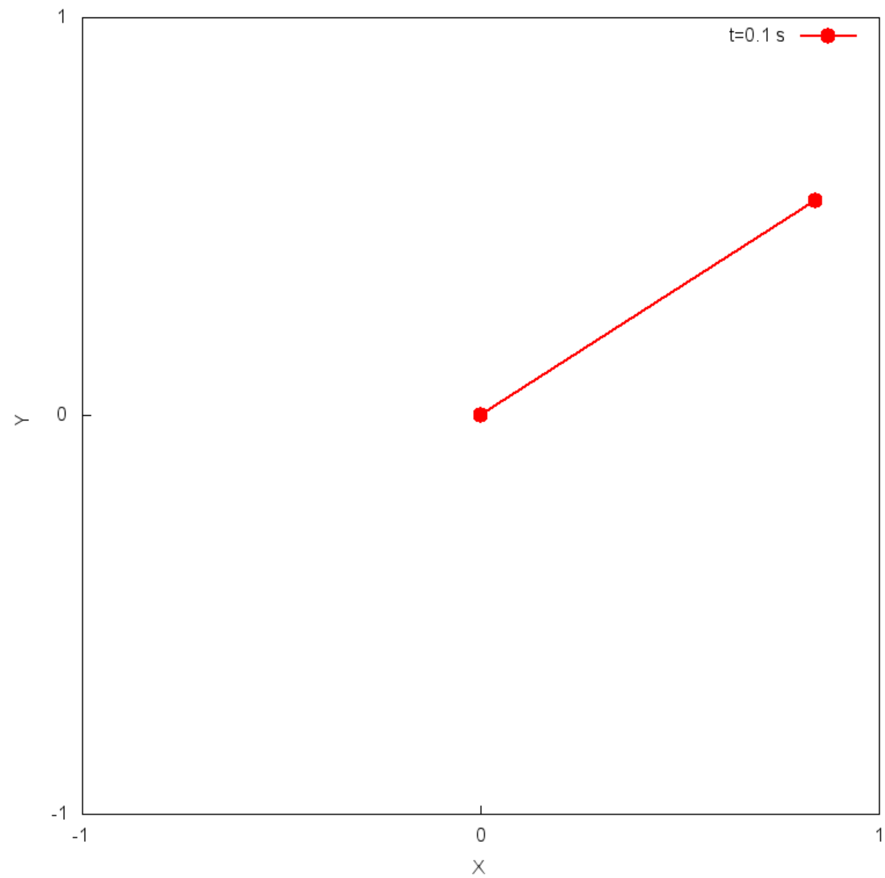
(%t9)

(%i10) `time(%);`

$$[0.797] \quad (\%o10)$$

(%i12) `print("Click the figure to start animation")$ wxanimate_draw( t,length(rksol), color = red, point_type = filled_circle, point_size = 2, points_joined = true, line_width = 2, key = sconcat("t=",float(t)/10," s"), points([[0.0,0.0], [sin(rksol[t][2]),cos(rksol[t][2])]])),params$`

Click the figure to start animation



(%t12)

(%i13) `time(%);`

$$[0.781] \tag{\%o13}$$