Geometric Algebra 9: Multivector Structure

Introduction

Welcome back, we are going to continue our study of Geometric algebra through the use of this paper and where we left off last time we were attacking the subject of multi-vectors and we had done mostly this, well we have basically done Section 3.3, I don't think we've really dug into their version of Section 3.1 yet so what we will do is we will begin today with a few errata I think that's going to be a habit here just a few small errata and then we will review and proceed so let's begin.

Double Dot Product

The first errata comes from my discussion of these common notations right I put in this commutator bracket here and that is by the way supposed to have meant a literal commutator bracket:

$$\begin{aligned} A\,B &= A:B + A\cdot B + A\wedge B \\ &= A\,.\,B + \langle A\,B\rangle_2 + A\wedge B \\ &= A\cdot B + [A,B] + A\wedge B \end{aligned} \tag{1}$$

That is supposed to be $[A,B]=A\,B-B\,A$, it's not notation for something else but there is supposed to be ½$[A,B]$ there and it's literally true that, I guess it's worth pointing out that if you do this calculation, you'll discover that everything except the grade two stuff drops out and it's actually worth having a look at and we can break this down:

$$\begin{aligned} A\,B &= (u\wedge v)(w\wedge s) \\ &= (v\cdot w)(u\cdot s) - (u\cdot w)(v\cdot s) \\ &\quad + (v\cdot w)(u\wedge s) - (u\cdot w)(v\wedge s) \\ &\quad - (v\cdot s)(u\wedge w) + (u\cdot s)(v\wedge w) \\ &\quad + u\wedge v\wedge w\wedge s \end{aligned} \tag{2}$$

We can execute this product, I showed how to do this a couple lessons ago about how to dig in and do the space-time product of two two blades or two bi-vectors and the grade zero part is these Minkowski contractions of every possible way of the two vectors that are closest to each other inside this space-time product, in this case v and w but we have to place each pair of vectors in there and sometimes we generate minus signs by flipping, that's just the mechanical way of doing this but when you do that you this is the what I'm calling the double dot product of A and B . There's this minus sign in here but this is a real number (2nd line in (2)) so this ends up being $\langle A\,B\rangle_0$ the grade zero part of $A\,B$. By the way the grade zero part of things sometimes they just ignore the zero they'll never ignore the ones twos and threes for higher bladed parts so the higher bladed parts will always have a number but the zero part may just be given without a number at all, if you don't see a number there, you assume it's the zero blade or the scalar part.

That's the scalar part and then you do the same process but stop halfway essentially and you get the bi-vector part so the bi-vector part (3rd and 4th lines in (2)), that's the grade two part of $A\,B$ and then the grade four part because remember you go up in increments of two so we start with $|p-q|-\ldots$ and then we go up in increments of two so in this case we have $2-2=0$ so we have a zero part and we go up to a two blade part and then we go to a four blade part and the four blade part is really simple (5[th] line in (2)). Now if we do this in reverse we get:

$$\begin{aligned} BA &= (w\wedge s)(u\wedge v) \\ &= (s\cdot u)(w\cdot v)-(w\cdot u)(s\cdot v) \\ &\quad +(s\cdot u)(w\wedge v)-(w\cdot u)(s\wedge v) \\ &\quad -(s\cdot v)(w\wedge u)+(w\cdot v)(s\wedge u) \\ &\quad +w\wedge s\wedge u\wedge v \end{aligned} \tag{3}$$

If you do this calculation first of all well it this part here (2[nd] line in (3)), if you look that's the same as this (2[nd] line in (2)), the order of this Minkowski dot product doesn't matter which is fine and the fact that the two are switched doesn't matter because real numbers commute so this grade zero part is actually the same and that makes some sense because the grade zero part is what we're going to be calling the magnitude of these things. The grade zero part if it was AA , the grade zero part would be the magnitude. Turns out these two, the grade two part is off by a $-$ where we flip everything around but flipping around the dot product doesn't matter, flipping around the wedge product generates this $-$ sign. Then the last term, the wedge product term, the grade four part, this grade four part (5[th] line in (3)) and this part (5[th] line in (2)) are actually the same in other words I can just write this part are equals.

The point being when we take these two and subtract $AB-BA$, the scalar parts are the same and the quad-vector part is the same so those go away so all that's left is twice the grade two part so we pull the two underneath and that's the two I forgot in the errata and what's left is just the bi-vector part of the product of A and B because we know that AB is the bi-vector part and when you subtract $AB-BA$ you get twice this bi-vector part and ergo this is a very neat way of writing just the bi-vector part which is why this notation is one of the things chosen it's a combination of an instruction of how to get the bi-vector part as well as just simple nice notation that's familiar.

$$AB-BA=2\left[(v\cdot w)(u\wedge s)-(u\cdot w)(v\wedge s)-(v\cdot s)(u\wedge w)+(u\cdot s)(v\wedge w)\right] \tag{4}$$

This is just a description, this is the bi-vector part with no instruction this gives you a little bit of instruction but you still have to know the mechanical procedure that is used here, the one that I taught a couple lessons ago. This is my new favorite notation which is, I should write down this notation:

$$\tfrac{1}{2}[A,B]=\tfrac{1}{2}(AB-BA)=\langle AB\rangle_2\equiv A\times B \tag{5}$$

That equals this bi-vector part but it turns out some authors actually like to create a new cross product form of this and I have seen this in a few pages in the past but this cross being a little confused regular cross product from standard vector analysis it shouldn't be because $A\in\Lambda_2(\mathsf{M}_{1,3})$, it's not a vector it's a bi-vector so again, using the language, this operator $\times$ is overloaded and you should actually look at what it's operating on, if it's operating between two bi-vectors it's defined this way (5), if it's operating between two, three-dimensional spatial vectors right from standard Physics, well then it in the standard way of cross products do. I'm not too alarmed by that usage but they do like to simplify, I prefer this notation because it literally tells you how to do it so I don't think this is very useful at my stage of understanding of all this material. that's the first errata, let me look at the second one.

Spacetime Product

Second errata happened when I was expounding on this beautiful notation where you basically are taking these bi-vectors γ_1 , γ_2 , you're talking about the space-time product and you realize hey if they're orthogonal, the space-time product is equal to the wedge product so we can write down the wedge product just by writing down the two orthogonal vectors next to each other and applying the space-time product, as long as they're orthogonal.

$$\begin{aligned}\gamma_0\gamma_3=\gamma_0\wedge\gamma_3=\gamma_{03}\\\gamma_1\gamma_2=\gamma_1\wedge\gamma_2=\gamma_{12}\\\gamma_1\gamma_3=\gamma_1\wedge\gamma_3=\gamma_{13}\\\gamma_2\gamma_3=\gamma_2\wedge\gamma_3=\gamma_{23}\end{aligned} \tag{6}$$

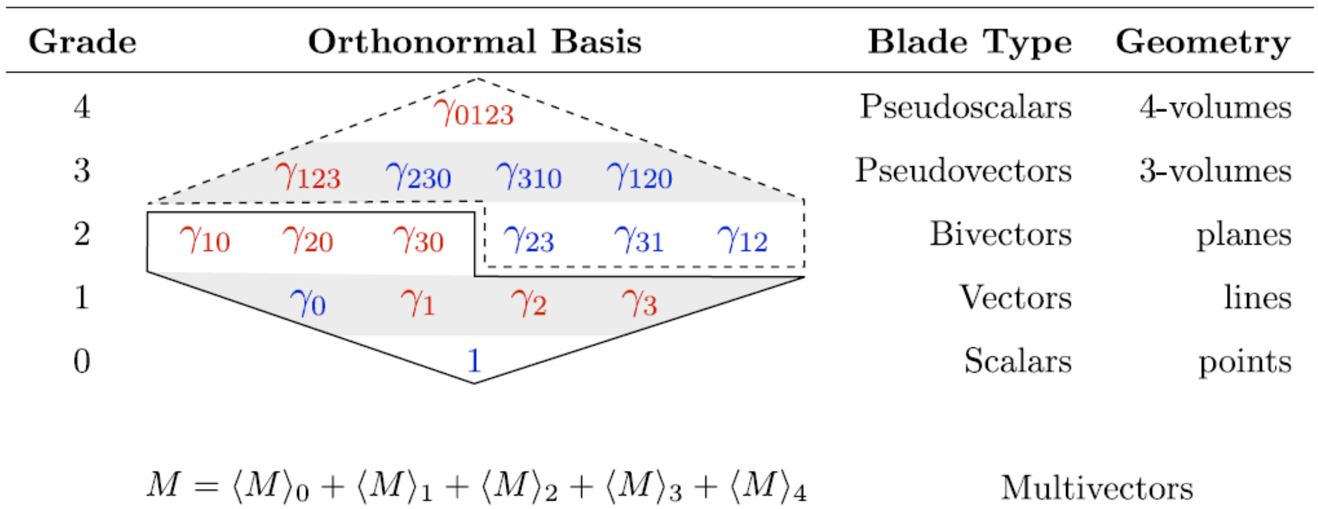
I can even squeeze that down and I don't have to write two γ , I could write one γ . I'm trying to be as like is few pen strokes as possible in this stuff and when I went down to that I was demonstrating this for a tri-vector and I showed $\gamma_0\gamma_1\gamma_2=\gamma_0(\gamma_1\wedge\gamma_2)$ because $\gamma_1\gamma_2$ as I just showed is the simple blade this $\gamma_1\wedge\gamma_2$ two blade and then I said well the scalar part of this is zero, we know this is going to be a scalar part and a tri-vector part and the scalar part is zero and I started writing down the scalar part and if you notice I got this backwards I wrote $(\gamma_0\cdot\gamma_2)\gamma_1-(\gamma_0\cdot\gamma_1)\gamma_2$ but the formula is this, it's the space-time product of a vector with a bi-vector:

$$a(b\wedge c)=(a\cdot b)c-(a\cdot c)b \tag{7}$$

You're changing the sign of that of this first vector and I actually was writing it that way and I confused myself and I switched it so this line here should read $(\gamma_0\cdot\gamma_1)\gamma_2-(\gamma_0\cdot\gamma_2)\gamma_1$. This is a really important formula (7), the more I've reviewed this material the more I realize this is one formula that's actually worth keeping in your hip pocket, the fact that the space-time product of a vector with a bi-vector the vector part of it is that projection of the vector a in the plane formed by bc and then rotated by 90º in the direction from b to c so that's how this orientation works in and that rotation is this $-$. We haven't spoken about arbitrary rotations yet, that's a big subject and I definitely want to treat it very carefully when we when we need it and so we'll get there soon but that's the second errata. This time we got away with two relatively small erratas, this first one isn't so much an errata, well I guess it's an errata because I forgot the -2 so that's definitely an errata but this notion of this new notation expounding on that is actually pretty interesting. Let's review now and move on.

Spacetime algebra

Where we left ourselves is we have the universe of the space-time algebra as presented by our authors in this beautifully tight form and what we're capable of understanding is we understand what all of this notation is we understand these are all abbreviations for space-time products when you have two or more indices you're looking at abbreviations for space-time products and so we now know that when say we deal with γ_{12} , we know that's γ_1 space-time product γ_2 and we've established that these vectors live in $M_{1,3}$ and are orthogonal with respect to the Minkowski metric which is our assumption, we've decided that we are going to be in a reference frame where this γ_0 represents the time-like a basis vector we know that in that circumstance this is $\gamma_1\gamma_2=\gamma_1\wedge\gamma_2$. It is a simple bi-vector and that's how this notation becomes so useful but it's all predicated on the fact that these vectors γ_0 , γ_1 , γ_2 , γ_3 satisfy this Minkowski metric $\eta(\gamma_\mu, \gamma_\nu)$ that has to equal $(1,-1,-1,-1)^T$ or zero depending on the relationship between μ and ν , $-1,-1,-1$ are the space-like guys and $+1$ is the time-like guy and this is when they're all zero.



That's how far we got with that and then we looked at this figure, we started foreshadowing what all of these things mean here because you have for example this dotted line which is different than this solid line and then you have some things that are red in color some things that are blue in color and you have this shaded area and these non-shaded areas and this is all relevant, this all means something in this this story and we alluded to the fact that this all has to do with highlighting the critical way this paper operates which is through the duality of various of these basis vectors and that duality can be looked at in several different ways, the way they talk about it here, there's there's two dualities, the Hodge duality is between this solid area in this dotted area so going from in here to in here that's the Hodge duality which if you study the "what is a tensor" section is really familiar but if not we're going to describe it here and it's beautifully simple in this architecture.

I wrote the Hodge duality a little bit more specifically than just the solid into the dotted, I specified that the Hodge duality took the scalars to the pseudo-scalars, took the vectors to the tri-vectors and took the bi-vectors to the bi-vectors, the bi-vectors are the Hodge dual of themselves. The way this captioning looks at it in terms of the Hodge duality just takes all of these guys down and moves them up here. I was being a little bit more specific and we'll see how that works but then there's also another duality, I don't know if they give it a name I seem to remember they do give it a name eventually but the other duality is a duality from, it's basically a duality from the shaded to the non-shaded within the blacks so the duality is from grade one into grade two. I think in the last lesson I made the duality from grade one to grade three, from shaded to shaded and that's true because these guys here γ_1 , γ_2 , γ_3 are Hodge dual to those guys γ_{230} , γ_{310} , γ_{120} but that's not what they mean by the shaded region, what they mean by the shaded region is, if I take everything in this shaded region and multiply by γ_0 on the right I end up in this non-shaded region here γ_{10} , γ_{20} , γ_{30} and this guy γ_0 ends up down here 1 , because $\gamma_0 \gamma_0 = 1$. Just to be clear $\gamma_0 \gamma_0 = 1$ because the space-time product of $\gamma_0 \gamma_0 = \gamma_0 \cdot \gamma_0 + \gamma_0 \cdot \gamma_0$ and in this case we're using the opposite of the notation that made γ_{12} simple, it's the wedge product that's zero when you wedge the same vector with itself and the dot product is non-zero because it's the Minkowski contraction of a vector with itself and in this case because it's the space-like one its contraction $\eta(\gamma_0,\gamma_0)=1$ so that's how you go from grade one to grade zero so the point is you have the shaded moves into the non-shaded under a certain duality, multiplication by a γ_0 . Likewise up here you go from grade two to grade three by multiplying by γ_0 .

We're getting ahead of ourselves anyway but the point is the reason I'm even dealing with this at all right now at this early stage is because I want you to see that this chart is designed to really tease out the relationships between things and as we learn about these duality relationships this chart is there to keep that in the forefront of your mind and really really help you remember these things and it's a brilliant chart and there's other charts in this thing that are equally as brilliant and I wish this was the first thing I rolled up on when I studied the subject that's one of the reasons I chose this paper is because I love the way they've organized everything for us visually and intellectually so any gaps that I suggest that are in this paper are simply because the paper is not a book, it's got to move through things and it knows that because it references other very famous books on the subject so that's where we stand right now and now we can move on with reading the paper all right beginning at the next paragraph.

Pseudo-scalars

“The four blade γ_{0123} geometrically signifies the same unit four volume in any basis and will be particularly important in what follows so we give it a special notation”.

$$I\equiv\gamma_{0123} \tag{8}$$

“This algebraic element is the *pseudo-scalar* for the space-time” so this is a big step this is an important paragraph, the geometric significance of the four blade is of a unit four volume, I think we understand that so this is a little unit hyper volume and it signifies the same uniform four volume in any basis and it will be particularly important what follows. Well the idea that is that this is a one-dimensional object there's only one way to create this four volume and so regardless of what basis you're in we're always going to end up constructing this pseudo-scalar the same way and it's this way and they call it the pseudo-scalar for the space-time so the reason it's a scalar is because the sub-space of four blades is one dimensional so it's one dimensional like a scalar they call it a pseudo-scalar because $\gamma_0\gamma_1\gamma_2\gamma_3$ will change sign if you flip the order of these basis vectors in this space-time product and this is why it's pseudo. In regular non-geometric algebra applications of Physics there are these weird things that are lesser known because we spend a lot of time with vectors so we know that there's a vector which is Axial and there's a vector which is Polar and it's these Polar vectors that we call pseudo-vectors but it's also true that there are and they we know that they switch direction under switching changing of handedness of the space-time, it turns out that there are scalars which do not change their value under a change of handedness of space-time but who knew there's also pseudo-scalars which do change their value if you flip the handedness of space-time.

The point is is in the past this has always been an ad hoc notion just like polar and axial vectors have been an ad hoc notion, you say well there's apparently there's two kinds of little pointy things out there those that do and those that don't change sign under a change of handedness and we just had to add something to our mathematical architecture to compensate for that. Now what we're doing here is we no longer have to do that because we are creating an object that literally does change its sign when you flip the orientation of space-time and let's make sure that's really clear on how that works.

Blowup

Let's start with this notation, our compressed notation γ_{0123} and we realize right away, we're going to blow it up, in our minds at least, step one of the blow up is to just show that it represents a space-time product in order of the four space-time basis vectors then we remember of course that we're dealing with vectors that are orthogonal according to the Minkowski metric and this means all dot products between these things will disappear except perhaps dot products between things and themselves and ultimately this completely cleans up to just this one four blade:

$$\gamma_{0123}=\gamma_0\gamma_1\gamma_2\gamma_3=\gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3 \tag{9}$$

We have to remember that for these vectors when you see space-time products, you can just replace them in your head with these blades by systematically going through the process of working this all out and calculating what's left behind for the zero blade, one blade, two blade, three blade and you realize they all go away and the only thing that's left is this four blade and once you have that four blade you quickly realize that now is the basis vector, this is now the basis vector for the $\Lambda_4(\mathsf{M}_{1,3})$ sub-space, it's got one dimension it's got one basis vector and it's a real vector space and this one basis vector is actually γ_{0123} now. We've compressed it in these two stages, we've compressed it, this is the obvious basis vector that we know (9), we've compressed it to this because we understand this property of this orthogonality and then we've compressed it to this because we are tired of writing too many γ matrices the only thing that matters is the order and the order is preserved in the index so because it's a one-dimensional object any member of this vector space is some real number a times γ_{0123} and now the point is $a\gamma_{0123}$ is our pseudo-scalar whereas a member of $\Lambda_0(\mathsf{M}_{1,3})$ is just a regular old real number in $\mathbb{R}$ or it's a number b times the basis vector 1 so now we've got these two, we've got these scalars and we've got these pseudo-scalars.

This is clearly part of our architecture, it's two separate kinds of numbers and if we exchange, if we say I don't like my right-handed system, I want to change this from to $\gamma_0\gamma_1\gamma_3\gamma_2$, I want to flip the orientation of those so all of a sudden I have now a left-handed system well if I flip 3 and 2 well then I have to introduce this $-$ because that's how blades that's how wedge products work so I have to introduce a $-$ everywhere and so boom, a $-$ appears right here and so when you flip these things you get that $-$ automatically as part of the process of the flip, of course no $-$ appears here $b(1)$ and so this is the reason I'm harping on this is because this is an example or probably the most prime example of why Geometric algebra really does have a claim to being the correct algebra which is not to say a better algebra or a better bookkeeping method but actually something that's correct because when you see this notion of ad hoc which I call the a dig, it's a dig because not because people are just being mean it's a dig because anything see any theory that has something that's thrown in there ad hoc has a weakness, if you find a theory where this ad hoc object has a natural place where it doesn't have to be inserted by hand you're definitely making progress or that's the conventional thought.

The last time we in this series saw something that was ad hoc was the Spinor formalism of Quantum mechanics where we try to introduce spin ad hoc by introducing a Spinor into a wave function so we would have some wave function then we just tack on this Spinor and then all of a sudden we have a way of describing spin one-half particles and it works great and it's not completely clear that this isn't necessarily an ad hoc edition it's a discovery we've discovered that we need to add this piece so it doesn't look ad hoc, it's just you realize well we had this great theory of a wave function but apparently we need this other thing to capture spin so you throw it in there and it works and then people say well is that ad hoc or have you just discovered something new, likewise you could say the same thing about pseudo-scalars, it's like, hey guys, there's another thing it changes sign, let's keep track of that and call it a pseudo-scalar, it changes sign when you flip the handedness, have you discovered pseudo-scalars or are you throwing in something ad hoc because there's a more broader theory which captures it naturally so I don't know the answer to that, I mean clearly in the case of spinners it took Dirac to come around with the Dirac equation which does very naturally in the context of Special relativity captures spin and all of a sudden this now looks ad hoc once you have Dirac theory then you say this is totally ad hoc and I think a similar thing happens here, pseudo vectors and pseudo-scalars have been around forever but it isn't until you see this structure of Geometric algebra that you realize this is why pseudo-scalars are a thing because they're not scalars with a special property, they actually live in a different space and they behave a little like scalars but significantly not ergo we're going to keep this old word pseudo-scalars and we're going to call this stuff pseudo-scalars but you could give it a totally different word.

Magnetic charge in Physics is supposedly a pseudo-scalar quantity that has physical reality, if magnetic charges existed it would be a pseudo-scalar so this question of ad hoc-ness is a little bit awkward because nothing is ad hoc until you suddenly have a better theory that encompasses it and in the case of the Dirac equation that better theory just fell out of thin air because without Dirac being there to do it I don't know if we'd even have it today but in Geometric algebra it was more of a realization that this Geometric algebra takes care of this question of having to label things as pseudo-scalars or scalars you no longer have to label things, you just have to track where it lives in the algebra and you could argue, I suppose, that that's a labeling under a different guide, I suppose, but regardless that's the significance of of this and it's so significant that we're even going to take an additional level of notation compression.

This thing here (8) is now officially I and that is like we're so committed to the significance of the pseudo-scalar that we're just going to assume that you remember that it is going to be γ_{0123} so that's almost like a new level of notation compression because γ_{0123} all the data is there, you do have to track everything I just described about this breakdown, going through (9). You do have to be tracking that but when you just write it as an I that's basically saying not only are we counting on you to track it but we're just going to give it a simple symbol and you just better keep you better hold on to your hat because that symbol is going to follow us around and you need to work with it very very well.

Let's go on, “that is, its scalar multiples αI act as scalars that flip sign under an inversion of the handedness of space-time”, which I just spoke about, I guess I rambled ahead of myself a little bit but that's important, we're seeing it again so it “flip side under the inversion of the handedness” that's because I will flip sign if you change the handedness. It's not like we actually do that a lot by the way, once we set this thing up you don't change the handedness of stuff but by carrying it around all of those properties that would exist for something whose handedness does change follows you. “The notation of I is motivated by the fact that $I^2=\gamma_0^2\gamma_1^2\gamma_2^2\gamma_3^2=(+1)(-1)(-1)(-1)=-1$ so $I^2=-1$ will perform a similar function to the scalar imaginary i ”, we definitely need to spend time on that so let's move over.

$$\begin{aligned}I^2=II&=\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3=-\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_3\gamma_2\\&=+\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_3\gamma_1\gamma_2=-\gamma_0\gamma_1\gamma_2(\gamma_3)^2\gamma_0\gamma_1\gamma_2\\&=-(\gamma_3)^2\gamma_0\gamma_1(\gamma_2)^2\gamma_0\gamma_1=+(\gamma_2)^2(\gamma_3)^2\gamma_0(\gamma_1)^2\gamma_0\\&=(\gamma_1)^2(\gamma_2)^2(\gamma_3)^2(\gamma_0)^2=(-1)(-1)(-1)(+1)=-1\end{aligned}\tag{10}$$

Here is the full breakdown of the statement that the base of pseudo-vector squares to -1 . First of all it's a space-time squaring so we write it as II and then because we have this all cached in our head we know that we can blow it up into this large space-time product and we know that this large space-time product is associative so I can look at this in any grouping I want, I can group these two together and or those two together and do the space-time product on those two before doing it on the rest so I'm going to take advantage of that and I'm going to look at the space-time product of just these last two and I could have picked really any two but because I know that $\gamma_2\gamma_3=\gamma_2\wedge\gamma_3=-\gamma_3\wedge\gamma_2$ because that's the property of the wedge product, I can write in general that $\gamma_{23}=-\gamma_{32}$. This process doesn't depend on what these numbers are, it's only that they're different, 2 has to be different from 3, you can't have $\gamma_2\gamma_2$ and go through this logic because as soon as you do $\gamma_2\gamma_2=(\gamma_2)^2$ which is in our system is going to be -1 , we will use this fact of course in the end but when they're different you end up with this wedge product and this anti-symmetry.

The point is I can replace $\gamma_2\gamma_3$ with $\gamma_3\gamma_2$ as long as I introduce a $-$ up front and then I do it again I now look at $\gamma_1\gamma_3$ and I interchange those two and I lose the $-$ because I've introduced yet another $-$ from flipping and then I do it again, I look at $\gamma_0\gamma_3$ and when I do $\gamma_0\gamma_3$ the first thing is I reintroduce a $-$ but now I have γ_3 next to γ_3 so I just write that as $(\gamma_3)^2$ but the square, the space-time square of any vector, any member of $M_{1,3}$, any one blade is a real number so I can take that real number and just pull it out in the front by linearity of this whole process so there it is, $(\gamma_3)^2$ comes out in front (3rd line of (10)). I take this process and I work on these two $\gamma_1\gamma_2$ and I move and I flip them and I flipped it twice I move those I moved the two over to here and then I flip the two and the zero until the two is next to this two and I have 4th line of (10) with $(\gamma_2)^2$ which is a number so I pull and that's two flips so there's no change of this sign this sign follows because, I've flipped γ_2 twice. I'm doing a little faster now here I demonstrated each flip now I'm going to just rush through it but so now I take γ_2 out front and then I flip these, I flip $\gamma_0\gamma_1$ so I have $(\gamma_1)^2$ so that flip is a single flip so now I do lose my $-$ it becomes a $+$. What's left behind is $(\gamma_0)^2$ and of course these are easy to calculate.

Notice how critical the Minkowski metric is here this $+1$ is what allows I^2 to actually square to -1 if this was a Euclidean metric these would all have been $+1$ and this would not have squared to -1 so that's an interesting fact, see that metric matters a lot, the flipping matters and the flipping depends on the dimensionality, I have four dimensions so I have basically this is a string of eight things, well if the dimensionality was say 5 or 11 the amount of flips that I would take in this process of $-$ switching to $+$ would be different so the dimensionality of the space matters but also the metric matters in deciding how I squared actually behaves so that's interesting and each dimensionality space is going to behave a little bit differently and spaces of odd and even dimensionality are going to have certain similarities but those similarities could be broken if you start changing the metric anyway I just want to make it clear that this process of going from I^2 is not quite as simple as the paper makes it look by just writing it this way, this is correct, they have $(\gamma_2)^2$ at the far left, I had it at the far right but at that point these are all real numbers so that doesn't matter.

Then they say we now know that the pseudo-scalar squared is going to have this -1 so the pseudo-scalar will perform a similar function to the scalar imaginary i so that remains to be seen but clearly anything that squares to -1 is going to have the properties of the complex imaginary number, the imaginary number i from standard complex analysis indeed the element i provides space-time with an intrinsic complex structure without any ad hoc introduction of the scalar complex numbers so what they're saying is that anytime we've done Physics, Electromagnetism in particular, you always see the imaginary number pop up for in a variety of ways and it's definitely there it's definitely shows up and wave propagation and calculation of impedance, it's always there and the introduction of complex numbers to students is like why is there all of a sudden an i here, in Quantum mechanics it shows up a lot, you see this i suddenly in the exponent of generators and it just pops up everywhere. Geometric algebra promises to explain why that i shows up.

Now I've explained why the i shows up without Geometric algebra for Quantum mechanics. I have a good feeling of why you need that i there but it took a lot of math to figure out exactly why that i shows up in Quantum mechanics and likewise for Electromagnetism when you start seeing i there if you really really break it down you see how i does the bookkeeping you needed to do. Geometric algebra promises … we've got it nailed down, there's no i show up, what shows up is vectors part of the Clifford algebra which is an inherent part of what space-time is because ultimately that is what this claim is, if I understand the claim correctly, that if you look at this thing (chart), we've always thought of space-time as just this stuff (lower part of chart) and he's saying no, you're wrong space-time is this whole thing and the lower part of chart is the part that we have good intuition of because we perceive in three dimensions and we can struggle to imagine time as a 4th dimension so even this part (lower) our brains can't really wrap around this stuff, this is all of normal Special relativity but we're missing these things in our thinking (upper) and we need to expand our notion of what space-time is from a bunch of four vectors doing their thing to this big architecture and once we do that complex numbers they live right there or some of them do right and I say some of them because there's other things here that multiply to -1 , let me give you an example let's look at say $(\gamma_{23})^2$ so that's $\gamma_2\gamma_3\gamma_2\gamma_3$, I want to do exactly what I did a moment ago so space-time product. I make one flip so I get:

$$(\gamma_{23})^2=\gamma_2\gamma_3\gamma_2\gamma_3=-\gamma_2\gamma_2\gamma_3\gamma_3=-(-1)(-1)=-1 \tag{11}$$

Because $(\gamma_2)^2=(\gamma_3)^2=-1$ but then I have that sign which is -1 so that equals -1 so $(\gamma_{23})^2=-1$ so the fact that this square to -1 isn't unique because I showed you that $I^2=-1$ but what about this guy:

$$(\gamma_{10})^2=\gamma_1\gamma_0\gamma_1\gamma_0=-\gamma_1\gamma_1\gamma_0\gamma_0=-(\gamma_1^2)(\gamma_0)^2=-(-1)(+1)=+1 \tag{12}$$

I make one flip but now I'm in trouble, this guy γ_{10} does not square to -1 and in fact that's true for all three of these γ_{10} , γ_{20} , γ_{30} , those three square to $+1$ and these three γ_{23} , γ_{31} , γ_{12} square to -1 but that's still significant because I've found three new objects that square to -1 so there's three so if I want to replace i , the complex number i , I've got a choice, I can choose and so I have to figure out well when i shows up in classical Electromagnetism which i is it? Which of these things are actually in there that is supposed to be the i because I have a bunch of choices to claim that any i you see in your classical Physics I have many choices to say if I claim that i has geometrical significance beyond what is represented in classical Electromagnetism, I have to know which of these it is so that's a challenge that this paper is going to have to deliver on and furthermore though don't walk away from this guy γ_{0123} , don't walk away from these three γ_{10} , γ_{20} , γ_{30} , they square to $+1$ that's important because what is $\sqrt{+1}$? Well, it is $+1$ but now I'm showing you that, wait a minute no, $+1$ has more than one square root, the square root of one is also γ_{10} , it's also γ_{20} and γ_{30} , these guys are all $\sqrt{+1}$ but in our world there's only one $\sqrt{+1}$, now all of a sudden there's four $\sqrt{+1}$, that's significant too in fact it's really just as significant as this, the fact that -1 has any square roots at all is amazing but the fact that it has four is like four times as amazing well, the fact that $+1$ has a square root doesn't amaze us but the fact that it has four square roots should amaze us. Now we do know that in complex numbers you have the square roots of unity or you have the roots of unity so the idea that unity would have a lot of different roots is not unfamiliar but that's not the same thing as this because when it comes to the square roots of unity, you have this complex number and you realize there's one you have all of these different complex numbers that are different roots of the unity but there's only one square root of unity well, I guess there's one square root of unity is $+1$, the other one is -1 since $(-1)^2=1$, I forgot about that one but there's two well-known things that square to $+1$ but now I'm offering you three more things that's square to $+1$ so that's a big deal too and we'll hopefully … I'm not sure when we'll actually focus on why that's important but we will, so let's move on.

“Indeed, the element I provides space-time with an intrinsic *complex structure* without any *ad hoc* introduction of the scalar complex numbers”. We talked about that. “We will detail this complex structure in section 3.5”. “It is worth emphasizing that there is no unique square root of -1 . There are many algebraic elements in C_{13} the square to -1 ”. I just said that, γ_{23} , γ_{31} , γ_{12} , “each with a distinct geometric significance”. “The scalar imaginary i of standard complex analysis does not specify any of this additional structure; hence, it is often enlightening to determine whether a particular $\sqrt{-1}$ is implied by a generic i that appears in a traditional Physics expression. We will show in what follows that I is indeed the proper physical meaning of i throughout the electromagnetic theory”.

All of my admonitions about being careful maybe maybe the I we're looking at is γ_{23} , γ_{31} , γ_{12} instead of γ_{0123} they're saying no, in Electromagnetic theory it's always going to be this guy γ_{0123} but that's not always true for other theories, obviously I've read this paragraph before because I absorbed what it said and I went off on a little rant about it in anticipation of it so that's the second time you see it

Notation

Let's read one more paragraph before we stop “any *multi-vector* in $M\in C_{13}$ if the space-time algebra may be written as the sum of independent k -blades” so here's the notation that they're introducing for a multi-vector basically they're going to go with Greek letters for scalars so those were real numbers right here, standard Roman italics letters for vectors, bi-vectors are going to be capitalized bold face letters, tri-vectors are going to be in this font, they call it Fraktur letters and now notice that they don't give a special symbol anymore for the pseudo-scalar what they do is the pseudo-scalar is going to be a real number which we've decided is a Greek lowercase and we're going to slap on the I so we're literally going to express pseudo-scalars in component form and remember vector spaces are all real so the components are real and then this is the basis vector, this I is the basis vector so they're actually doing pseudo-scalars in component form but it is still true that this β , the β is not the pseudo-scalar.

$$M=\alpha+v+\boldsymbol{F}+\mathscr{T}+\beta I \tag{13}$$

The pseudo-scalar is the whole thing the β is a real number, I is what makes it a pseudo-scalar, it's a vector space so β is the component of the single bases vector just like α can be viewed not as a real number per se but the bin of scalars $\mathbb{R}$ that fills the vector space Λ_0 and the basis vector is the number 1 and so to be symmetric you could write $\alpha(1)$ here for scalars and βI for pseudo-scalars and that would provide a little symmetry but they don't want to do that they don't want to carry this 1 around and I don't blame them so that is what they're going with so that's their notation for multi-vectors.

Electromagnetic Theory

This notation (13), as this paper goes on, changes, it changes not because they're just arbitrarily changing things but because the deeper you understand the relationship between these things and the relationship between them and Electromagnetic Theory there's just better ways of writing the multi-vector than as the sum of its individual blade parts because those blades have relationships with each other that to highlight we blend them together a bit so but you've got to build up to it and this paper does a great job of building up to it, I really love the way they do that so what did it say?

Bi-vector

They said “the Greek letters α , β are real numbers, the lowercase Roman letter v is a vector with four real components”, that's always true, four vectors have four real components, “the boldface Roman letter $\boldsymbol{F}$ is a bi-vector with six real components” so when they say six real components of course they are just referencing the fact that that bi-vectors have six basis, it's a six-dimensional sub-space so it's got six basis vectors and each basis vector has a component and each component is real because all of our vector sub-spaces are real so that's all they're saying there so that's six real components, “and the Fraktur” love that word Fraktur “letter $\mathscr{F}$ is a tri-vector with four real components”, for the same reason there's four tri-vectors, “each of these independent grades are distinct and proper geometric objects” so when they say this “geometric objects” thing they're talking about our interpretation of how this algebra goes and we've decided that these are simple real scalars α and these are vectors v or little pointy things in our four vectors, bi-vectors are these plane things, tri-vectors are these little volume things, these little three-dimensional volume things and this last thing, the pseudo-scalar is some hyper cubic thing which if you drew it aggressively you would have two little cubes inside each other looking like this little Tesseract but that's what it is, it is literally a tesseract, to use a poppy science word for what a four-dimensional volume element might look like.

They're proper geometric objects that's what that all means “we shall see in section 3.5 that in practice we can dramatically simplify the description of multi-vectors by exploiting the dualities of the algebra” that's what I was talking about before this (13) is going to change once we understand these dualities really well, “in particular we'll be able to dispense with the tri-vectors (and their elaborate notation)” I think the elaborate notation they're talking about is, it's just a pain in the ass to write γ_{123} , who's got time to write three indices on the letter γ so we're going to get rid of that. We've already gotten rid of this one βI , I mean who's got time to write $\beta\gamma_{0123}$ who's got time for that we just go boom right there to I . Turns out we can do a similar trick with the tri-vectors and that's going to be really sweet, it's going to be vectors and tri-vectors are going to be dual related, a Hodge duel, that'll be a Hodge duality if I recall correctly.

Notations

“To keep these distinctions conceptually clear, we shall make an effort to maintain useful notational conventions throughout this work that are summarized in table two for reference”. Let me quickly check table two so we'll look at table two a little later we're not quite ready for it, this is good enough for today's lesson so “each k -blade in a multi-vector can be extracted by a suitable grade projection” so we've already talked about this, this notation here $\langle M\rangle_k$ that's a projection operator where you're projecting basically the k component say the vector component of a multi-vector out, you get rid of everything else you leave this behind so that's what they mean by this “grade projection” for example the bi-vector projection is $\langle M\rangle_2=\boldsymbol{F}$. “Notably, the scalar projection satisfies the cyclic property”:

$$\langle M\,N\rangle_0=\langle N\,M\rangle_0 \tag{14}$$

They talk about the scalar projection MN then if you cycle M and N for any two multi-vectors it the scalar part is always the same. Cyclic means that this extends to $\langle MNOP\rangle_0 = \langle PMNO\rangle_0$. If as long as it's cyclic you're good so this “is an algebraic *trace* operation”. and I showed you how to do this calculation for bi-vectors, you're driving this whole thing down by using a Minkowski contraction ultimately to a single number and that is what a trace does, a trace takes a matrix and it grinds it down to a real number. “This trace is entirely equivalent to the matrix trace operation, which can be verified by considering a (Dirac) matrix representation of the space-time algebra C_{13} that simulates the non-commutative vector product using the standard matrix product. We will comment more on such a matrix representation in Section 3.8, but we shall not require it in what follows”.

Here's where they actually say, these γ matrices are the 4×4 Dirac matrices that we know about from the Dirac algebra and this, well, I shouldn't say they are, these are our basis, this is terrible the way I've done this. This does not equal a matrix, let me say that this can be represented by a matrix if we like and if we use the right matrices then all of this $\gamma_\mu \gamma_\nu$ properties, the properties of this space-time product get mimicked by these matrices so a representation of the γ matrices but space-time algebra says that's a shame too because these are a pain in the butt, γ_μ are really simple, it just so happens that Dirac discovered all this using the pain in the butt way and now we're trying to use Geometric algebra to take all of this stuff and to say, we figured it out with these γ matrices but nobody's quite realized that they're just a representation of a very simple idea of basis vectors in C_{13} and these collapses (14) of multi-vector products to their scalar form, can be understood in the matrix version if you just take traces of these matrices so they don't go into that very much, I don't recall them going into that a lot in this paper but that is the point of this whole thing is to say we're going to do this all algebraically not linear algebraically.