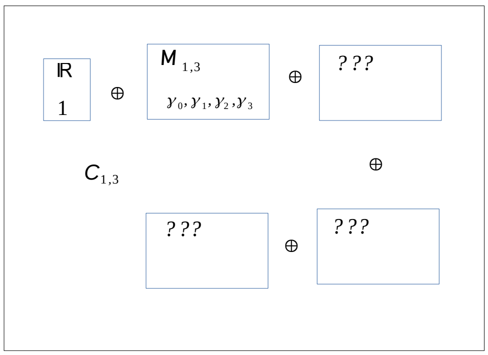
# Geometric Algebra 4: The anti-symmetric part

Let us continue our exploration of the space-time algebra I'm going to try to talk about Geometric algebra in general as well in this lesson in these lessons the space-time algebra is always going to be our anchor as one our favorite exemplar but there are other ones that there are other Geometric algebras that are important, probably the most important is the just the standard three-dimensional Geometric algebra and I do want to take a quick diversion to understand what this word dimension means in the story of Geometric algebra so we'll probably do that first and then we'll continue on because we were working on the space-time product, we're working on this guy, we're on Section 3.2 and we've done half of that so let's begin.




This is our picture of our total space-time algebra the total algebra is everything all the elements are inside this outer box and this box is eventually, this algebra is called $C_{13}$ eventually the paper is going to call it $C_{13}$ we haven't gotten there yet but that $C$ stands for William Kingdon Clifford, the historical figure that's most associated with this analysis and inside of the Clifford algebra we have four, essentially they are sub algebras meaning each one is actually an algebra and therefore a vector space unto itself and the Clifford algebra is the literal direct sum of these four sub-algebras and ultimately the word for this, by the way, is a Graded algebra, that means if you can take an algebra and make it the direct sum of several sub algebras you call that thing a Graded algebra so that's an important point we're going to study how this sum works a little bit later once we've understood all of these different vector spaces that are inside the Clifford algebra and the Minkowski vector space, this is the vector space of four vectors and we already learned that we need to augment the vector space of four vectors with real numbers and we haven't quite looked at the other guys yet or we're going to hopefully learn at least one of them today but probably not these other two.

What makes the Clifford algebra a Clifford algebra is that we can take any two elements of the algebra and multiply them together to get a third element of the algebra and in particular right now we are studying just the space-time multiplication $ab=c$ is called space-time multiplication and right now we are just considering the space-time multiplication of objects in $M_{1,3}$. The way the paper is presenting this is it's telling us that if we take the space-time product of two elements of $M_{1,3}$ we can think of it in terms of this dot product $\cdot$ and this other thing that we're going to study today the wedge product $\wedge$ and we can associate those pretty unambiguously with the symmetric part and the anti-symmetric part.

$$\begin{aligned} ab &= a\cdot b + a\wedge b \\ &= ½(ab+ba) + ½(ab-ba) \end{aligned} \quad \text{where } a,b \in M_{1,3} \tag{1}$$

This rule does not apply to all things in $C_{13}$ if I took a vector $A \in C_{13}$ and a vector $B \in C_{13}$ just a general object of $C_{13}$ that is not necessarily living in $M_{1,3}$ but it could live in either one of those other sub spaces of $M_{1,3}$ like it could live in in here which we'll talk about today or here or here or here or it could be some sum of those things. The space-time product is not so easily broken up into this, there's some instances where things are switched where the scalar part is actually associated with the anti-symmetric part and the wedge part is associated with the symmetric part and then there's some that are just general enough that it's really not very valuable to break it up like this but it is very valuable to break it up when you have two elements of $\mathsf{M}_{1,3}$ and then we build everything up from there.

Now the idea of this notation (1) was to say that the symmetric part can be associated with a something that we're going to call a dot product $\cdot$ and the idea of course is that this is supposed to be familiar this is supposed to be the Minkowski contraction of $a$ and $b$ i.e. $\eta(a,b)$ that's why they want to use the familiar dot product notation, they want this familiarity to surface again here there's the wedge product and that's maybe less familiar to the general audience but it's very familiar to anybody who's studied these kinds of algebras before and we'll get to that a little later but this familiarity was what they're after so they had to show that this symmetric part of this space-time product in fact equals the Minkowski contraction and we demonstrated it in a very straightforward way we just took the square of a sum of two Minkowski vectors and threw it into this expression here which comes from the axioms of the foundational axioms of our Clifford algebra and we just strolled along and we were able to show that the symmetric part of the space-time product did in fact equal the Minkowski contraction of the two factors. We simplify that with the dot notation.

This is the coordinate free version of everything, now I wanted to have a look at that also and I tried at the end of the last lesson to give you a version of this in a real coordinate system and that proof looked like this, this is what's left of that proof but I ended up leaning on the fact that we already had proven it coordinate free so this really wasn't a proof as much as it was a demonstration of how to cast something into a coordinate system and the hint that you're casting something into a coordinate system was when you take the vectors and you replace them with basis vectors, you take these coordinate free vectors $a, b$ and you replace it with something that's cast in a certain basis $e_\mu$ in this case:

$$ab+ba=2\left(a^0b^0e_0e_0+a^1b^1e_1e_1+a^2b^2e_2e_2+a^3b^3e_3e_3\right)+\sum_{\mu\neq\nu}a^\mu b^\nu\left(e_\mu e_\nu+e_\nu e_\mu\right) \tag{2}$$

Once you start doing that you're leaning on your understanding of the orthogonality of these basis vectors to make various cancellations now this proof wasn't a very good proof well it wasn't really a proof at all it was just a demonstration that I thought might be a proof until I realized that there was a problem but it turns out in this case the proof is that even a little bit simpler than what I suggested and in fact this is an example where the coordinate free proof is actually easier but we're trying to show that the symmetric part of the space-time product is equal to this dot product:

$$\begin{aligned}(a+b)(a+b)&=a^2+b^2+ab+ba\\ a^2+b^2+ab+ba&=a^\mu a^\nu e_\mu e_\nu+b^\mu b^\nu e_\mu e_\nu+a^\mu b^\nu e_\mu e_\nu+a^\mu a^\nu e_\nu e_\mu\\ &=\eta(a^\mu e_\mu,a^\nu e_\nu)+\eta(b^\mu e_\mu,b^\nu e_\nu)+\eta(a^\mu e_\mu,b^\nu e_\nu)+\eta(b^\nu e_\nu,a^\mu e_\mu)\\ &=a^\mu a^\mu\eta(e_\mu,e_\mu)+b^\mu b^\mu\eta(e_\mu,e_\mu)+a^\mu b^\nu\eta(e_\mu,e_\nu)+b^\nu a^\mu\eta(e_\nu,e_\mu)\end{aligned} \tag{3}$$

You have to start with the space-time product so you do blow this up but as soon as you blow it up over here you'll see we have the full algebraic expansion using the linearity of the space-time product and you immediately could cast it straight away into a relative form so this is called the proper form (first line of (3)) and this stuff here is called relative form (second line of (3)) so this stuff here is all relative once you cast it in a basis we're calling it relative and when you don't cast into base it's called proper, that's my understanding of the paper's use of the word relative and proper so now we have this relative form and we could also blow up this guy to the blowing up the metric form because we know that this thing up here has to equal $\eta(a+b, a+b)$ and this is the proper form of applying the Minkowski metric and then we just substitute the relative forms in there and we expand it out this way that's what we basically did that in the last lesson.

Then using our definitions, once we do this expansion our definitions immediately allow us to cancel $a^2, b^2$ leaving us with twice the symmetric part of the space-time product and then we recognize that:

$$½(ab+ba)=a^{\mu}b^{\nu}\eta(e_{\mu},e_{\nu})=\eta(a,b) \tag{4}$$

That is how you might do this proof that the symmetric part of the space-time product is in fact equal to the Minkowski contraction and this is using the coordinate form. It's not very enlightening or even easier in this case but that is how you would do it using the relative form. With that review let us now move on to the anti-symmetric part of this space-time product.

The first thing we can do is we go back to this expansion (1), the space-time product, we've got this symmetric part down, well what's left? Everything left in the space-time product must be the anti-symmetric part and we're looking for some simplification of this anti-symmetric part. Now it's very suggestive to throw that little wedge in there but we could just say for the time being that $a*b$ is identified with this anti-symmetric part $½(ab-ba)$ and try to understand how we might interpret this piece of the space-time product. Well there's several important things but one critical thing is that well clearly $a*b$ , whatever that asterisk is must equal $-b*a$ so it is anti-symmetric this $*$ operator is anti-symmetric. That's an important hint, if it's anti-symmetric what's really important too is that $a*a=0$ and between those two things we are lead pretty quickly to say hey you know what the wedge product of two vectors has those two properties it's anti-symmetric and clearly the wedge product of a vector with itself must be zero because of the anti-symmetry.

Now we are going to take that sense and then go ahead and fully interpret this as the wedge product and as a geometric object, now when we study the wedge product in our previous classwork our previous work on this stuff we usually are talking about forms and we would call $a\wedge b$ a two form, a two form is what we usually see wedged together but $a$ and $b$ are not forms here, $a$ and $b$ are elements of the Minkowski vector space so they are in fact vectors so this little object, if we're going to interpret it as the standard product is actually something that we speak a little bit less of in the subject of Exterior algebra, it's actually a two vector right and this brings us into another form of language ambiguity because remember we are saying that everything that lives in $\mathsf{M}_{1,3}$ we want to call those guys four vectors because that's what we call them in Relativity we call them four vectors and when we study Relativity we call them four vectors because students understand the word vector to mean three-dimensional vectors, your little pointy things and if we go to Minkowski space we're still dealing with little pointy things as vectors but they have four dimensions and we have to remember they're not Euclidean they're Minkowski so they obey the Minkowski metric.

Students need to be able to distinguish between three and four vectors so we call these guys four vectors however in the language of differential forms a four vector would have been an object that looks like $a\wedge b\wedge c\wedge d$ , that's a four vector, a three vector is $a\wedge b\wedge c$ and a two vector is $a\wedge b$ where $a, b, c, d$ where each vector is some member of some vector space and it's an arbitrary vector space could be any vector space and if it's a co-vector space which is strangely enough also a vector space it's just full of co-vectors then we call it two form, three form, four form but if it's the a vector space it's two vector, three vector, four vector but this guy is not to be confused with four vectors in $\mathsf{M}_{1,3}$ so we could have a four vector of four vectors which is a complete and absolute mess of language.

Sadly we do have a four vector in our Minkowski space this last one is actually a four vector in this sense in this sense here $a \wedge b \wedge c \wedge d$ so we do actually have this problem of four vectors of four vectors so we just have to deal with it and I'll probably be repeating this over and over again but the point is, if we have to study right now we're faced with the problem of what is this two vector idea $a \wedge b$ so it's time to go back to the paper and they write the anti-symmetric part of the product that's this part right here $\frac{1}{2}(ab-ba)$ , they are defining to be this notion of a wedge product, it is called the wedge product and it is the proper generalization of the cross product to relativistic four vectors so this is a little bit of an awkward sentence I mean no it's a very good sentence but it's there's a lot in there I get when I say awkward I mean wow you're processing a lot and it's not clear what's supposed to be obvious and what is supposed to be something that they're expressing and teaching but the proper generalization first of all we use proper to mean coordinate free but in this case proper means the correct generalization of the vector cross product to relativistic four vectors.

Here's the deal if you understand the vector cross product to mean that if I have a vector $a$ and I take its cross product with a vector $b$ , if you understand that the result is supposed to be orthogonal to $a$ and $b$ and it's supposed to be unambiguous meaning there's only one answer, well in three-dimensional space this answer will be another vector $c$ but in any other dimension you have a much bigger space that's orthogonal to any two skimpy little pointy thing vectors right and $a$ and $b$ are assumed to be little pointy thing, vectors so this guy here will not be a vector it'll be a bigger geometric object it could be a plane or it could be a volume, it could be a hyper volume so if that's your understanding of a cross product which means everything that's orthogonal to the vectors $a$ and $b$ the plane created by the vectors $a$ and $b$ then yes the wedge product is absolutely the proper generalization of the cross product if you have the mistaken idea that a cross product between any two vectors of any dimensionality regardless, three or any other number is supposed to be another vector then you're in the wrong place because that's not possible to do.

That's what this means "the proper generalization of the cross product", they're trying to say you're supposed to think of the cross product as everything that's orthogonal to the plane spanned by the two vectors $a$ and $b$ , let's use the right language, if I have a vector $a$ and $b$ they span a plane I want to know everything that is orthogonal to that plane well everything that is orthogonal to that plane is something that has no component along $a$ and no component along $b$ and typically well that's just another vector $c$ but in four-dimensional space it's going to be a vector $c$ and another vector $d$ and the whole plane contain containing $c$ and $d$ will be orthogonal to the whole plane containing $a$ and $b$ so that's what that notion of the proper generalization of the cross product means and what's interesting is they specify to relativistic four vectors this notion of the wedge product being the proper generalization of the vector cross product applies to any Geometric algebra we're doing so any vector space not just $\mathsf{M}_{1,3}$ , this paper is all about $\mathsf{M}_{1,3}$ but this generalization is very broad and applies to the Geometric algebra as a whole.

Anyway, the wedge project produces a qualitatively new type of object called a bi-vector so this is a good job because I just went through this whole long rant about why there's a language ambiguity using the notion of a two vector, they don't even bother with that, they go right to the idea we're going to call this thing a bi-vector and a bi-vector, as you're going to see in a moment, is this little plane segment so they've gotten rid of this language ambiguity early on the problem is they don't use tri-vector and quad-vector, I bet but I'm not sure how that's going to be dealt with, I've read this paper before, I just don't remember how they deal with that but I think the way they deal with it is they turn everything into bi-vectors eventually, that's right, ultimately real numbers, vectors and bi-vectors are pretty much all you need in the end, despite the fact that there are tri-vectors and quad-vectors also involved in this and you'll see why later but for right now we're trying to create everything separately.

$$a \wedge b \equiv \tfrac{1}{2}(ab - ba) = -b \wedge a \tag{5}$$

Now this bi-vector (5), this thing is not in $\mathsf{M}_{1,3}$ so that goes back to this whole notion that we're expanding $\mathsf{M}_{1,3}$ to get $C_{13}$ . "A bi-vector $a \wedge b$ produced from a space-like vectors $a$ and $b$ has their geometric meaning of a *plane segment* with magnitude equal to the area of the parallelogram bounded by $a$ and $b$ and a surface orientation (handedness) determined by the right hand rule" so the only thing I want to say there is we're going to describe that in a moment when we look at figure one but this word space-like gives me the willies right here. When they say space-like you might jump on the notion that given any two space-like vectors I'm thinking $\eta(a,b)<0$ in the convention we're using, a space-like relationship between two vectors is less than zero or space-like vectors $a$ and $b$ would imply that their magnitude $\eta(a,a)<0$ but that's not what they really mean here I think what they really mean here is space-like in the sense that it's an element of $\mathsf{M}_{1,3}$ which is our space time so unless they're trying to say well it's got to be space-like in order to have an area because the time component of a four vector how do you interpret that as an area? But the answer is the time component is measured in the same units as the space component, $ct$ has the same units as $x, y, z$ or $X^0, X^1, X^2, X^3$ . We're expecting all of these to have the same units so a little piece of plane that's going off in the space-like direction $X^1$ and the time-like direction $X^0$ , that plane still has an area given in $\text{cm}^2$ so I'm pretty sure that their uses of space-like is just to say we're creating a bi-vector from two elements of $\mathsf{M}_{1,3}$ .

Now understand that this idea that $a \wedge b$ can be interpreted as a plane element that is an interpretation of the algebra all we really know is that the anti-symmetric part of the algebra is a thing and we happen to understand that we're basically defining the wedge product to be the anti-symmetric part of the space-time product and because it's anti-symmetric we immediately think of the one anti-symmetric thing we really understand well which is this little bi-vector thing which can be understood as this vector $a$ being swept along $b$ to produce a little plane and that plane has an area which we're going to call $|a \wedge b|$ , the magnitude of $a$ wedge $b$ and it has a circulation which is given by the right hand rule of taking $a$ into $b$ . We take our fingers put them in the direction of $a$ curl towards the arrowhead of $b$ and our thumb would normally give us the direction of the cross product but we can't rely on that thumb anymore right now it's just the curl of our fingers that matters, oh that was a close call, I almost went with the thumb on that but no it's not the thumb, the thumb doesn't have anything to do with it that's that's the whole point is the thumb is no longer any good but the curling of the fingers is still fine so the circulation is entirely defined inside the plane itself and then ultimately you realize well this notion of a parallelogram that's that's out the window too we're fine with just an arbitrary area an undescribed area, an area that has no specific boundary, it's just an area as long as it also has a circulation and this is actually similar to the notion of, if you have a vector what matters? Well what matters is the vector's length and the direction it's pointing, this plane all that matters is its area and its orientation in space and it's circulation that I guess that we call this the circulation.

This is actually the orientation but what doesn't matter is the shape, just like it doesn't matter how thick we draw this arrow, it's still just a vector, the thickness of the arrow is irrelevant in this case the shape of the plane segment is irrelevant so in principle the core idea is that this symbol $a \cdot b$ when $a, b \in \mathsf{M}_{1,3}$ is the symmetric part and the wedge product $a \wedge b$ is defined as the anti-symmetric part, we've taken this leap of faith, well this one $a \cdot b$ we've able to show was a real number very easily and that really flowed from our fundamental axiom that the square of any vector is a real number so that we leaned on that and we were very able to very easily see that this thing had to be a real number, To show that $a \wedge b$ has to be a little line or an oriented area, that's a bit of an inference, we know it's not a real number and we know through its anti-symmetry and the fact that if $a$ and $b$ are colinear it's zero in case that wasn't obvious by the way.

$$\begin{aligned} a \cdot b &= ½(ab+ba) \\ a \wedge b &= ½(ab-ba) \end{aligned} \tag{6}$$

Just to be clear if we're wedging vector $c$ with vector $d$ but if $d$ is actually co-linear with $c$ which we will write as some real number factor times $c$ , the real number factor comes out because everything here is linear, we get:

$$c \wedge d = c \wedge \varepsilon c = \varepsilon c \wedge c = 0 \tag{7}$$

The fact that co-linear vectors wedge to zero and the whole thing is anti-symmetric it lets us think in terms of this swept up plane but we can do a little bit better in getting our interpretation of the general wedge product to be the swept up plane by looking at it in component form so let's take this structure here (second line of (6)) and let's break it down in component form so here's the anti-symmetric or twice the anti-symmetric part and we immediately go into a relative form:

$$\begin{aligned} ab - ba &= (a^{\mu} e_{\mu})(b^{\nu} e_{\nu}) - (b^{\alpha} e_{\alpha})(a^{\beta} e_{\beta}) \\ &= a^{\mu} b^{\nu} e_{\mu} e_{\nu} - a^{\beta} b^{\alpha} e_{\alpha} e_{\beta} \\ &= a^{\mu} b^{\mu} e_{\mu} e_{\mu} + \sum_{\mu \neq \nu} a^{\mu} b^{\nu} e_{\mu} e_{\nu} \\ &\quad - a^{\beta} b^{\beta} e_{\beta} e_{\beta} - \sum_{\beta \neq \alpha} a^{\beta} b^{\alpha} e_{\alpha} e_{\beta} \\ &= \sum_{\mu \neq \nu} a^{\mu} b^{\nu} e_{\mu} e_{\nu} - \sum_{\beta \neq \alpha} a^{\beta} b^{\alpha} e_{\alpha} e_{\beta} \end{aligned} \tag{8}$$

We just blow it up the way we did before. The linearity of the real numbers $a^{\mu}$ and $b^{\nu}$ come out on both cases and then we end up with this difference which we work up we work over this difference a little bit what we're going to do is work for this piece here we're going to take out the ones where $\mu$ and $\nu$ are equal to each other and we're going to separate it for the ones where $\mu$ and $\nu$ do not equal each other likewise for the other piece, we're going to take out the one the terms were $\beta$ and $\alpha$ are the same and we're going to leave the summation together for when $\beta$ and $\alpha$ are different and then when we do that we realize right away that the first term and third term cancel each other out and so all we're left with are these sums so we rewrite it in terms of just those sums where $\mu \neq \nu$ and $\beta \neq \alpha$ :

$$\begin{aligned} ab - ba &= \sum_{\mu \neq \nu} a^{\mu} b^{\nu} e_{\mu} e_{\nu} - \sum_{\mu \neq \nu} a^{\nu} b^{\mu} e_{\mu} e_{\nu} \\ &= \sum_{\mu \neq \nu} (a^{\mu} b^{\nu} - a^{\nu} b^{\mu}) e_{\mu} e_{\nu} \end{aligned} \tag{9}$$

Then these are all dummy variables so we just change $\beta$ and $\alpha$ to $\nu$ and $\mu$ and now of course we're going to combine it into a single sum because we're dealing with the same indices and so that's a completely allowed maneuver so we have a single sum where $\mu \neq \nu$ so nothing's ever the same and so my next idea is to take this sum where $\mu \neq \nu$ and just do it over the terms where $\mu < \nu$ and then add an additional term to cover the other side where $\mu > \nu$ .

$$ab - ba = \sum_{\mu<\nu} \left[ a^\mu b^\nu e_\mu e_\nu + a^\nu b^\mu e_\nu e_\mu - a^\nu b^\mu e_\mu e_\nu - a^\mu b^\nu e_\nu e_\mu \right] \tag{10}$$

That's this first one but then I flip it and then I'll capture the one that I've thrown away by only taking $\mu < \nu$ , I've already thrown away all the ones where $\mu = \nu$ and I do that for both terms of (10) which are equivalent to this first term in (9) so once I've done that then the next thing is I'm going to combine the terms with the same ordering of the space-time product, remember $a^\mu, b^\mu$ are just real numbers being multiplied together but these guys $e_\mu, e_\nu$ are space-time products of unit vectors.

$$ab - ba = 2 \sum_{\mu<\nu} \left( a^\mu b^\nu - a^\nu b^\mu \right) e_\mu e_\nu = 2 \sum_{\mu<\nu} \begin{vmatrix} a^\mu & a^\nu \\ b^\mu & b^\nu \end{vmatrix} e_\mu e_\nu \tag{11}$$

This is nothing more than the $2 \times 2$ determinant of the components of the vector $a$ and $b$ and what's important to understand is, if I have a vector $a$ and $b$ , if I take its components and I produce the determinant and I find that that determinant equals the area of this parallelogram so that gives me a big hint because remember what I started with I started with the anti-symmetric part of $a$ and $b$ and I've shown that it equals the sum of the space-time product of $e_\mu e_\nu$ multiplied by an area component so that makes me think these might be basis areas, it's always $\mu < \nu$ so I know that $e_\mu e_\nu = e_\mu \wedge e_\nu$ because I never have the case where $\mu$ and $\nu$ are the same and I know that these are orthogonal unit vectors so the space-time product equals the wedge product because $e_\mu \cdot e_\nu = 0$ always because of the way the sum is done so I'm only dealing with the wedge product and I guess the last thing I have to say is:

$$e_\mu e_\nu = e_\mu \cdot e_\nu + e_\mu \wedge e_\nu \tag{12}$$

That's our premise, that's the symmetric part and them the anti-symmetric part. The basis vectors are orthogonal and I've already shown that the symmetric part is equal to the Minkowski contraction so we know that part is zero so now when I see the space-time product of any two unit vectors I immediately think of the bi-vector that they create well so this is a little bi-vector this is $e_\mu \wedge e_\nu$ and that bi-vector has as its component the area in front of it so now I'm basically adding up a bunch of areas between all of the different basis bi-vectors and how many bases by vectors are there? There are exactly six, there are six cases where $\mu < \nu$ and those six bi-vectors are are basis bi-vectors and they all represent little plain elements in the zero one plane the zero two plane the zero three plane so these are all planes that have an extension in the time-like direction and then the one two plane and the two three plane:

$$e_0 \wedge e_1 \ \ , \ \ e_0 \wedge e_2 \ \ , \ \ e_0 \wedge e_3 \ \ , \ \ e_1 \wedge e_2 \ \ , \ \ e_1 \wedge e_3 \ \ , \ \ e_2 \wedge e_3 \tag{13}$$

There are planes purely in the spatial directions (last three in (13)) like the planes that contains $xy, yz, xz$ and there are planes that have a little piece in the time axis but there are six of them there are six possible basis vectors and they combine with their coefficients that represent an area so this demonstration seals the deal on the interpretation for me but the fact you have to know is that the area of a parallelogram is this determinant of the components so this is an area of one of the unit bi-vectors that are out there so that's pretty good so where does that leave us?

Well I think that leaves us with a full interpretation of this space-time product for two vectors in the Minkowski subspace of our Clifford algebra and we've got the symmetric anti-symmetric part we fully understand the symmetric part now is this is simply the dot product and we understand that the anti-symmetric part is to be interpreted as a little piece of of area basically called a bi-vector and it's this wedge product that we are going to study, it is the wedge product by definition, we're defining the wedge product to be this anti-symmetric part but the properties of this wedge product lead us to think of it and interpret it in our mind as these little unit areas and it's very tempting of course to constantly think of it as a little parallelogram and what I just did the demonstration I done doesn't help because that area is literally the calculated area of a parallelogram but once you understand that this parallelogram is just as legit in shape as any arbitrary shape you realize that we're dealing with bi-vectors is another vector space where you can add them together and create arbitrary size and shape plane elements using the unit vectors (13) and there's six new dimensions and so we have achieved a certain amount of … we're getting there, we've got four dimensions, five dimensions, 11 dimensions so far so this is going to give us a 12 dimension and this will be four more so the total thing will have 16 dimensions when we're done.

All right so we still need to understand how these additions work because clearly you see the problem here is that we are adding a little piece of plane right we're adding a little plane area with circulation that's this guy to a number so we have seven plus a plane area and that's what our space-time product equals so we have to understand how to interpret this **+** we still have to work on that but for now we're good, we are now going to move on a little, we're going to try to finish this paragraph 3.2 next time and I think what I'll do is we're we'll probably talk about multi-vectors and then we'll deal with that addition question so I'll see you next time.