[Lesson 15 The Covariant derivative of a (p,q)-rank tensor]

Today we're going to describe how to do the calculation of the Covariant derivative of a general (p,q) rank tensor so it's worthwhile to just review all the things that we've done notation wise up to this point because we're gonna actually start using everything in order to finally prove the metric connection, the issue between connecting the connection coefficients to the metric, you'll notice nothing in this has a metric right everything here is just the manifold, some functions on the manifold some vector and co-vector fields on the manifold and a connection and the connection really is the geometry of the manifold because it describes parallelism and that's by the way when we talk about differential geometry this is really where it all lives it all lives in this connection and our Physics is going to ultimately give us this thing so let's see what we have:

$$\nabla_B A = \nabla(\quad, A(x), B(P)) \tag{1}$$

We have here the Covariant derivative of a vector field A in the direction of the vector B and this is the CFREE notation and in the real stripped down CFREE notation of [MTW] (Misner, Thorne and Wheeler), ∇ is, the they call the gradient operator which takes three different elements to it, one is a vector field $A(x)$, one is a vector at a point $B(P)$ and then this (empty space) would be a 1-form but they leave this blank, this is defined as this object with that 1-form left blank and the way we attack this thing is we broke it down into the principles of what we want it to behave like in order to be a derivative and we wanted it to be linear in this vector B so that's expressed by this equation:

$$\nabla_{\mathrm{f}B} A = \mathrm{f}\,\nabla_B A \tag{2}$$

That function or value multiplied by B just a real number by B will come out front, now this we express this as a function because B could be a vector field but the point is, it's its value at one point is all that matters and we went to some pains to show that so if this is a function and this is a vector at a point then we're talking about the value of the function at that same point but regardless it will come out front and then you are left back with this original definition so it has to have that property.

$$\nabla_B \mathrm{f}\,A = \mathrm{f}\,\nabla_B A + B(\mathrm{f})\,A \tag{3}$$

Likewise if you have a function now this time you do need a function on the space-time because A is a field and it has values all over the space-time for sure and it must in order to fit into this position on the gradient operation then that comes out in a bit of a Leibniz fashion, it comes out in front and then the derivative is taken on A and then the derivative is taken on f. When I say the derivative is taken on f, B is a derivation on the manifold which we've decided to establish, the basis to be of all linear operators so that is a linear differential operator that can act on the function so $B(\mathrm{f})$ is a well calculable function on the space-time so B acting on f will be another function on the space-time times the field A so that's an important property.

$$\nabla_B \mathrm{f} = B(\mathrm{f}) \tag{4}$$

Then in order to understand what this is the definition of the Covariant derivative in the direction B of a function simply is B acting on that function that's something we have to establish as well, you have to establish how this Covariant derivative derivative operator works to the lowest level object which would just be a $(0,0)$ rank tensor and this is the only plausible choice then from there we established the relationship between the CFREE notation and the component notation that relationship is:

$$\nabla_{\partial_i}\partial_j \equiv \Gamma^k_{ji}\partial_k \tag{5}$$

In fact we could even put three lines here to say that it's a definition really so the Covariant derivative in the direction of the basis vector i of the basis vector j is given by this little vector here, and it's a vector because it has a partial here at the end so it's a differential operator and these connection coefficients are have to be specified, we have to figure out what they are or we have to be told what they are or we just have to arbitrarily choose them we have a lot of choices for this but this is the abstract notation (left-hand side) and this is the component notation (right-hand side) and actually this is almost weakly the abstract notation because we are here we have really cast a basis, we've established a basis, I guess the true abstract notation would look like this:

$$\nabla_{e_i} e_j = \Gamma^k_{ji} e_k \tag{6}$$

Whatever basis you pick, this is what you got, that would be a little bit more abstract but even there you're still picking out a basis right to stay truly abstract you would have to always be in this vector and vector relationship without specifying a basis. (5) is the one that we use all the time. Sometimes you'll see in textbooks this relationship (5) actually defined this way:

$$\left\langle \mathrm{d}x^k, \nabla_i \partial_j \right\rangle = \Gamma^k_{ji} \tag{7}$$

Where you take the dual space mapping with a co-vector basis and then you just get connection coefficients so if you want to literally just define the connection coefficient (7) is the actual true definition of just the coefficient, this is a number or a function on space-time and (5) is a vector so there's the difference that's a good important point. The Covariant derivative in the direction B of a vector A is always going to be another vector and that can be traced to this definition (5).

Alert: there is a convention where the j and the i are switched in (7) so the i and j of the left-hand side matches the i and j in the ordering of right-hand side, that confused me for quite a while in the Straumann book that I referenced earlier (General Relativity), they use that reverse convention which is ironic to call it the reverse convention because in their book the convention on the left side here and the right side are the same as the order is the same i,j and i,j , it just so happens that most books including MTW and d'Inverno and the elementary books seem to have this convention (7) where you take the coefficient i,j and you reverse them and I think that's ultimately because it's much nicer to have things matching up in this type of notation where you start introducing these slashes $|,\|$, you want the order of things to agree up μ , down ν , you would have the inner and the outer index are the same and that's nice also, I can understand why we might want to do that if you didn't, you wouldn't be able to use this notation here and then match it up with vectors here so I like the reverse notation that's fine or flipping these things so this is the notation we're gonna use then if you start really going into the full component form the straight-up Covariant derivative of A is given by this formula:

$$\nabla A \equiv \left[A^{\mu}{}_{|\nu} + \Gamma^{\mu}_{\sigma\nu} A^{\sigma} \right] \partial_{\mu} \otimes \partial^{\nu} \equiv A^{\mu}{}_{\|\nu} \partial_{\mu} \otimes \partial^{\nu} \tag{8}$$

where $A^{\mu}{}_{|\nu}$ represents that abbreviated form of the partial derivative of the vector field components, the components of the vector field with respect to the different basis vectors so that symbol is actually $\partial A^{\mu}/\partial x^{\nu}$ that symbol is what that means and then $A^{\mu}{}_{||\nu}\partial_{\mu}\otimes\partial^{\nu}$ is the super simplification of this guy so all of that is in this symbol with that double bar you start getting everything in there and of course I've presented it with the basis here but most textbooks and even we eventually or as we go sometimes we will sometimes we won't include the basis vectors and we'll just say the Covariant derivative with respect to A is just the expression in brackets, we will just talk only about the components but strictly speaking the Covariant derivative of a vector A has got to be a $(1,1)$ tensor so that's why we have it here the Covariant derivative of the vector A in the direction B is just this thing contracted with the components of the vector B in the comp system and so you lose this co-vector basis $\mathrm{d}x^{\nu}$ and you end up with just a vector again:

$$\nabla_{B} A=B^{\nu} A^{\mu}{}_{||\nu}\partial_{\mu} \tag{9}$$

Then we moved on and we define this abstract operator notation which was really pretty interesting I thought we are now going to define the Covariant derivative of the vector A in the direction of a curve so now I replace this vector B with $\dot{\gamma}$ where γ is the curve and $\dot{\gamma}$ is just the tangent vector to the curve at some point, it's always at some point and we define that to be this thing:

$$\nabla_{\dot{\gamma}} A=\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=\tau}\Omega_{\tau,s} Y[\gamma(s)] \tag{10}$$

Where we introduce this operator $\Omega_{\tau,s}$ which translates any vector from one point s to its parallel vector at a point t , we just defined it that way and we know we can do it because we knew about the Parallel transport equation and that it was unique the result answer for the Parallel transport equation existed and was unique based on our understanding of differential equations so we know that such an operator could in principle be constructed and the Parallel transport of Y from s to t the derivative of that with respect to the variable s or the parameter s along the curve and then evaluated at $s=t$ and that was the Covariant derivative of the vector A along a curve γ and then we define the same thing we changed up our notation a bit, we define the same thing for an arbitrary tensor Q :

$$\nabla_{\dot{\gamma}} Q=\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\Omega_{s}^{-1} Q(\gamma(s)) \tag{11}$$

We're gonna say for an arbitrary tensor Q , it's the same thing but we invented this slightly new operator notation where Ω_{s} is taking from s to zero. Here's how I break it down, I know that this guy is at s right it's the value of the tensor field at $\gamma(s)$ so I would want to move that to zero so this thing Ω_{s}^{-1} must move it from s to zero which means the non inverted form moves it from zero to s .

That's the definition of how the tensor Covariant derivative along a curve will be defined and then we have these relations that were really important. The notion of parallelism should mean to us that if I take a vector at s and a co-vector at s and I Parallel transport both of them to t , the dual space mapping of those parallel transported vectors should equal the dual space mapping of the original pair at the other point.

$$\left\langle \Omega_{\tau,s}\alpha(s), \Omega_{\tau,s}X(s)\right\rangle = \left\langle \alpha(s), X(s)\right\rangle \tag{12}$$

This also is likewise if I take a co-vector at s and move it to t and take the dual space mapping I should get the same result as if I took the vector at t and Parallel transported it to s and took the dual space mapping so this is these are really two different ways of saying the same thing (12) and (13).

$$\left\langle \Omega_{\tau,s}\alpha(s), X(t)\right\rangle = \left\langle \alpha(s), \Omega_{\tau,s}^{-1}X(t)\right\rangle \tag{13}$$

Then what did we do after that, after that we pushed this whole thing to the limit and we did the definition with tensors we said that a tensor field at s moved along the curve and there's always an implied curve here, by the way, we can only do Parallel transport along a curve and it's really critical to know that the result, I don't know if I said that enough last time but the result of the Parallel transport along curve is curve dependent and this is really critical, we'll be talking about it a lot more when we talk about the Riemann tensor.

If you're at one point and then there's another point and you do Parallel transport along that curve and Parallel transport along another curve going back to the original point, the resulting parallel vectors here, even if I start with the same vector here and do Parallel transport this way and then Parallel transport that way, I'm gonna get two different answers, I need to know exactly what that path is otherwise we wouldn't be so crazy about identifying the path all the time, the path would have dropped out somewhere in the mathematics but it didn't, it was always persistent and the reason is because it matters, it matters very strictly and the reason it matters is that, just to remind you, Γ^{k}_{ji} is a function on the manifold, it'll take different values at different points on the manifold and if you go through different points this is going to change in different ways along different paths.

We said that a tensor field at s Parallel transport along some curve to t it's going to act on a bunch of some p number of co-vectors and q number of vectors to give a real number so this would be a (p,q) rank tensor, it takes p co-vectors and q vectors:

$$\begin{aligned}&\left(\Omega_{\tau,s}Q(s)\right)\left[\sigma_1(\tau)\cdots\sigma_p(\tau), Z_1(\tau)\cdots Z_q(\tau)\right]\\ &\quad = Q(s)\left[\Omega_{\tau,s}^{-1}\sigma_1(\tau)\cdots\Omega_{\tau,s}^{-1}\sigma_p(\tau), \Omega_{\tau,s}^{-1}Z_1(\tau)\cdots\Omega_{\tau,s}^{-1}Z_q(\tau)\right]\end{aligned} \tag{14}$$

Whatever this parallel transported value of Q is it should equal the value of Q at the unparalleled point I mean the value of Q at its original point and then I just drag each of these p co-vectors and q vectors which lived at t , the new point and I drag them back to s which is what these inverse operators do $\Omega_{\tau,s}^{-1}$, that is an important relationship about Parallel transport for full-on (p,q) tensor fields and we're going to talk a lot more about that in this lesson. Then at the end I had to show that the Covariant derivative operator in the direction of X is Leibniz with respect to the tensor product.

$$\nabla_X[Q\otimes S] = (\nabla_X Q)\otimes S + Q\otimes(\nabla_X S) \tag{15}$$

Then we showed that the contraction operator and the Covariant derivative operator just commute so whatever you put in here doesn't matter you can always commute those two as long as this is some tensor field, you can only contract on the tensor field we know that this is going to result in a tensor field what we have to study how to get a tensor field:

$$C\nabla_X[\ ]=\nabla_X C[\ ] \tag{16}$$

I guess that's a good time to introduce today's topic, the Covariant derivative in the direction of the vector X of some arbitrary tensor Q what is that equal? How do we know what that is? We know what it is for a vector and we know what it was for a co-vector because we did that in the last lesson, we did the co-vector in the last lesson and I think I wrote that down up here:

$$\nabla\alpha=\left(\alpha_{\kappa|\delta}-\Gamma^{\sigma}_{\kappa\delta}\alpha_{\sigma}\right)\mathrm{d}x^{\kappa}\otimes\mathrm{d}x^{\delta} \tag{17}$$

Then last lesson we ended up with the Covariant derivative of a co-vector was equal to this expression (17) and then the Covariant derivative in the direction X of the co-vector ∇_X is just going to be that expression (17) contracted with X^{δ} and then you will end up with another co-vector. Now we can begin to study the general idea of how to calculate the Covariant derivative and the director X of an arbitrary tensor Q where Q is a (p,q) tensor. We pick a tensor $Q\in J^p_q$, it's a (p,q) rank tensor so it's an element of the (p,q) rank tensor products base and we presume it's a tensor field, it's got to be a tensor field so its components are functions of space-time so a (p,q) rank tensor as a reminder:

$$Q^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_p}\otimes\mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_q} \tag{18}$$

If you broke it down into the comp notation would have p contravariant indices and q covariant indices and the basis, I'll just throw it down here, will be $\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_p}$. Then all the tensor products along of the covariant 1-forms $\mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_q}$ so that's the tensor and it's full on basis. I'm going to try to flow between the two notations because we're ultimately trying to find expressions in the CFREE notation but it's just you've got to be able to move between the two very quickly so we'll just do it all.

$$S=\alpha_1\otimes\cdots\otimes\alpha_p\otimes Y_1\otimes\cdots Y_q\otimes Q \tag{19}$$

The way we begin to figure out how to calculate this Covariant derivative if this is going to parallel exactly what we did to calculate (17), to figure what this was, we did that in the last lecture this is going to be the same thing just a lot bigger so the first thing we start with is we create a new tensor S or a new multi-linear mapping, I'll call it a multi linear mapping because strictly speaking it's not a tensor but it behaves exactly like one, it's just the name is not a tensor the reason is that this multi-linear mapping we're gonna add p co-vectors, tensor product all together with q vectors, tensor product all together and then we're gonna take a tensor product with Q itself (19).

Now this makes perfect sense remember Q is just a bunch of vectors and co-vectors in its own right so Q will always be a tensor product of p co-vectors and q vectors so you can end up with an index structure like this:

$$S_{i_1\cdots i_p}{}^{j_1\cdots j_q\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}\,\mathrm{d}x^{i_1}\cdots\mathrm{d}x^{i_p}\,\partial_{j_1}\cdots\partial_{j_q}\,\partial_{\mu_1}\cdots\partial_{\mu_p}\,\mathrm{d}x^{\nu_1}\cdots\mathrm{d}x^{\nu_q} \tag{20}$$

S is going to have p covariant indices q contravariant indices, that's going to take care of these guys and these guys and then Q is right behind it so Q is gonna have p contravariant indices and q covariant indices and the reason we do this is because our next move is to contract all of these things we're going to contract all of the first string of covariant indices with the first with the second string of contravariant indices and the first string of contravariant indices with the second string of covariant indices and so the object we're actually going to create is this contraction:

$$S_{i_1\cdots i_p}{}^{j_1\cdots j_q i_1\cdots i_p}{}_{j_1\cdots j_q} \tag{21}$$

That's a huge contraction but that's the contraction we're trying to build and I'm saying that a little early because we're not going to build it yet but we're gonna get there so let's start with this expression (20) and imagine that we tried to calculate the Covariant derivative of S . The Covariant derivative in the direction X of S , that is going to be:

$$\nabla_X S=\nabla_X\left[\alpha_1\otimes\cdots\otimes\alpha_p\otimes Y_1\otimes\cdots Y_q\otimes Q\right] \tag{22}$$

That's the Covariant derivative we're going to see now the good news is that we know that the Covariant derivative acts Leibniz if the derivation on the tensor algebra so it acts with the Leibniz rule because we've already proven that, we proved that way way back in (15). We decided that that's something we can actually lean on, because this is true for any two tensors. With that in mind I can now write this is equal to:

$$\nabla_X S=\nabla_X\alpha_1\otimes\cdots\otimes Q+\alpha_1\otimes\nabla_X\alpha_2\otimes\cdots\otimes Q+\alpha_1\cdots\nabla_X Q \tag{23}$$

That that is simply the Leibniz rule applied so now if you can remember, we're just doing the same thing now for a (p,q) tensor so what did we do there? We took the contraction so the contraction of the Covariant derivative in the direction of X of S is going to be the contraction of (23). The contraction of a sum is equal to the sum of the contractions, that's an important point.

$$\begin{aligned} C\nabla_X S&=C\{\nabla_X[\alpha_1\otimes\cdots\otimes Q]\}\\ &=C\left[\nabla_X\alpha_1\otimes\cdots\otimes Q\right]+C\left[\alpha_1\otimes\nabla_X\alpha_2\otimes\cdots\otimes Q\right]+\cdots+C\left[\alpha_1\otimes\cdots\otimes\nabla_X Q\right]\end{aligned} \tag{24}$$

I'm really abbreviating now. I'm just doing the first one just to show the pattern. I'll show the last one. That is just taking the contraction of (23) according to Leibniz and expansion of this Covariant derivative, I take the contraction of the whole thing, it's the sum of the contraction so I end up with that. Now we commute the contraction with the Covariant derivative in the direction of X :

$$\nabla_X\left[C\left(\alpha_1\otimes\cdots\otimes Y_p\otimes Q\right)\right] \tag{25}$$

Now we've commuted these two and so now I have the Covariant derivative of a contraction now this is a full contraction, that's why we designed S this way (20), we can contract every index because S is such a big thing, we've added one contractible tensor or vector for every component of Q so we can contract this whole thing so ultimately (25) is a function and the value of that function is simply

$Q(\alpha_1 \cdots Y_q)$, that's how you execute the contraction, that's the definition of how contractions work what contractions are, you're feeding the various constituents of the tensor into itself and you pair it up in one way or another. On a full contraction, every one of the contravariant indices is gobbling up every one of the covariant indices so you end up with $Q(\alpha_1 \cdots Y_q)$ but that is a real number so you end up here with the Covariant derivative with respect to X of this thing:

$$\nabla_X \{Q(\alpha_1 \cdots Y_q)\} = X Q(\alpha_1 \cdots Y^q) \tag{26}$$

This is a function. We've used almost every one of our rules, we use the commutation of the contraction with the Covariant derivative in the direction of X , we've used the Leibniz rule and then we use this fundamental definition that the Covariant derivative in the director B of a function is just acting on the function (4) so we use three rules in one fell swoop right there. Notice that this isn't exactly what we're looking for, this is the Covariant derivative of a function and it almost looks like what we're after the Covariant derivative and direction X of Q that's our goal, our goal was to find the Covariant derivative of this Q , the creation of S is an intermediate step but we're looking how do you understand the Covariant derivative in the direction X of Q and it almost looks like we have it but that's not what this is, this is the Covariant derivative with respect to X of a fully contracted S which is this real number, which is this real function on the manifold, that's not it. We're interested in is over here $\nabla_X Q$, that is the Covariant derivative of Q in the Leibniz expansion, that's what we're after that's the guy we want.

We want this contraction $C[\alpha_1 \otimes \cdots \otimes \nabla_X Q]$. The good news is this thing is fully contractible because remember the Covariant derivative in the direction X of a (p,q) tensor is still going to be a (p,q) tensor, we learned that that this thing for a vector, Covariant derivative in the direction X of a vector A was also a vector, it lived in the $(1,0)$ tensor product space. The Covariant derivative of A was a $(1,1)$ tensor but you contract a $(1,1)$ tensor, if you remember the definition of things $X^\alpha A^\beta{}_{\|\alpha}$ which is the definition of $\nabla_X A \in J_0^1$, that contracts out and all you're left with is a contravariant index indicative of a vector, likewise with co-vectors. It's always going to be true for anything, if ultimately we're gonna write Q something $\|\delta$ and if you contract it with X^δ you're gonna to end up with the something which is unchanged, which doesn't add a covariant index like you usually do on Covariant derivatives so we know ultimately hiding right in here is the Covariant derivative of the tensor Q in the direction of X .

We still have to take all of these contractions but they're not hard to take because it's the same principle we just feed Q each of the vectors, it just so happens one of the vectors is going to always be a Covariant derivative vector but either the Covariant derivative of a 1-form or the Covariant derivative of a vector. There will be a term like $C[\alpha \otimes \cdots \otimes \nabla_X Y_q \otimes Q]$, there will be q terms just like that but it's still the same thing, it's still a vector and these are still co-vectors, in other words that's still a co-vector and that's still a vector therefore you can still execute the full contraction and so what is that full contraction look like? The full contraction of any one of those terms is going to look like any one of those terms except for the last term, like let's look at this term $C[\alpha_1 \otimes \nabla_X \alpha_2 \otimes \cdots \otimes Q]$. That term right there is going to be $Q(\alpha_1, \nabla_X \alpha_2, \cdots, Y_q)$. That's what that contraction would be. The last term here, that's the important one, that's going to be $\nabla_X Q(\alpha_1, \ldots, Y_q)$. That's what that's gonna be.

$$\begin{aligned} &Q(\nabla_X \alpha_1, \alpha_2, \cdots, Y_q) + \cdots + Q(\alpha_1, \cdots, \alpha_p, \nabla_X Y_1, \cdots, Y_q) \\ &+ \cdots + Q(\alpha_1, \cdots, \alpha_p, Y_1, \cdots, \nabla_X Y_q) + \cdots + \nabla_X Q(\alpha_1, \ldots, Y_q) \end{aligned} \tag{27}$$

This is the function, this is the left-hand side of (26), Covariant derivative in the direction X of this function Q which we know by the definition is just X operating on this fully contracted tensor Q which is a function on the space-time so we know what that thing is going to equal.

These terms here are the various contractions (24) I discussed before so I wrote them out a little more cleanly in (27), it's going to be Q being fed the Covariant derivative of the first 1-form and then all the rest plus and then I accelerated it so you do that for the first 1-form the second, the third, each one that's all these $\cdots$ and then you arrive at the first vector and then it's just no big change it's just Q being fed all of those 1-forms and then the Covariant derivative of the first vector and then all the rest of the vectors and then you do that until you've gotten every single one of these Leibniz expanded terms and then the last term though, is the more interesting one, because whatever the Covariant of the derivative of the vector X is being fed all of those vectors and co-vectors and so on. Now you solve for the last term in (27) and that solution is not going to be very difficult:

$$\begin{aligned}\nabla_X Q(\alpha_1,\ldots,Y_q) &= X\,Q(\alpha_1\cdots Y_q) - Q(\nabla_X\alpha_1,\alpha_2,\cdots,Y_q) - \cdots \\ &\quad - Q(\alpha_1,\nabla_X\alpha_2,\cdots,Y_q) - \cdots - Q(\alpha_1,\cdots,\alpha_p,Y_1,\cdots,\nabla_X Y_q)\end{aligned} \tag{28}$$

That's our answer, we know what's going on, this is a (p,q) tensor so it's fully defined by operates on the appropriate co-vectors and vectors and this is how it's done, we know every one of these terms, we can definitely fully contract Q with α_1 through Y_q and take its derivative and operate on it with the differential operator X . We can definitely calculate that, we know how to calculate this and we certainly know how to calculate this full contraction. We know how to calculate that so we can calculate the full contraction, we know how to calculate that.

Everything on the right-hand side of (28) we know how to calculate and what we're looking for is on that left-hand side. This is the CFREE result, this is the CFREE definition of how that guy works so all that's left now is to translate this into the comp notation and therefore this becomes:

$$X^\mu Q\left(-\Gamma^{i_1}_{\alpha\mu}\,\mathrm{d}x^\alpha,\mathrm{d}x^{i_2},\ldots\right) = -X^\mu\Gamma^{i_1}_{\alpha\mu}Q\left(\mathrm{d}x^\alpha,\mathrm{d}x^{i_2},\ldots\right) \tag{29}$$

That's how these things simplify, now eventually this Covariant derivative term goes into the vector part in which case the substitution here will no longer utilize this pattern:

$$\nabla_i\,\mathrm{d}x^j = -\Gamma^j_{\alpha i}\,\mathrm{d}x^\alpha \tag{30}$$

It'll no longer use that pattern (30), it'll use this pattern:

$$\nabla_i\partial_j = \Gamma^\alpha_{ji}\partial_\alpha \tag{31}$$

You make those substitutions and then after you've completed all of those substitutions you're left with:

$$Q\left(\mathrm{d}x^{i_1}\cdots\nabla_X\partial_{j_q}\right) \tag{32}$$

We can break down each one of those terms in this manner (29) either using the appropriate definition for individual co-vectors and individual vectors, that's why we had to prove the statement for each one individually so we could actually do it for the tensor as a whole so we'll have this term here:

$$X^{\mu}\frac{\partial}{\partial x^{\mu}}Q(-) \tag{33}$$

which we can solve because we it's just gonna be simply this expression right there:

$$X^{\mu}Q^{i_1\cdots i_p}{}_{j_1\cdots j_q|\mu} \tag{34}$$

Remember Q with all of these guys in, it is just going to be the real number of that is equal to its component so then you take that real number which is equal to its component you take its derivative with respect to μ and you get that little μ there and it's regular partial derivative so it's only one line and then of course the X^{μ} is hanging out in front ready to contract with that μ so in these cases you will end up with so let's go ahead and write that down:

$$X^{\mu}Q^{i_1\cdots i_p}{}_{j_1\cdots j_q|\mu}+X^{\mu}\Gamma^{i_1}{}_{\alpha\mu}Q^{\alpha\cdots i_p}{}_{j_1\ldots j_q}\cdots-X^{\mu}\Gamma^{\alpha}{}_{J_1\mu}Q^{i_1\cdots i_p}{}_{\alpha\cdots j_p} \tag{35}$$

Then you just you complete this substitution and you can get rid of the X^{μ} because it's working against everything and you can use that basic rule that is good for X^{μ} any direction and then you can develop the basic expression for the Covariant derivative of a tensor ∇Q which we define as:

$$\nabla Q\equiv Q^{i_1\cdots i_p}{}_{j_1\cdots j_q\|\delta}=Q^{i_1\cdots i_p}{}_{j_1\cdots j_q|\delta}+\Gamma^{i_1}{}_{\sigma\delta}Q^{\sigma\, i_2\cdots i_p}{}_{j_1\cdots j_q}\cdots-\Gamma^{\sigma}{}_{j_1\delta}Q^{i_1\cdots i_p}{}_{\sigma\cdots j_q}\cdots \tag{36}$$

Which is the pattern you see in almost every textbook and so ultimately this is the final component based analysis of ∇Q . On the CFREE notation this is the expression right here (28) this is the definition of how you take the Covariant derivative of a (p,q) tensor and then when you break that down into the component notation you end up with this expression here (36). We've made that connection Here I've actually cleaned it up and summarized it right in the abstract notation this is the definition of the Covariant derivative of a tensor (p,q) tensor Q , in the direction of X . Now that tensor operates on co-vectors and vectors with the following formula. Remember the right-hand side is stuff we all know we know how to calculate the tensor Q operating on a bunch of vectors and co-vectors and understanding that that's just a vector and that's just a vector and that's just a co-vector and that's just a co-vector and I put in enough for you to see the pattern.