Cosmology Lecture 1: Friedmann–Lemaître–Robertson–Walker Metric

Space of Geometry

Now let us talk about some of the possible spatial geometries under the condition of homogeneity and isotropy. To determine the possible space of geometry so we first have to write γ_{ij} . Since γ_{ij} needs to be isotropic around any choice of origin O , the metric can be written in spherical coordinates in the following manner:

$$\mathrm{d}s_3^2=\mathrm{d}\chi^2+\mathrm{f}(\chi)\left(\mathrm{d}\theta^2+\sin^2\theta\ \mathrm{d}\varphi^2\right) \tag{1}$$

We must find functions $\mathrm{f}(\chi)$ that lead to a homogeneous isotropic 3-manifold and to do so we let us compute The Ricci tensor and there are many ways to do this but my preferred way is to use differential forms as it is relatively faster than calculating the Christopher symbols, we will calculate the $\boldsymbol{R}_{\chi\chi}$ component as an example and the rest will be listed in subsequent slide as the same methodology can be applied. To compute the Ricci tensor using differential forms we first have to write the orthonormal basis. The easiest orthonormal basis one can write is as follows:

$$\begin{aligned}
&e^{\hat{\chi}}=\mathrm{d}\chi\\
&e^{\hat{\theta}}=\sqrt{\mathrm{f}(\chi)}\ \mathrm{d}\theta=\mathrm{h}(\chi)\ \mathrm{d}\theta\\
&e^{\hat{\varphi}}=\sqrt{\mathrm{f}(\chi)}\sin\theta\ \mathrm{d}\varphi=\mathrm{h}(\chi)\sin\theta\ \mathrm{d}\varphi
\end{aligned} \tag{2}$$

where the hats denote the coordinates in the orthonormal bases. Now let us take the exterior derivatives of these orthonormal bases and we yield the following, where the primes denote the derivatives with respect to χ :

$$\begin{aligned}
&\mathrm{d}e^{\hat{\chi}}=0\\
&\mathrm{d}e^{\hat{\theta}}=h'\ \mathrm{d}\chi\wedge\mathrm{d}\theta\\
&\mathrm{d}e^{\hat{\varphi}}=h'\sin\theta\ \mathrm{d}\chi\wedge\mathrm{d}\theta+h\cos\theta\ \mathrm{d}\theta=\frac{h'}{h}e^{\hat{\chi}}\wedge e^{\hat{\varphi}}+\frac{\cot\theta}{h}e^{\hat{\theta}}\wedge e^{\hat{\varphi}}
\end{aligned} \tag{3}$$

Robertson Walker Metric

From this we can now introduce the Cartan's first equation of structure to calculate the spin connections

$$\mathrm{d}e^{\hat{\alpha}}+\omega^{\hat{\alpha}}_{\ \hat{\beta}}\wedge e^{\hat{\beta}}=0 \tag{4}$$

The equality above is zero because there is no torsion in General relativity and by inspection the relevant spin connections are as follows:

$$\omega^{\hat{\theta}}_{\ \hat{\chi}}=\frac{h'}{h}e^{\hat{\theta}}=h'\ \mathrm{d}\theta \tag{5}$$

$$\omega^{\hat{\varphi}}_{\hat{\chi}} = \frac{h'}{h} e^{\hat{\varphi}} = h' \sin\theta \; \mathrm{d}\varphi \tag{6}$$

$$\omega^{\hat{\varphi}}_{\hat{\theta}} = \frac{\cot\theta}{h} e^{\hat{\varphi}} = \cos\theta \; \mathrm{d}\varphi \tag{7}$$

Now to calculate the Ricci tensor in orthonormal bases we use the Cartan's second equation of structure

$$\boldsymbol{R}^{\hat{\alpha}}_{\hat{\beta}} = \mathrm{d}\omega^{\hat{\alpha}}_{\hat{\beta}} + \omega^{\hat{\alpha}}_{\hat{\gamma}} \wedge \omega^{\hat{\gamma}}_{\hat{\beta}} \tag{8}$$

Therefore the relevant Ricci tensor in orthonormal basis are as follows:

$$\boldsymbol{R}^{\hat{\theta}}_{\hat{\chi}} = h'' \; \mathrm{d}\chi \wedge \mathrm{d}\theta \tag{9}$$

$$\boldsymbol{R}^{\hat{\varphi}}_{\hat{\chi}} = h'' \sin\theta \; \mathrm{d}\chi \wedge \mathrm{d}\varphi \tag{10}$$

Moreover, since the Riemann tensor is related to the Ricci tensor in orthonormal bases:

$$\boldsymbol{R}^{\hat{\alpha}}_{\hat{\beta}\,\gamma\,\delta} \; \mathrm{d}\gamma \wedge \mathrm{d}\delta = \boldsymbol{R}^{\hat{\alpha}}_{\hat{\beta}} \tag{11}$$

The Riemann tensor is related to the Riemann tensor in component form in the following manner:

$$\boldsymbol{R}^{\alpha}_{\beta\,\gamma\,\delta} = e^{\alpha}_{\hat{\alpha}} e^{\hat{\beta}}_{\beta} \boldsymbol{R}^{\hat{\alpha}}_{\hat{\beta}\,\gamma\,\delta} \tag{12}$$

Where $e^{\alpha}_{\hat{\alpha}}$ is the connection 2-form. Therefore:

$$\boldsymbol{R}_{\chi\,\chi} = \boldsymbol{R}^{\varphi}_{\chi\,\varphi\,\chi} + \boldsymbol{R}^{\theta}_{\chi\,\theta\,\chi} = \frac{2h''}{h} = \left(\frac{\mathrm{f}'}{\mathrm{f}}\right)' - \frac{\mathrm{f}'^2}{2\,\mathrm{f}^2} \tag{13}$$

The other Ricci tensors are listed as follows:

$$\boldsymbol{R}_{\theta\,\theta} = -½\mathrm{f}'' + 1 \tag{14}$$

$$\boldsymbol{R}_{\varphi\,\varphi} = \boldsymbol{R}_{\theta\,\theta} \sin^2\theta \tag{15}$$

For the 3-manifold to be spatially homogeneous and isotropic, it is necessary (though may not be sufficient) for the manifold to be an Einstein manifold that is, $\boldsymbol{R}_{i\,j} = K\,\mathrm{g}_{i\,j}$ where K is some constant therefore one requires:

$$K=\left(\frac{f'}{f}\right)'-\frac{f'^2}{2f^2}=\frac{-½f''+1}{f} \tag{16}$$

As a result, we have the following second-order differential equation and it is relatively easy to solve:

$$f''=-2Kf+2 \tag{17}$$

The equation in the previous slide (17) is a simple ordinary differential equation that can be solved by using the ansatz:

$$f(\chi)=c_1\sin(\sqrt{2K}\chi)+c_2\cos(\sqrt{2K}\chi)+1/K \tag{18}$$

Where c_1 and c_2 are arbitrary concepts of integration. However regularity at O requires that $f(\chi=0)=f'(\chi=0)=0$ thus the solution simplifies to:

$$f(\chi)=\frac{2}{K}\sin^2\left(\sqrt{\frac{K}{2}}\chi\right) \tag{19}$$

Therefore the spatial metric for a homogeneous and isotropic universe is as follows:

$$ds_3^2=d\chi^2+\frac{2}{K}\sin^2\left(\sqrt{\frac{K}{2}}\chi\right)(d\theta+\sin^2\theta\ d\varphi) \tag{20}$$

From this the metric in equation (20) it is easy to check homogeneity and isotropy. Equation (20) is the metric for a 3-sphere of radius $\sqrt{2/K}$. In the limit, when $K\to 0$, Equation (20) becomes the flat 3D Euclidean metric in spherical coordinates. Therefore the spatial metric $K>0$, then we say that the space-time is closed and if $K=0$ we say that the space-time is spatially flat. There is another interesting property when $K<0$. In this case spatial geometry is open and one can take the analytic continuation of the sine function to obtain the following:

$$ds_3^2=d\chi^2-\frac{2}{K}\sinh^2\left(-\sqrt{\frac{K}{2}}\chi\right)(d\theta+\sin^2\theta\ d\varphi) \tag{21}$$

Now let us summarize some of the spatial geometries:

- The first of which is $\mathbb{R}^3$ or the specially flat space-time. This is standard Euclidean space, with infinite volume and the rules of geometry. In fact, our universe appears to be spatially flat to a very good approximation.
- The second geometry S^3 is the closed space-time. This space time has the geometry of a 3-sphere and the usual coordinate system correspond to hyper-spherical coordinates. The maximum distance from the origin is the anti-podal point at $\chi=\pi(K/2)^{-½}$ where there is a

coordinate singularity. The volume of the sphere is finite: $V_3 = 2\pi^2 (K/2)^{-3/2}$. The closed space exhibits non-euclidean features for example the interior angles of a triangle add to $>\pi$ and the Pythagorean theorem reads $a^2 + b^2 > c^2$.

- The third of which is H^3 or the open space-time. This is known as the hyperbolic space, with the same topology as Euclidean space. However it exhibits non-Euclidean features. For example the interior regions of a triangle add to $<\pi$ and the Pythagorean theorem reads $a^2 + b^2 < c^2$. Not only is the volume of an open space infinite, but it's exponentially infinite in the sense that the volume of a sphere of radius r is $V_3 = 2\pi(-K/2)^{-3/2}$ which increases exponentially with the radius.
- The final of which is $\mathbb{R}P^3$ or projector space. This is an alternative topology of the closed universe in which the anti-podal points are: $(\chi, \theta, \varphi) = (\pi K^{-1/2} - \chi, \pi - \theta, \pi + \theta)$. Locally it looks like a closed universe but has half the volume $V_3 = \pi^2 (K/2)^{-3/2}$.

Summary

The unique region of the projection space has $\chi < \pi / \sqrt{2K}$ i.e. is the region between the North Pole (origin) and the equator; if one passes the equator one reappears on the opposite side of the sphere. This is the only non-trivial topology of any of these spaces we have considered that is globally homogeneous and isotropic.

Cosmology Lecture 2: The Friedmann Equations and Single-Component Universes

Intro

The following is part of a series of lectures on Cosmology in this particular lecture we will derive the Friedman equations and explore some universes with single components such as matter, radiation and dark energy. The outline of this lecture is as follows, first we will summarize what we have done in the last lecture then we will proceed to derive the Friedman equations, next we will write the Friedman equations in terms of the density parameters afterwards we will explore four types of single component universes and finally we will summarize the four types of single component universes using some plots

Summary of Last Lecture

In the last lecture we first mathematically formalized the notion of homogeneity and isotropy. Thereafter we derived the Friedmann–Lemaître–Robertson–Walker metric in the following form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + [a(t)]^2 \gamma_{ij}(x^k)\,\mathrm{d}x^i \mathrm{d}x^i \tag{22}$$

Finally, we derived the possible spatial geometries under the condition of homogeneity and isotropy. We found that if $K>0$ then we have a closed universe with the following metric:

$$\mathrm{d}s_3^2 = \mathrm{d}\chi^2 + \frac{2}{K}\sin^2\left(\sqrt{\frac{K}{2}}\chi\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{23}$$

Next we found that if $K=0$, we have the flat universe with the flat Euclidean metric:

$$\mathrm{d}s_3^2 = \mathrm{d}\chi^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 \tag{24}$$

And finally if $K<0$ we take the analytic continuation of the sine function and we obtain the open universe with the following metric:

$$\mathrm{d}s_3^2 = \mathrm{d}\chi^2 - \frac{2}{K}\sinh^2\left(-\sqrt{\frac{K}{2}}\chi\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{25}$$

There are also non-trivial topologies but they will not be discussed as they are beyond the scope of this lecture.

The Friedmann Equations

The Friedman equations are a set of equations in physical Cosmology that govern the expansion of space in a homogeneous and isotropic models of the universe. To derive the Friedman equations we need to solve the Einstein field equations using the Friedmann–Lemaître–Robertson–Walker metric. To do this let us first write the Friedmann–Lemaître–Robertson–Walker metric in an alternative form namely the following:

$$\mathrm{d}s^2=-\mathrm{d}t^2+[a(t)]^2\left[\frac{\mathrm{d}r^2}{1-kr^2}+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2\right)\right] \tag{26}$$

Like the last lecture, we will use the method of differential forms to compute the $\boldsymbol{R}_{tt}$ component for pedagogical purposes. Then the rest will be listed out, as the same methodology can be used to obtain the other components of the Ricci tensor.

To compute the Ricci tensor using differential forms, we first have to write the orthonormal bases. The easiest orthonormal one can write is:

$$\begin{aligned}
e^{\hat{t}}&=\mathrm{d}t\\
e^{\hat{r}}&=\frac{a(t)}{\sqrt{1-kr^2}}=a(t)\mathrm{f}(r)\mathrm{d}r\\
e^{\hat{\theta}}&=a(t)r\,\mathrm{d}\theta\\
e^{\hat{\varphi}}&=a(t)r\sin\theta\,\mathrm{d}\varphi
\end{aligned} \tag{27}$$

Where the hats denote the coordinate in the orthonormal bases. Now let the prime and over dot denote the derivative with respect to r and t respectively. Taking the exterior derivative of the orthonormal basis yields the following:

$$\begin{aligned}
\mathrm{d}e^{\hat{t}}&=0\\
\mathrm{d}e^{\hat{r}}&=\dot{a}\,\mathrm{f}\;\mathrm{d}t\wedge\mathrm{d}r=\frac{\dot{a}}{a}e^{\hat{t}}\wedge e^{\hat{r}}\\
\mathrm{d}e^{\hat{\theta}}&=\dot{a}r\;\mathrm{d}t\wedge\mathrm{d}\theta+a\;\mathrm{d}r\wedge\mathrm{d}\theta=\frac{\dot{a}}{a}e^{\hat{t}}\wedge e^{\hat{\theta}}+\frac{1}{ar\,\mathrm{f}}e^{\hat{r}}\wedge e^{\hat{\theta}}\\
\mathrm{d}e^{\hat{\varphi}}&=\dot{a}r\sin\theta\;\mathrm{d}t\wedge\mathrm{d}\varphi+a\sin\theta\;\mathrm{d}r\wedge\mathrm{d}\varphi+ar\cos\theta\;\mathrm{d}\theta\wedge\mathrm{d}\varphi\\
&=\frac{\dot{a}}{a}e^{\hat{t}}\wedge e^{\hat{\varphi}}+\frac{1}{ar\,\mathrm{f}}e^{\hat{r}}\wedge e^{\hat{\varphi}}+\frac{\cot\theta}{ar}e^{\hat{\theta}}\wedge e^{\hat{\varphi}}
\end{aligned} \tag{28}$$

Now we can use Cartan's first equation of structure to calculate the spin connections:

$$\mathrm{d}e^{\hat{\alpha}}+\omega^{\hat{\alpha}}_{\hat{\beta}}\wedge e^{\hat{\beta}}=0 \tag{29}$$

Note that this equality is equal to zero because in General relativity there is no torsion. By inspection the relevant spin connections are as follows:

$$\omega^{\hat{r}}_{\hat{t}}=\frac{\dot{a}}{a}e^{\hat{r}}=\dot{a}\,\mathrm{f}\,\mathrm{d}r \tag{30}$$

$$\omega^{\hat{\theta}}_{\hat{t}} = \frac{\dot{a}}{a} e^{\hat{\theta}} = \dot{a} r \, \mathrm{d}\theta \tag{31}$$

$$\omega^{\hat{\theta}}_{\hat{r}} = \frac{1}{a r \mathrm{f}} e^{\hat{\theta}} = \frac{1}{\mathrm{f}} \, \mathrm{d}\theta \tag{32}$$

$$\omega^{\hat{\varphi}}_{\hat{t}} = \frac{\dot{a}}{a} e^{\hat{\varphi}} = \dot{a} r \sin\theta \, \mathrm{d}\varphi \tag{33}$$

$$\omega^{\hat{\varphi}}_{\hat{r}} = \frac{1}{a r \mathrm{f}} e^{\hat{\varphi}} = \frac{\sin\theta}{\mathrm{f}} \, \mathrm{d}\varphi \tag{34}$$

$$\omega^{\hat{\varphi}}_{\hat{\theta}} = \frac{\cot\theta}{a r} e^{\hat{\varphi}} = \cos\theta \, \mathrm{d}\varphi \tag{35}$$

Now recall Cartan's second equation of structure is as follows:

$$\boldsymbol{R}^{\hat{\alpha}}_{\hat{\beta}} = \mathrm{d}\omega^{\hat{\alpha}}_{\hat{\beta}} + \omega^{\hat{\alpha}}_{\hat{\gamma}} \wedge \omega^{\hat{\gamma}}_{\hat{\beta}} \tag{36}$$

The Ricci tensor is defined in the following manner:

$$\boldsymbol{R}_{tt} = \boldsymbol{R}^{t}_{ttt} + \boldsymbol{R}^{t}_{trt} + \boldsymbol{R}^{\theta}_{t\theta t} + \boldsymbol{R}^{\varphi}_{t\varphi t} \tag{37}$$

Now recall that the Riemann tensor in orthonormal basis is related to the Ricci tensor in orthonormal basis in the following form:

$$\boldsymbol{R}^{\hat{\alpha}}_{\hat{\beta}\gamma\delta} \, \mathrm{d}\gamma \wedge \mathrm{d}\delta = \boldsymbol{R}^{\hat{\alpha}}_{\hat{\beta}} \tag{38}$$

Thus we only require $\mathrm{d}\omega^{\hat{r}}_{\hat{t}}$, $\mathrm{d}\omega^{\hat{\theta}}_{\hat{t}}$ and $\mathrm{d}\omega^{\hat{\varphi}}_{\hat{t}}$. By inspection we find that the Riemann tensors in orthonormal basis are as follows:

$$\boldsymbol{R}^{\hat{r}}_{\hat{t}rt} = -\ddot{a}\mathrm{f} \tag{39}$$

$$\boldsymbol{R}^{\hat{\theta}}_{\hat{t}\theta t} = -\ddot{a} r \tag{40}$$

$$\boldsymbol{R}^{\hat{\varphi}}_{\hat{t}\varphi t} = -\ddot{a} r \sin\theta \tag{41}$$

Now recall that the Riemann tensor in coordinate basis is related to the Riemann tensor in in orthonormal basis in the following manner:

$$\boldsymbol{R}^{\alpha}_{\beta\gamma\delta}=e^{\alpha}_{\hat{\alpha}}e^{\hat{\beta}}_{\beta}\boldsymbol{R}^{\hat{\alpha}}_{\hat{\beta}\gamma\delta} \tag{42}$$

Where $e^{\alpha}_{\hat{\alpha}}$ is the connection 2-form. Therefore:

$$\boldsymbol{R}_{tt}=-3\frac{\ddot{a}}{a} \tag{43}$$

Listing out the other Ricci tensor components yields:

$$\boldsymbol{R}_{rr}=\frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} \tag{44}$$

$$\boldsymbol{R}_{\theta\theta}=r^2(1-kr^2)\boldsymbol{R}_{rr} \tag{45}$$

$$\boldsymbol{R}_{\varphi\varphi}=\boldsymbol{R}_{\theta\theta}\sin^2\theta \tag{46}$$

Moreover, the Ricci scalar is as follows:

$$\boldsymbol{R}=6\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right) \tag{47}$$

Now, let the energy momentum tensor be that of a perfect fluid:

$$\boldsymbol{T}_{\mu\nu}=(\rho+p)u_{\mu}u_{\nu}+p\,g_{\mu\nu} \tag{48}$$

Where p is the pressure, ρ is the energy density and u is the four-velocity which is defined in the following manner: $u^{\mu}=u_{\mu}=(1,0,0,0)^T$. Substituting the relevant Ricci tensors and scalars into the tt component of the Einstein field equation yields the following:

$$\frac{\dot{a}^2+k}{a^2}=\frac{8\pi\rho+\Lambda}{3} \tag{49}$$

Now, taking the trace of the Einstein field equation yields:

$$\frac{\ddot{a}}{a}=-\frac{4\pi}{3}(\rho+3p)+\frac{\Lambda}{3} \tag{50}$$

Equations (49) and (50) are collectively known as the Friedman equations. These equations govern the dynamics of an expanding homogeneous and isotropic universe.

Now let us discuss the Friedman equations. The density parameter Ω is defined as the ratio of the actual (or observed) density ρ to the critical density ρ_c of the Friedman universe. The relation between the actual density and the critical density determines the overall geometry of the universe. Critical density ρ_c is found by setting $\Lambda=k=0$. Moreover, if we set the Hubble parameter $H=\dot{a}/a$ then $\rho_c=3H^2/8\pi$ Thus $\Omega=8\pi\rho/3H^2$. If the density parameter $\Omega<1$, there is too little matter to overcome the dark energy and therefore the universe is open. If the density parameter $\Omega=1$ there is sufficient matter to counter the repulsive effects of the dark energy and thus the universe is Euclidean flat. If $\Omega>1$, then there is too much matter and thus the universe is closed. Subsequently we will explore single-component universes. From what we know the universe consists of four major components: ordinary matter, dark matter, radiation and dark energy.

Although dark matter is much denser, ordinary matter and dark matter contributes to the favor of the contraction of the universe. Dark energy does not lead to the contract of the universe but rather the accelerated expansion. Now to explore single-component universes, the Friedman equations can be written in terms of the density parameters and the cosmic scale factor $a(t)$ in the following manner:

$$\frac{H^2}{H_0^2}=\Omega_{0,R}a^{-4}+\Omega_{0,M}a^{-3}+\Omega_{0,k}a^{-2}+\Omega_{0,\Lambda} \tag{51}$$

Where H_0 is the Hubble's constant, $\Omega_{0,R}$ is the radiation density today, $\Omega_{0,M}$ is the matter density today which include both dark and baryonic matter, $\Omega_{0,k}$ is the spatial curvature density today and $\Omega_{0,\Lambda}$ is the cosmological constant or vacuum density today.

The first universe we will explore is an empty universe absent of mater, radiation and dark energy. With this, $\Omega_{0,k}\gg\Omega_{0,R}$, $\Omega_{0,k}\gg\Omega_{0,M}$ and $\Omega_{0,k}\gg\Omega_{0,\Lambda}$. Therefore $\Omega_{0,R}\approx\Omega_{0,M}\approx\Omega_{0,\Lambda}\approx 0$ and $\Omega_{0,k}\approx 1$ and from this we obtain the following Friedman equation:

$$H^2=\left(\frac{\dot{a}}{a}\right)^2=H_0^2a^{-2} \tag{52}$$

Therefore solving the above differential equation yields the following:

$$a(t)=H_0t+c_1 \tag{53}$$

We see that the cosmic scale factor is related to the time in a linear fashion where c_1 is some constant of integration. If we use initial condition $a(0)=0$ then $c_1=0$ and therefore:

$$a(t)=H_0t \tag{54}$$

The second universe we will explore is the matter-dominated universe with this, $\Omega_{0,M} \gg \Omega_{0,k}$, $\Omega_{0,M} \gg \Omega_{0,R}$ and $\Omega_{0,M} \gg \Omega_{0,\Lambda}$. Therefore the density parameters $\Omega_{0,R} \approx \Omega_{0,k} \approx \Omega_{0,\Lambda} \approx 0$ and $\Omega_{0,M} \approx 1$. From this we obtain the following Friedman equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 a^{-3} \qquad (55)$$

Solving the above differential equation yields the following and using $a(0)=0$:

$$a(t) = \left(\frac{3}{2} H_0 t\right)^{\frac{2}{3}} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} \qquad (56)$$

As we can see that $a(t)$ is proportional to $t^{2/3}$. We add a constant to compensate t_0 which is the age of the universe.

The third universe we will explore is the radiation-dominated universe. With this, $\Omega_{0,R} \gg \Omega_{0,k}$, $\Omega_{0,R} \gg \Omega_{0,M}$ and $\Omega_{0,R} \gg \Omega_{0,\Lambda}$. Therefore, the density parameters $\Omega_{0,m} \approx \Omega_{0,k} \approx \Omega_{0,\Lambda} \approx 0$ and $\Omega_{0,R} \approx 1$. From this we obtain the following Friedman equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 a^{-4} \qquad (57)$$

Solving the above differential equation yields and using $a(0)=0$:

$$a(t) = \sqrt{2 H_0 t} = \left(\frac{t}{t_0}\right)^{\frac{1}{2}} \qquad (58)$$

Here $a(t)$ is proportional to $t^{1/2}$ and t_0 is again the age of the universe.

The final universe we will explore is the Λ -dominated universe. This is a little bit different because it's not related to t as a power law but it is instead related as an exponential. With this, $\Omega_{0,\Lambda} \gg \Omega_{0,k}$, $\Omega_{0,\Lambda} \gg \Omega_{0,M}$ and $\Omega_{0,\Lambda} \gg \Omega_{0,R}$. Therefore, the density parameters $\Omega_{0,M} \approx \Omega_{0,k} \approx \Omega_{0,R} \approx 0$ and $\Omega_{0,\Lambda} \approx 1$. From this we obtain the following Friedman equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \qquad (59)$$

Solving the above differential equation and using the fact that the second derivative with respect to time is positive:

$$a(t) = a_0 E^{H_0 t} \qquad (60)$$

We find that the cosmic scale factor is related to time as an exponential function where a_0 is some constant that represents some initial cosmic scale factor at $t=0$.

To summarize we plot the cosmic scale factor a versus some conformal factor times time. The plot has been normalized to have the same scale factor and expansion rate in the current epoch for all cases so as we can see the cosmic scale factor increases more rapidly with time for the Λ -dominated universe compared to the radiation and matter only dominated universes. The slope of the matter-dominated universe and the radiation-dominated universe is less steep than that of the empty universe this indicates that the radiation and matter hinders the universe's ability to expand.

That is all for this lecture and I hope you have learned something, see you in the next lecture thank you.