The basics of gauge transformations

Introduction

Today we're going to continue our discussion of the formal structure of Maxwell's equations. We're still in real space, I titled this in real space, in reciprocal space because to me this is the important concept I kind of want to get to but there's a lot of structure to Maxwell's equations we have to know as prerequisites to QED and it's mostly a review and we're doing it in a very formal way which is how you need to know it for QED and a lot of this can be done in real space and then converted into reciprocal space so we will persist with our work in real space and then convert to reciprocal space.

Today we're on the big topic of gauge invariance and this is an important issue, it's a fascinating issue, and I'm not sure how many lectures we're going to do on it but we'll at least do this one is absolutely critical prerequisite lecture the subsequent ones will be important and interesting but less and less critical as a prerequisite so this time this lecture it's really just the basics of what we mean by gauge invariance in real space. Let's begin.

Maxwell's equations

As it has become my habit I've now created a bit of a mind map for this lesson so we want to start with Maxwell's equations themselves again these are all vector fields (1) that I write without vector notation, I've gone through this before and Maxwell's equations we've talked about in detail and we're comfortable with them.

$$\boldsymbol{E}=\vec{E}(\vec{r},t)\quad,\quad \boldsymbol{B}=\vec{B}(\vec{r},t)\quad,\quad \rho=\rho(\vec{r},t)\quad,\quad \boldsymbol{j}=\vec{j}(\vec{r},t) \tag{1}$$

$$\begin{cases} \nabla\cdot\boldsymbol{E}=\dfrac{1}{\varepsilon_0}\rho \\ \nabla\cdot\boldsymbol{B}=0 \\ \nabla\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t} \\ \nabla\times\boldsymbol{B}=\dfrac{1}{c^2}\dfrac{\partial\boldsymbol{E}}{\partial t}+\dfrac{1}{\varepsilon_0 c^2}\boldsymbol{j} \end{cases} \tag{2}$$

I've written them as the two divergence equations on top and the two curl equations on the bottom an alternative way of doing this is to write the two source equations these have this is a source equation (the first) and this is a source equation (the fourth) and pair them up and have the source equations and the source-less equations paired up, we could pair those two up that's another way of pairing them up but either way you have to understand that we're dealing with divergence equations, curl equations, equations with sources and equations without sources and there are four of them that's the important topic to nail down right now and we've also prepared ourselves with the notion of the vector potentials, the vector potential and the scalar potential and we understand that the electric field can now be defined in terms of the vector and scalar potential and the magnetic field a little simpler it is just the curl of the vector potential.

$$\begin{cases} \boldsymbol{E} = -\nabla V - \dfrac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} = \nabla \times \boldsymbol{A} \end{cases} \tag{3}$$

Now what we did a little bit last time I made a proof of how any magnetic field can be derived from the curl of some vector field and that proof actually had a step in it where we determined that the z component of A i.e. A_z , I think I chose the z component you could have done the proof with any of the components. I invented a condition where the z component of the vector potential was zero and inside that process I exploited something called Gauge invariance and the notion was that I could add to $\boldsymbol{A}$ any gradient of a function F where F is a scalar function of space $x\,y\,z$ and time or $\hat{r}$ and time and as long as that was a gradient of a scalar function, I pick some scalar function calculate its gradient add it to a the curl of $\boldsymbol{A}$ would still be $\boldsymbol{B}$ but the curl of the gradient is zero because the curl of the gradient of any scalar function always equals zero.

I exploited this fact in that proof but what's interesting is this is a very profound statement that and it's and it's actually it should be confusing, you should be confused by this in the sense that if you were just told that I can find a vector field $\boldsymbol{A}$ that gives you a magnetic field $\boldsymbol{B}$ you think okay $\boldsymbol{B}$ is a physical thing, $\boldsymbol{A}$ must be a physical thing too because simple process on this physical thing gives you a physical thing but now we see that $\boldsymbol{A}$ has a problem, it is ambiguous. There are many different $\boldsymbol{A}$'s which satisfy this relationship, but $\boldsymbol{B}$ is a physical thing so $\boldsymbol{B}$ has to be unique so on this side we have something that's unique but it's defined through this potential formulation by something that's ambiguous and I'm not an expert on the history of Physics, I do find it kind of fun to see how these ideas were created but this was a big puzzle around the turn of the last century so between the 1800s and the 1900s, this was like a real question that people scratch their heads over and ultimately the answer that has been settled on right the settled science in the sense that this is the way we describe it people know that there's some questions about it but those questions are very rarely really problematic but it goes like this: since the force on a particle which has to do with the acceleration of a particle which has to do with motion, so force goes to motion:

$$\boldsymbol{F} = m\boldsymbol{a} = q\left(\boldsymbol{v} \times \boldsymbol{B}\right) \tag{4}$$

The way we created this notion of force is through the magnetic field, the notion of force is tied to motion and what is Physics if not the study of motion so because of this we want to call $\boldsymbol{B}$ real meaning it exists not real numbers but it exists, it's actually something in the world. Now this is all really weird because it's still a field in space and we're sitting in space we can't taste it we can't see it, we can't touch it and that causes problems for people but what we can see is things moving so $\boldsymbol{B}$ is real because it immediately gives us something that moves through this relation (4).

Now there's a lot of problems with this but that's the general idea of this ambiguity because $\boldsymbol{A}$ is ambiguous we are going to say that it is not real and because $\boldsymbol{B}$ is not ambiguous we are going to say that it is real and we'll sort of fine-tune this a little bit and of course Quantum mechanics comes in and causes trouble with this entire theory, with this entire philosophy, this entire ontology I guess is the way to say it, ontology is the notion of what is real and what is not, it's a philosophy concept though it's not a Physics concept and I'm certainly no expert in it but the point is we are going to adopt the position that $\boldsymbol{A}$ is a mathematical notion that helps us understand how to model $\boldsymbol{B}$, likewise $\boldsymbol{v}$ is a mathematical notion that when combined with $\boldsymbol{A}$ helps us model $\boldsymbol{E}$. For the purposes of this lecture that's the ontology we're going to adopt and it's the standard ontology it's not like some weird version of things.

Derivative expressions

Now that we've gotten the potential formulation down and we have Maxwell's equations down, now we want to ask the question well what if we substitute the potential formulation into Maxwell's equations and we can see a couple things are definitely going to happen right away one is since the potential formulations already have derivatives, everything is a derivative of something on the right hand side here and Maxwell's equations are all derivative equations, we're going to get an additional derivative so these are all first derivative expressions where we're going to end up with second derivative expression so let's just do it.

$$\nabla\cdot\left[-\nabla V-\frac{\partial \boldsymbol{A}}{\partial t}\right]=\frac{1}{\varepsilon_0}\rho \tag{5}$$

We just have to calculate the divergence of a gradient, well that's the Laplacian, remember V is a scalar field so it's the Laplacian of that scalar and the time derivative of $\boldsymbol{A}$, is the time derivative of a vector field you do that component by component and you end up with this expression:

$$-\nabla^2 V-\frac{\partial}{\partial t}[\nabla\cdot\boldsymbol{A}]=\frac{1}{\varepsilon_0}\rho \tag{6}$$

The divergence of the vector potential comes by distributing this divergence right into here and then swapping the derivative orders so that's what gives us (6), that's Maxwell's first source equation written in terms of the vector and scalar potential. Now we can turn to the second equation, well that's really easy because that's just the divergence of the curl of the vector potential:

$$\nabla\cdot[\nabla\times\boldsymbol{A}]=0 \tag{7}$$

We actually use this to define the vector potential so this is automatically satisfied via the very mechanics we use to create the notion of the vector potential. We'd learn nothing new there, likewise we learn nothing new from the third equation because we use that to understand and create the notion of the scalar potential.

$$\nabla\times\overbrace{\left[-\nabla V-\frac{\partial \boldsymbol{A}}{\partial t}\right]}^{\boldsymbol{E}}=-\frac{\partial}{\partial t}\overbrace{[\nabla\times\boldsymbol{A}]}^{\boldsymbol{B}} \tag{8}$$

That's automatically satisfied by the very definition of how V and $\boldsymbol{A}$ work and you can see if you take the time derivative of the curl of $\boldsymbol{A}$ is going to cancel with the curl of the time derivative of $\boldsymbol{A}$, you're going to get it on both sides of the equal signs those will cancel and then all you have left is the curl of the gradient of V and of course that's zero because the curl of a gradient is zero so this is automatically satisfied so there's nothing new there it's all part of our definition of what the potentials are.

That was from our previous lessons so these two guys don't really give us anything new, the third one however does so basically the two equations that have sources in them give us new information or are actually still part of what's left over that has new information:

$$\nabla\times\overbrace{[\nabla\times\boldsymbol{A}]}^{\boldsymbol{B}}=\frac{1}{c^2}\frac{\partial}{\partial t}\overbrace{\left[-\nabla V-\frac{\partial\boldsymbol{A}}{\partial t}\right]}^{\boldsymbol{E}}+\frac{1}{\varepsilon_0 c^2}\boldsymbol{j} \tag{9}$$

If you wrestle with this a little bit and you use your vectors identities:

$$\nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}=-\frac{1}{c^2}\frac{\partial}{\partial t}[\nabla V]-\frac{\partial^2\boldsymbol{A}}{\partial t^2}+\frac{1}{\varepsilon_0 c^2}\boldsymbol{j} \tag{10}$$

Notice that the Laplacian of a vector field is a vector but the Laplacian of a scalar field is a scalar so you need to understand that and that comes from the vector identity that links the curl of a curl to these two things (left hand side of (10)). You do have to know this vector identity exists and then this (right hand side of (10)) is just distributing derivative operator into the time derivative operation into this bracket you get the time derivative of the gradient of the scalar V and then you get the second time derivative of $\boldsymbol{A}$, this is actually a new equation and ultimately this leads to the compression of all of Maxwell's equations where these two (source-less) are compressed by the very definitions of $\boldsymbol{A}$ and V and then these two (source-full) are compressed just as we've executed them a moment ago and you're left with these two guys:

$$\begin{cases} -\nabla^2 V-\dfrac{\partial}{\partial t}[\nabla\cdot\boldsymbol{A}]=\dfrac{1}{\varepsilon_0}\rho \\ \left[\dfrac{1}{c^2}\dfrac{\partial}{\partial t}-\nabla^2\right]\boldsymbol{A}=\dfrac{1}{\varepsilon_0 c^2}-\nabla\left[\nabla\cdot\boldsymbol{A}+\dfrac{1}{c^2}\dfrac{\partial V}{\partial t}\right] \end{cases} \tag{11}$$

These are now Maxwell's equations in potential form and so this is where we're going to start our motivation of this notion of gauge and exploiting the ambiguity of $\boldsymbol{A}$ and V .

Second order differential equations

When I first learned this subject I didn't like it very much, I was uncomfortable because they didn't do enough hyperlinking for my taste so I'm going to do all the hyperlinking for you and the issue is when we looked at these two that Maxwell's equations compressed (11) as two second order differential equations as opposed to four first order differential equations. It's hard to call that a compression, by the way, because two second-order differential equations with a lot of scribbles is kind of equal to four first-order differential equations with fewer scribbles I mean it's not really compression it's just in a different form. When we look at these we say okay we're going to these are differential equations we're going to solve them well we have a scalar equation and a vector equation. The scalar equation, the Laplacian of a scalar is a scalar, the time derivative of a divergence of a vector field is a scalar and the charge distribution is a scalar so the first equation of (11) is a scalar equation and the second equation of (11) is a vector equation for opposite reasons, $\boldsymbol{A}$ is a vector, the Laplacian of a vector is a vector, the second time derivative of a vector field is a vector, $\boldsymbol{j}$ is a vector field, a source vector field, the gradient of a divergence, well divergence is a scalar, the gradient of a scalar is a vector, etcetera.

The point being that these are second order differential equations but they're complicated by the fact that they're coupled, you have V this would normally be an unknown function V and an unknown vector function $\boldsymbol{A}$ and they are coupled with an unknown vector function $\boldsymbol{A}$ and an unknown scalar function V so normally the way you'd kind of do this is you solve one in terms of the other and then you substitute into the remaining equation and you can imagine how difficult this might become for any given circumstance however we're working in a very formal rarefied environment right now and we've already learned something interesting we've learned that $\boldsymbol{A}$ is ambiguous and so is V hold on

Gauge fixing

$\boldsymbol{A}$ is ambiguous up to the gradient of a scalar function F . Remember this whole thing depends critically on the fact that $\boldsymbol{B}$ remains the same well the same is true for $\boldsymbol{E}$, it has to remain the same if you change $\boldsymbol{A}$ but if we change $\boldsymbol{A}$ here we get plus the gradient of F and so we've actually added the time derivative of the gradient of F so in order for $\boldsymbol{E}$ to be the same we have to adjust V which has to become V minus the time derivative of F because you'll now have the gradient of V which is what this term is and then you have the gradient of the time derivative of F and you have that minus sign here which cancels that minus sign here so you have a plus so it ends up being you add a term that's plus the time derivative of F and it'll be plus the gradient of the time derivative of F which will cancel the minus gradient of the time derivative of F here so V is ambiguous as well but it's ambiguous based on the same F so once you pick F to change $\boldsymbol{A}$ you've the same F will change V so $\boldsymbol{A}$ and V are not independently ambiguous but they're both ambiguous based on choosing F . If I choose F it's called gauge fixing, it establishes V as well as $\boldsymbol{A}$. If I choose F , I'm converting from one gauge $\boldsymbol{A}$ to another gauge $\boldsymbol{A}'$ and V to another gauge V'

$$\begin{cases} \boldsymbol{A}' = \boldsymbol{A} + \nabla F \\ V' = V - \dfrac{\partial F}{\partial t} \end{cases} \tag{12}$$

When I'm down here (11) I want to ask myself, wouldn't it be nice if I could use that ambiguity of $\boldsymbol{A}$ and its attendant ambiguity on V ? Could I get it so that the divergence of $\boldsymbol{A}$ was zero and then $\nabla\cdot\boldsymbol{A}$ would go away and all that would be left would be the Laplacian of V equals this source term so it would be an in-homogeneous Laplace equation for V that's simple to solve in principle remember this is very formal so I'm assuming we can solve that in principle and then I can substitute that into the second equation of (11) and now I have just a wave equation for $\boldsymbol{A}$ because this is a wave operator this gives me the wave equation for $\boldsymbol{A}$ and this just becomes an in-homogeneous wave equation for $\boldsymbol{A}$ which could be solved in principle but it's uncoupled to V because I now know what V is I've solved for V and V goes away as an unknown so that would be nice.

If the ambiguity for $\boldsymbol{A}$ allows it and that's the part I had trouble as a student. As a student they just said oh because of the ambiguity of $\boldsymbol{A}$ we can assume that $\nabla\cdot\boldsymbol{A}=0$ and we call that the Coulomb gauge, but the problem that I didn't understand very well was that in order to have a gauge I've got to fix it, in order to fix it I've got to have F . Well, what's the F that gives me the Coulomb gauge? I had trouble understanding that I don't need to demonstrate an F , I just need to prove that given any $\boldsymbol{A}$, any vector potential I can always find an F to convert it to a new vector potential $\boldsymbol{A}'$ such that $\nabla\cdot\boldsymbol{A}'=0$ and all I need to know is that there exists an F that can do that. I'm going to show you this proof, it's not a long proof and actually some books do do this, I just was not lucky enough to have that book and I was always confused I just was giving the presumption, it does take a little time to do and there is a lot to cover in an EM course so I have that sympathy.

Likewise there's another choice you can make, can I use the ambiguity of $\boldsymbol{A}$ and V to make the last term go to zero, i.e. can I make

$$\left[\nabla\cdot\boldsymbol{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}\right]=0 \tag{13}$$

Can I get that to go to zero and if I could do that, well then what would happen now, well this term goes away and I just get this very much simpler wave equation for $\boldsymbol{A}$, it's just a straight up wave equation for $\boldsymbol{A}$, an in-homogeneous granted which definitely makes it hard but it's still just a wave equation for $\boldsymbol{A}$ and if this expression in (13) actually equals zero then $\nabla\cdot\boldsymbol{A}$ can be calculated from (13), which now would become, if I substituted that into the first equation, I would get:

$$-\nabla^2 V+\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=\frac{1}{\varepsilon_0}\rho \tag{14}$$

Now I have a wave equation for V also and notice they're also decoupled right I have no expression of V here and I have no expression of $\boldsymbol{A}$ there so this choice completely decouples both equations and they're both the same equation one is a scalar wave equation for V and one is a vector wave equation for $\boldsymbol{A}$ so that's another question can I make a choice of gauge such that (13) is true and the answer is in both quick cases a pretty simple yes and it's simple to see that you can always find a gauge function F that does this so let's have a quick look at the condition where the divergence of the gauge we'll call it $\boldsymbol{A}'$. (11) is the expressions in terms of a some vector potential $\boldsymbol{A}$ this is going to be true and if we literally solve these equations we would discover that we would need some additional condition to nail down $\boldsymbol{A}$, it's partial differential equation so we would end up with arbitrary functions as constants that we would have to nail down somehow. Well let's see if we can nail down those functions. Those would be that's equivalent of choosing an F , we can choose this F and if we chose this F aggressively then we nail down all of the unspecified constants of this partial differential equation. Can we find a condition where $\nabla\cdot\boldsymbol{A}=0$ can we find an F where $\nabla\cdot\boldsymbol{A}=0$? These is what Maxwell's equations reduce to in potential form:

$$\nabla\cdot\boldsymbol{A}'=0 \rightarrow \begin{cases} \nabla^2 V=\frac{1}{\varepsilon_0}\rho \\ \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right]\boldsymbol{A}=\frac{1}{\varepsilon_0 c^2}\boldsymbol{j}-\frac{1}{c^2}\nabla\frac{\partial V}{\partial t} \end{cases} \tag{15}$$

Laplace's equation for V and an in-homogeneous wave equation for $\boldsymbol{A}$ where now the V part can be immediately substituted from the Laplace equation. There is some interesting facts here. The first turns out to be what they call instantaneous. Notice this is all spatial right so V is going to be something that changes instantaneously with time meaning as the charge density, if it changes because it's a function of time. Remember charge density is going to be a function of $\boldsymbol{r}$ and t. As the charge density changes as a function of time, V changes instantaneously all throughout space because there is no time derivative term here and that's an interesting problem that has to be fully understood which we'll talk about later but also if I choose this gauge here or when I say choose a gauge meaning I find a gauge function that creates a gauge transformation where the new gauge has this property:

$$\nabla\cdot\boldsymbol{A}'+\frac{1}{c^2}\frac{\partial V'}{\partial t}=0 \rightarrow \begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2}-\nabla^2 V=\dfrac{1}{\varepsilon_0}\rho \\ \dfrac{1}{c^2}\dfrac{\partial^2 \boldsymbol{A}}{\partial t^2}-\nabla^2\boldsymbol{A}=\dfrac{1}{\varepsilon_0 c^2}\boldsymbol{j} \end{cases} \tag{16}$$

We call this the Lorenz gauge. The previous one is called the Coulomb gauge meaning I found a function F that converts the vector potential into a new vector potential where the corresponding property is true and when I do that in the second case, you end up with this symmetric pair of wave equations one of which is scalar and one of which is vector.

Now let's show that this can be done. Before we move on this is me from the future I wanted to make one little thing clear on review I noticed. If we choose this Coulomb gauge which we're going to explain is always possible in a minute if we choose this Coulomb gauge, it is true that this becomes a scalar in-homogeneous Laplace equation and I said that you could solve it and you plug it in here and you end up with a wave equation but I kind of extended it all the way over here and if $\boldsymbol{A}$ was on this side it wouldn't be a wave equation, the wave equation would only make sense if this is the only differential operator but if you have this stuff left over with $\boldsymbol{A}$, it's not a wave equation but remember it's the Coulomb gauge so this term after the gauge transformation is equal to zero so this goes away too and I didn't emphasize that point this actually goes away as well and so this whole thing goes away which is what I correctly showed over here (16), it's just I didn't draw my lines very clearly over here to show how this whole thing goes down and then the other thing that I want to mention is that you know so these are second order differential equations as I was saying before, to nail them all down you need, it's just your standard stuff, you need boundary conditions or initial conditions I mean they're functions of time in here so you need initial conditions on the derivatives and whatever structure for a given problem will articulate the boundary conditions. The gauge condition is actually a different condition so when I was talking about the constant functions in the partial differential equations that will emerge that have to be dealt with those are actually dealt with by boundary conditions, the gauge condition is altogether a separate issue. With those two caveats to clarify what I was trying to get across let's move on and begin these proofs that you can always find these gauge functions that will allow you to move into either the Coulomb gauge or the Lorenz gauge or any other gauge that's out there, in fact we're going to start by talking about the axial gauge.

Gauge invariant

When we talk about a gauge transformation we are switching from one gauge to another by adding this gradient of a scalar field F and this is the gauge transformation and notice that $\boldsymbol{A}$ is not gauge invariant, obvious because you're adding something to it and $\boldsymbol{A}'$ does not equal $\boldsymbol{A}$, things are gauge invariant where if you make this transformation on $\boldsymbol{A}$ that property remains the same which is what $\boldsymbol{B}$ is, $\boldsymbol{B}$ is gauge invariant because $\nabla\times\boldsymbol{A}$ and $\nabla\times\boldsymbol{A}'$ give you the same $\boldsymbol{B}$ so if you make a gauge transformation from $\boldsymbol{A}$ to $\boldsymbol{A}'$, $\boldsymbol{B}$ does not change so $\boldsymbol{B}$ is considered gauge invariant.

$$\begin{cases} \boldsymbol{A}'(\boldsymbol{r},t)=\boldsymbol{A}(\boldsymbol{r},t)+\nabla F(\boldsymbol{r},t) \\ \boldsymbol{B}=\nabla\times\boldsymbol{A}'=\nabla\times\boldsymbol{A}+\nabla\times\nabla F \\ V'=V-\frac{\partial F}{\partial t} \\ E=-\nabla V-\frac{\partial \boldsymbol{A}}{\partial t}=-\nabla V+\nabla\frac{\partial F}{\partial t}-\frac{\partial \boldsymbol{A}}{\partial t}-\nabla\frac{\partial F}{\partial t} \end{cases} \tag{17}$$

Now this is different than symmetry, symmetry is a property where you have translation symmetry, rotational symmetry, symmetry under parity, symmetry under reflection, symmetry under time reversal all of those things make perfect sense in the context of Electrodynamics and they are firmly embedded usually in Noether's theorem and those symmetries are not this so we're not going to talk about gauge symmetry, we're going to talk about gauge invariance which is a different idea. Symmetry is actually like a physical property of the system again the gauge issue is this is some mathematical artifact of the redundancy embedded in our math is the approach we're taking the rule set we're using and likewise a gauge transformation of the scalar potential is similarly executed using the same gauge function and the electric field is also preserved under gauge transformations because when you execute the gauge transformation and convert V to V' and $\boldsymbol{A}$ to $\boldsymbol{A}'$ you end up with these scalar function terms canceling out to zero leaving it unchanged. Now we're going to talk about some gauge transformations that, three of them in particular one that we've already looked, the axial gauge and then we're going to show that you can always find an F to transform any gauge into the Axial gauge, the Coulomb gauge or the Lorenz gauge so let's do that.

Axial gauge

Let's first look at this axial gauge which we did last time if we assume that our vector potential has three components $\boldsymbol{A}=(A_x, A_y, A_z)$ we understand that each of those components are scalars because they're the components of vectors and they are each functions of $A_i(x,y,z,t)$ so I'm looking for a gauge function of x, y, z, t. The time should be in there too so we're looking for a scalar function and we're going to take its gradient and we're going to get the partial derivative of F:

$$F(x,y,z,t)\rightarrow\nabla F=[\partial_x F, \partial_y F, \partial_z F] \tag{18}$$

Those are the components of the gradient of F which is a vector so now I'm going to choose F so that it's the anti-derivative of the A_z component with respect to z which will introduce an arbitrary function $\phi(x,y)$ when I do that so F still has this ambiguity into it but the fact that the partial derivative of F with respect to z is going to be $-A_z$ by the fundamental law of calculus:

$$\text{Choose}\rightarrow\ F=-\int_0^z A_z\,\mathrm{d}z+\phi(x,y) \tag{19}$$

When I transform the vector potential $\boldsymbol{A}$ to $\boldsymbol{A}'$ by adding this gradient ∇F , I'm going to get:

$$\boldsymbol{A}'=\boldsymbol{A}+\nabla F=\left(A_x+\partial_x F, A_y+\partial_y F, A_z+\partial_z F\right)=\left(A'_x, A'_y, 0\right) \tag{20}$$

The last term, the partial derivative of F with respect to z is going to be A_z and this is going to go away and that will cancel it, that I included in the definition of F , (19) and so $A'_z=0$ and this I just call $A'_x=A_x+\partial_x F$ and this I just call $A'_y=A_y+\partial_y F$ so I have found a condition where $A'_z=0$ always equals zero and this integral can always be done, the integral for F I can always be done and ergo I can always presume a gauge where $A'_z=0$. That's one down, let's look at the other two.

Coulomb gauge

Now the Coulomb gauge, I'm looking for a gauge where the divergence of the vector potential is zero so I presume I'm in some gauge already, I've already got some vector potential, I'm now looking for a function whose gradient when I add it to the vector potential which gives me a new vector potential $\boldsymbol{A}'$, that's what this is:

$$\nabla\cdot\boldsymbol{A}'=\nabla\cdot\left[\boldsymbol{A}+\nabla F\right]=\nabla\cdot\boldsymbol{A}+\nabla^2 F \tag{21}$$

$$\text{Choose}\rightarrow\ \nabla^2 F=-\nabla\cdot\boldsymbol{A}\ \text{ so }\ \nabla\cdot\boldsymbol{A}'=0 \tag{22}$$

I want that divergence to equal zero so I'm just going to distribute the divergence through and I get the divergence of $\boldsymbol{A}$ and the Laplacian of F so if the Laplacian of F equals the opposite of the divergence of $\boldsymbol{A}$, then sure enough this whole thing will indeed equal zero and that's exactly the condition. I have this in-homogeneous Laplace equation which is scalar, because $\nabla\cdot\boldsymbol{A}$ is a scalar, I solve this in-homogeneous Laplace equation for F , which can always be done and once I have that F I create my $\boldsymbol{A}'$ and the divergence of $\boldsymbol{A}'$ is 0. This is why you can always shift into the Coulomb gauge, you don't have to hunt down F you just say, well whatever gauge you would do it in, an F exists where you can be in the Coulomb gauge so you just shift into the Coulomb gauge based on this principle (22) that you can solve this differential equation in principle and likewise for the Lorenz gauge, the logic is exactly the same.

Prime gauge

I'm now looking for a gauge $\boldsymbol{A}'$, V' where:

$$\begin{aligned}\nabla\cdot\boldsymbol{A}'+\frac{1}{c^2}\frac{\partial V'}{\partial t}&=\nabla\cdot\left(\boldsymbol{A}+\nabla F\right)+\frac{1}{c^2}\left[\frac{\partial V}{\partial t}-\frac{\partial^2 F}{\partial t^2}\right]\\&=\nabla\cdot\boldsymbol{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}+\nabla^2 F-\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}\end{aligned} \tag{23}$$

This term equals zero, the sum of the divergence of the vector potential and the one over c squared times the time derivative of the scalar potential that sums got to be zero.

Well, whatever gauge I'm in, I'm looking for a function F, I'm in this gauge, I'm looking for a function F and this guy here equals $\boldsymbol{A}'=\boldsymbol{A}+\nabla F$, that's the gauge transformation for $\boldsymbol{A}$, $V'=V-\dot{F}$, that's the transformation for V. With that in mind, I'm now looking for the function F that satisfies this equation (23) equaling zero and it's not hard to do, I just separate the old gauge pieces that and I put that over here on the left and I look for this function F on the right and the stuff on the right has to equal this stuff on the left because I need this to cancel that. Again not hard to do because this is simply the D'alamertian of F, this is the wave operator:

$$\text{Choose}\rightarrow\ \Box F=-\left[\nabla\cdot\boldsymbol{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}\right]\ \text{so}\ \nabla\cdot\boldsymbol{A}'+\frac{1}{c^2}\frac{\partial V'}{\partial t}=0 \tag{24}$$

I have the D'alambertian coming in the opposite sign as the Laplacian and that's usually written by a box so this box operator is this D'alambertian so how would I do that I would have to go:

$$\Box F \stackrel{\text{def}}{=} \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right]F \tag{25}$$

Now I have this equation to solve (24). I'm starting from a gauge that I presumably know so I can always solve this equation and once I solve that equation and I use F to make the transformation from $\boldsymbol{A}$ to $\boldsymbol{A}'$, I have this expression (24) so I can always choose my F that solves this equation just I choose the F to solve the in-homogeneous Laplace's equation (22). Now I guess I have basically an in-homogeneous wave equation here is what this is all about but they can always be done. I could always find a way to move into the Coulomb gauge and move into the Lorenz gauge and if I'm in the Lorenz case I can shift to the Coulomb gauge and I can find an F to do that.

Summary

That wraps up what I wanted to talk about today, we reviewed Maxwell's equations in their pure form we took our our hard one potential formulation stuck them into Maxwell's equations and generated Maxwell's equations in potential form which are now two second order differential equations and then we discovered that the ambiguity of these potential formulations, the gauge ambiguity allows us to choose these conditions, these gauge conditions and we can choose the Coulomb gauge condition because, we can always find an F to transform us into the Coulomb gauge and we can always find an F to transform us into the Lorenz gauge where we get these nice simple versions of Maxwell's equations simple in a formal sense but they're dissimilar but they are simple and you can quantize Electrodynamics in either one of these two gauges and we'll you know maybe we'll explore that later but the point is is is you never know what book you're going to land in when you study QED you might study the quantization of it in the Coulomb gauge you might study the quantization of it in the Lorenz gauge and then I just went through the proof to show how you can always find this F and the reason I did that is because I found that personally to be very difficult to understand when I first learned it because the presumption was offered to us that because of this gauge freedom you could always get to the gauge you want to be in but they never took me through this very simple step.

It wasn't much of a step was it? Wasn't too hard you just have to understand that you can solve these differential equations in principle given that all the fields are well behaved so this is the core concepts of gauge transformations that are essential for QED. I think we're going to explore this a little bit more we're going to talk about this in the context of quantum mechanics and we'll discuss that an ontology question a little bit more but those lessons will be less and less critical for QED prerequisite and more for completeness of really coming to an understanding of the elementary notions of gauge and invariance and the significance of it. I'll see you next time