QED Prerequisites Geometric Algebra 16: Canonical Bivectors

Thank you welcome back as we continue our study of space-time algebra and Geometric algebra in general and we stopped last lesson at, where were we? We had done bi-vectors, we've done reversion and inversion we've done reciprocal basis and tensors, we were working on the canonical form of bi-vectors Section 3.5.1 *Bi-vectors: canonical form* and for this one section I plan to diverge from the text of this paper and go to another paper where I was a little bit more comfortable with the explanation this is the section of the paper that we're reading that talks about the canonical form of a bi-vector and basically what they're after is, if you have a bi-vector $\boldsymbol{F}$, you want to write it as a canonical bi-vector and a phase factor and we understand this phase factor to be exactly the same as you would for a complex number for example they show it right here:

$$\boldsymbol{F}=\mathrm{f}\exp(\varphi I)=\mathrm{f}(\cos\varphi+I\sin\varphi) \tag{1}$$

Where I is our pseudo-scalar basis vector so this is grade zero and this is grade one (in parenthesis) so when you look at $\epsilon_F=\exp(2\varphi I)=\cos 2\varphi+I\sin 2\varphi$, you're actually looking at a multi-vector, it's not written in our standard multi-vector form but that's what this is, this is a grade zero plus grade four multi-vector and what they're trying to say is that if you have a bi-vector $\boldsymbol{F}$, any bi-vector $\boldsymbol{F}$, you always can have this grade zero plus grade four multi-vector multiplying a canonical $\boldsymbol{F}$ and that canonical $\boldsymbol{F}$ will always have a negative signature in this convention and then they show you how to calculate φ in terms of the grade zero part of the magnitude of $\boldsymbol{F}^2$ and the grade four part of the magnitude of $\boldsymbol{F}^2$.

$$\varphi=\frac{1}{2}\tan^{-1}\frac{\ell_1}{\ell_2}\quad,\quad \mathrm{f}=\boldsymbol{F}\exp(-\varphi I) \tag{2}$$

Where $\ell_1=\langle\boldsymbol{F}^2\rangle_0=|\boldsymbol{F}|^2\cos 2\varphi$ and $\ell_2=\langle\boldsymbol{F}^2\rangle_4 I^{-1}=|\boldsymbol{F}|^2\sin 2\varphi$. The grade zero part of the space-time square of $\boldsymbol{F}$ and the grade four part of the space-time square of $\boldsymbol{F}$. Now the grade four part of course is a real number times I so you have to multiply by this I^{-1} to clear it to get it to be a real number so you can take this ratio, calculate the inverse tangent, divide that by two, get φ and that φ will become part of this unit multi-vector, grade zero and grade four multi-vector that has unit magnitude and then you can calculate the canonical form of $\boldsymbol{F}$ by just running this equation in reverse, inverting this equation. Unfortunately, I had a little trouble getting through this, I thought it was a little terse and also there's some real confusion for me regarding what they mean by $|\boldsymbol{F}|^2$, the magnitude of $\boldsymbol{F}$ squared. This I understand $\langle\boldsymbol{F}^2\rangle_0$, I understand, the 0^{th} order of the space-time square of $\boldsymbol{F}$, the 0^{th} grade of the space-time square of $\boldsymbol{F}$ but here you see the magnitude of $\boldsymbol{F}$ squared and in the paper they define the magnitude of a multi-vector is its reversion the 0^{th} grade, zero part of the reversion, its magnitude squared plus the grade four part of the reversion of the square of the reverse square I guess you would call this not the reversion but the reverse square grade zero part the grade four part of the reverse square which is going to be a grade four object but you clear the quad-vector basis vector out this ends up being a real number you square it you add these two things together:

$$\text{Positive Magnitude}\rightarrow\ |M|^2=|\langle\widetilde{M}\,M\rangle_0|^2+|\langle\widetilde{M}\,M\rangle_4 I^{-1}|^2 \tag{3}$$

I couldn't quite reconcile this with the work in the paragraph that describes all of this so what I did is I went to a different paper, a paper by Hestenes himself. Let's actually begin with a little bit of errata, some viewers have pointed out some errors, in particular in Lesson 13 I miswrote the two tensor or the $(0,2)$ tensor $\boldsymbol{F}$ and I had two superscript μ on the two different one forms and in other words I had this ν was written as a μ so there would be two μ and that those two μ would not marry up with the μ and the ν in the Einstein sum that's required:

$$\text{Lesson 13} \rightarrow \; F_{\mu\nu}\, dx^{\mu} \otimes dx^{\nu} \tag{4}$$

That's an easy one, that's an easy error and the way I've written it here is correct. In Lesson 12 I went over to a chart in the paper and I was fumbling around trying to find the grade four part so let me go back to that. You may remember I was looking at this

$$\begin{array}{ll} \text{bi-vector contraction} & a\cdot\boldsymbol{F}=½(a\boldsymbol{F}-\boldsymbol{F}a)=-\boldsymbol{F}\cdot a \\ \text{bi-vector inflation} & a\wedge\boldsymbol{F}=½(a\boldsymbol{F}+\boldsymbol{F}a)=\boldsymbol{F}\wedge a \\ \text{bi-vector dot product} & \boldsymbol{F}\cdot\boldsymbol{G}=½(\boldsymbol{F}\boldsymbol{G}+\boldsymbol{G}\boldsymbol{F})=\boldsymbol{G}\cdot\boldsymbol{F} \\ \text{commuatator bracket} & [\boldsymbol{F},\boldsymbol{G}]=½(\boldsymbol{F}\boldsymbol{G}-\boldsymbol{G}\boldsymbol{F})=-[\boldsymbol{G},\boldsymbol{F}] \\ \text{bi-vector cross product} & \boldsymbol{F}\times\boldsymbol{G}=[\boldsymbol{F},\boldsymbol{G}]I^{-1}=-\boldsymbol{F}\times\boldsymbol{G} \end{array} \tag{5}$$

I was saying, this is the symmetric part (3rd line in (5)) and this is the anti-symmetric part (4th line in (5)) and I said well hey this is the grade zero and this is the grade two, well what's important is that this dot product part this symmetric part is actually the grade zero and the grade four part because I actually was trying to think that this bi-vector cross product thing which we'll talk about later I was thinking was that the vector four part, where's the vector four part? But of course the answer is astonishingly obvious but whenever you take the multi-vector product well you have a multi-vector M and a multi-vector P and you take the space-time product of that, you can always write this as:

$$M\,P=½(M\,P+P\,M)+½(M\,P-P\,M) \tag{6}$$

You can always break a space-time product between any two multi-vectors into a symmetric part and an anti-symmetric part and what's what's going on is when you do that, the symmetric part will take up a certain number of the relevant blades so if M was 3-blade and P was a seven blade so you would expect the space-time product to contain in generality, a seven minus three, 4, all the way up to seven plus three 10 blades and those will be in steps of two, they'll each be in steps of two so you would expect a four blade, a six blade, an eight blade and a ten blade in this product so the question is where do these blades live inside the symmetric and anti-symmetric piece? The symmetric piece will absorb some of these blades and the anti-symmetric piece will absorb the others and which ones go where depend on the degree or the grade of M and the grade of P assuming it's a pure grade, I'm calling a pure grade thing a blade, which doesn't necessarily have to be. There's that whole issue of the definition of a blade but a grade three multi-vector M and a grade seven multi-vector P would end up with a grade four part, a grade six part, a grade eight part and a grade ten part and those parts will be distributed in these two, in these symmetric piece and the anti-symmetric piece of this a space-time product (6), Well if M and P are two blades, it turns out and I've worked it out here, I have if A and B are two grade two bi-vectors, well A and B are bi-vectors and I blow them up in components, you'll see that there's a grade four part, a grade two part, this is a grade two part because these guys right here are all bi-vector unit vectors and then there's a grade zero part, a grade zero part everything here is a real number and that is of course a quad-vector and if I reverse the order of those two, if I take the product of A and B in the reverse order, the grade four part is actually the same, the grade four part doesn't change, the grade two part actually change into its negative and the way you can see that is the bi-vector $\gamma_{\mu\beta}$ ends up down here as $\gamma_{\beta\mu}$ so you're going to get a sign change and so the grade two part in the commuted multi space-time product is going to have a different sign but the grade zero part is not going to have a different sign:

$$\begin{aligned} B^{\alpha\beta}A^{\mu\nu}\gamma_{\mu\nu}\gamma_{\alpha\beta} &= \gamma_{\mu\nu\alpha\beta} \\ &+(\gamma_\nu\cdot\gamma_\alpha)\gamma_{\mu\beta}-(\gamma_\nu\cdot\gamma_\beta)\gamma_{\mu\alpha}-(\gamma_\mu\cdot\gamma_\alpha)\gamma_{\nu\beta}+(\gamma_\mu\cdot\gamma_\beta)\gamma_{\nu\alpha} \\ &+(\gamma_\nu\cdot\gamma_\alpha)(\gamma_\mu\cdot\gamma_\beta)-(\gamma_\nu\cdot\gamma_\beta)(\gamma_\mu\cdot\gamma_\alpha)-(\gamma_\mu\cdot\gamma_\alpha)(\gamma_\nu\cdot\gamma_\beta)+(\gamma_\mu\cdot\gamma_\beta)(\gamma_\nu\cdot\gamma_\alpha) \end{aligned} \tag{7}$$

$$\begin{aligned} B^{\alpha\beta}A^{\mu\nu}\gamma_{\alpha\beta}\gamma_{\mu\nu} &= \gamma_{\alpha\beta\mu\nu} \\ &+(\gamma_\beta\cdot\gamma_\mu)\gamma_{\alpha\nu}-(\gamma_\beta\cdot\gamma_\nu)\gamma_{\alpha\mu}-(\gamma_\alpha\cdot\gamma_\mu)\gamma_{\beta\nu}+(\gamma_\alpha\cdot\gamma_\nu)\gamma_{\beta\mu} \\ &+(\gamma_\nu\cdot\gamma_\alpha)(\gamma_\mu\cdot\gamma_\beta)-(\gamma_\nu\cdot\gamma_\beta)(\gamma_\mu\cdot\gamma_\alpha)-(\gamma_\mu\cdot\gamma_\alpha)(\gamma_\nu\cdot\gamma_\beta)+(\gamma_\mu\cdot\gamma_\beta)(\gamma_\nu\cdot\gamma_\alpha) \end{aligned} \tag{8}$$

As a consequence of this if I take the symmetric sum, I get the grade four part plus the grade zero part:

$$\begin{aligned} ½(AB+BA) &= \gamma_{\mu\nu\alpha\beta} \\ &+[(\gamma_\nu\cdot\gamma_\alpha)(\gamma_\mu\cdot\gamma_\beta)-(\gamma_\nu\cdot\gamma_\beta)(\gamma_\mu\cdot\gamma_\alpha)-(\gamma_\mu\cdot\gamma_\alpha)(\gamma_\nu\cdot\gamma_\beta)+(\gamma_\mu\cdot\gamma_\beta)(\gamma_\nu\cdot\gamma_\alpha)] \end{aligned} \tag{9}$$

If I take the anti-symmetric sum I get the grade two part so the anti-symmetric sum is the grade two part all by itself but the symmetric sum is the grade four part and the grade zero part together.

$$\begin{aligned} &½(AB-BA) \\ &=[(\gamma_\nu\cdot\gamma_\alpha)\gamma_{\mu\beta}-(\gamma_\nu\cdot\gamma_\beta)\gamma_{\mu\alpha}-(\gamma_\mu\cdot\gamma_\alpha)\gamma_{\nu\beta}+(\gamma_\mu\cdot\gamma_\beta)\gamma_{\nu\alpha}] \end{aligned} \tag{10}$$

That's where the missing grade four part went when I was fumbling around in that lesson so that answers that question:

$$\text{Lesson 12}\rightarrow\ [\boldsymbol{F},\boldsymbol{G}]\equiv ½(\boldsymbol{F}\boldsymbol{G}-\boldsymbol{G}\boldsymbol{F}) \tag{11}$$

Then there's what this other errata really annoys me because I should have caught it but I was really struggling with how to describe the inverse of a basis bi-vector so if we have $\gamma_{\mu\nu}$ which is defined as the space-time product of γ_μ and γ_ν which because of orthogonality is equal to $\gamma_\mu\wedge\gamma_\nu$, we all know that but I want to talk about the inverse and what I said in the lesson is that the inverse of that is just going to be $\gamma^{\mu\nu}$, you just raise the two indices and you should get by definition the inverse and that's only true for the basis vectors themselves, the basis vectors themselves it is definitely true that:

$$\gamma^\mu=(\gamma_\mu)^{-1} \tag{12}$$

For bi-vectors we really should use this definition of the inversion and that is the reversion of the bi-vector divided by the reversion square and the reversion of the bi-vector $\gamma_{\mu\nu}$ is $\gamma_{\nu\mu}$ so to get this inversion we have to write, well we literally write this down and when we do this calculation you know this becomes a real number down below, note there's no absolute values here and you get:

$$(\gamma_{\mu\nu})^{-1}=\frac{\tilde{\gamma}_{\mu\nu}}{\tilde{\gamma}_{\mu\nu}\gamma_{\mu\nu}}=\frac{\gamma_{\nu\mu}}{\gamma_{\nu}\gamma_{\mu}\gamma_{\mu}\gamma_{\nu}}=\frac{\gamma_{\nu}\wedge\gamma_{\mu}}{\gamma_{\nu}^{2}\gamma_{\mu}^{2}}=\frac{\gamma_{\nu\mu}}{\gamma_{\nu}^{2}\gamma_{\mu}^{2}} \tag{13}$$

The denominator will often be 1 but it could be -1 if μ or ν are zero, well they can't both be zero because then the numerator would be zero but this will either be 1 or -1 but notice the top is the reverse it's now $\gamma_{\nu\mu}$, you flip the indices around to get that inversion so the inverse of $\gamma_{\mu\nu}$ is $\gamma_{\nu\mu}$ divided by this factor which may be -1 . Now we just look at the same definition for the inverse of an arbitrary reference vector it's the reversion of that reference vector divided by the reversion squared, the reversions or the square reversion but of course the reversion is equal to itself so you just get:

$$\gamma_{\mu}^{-1}=\frac{\tilde{\gamma}_{\mu}}{\tilde{\gamma}_{\mu}\gamma_{\mu}}=\frac{\gamma_{\mu}}{\gamma_{\mu}^{2}}=\gamma^{\mu} \tag{14}$$

The point is that's this pattern here and this pattern here (13) and we are defining this, by definition is γ^{μ} what this means is that the final equal sign here is in fact:

$$(\gamma_{\mu\nu})^{-1}=\gamma^{\nu}\wedge\gamma^{\mu}=\gamma^{\nu\mu} \tag{15}$$

The correct way to write this inverse is $\gamma^{\nu\mu}$ with these indices flipped from the original indices so that's an error and it's the second time I've been caught by that flipping of the indices with the bi-vectors, you may remember I discovered I was caught by that one we did the demonstration of taking components so that's enough for errata so let's now move on.

When I say David Hestenes himself, we're treating this man as the spokesperson of this whole Geometric algebra way of looking at space-time Physics and he wrote this paper called "*Spacetime Physics with Geometric Algebra*" we're certainly not going to go through this whole paper, I'm just going to go through this one section to get us through, that's the equivalent section in the paper we're reading. You can find this paper at Dr Hestenes website which is given right here. You can just look up this paper and find the link directly and the part of the paper that we're going to study begins here, I'm not sure what page it is but it begins in this section right here, Section II. Spacetime Algebra, Page 3..

Let's begin read this and see if we can make contact with the material in our paper so it starts with a very simple definition, the magnitude square of a multi-vector is the grade zero part of the reversion squared, this definition, that I understand, that's really clear to me because this object is a square:

$$|M|^{2}=|\langle M\,\tilde{M}\rangle| \tag{16}$$

You wouldn't see, you wouldn't expect to see a square here, right hand side of (16), because you're dealing with something squared, you're dealing with a second order expression in a space-time product and so this I understand and because it's got no index here, when there's no index you assume it's the grade zero part so it's the grade zero part of the reversion squared is what they're calling the magnitude squared for any multi-vector. Imagine you could have a multi-vector with all of these parts in it:

$$\tilde{M} = \alpha + v - F - w I + \beta I \tag{17}$$

This is the reversion of M but M itself, we would just write M itself with all these plus signs:

$$M = \alpha + v + F + w I + \beta I \tag{18}$$

You can have any multi-vector, you multiply it together you get all these different parts back but the magnitude is all focused on α according to this definition (16), well after you do the reverse square, you're going to end up with some multi-vector because M times its reversion, is a multi-vector and whatever the zero order term of that multi-vector is, that's the magnitude so then he says "Any multi-vector M can be decomposed into the sum of an *even* part M_+ and an *odd* part M_- ", and that's pretty straightforward, we know that the even parts of any multi-vector is the grade zero part the bi-vector part and the quad-vector part or the grade four part. Now notice in this paper they don't use a capital I , they use a little i and I much prefer the capital I because the little i just literally looks like a complex number but that does go to show you how literal they want you to take our ability to treat I as though it's a complex number, here they don't even hide behind some special symbol, capital I , they're just all in with complex numbers on this but despite that fact when you see that complex number there it is to be understood exactly like in our paper as capital I and so this I is defined by that capital I which is literally defined as:

$$I \equiv \gamma_{0123} \tag{19}$$

Make no mistake about it, we're switching papers, the notation moves over a little bit but it's still the same thing. Now the odd part of this algebra is given by the vectors and by the pseudo-vectors:

$$\begin{aligned} M_+ &= \alpha + \boldsymbol{F} + \beta I \\ M_- &= a + b I \end{aligned} \tag{20}$$

or, equivalently, by:

$$M_\pm = ½\left(M \mp I M I\right) \tag{21}$$

If you add the vectors to the pseudo-vectors, that's a grade one part a , that is a grade three part b . Notice this a does not change under reversion but $b I$ doesn't change under reversion nor does a change under reversion. βI and $\boldsymbol{F}$ change under reversion but α does not so just because you're in the even part doesn't mean everything in here changes sign under reversion, in fact for a general multi-vector M , the only things that change under reversion is the sign of the bi-vector part and the sign of the tri-vector part, all these other three parts stay the same and here's a little happy equation you can use to figure out the given the multi-vector itself you can use this to separate out the even part and the odd part if you ever needed to (21). Let's move on, the set of all multi-vectors that are only of this form, all even multi-vectors, "The set $\{M_+\}$ of all even multi-vectors forms an important sub-algebra of the space-time algebra called the *even sub-algebra*.", so now the reason it's a sub algebra of course is because any element of this set with the space-time product with another element of the set will produce another element of this set so even multi vectors space-time product another even multi-vector will give you an even multi-vector so that's called the even sub-algebra of our space-time algebra. The elements of the even sub-algebra only have scalars bi-vectors and pseudo-scalars, they have no vectors and no tri-vectors and when I say tri-vectors I really should start using the word pseudo-vector, if I'm going to call this a pseudo-scalar all the time I should call this a pseudo-vector but notice that the way they write it, you always see it as the dual of a vector but when you see the dual of a vector you now need to be thinking pseudo-vector and when you hear a pseudo-scalar you think tri-vector and so that's part of the conventions that you have to sort of work through in order to understand all this.

They go on in this paper, "If ψ is an even multi-vector meaning a member of $\{M_+\}$, "then $\psi\tilde{\psi}$ is also even, but its bi-vector part must vanish according to (17). Well (17) shows that the reversion of any multi-vector is the same multi-vector where the sign changed for the bi-vector and the sign change for the tri-vector part but since the reversion of a reversion square is equal to the reversion times the reversion is if the reversion of a reversion squared is equal to the reversion square, the only way that can be is if the reversion square does not have anything that changes sign and if you're in the even sub-algebra, this isn't even there you don't even have the pseudo-vector part, he would only have a bi-vector part but in the reversion square that bi-vector part must go away or else this expression wouldn't be true

$$(\psi\tilde{\psi})^{\sim}=\psi\tilde{\psi} \tag{22}$$

We can actually check this, let's take an even multi-vector M , just scalar, pseudo-scalar and bi-vector and let's just go ahead and multiply it by its reversion and see what we get and well find the reversion square of an arbitrary even multi-vector and I came out with this collection of things:

$$\begin{aligned} M&=\alpha+\boldsymbol{F}+\beta\, I \\ \tilde{M}&=\alpha-\boldsymbol{F}+\beta\, I \end{aligned} \tag{23}$$

$$M\,\tilde{M}=\alpha^2-\alpha\,\boldsymbol{F}+\alpha\,\beta\, I+\alpha\,\boldsymbol{F}-\boldsymbol{F}\,\boldsymbol{F}+\beta\,\boldsymbol{F}\, I+\alpha\,\beta\, I-\beta\,\boldsymbol{F}\, I+\beta^2 I^2 \tag{24}$$

It has this scalar part here α^2 and it has a bi-vector part here $\alpha\boldsymbol{F}$ which has to get go away so it's canceled with this bi-vector part there $-\alpha\,\boldsymbol{F}$ so that cancellation eliminates the bi-vector part that's a pseudo scalar part $\alpha\,\beta\, I$ which is fine and this is a bi-vector bi-vector product that we're going to have to look at $\boldsymbol{F}\boldsymbol{F}$ but here's another bi-vector because this is the dual of bi-vector $\beta\boldsymbol{F}\, I$, where I've done the cancellation is $\boldsymbol{F}\, I$ and that is the dual of a bi-vector which is a bi-vector but it's canceled by this corresponding bi-vector part here $-\beta\boldsymbol{F}\, I$, that $-$ is really saving the day, this is a scalar part $\beta^2 I^2$ because $I^2=-1$ so this is another scalar part and then this $\alpha\,\beta\, I$ pseudo-scalar part and this pseudo-scalar apart $\alpha\,\beta\, I$, they live so you have the scalar part that lives in α^2 and β^2 and you have a pseudo-scalar part that lives in $\alpha\,\beta\, I$. All we have left to worry about is this guy $\boldsymbol{F}\boldsymbol{F}$, a bi-vector times a bi-vector in principles should produce a grade zero object $\langle\boldsymbol{F}\boldsymbol{F}\rangle_0$, a grade two object $\langle\boldsymbol{F}\boldsymbol{F}\rangle_2$ and a grade four object $\langle\boldsymbol{F}\boldsymbol{F}\rangle_4$, this is our rule of multiplication, I'm applying the rule of space-time multiplication for two blades but using blade in the pure sense, a pure blade as our paper describes it. In this product potentially, there is this bi-vector piece $\langle\boldsymbol{F}\boldsymbol{F}\rangle_2$ that's a scalar piece $\langle\boldsymbol{F}\boldsymbol{F}\rangle_0$, that's a pseudo-scalar piece $\langle\boldsymbol{F}\boldsymbol{F}\rangle_4$ so a bi-vector piece could end up there, however now we can apply this fact that we learned from this paper's organization of notation is that the bi-vector part of a space-time product of two bi-vectors, the bi-vector part is given by this commutator which this paper defines:

$$\langle \boldsymbol{F}\boldsymbol{G}\rangle_2=[\boldsymbol{F},\boldsymbol{G}]\equiv ½(\boldsymbol{F}\boldsymbol{G}-\boldsymbol{G}\boldsymbol{F}) \tag{25}$$

Of course, if $\boldsymbol{F}$ and $\boldsymbol{G}$ are the same, this is zero:

$$\langle \boldsymbol{F}\boldsymbol{F}\rangle_2=[\boldsymbol{F},\boldsymbol{F}]\equiv ½(\boldsymbol{F}\boldsymbol{F}-\boldsymbol{F}\boldsymbol{F})=0 \tag{26}$$

That explains that the bi-vector part of this product is zero and ergo everything that's left in this reversion square is in fact just scalar and pseudo-scalar where M now is an arbitrary member of, remember this is completely arbitrary member of the even sub-algebra which I guess the way the paper would write this is is an element of $\{M_+\}$ so we have explained that statement.

Continuing on, "Therefore $\psi\tilde{\psi}$ ", the version square of an even multi-vector, remember ψ has been defined as an even multi-vector so the reversion square of an even multi-vector, "has only scalar and pseudo-scalar parts", only scalar parts and pseudo-scalar parts which means it can be expressed as a complex scalar so we can say $\langle\psi\tilde{\psi}\rangle$, we don't need a zero there because the notation says without a zero we just assume a zero that equals some real number $\alpha+\beta I$. Now they use i here remember but we're going to use I just for our own continuity sake and so once we agree that this is basically just like a complex number it has a polar form so they write it as:

$$\psi\tilde{\psi}=\rho e^{i\beta}=\rho(\cos\beta+i\sin\beta)=\rho\cos\beta+i\rho\sin\beta \tag{27}$$

This is a polar form where the imaginary number $I=\rho e^{(\gamma_{0123}\beta)}$, that's the imaginary number I which they write as i and some angle β where β and ρ are real numbers and just like exponentiation in Lie algebras where we frequently write e and we have some matrix up there, literally a matrix, it could be a 10×10 matrix, that's in the exponent of e , it's very puzzling to people but immediately we just understand that this is to be understood as its Taylor expansion, it's just a Taylor series and also we can apply that Euler's formula and just assume that $e^{i\beta}$ can be broken into $\cos\beta+i\sin\beta$ and if you end up with a little $-$ in here, if you end up with a $e^{-i\beta}$ then this just becomes $\cos\beta-i\sin\beta$.

All of the rules of complex analysis where we put things into Polar form apply here just fine. Now, let's take a quick look to make sure this Polar form is consistent, well ρ is a real number so this is going to equal $\rho\cos\beta$, well β is a real number so this first part is all a real number. The second part is going to be $i\rho\sin\beta$, well ρ and $\sin\beta$, those are real numbers and I is a four-blade so $I=\gamma_{0123}$. It's the basis vector four blades so that's a real number times I so this whole thing is a pseudo-scalar so when you look at this you see it as a pseudo-scalar, the reversion square of any even multi-vector has to be a pseudo-scalar and so now the question is well we want to calculate θ and we want to calculate ρ so we can write this reversion square (27) in this form at will so let's learn about how to do that.

Let's start by saying you took the reversion square of ψ and you ultimately wrote it as $\psi\tilde{\psi}=c+dI$ where these are real numbers, I'm running out of Greek letters here that don't have some ambiguity to it so I'm just going with $c\in\mathbb{R}$ and $d\in\mathbb{R}$ and now we just use regular complex analysis to put this in polar form ρ is the magnitude of this thing which is $\sqrt{c^2+d^2}$ and you raise this to the power of $I\tan^{-1}(d/c)$. This is elementary complex analysis. d we can write down as the grade four part of the reversion square multiplied by I^{-1} in order to clear the grade four basis vector and c is just the grade zero part of $\psi\tilde{\psi}$:

$$\psi\tilde{\psi}=c+dI \text{ where } c,d\in\mathbb{R} \tag{28}$$

$$\psi\tilde{\psi}=\rho e^{I\beta}=\sqrt{c^2+d^2}\exp\left[I\tan^{-1}(d/c)\right] \tag{29}$$

$$d=\langle\psi\tilde{\psi}\rangle_4 I^{-1} \quad , \quad c=\langle\psi\tilde{\psi}\rangle_0 \tag{30}$$

We also know by the definition that we had earlier in the paper, we had this definition, the magnitude squared of a multi-vector M was given by the reversion squared grade zero part, the absolute value.

$$|M|^2=|\langle M\tilde{M}\rangle| \tag{31}$$

This is what would go inside here so the absolute value of this is called the magnitude of our reversion square or it is called the magnitude of ψ . That's not so important, now it's just sometimes this the zero blade part of things shows up in a lot of different contexts but now that we've got this, we're going to identify ρ with this number $\sqrt{c^2+d^2}$ and β with this inverse tangent and so now we have expressed we have formulas for this. Now, if you remember in our the paper we're working with they define β with a factor of ½ in here, $I\tan^{-1}(d/c)$ and we might see I think why this ½ is a way of going then this part here $e^{I\beta}$ will have a factor of two in the exponent but so there is this variation in this process of finding the polar form of a reversion square, there is this variation that has a ½ in this definition, we'll try to flush out exactly why that's there in a moment.

$$\beta=½\tan^{-1}(d/c) \tag{32}$$

The next thing they notice is that once we've understood this Polar form idea, "If $\rho\neq 0$ we can derive from ψ an even multi-vector $R=\psi(\psi\tilde{\psi})^{-½}$ is our even multi-vector multiplied by our reversion square to the power $-½$ and this property should exist for R :

$$R\tilde{R}=\tilde{R}R=1 \tag{33}$$

They're saying, given any even multi-vector, we know that we can put its reversion square in this form and once we've done that we can define a new even multi-vector whose reversion square is 1 and this derivation is directly related to the even multi-vector we were given in the beginning, this arbitrary R . Let's see if we can work through this, now at first glance this looks weird, we're throwing into the denominator, the square root of a multi-vector, this is a space-time product of ψ with its reversion. We know this has a complex form and we're taking the complex scalar, as I guess that is what we call, we know this is a complex scalar but we're going to take that to the $-½$. Well now that we have Polar form just as with complex numbers now that we have this form (29) for this reversion, this process is easy to do, we just write down:

$$R\equiv\psi(\psi\tilde{\psi})^{-½}=\psi\left[\rho e^{I\beta}\right]^{-½}=\psi\left[\rho^{-½}e^{-½I\beta}\right] \tag{34}$$

Now we understand this $-I$, as I mentioned before, this $-$ here simply comes in as, once you blow (29) up into its Euler form, this $+$ becomes a $-$ because of this $-$ down here (34) so this is actually a very easy form to write down and then the claim is that the reversion square of R equals one. Well we know that it will equal, just wrote it all out at once:

$$R\tilde{R} \stackrel{?}{=} 1 = \psi\left[\rho^{-½} e^{-½ I\beta}\right]\left[\rho^{-½} e^{-½ \tilde{I}\beta}\right]\tilde{\psi} \tag{35}$$

This is R , this is how R is literally defined, this is the reversion of R so notice we take our bi-vector, we reverse it and we reverse it with the quad-vector, well we basically reverse it with the quad-vector piece or the complex piece and it's recursively reversed so you reverse the two multi-vectors and then you take the reversion of the two multi-vectors but the good news, of course, is that the reversion of I is still I i.e., $\tilde{I}=I$ so this allows us to rewrite this as:

$$R\tilde{R} = \psi\rho^{-1} e^{-I\beta}\tilde{\psi} = \left[\rho^{-1} e^{-I\beta}\right]\psi\tilde{\psi} = \left[\rho^{-1} e^{-I\beta}\right]\rho e^{I\beta} = 1 \tag{36}$$

That action brings the multi-vector with its reversion like this, you combine those two but this is by definition or by our calculation actually, we've already calculated the reversion of ψ with itself, it's reversion squared is given by $\rho e^{I\beta}$ so this becomes $\rho e^{I\beta}$, these two things cancel and you get the scalar one. The point of all this is that R is something implied directly from the existence of an even multi-vector but what makes R canonical is that we're forcing it to equal this very unique thing, the scalar unit which we say that's special enough that we're going to call this a canonically related to ψ so that is how R is calculated. Then once we've decided R is this special multi-vector that we can generate from any even multi-vector, we can then reverse the process and say that any even multi-vector has this canonical R and a magnitude and phase but in particular a phase that connects it to this canonical R whose defined by $R\tilde{R}=1$.

$$\psi = \rho^{½} e^{½ I\beta} R \tag{37}$$

Now what's interesting, I guess, is to think about is, if we think of the Electromagnetic field is ultimately going to be an even multi-vector, the claim is that this form of the Electromagnetic field it's hard to discover in tensor calculus but it's very evident in this Geometric algebra form but (37) is now the canonical form of an even multi-vector ψ and that's the way this proceeds. It says, we've defined R using this technique and we now have the canonical form of any even multi-vector ψ as being this:

$$\psi = \left(\rho e^{I\beta}\right)^{½} R \tag{38}$$

Just to make sure that that's how we defined it and that's exactly how we defined it, we just brought in this square root part, then they say "We shall see that this *invariant decomposition* has a fundamental figure significance and Dirac Theory" so this decomposition, that's the idea, any given multi-vector ψ can be decomposed into this form. Then it talks about "An important special case of this decomposition is its application to a bi-vector $\boldsymbol{F}$ " so remember ψ is any even multi-vector but now what if we just do it with a bi-vector $\boldsymbol{F}$, a straight up bi-vector and in this case they actually make some symbolic modifications they say "it is convenient to replace $\beta/2$ by $\beta+\pi/2$ " and write down this form of bi-vector $\mathrm{f} = \rho^{½} R i$:

$$\boldsymbol{F}=\mathrm{f}\,e^{i\beta}=\mathrm{f}(\cos\beta+i\sin\beta) \tag{39}$$

Let's understand how that transition works, I've written that all down here where $\boldsymbol{F}$ is our arbitrary bi-vector, the only thing we know about $\boldsymbol{F}$ is that it's a bi-vector but we know it has this canonical form there's some β out there where you can write this bi-vector as its canonical R and this phase factor, it's not purely a phase factor this is the phase factor because it's got magnitude one but it's also got this overall magnitude factor as well, (note $\beta/2\Rightarrow\phi+\pi/2$ then $\beta=2\phi+\pi$:

$$\boldsymbol{F}=\rho^{+½}e^{½I\beta}R=\rho^{½}e^{I\phi+I\pi/2}R=\rho^{½}e^{I\phi}RI=\rho^{½}RI\,e^{I\phi}=\mathrm{f}\,e^{I\phi} \tag{40}$$

Where $\mathrm{f}\equiv\rho^{½}RI$. In his paper, Hestenes writes, we're going to take $\beta/2$ and we're going to replace it with some other angle $\phi+\pi/2$. In his paper they use ψ and β , they just redefine the symbols and I find that a little bit confusing so I'm going to change it to everywhere we see $\beta/2$, we're going to say that equals $\phi+\pi/2$ so $\beta=2\phi+\pi$ and then we just substitute that in for $\beta/2$. We're left with (40). Recall $e^{½\pi I}=I$ and I commutes with R so this I pops over to here and lands over there so that's easy enough and then what's left behind is $e^{I\phi}$ and the square root of this magnitude, then since bi-vectors commute with the quad-vector basis vector I can move R all the way over, I can move RI all the way over and I can write it in this form where I have the dual of R of this canonical vector sandwiched in the middle and then I just define this canonical form $\mathrm{f}\equiv\rho^{½}RI$ and I put that out in front and I'm left with this $\mathrm{f}\,e^{I\phi}$ factor and now I really do have a canonical bi-vector and a pure phase so that's the process here. It's not that confusing but if you do check it's pretty straightforward, it's not something I would have thought of but that's why you have researchers who study mathematics for a living, they come up with these things. Then you take a look at what is the f , what is the taking this canonical version, what is the reversion square of f ? We can write down:

$$\mathrm{f}\,\tilde{\mathrm{f}}=\rho RI\tilde{I}\tilde{R}=\rho R(-1)\tilde{R}=-\rho R\tilde{R}=-\rho \tag{41}$$

You have this recursive reversion of I and R but I reversion is just I so you end up with a -1 there so that's easy and then you end up with $-\rho R\tilde{R}$ but $R\tilde{R}=1$ because we've defined it that way, we built R so that its reversion square is one so you end up with $-\rho$. Now $\rho>0$, it's always greater than zero so $-\rho<0$ so if we calculated the signature of f we take its reversion square over its magnitude and remember we defined its magnitude very cleanly in the beginning as the absolute value of the zero grade portion of $\mathrm{f}\,\tilde{\mathrm{f}}$ so that ends up being:

$$\varepsilon_{\mathrm{f}}\equiv\frac{\mathrm{f}\,\tilde{\mathrm{f}}}{|\mathrm{f}|^2}=\frac{-\rho}{|\langle\mathrm{f}\,\tilde{\mathrm{f}}\rangle|}=\frac{-\rho}{\rho}=-1 \tag{42}$$

The signature of f is -1 . That's about as far as we need to go with the Hestenes paper, we have achieved this story right here, we've achieved an understanding of this expression (39) and if we go back to the paper that we are studying ourselves, what we see is basically the same process, they end up in exactly the same place. Now you'll notice that they define φ with that factor of ½ (2) but we saw plenty of places where that, we saw that final substitution that Hestenes made to get rid of the ½ that floated into the problem and this paper just gets rid of it at the front end basically and then of course there's this expression here (2) which is literally just the inverse of calculating $\boldsymbol{F}$, these two are inversely related so the point is, that the idea is that we can always take any bi-vector $\boldsymbol{F}$ create a canonical form of $\boldsymbol{F}$ that has a definite negative signature f and they choose a negative signature for some reason, now canonical is interesting, when I use the word canonical in previous work what I usually mean, is it's forced upon us, in this case there are some decisions that are being made like for example this whole idea like we really dig this R because $R\tilde{R}=1$, we like this one so it's certainly possible to do a hundred percent of the time but being canonical is more of a being a convenience, in this case but I don't think there's any superficial meaning for the word canonical in Math and Physics necessarily, it just means something that you can do all the time and sometimes there's only one thing that you can do and other times there are many things you can do but one of them is the best or the most convenient and I think that's what's going on here.

Now we understand that any bi-vector, we have this canonical form that has a phase factor and we know how to calculate those phase factors and this factor of two is not so hard to understand anymore and now we can move on to section 3.5.2 "*Bi-vectors: products with bi-vectors*" and I think we'll do that in our next lesson this is a good place to stop and the next lesson we will take up the subject of bi-vectors products with bi-vectors which you've actually already studied right we know (43), we know this statement for example but we don't know it in this context with the canonical form so we've learned something important and I'll see you next time.

$$\boldsymbol{F}\boldsymbol{G}=\langle\boldsymbol{F}\cdot\boldsymbol{G}\rangle_0+[\boldsymbol{F},\boldsymbol{G}]+\langle\boldsymbol{F}\cdot\boldsymbol{G}\rangle_4 \tag{43}$$