Maxwell's Equations via Differential Forms Part I

We're going to begin a review of differential forms which is just enough of a review to get our heads back in the space of how differential forms work in general and this will allow us to create this new mathematical architecture for the Maxwell's equations. I want to point out however that this is a review if you don't know differential forms, if this is the first time you've seen it then you're really not eligible for this lecture yet and I recommend that you go back and you study the lecture series in the playlist “What is a tensor?” which if you haven't studied it you're in for a treat because understanding what is a tensor in the way that we present it on this channel is really I think fun, it's a delightful experience.

We basically were using the notion of a tensor that really decomposes the idea into components and its basis elements, its bases vectors, tensors as vector spaces which is confusing just to say totally needs to be done but if you're here I assume that you've studied pretty much everything that's in that playlist but as a math check review meaning I'm going to run through a few statements right now and if these statements don't make a lot of sense to you then that's an indication that you need to learn the material, the prerequisite to this lesson which is presumably a bunch of prerequisites to QED however this material is not specifically prerequisites to QED so this is prerequisites to tangential material, how about that? I like that idea, that's what this is right right now we are going to engage in prerequisites to tangential material anyway.

The idea is first of all we are going to consider some kind of manifold right and at each point of the manifold we are going to set up at every single point we're going to set up a vector space V , this is the tangent space at the manifold and that tangent space if you understand tangent spaces has a basis vector ∂_μ because there is a coordinate system here x^0, x^1 and all the others to the x^{N-1} dimension so we have N dimensions so the manifold has N dimensions and each point has a tangent space which is a vector space, the basis of the vector space of these differential operators, every tangent space has dimensionality N , every vector space for our purposes will be a real vector space $\mathbb{R}$ and the vector space will be endowed with an inner product (v, w) which produces a real number.

Every vector space immediately spawns into existence a co-vector space V^* and the co-vector space has a set of basis vectors of one forms $\mathrm{d}x^\mu$, these are the big objects of study that are going to come up in this lesson, these notions of one forms. The co-vector space has also N dimensions, it's real and the co-vector space inherits an inner product from the vector space and then also the co-vector space.
I should write here an arbitrary vector $v \in V$ can be expressed in the basis with contravariant components by $v = v^\mu \partial_\mu$ and this is what would be known as a contravariant vector in the standard literature but a contravariant vector is always connected to a basis vector in the underlying tangent space and likewise we should be able to recognize that an arbitrary one form $\omega \in V^*$, can be expressed as $\omega = \omega_\nu \mathrm{d}x^\nu$ and where this now is a covariant components and we in the literature you would call this a covariant vector you may not call it a co-vector, we will, I mean we'll mix it up I mean the point is if you understand it you can say co-vector or covariant components of the vector.

Because you have this inner product that can be brokered into a one-to-one mapping between vectors and co-vectors, meaning you can have an unambiguous canonical map from this vector space V to the co-vector space V^* as long as the vector space itself is an inner product space and this will also give rise to the notion of a metric $\mathrm{g}(v, w) \in \mathbb{R}$ which is a $(0, 2)$ rank tensor which is the metric and this metric exists also on the manifold, the metrics purpose in life is to take two vectors and return a real number. Everything that I just said should make perfect sense to you. One thing missing I suppose it the notation we want to use for the dual space mapping would be:

$$\langle \mathrm{d}x^{\mu}, \partial_{\nu} \rangle = \delta^{\mu}_{\nu} \tag{1}$$

This is the notation of the co-vector map identified by this particular basis one form acting on the basis vector of the underlying vector space. Here is a dual space mapping, this is an inner product which takes two vectors, the dual space mapping takes a co-vector and applies the co-vector mapping to the vector and we choose the basis of the co-vector space to be the maps that make (1) true, this is what you would call this dual basis set.

If you know this stuff that I just said very well then you're ready for today's review so let's begin. I've written down a bunch of formulas here that we want to review and maybe by this will be the goal of this lecture is to make sure that we review these formulas and then we're going to be in a good position to get our work done. We're going to try to just get through this. The first one I think we've already done, an arbitrary one form can be expressed as this Einstein sum so that's in the bag already:

$$\omega = \omega_{\mu} \mathrm{d}x^{\mu} \tag{2}$$

This is closely related to:

$$\alpha = \sum_{I} \alpha_{I} \mathrm{d}x^{I} \tag{3}$$

This formula here (3) which expresses an arbitrary k form, let's assume this is a k form so a k form is by definition it's a wedge product of one form so you would have $a \wedge b \wedge c \wedge d$ where each of these guys here are one forms so this would be a general four form right and this four form of course has all of those properties of anti-symmetry and everything else you would expect from a bunch of wedge products but any arbitrary k form can be expressed this way. What's important here is this I . This I is an index set and you're summing over this index set and if you're dealing with a k form then I is all the permutations from 0 to $N-1$, well I guess it's all the permutations of the numbers 0 to $N-1$ chosen k at a time such that the permutations are always increasing, you dip into zero to $N-1$ objects you pull out k then you take those k objects and you put them in order and what you should end up getting is $i_1, i_2, i_3, \ldots, i_k$ because you've pulled out k items and you've always got to arrange them so that they are in increasing order and then you take the set of all of those increasing sequences of k indices and that is your index set I and you're summing over that index set I .

The way of saying that is that the basis vectors for a k form, let's just to make it simple for ourselves let's do four forms because we're going to work a lot with four forms because we're going to work a lot with four-dimensional vector spaces so this would be well let's just imagine we have four forms in an N -dimensional vector space so you could have $\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, that would be a basis vector for the set of all four forms. Another possibility would be $\mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$, that would also count because you are you've chosen four one forms and you have an increasing set of indices $0, 2, 3, 4$ so all of these become the basis vectors of the vector space of four forms and therefore they each get a component right and that component is labeled here so that's what this sum represents.

There's other ways of doing this by the way, you can release this restriction of having these increasing indices the problem is that not all of your list of basis vectors are linearly independent and so you're over counting and so you have to include a factor in front of this sum actually and if you do that well then there's this number that follows you around. The price you pay for losing the factor to count for the over counting, is you end up with this index set idea that you have to keep track of, so that concept is reviewed and so the next concept we're going to talk about is this d operator. This d operator is the Exterior derivative and we need to know how this Exterior derivative acts on functions on the manifold and we need to know how this Exterior derivative operates on one forms and we need to know how this Exterior derivative operates on k forms and that's these three different formulas so the first formula:

$$\mathrm{d}\,\mathrm{f}=\partial_\eta \mathrm{f}\,\mathrm{d}x^\eta \tag{4}$$

Remember now we have our manifold on space time with coordinate system x^μ so we can imagine a function on that space time that takes a point x which I could write as just x because we know it's a full on coordinate, I could write it as $\mathrm{f}(x^0, x^1, \ldots, x^{N-1})$ to show it's a function whose arguments are the coordinates of the manifold or I could abuse notation write $\mathrm{f}(x^\mu)$ and the reason this is abusive is because it almost makes you think like f only takes one argument and that argument is the μ -th coordinate of a point. These are all supposed to be the same I could do those or I could do $\mathrm{f}(x)$ so all three of these notations are really supposed to be the same.

The point is that f is a function on the manifold, because it's a manifold it's a function of coordinates so if I want to take the Exterior derivative of a function $\mathrm{f}(x)$ it is defined in the most elegant and simplest way namely:

$$\mathrm{d}\,\mathrm{f}=\frac{\partial\,\mathrm{f}}{\partial\,x^0}\,\mathrm{d}x^0+\frac{\partial\,\mathrm{f}}{\partial\,x^1}\,\mathrm{d}x^1+\ldots+\frac{\partial\,\mathrm{f}}{\partial\,x^{N-1}}\,\mathrm{d}x^{N-1} \tag{5}$$

I just repeat this process for each of the coordinate values and voila we have constructed the Exterior derivative and we've constructed very quickly the one form that is the Exterior derivative of a function f and notice, that's an important point, a function f is itself a zero form and then $\alpha_\mu \mathrm{d}x^\mu$ that's a one form so this is clearly in the form of $\alpha_\mu \mathrm{d}x^\mu$ where each of these would be our α and so you've raised the grade of f from zero to one so the index is called the grade of the form, a zero grade form, a one grade form etc. You don't use the word grade quite as much in the language of differential forms because you call them forms, you call them zero forms or one forms etc. The operation of taking the Exterior derivative involves raising the grade of the form, now we'll show that that's true in general, taking the Exterior derivative of an arbitrary grade form raises it but here we've just given it by definition, this is our definition of how the Exterior derivative works.

Now we're not going to go into why the Exterior derivative is a thing or what its interpretation is, you may remember there are three different derivatives you can take because we're dealing with a manifold and we're dealing with functions on a manifold and forms and there's essentially three different kinds of derivatives, there's the Lie derivative, there's the Exterior derivative and then there's the Covariant derivative and they all require some different special property in order to work, the Lie derivative requires a preferred vector field, the Covariant derivative depends on the existence of a connection and the Exterior derivative depends on the anti-symmetry of what you're taking the derivative of so these these different derivatives do different things they have different purposes they exist under certain conditions require certain assumptions or a certain extra elements added on top of the manifold but we're not gonna study that right now right we're just trying to basically get the notation down in order to make progress. That's the Exterior derivative of a function.

Now we want to study the Exterior derivative of a one form. The way it's presented here is basically as a rule, the Exterior derivative of a one form you execute this sum but I guess it is worth reviewing

$$d\omega = \partial_\eta \omega_\mu (dx^\eta \wedge dx^\mu) \tag{6}$$

where this sum comes from. What I've done over here is I've pulled from the Wikipedia page in the section on Exterior derivatives and I pull down the axioms of Exterior derivatives this is how we define Exterior derivatives and df is the differential f for a zero form we've already gone through that this here is what they mean by the differential f :

In terms of axioms

The exterior derivative is defined to be the unique $\mathbb{R}$ -linear mapping from k forms to $(k+1)$ forms that has the following properties:

1. df is the differential of f for a 0 -form f .
2. $d(df)=0$ for a 0 -form f
3. $d(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^p(\alpha\wedge d\beta)$ where α is a p -form. That is to say, d is an anti-derivation of degree 1 on the exterior algebra of differential forms.

This is the axiom for Exterior derivatives that applies to all forms so if we applied this to a one form and this is the confusing part a one form is $\omega_\mu dx^\mu$, that's a general one form so if we take the Exterior derivative of a general one form where actually this general one form really is $\omega_\mu(x)\wedge dx^\mu$ so I'll just put that as a function of x , that wedge product belongs in there where you're wedging a zero form with a one form it just so happens when you do that wedging this notion of the wedge product becomes degenerate essentially with regular scalar multiplication so this is what ends up being in here and then we apply this rule (#3) so we're actually looking for the Exterior derivative of this thing $\omega_\mu(x)\wedge dx^\mu$ and now that just equals the we've now followed the formula where $\alpha=\omega_\mu(x)$ and $\beta=dx^\mu$ so β is a one form and α is a zero form so now following that formula is (6).

The key to understanding how this works is that the Exterior derivative of an Exterior derivative is equal to zero by axiom (#2). This is the formula for the exterior derivative of a one form and it's derived from the axiomatic recursive definition I say this is recursive because do you have the Exterior derivative of the form α and the Exterior derivative of the form β and that's a p form so you're going to have to now go through this process for the p form but for one form we can write down this nice little neat formula (6) and that's what this formula is so we've now covered the Exterior derivative of a one form. We've covered the Exterior derivative of zero form, of a one form, we've defined a one form, we've also defined an arbitrary vector is $v=v^\mu\partial_\mu$, just like an arbitrary one form. What's next so now we want to look at the Exterior derivative of basically an arbitrary k form.

$$d\alpha = \sum_I \partial_\mu \alpha_I \, dx^\mu \wedge dx^I \tag{7}$$

You get this nice formula (7) which is quite reminiscent of this formula (6). You have this partial derivative, summing with an index that sums into this first term and you have this sum over these ordered indices now so let's see if we can understand this a simple expression using our basic algorithms over here so I'll begin just by writing down (3), is this sum over this index set that we've already discussed. We're not specifying the index set very particularly, we're just saying it's that set of increasing indices of k coordinates pulled out of an N -dimensional or a possible choice of N

different coordinates. I'm going to ignore the sum sign and we're going to think of this as an Einstein sum so we're going to get rid of this summation. Now we'll just apply this general idea of how to create an Exterior derivative of of the wedge product of two forms and we'll do the same thing we've got this function on the space time because remember these are all fields right this is a form field so there's one form at every point in space time so this is a function on the space time it's hard to remember that underneath all of this is a manifold which in our case is going to be space time and the manifold has coordinates x^{μ} you've got to always remember that and at every point you have a value of this form and ergo you end up having a form field and that is why this $\alpha_I(x^0, x^1, \ldots, x^{N-1})$ is actually a function of the coordinates I know I said that before but the problem is as you move away from it your mind drifts off of it and you tend to forget so saying it again can't possibly hurt so that's the reminder, this is a form field and we're going to express it just like this without the summation. Now we take this guy (3), and we treat it very much the same way we did before. We're looking for $\mathrm{d}\alpha$:

$$\mathrm{d}\alpha = \mathrm{d}\left(\alpha_I \mathrm{d}x^I\right) = \mathrm{d}\left(\alpha_I \wedge \mathrm{d}x^I\right) = \mathrm{d}(\alpha_I) \wedge \mathrm{d}x^I + (-1)^0\left[\alpha_I \wedge \mathrm{d}(\mathrm{d}x^I)\right] \tag{8}$$

This is a zero form wedged with $\mathrm{d}x^I$ which is the k form and this sets us up to use (#3). Recall $\mathrm{d}(\mathrm{d}x^I) = 0$ according to (#2). Now we are left with the first term itself but we know how to do it:

$$\mathrm{d}\alpha = \partial_\eta \alpha_I \mathrm{d}x^\eta \wedge \mathrm{d}x^I \tag{9}$$

This with the implied sum over I which comes up front this is $\mathrm{d}\alpha$. This is the glorious final formula. Now we can take the Exterior derivative of an arbitrary an arbitrary form α . What next? Well, I think next we should just do an example right so let's take a look at this particular two form:

$$\alpha = a\ \mathrm{d}x^0 \wedge \mathrm{d}x^1 + b\ \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{10}$$

We are going to say that $a(x^0, x^1, x^2, x^3)$ and $b(x^0, x^1, x^2, x^3)$ are function in space, I'm going to say we're in four dimensional space. It is in the form $\alpha_I \mathrm{d}x^I$ because we've indexed things correctly and it just so happens that the other combinations that are allowed, those coefficients are just zero, in this sum those α_I are zero. We've chosen a two form and we're going to take the Exterior derivative of this two form so we'll just write that down we're gonna take the Exterior derivative of our two form now this is linear so we're going to break up this sum right that's the equivalent of taking this big old summation symbol and breaking it up into its explicit parts and the Exterior derivative is linear so this is a good opportunity to point that out anything that that's pretends to be a derivative is linear and so it's going to be the Exterior derivative of the total to form is the Exterior derivative of this first component and the Exterior derivative of the second component:

$$\begin{aligned}\mathrm{d}\alpha &= \mathrm{d}\left[a \wedge \left(\mathrm{d}x^0 \wedge \mathrm{d}x^1\right)\right] + \mathrm{d}\left[b \wedge \left(\mathrm{d}x^2 \wedge \mathrm{d}x^3\right)\right] \\ &= \mathrm{d}a \wedge \left(\mathrm{d}x^0 \wedge \mathrm{d}x^1\right) + (-1)^0\left[a \wedge \mathrm{d}\left(\mathrm{d}x^0 \wedge \mathrm{d}x^1\right)\right] + \cdots \\ &= \left[\frac{\partial a}{\partial x^0}\mathrm{d}x^0 + \frac{\partial a}{\partial x^1}\mathrm{d}x^1 + \frac{\partial a}{\partial x^2}\mathrm{d}x^2 + \frac{\partial a}{\partial x^3}\mathrm{d}x^3\right] \wedge \left(\mathrm{d}x^0 \wedge \mathrm{d}x^1\right) + \cdots\end{aligned} \tag{11}$$

$$d\alpha = \frac{\partial a}{\partial x^2}\, dx^2 \wedge dx^0 \wedge dx^1 + \frac{\partial a}{\partial x^3}\, dx^3 \wedge dx^0 \wedge dx^1 + \cdots \tag{12}$$

Now the only problem is is our final form doesn't really satisfy our increasing index rule, we want these indices to go up but the only possible way is you're off by a sign so I can rearrange these things as long as I keep track of how many swaps I make that's an important thing to remember and this is a good chance to review it I'm glad we did this example:

$$d\alpha = \frac{\partial a}{\partial x^2}\, dx^0 \wedge dx^1 \wedge dx^2 + \frac{\partial a}{\partial x^3}\, dx^0 \wedge dx^1 \wedge dx^3 + \cdots \tag{13}$$

$$d\alpha = \left(\frac{d}{dx1} b\right)\ dx1\ dx2\ dx3 + \left(\frac{d}{dx0} b\right)\ dx0\ dx2\ dx3 + \left(\frac{d}{dx3} a\right)\ dx0\ dx1\ dx3 + \left(\frac{d}{dx2} a\right)\ dx0\ dx1\ dx2$$

This is the final answer, this is the Exterior derivative of α , notice that α is a two form (10) and down here its differential is a three form (13) and that's because the Exterior derivative always boosts your form by one. Let's see what were these other two this is just maybe I should have done this one earlier. If ω is a k form (using this Einstein sum notation) over an index set of increasing indices:

$$\text{If } \omega = \omega_I dx^I \text{ and } \lambda = \lambda_\mu dx^\mu \text{ then } \omega \wedge \lambda = \omega_I \lambda_\mu dx^I \wedge dx^\mu \tag{14}$$

If you don't just bite the bullet here and you can review the actual language so here's the way this works, if you have a form ω and $d\omega = 0$ then ω is called closed. Consider $\gamma = d\omega$, if γ is a k form then ω has to be a $k-1$ form then γ is called exact. that's the two words we're using. The Exterior derivative of any k form ω is closed because the Exterior derivative of the Exterior derivative of anything is zero. However if I had written this as $\gamma = d\omega$ and then I wrote $d\gamma = 0$, I would say two things about γ , I would say γ is closed because its Exterior derivative is zero but γ is also exact because γ is the Exterior derivative of a $k-1$ form and this basically is saying that all exact forms are closed. The profundity of this goes way beyond my little daft explanation here, I'm not even explaining things I'm just defining things. These little definitions and these words are not trivial this is a very very profound concept in differential geometry but for our purposes we just need to understand that exact forms are closed.

What am I missing here, this review so far is missing the notion of Hodge duality and that's too big a topic to proceed with now so we will begin our next lecture with a review of Hodge duality and then we'll be set up to do everything we need to do Maxwell's equations all right so I'll see you next time.