Geometric Algebra 3: The symmetric part

Let's continue our review of this excellent paper “Space-time algebra as a powerful tool for electromagnetism” and we have made it, well we started section 3.2 we've started as far as the space-time product which is absolutely a very vital concept to this entire endeavor, the space-time product is what makes the space-time algebra a thing, without a product you don't have an algebra and the space-time product is the specific product that turns a Geometric algebra into the space-time algebra, it really defines the unique Geometric algebra because the space-time product demands a few things it's first of all it's talking about four dimensions we got to talk about dimensions, by the way and it also establishes the Minkowski metric which is another thing we glossed over a little bit in the last lesson. Let's review a little bit of where we're standing emphasize a few points and then move into this critical subject of the space-time product so let's begin.

We will begin our work today where we almost left off last time which is these fundamental properties of the space-time product and there's a few things I want to emphasize about this. The first is when we're talking about a,b,c , notice this very important point, a,b,c these vectors that we're using as exemplars for these axioms that establish the properties of the space-time product, those vectors are in Minkowski space they're in this vector space $\mathsf{M}_{1,3}$, this inner product space $\mathsf{M}_{1,3}$. Now I've already said that the algebra that we seek, the Geometric algebra which I'm drawing here to be contained in this box right is we're going to specialize all of the Geometric algebra to one particular algebra the space-time algebra and through my explanations here I am going to occasionally do some demonstrations of why that's the case basically what I plan to do is from time to time drop into your regular three dimensional vector mathematics that we've all learned in high school and first year college and talk about the Geometric algebra associated with those vectors.

Of course, the space-time algebra is dealing with the notion of four vectors of Special relativity so that's a four-dimensional space-time Geometric algebra and also importantly the metrics between these two are different and we're going to talk about that in a moment, actually we're headed there to that subject right now so the general idea is we're working on we're taking this Geometric algebra and it's really we're going to focus on just the space time algebra and that first step of focusing on the space-time algebra is to acknowledge the fact that a big piece of this Geometric algebra, remember the Geometric algebra it is in algebra but it's also a vector space and the vector space contains vectors and we went through this last time and some of those vectors are the same vectors that live in $\mathsf{M}_{1,3}$ which is of course the four vectors and those four vectors we're calling for right now or the paper is called a,b,c . a,b,c are exemplars of these four vectors that live in $\mathsf{M}_{1,3}$ so they don't do anything like $(t,\boldsymbol{X})$ which they could, I mean that is a four vector but they're saying no no we're just going to call these things a,b,c and we're going to give them just straight up names.

Part and parcel with this is those four vectors if they're in this Geometric algebra they have to have a basis system and we know what that basis system is we we call the basis system e_0,e_1,e_2,e_3 , that's our typical basis system of four vectors however we are going to change this notation and that's because the paper will eventually change the notation and it's very important in telling and I think it's exciting because I this is an example of a notation change that sparks the imagination. We're going to change this notation from e_0,e_1,e_2,e_3 to $\gamma_0,\gamma_1,\gamma_2,\gamma_3$. Now if you've been following the lectures on prerequisites for QED, you'll immediately recognize that these are the same symbols that are used for the γ matrices however, and that's important that's why we're doing this, we are doing this because don't those look like the γ matrices? But the answer is no they don't because these are just the four basis vectors in $\mathsf{M}_{1,3}$ that specify a spanning set of orthogonal basis vectors.

We'll talk about the orthogonality in a moment because that's critical and we're just literally renaming them so these are meant to be basis vectors exactly like this, this is a renaming, it's literally just like changing the typography. Now the reason we're doing this is because yes, down the road we're going to discover that the Dirac matrices of the Dirac equation are indeed equivalent to or associated with the very basis vectors of the four vectors that live inside the space-time algebra and when you have a space-time algebra you can interpret these basis vectors of just this piece of it to be the Dirac matrices and what we say is the Dirac matrices are a representation of the basis vectors of the simple straightforward basis vectors but how much easier is it to think of four basis vectors than it is of these four bizarre Dirac matrices that really came to us through a very strange angle.

Hopefully when we get there you'll be like wow that's really really fascinating so we end up with these four vectors but we also realized that we need to also augment the space with the real numbers, now when you do that obviously the real numbers cannot be constructed out of these basis vectors but the real numbers $\mathbb{R}$ is a vector space in itself, remember this is the vectors space $\mathbb{M}_{1,3}$, $\mathbb{R}$ is another vector space that we're adding into our big vector space so we're putting these two vector spaces together now the way you put two vectors pastes together in the form that we plan is something called the [Direct sum] so we're not going to jump on that right now but keep that in mind but the real numbers have their own basis vector and it's the unit 1 , the number one is now the basis vector of this one-dimensional vector space that we've added to build up this algebra and unfortunately it's actually pretty interesting we have other vector spaces that we need to also Direct sum into this and I think in the case of the space-time algebra there should only be two more, this one, the first one we add it should be four more and we'll be building each one of these as we need them.

What's important to understand is, remember we're combining these vector spaces together and each time we do it each of these little sub vector spaces are going to have their own basis vectors that are going to be added into this this one $\mathbb{R}$ only has one basis vector, turns out this last one will also only have one basis vector this one will have four basis vectors and this one will have six so you have one basis vector, four, six, four, one and for those of you who've studied with me about differential forms and the exterior algebra, the exterior calculus, those numbers will sound really familiar and I guess just so we're consistent with the future this $\mathbb{R}$ really should be listed first. You want the low dimension one dimension, four, this will be dimension six, dimension four, and then dimension one again so you have to symmetry: one, four, six, four, one.

We'll see why that is later we'll also understand exactly why these direct sums are here these Direct sums are really important, Direct sums are used to make bigger vector spaces from smaller vector spaces but what's interesting so the point of this in a way the point of getting down this little descriptive tangent of where we're headed is to focus on this box right here because these four vectors they're a bit of an anchor to the entire theory of Geometric algebra and it took me a while to figure this out so I want to save you the trouble and the point being that these vectors these vectors a, b, c , that's what this is talking about, when this assumptions are made about the Geometric algebra these properties that they lay down as fundamental and foundational to this theory of which everything flows from them:

$$\begin{aligned} a(bc) &= (ab)c \quad (\text{Associativity}) \\ a(b+c) &= ab+ac \quad (\text{Left Distributivity}) \\ (b+c)a &= ba+ca \quad (\text{Right Distributivity}) \\ a^2 = \eta(a,a) &= \epsilon_a |a|^2 \quad (\text{Contraction}) \end{aligned} \tag{1}$$

The whole theory flows from these four properties but these are not members of the Geometric algebra in its entirety, they're only members of the little sub piece of the Geometric algebra $M_{1,3}$, (1) are rules for four vectors they're not rules for any of the other elements of the Geometric algebra now maybe at this point that seems a little obvious but if you study the subject raw you have to come back and remember, wait a minute this whole thing is really built on the properties of these guys and knowing how we want these guys to work we start realizing exactly what we need to add to the theory and it all flows from this and in particular when we talk about this one the last (contraction) we realized this one, this contraction axiom forced us to add the real numbers we went through that last time and so that was the first implication of the power of this to expand our vector space of four vectors into something that looks like a Geometric algebra.

One thing I did not emphasize although we stated it and it's very obvious right here is the presence of the metric in this right here and what that's what's a really critical fact and this is true for any Geometric algebra, the fact that they wrote η here in our case η represents the Minkowski metric where you take two four vectors and you jam it into the Minkowski metric and you get a number back that's either space-like, time-like or light-like and in the method that they're using a negative number coming out of this process is a space-like vector and a positive number is a time-like vector but the point is this axiom right here, it's the core axiom at the foundation of the whole thing and what do you see in the middle of it? Right there you see the metric, the metric is literally built in to this axiom of this space-time algebra. It's part of the fundamental premise of this algebra and this metric could be very general it could have been the case, it could easily have been the case that this this metric $\eta(a,a)$ or remember it's a metric so $\eta(a,b)$, it doesn't really matter it could have been that this is the euclidean metric and it would be positive definite so this could have been $\eta(a,a)=\epsilon_a|a|^2$. Where this $|a|$ magnitude of a is what the metric defines.

If it was Euclidean then we wouldn't need ϵ_a because it would always be positive but because our metric is Minkowski we include this ϵ_a so we can isolate the sign from the magnitude of the metric contraction or the contraction between a and b but what I'm trying to emphasize is that this metric is part of this four vector space metric and because it's the Minkowski metric that's what makes a general Geometric algebra the space-time algebra, it's because of this, this metric is what takes this Geometric algebra and turns it into the space-time algebra. The two things. the fact that this is a four-dimensional vector space and that it has a Minkowski metric, that is the two things, those are the only two things you need to take your general geometric algebra and get a space-time algebra because I could say well forget $M_{1,3}$, let's say I want a seven dimensional vector space, I want this to be seven dimensional and my metric is going to be some weird seven dimensional metric, that'd be fine.

I would have $e_0, e_1, e_2, e_3, e_4, e_5, e_6$ and I would have to add a few more vector spaces it turns out to make this whole thing complete but I would have a seven dimensional Geometric algebra with a particular metric and it would be a very unique, interesting Geometric algebra it just wouldn't represent anything in Physics that we know about or I should better be careful there, that I know about but the one that we do know about is, if you take this concept of this Geometric algebra which is just picking a vector space and using these same axioms, no matter what Geometric algebra you take, these axioms will always apply. I pick a vector space and I pick a metric and I've got a particular Geometric algebra and when this vector space is $M_{1,3}$ and when this metric is Minkowski I have the space-time algebra. Now we haven't gotten there yet, we haven't finished it, we won't fully finish the space-time algebra until we know everything there is to know about these things and there is a few other key principles we have to deal with but that is the program here.

The rest of today we are going to leave this subject of these axioms because I think we've spoken about them plenty understanding that they apply only to the four vectors of our Geometric algebra which is I think fascinating remember this product has to be good, that's an important point the product that we're trying to create has to be good for everything in the Geometric algebra that means if I take anything out of this algebra, I go into here and I pull out something and I'll call it X and X may be I pull out at something X and then I pull out another thing Y , I definitely I have to be able to write the geometric product or the space-time product as we're going to call it of XY I have to be able to write that and XY is going to have to equal Z and Z has got to be inside the algebra. X and Y do not have to be from $\mathbb{M}_{1,3}$, X and Y can be from $\mathbb{R}$ or it can be from this thing that we've yet to create or this thing or this thing or it can be the sum, remember because you're supposed to be able to add anything in here too so I should be able to take something from here and add it to something from here and that that counts as X and then Y could be something from here added to something there and that counts as Y

I need to be able to take the geometric product of of those two sums as well, so the geometric product is not forced into this little box what is forced in this little box is the axioms that the geometric product must satisfy so whatever this product is we force it to satisfy certain rules for these guys and then we accept whatever we're forced to accept for everything else, we accept whatever we need to expand the vector space $\mathbb{M}_{1,3}$ to create the overall space-time algebra and then we accept whatever rules are forced upon us for these other geometric products of arbitrary members of the space-time algebra and that process is actually pretty tough but we're leaving this four vector set of axioms and now we are moving on to something else we're going to talk about the geometric product in a little more detail and this paragraph here is the next paragraph of our study, there's a lot to talk about inside this paragraph and this paper, it really wants to get to the notion of electromagnetism very quickly and so it offers this up this strange language with these new symbols this $\cdot$ (dot) and this $\wedge$ (wedge) which are familiar symbols to anybody who's dealt with a lot of vector analysis and certainly well this is the forms and exterior calculus, exterior algebra but this paper doesn't have the time to really dig into why these things are what they are how they're proven and we are going to spend that time this is one of the gaps that we're going to fill but this is all you really need to know, I mean you could understand this formally and proceed through the paper but we're going to spend some time here.

We're realizing still a couple things $a, b \in \mathbb{M}_{1,3}$ so we're still looking at these formulas whenever we see a and b we're still stuck in $\mathbb{M}_{1,3}$, this little sub piece of our ultimate final algebra and again we will build up the rest of the algebra from that so let's start looking at at first of all, let's just look at this thing, it's basically saying well it's literally saying that this space-time product between two vectors in $\mathbb{M}_{1,3}$ can be decomposed into two parts:

$$ab = a \cdot b + a \wedge b \tag{2}$$

The first part is this $\cdot$ (dot) product thing and the second part is this $\wedge$ (wedge) product thing and now we have to say well so what is he going for? What are we going for here. Well this dot and wedge product are supposed to be familiar, they're saying this is easy, don't worry about this geometric product this space-time product because it's really the product the sum of two things you already know about, you know about the dot product of two four vectors, that's easy because the dot product of two four vectors is just $\eta(a,b)$, it's the Minkowski contraction of a against b and that's a scalar. Now that being a scalar is very familiar this other part $a \wedge b$ is probably less familiar but they're hoping that oh no you've studied the exterior calculus so you probably even know what that is too. Now the paper is very good about saying well a lot of our readers probably don't know about this and then they they talk about it here and then they go through these geometric analyzes later which we will follow, we're going to follow this paper literally but many people will find both of these pieces familiar, what a lot of people will have trouble with if you do find both of these people's pieces familiar what you might immediately have trouble with is saying well wait a minute this is a scalar and this is what's called a two vector, it's a directed area how do you add scalars to two vectors so that is something that people will scratch their heads about and I'm not going to let that sit I'm going to fill that gap for sure.

For right now we are going to just understand that they're offering this up as a definition that they're going to have to explain these two things and they leave it up to the reader to dig in a lot on this this is this is the essence of geometric algebra and they're saying you know they don't have a lot of time in this paper to really build it up from total scratch so they offer a reference for example but they do build up a lot, they build up these key points and you could follow this paper by literally accepting these things but we're going to focus on studying this paragraph a little bit more so right now we're going to focus on this part $a \cdot b$ and for the rest of this lesson we're just going to focus on this, what I'm going to call the scalar part of this expression and then in the next lesson we'll either finish that up or focus on $a \wedge b$ this two vector part this wedge product part so we should just read what they say, we should read the literal words of the author because the authors have thought a great deal about this, “decomposing the resulting associative vector product into symmetric and anti-symmetric parts produces the proper space-time generalizations to the three vector dot and cross products that we are seeking” so what they want to say is that they are trying to generalize the three vector dot and cross products well here's the dot product so this must have something to do with the generalization of the cross product I presume.

Just to try my best to be clear, this space-time product is not scalar, the only time it is scalar is when you're doing the space-time product with a vector by itself $a a$, we would expect everything here to be scalar because of this axiom (1) so if we're dealing with $a a$, we're expecting $a^2 = \eta(a, a)$ the space-time contraction, the Minkowski contraction of a with itself which would mean, based on this formula

$$a a = a \cdot a + a \wedge a \tag{3}$$

But $a a$ and $a \cdot a$ by the axiom are the same thing so this would have to be zero for $a \cdot a$ so that that's an important point is to understand that this rule applies to any two vectors, we're going to have to learn why this piece is zero a little bit later but this contraction rule does not apply to $a b$ it only applies to a^2 , a literally squared so continuing “the symmetric part of the product”

$$a \cdot b \equiv ½ (a b + b a) = b \cdot a = \eta(a, b) \tag{4}$$

The symmetric part of the product meaning this product here $a b$, this product has a symmetric part and an anti-symmetric part and they want to say the symmetric part which what they really mean is (4) is this part right here, this is the symmetric part of the product is this piece right here $½(a b + b a)$, that symmetric part of the product must equal $\eta(a, b)$ and it must equal this dot product $a \cdot b$ so that's a new concept, the symmetric part of the product is not obvious from here and it is a bit of a jump so we have to understand the following obvious formula we write:

$$a b = ½ (a b + b a + a b - b a) \tag{5}$$

The ba cancel, the ab double but you divide by two so we've written this out in this very plain way and then the idea is that this first part is the symmetric part and this second part is the anti-symmetric part. Now notice we've not introduced basis vectors here yet, all this talk we've got is is without basis vectors that's what we mean by coordinate free, these are all what we would call proper vectors in the sense that we have no coordinate frame and this is probably our first example of a little bit of a demonstration that uses purely proper vectors, no basis vectors we've just split we can take any two vectors and just immediately break it up into a symmetric and anti-symmetric part.

What they want us to see is that this symmetric part has to be identified with this dot product $b \cdot a$ which by the way commute so these two dot products so that's what they want us to understand so can we demonstrate that? Well we got to go to the last sentence of the paragraph. “The symmetric part of the product” which we've now identified, “is precisely the scalar dot product inherited from the space-time structure of $\mathsf{M}_{1,3}$ ”. This is really a great statement, “the space-time structure of $\mathsf{M}_{1,3}$ ” defines the symmetric part of the space-time product which is the scalar dot product and why is it inherited from that? Well, because you've got this η and that η has the space-time structure built right into it because what is space-time structure? Space-time structure is the metric and when people study General relativity it takes them a while to get their head around the idea that space time is the metric, in General relativity but it's also true in Special relativity, the notion of space-time is completely wrapped up in the Minkowski metric all of the Lorenz Transformations all of the rules about speed of light being as fast as you can go and no signaling and all of that great stuff, it's all about the metric.

$$a \cdot b \equiv ½(ab+ba) = b \cdot a = \eta(a, b) \tag{6}$$

They want to emphasize, the authors, I feel, want to emphasize that the space-time structure of $\mathsf{M}_{1,3}$ is automatically brought into the whole problem through this contraction axiom but this is still just a statement “the symmetric part of the product”, “is precisely a scalar (dot) product inherited from space-time structure of $\mathsf{M}_{1,3}$. The last equivalence follows from the contraction relation”:

$$(a+b)^2 = \eta(a+b, a+b) \tag{7}$$

That's our hint on how to show this, the contraction relation (7) demanded by contraction property (1) which is of course the all-important contraction property. Let's unpack (7) and make sure we're very comfortable with this idea and then call it a lesson. We're now going to see if we can understand how to take the space-time product of any two vectors that live in this portion of our Geometric algebra with the understanding that we still don't understand this portion (four vectors) and maybe we don't fully understand these additions but we don't have to worry about that because we're just looking at these guys themselves, there's no additions involved with other pieces of the algebra and none of the material that's in these other sections of the algebra even matter to us yet so what are we looking at?

We're going to start with the notion that we have two vectors a and b , we definitely know that we can add a and b together so we can definitely add a and b and we still are inside this space $\mathsf{M}_{1,3}$ and because of that we can take the metric from this space and we can apply the metric to a and b if we want, we can also take the geometric product of a and b with itself so a and b is a vector in the space so $(a+b)(a+b)$ is also, well it is now subject to that contraction rule, this has to equal $\eta(a+b, a+b)$ because remember a and b are both vectors in $\mathsf{M}_{1,3}$ so $a+b$ is a vector in $\mathsf{M}_{1,3}$ so I'm basically taking the space-time product of a vector with itself and this is the rule, this is the rule for doing that so now I can look at the left side and the right side of this separately. Everything's linear so the next step is pretty darn obvious we write this as:

$$(a+b)(a+b)=a^2+ab+ba+b^2 \tag{8}$$

That's definitely the left hand side and if you look you can already see the inklings of the symmetric part of the product because remember what we had over here $½(ab+ba)$ is the symmetric part so you clearly see that that's showing up already in our expansion and our very linear expansion of this space-time product. Both sides of this are linear because that operator over there is linear so this becomes:

$$\eta(a+b,a+b)=\eta(a,a)+\eta(a,b)+\eta(b,a)+\eta(b,b) \tag{9}$$

We know that $\eta(a,a)$ and $\eta(b,b)$ we know that η being the Minkowski metric as it is, we know that this is symmetric, in fact all metrics relevant to General relativity and relevant to Special relativity at least the elementary versions are symmetric metrics so we can combine these two and we would get:

$$\eta(a+b,a+b)=\eta(a,a)+\eta(b,b)+2\eta(a,b) \tag{10}$$

Then we immediately can do some cancellation between (8) and (10) because $a^2=\eta(a,a)$ and $b^2=\eta(b,b)$ by definition, by the axiom that we apply to the four vectors, the axiom I keep referring to it because it's so important this contraction axiom and so that leaves us with this:

$$ab+ba=2\eta(a,b)\rightarrow ½(ab+ba)=\eta(a,b)=a\cdot b \tag{11}$$

We now get the symmetric part of the space-time product ab which is equals the contraction between the two and that is written in shorthand as $a\cdot b$, that's the shorthand version and that completes the demonstration that they're alluding to right here. “The last equivalence falls from the contraction relation (7), that's the key equivalence that they're interested in making sure we understand is that the symmetric part of the product is equal to this Minkowski contraction so this dot product is it's the same as this symmetric piece so that means now if we look if we go back one little step and we look at ab and we realize that it's equal to this dot product plus this wedge product (2) and we now know that this dot product part $a\cdot b$ is the symmetric piece, well what's left? The only thing left is this wedge product this must equal the anti-symmetric piece because $a\cdot b$ accounts for the entire symmetric piece here this the dot product can counts for the entire symmetric piece you need to get to ab so all that you have left is the anti-symmetric piece and so this anti-symmetric piece is identified as the wedge product.

It's really interesting because we know $a\cdot b$ is a scalar and $a\wedge b$ is not a scalar, this wedge product as we're going to see is not a scalar so this product ab cannot be a scalar either, now if it was just aa it would be a scalar so there's some weird product out there that when you take the space-time product with itself, this weird space-time product is when you take a vector's space-time product with itself you get a pure scalar and the anti-symmetric part goes away but when you take it with any other vector you get a scalar piece and a non-scalar piece so you've got this product that produces things that are mixed character, there's a little scalar and a little not scalar. That's what we need to get into next and sure enough the next thing they talk about is well what is the anti-symmetric part of this product?

Now you'll notice that we did this proof here without any reference to basis vectors which means we never really selected a particular reference frame and when we write the metric like this η , that's very abstract, that's true in every single reference frame because $\eta(a,b)$ in reference frame one is equal to $\eta(a',b')$ in reference frame two as long as you do the transformation correctly so we can write this this is in full generality, no reference frame was chosen here to make this calculation a little bit easier we and in this particular case it this non-reference frame version of the of the proof was actually pretty easy but there is a version of this proof that does in fact use a reference frame and I want to show you that now before we go because we're going to be doing a lot of proofs and sometimes we will in fact discover that it's actually easier to do the proof if we establish a reference frame and then come to understand that we could have done that proof in any reference frame as opposed to starting a proof like this where we never even discuss reference frames so let's have a look at that other method.

The first thing we need to do is establish a basis so we're going to establish some basis vectors now for this purpose I'm still going to use the e_0, e_1, e_2, e_3 so we are now calling this the basis vectors of this subspace of our overall space-time algebra and I haven't really gone into the Dirac stuff yet, we'll probably take that up maybe well we'll probably take it up as soon as the authors do so. We have these four basis vectors now we're going to assert a couple things, we're going to assert that they're orthonormal which basically means we can make some statements about the Minkowski inner product

$$\begin{aligned} &\eta(e_0,e_0)=+1 \\ &\eta(e_i,e_j)=-\delta_{ij} \text{ where } i,j=1,2,3 \\ &\eta(e_0,e_i)=0 \end{aligned} \tag{12}$$

This is the Minkowski contraction of these basis factors with each other and with themselves. Whenever you have these Roman subscripts, it's $1,2,3$. If you had a Greek subscripts it would go from $0,1,2,3$ so anytime you see a basis vector contracted with itself then these are actually orthonormal, they're going to be unit vectors as well yeah so they're orthonormal because they're unit vectors but they're also clearly orthogonal i and j have to be the same and I guess for completeness e_0 with any of the three spatial, the time like unit vector with any of the spatial unit vectors is also 0. It's essentially a diagonal metric so this is what we mean when we say we're picking a basis we pick a basis set of basis vectors and we fully understand how the metric operates on those basis vectors and we're free to choose this basis, a basis that has nice Cartesian orthogonal basis vectors but non-Euclidean, we're using this as Minkowski metric so once we've done that, now we can say well these are vectors right these are four vectors in this thing so all of these axioms apply so we can immediately write down what is the space time product of $e_\mu e_\nu$ and the rule is that it has a symmetric and anti-symmetric part so the symmetric part we already know is $e_\mu \cdot e_\nu$ which just as a reminder is $\eta(e_\mu, e_\nu)$ plus this anti-symmetric part which I'm just going to write down as $½(e_\mu e_\nu - e_\nu e_\mu)$ because we know we've already shown that this is the symmetric part and this is the anti-symmetric part:

$$e_\mu e_\nu = \overbrace{e_\mu \cdot e_\nu}^{\eta(e_\mu, e_\nu)} + ½(e_\mu e_\nu - e_\nu e_\mu) \tag{13}$$

We also know that $\eta(e_\mu, e_\nu)$ is going to be zero whenever μ and ν are not equal and it's going to equal either $+1$ or -1 when μ and ν are in fact equal but when μ and ν are in fact equal this term goes away because if μ and ν are equal you get something minus itself, for example you might get you know $½(e_0 e_0 - e_0 e_0) = 0$ so what's interesting here is because of the basis we've chosen this part is if μ and ν are the same everything in this space-time product of these two basis vectors lives here (symmetric part) and this anti-symmetric part goes away. On the other hand if the basis vectors are different then $\eta(e_\mu, e_\nu)$ is driven to zero and that symmetric part goes away and the space-time product between two different unit vectors has to live entirely in $½(e_\mu e_\nu - e_\nu e_\mu)$ in the anti-symmetric part.

Those are a few of the lessons we've learned about how these basis vectors work in the space-time algebra given that we choose this metric relationship (12), actually this would apply to any metric relationship. This point here about the symmetry and the anti-symmetry is completely general so with that in mind, let's have a look at breaking down one of our rules in terms of basis vectors. We're going to look at this sum of $ab+ba$ and what we do is we assert straight up that $a = a^\mu e_\mu$ and $b = b^\nu e_\nu$ and once we've done that we've chosen our basis and we're using the basis to be the one we just discussed. We can create this symmetric product, the symmetric part of the product ab , remember this is twice the symmetric part of the space-time product ab and we just write it out, we just blow it up:

$$ab + ba = a^\mu b^\nu e_\mu e_\nu + a^\mu b^\nu e_\nu e_\mu = a^\mu b^\nu (e_\mu e_\nu + e_\nu e_\mu) \tag{14}$$

The real numbers can pop out in front and you're left with the space-time products of these unit vectors and then we pull out these two real parts right up front and these are space-time products now we just put the vectors right next to each other these are space-time products of unit vectors and I can blow this sum up into all the parts where the unit vectors are the same in the space-time product and where they are all different in the space-time product:

$$ab + ba = 2(a^0 b^0 e_0 e_0 + a^1 b^1 e_1 e_1 + a^2 b^2 e_2 e_2 + a^3 b^3 e_3 e_3) + \sum_{\mu \neq \nu} a^\mu b^\nu (e_\mu e_\nu + e_\nu e_\mu) \tag{15}$$

If $\mu = 1$ and $\nu = 1$ I get $a^1 b^1 (e_1 e_1 + e_1 e_1)$ so these two add up to give me a two and so I get this section here and this section here (summation symbol). The problem is on this section (summation symbol) I know that μ and ν are different, I know that they're different and because I know that they're different I go back to what I just did and I showed that well if they're different we know that the symmetric part goes away because the symmetric part is equal to the inner product which we've established as being Minkowski so if they're different this part goes away so everything that's left is the anti-symmetric product so I know that the space-time product between two of our basis vectors when they are not the same basis vector is in fact just the anti-symmetric part of that space-time product it's these two things $½(e_\mu e_\nu + e_\nu e_\mu)$ but the key idea is that the symmetric part is zero, that we know that I should just to be clear right write that down we know that $½(e_\mu e_\nu + e_\nu e_\mu) = 0$ for this case.

That's exactly what we have here we have this symmetric piece sitting in front of us so we know that that symmetric piece equals zero so what does that leave us with? Well that leaves us with we we take this down and we write:

$$½(ab + ba) = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = \eta(a, b) \tag{16}$$

That's exactly the definition of this whole thing so we've re-verified through using the basis vectors, we've re-verified that this in fact equals this (16), well this is what we mean when we say we're going to throw down basis vectors we take these geometric or space-time products and we cast them in terms of basis vectors and then knowing how the Minkowski metric works on the basis factors we simplify things and we come up with expressions for the components that are something that we're familiar with so this is just another approach of handling that demonstration. Now admittedly having gone through this I sold this and beginning as an alternative way of showing that the symmetric product equals the Minkowski contraction but that's not really true. We're going to use basis vectors instead of total generality but take note that when we did this part $e_\mu \cdot e_\nu$ I derived that fact from our original proof our basis vector free proof so I think in this particular case you really do have to start from fundamentals in order to make this connection because making this connection was what was so critical so what I did was in order to say that the symmetric part was in fact the dot product right I used this proof but then I applied that right here to say that oh the symmetric part of this is zero therefore it's just the anti-symmetric part and then I use the fact that the symmetric part was zero to eliminate this piece which of course led to the conclusion that I essentially had already assumed so this isn't an exact proof, this is just an alternative way of understanding it and a good introduction on how to throw basis vectors into the problem use the simplifications afforded by the Minkowski metric to see how things flush out but this isn't really an alternate proof, there are circumstances however we're casting things in terms of easy to understand basis vectors do make proofs a lot simpler and I hope to give you a few examples of those as we move forward.

That takes care of what? We're unpacking this story and we unpacked it by breaking down this product (7) exactly as they asked and made this conclusion that the symmetric part was in fact the classic dot product using the Minkowski metric, we validated this statement (7) and the next part is to understand this second piece, this wedge product piece which is obviously the anti-symmetric part of the product and we'll start getting into that next time.