Geometric Algebra: Spacetime

Let's get back to work at our study of Space-time algebra which is going to be a study of Geometric algebra, Geometric Space-time algebra is an instance of Geometric algebra I guess that's the best way to say it. In our last lecture we basically got through this introductory section on the space-time algebra so now we are going to move into these two sections space-time and space-time product. Let's begin, section 3.1 is beginning to describe the Special relativity of space-time so this is where this paper really focuses on the Physics and application of Geometric algebra to Physics so it leads with Physics, it's leading with the subject of space-time, most treatments of Geometric algebra lead with the mathematics of the Geometric algebra, being a physicist I just prefer it this way however I've read plenty of papers that go the other direction, it starts with a study of Geometric algebra in total generality and then it applies it to space-time so but this going the other way I feel like it actually slips into the subject a little bit better and it starts very obviously, it says that we have in Special relativity you postulate is an important word here because Special relativity is a presumption about space-time, you can't develop it from simpler principles really, you have to make assumptions although to be honest that's true of any physical theory but in the postulation of space times you have a scalar time and they like to say vector spatial coordinates $\boldsymbol{x}$, that's just a way to say the spatial coordinates are bundled up into a group of three and that those three can be treated as vectors in a particular reference frame.

You always have the scalar time involved and then Special relativity has an invariant interval $(c\,t)^2-|\boldsymbol{x}|^2$ and right there we immediately answer the question about the signature clearly this signature for this is going to be $(1,-1,-1,-1)$ which is the opposite of the signature we used in the class on “what is general relativity” which is not a big deal you we should all be familiar with both of them. So far this is a very simple explanation of Special relativity and then they want to say an elegant way of coding this physical postulate is to combine the scale time and spatial components into a four vector $x=(c\,t\,,x_1,x_2,x_3)$, this is a four vector that locates an event in space-time with a squared length equal to the invariant interval so when you square the length of x this four vector you're supposed to get $(c\,t)^2-|\boldsymbol{x}|^2$ and of course the only way that's true is if you use the Minkowski metric to do the squaring, that's what they describe down here in terms of these two four vectors $a=(a_0,a_1,a_2,a_3)$ and $b=(b_0,b_1,b_2,b_3)$ here in a certain particular inertial frame we say:

$$a\cdot b\equiv b\cdot a\equiv\eta(a,b)=a_0\,b_0-a_1\,b_1-a_2\,b_2-a_3\,b_3 \tag{1}$$

Where $a\cdot b$ is commutative, it is the Minkowski metric applied to $a\cdot b$ which is (1) and then they reaffirm the signature $(+,-,-,-)$. Now they have a vector space because you can add the position of two events in space time you can subtract them as long as you understand you're dealing with a scalar time and a vector position and it's got this Minkowski metric of $(+,-,-,-)$ so they want to say the vector space of all these four vectors, they're going to give it a name and that name is $\mathsf{M}_{1,3}$ it is just the vector space, is space time and it's an inner product space with a Minkowski metric, so no Geometric algebra so far. In the next paragraph “all physical vector quantities and Special relativity are postulated to be four vectors in $\mathsf{M}_{1,3}$, this new vector space that we've just identified. All physical quantities four vectors in $\mathsf{M}_{1,3}$ that satisfy the Minkowski metric. That's of course, everything must otherwise it would literally violate Special relativity.

They want to say that these vectors are called geometric quantities well these vectors are geometric quantities but they do not depend on the choice of reference frame and that's true, that's the whole deal with Special relativity you want to find those things that are independent of reference frame and as such without a reference frame we're going to call them proper relativistic objects so we'll look for this language as we read this paper proper relativistic objects and then they talk a little bit about how the Minkowski metric is different from the euclidean metric which I think we all know we have time-like space-like and light-like, that's fine but let's get to this definition that they use in this paper that isn't very typical and that's this notion of the pseudo norm. The pseudo norm of a four vector and that's given as $a \cdot a$ which is just the magnitude of the vector but they want it to break this up into this structure here they want to have it as $a \cdot a = \epsilon_a |a|^2$, an absolutely positive part which is this magnitude of $|a|^2$ and they want this ϵ_a which is either $+1$ or -1 so they'll give it a sign separate from its magnitude and that sign is going to be called the signature right and they use this a bunch in this paper.

The magnitude of a four vector they're pointing out it can be positive, zero or negative and the positive and negative is captured by the signature of this magnitude of the pseudo norm they're going to call this a pseudo norm and they break it into these two parts, the signature and this positive magnitude. Now in the next paragraph they present a conundrum, takes them a while to get there but they do get there what they're basically saying here is that now we have this vector space $\mathsf{M}_{1,3}$ of all these four vectors and there are certain things that it does very well, you can model space-time events, the space time itself in $\mathsf{M}_{1,3}$ and you can model the energy momentum vector $\boldsymbol{p}$ this is the energy piece of a four momentum E those do very well but you cannot take they notice right away you can't take the electric field and the magnetic field and somehow create a four vector out of the electric and magnetic field and point out that in order to handle the electric and magnetic field relativistically you've got to create these this tensor $F^{\mu\nu}$, the electromagnetic field tensor F and we've done that already together in our prerequisite class so we've seen this and we understand how this this tensor is manifestly invariant and it represents the only way to express the electromagnetic field independent of coordinates.

Now they argue that's okay, it's formally correct is what they're saying but it's “conceptually opaque”, they've decided and then they ask the questions why does a single rank two tensor $F^{\mu\nu}$ decompose into two three-vector quantities $\boldsymbol{E}$ and $\boldsymbol{B}$ in a relative frame so let's look at that question, you have a rank two tensor, we have $F^{\mu\nu}$ and then $F^{\mu\nu}$ is the only way to speak about the electromagnetic field in total generality regardless of what frame you're in but if you do choose one particular frame then you can peel this apart and you can create a three-vector $\boldsymbol{B}$ and a three vector $\boldsymbol{E}$ that's good for that particular frame so they're asking well why does this two rank tensor decompose like that and then that's a question, now we understand how the decomposition is done, we understand that when you get this object, you put it in a particular frame, you can read off the $\boldsymbol{B}$ and you can read off the $\boldsymbol{E}$.

$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{2}$$

Remember how this thing worked right, you had E_x, E_y, E_z and then you had a section that had $\boldsymbol{B}$. Let me just write it a little neater, we know that this these are the components you can read off the components of the three of the of the magnetic field $\boldsymbol{B}$ out of this bulk of the matrix and you can read off the components of the electric field $\boldsymbol{E}$ off of these columns and once you just transform this into the right frame you can read off these three-vectors so what they're saying is you know well why is that? It seems pretty obscure or difficult I guess is what they're after and they want to know is there some deeper significance to the mathematical space in which the tensor $F^{\mu\nu}$ resides.

Notice this tensor does not reside in $\mathbb{M}_{1,3}$. $\mathbb{M}_{1,3}$ is a vector space of four vectors, only four vectors live there, $F^{\mu\nu}$ is not a four vector, it's a rank two tensor so it lives in some other mathematical space and then he asks "do the proper tensor descriptions have any geometric significance in Minkowski space? Well, Minkowski space is what this $\mathbb{M}_{1,3}$ is supposed to be so what is the geometrical significance of this guy $F^{\mu\nu}$ inside of this space $\mathbb{M}_{1,3}$? It seems strange that we have to introduce something else in Minkowski space to model something so basic as the electromagnetic field so he's upset about that and I get it, I see the point, it's a question I never asked, I just learned the math and kept going but so evidently the vector space $\mathbb{M}_{1,3}$ does not contain the complete physical picture implied by Special relativity since it must be augmented by quantities like $F^{\mu\nu}$ which are outside the space so the simple vector space of four vectors isn't enough, you've got to add new quantities to make it work. Frankly what he says there is somewhat obvious but it's not typically something we would articulate to ourselves, we learn things move on realize it's this two rank tensor how cool.

Now also in this paragraph he alludes to some another example of Angular momentum which he calls $M^{\mu\nu}$ and he uses $\boldsymbol{L}=\boldsymbol{x}\times\boldsymbol{p}$. We're going to skip that for now, it's not particularly necessary so the foundation of this section 3.1 is it tells us hey we're using this thing we're basically creating Minkowski space $\mathbb{M}_{1,3}$, the space of all four vectors and it has this signature $(+,-,-,-)$ and then he gives us a little bit of his notation conventions and then he poses the question how do we expand or how do we broaden Minkowski space in order to contain all of the things it needs to describe electromagnetism so then we will now move on to the next section.

That next section 3.2, is the space-time product so we continue. "To obtain a complete picture of Special relativity in a systematic and principled way, we make a critical observation: any physical manipulation of vector quantities uses not only addition but also *vector multiplication*". Here we're talking about turning something into an algebra, remember $\mathbb{M}_{1,3}$ is just a vector space, that's only a vector space it is not an algebra because this notion, we allowed it to have an inner product so it does have an inner product but the way we've defined this vector space we haven't defined any form of multiplication of two four vectors that would give you another four vector so we don't have a vector multiplication yet in $\mathbb{M}_{1,3}$ so that's what's going to be added here so "indeed standard treatments of electromagnetism involving relative 3-vectors". When he uses the word "relative 3-vectors", whenever you see that in this paper, this word relative means a particular frame of reference that's what he means by "relative 3-vectors", we've chosen a basis we've got a frame of reference and with that basis we can use dot products and cross products to discuss the physical implications of the theory.

Now the physical implications of the theory, when he's talking about that, think Maxwell's equations, you have cross products, you have dot products, you have curl, and divergences which are the calculus version of the vector cross product and vector dot product but as you learned electromagnetism you learned how these relative three vectors, because whenever we studied it initially, we were in a definite reference frame a definite frame of reference and we knew how to take dot products we knew how to take cross products and then when we got good we learned how to take curls and divergences and all kinds of calculus so "the symmetric dot product and the anti-symmetric vector cross product to discuss the implications of the theory", so that means to discuss Maxwell's equations.

However, they point out, "the vector space $\mathbb{M}_{1,3}$ only specifies the relativistic version of the dot product in the form of the Minkowski metric" so there is a dot product in $\mathbb{M}_{1,3}$, it's the inner product, it's not vector multiplication like we need for an algebra, remember it is just vector multiplication, well it's not even vector multiplication it is a bilinear product that takes two vectors and produces a scalar and in this case it's through the Minkowski metric so it's an inner product. "Without introducing the proper relativistic notion of the cross product the physical picture of the space time is incomplete". So $M_{1,3}$ as it stands has no type of cross product which would be one vector crossed into another vector yielding a third vector $\boldsymbol{v}\times\boldsymbol{w}=\boldsymbol{y}$. $M_{1,3}$ is full of four vectors and we don't have a cross product introduced on the whole thing and that's a big omission because without having that ability we can't study Maxwell's equations or anything else. Mathematically, they they go on to say, "mathematically, the introduction of a product on a vector space creates an *algebra"*.

Now we we talked about that in the last lesson so that's clear, see that's what I mean I'm going to try to fill in the holes a little bit, I'm going to expand in these sentences "hence, we seek to construct the appropriate algebra for space-time from the vector space $M_{1,3}$ by introducing a suitable vector product", so this is the whole program right here, "we seek to construct the appropriate algebra for space-time" so space-time, they're basically saying look $M_{1,3}$ not good enough, the vector space of all four vectors not good enough, first of all you need to promote this thing to some algebra right and then when you have this algebra it'll understand everything we want to know about space-time or everything we do know about space-time will be embedded inside this algebra so all manipulations all physical things that exist in space-time should be modeled inside the algebraic structure we create and that is what we're after, we're trying to create an algebraic structure that can model anything we know about in space-time that has been amenable to our analysis which is the electromagnetic field and movement and things like that. Now I got it with the caveat, I'm not sure how General relativity will ultimately fit into all of this, I have no doubt there's somebody out there who's taken this work and shown how it applies to General relativity but right now for everything we do here we're not assuming any gravity so if we assume later on massive objects moving around we're not going to consider their effect on the metric, the metric is always going to be flat Minkowski metric everywhere.

The way they say they're going to do this is they're going to take $M_{1,3}$ and introduce "a suitable vector product", now it's really interesting how this is done you'll see it in a moment it's so unbelievably simple it's mindbogglingly simple which I find really really impressive but anyway "we expect this vector product to be non-commutative since the familiar cross product is *non-commutative"* so what they're getting at here is, they're basically running on instinct here they're saying hey the cross product is so important in how we've learned and we understand electromagnetism that we know its properties have got to surface somewhere in this program that we have of creating the appropriate algebra for space-time so if we're going to introduce a vector product, we know that the cross product is not-commutative so this vector product, it's the only new thing we're adding, we're only adding one thing to our vector space $M_{1,3}$ and so we're going to speculate that, whatever it is, it's non-commutative because it's got to be able to capture whatever cross product has given us, it's got to be captured by this so ultimately, since it's the only thing we're adding and the dot product or the inner product is already commutative, well this thing better be non-commutative.

I guess the theory there is that if you add a commutative vector product it's hard to imagine how it would be since everything's linear it's hard to imagine how anything any non-community would ever show up so that's why they're going with non-commutative, we also expect the vector product to enlarge the mathematical space now this is an important point I touched on it last lesson. It's going to enlarge the mathematical space in order to properly accommodate quantities like the electromagnetic field tensor. We're going to introduce a multiplication operation which for example we called star say or we can take a vector and we can star it to another vector now as I said before this has got to yield us yet another vector right that's what an algebra does and algebra has an operator that's this is the appropriate multiplication that we need and yet we're seeking to enlarge the mathematical space so what does that mean? Well we've enlarged it by adding this multiplication but that's not what they mean. What they mean is when we talk about the vector space $\mathsf{M}_{1,3}$, it's got four vectors in it, it contains four vectors so a good four vector might be we'll call it $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$. These are four vectors inside $\mathsf{M}_{1,3}$ and of course we've got the addition operation $+$, and each of these four vectors has this scalar time part t and this three vector space part $(t,\boldsymbol{X})$ that's what everything in here has. To enlarge the space means that not only are we going to have these four vectors in there, we're actually going to add other mathematical objects that we yet don't fully know about and those things are not four vectors but they're going to be part of this space which means you should be able to take a four vector and add it to one of these unknown enlarged things and get something else that is also inside the space.

This vector addition must accommodate the four vectors that are already there and whatever this enlargement that they're teasing is going to be so suddenly space time $\mathsf{M}_{1,3}$ it's no longer $\mathsf{M}_{1,3}$, it's got to be we gotta have to rename it to something else because the set is different $\mathsf{M}_{1,3}$ was the set of all these four vectors and this new thing that we're making is four vectors plus this stuff and it's just simply bigger so that's what they're teasing, we're going to be forced and you'll see why in just a second we're going to be absolutely forced to enlarge the space after we introduce this non-commutative product and that in large space should have room for this guy it should have room for this $F^{\mu\nu}$ electromagnetic field tensor or something equivalent to it, so let's see what are they going to do and well here it is.

“To accomplish these goals we define an appropriate space-time product” so this is what they were calling the Clifford product. The Clifford product is the general product for all Geometric algebra so that would be the Clifford product, when you particularize everything to the study of the particular Clifford geometric algebra that is applied to space-time he's going to call it the space-time product or the authors here will call it the space-time product so instead of calling it the Clifford product I'll try to call it the space-time product so “we define the appropriate space-time product to satisfy the following four properties for any four vectors $a,b,c\in\mathsf{M}_{1,3}$ ”:

$$
\begin{aligned}
a(bc)&=(ab)c && (\text{Associativity})\\
a(b+c)&=ab+ac && (\text{Left Distributivity})\\
(b+c)a&=ba+ca && (\text{Right Distributivity})\\
a^2&=\eta(a,a)=\epsilon_a|a|^2 && (\text{Contraction})
\end{aligned}
\tag{3}
$$

This is why it's called bilinear but this last one that's where all the magic happens right here because what this is saying is that the space-time product of a with itself which is $aa=a^2$ what we're going to call a^2 , that is equal to $\epsilon_a|a|^2$ and notice what this guy, is the magnitude of a which is $\eta(a,a)$ which is a number so we have now the problem that well we have the fact, I guess that this is an assertion we are asserting this we are saying this is a property of the multiplication that we are choosing that we are inventing, we are inventing the space-time product and I want to be associative, I want it to be bilinear but I also wanted to be that when you take the space-time product of a four vector with itself you end up with a number, a real number which violates this tenant that a vector times a vector must equal another vector and there's our first degree of expansion.

We need to take $\mathsf{M}_{1,3}$ and to that we are attaching the real numbers, we're binding them together and this new thing our final space-time algebra has to have at least all of the vectors that could possibly exist in $\mathsf{M}_{1,3}$ and all of the real numbers that could possibly exist in $\mathbb{R}$. I'll say $\mathbb{R}$ I'll give them a little I call these vectors here a,b,c because that's what they did here, he doesn't really have a real number here so I'll just throw down $\epsilon \in \mathbb{R}$, all the possible real numbers so this vector space is now a bigger object and it's a bigger object exclusively because, so far, of this statement here, contraction in (3) has forced us to add real numbers to this vector space $a^2 = \eta(a,a) = \epsilon_a |a|^2$, we're talking about, then after that he goes on to offer a little bit of anticipation, a little bit of insight of what's to come but he's saying that "the contraction property (3) distinguishes the resulting space-time algebra as an "*orthogonal Clifford algebra* that is generated by the metric η and the vector space $\mathsf{M}_{1,3}$ ", so he's basically just saying, hey look there's a lot of prior work on this, he didn't invent a lot of this, I think he spent a lot of time flushing it out and applying it to its full potential.

I don't know how much pure research he did as he developed this theory for us, I should check into the history of it but what he's basically saying is this thing here that's been around for a while and even Clifford was the guy who came up with it and it also states without proof or anything that this Clifford algebra is the largest associative algebra that can be constructed solely from space-time so starting with $\mathsf{M}_{1,3}$ and this provision, this contraction assumption of the multiplication property, starting with those two things the algebra that you're gonna we're ultimately going to build here is the largest one you can possibly make that's also associative. Not something that I'm going to hunt down tempted but not going to. The point is, all other algebras that could ever be relevant must be buried inside this largest algebra and therefore we should see every algebra we've ever seen before tucked away as a sub-algebra of this algebra. Obviously the notion of a sub algebra is just like a sub space, it's a completely closed subset of our space-time algebra, that's closed under obviously all the vector operations but also closed under all of the these multiplication operations as well.

We'll see in this paper they really do go out of their way to show how these algebras nest together and so that's the point where we want to stop for now is we have actually now introduced the key point from which everything else really flows which is this contraction property of the space-time product and we've already shown a little bit about how we can expect that to take $\mathsf{M}_{1,3}$ and blow it up into something bigger that includes all the four vectors of $\mathsf{M}_{1,3}$ and notice how the language works, $\mathsf{M}_{1,3}$ is a vector space so everything in it is a vector, however the things inside $\mathsf{M}_{1,3}$ we've been used to call we usually call four vectors because four vectors are the things that we learn about in Relativity so the vectors of $\mathsf{M}_{1,3}$ are actually called four vectors but that doesn't mean they're special vectors, everything in $\mathsf{M}_{1,3}$ is a vector and in fact we're blowing up $\mathsf{M}_{1,3}$ to be even bigger so the scalars are part of this new algebra so the vector space that we've expanded to includes scalars so now scalars are vectors in the vector space as well as four vectors so you can see how the language just spirals out of control here. Maybe, just before we step away, we're I'm going to lock this down all right hold on all right.

$$\left(ct, x^1, x^2, x^3\right) \ , \ \left(x^0, x^1, x^2, x^3\right) \ , \ \left(ct, \boldsymbol{x}\right) \ , \ \left(E/c, p_x, p_y, p_z\right) \tag{4}$$

The idea is $\mathsf{M}_{1,3}$ contains all of the four vectors we can ever think of, now $\mathsf{M}_{1,3}$ M13 is a vector space so every element of the vector space are called vectors however in Relativity the objects that we use to add and subtract and model physical reality, those are called four vectors so the vectors of the $\mathsf{M}_{1,3}$ vector space are four vectors, that's just an unfortunate overloaded use of the word vector and it gets worse so now what we're learning is that we want to we need to create our algebra here so we introduce this product where you take any two vectors a, b and you multiply them together using what we're calling the space-time product and that result, whatever that result is, it has to equal something that's inside the algebra so if the algebra was limited to $\mathsf{M}_{1,3}$ it would have to be another four vector, it would have to be some four vector c but what we've discovered right away is that aa is actually just a real number I'll call it σ , it's a real number σ which is an element of the real numbers $\mathbb{R}$ so the problem is of course that this is not in $\mathbb{M}_{1,3}$ so if we believe that this is how our vector product works then these real numbers have to suddenly become part of the vector space and now we no longer have $\mathbb{M}_{1,3}$ as we're now dealing with a different vector space we're dealing with a vector space that not only contains four vectors where the four vectors is referent to the vectors of relativity but also contains real numbers so we've now added $\sigma, \pi, \eta, \ldots$ all of these real numbers are now part of this as well so now whatever the vector space is that, we've expanded to in order to create this algebra the underlying vector space that's creating the algebra has four vectors and real numbers.

Remember everything in a vector space is called a vector so these guys are also vectors, they're just vectors that we typically call we typically call these things real numbers but because they're members of the vector space they are vectors in that sense, but what's even more disturbing is if I have a four vector which I call a and a real number which I call σ I have to be able to write $a+\sigma$ which is crazy because I'm writing (x^0, x^1, x^2, x^3) and then I've got this vector addition which normally just allowed me to add four vectors to one another but I've got this vector addition and I'm adding to it this real number σ which has got to be legit I got to be able to do it because a is a vector from the vector space and σ is a vector from the vector space $+$ operates on all vectors on the vector space so I've got to get something else over here that's also a member of the vector space.

That's where I just wanted to emphasize this language, we use the word vector in a lot of ways here in order to get on top of this but so far I don't know if we fully expanded, we know that we've added one thing to the vector space in order to create our algebra but we may have to add other things indeed we definitely do have to add other things to the vector space in order to make the algebra. Good news is we don't have to add an infinite number of things, well we do add it like the real numbers, there's an infinite number of real numbers so we have in some sense already added an infinite number of things but we don't have to add an infinite number of different kinds of things, I guess is the way to say it, we're going to add real numbers and we're going to add a whole bunch of other stuff in the next lesson and in the next lesson we will actually start exploring all the implications of this contraction property we will begin to explore and study in detail and that's when we start getting into the meat of the subject so I'll see you next time.