QED Prerequisites Geometric Algebra 17: bi-vector products

Introduction

Welcome back, we are going to continue our review of this paper: "*Space-time algebra as a powerful tool for electromagnetism*", keeping in mind that we are also studying Geometric algebra in general using this particular algebra as our basis which is a little unusual, typically when you study the subject you study 3D Geometric algebra, 2D Geometric algebra, sometimes you'll study a certain Geometric algebra called Projective 3D dimension Geometric algebra which has a non-euclidean metric but not one equivalent to a Minkowski metric so we're actually starting at a really hard place, this is a little bit advanced, it's 4D and it's got a Minkowski metric, it's got the $\eta_{\alpha\beta}$ metric which is all unusual however, there's no reason why, it's all a bunch of formal rules at some level and gaining this deep insight into the geometry would be nice but if we did that we would really have to focus on this 3D thing, there will be a time to do that by the way when we get to Lorentz transformations and maybe when we get to this stuff on para-vectors or relative vectors then we'll have a chance to start doing that and we'll have to lean on 3D because the 3D Euclidean Geometric algebra is embedded inside the space-time algebra so I'll probably find a way to evoke that and then study just that sub-algebra so that'll be interesting when we get there. Right now we are ready to study Section 3.5.2 "*Bi-vectors: products with bi-vectors*" a subject we've already touched on so let's begin.

Errata

Let me start with an Errata from our last lecture, I think in the last lecture somewhere right around here I wrote down this expression:

$$\langle\psi\tilde{\psi}\rangle=\alpha+\beta I \tag{1}$$

I showed that this reversion was some kind of pseudo-vector and this is notationally incorrect, when you see these angle brackets without any number, you assume the number is zero and you're always talking about the zero-blade and the zero-blade is always the scalar part so it does not include that pseudo-scalar part so when I wrote that down, I think it was around 23 minutes in the last lesson, there was this erroneous pseudo-scalar piece in the notation so whenever you see these angle brackets, do not confuse them with some parenthesis like this $(\psi\tilde{\psi})$, this would actually be a complex scalar and remember when I write $a+b\,i$, this is what I mean by a complex scalar, a complex scalar always has a grade zero part plus a grade four part in the space-time algebra, which is a scalar part and a pseudo-scalar part, all right so much for the errata now let's move on.

Bi-vector products

All right, so let's proceed with Section 3.5.2 "*bi-vectors: products with bi-vectors*". Now to be clear, we've actually done this, we've already talked about how to take a bi-vector space-time product and we have generated this very thing:

$$\boldsymbol{F}\boldsymbol{G}=\langle\boldsymbol{F}\cdot\boldsymbol{G}\rangle_0+[\boldsymbol{F},\boldsymbol{G}]+\langle\boldsymbol{F}\cdot\boldsymbol{G}\rangle_4 \tag{2}$$

Where we understand that the bi-vector bi-vector space-time product is going to have, well it's going to have $|p-q|$ all the way up to $|p+q|$ blades in increments of two so p and q for two bi-vectors is 2 and 2 so clearly we're going to have a zero-blade component $|p+q|=2+2=4$ so we'll have a four-blade component and $0+2=2$ means that whatever is in here this must be the blade two component. now they don't have that written down the way we're used to seeing it here but this would or to be consistent with this, this would read $[\boldsymbol{F},\boldsymbol{G}]=\langle \boldsymbol{F},\boldsymbol{G}\rangle_2$. Now you'll notice what they're doing here is definitely a little bit different than that, they're writing $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_0$, $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_4$ and then they're using this notation for this commutator bracket $[\boldsymbol{F},\boldsymbol{G}]$ that also has this extra ½ in its definition but to be clear, as far as notation goes this could definitely be written as:

$$\boldsymbol{F}\boldsymbol{G}=\langle \boldsymbol{F}\boldsymbol{G}\rangle+\langle \boldsymbol{F}\boldsymbol{G}\rangle_2+\langle \boldsymbol{F}\boldsymbol{G}\rangle_4 \tag{3}$$

As this $\langle \boldsymbol{F}\boldsymbol{G}\rangle$ with no number means the blade zero component, $\langle \boldsymbol{F}\boldsymbol{G}\rangle_2$ with angle brackets with a second subscript is the blade two component and this is $\langle \boldsymbol{F}\boldsymbol{G}\rangle_4$ the blade four component where the $\boldsymbol{F}\boldsymbol{G}$ inside here is the space-time product of the bi-vector $\boldsymbol{F}$ and the bi-vector $\boldsymbol{G}$ and then of course we just equate $[\boldsymbol{F},\boldsymbol{G}]=\langle \boldsymbol{F},\boldsymbol{G}\rangle_2$ this is the blade two part, this thing we're going to define is $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_0=\langle \boldsymbol{F}\boldsymbol{G}\rangle$ and $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_4=\langle \boldsymbol{F}\boldsymbol{G}\rangle_4$ goes out here so there's this new operation that they're talking about which is the dot product of two bi-vectors and you'll notice that their version of this dot product clearly has a blade zero part and a blade four part so if you look at what they're intending they're intending $\boldsymbol{F}\cdot\boldsymbol{G}=\langle\rangle_0+\langle\rangle_4$ to have a blade zero part and a blade four part only and this is definitely different than the way I introduced it earlier where I said the space-time product of $\boldsymbol{F}$ and $\boldsymbol{G}$ was:

$$\boldsymbol{F}\boldsymbol{G}=\boldsymbol{F}:\boldsymbol{G}+\boldsymbol{F}\cdot\boldsymbol{G}+\boldsymbol{F}\wedge\boldsymbol{G} \tag{4}$$

Where $\boldsymbol{F}\wedge\boldsymbol{G}$ is the blade four part, $\boldsymbol{F}\cdot\boldsymbol{G}$ is the blade two part and $\boldsymbol{F}:\boldsymbol{G}$ was the blade zero part so be very careful, the $\boldsymbol{F}\cdot\boldsymbol{G}$ that I spoke about earlier is not the same $\boldsymbol{F}\cdot\boldsymbol{G}$ that they're talking about here and this is just one of the things about the notation that is out there, there's different notations out there, I do want to say that the notation in the paper here, their notation is more in common use than this $\boldsymbol{F}:\boldsymbol{G}$ and this $\boldsymbol{F}\cdot\boldsymbol{G}$ I found in another really good paper that I learned a lot from and I just glommed onto it but it's clearly an outlier as far as notation goes so what do they mean by $\boldsymbol{F}\cdot\boldsymbol{G}$? Well, first of all they define it right here so there's not too much to argue about it, it's clearly the symmetric part of the space-time product of the bi-vector $\boldsymbol{F}$ with the bi-vector $\boldsymbol{G}$:

$$\boldsymbol{F}\cdot\boldsymbol{G}=½(\boldsymbol{F}\boldsymbol{G}+\boldsymbol{G}\boldsymbol{F})=(\mathbf{f}\cdot\mathbf{g})\exp[(\vartheta+\varphi)I] \tag{5}$$

Where $\boldsymbol{F}=\mathbf{f}\exp(\vartheta I)$,, and to have this dot being used for this symmetric part of this bi-vector product because, well what I'm trying to say is, this is an overloaded operator because we know that $\boldsymbol{F}\cdot v$ where v is a vector, we know that that's an anti-symmetric product so the point being that we understand the definition of this dot product to be dependent on what are the two arguments so the arguments here are two bi-vectors when you see two bi-vectors we know we're going to use this definition (5), the symmetric part and any space-time product always has a symmetric part and an anti-symmetric part but it is not always true that the symmetric part and the anti-symmetric part have just one blade, in this case (2), the anti-symmetric part does in fact have one blade, it's the blade two piece so the entire blade two piece $[\boldsymbol{F},\boldsymbol{G}]$ is this anti-symmetric part of the space-time product, however the symmetric part has a blade zero piece $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_0$ and a blade four piece $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_4$ and this was very different than with the vectors, remember with vectors, if I have a vector a dot with a vector b that is entirely the blade zero piece $a\cdot b=\langle\rangle_0$.

You have to stay on your toes about how these things are defined but it's very clear how they're defined $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_0+\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_4$ is the symmetric part of the space-time product for two bi-vectors (5), when $\boldsymbol{F}$ and $\boldsymbol{G}$ are both bi-vectors so we should read "A product between two bi vectors $\boldsymbol{F}$ and $\boldsymbol{G}$ can be split into *three* distinct grades (scalar, bi-vector and pseudo-scalar)", we've covered that, the scalar bi-vector and pseudo-scalar, "in contrast to the two grades in vector vector and vector bi-vector products of (3.2) and (3.9)" (2) and just as a reminder this is the vector vector form:

$$\boldsymbol{a}\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\wedge\boldsymbol{b} \tag{6}$$

That's going to be blade zero plus blade two and this is the vector bi-vectors space-time product:

$$\boldsymbol{a}\boldsymbol{F}=\boldsymbol{a}\cdot\boldsymbol{F}+\boldsymbol{a}\wedge\boldsymbol{F} \tag{7}$$

That's going to be blade one, a vector and this is going to be blade three, a tri-vector. We just reviewed (3.2) and (3.9) so the bi-vector bi-vector product is going to be blade zero $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_0$, blade two $[\boldsymbol{F},\boldsymbol{G}]$ blade four $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_4$, what's nice is this gives you a little formula for calculating the blade two part $[\boldsymbol{F},\boldsymbol{G}]$, it's that commutator and you can calculate the complex scalar part that's another way to say it because $\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_0+\langle \boldsymbol{F}\cdot\boldsymbol{G}\rangle_4$ is a scalar plus a pseudo-scalar which that equals a complex scalar so the complex scalar part is the symmetric part of the space-time product and we know that because if $[\boldsymbol{F},\boldsymbol{G}]$ is the anti-symmetric part everything left must be the symmetric part. "We will find it instructive to consider the scalar and pseudo-scalar parts together as the same complex scalar $\boldsymbol{F}\cdot\boldsymbol{G}$ ". They're going to now say that $\boldsymbol{F}\cdot\boldsymbol{G}$, bi-vector dot bi-vector is the complex scalar part of the space-time product that's what they want this to mean.

Canonical decompositions

"The symmetric part of the product produces this complex scalar, which we can better understand by making these canonical decompositions $\boldsymbol{F}=\mathbf{f}\exp(\vartheta I)$, $\boldsymbol{G}=\mathbf{g}\exp(\varphi I)$ " so we know we're talking about a bi-vector $\boldsymbol{F}$ times a bi-vector $\boldsymbol{G}$ and we know we can always take its canonical decomposition into a canonical form $\mathbf{f}$ and a canonical form $\mathbf{g}$ with a phase here, this exponential phase so they do that and they just write it down as (5), $\boldsymbol{F}\cdot\boldsymbol{G}$ is the symmetric part of the space-time product of $\boldsymbol{F}$ and $\boldsymbol{G}$ which is going to be $\mathbf{f}\cdot\mathbf{g}$ which is the same, remember $\boldsymbol{F}$ and $\boldsymbol{G}$ are bi-vectors so that little dot is the same as this little dot $\boldsymbol{F}\cdot\boldsymbol{G}$ these little dots are the same because $\mathbf{f}$ and $\mathbf{g}$ are bi-vectors. they're canonical bi-vectors but they're bi-vectors and then it's just times this exponential factor. Let's just look at this decomposition (5) very quickly, it's easy and I've written it out right here:

$$\begin{aligned}½(\boldsymbol{F}\boldsymbol{G}+\boldsymbol{G}\boldsymbol{F})&=½[\mathbf{f}\exp(\vartheta I)\mathbf{g}\exp(\varphi I)+\mathbf{g}\exp(\varphi I)\mathbf{f}\exp(\vartheta I)]\\&=½[\mathbf{f}\mathbf{g}+\mathbf{g}\mathbf{f}]\exp(\vartheta I)\exp(\varphi I)\\&=(\mathbf{f}\cdot\mathbf{g})\exp([\vartheta+\varphi]I)\end{aligned} \tag{8}$$

This is the symmetric part of the space-time product of the bi-vector $\boldsymbol{F}$ times bi-vector G so we simply make $\boldsymbol{F}$, we replace it with its decomposition and we take $\boldsymbol{G}$ and we replace it with its decomposition and we do that everywhere and it's pretty obvious what happens, this scalar decomposition commutes with everything because $\mathbf{f}$ and $\mathbf{g}$, these are bi-vectors, $\mathbf{f}$ and $\mathbf{g}$, being both bi-vectors they commute with the pseudo-scalar so we can pull these exponentials right out and we end up with $[\mathbf{f}\,\mathbf{g}+\mathbf{g}\,\mathbf{f}]$, that's what left behind after you pull out these exponentials multiplied by $\exp(\vartheta I)\exp(\varphi I)$ and these exponentials combine just like they would in regular complex analysis and so you get that combination and then this one half plus this symmetric sum that's the definition of $\mathbf{f}\cdot\mathbf{g}$ so that explains exactly how this formula (5) surfaces. Now this next line is worth a little bit of thinking.

Relative 3 vectors

"Since $\mathbf{f}$ and $\mathbf{g}$ have fixed signature, $\mathbf{f}\cdot\mathbf{g}=½(\mathbf{f}\,\mathbf{g}+\mathbf{g}\,\mathbf{f})$ " which is the symmetric part of the space-time product of $\mathbf{f}$ and $\mathbf{g}$, "is a real scalar". We need to actually probe this a little bit, "In fact, it will become clear after we introduce relative three vectors", they're presaging this notion of relative three vectors which is really important, "that this real scalar is completely equivalent to the usual (Euclidean) dot product from non-relativistic three vector analysis, which motivates this choice of notation". Ultimately when we introduce this notion of relative three vectors in the space-time algebra it turns out that it's a very important observation is that these relative three vectors form a sub-algebra inside the space-time algebra and that sub-algebra is regular 3D Geometric algebra with a Euclidean metric so when I say regular 3D I mean three-dimensional Geometric algebra with a Euclidean metric and that lives inside the space-time algebra and seeing where that lives is really important but before we go there we have to we have to understand a little bit about what this $\mathbf{f}\cdot\mathbf{g}$ is all about, $\mathbf{f}\cdot\mathbf{g}$ is a real scalar, they're fixed signature so let's probe this a little bit because we need to understand the nature of these $\mathbf{f}$ and $\mathbf{g}$, we've studied and we know that any bi-vector $\boldsymbol{F}$ can be put into this form $\boldsymbol{F}=\mathbf{f}\exp(\vartheta I)$ but we don't know much about what makes $\mathbf{f}$ and $\mathbf{g}$ interesting in that regard:

$$\mathbf{f}\,\widetilde{\mathbf{f}}=\rho\,R\,I\,\widetilde{I}\,\widetilde{R}=\rho\,R(-1)\widetilde{R}=-\rho\,R\,\widetilde{R}=-\rho \tag{9}$$

We're going to probe that right now to get a good sense of what this first sentence of this paragraph means, now you may remember from our last lesson when we created this little canonical bi-vector $\mathbf{f}$, we calculated $\mathbf{f}\,\widetilde{\mathbf{f}}$ (9) and in full generality it worked out that $\mathbf{f}$ is always going to have a negative signature, ρ is always going to be positive and you're going to get this negative in front of it and so the signature of $\mathbf{f}$ which is defined as:

$$\varepsilon_{\mathrm{f}}\equiv\frac{\mathbf{f}\,\widetilde{\mathbf{f}}}{|\mathbf{f}|^2}=\frac{-\rho}{|\langle\mathbf{f}\,\widetilde{\mathbf{f}}\rangle|}=\frac{-\rho}{\rho}=-1 \tag{10}$$

This is always going to be -1 so aside from the fact that the signature is -1 , what does that tell us about $\mathbf{f}$ other than this definition of its signature (10)? Let's have a look at that and the way we're going to look at it is let's look at the signatures of the various bi-vectors that are out there.

$$\begin{aligned}\gamma_{10}\widetilde{\gamma}_{10}&=\gamma_{10}\gamma_{01}=\gamma_0{}^2\gamma_1{}^2=-1\\ \gamma_{20}\widetilde{\gamma}_{20}&=\gamma_{20}\gamma_{02}=\gamma_0{}^2\gamma_2{}^2=-1\\ \gamma_{30}\widetilde{\gamma}_{30}&=\gamma_{30}\gamma_{03}=\gamma_0{}^2\gamma_3{}^2=-1\end{aligned} \tag{11}$$

Our basis bi-vectors, if we look at the basis bi-vector γ_{01} and we say what's its reversion square, the reversion square that's the numerator part of the signature and that's a good point. Remember the definition of the signature is this thing:

$$\varepsilon_{\boldsymbol{F}}=\frac{\boldsymbol{F}\widetilde{\boldsymbol{F}}}{|\langle\boldsymbol{F}\widetilde{\boldsymbol{F}}\rangle|} \tag{12}$$

It's a bi-vector times its reversion divided by the 0-blade part of its reversion square, now in this case we're dealing with this, the absolute value of that so this our absolute value of this will always be 1 when we're dealing with basis vectors because you'll see, the reversion square of the basis vector γ_{10} is going to be $\gamma_{10}\gamma_{01}$, the γ_0 come together to form $\gamma_0{}^2$ and $\gamma_1{}^2$ and it's the fact that this $\gamma_0{}^2$ is equal to 1 and all of the $\gamma_1{}^2$, $\gamma_2{}^2$ and $\gamma_3{}^2$ are -1 that means that γ_{10} will always have this negative signature and the same is true for γ_{20} and γ_{30} and these negative signatures while the absolute value is positive so this denominator here is always going to be one so we're just going to focus on just the numerator for these basis vectors. The signature of these three guys is -1 and what's special about these three basis factors? Well they're all the basis vectors that are bi-vectors of $\gamma_i\wedge\gamma_0$ and γ_0 is distinct because it has the signature of 1 , not -1 or $\gamma_0{}^2$ is 1 , well $\gamma_i{}^2$ is -1 because we're presuming $i=1,2,3$ because if $i=0$ then $\gamma_i\wedge\gamma_0=0$ so we don't have to consider $i=0$ because $\gamma_0\wedge\gamma_0=0$. These three basis vectors here are the time-like basis vectors for the bi-vector sub-space so these are the time-like bi-vector bases. Now likewise, we can do this for the three space-like ones where zero does not appear anywhere in the subscript so we're not dealing with any zeros but all of those, if you just do the same reversion process, all of those have signatures that are equal to $+1$.

$$\begin{aligned}\gamma_{12}\widetilde{\gamma}_{12}&=\gamma_{12}\gamma_{21}=\gamma_1{}^2\gamma_2{}^2=+1\\ \gamma_{23}\widetilde{\gamma}_{23}&=\gamma_{23}\gamma_{32}=\gamma_2{}^2\gamma_3{}^2=+1\\ \gamma_{31}\widetilde{\gamma}_{31}&=\gamma_{31}\gamma_{13}=\gamma_3{}^2\gamma_1{}^2=+1\end{aligned} \tag{13}$$

This is a key distinction that we have to remember, the point of this narrative is that the bi-vectors, this time-like bi-vectors have signatures of -1 , the space-like bi-vectors have signatures of $+1$.

Time-like bi-vectors

Now we imagine our canonical little vector, imagine just a bi-vector and we say $\mathbf{f}$ which has negative signature, well if it has negative signature imagine a bi-vector that is just got time-like components, imagine all of the space-like components are equal to zero so this is a general space-like bi-vector:

$$\mathbf{f}=a\gamma_{10}+b\gamma_{20}+c\gamma_{30} \tag{14}$$

Well, the reversion square of a general time-like bi-vector basis can be calculated, I just did it explicitly, I went out and did the whole long thing in full generality:

$$\begin{aligned}\mathbf{f}\widetilde{\mathbf{f}}&=(a\gamma_{10}+b\gamma_{20}+c\gamma_{30})(a\gamma_{01}+b\gamma_{02}+c\gamma_{03})\\ &=a^2\gamma_{1001}+ab\gamma_{1002}+ac\gamma_{1003}\\ &+ba\gamma_{2001}+b^2\gamma_{2002}+bc\gamma_{2003}\end{aligned} \tag{15}$$

$$
\begin{aligned}
&+ca\,\gamma_{3001}+cb\,\gamma_{3002}+c^2\,\gamma_{3003}\\
&=-a^2-b^2-c^2
\end{aligned}
\tag{16}
$$

I'm squaring it so I have a , b and c but I've reversed because we're using $\mathbf{f}$ reversion so these are the reversions of the time-like basis bi-vectors and if you do this all out you end up with these terms right here (15) and (16) and you'll notice that the diagonal terms, the γ_{00} part squares to … remember we write γ_{1001} , you've got to think $\gamma_1\gamma_0\gamma_0\gamma_1$ where everything is space-time multiplied and you don't have to worry about when your space-time multiplying these things associatively, you can associate them and you don't have to worry about the bi-vector parts of these vector vector space-time products because they are, well you either worry about the bi-vector part or only the dot product part because they're orthogonal that's the point but anyway this is becomes $\gamma_0{}^2$ and that's a real number and then it's $\gamma_1{}^2$ which is $\gamma_{1001}=1\cdot(-1)=-1$. Likewise this guy $\gamma_{2002}=-1$ and this guy $\gamma_{3003}=-1$. In each of those cases but what about the remaining cases? Well if you look at this one γ_{1003} , what you're expecting is $\gamma_1\gamma_0\gamma_0\gamma_3$. Well that ends up being the middle two space-time multiply to -1 so you get minus $\gamma_1\gamma_3$ which is $-\gamma_{13}$ which is a bi-vector so this guy here gives you a bi-vector, that what you need to understand is that will exactly cancel with this bi-vector down here γ_{3001} which we'll end up with as γ_{13} so those two actually will cancel and all of these will cancel all of the ones that would give you bi-vectors cancel completely leaving only the diagonal terms and those diagonal terms as I said all carry a $-$ through and you end up with a real number and it's also a negative real number. If you start with a time-like bi-vector a purely time-like bi-vector, its signature will be -1 because you'll divide by the absolute value of this and you'll end up with -1 so the point is that when we say something has a purely negative signature, it's a bi-vector with a negative signature right away that it's a time-like bi-vector meaning all of its components that include space-like bi-vectors are zero and the only non-zero components are on $\gamma_{10},\gamma_{20},\gamma_{30}$.

Space-like bi-vectors

Now we can also do the same thing for the space-like bi-vectors so we know that the space like bi-vectors are the duals of the time-like bi-vector so we're going to start using that dual so if $\mathbf{f}$ is just as I said before, if $\mathbf{f}$ is defined exactly as it always was this way right here (14), the way we just defined it, if I just take its dual by right multiplying by I , I take the dual of each of these basis vectors:

$$
\begin{aligned}
\mathbf{f}I&=a\,\gamma_{10}I+b\,\gamma_{20}I+c\,\gamma_{30}I\\
&=a\,\gamma_{32}+b\,\gamma_{13}+c\,\gamma_{21}
\end{aligned}
\tag{17}
$$

which are the three space-like bi-vectors and I can calculate its reversion square because $\mathbf{f}I$ times its reversion squared, it's a recursive reversion but what's really nice is the pseudo-scalar basis vector is its own reversion so that becomes:

$$
\mathbf{f}I\tilde{I}\tilde{\mathbf{f}}=-\mathbf{f}\tilde{\mathbf{f}}=a^2+b^2+c^2 \tag{18}
$$

Recall: $I\tilde{I}=-1$, that's where this $-$ comes from and then you get $\mathbf{f}\tilde{\mathbf{f}}$ which we already know and that $-$ makes it all positive. It's space-like, this space like bi-vectors, the purely space-like bi-vectors have real number signatures that are positive. Now it is possible to get, of course, a bi-vector that has a signature of zero and it is also possible to have bi-vectors with a signature that is complex, we demonstrated that in a previous lecture but you have to mix time-like with space-like components to get that effect for example $\gamma_{10}+\gamma_{13}$, if you calculate this you will discover that it has a signature of zero, it actually has a signature of zero and if you have $\gamma_{10}+\gamma_{23}$, the distinction in this second case being that γ_{23} is completely orthogonal to γ_{10} , this will have a signature that's complex. Now having said that, there is definitely some ambiguity, although $\gamma_{10}+\gamma_{13}$ has a signature of zero what about this guy, where you mix non-orthogonal time-like and space-like parts and you have some coefficient where it doesn't balance out nice and easily like $\gamma_{10}+\gamma_{13}$ does and so what if we take that multiply it by its reversion. We're going to calculate its reversion square:

$$(\gamma_{10}+2\gamma_{13})(\gamma_{01}+2\gamma_{31})=\gamma_{10}\gamma_{01}+2\gamma_{10}\gamma_{31}+2\gamma_{13}\gamma_{01}+4\gamma_{13}\gamma_{31} \tag{19}$$

The first term $\gamma_{10}\gamma_{01}$ ends up being -1 , a real number. γ_0 squares to $+1$, γ_1 squares to -1 leaving over a -1 , the last term $4\gamma_{13}\gamma_{31}$ but γ_3 is in the middle, square to -1 , the γ_1 outside square to -1 so you end up with a $+1$ so you end up with $+4$ and -1 . These two in the middle actually cancel, by the time you reconcile the $\gamma_{10}\gamma_{31}$ with the $\gamma_{13}\gamma_{01}$ you realize that this is the opposite of this and so those cancel so there's no bi-vector piece at all, it's still a real number but in this case it's going to end up being $-1+4$ so it's going to have a positive signature, it's going to trend more towards the spatial part, it's got a larger signature so here's one where you have a mixture of components space-like and time-like and the signature will favor, through this reversion calculation being positive or negative and ultimately you would say that this is a space-like bi-vector because its signature is actually positive. We're not done talking about this, unfortunately, I want to go into the next subject regarding this.

Real scalars

I want to flush out this statement "Since $\mathbf{f}$ and $\mathbf{g}$ have a fixed signature $\mathbf{f}\cdot\mathbf{g}=½(\mathbf{f}\,\mathbf{g}+\mathbf{g}\,\mathbf{f})$ " or the symmetric part of the space-time product of $\mathbf{f}$ and $\mathbf{g}$, "is a real scalar". We're just going to demonstrate this, we're just going to go through the math and you're going to see that $\mathbf{f}\cdot\mathbf{g}$ as defined, the way we define $\mathbf{f}$ and $\mathbf{g}$ has to be a real scalar. As a quick prelude to the next few sections of this lecture, in case you may want to skip it, I'm going to explore some various products of two bi-vectors $\mathbf{f}$ and $\mathbf{g}$ and the first one I'm going to explore is when $\mathbf{f}$ and $\mathbf{g}$ are both entirely time-like in the sense that all of their basis vectors are these purely time-like basis vectors that is, there's no γ_{ij} where $i,j=1,2,3$, none of those. What we'll see is that the signature of this product is also going to be entirely time-like and we'll see how that happens, then I'm going to switch it to where $\mathbf{f}$ is still time-like but you'll see that $\mathbf{f}$ is time-like despite the fact that it has a space-like component its signature is still negative, likewise here $\mathbf{g}$ its signature is still negative then we'll take the product of $\mathbf{f}$ and $\mathbf{g}$ and what we'll discover is that this product is zero because $\mathbf{g}$ is actually entirely orthogonal to $\mathbf{f}$ in some sense, they don't share any basis vectors and as such $\mathbf{g}$ and $\mathbf{f}$, well their product won't be zero but there's the symmetric part of their product will turn out to be zero.

$$\mathbf{f}\,\mathbf{g}=(a\gamma_{10}+b\gamma_{20}+c\gamma_{30})(d\gamma_{10}+e\gamma_{20}+h\gamma_{30}) \tag{20}$$

Then I start changing $\mathbf{g}$ a little bit, I give $\mathbf{g}$ the same basis vector that lives in $\mathbf{f}$ so we'll see what that product is like and what you'll discover in that case is that $\mathbf{g}$ actually as I've written it here isn't really space-like, it has a complex signature so we skip that and then the last one is the case where you end up with two truly … I'm sorry $\mathbf{g}$ isn't time-like, it's actually complex and in this case this is where you have $\mathbf{f}$ and $\mathbf{g}$ they're both time-like and they share a common basis vector. We're going to go through those three examples and we're just going to work them straight out and catch what the symmetric product is in all those cases except this case (2nd) where $\mathbf{g}$ is actually not even time-like, I just want to illustrate how you can get a non-time-like, what happens to make this not time-like, even though the time component is substantially larger than the space component. If you're interested in that, follow on, if not then you can just move on to the next lecture.

$$\begin{aligned}\mathbf{f}&=(2\gamma_{10}+\gamma_{13}), \quad \mathbf{g}=(3\gamma_{20}+\gamma_{21})\\ \mathbf{f}&=(2\gamma_{10}+\gamma_{13}), \quad \mathbf{g}=(3\gamma_{20}+\gamma_{13})\\ \mathbf{f}&=(2\gamma_{10}+\gamma_{13}), \quad \mathbf{g}=(3\gamma_{30}+\gamma_{13})\end{aligned} \tag{21}$$

Let's just explore this a little bit, I don't think I'm going to come up with a formal proof here but I do want to get comfortable with understanding a little bit how these signatures work a little even better than we just did. If we consider two general, totally time-like bi-vectors and $\mathbf{g}$ and I write $\mathbf{f}$ and $\mathbf{g}$ as (20), these are all time-like basis vectors, if I execute this product which is the most general product between two bi-vectors that only have the time-like bases vectors so we're leaving out γ_{23} , γ_{31} and γ_{12} and if we did execute that product you're going to get this expression:

$$\begin{aligned}\mathbf{f}\,\mathbf{g}=&ad\,\gamma_{1010}+ae\,\gamma_{1020}+ah\,\gamma_{1030}\\ &+bd\,\gamma_{2010}+be\,\gamma_{2020}+bh\,\gamma_{2030}\\ &+cd\,\gamma_{3010}+ce\,\gamma_{3020}+ch\,\gamma_{3030}\end{aligned} \tag{22}$$

It's very straightforward, let me make multiplication and these diagonal row elements, you can see are all going to all of these diagonal elements, these guys are going to become, let's see they'll become -1 because it'll be one flip it'll be a -1 , I don't know, it would be positive so each of these are going to become $+1$ but these don't cancel, $\mathbf{f}\,\mathbf{g}$ will have all of these components in it and these guys will end up being bi-vectors because you'll flip for example $bd\,\gamma_{2010}$, in this case you'll flip the 1 and the 0 , you'll pick up a $-$ but the γ_{00} part will go away to $+1$ and you'll end up with γ_{21} . Likewise here $bd\,\gamma_{2010}$, you'll end up with γ_{31} but then if you reverse this product instead of doing $\mathbf{f}\,\mathbf{g}$ you do $\mathbf{g}\,\mathbf{f}$ you end up with similar matrix:

$$\begin{aligned}\mathbf{g}\,\mathbf{f}=&ad\,\gamma_{1010}+ae\,\gamma_{2010}+ah\,\gamma_{3010}\\ &+bd\,\gamma_{1020}+be\,\gamma_{2020}+bh\,\gamma_{3020}\\ &+cd\,\gamma_{1030}+ce\,\gamma_{2030}+ch\,\gamma_{3030}\end{aligned} \tag{23}$$

In each of the cases the off-diagonal terms where I'm calling these the diagonal terms here, you'll pick up the opposite sign, the ah term which is related to the first and the last, for example, is γ_{1030} in (22) and it's γ_{3010} in (23) so if you try to reconcile this and you try to bring you just do the permutation you try to bring $1,0,3,0$ and your goal is to make it $3,0,1,0$ and you're asking, do I pick up a $-$ sign so that if I were to add $\mathbf{f}\,\mathbf{g}$ to $\mathbf{g}\,\mathbf{f}$, if I were to form the symmetric sum, would these off-diagonal terms cancel? It's a short exercise, you realize oh I got one flip to so I get $-\gamma_{1003}$ and that becomes $-\gamma_{13}$

and the other becomes $-\gamma_{3001}$ which is now $-\gamma_{31}$, well $-\gamma_{31}$ equals γ_{13} so if you were to add $\mathbf{f}$ to $\mathbf{g}$ you know that this term $ah\gamma_{1030}$ would cancel $ah\gamma_{3010}$ and if you check it works for all of the off-diagonal terms so the only things that survive when you do this operation is just these diagonal elements ad , be and ch . They don't cancel, they actually add up. What you know is that if you multiply two time-like vectors together you are going to get a real number because these diagonal elements here, this sum is going to be a real number and that real number is:

$$½(\mathbf{f}\,\mathbf{g}+\mathbf{g}\,\mathbf{f})=ad+be+ch \tag{24}$$

In some sense that was the explanation that I'm after here, is that if you have two of these purely time-like, when I say purely time-like I mean it only has time-like basis vectors, you're clearly gonna get from the symmetric sum, you're just going to get a real number. Let's go ahead and explore this just a little more just just to get some exercises done here. Say $\mathbf{f}$ is not a purely time-like bi-vector, it is a time-like bi-vector, let's take those two:

$$\mathbf{f}=(2\gamma_{10}+\gamma_{13}) \quad , \quad \mathbf{g}=(3\gamma_{20}+\gamma_{21}) \tag{25}$$

First of all, is $\mathbf{f}$ time-like even though it has this space-like basis vector? Well what's $\mathbf{f}\tilde{\mathbf{f}}$ reversion? We calculate $\mathbf{f}\tilde{\mathbf{f}}$ and when I did so I got this expression:

$$\begin{aligned}\mathbf{f}\tilde{\mathbf{f}}&=(2\gamma_{10}+\gamma_{13})(2\gamma_{01}+\gamma_{31})\\&=-4+2\gamma_{1031}+2\gamma_{1301}+\gamma_{1331}\\&=-4+1=-3\end{aligned} \tag{26}$$

Now you can tell right away because all the indices match in pairs, that's going to be either $+1$ or -1 , it turns out it is $+1$. This first term is γ_{1001} so likewise it's going to be a real number and it is in fact -1 so the twos come together so you get negative -4 so overall you get $-4+1=-3$. Since $-3<0$ the signature of $\mathbf{f}$ is going to be -1 because remember the signature of $\mathbf{f}$ is (26) divided by the magnitude of this so it's going to be $-3/3=-1$ you get a signature of -1 for $\mathbf{f}$. For $\mathbf{g}$ you do the same calculation the way we've set up $\mathbf{g}$ and you get a result of -8 , you get a signature of -1 .

$$\begin{aligned}\mathbf{g}\tilde{\mathbf{g}}&=(3\gamma_{20}+\gamma_{21})(3\gamma_{02}+\gamma_{12})\\&=-9+3\gamma_{2012}+3\gamma_{2102}+\gamma_{2112}\\&=-9+1=-8\end{aligned} \tag{27}$$

Let's do this $\mathbf{f}\,\mathbf{g}$, $\mathbf{g}\,\mathbf{f}$ calculation here and what are we going to get is:

$$\begin{aligned}\mathbf{f}\,\mathbf{g}&=(2\gamma_{10}+\gamma_{13})(2\gamma_{20}+\gamma_{21})\\&=4\gamma_{1020}+2\gamma_{1021}+2\gamma_{1320}+\gamma_{1321}\end{aligned} \tag{28}$$

I get this expression which you can break down the usual way, I see there are two zeros in the 1st term so I know that this is $\gamma_1\gamma_0\gamma_2\gamma_0=-\gamma_1\gamma_0\gamma_0\gamma_2$ because they're orthogonal, they anti-commute straight up like that, that's the beauty of using basis vectors then I use the associative rule and I know that $(\gamma_0)^2=1$ so I know it is $-\gamma_1\gamma_2=-\gamma_{12}$ so I do this operation for each of these basis vectors this is just a review, I know at this point, everything I've done here should be pretty familiar in second nature but it never hurts to review so you do this for each of these basis vectors and you see that the 4th term will collapse because you have a pair of ones, the 3rd term will not collapse because they're all different so when that happens this is going to be some form of the pseudo-scalar $\pm I$ so that's important to note and then the 2nd term will collapse into a bi-vector so this will be a bi-vector so you'll end up with three bi-vectors and a pseudo-scalar so $\mathbf{f}\,\mathbf{g}$ it turns out I've got two arbitrary $\mathbf{f}$ and $\mathbf{g}$, well I end up with two bi-vectors and a pseudo-scalar. Notice I don't end up with a scalar, where did the scalar go? Well there's no scalar because these two would have to share a basis vector for a scalar to appear here because you would need something that would look like $abab$ so a and b would fully contract into a scalar. You have the opposite here where none of them contracts, you end up with a pseudo-scalar so that's why there's no scalar there but if my calculation is correct on this I ended up with:

$$\mathbf{f}\,\mathbf{g}=-4\gamma_{12}-2\gamma_{02}-2I-\gamma_{32} \tag{29}$$

If I calculate $\mathbf{g}\,\mathbf{f}$, well it turns out if I calculate $\mathbf{g}\,\mathbf{f}$ I get the same thing with an opposite sign:

$$\mathbf{g}\,\mathbf{f}=4\gamma_{2010}+2\gamma_{2110}+2\gamma_{2013}+\gamma_{2113} \tag{30}$$

For example, let's look at this 1st term, $4\gamma_{2010}$, well if I flip the one and the zero I pick up a $-$, the two zeros at the end square to $+1$ so I have $-4\gamma_{21}$. Well what did I have up here (29)? I had $-4\gamma_{12}$. Well, those are opposites because if I flip the two and the one I lose the $-$ and I pick up a $+$ if I flip those two and then when I add these two together those two will cancel so this bi-vector will cancel. Likewise, if I compare the 2nd terms, what will I end up with? I want to get these two ones together so you pick up a $-$ on the first flip, you lose the $-$ in the second flip but you end up with $\gamma_1{}^2=-1$ so you pick up a $-$ there because of the signature of the metric so you end up with $-\gamma_{02}$ and what happens down here, well they're already next to each other so you immediately pick up a -1 so you end up here with a $-\gamma_{20}$ and for the same reason when you flip these two to make them match you lose the $-$ and when you add everything together those two cancel. Well, it's interesting so why is all this cancellation happening? It will happen again for the 3rd and 4th terms, those will cancel. What's important about these two canceling is the 3rd terms are $2I$ and $-2I$, those will cancel so everything's going to cancel. In fact, let me put these together, when you work this out you end up with this:

$$\mathbf{g}\,\mathbf{f}=-4\gamma_{21}-2\gamma_{20}+2I-\gamma_{23} \tag{31}$$

When you calculate:

$$½(\mathbf{f}\,\mathbf{g}+\mathbf{g}\,\mathbf{f})=0 \tag{32}$$

Orthogonality

You end up with zero because these two cancel so what happened? How is it that we happen to pick $\mathbf{f}$ and $\mathbf{g}$ that the symmetric sum exactly canceled? If you look at it, this exercise, you see, well $\mathbf{f}$ and $\mathbf{g}$ are actually orthogonal as bi-vectors, they don't share any basis vectors so as bi-vectors go, they are orthogonal which shows that there's a different notion of orthogonality here γ_{10} and γ_{20} are orthogonal bi-vectors because well they're not the same but they do share this time like piece, so there is this notion of completely detached and separate and orthogonal that we still need to clear up a little bit but for example γ_{10} and γ_{23} don't share any lines, these two planes never intersect anywhere in four-dimensional space, notice they're not parallel in the context of three-dimensional space where you can have two planes that never intersect but they're parallel but this is a different thing, they're parallel in the sense that they never intersect but they're very distinct in that they're they're separate they're completely separate bi-vectors this is why any bi-vector in four dimensions can always be broken down into the product of two bi-vectors that are separate at most, you can always get it down to at least two if not one and that's a different issue but the point is $\mathbf{f}$ and $\mathbf{g}$ are completely orthogonal and that they share no basis vectors so there turns out that there's symmetric sum is what captures that and it goes to zero so because of this we can say, if we define $\mathbf{f}\cdot\mathbf{g}$ as that symmetric sum, it is zero and they share no bi-vectors in their bases so this makes sense.

Let's try this again and change $\mathbf{g}$ to have γ_{13} , now they do share a basis vector so let's see what happens now well first of all the signature for $\mathbf{f}$ doesn't change but let's calculate the signature of $\mathbf{g}$:

$$\begin{aligned}\mathbf{g}\tilde{\mathbf{g}}&=(3\gamma_{20}+\gamma_{13})(3\gamma_{02}+\gamma_{31})\\&=-9+3\gamma_{2031}+3\gamma_{13}\gamma_{02}+\gamma_{13}\gamma_{31}\\&=-9+1-6I=-8-6I\end{aligned}\tag{33}$$

You end up with $-8-6I$ so this guy has a complex signature. I would have to divide this by 8 to get the signature. This one we don't want to deal with because we know that when we create a canonical basis vector, the canonical basis vector always has a negative signature so why did this guy come up with a complex signature and the answer is, it came up with a complex signature because it's two basis vectors are completely detached from one another so when you do this reversion product you end up with these pseudo-scalars that actually add up, they don't cancel out so this is not eligible to be the lead factor of the canonical form of an arbitrary bi-vector. I'm not going to study $\mathbf{f}\,\mathbf{g}-\mathbf{g}\,\mathbf{f}$ for this particular case. Then we can do one more case, right now I'm going to change this to $\mathbf{g}=3\gamma_{30}+\gamma_{13}$:

$$\begin{aligned}\mathbf{g}\tilde{\mathbf{g}}&=(3\gamma_{30}+\gamma_{13})(3\gamma_{03}+\gamma_{31})\\&=-9+3\gamma_{3031}+3\gamma_{1303}+\gamma_{13}\gamma_{31}\\&=-9+3\gamma_{01}+3\gamma_{10}+1=-8\end{aligned}\tag{34}$$

Now in this circumstance $\mathbf{f}=(2\gamma_{10}+\gamma_{13})$ and $\mathbf{g}=3\gamma_{30}+\gamma_{13}$ do in fact share at least one basis vector and these two basis vectors in $\mathbf{g}$ and the two basis vectors in $\mathbf{f}$ are not completely detached from one another meaning they both have an index of 1 here and they both have an index of 3 here. There must be a word for this, there must be a word for γ_{ab} and γ_{cd} where neither c nor d equals a or b . I presume the word is orthogonal, I think that's the right word but all these basis factors are orthogonal in some sense so I'm going to be on the lookout for the right word for that kind of distinction between these bi-vectors. If we calculate the signature of this $\mathbf{g}$ (34) once again, it comes up to be -8 so it's negative, well the signature is -1 but the fact that it's negative is important because now I have a time like bi-vector $\mathbf{g}$ and we know $\mathbf{f}$ is time like because we've done $\mathbf{f}$, $\mathbf{f}$ is not changing so now if I calculate $\mathbf{f}\,\mathbf{g}$ I end up with this bi-vector, it has a bi-vector part and a real part notice $\mathbf{f}\,\mathbf{g}$ won't have a pseudo-scalar part because there's no components over in $\mathbf{g}$ that are entirely detached from the components in $\mathbf{f}$, I don't have a γ_{23} for example over here.

$$\begin{aligned}\mathbf{f}\,\mathbf{g} &= (2\gamma_{10}+\gamma_{13})(2\gamma_{30}+\gamma_{13}) \\ &= 4\gamma_{1030}+2\gamma_{1013}+2\gamma_{1330}+\gamma_{1313} \\ &= -4\gamma_{13}+2\gamma_{03}-2\gamma_{10}-1\end{aligned} \tag{35}$$

$$\begin{aligned}\mathbf{g}\,\mathbf{f} &= 4\gamma_{3010}+2\gamma_{3013}+2\gamma_{1310}+\gamma_{1313} \\ &= -4\gamma_{31}-2\gamma_{01}+2\gamma_{30}-1\end{aligned} \tag{36}$$

When you calculate $\mathbf{g}\,\mathbf{f}$ what you're going to see is that all of the bi-vector parts will ultimately cancel when you add them together so you end up with just the scalar part and you'll see it's negative. This is showing you how these signatures work, I have these two time-like bi-vectors and their symmetric sum is also a time like, well the symmetric sum is actually just a real number, it's just a negative number.

$$\mathbf{f}\,\mathbf{g}+\mathbf{g}\,\mathbf{f}=-2 \tag{37}$$

Outro

That is what they're saying in the paper, in the paper they want to say: “Since $\mathbf{f}$ and $\mathbf{g}$ have fixed signature, $\mathbf{f}\cdot\mathbf{g}=½(\mathbf{f}\,\mathbf{g}+\mathbf{g}\,\mathbf{f})$ is a real scalar”. They want to say $\mathbf{f}\cdot\mathbf{g}$ which is this part is a real scale so we've worked through that and demonstrated a few cases just started to exercise this process so let's move on. Well I think that's actually enough for today we spent a lot of time really playing around with different bi-vector products, I mean we did a lot of bi-vector products in different forms and I think we're you can't spend enough time doing it good good some good exercises so next time we'll pick up right here and we'll move through Section 3.5.3 Complex conjugations. I'll see you next time.