Maxwell's Equations via Differential Forms Part 2

Introduction

We are going to continue our review we've hashed through the basic notation and the notion of the Exterior derivative in a very mechanical way because again this is a review so we have to just get the notation under our belts and hopefully all of this stuff the meaning and interpretation is percolating back up into your mind we've obviously done the basic review of how this all pertains to a manifold with tensor fields of various kinds in particular form fields one form fields vector fields tangent spaces all that stuff and now we need to review this critical notion of Hodge duality so with that, let's begin.

Basis vectors

I drew up this chart to somewhat show the different basis vectors for that we can create using the wedge product and the underlying vector space V here I've chosen V to have four dimensions so the basis vectors of V are these four differential operators and the basis dimension of the dual space of V are these four one forms so V itself is constructed from these four basis vectors and the dual space of V is constructed from these four basis vectors and we give this a name we call this the first exterior power of the dual space $\Lambda^1(V^*)$ and we call this the first exterior power of the vector space $\Lambda^1(V)$ and that those are actually equal to the vector space itself, this is equal to the vector space itself and this is equal to the dual space itself. All linear combinations of these one forms create the one form space which is the first exterior power of the dual space. Then we can do the same thing with the second exterior power so this is the basis set of all two forms and all two forms can be constructed from these six basis vectors and that is given the name the second exterior power of the dual space. This notation comes from Wikipedia. I'm using the Wikipedia notation here, I've seen other notations that play with the superscript or subscript of the degree of the form but we're going to use the Wikipedia notation.

$$\Lambda^1(V) \to \partial_0, \partial_1, \partial_2, \partial_3 \tag{1}$$

$$\Lambda^1(V^*) \to dx^0, dx^1, dx^2, dx^3 \tag{2}$$

$$\Lambda^2(V) \to \partial_0 \wedge \partial_1, \partial_0 \wedge \partial_2, \partial_0 \wedge \partial_3, \partial_1 \wedge \partial_2, \partial_1 \wedge \partial_3, \partial_2 \wedge \partial_3 \tag{3}$$

$$\Lambda^2(V^*) \to dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3, dx^1 \wedge dx^2, dx^2 \wedge dx^3, dx^2 \wedge dx^3 \tag{4}$$

$$\Lambda^3(V) \to \partial_0 \wedge \partial_1 \wedge \partial_2, \partial_0 \wedge \partial_1 \wedge \partial_3, \partial_0 \wedge \partial_2 \wedge \partial_3, \partial_1 \wedge \partial_2 \wedge \partial_3 \tag{5}$$

$$\Lambda^3(V^*) \to dx^0 \wedge dx^1 \wedge dx^2, dx^0 \wedge dx^1 \wedge dx^3, dx^0 \wedge dx^2 \wedge dx^3, dx^1 \wedge dx^2 \wedge dx^3 \tag{6}$$

$$\Lambda^4(V) \to \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3 \tag{7}$$

$$\Lambda^4(V^*) \to dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{8}$$

Importantly, the basis vector just the number one and all the functions on the manifold that constitutes the 0th exterior power of the vector space and the 0th exterior power of the dual space. Interesting to note is obviously these two contain the same things, functions on the manifold so the 0th exterior power of the vector space is somewhat degenerate with the 0th exterior power of the dual space. The first exterior power is just a way of restating the existence of the vector space itself because this obviously equals the vector space V and is equals to the dual space V^* , they're the same.

The 2nd, 3rd and 4th exterior powers, those are different, they have different basis vectors so they're different vector spaces. There's a couple different ways to move between the different vector spaces in this diagram and that is understanding how that movement occurs is actually a pretty important step in understanding how all of this works but for example we can move from the left to the right by raising and lowering indices, if we have a metric or an inner product and a metric that connects V and V^* , you can move in this direction pretty smoothly you can there's a one-to-one correspondence between vectors and one forms between two vectors and two forms between three vectors and three forms and between four vectors and four forms and this goes back to what raising and lowering indices is all about the ability to raise the lower indices allows you to move in this direction. We're not actually going to talk about that here but suffice it to say that well let's quickly just to make sure you can make contact with what we've studied before let me quickly mention how it works. If we have a two form written out using the Einstein summation convention:

$$B_{\mu\nu}\,\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu} \tag{9}$$

Where $\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}$ is the basis vector and $B_{\mu\nu}$ is the coefficient. We're using Einstein's sum notation so we're actually dealing with double counting a little bit but this has to be anti-symmetric for this to actually make sense so the idea is (9) lives in this vector space $\Lambda^2(V^*)$ so given that we want to move to this vector space $\Lambda^2(V)$ well we need to actually raise the indices on it and we use:

$$B^{\alpha\beta}=\mathrm{g}^{\alpha\mu}\mathrm{g}^{\beta\nu}B_{\mu\nu} \tag{10}$$

Likewise this basis vector $\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}$ is going to have to change $\partial_{\mu}\wedge\partial_{\nu}$. That so that is how you move between vector spaces from the left and the right from vector spaces, these exterior powers constructed from the vector space to exterior powers constructed from the dual space. You must have this metric and that implies that this vector space V must have an inner product, we've discussed this extensively in the “What is a tensor series?” and I'm just reminding you that this movement here is actually quite trivial. What we're interested in is understanding how to move in the vertical direction. That's what we're going to talk about now, that's what Hodge duality is all about.

Hodges duality

The notion of Hodge duality exploits the fact that the dimensionality of the k -th exterior power is the same as the $n-k$ exterior power so there's four dimensions in the first exterior power and there's also four dimensions and the third exterior power so because these two vector spaces have the same dimensionality it's a reasonable question to say can I create a correspondence between $\Lambda^1(V^*)$ and $\Lambda^3(V^*)$ likewise can I create a correspondence between the $\Lambda^0(V^*)$ exterior power and the $\Lambda^4(V^*)$ exterior power and as far as the $\Lambda^2(V^*)$ exterior power goes can I create a correspondence between vectors of that exterior power and itself. Now if I had five dimensions here and I ended up with a fifth exterior power of V^* then there wouldn't be a central exterior power right there's two below it and two above it there would bean even number of exterior powers and you wouldn't have a central one and so you would you wouldn't have this lone central one but when you do have this lone central one you want to ask the question can you create a mapping between it and itself?

That's the question that we're going to answer with Hodge duality and for Hodge duality to make a lot of sense you have to establish what these vector spaces are meaning you have to decide what your basis vector is of the highest exterior power vector space in this case it's 0 1 2 and 3. We've established this by choosing that we want all of our vectors to have increasing indices and when we've done that now we can create these connections between the first exterior power and the third in particular that's the one that's of most interest but also also this one the one between the second exterior power and itself.

Now I'm going to provide you the formula that allows you to make this connection and that formula will take a vector in the first exterior power and find its Hodge dual which is a vector that lies in the third exterior power and likewise it'll take a vector in the second exterior power and find its Hodge tool which will also lie in the second exterior power so let's let's examine that.

The formula

$$*\left(\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}\wedge\cdots\wedge\mathrm{d}x^{i_p}\right)=\mathrm{sign}\left(i_1,i_2,\ldots,i_N\right)\varepsilon\left(i_1\right)\varepsilon\left(i_2\right)\cdots\varepsilon\left(i_p\right)\,\mathrm{d}x^{i_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{i_N} \tag{11}$$

This is the formula that we need and it's quite a grand looking line of text and I want, it's very simple actually but I do need to break down what these things mean, what these ε mean, what this sign term means and how to figure out what to put on the left and the right hand side. Let's attack it piece by piece using this fancy index notation that we are going to apply here. First of all I'm just going to teach the formula so this is a review so the nature of the Hodge dual and how it actually sits that's for a more complex lecture. This is the formula to help you understand that we can quickly derive the Hodge dual using a very rote method but the actual nature of the Hodge tool is quite profound and in a different rabbit hole we'll go down and a whole different way to approach the Hodge dual later on in geometric algebra but right now this is just a formula and how to apply it.

Let's look at the left hand side first. The first thing to see is that this star operator is called the Hodge star operator and the Hodge star operator its job is to take whatever is to its right which is always going to be a form of some kind and turn it into its Hodge dual so the left hand side is the form that's being converted and the right hand side is what it's being converted into. Now let's look at the way we I've expressed the form that's being converted first of all you'll notice that it's not a general form it is just a basis vector and we're following the idea that yeah as a practical matter if you can find the Hodge dual of the basis vectors you can find the Hodge dual of anything, you just express whatever it is in the appropriate basis and convert each basis vector to its Hodge dual and you've converted the whole form into its Hodge dual, that's very classic vector space stuff, if you know how to handle a certain manipulation on the basis vectors then you can do it for anything.

Now how do we express this basis factor? Well in this case there's obviously p of these different basis vectors of the original vector space wedged together so this is a p form and the index, each p form for example four-dimensional space is either $\mathrm{d}x^0,\mathrm{d}x^1,\mathrm{d}x^2,\mathrm{d}x^3$. Now for N -dimensional space this could go on to $\mathrm{d}x^{N-1}$ if you start with zero. This is the total number of vectors in the vector space or basis vectors in the vector space V , obviously if you're dealing with the p exterior power, you end up with this p form so I guess the way to say that this is a p form basis vector and these indices could be anything but there has to be p of them and you can't repeat obviously because if you repeat it, it would be zero so you pick any p of these and you put it in here and you're asking the question what is the Hodge dual of that so you pick the particular p you want, you may be interested in $\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2$ in which case you're looking for the Hodge dual of a particular three form, the 0 1 2 basis vector.

That explains the left hand side so now on the left-hand side notice you've used up p different basis vectors which means you have $n-p$ left that are not showing up on the left hand side, all of those $n-p$ remaining ones must show up on the right hand side and in this piece so whatever is not used here (left) must be used over here (right) but it doesn't matter what order it's in that's very important we have a standard preferred order where we want the indices to be increasing so in our preferred order it would always be the case that $i_1 < i_2 < \cdots < i_p$ and likewise we would expect this index i_{p+1} to be less than this ones on the right, all the way to be less than i_N but it doesn't have to be, it turns out this formula will work regardless, in fact all of this machinery over here is just to get the sign of everything correct which has to do with essentially reordering everything one way or another.

Whatever is missing from here (left) must be here (right) and that shows you the correspondence we're talking about, if this is a p form (left) then it must be that this over here is an $n-p$ form (right) which is exactly what I advertised would happen, this is a one form you end up with a four minus one or a three form. If it's a two form you end up with another two form (four minus two is two), if it's a zero form you better end up with a four form or likewise it can go the other way, if you started with a three form what's left over is going to be a one form, if you started with a two form you're going to get another two form, you started with a four form you're going to get a zero form. That's enforced by this equation, whatever is not used over here must (left) be used over here (right).

The easy part

Then all that remains is to understand this (middle) and this comes in two parts the easy part and the more subtle part. The easy part is you just line up all the indices in the actual order that you laid them out in the formula $i_1 \cdots i_p$ although it goes in the first p slots and then $p+1$ goes in the remaining slots so this expression here is actually $\operatorname{sign}\left(i_1 i_2 \cdots i_p i_{p+1} \cdots i_N\right)$. This sequence here is exactly this sequence (left) followed by this sequence (right). Whatever sequence you chose for your basis vectors you wish to seek the Hodge dual, you line those up as the first p and then the remaining ones that you've chosen that go $p+1$ to N, you line up as the remaining $p+1$ to N or $n-p$ indices so then you have a string of numbers here and these numbers might be, I'll just give an example.

This might end up being, for example, $\operatorname{sign}(2,0,1,3)$ in which case you are looking for the Hodge dual of $\mathrm{d}x^2 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^1$ and on the right side you end up with what's left over, well the only thing that's left over in a four dimensional vector space is $\mathrm{d}x^3$ so you end up with this question of what is the sign of this sequence $(2,0,1,3)$ and the answer is it's ± 1. It depends on whether this is an even or odd sequence relative to the standard sequence $(0,1,2,3)$. It's just how many flips does it take to get $(2,0,1,3)$ into this order $(0,1,2,3)$ and if it's an even number of flips you get a $+1$, if it's an odd number of flips you get a -1. In this case I have to move the two over once and over twice so this would be an example of $\operatorname{sign}(2,0,1,3) = +1$. This term comes as a plus or minus one to compensate for how this overall sequence compares to zero through N in order and what that's basically saying is when you go back here you've chosen you've chosen this highest order k form the four form you've chosen that as your definitive basis vector so this becomes the reference sequence of numbers $(0,1,2,3)$. Once you've chosen this everything needs to conform to that and in order to conform to this, because this is an arbitrary choice, I could have chosen $\mathrm{d}x^1 \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ if I wanted to, but we like to do this increasing order sequence thing so when we do that we end up introducing this sign term to keep track of how you may be organizing your basis vectors on the left and right of this equation. That takes care of this term, it'll give you a plus or minus one.

Now these ε are a little bit more subtle, the ε are concerned with what is the value of the inner product of a basis vector with itself so if I go back, I will show you how this is defined here:

$$\varepsilon(\mu)=(\mathrm{d}x^{\mu},\mathrm{d}x^{\mu}) \tag{12}$$

e.g. $\varepsilon(0)=(\mathrm{d}x^0,\mathrm{d}x^0)$ which in our convention is equal to -1 but $\varepsilon(1)=\varepsilon(2)=\varepsilon(3)=+1$ in our convention. In Minkowski space the metric drives $\varepsilon(0)=(\mathrm{d}x^0,\mathrm{d}x^0)=-1$ in our convention and these $\varepsilon(1)=\varepsilon(2)=\varepsilon(3)=+1$ and so that's the meaning of $\varepsilon(\mu)$ and so you see that the formula that we're working with includes the $\varepsilon(\mu)$ values of all of the basis vectors that are on the left-hand side i_1 through i_p so you have $\varepsilon(i_1)$ all the way to $\varepsilon(i_p)$ and these will only give you a -1 , if at most this can give you either -1 one $+1$ and because remember, you can't repeat an index so $\varepsilon(i_j)$ is going to appear only once here if you're dealing with Minkowski space-time. The beauty of this formula is that's completely general, it's leaving open what your metric is, you might have a metric with, I don't know you might be playing with some strange space time or a strange concept where there's two time directions so there's two different basis vectors with a negative sign and two with a positive sign or something crazy but in Minkowski space you're not going to deal with that so that is how the formula is structured and now we will apply it with some examples I've written out a whole bunch of examples right here so let's begin with that.

The chart

On this big old chart I did a bunch of examples. I've chosen the examples from Minkowski space in $\mathbb{R}^4$ so we have a four-dimensional vector space V and noting that in Minkowski space the 0$^{\text{th}}$ basis vector is going to come in with a $\varepsilon(0)=-1$ meaning its inner product with itself is -1 but all the others $\varepsilon(1)=\varepsilon(2)=\varepsilon(3)=+1$ i.e. the epsilon values for $\mathrm{d}x^1,\mathrm{d}x^2,\mathrm{d}x^3$ those are all going to come in with a $+1$. I also wrote it down for Euclidean space and in Euclidean space they're all going to have a $+1$ so there's gonna be no -1 there and Euclidean space also I'm just considering three dimensions not four dimensions. In Euclidean space $(\partial_i,\partial_j)=(\mathrm{d}x^i,\mathrm{d}x^j)=\delta^i_j$. Let's look at some examples of exploiting this formula (11), this is the actual formula that we learned and let's just begin with looking for the Hodge dual in Minkowski space.

Examples

$$*(\mathrm{d}x^0)=\mathrm{sign}(0,1,2,3)\,\varepsilon(0)\ \mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3=-\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \tag{13}$$

$$*(\mathrm{d}x^1)=\mathrm{sign}(1,0,2,3)\,\varepsilon(1)\ \mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3=-\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \tag{14}$$

$$*(\mathrm{d}x^2)=\mathrm{sign}(2,0,1,3)\,\varepsilon(2)\ \mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^3=+\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^3 \tag{15}$$

$$*(\mathrm{d}x^3)=\mathrm{sign}(3,0,1,2)\,\varepsilon(3)\ \mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2=-\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 \tag{16}$$

That's the art of this, the right hand side contains the wedge product of everything that's not on the left-hand side with some sign that's calculated by this formula (11). Now if I was in Euclidean three-dimensional space it's a lot easier, well it's not a lot easier it's not a lot easier or harder the point is these ε don't even matter because they're always one right there is no such thing as a negative one because the metric is so simple the metric is always positive so I can almost ignore those ε now but now I'm only dealing with three dimensions so the only thing that's missing, I guess I called the $1,2,3$ not zero and that's fine because when you have a zero there you usually are implying Minkowski space time and $\varepsilon(0)$ should be -1 .

$$*(\mathrm{d}x^1)=\mathrm{sign}(1,2,3)\varepsilon(1)\ \mathrm{d}x^2\wedge\mathrm{d}x^3=+\mathrm{d}x^2\wedge\mathrm{d}x^3 \tag{17}$$

$$*(\mathrm{d}x^2)=\mathrm{sign}(2,1,3)\varepsilon(2)\ \mathrm{d}x^1\wedge\mathrm{d}x^3=-\mathrm{d}x^1\wedge\mathrm{d}x^3 \tag{18}$$

$$*(\mathrm{d}x^3)=\mathrm{sign}(3,1,2)\varepsilon(3)\ \mathrm{d}x^1\wedge\mathrm{d}x^2=+\mathrm{d}x^1\wedge\mathrm{d}x^2 \tag{19}$$

Likewise I can find the Hodge dual in Euclidean space of this two form:

$$*(\mathrm{d}x^1\wedge\mathrm{d}x^2)=\mathrm{sign}(1,2,3)\varepsilon(1)\varepsilon(2)\ \mathrm{d}x^3=+\mathrm{d}x^3 \tag{20}$$

$$*(\mathrm{d}x^1\wedge\mathrm{d}x^3)=\mathrm{sign}(1,3,2)\varepsilon(1)\varepsilon(3)\ \mathrm{d}x^2=-\mathrm{d}x^2 \tag{21}$$

$$*(\mathrm{d}x^2\wedge\mathrm{d}x^3)=\mathrm{sign}(2,3,1)\varepsilon(2)\varepsilon(3)\ \mathrm{d}x^1=+\mathrm{d}x^1 \tag{22}$$

The Hodge dual of a function:

$$*\,\mathrm{f}=\mathrm{f}\ \mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \tag{23}$$

In Minkowski space you have the only complication that you end up with these $\varepsilon(0)$

$$*(\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2)=\mathrm{sign}(0,1,2,3)\varepsilon(0)\varepsilon(1)\varepsilon(2)\ \mathrm{d}x^3=-\mathrm{d}x^3 \tag{24}$$

$$*(\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3)=\mathrm{sign}(0,2,3,1)\varepsilon(0)\varepsilon(2)\varepsilon(3)\ \mathrm{d}x^1=-\mathrm{d}x^1 \tag{25}$$

$$*(\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^3)=\mathrm{sign}(0,1,3,2)\varepsilon(0)\varepsilon(1)\varepsilon(3)\ \mathrm{d}x^2=+\mathrm{d}x^2 \tag{26}$$

$$*(\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3)=\mathrm{sign}(1,2,3,0)\varepsilon(1)\varepsilon(2)\varepsilon(3)\ \mathrm{d}x^0=-\mathrm{d}x^0 \tag{27}$$

Now what I didn't show here is the Hodge dual of a two form so let me do that real quick so just just to make it interesting I'm going to take the Hodge dual of:

$$*\left(dx^2\wedge dx^0\right)=\text{sign}(2,0,3,1)\,\varepsilon(2)\,\varepsilon(0)\;dx^3\wedge dx^1=dx^3\wedge dx^1=-dx^1\wedge dx^3 \tag{28}$$

This is not a standard basis right because we have a decreasing set of indices here but just to prove to you that it doesn't matter, let's work it out. What's going to be on the right hand side? Well the first thing is what's remaining on the right hand side well we're missing one and three. I'm not choosing the correct canonical form, I don't know if canonical forms the right way to say it, I'm not choosing the increasing convention here just to show that it works fine. We have $\text{sign}(2,0,3,1)=-1$, $\varepsilon(2)=+1$ and $\varepsilon(0)=-1$. Just to shows you that now this the Hodge dual of a two form is an $N-2$ form so it gives me another two form. Notice these signs all are taken care of in this in these coefficients here so I don't have to worry about this increasing order element of things although it is probably good practice to stick with it. That is how we find the Hodge dual of all this stuff how we move between N forms and $N-p$ forms using the dual Hodge to jumps you from one forms to three forms two forms it keeps you inside two forms and it goes from four forms to zero forms.

This is very important because now we're going to take electromagnetic field elements and we're going to associate them with vectors, we're going to pop over then associate them with dual vectors and then we're going to freely move between these dual vectors and their Hodge dual so this is dual vectors and this is Hodge dual right so there's two different uses of the word dual here dual is always this notion usually of moving between different vector spaces so and then we're going to broker all that to decide how to call magnetic fields and what to call magnetic fields and what to call electric fields and we're going to create some expressions for Maxwell's equations using the language of forms and the language of Hodge duality, well Hodge duality is part of the language of forms I guess so that is our next project.

Our next project is to take what we understand about forms now that we understand Hodge duality we're going to start doing this and take Maxwell's equations and turn it into expressions involving nothing more than forms and Hodge dual forms and so that will help us create a slightly simpler form of Maxwell's equations one step simpler than the tensor form that we learned in the two lessons ago alright so I'll see you next time.