Maxwell's Equations in Tensor Form

Introduction

We are going to continue our examination of Maxwell's equations in the context of prerequisites to QED so it's a very formal examination of Maxwell's equations, as I've been saying over and over again we're going to do it in reciprocal space eventually but we've been stuck in real space because it's a really rich topic and I want to cover I want to squeeze all the prerequisite juice out of it that we can and now that we've gotten the potential formulation down, that is, we've written the Maxwell's equations as two second order differential equations that counts as a different form of Maxwell's equations. The question is though, is it how different, is it really because if you look at regular old Maxwell's equations and when I say regular what I mean is we're treating electric field and the magnetic field as vector fields, regular vector fields and we've got four first order differential equations now first order and time first order in space because these are differential operators:

$$\boldsymbol{E}=\vec{E}(\vec{r},t) \ \ , \ \ \boldsymbol{B}=\vec{B}(\vec{r},t) \ \ , \ \ \rho=\rho(\vec{r},t) \ \ , \ \ \boldsymbol{j}=\vec{j}(\vec{r},t) \tag{1}$$

$$\begin{cases} \nabla\cdot\boldsymbol{E}=\dfrac{1}{\varepsilon_0}\rho \\ \nabla\cdot\boldsymbol{B}=0 \\ \nabla\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t} \\ \nabla\times\boldsymbol{B}=\dfrac{1}{c^2}\dfrac{\partial\boldsymbol{E}}{\partial t}+\dfrac{1}{\varepsilon_0 c^2}\boldsymbol{j} \end{cases} \tag{2}$$

We have compressed it into two equations, there's no doubt about that, but they're two 2nd order differential equations so the actual mathematical content, complexity and sophistication is arguably not improved:

$$\begin{cases} -\nabla^2 V-\dfrac{\partial}{\partial t}[\nabla\cdot\boldsymbol{A}]=\dfrac{1}{\varepsilon_0}\rho \\ \left[\dfrac{1}{c^2}\dfrac{\partial}{\partial t}-\nabla^2\right]\boldsymbol{A}=\dfrac{1}{\varepsilon_0 c^2}-\nabla\left[\nabla\cdot\boldsymbol{A}+\dfrac{1}{c^2}\dfrac{\partial V}{\partial t}\right] \end{cases} \tag{3}$$

Today we're actually going to see about actually improving the knowledge density of these equations and we're going to take Maxwell's equations and write these four equations in tensor form and the goal here is to understand this notion this principle: I can take the electric field and the magnetic field and I can transform them from one reference frame to another reference frame from one Lorentz frame to another, I can make these transformations and if I do that and I substitute the transform fields into this these four expressions I'm going to get the exact same four expressions in terms of these transformed fields. The problem is that's certainly not obvious we would have to execute this transformation and then plow through the mathematics and simplify things to get all of this written in prime vector and scalars like we'd have to do that and it would take a long time.

The fact that it is true is a big hint that we can create a relativistic form of this that uses what we call manifest Lorentz invariants which means that these equations will be not only, well they are Lorentz invariant but we are going to be able to write them down in a form that demonstrates the Lorentz invariance just by their visual structure ultimately there is an improvement, a mathematical density improvement so to speak and the idea is well with mathematical improvement should come some interpretation improvement so that's what we're going to do today and let's begin.

The Four Vector

We have already worked pretty hard to understand the vector potential which is these three components of the vector potential A_x, A_y, A_z which we write as $\boldsymbol{A}$ now that we've mastered this vector potential idea and we also learned about the scalar potential now that we have these two potentials and we understand the relationship to $\boldsymbol{B}$ and $\boldsymbol{E}$, well our first step into unifying Maxwell's equations into a tensor expression, is to first unify these two potentials together into a four vector and that four vector we call the four potential and its first component in SI units is the scalar potential over c and then A_x, A_y, A_z the three components of the vector potential:

$$A^{\mu}=\left(\phi/c, A_x, A_y, A_z\right)=\left(\phi/c, \boldsymbol{A}\right) \tag{4}$$

This is in one frame of reference so we're in some reference frame some inertial reference frame and in that inertial reference frame we have a vector potential that's a function of x, y, z, t and we have a scalar potential likewise it's a function of x, y, z, t and we're going to combine these two together into a four potential. now first to be clear that's not an arbitrary step you can't just take any four things and make a four percent and make a four vector out of it or line them up and call it a four vector. We have to understand what it means to be a four vector and the idea of course is that if I switch to a different Lorentz reference frame that that's going to be a Lorentz transformation of exactly the form that we learned about in earlier sections of this lecture:

$$\boldsymbol{A}_{\mu}=\left(\phi/c, -A_x, -A_y, -A_z\right)=\eta_{\mu\nu}\boldsymbol{A}^{\nu} \tag{5}$$

$$\boldsymbol{A}^{\mu'}=\Lambda_{\mu}{}^{\mu'}\boldsymbol{A}^{\mu} \tag{6}$$

We now know that we can take this Lorentz transformation matrix take our four potential and transform it to a new Lorentz frame using the architecture of this Lorentz transformation matrix that's what it means to be a four potential. Now you could ask the question well how do we know that this is a four potential, that we can make this construction that we know that this is the case? That's a good question and the answer is actually tricky but we're not going to go into it, that's one of the rabbit holes we are not going to go down. You can google it, the proof is actually a little bit more subtle than you would imagine or the understanding is a little more subtle than you would imagine but for the purposes of getting into QED (4) is a four potential and this transformation (6) is true. We're going to do this again for the four current, we're going to write the four current down as the three dimensional current density $\boldsymbol{j}$, which we've already discussed, which is a function again of x, y, z, t and the charge density ρ which is also a function of x, y, z, t and we're going to combine those into a four current as well and when we do that, what we mean is that we can transform it using a Lorentz transformation matrix and this is the matrix that we spent so much time getting familiar with earlier and now we're getting the payoff.

With that first step we now need to understand that the way we're going to construct this is with four potential components so (4) are the contravariant components of the vector four potential, (5) are the covariant components of the vector four potential and these minus signs are introduced because we are assuming a mostly minus metric for our work. We're going to use SI units, the SI units is really ultimately what this c , this speed of light constant comes from and we are going to use this mostly minus metric which is constant so we're in flat space time and mostly minus metric, even if you're in spherical or cylindrical coordinates you could always transform to Cartesian coordinates and every point in space time would have this value for the metric and when you lower this index using this metric you're going to introduce these minus signs so that's using this very classic standard formula depicted right here (5) so that's the first step is we have to now in order to simplify Maxwell's equations we have to realize that our vector and our scalar thing is actually a unified object called the four potential and that little unification is what opens the door to creating the electromagnetic field tensor which is our next subject.

Transformation Matrix

Before we move on I actually wrote this out a little bit more thoroughly so you could see it but this notion of transforming using the Lorentz transformation matrix. I'll remind you we went through in earlier lessons derived the most general form of this Lorenz transformation matrix and that general form, I suppose I should take these β in the denominator and give them parentheses because they're meant to be the magnitude of β the velocity β whereas this β^2 here is the second component of β

$$\begin{pmatrix} \gamma & -\gamma\beta^1 & -\gamma\beta^2 & -\gamma\beta^3 \\ -\gamma\beta^1 & 1+(\gamma-1)\frac{(\beta^1)^2}{(\beta)^2} & (\gamma-1)\frac{\beta^1\beta^2}{(\beta)^2} & (\gamma-1)\frac{\beta^1\beta^3}{(\beta)^2} \\ -\gamma\beta^2 & (\gamma-1)\frac{\beta^1\beta^2}{(\beta)^2} & 1+(\gamma-1)\frac{(\beta^2)^2}{(\beta)^2} & (\gamma-1)\frac{\beta^2\beta^3}{(\beta)^2} \\ -\gamma\beta^3 & (\gamma-1)\frac{\beta^1\beta^3}{(\beta)^2} & (\gamma-1)\frac{\beta^2\beta^3}{(\beta)^2} & 1+(\gamma-1)\frac{(\beta^3)^2}{(\beta)^2} \end{pmatrix} \cdot \begin{pmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \phi'/c \\ A'_x \\ A'_y \\ A_z \end{pmatrix} \tag{7}$$

If $\boldsymbol{\beta}$ is a vector then $\boldsymbol{\beta}=\beta^1\hat{x}+\beta^2\hat{y}+\beta^3\hat{z}$. It's a little bit sloppy the way I did it but now I've cleaned it up with these parentheses but the point is this is a mess of a matrix, you've got these γ term of course and then you've got the components of the velocity and you clearly are going to mix up all these components of the four potential together to get the four potential in the new reference frame

$$\begin{pmatrix} \gamma & -\gamma\beta^1 & 0 & 0 \\ -\gamma\beta^1 & 1+(\gamma-1)\frac{(\beta^1)^2}{(\beta)^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \gamma\phi-\gamma\beta^1 A_x \\ -\gamma\beta^1\phi+\left[1+(\gamma-1)\frac{(\beta^1)^2}{(\beta)^2}\right]A_x \\ A_y \\ A_z \end{pmatrix} \tag{8}$$

What's interesting about this is and if we just did a boost in the x direction for example this is how it would ultimately look if you calculated the new components (8) but note by the way this is a there's a little bit of a subtle trick here remember that A_x is a function of (x, y, z, t) and when I do this transformation, when I multiply these matrix elements by this column vector, I end up with A_x, A_y, A_z on the right hand side but what's interesting about it is this expression, for example (second from the top of the result) A'_x or I guess you could say it's $A^{1'}$ using our component notation, means the same thing. This is still going to be a function of (x, y, z, t) but notice these guys are in the un-primed frame so this is an annoying little problem right because you're transforming the potential from the un-primed to the primed system but you're left with these functions that are still in terms of the un-primed system so you have to transform all of the (x, y, z, t) inside these functions to (x', y', z', t') so when you study this, don't forget that and the idea is why would you want to study it? Well you might want to prove to yourself that the Coulomb gauge in the (x, y, z) frame, if you transform to the primed frame, this is not necessarily equal to zero, in fact it probably won't be but in order to do this check you have to take derivatives with the primed variables and you have to make sure you convert this (x, y, z) to (x', y', z') not just $\boldsymbol{A}$ to $\boldsymbol{A}'$. That is the point of the value of being a four vector. Now with this four vector potential we are going to now derive an Electromagnetic field tensor.

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{9}$$

Wikipedia gives us this beautiful definition of the electromagnetic field tensor $F^{\mu\nu}$ and it does it component by component as long as you understand that $\boldsymbol{A}$ now is the four potential and these indices run from these indices run from $0, 1, 2, 3$ and when you do this and you execute this for each of the possible combinations for example F^{01} all the way to I guess F^{23} , you will generate a tensor that has components and each of the components can be expressed as the components of the electric or magnetic field and with the caveat that you have a few minus signs in there and a couple factors of c and the first thing you can notice from this definition is that $F^{\mu\nu}$ is anti-symmetric right if you switch μ and ν you switch the sign of this thing and indeed this whole thing is obviously anti-symmetric $F^{\mu\nu} = -F^{\nu\mu}$, the diagonals are zero, that's important and this is the definition of the electromagnetic field tensor and the metric that we're using here for this is the mostly minus metric $(+,-,-,-)$. If you use the mostly the mostly positive metric $(-,+,+,+)$, all the sign switch but we're not going to do that.

Now you know right away that this is a tensor and it's Lorentz invariant because if you look at this you see this is a fully indexed object so clearly it's every one of those things is subject to a Lorentz transformation and you'll just change μ to μ' and ν to ν' and you'll have an expression that's true in the new frame $F^{\mu'\nu'}$ will be true so this is an electromagnetic field tensor. Now what you don't have to worry about so much in this prerequisite work is this whole notion of a tensor should never be separated from its unit vector part it's the part that makes it a tensor $F^{\mu\nu} \partial_{\mu} \otimes \partial_{\nu}$ this in other words is simply a component a real number, this guy here is the actual tensor piece that makes it a vector space and if in the what is a tensor series I spend a lot of time on that but you remember even in the what is a tensor series I eventually say now that you understand this we can actually see why we can give this up and do everything in terms of the components. I'm not a fan of that, I mean anybody who did that course knows I rail against it on the other hand, you can't follow me around everywhere and do what I say, you've got your own books to read and clearly Wikipedia has no trouble with it, as we see (9). Wikipedia does not have $\partial_\mu \otimes \partial_\nu$ stuck next to $F^{\mu\nu}$ that so clearly you can't live and die by that ideal that pedagogical ideology is I guess what I'm saying.

There's the tensor and clearly you see already we have now unified these fields into a single mathematical object and they really do lose their identity because once we have an intensive form we think of it as universal to any possible Lorentz transformation that's out there that's interesting and once we you start Lorenz transforming this guy the $\boldsymbol{E}$ field and the $\boldsymbol{B}$ field completely get mixed up and they lose their independent nature so once you start thinking of the Electromagnetic field tensor as something that has validity in any Lorentz frame clearly you realize that the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields do not have meaningful invariant natures and $F^{\mu\nu}$ actually becomes the fundamental physical object and now arguably you could say well are we elevating the physical nature of the four potential, if $\boldsymbol{E}$ and $\boldsymbol{B}$ is completely scrub now well this is a tensor but so is this expression, maybe these vector potentials are more important part of nature than we thought and yeah that this would definitely imply that this would definitely lead one to start thinking well maybe the four potentials are fundamental, the fields are not. You still have the problem of ambiguity though, you have this ambiguity of the four potentials and that just doesn't go away it just so there's a lot of debate to be had here but it's interesting to track the debate because you start understanding the nuts and bolts of what you're doing. Now let's see how does this for potential actually give us these fields.

Magnetic Field

The only way to exemplify this is just to do it. The magnetic field is the curl of the vector potential now this is the vector three potential right so this is our elementary form where we have a vector and a vector and a curl of a vector is a vector so this is all good.

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (\partial_2 A_3 - \partial_3 A_2)\hat{x} + (\partial_3 A_1 - \partial_1 A_3)\hat{y} + (\partial_1 A_2 - \partial_2 A_1)\hat{z} \tag{10}$$

If we look at $F_{\mu\nu}$ and we look at its definition we realize that okay there's only so many components that aren't zero, we've got the diagonal of zero so let's forget those those are clearly going to be zero:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{11}$$

$$\begin{cases} F_{12} = \partial_1 A_2 - \partial_2 A_1 = B_z = F^{12} = -F_{21} \\ F_{13} = \partial_1 A_3 - \partial_3 A_1 = -B_y = F^{13} = -F_{31} \\ F_{23} = \partial_2 A_3 - \partial_3 A_2 = B_x = F^{23} = -F_{32} \end{cases} \tag{12}$$

If you raise F_{12} two indices using the metric you learn that B_z is not only the F_{12} component of the fully covariant form of the electromagnetic field tensor, it's also the fully contravariant form F^{12} , in other words, raising these two indices doesn't change a sign. Sometimes you get a sign change actually for the electric fields you do get a sign change as you'll see in a moment so you can calculate this for all three of these and you do end up with this minus $-B_y$ and remember it's anti-symmetric. You have a lot of these little weird index things going on but the point is that the spatial part of the tensor meaning the part that has no index that's zero are all related to the magnetic field except for the diagonal ones which are zero because of the anti-symmetry of our definition.

The electric parts are a little bit more interesting, you still just run the same definition straight up:

$$\begin{cases} F_{01}=\partial_0 A_1-\partial_1 A_0=\dfrac{\partial A_1}{\partial x^0}-\dfrac{1}{c}\dfrac{\partial \phi}{\partial x^1}=-\dfrac{1}{c}\dfrac{\partial A_x}{\partial t}-\dfrac{1}{c}\dfrac{\partial \phi}{\partial x}=E_x/c=-F^{01} \\ F_{02}=\partial_0 A_2-\partial_2 A_0=\dfrac{\partial A_2}{\partial x^0}-\dfrac{1}{c}\dfrac{\partial \phi}{\partial x^2}=-\dfrac{1}{c}\dfrac{\partial A_y}{\partial t}-\dfrac{1}{c}\dfrac{\partial \phi}{\partial y}=E_y/c=-F^{02} \\ F_{03}=\partial_0 A_3-\partial_3 A_0=\dfrac{\partial A_3}{\partial x^0}-\dfrac{1}{c}\dfrac{\partial \phi}{\partial x^3}=-\dfrac{1}{c}\dfrac{\partial A_z}{\partial t}-\dfrac{1}{c}\dfrac{\partial \phi}{\partial z}=E_z/c=-F^{03} \end{cases} \tag{13}$$

The reason we did that by the way is so we could use our definition of the electric field which is:

$$\boldsymbol{E}=-\frac{\partial \boldsymbol{A}}{\partial t}-\nabla\phi \tag{14}$$

Remember, to raise it I use this formula right here:

$$F^{\mu\nu}=\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} \tag{15}$$

If I want the raised F_{12} , I execute this sum and what makes the sum simple is that all the off diagonal terms of the η 's are zero, so I don't have many terms in the sum I only have one term that survives but g^{11} and g^{22} they're both -1 in the mostly minus convention, these two guys are -1 so they cancel:

$$F^{12}=g^{11}F_{12}g^{22}=F_{12}=B_z \tag{16}$$

For any totally spatial piece of the Electromagnetic field tensor the two will cancel and the covariant and the contravariant components are going to be the same but when you have only one spatial part you have a time piece g^{00} is 1 and g^{11} is -1 so you don't get a cancellation of minus sign so you introduce a minus sign here and that explains why these minus signs appear on the right hand side of (13).

I've shown you the compact construction of the Electromagnetic field tensor and how using these potentials in their four potential form which takes our magnetic vector potential and our scalar potential and combines it into a four vector we use that to define the Electromagnetic field tensor and you can see how the components of the Electromagnetic field tensor as defined this way (11) gives you the fields embedded in this object and now the fields are in the rear view mirror and it's just this tensor that matters in our Physics so hopefully we're going to be able to write all of our Physics in terms of this tensor, well that would be that would be the goal, all of our laws of physics could be written in the terms of the tensor, so what do we do next?

Lorenz Gauge

One thing we should do is since we discussed the idea that the coulomb gauge is not going to survive a Lorentz transformation but if we look at the Lorenz gauge and explicitly executing the sum you get this string of this expression of four of these partial derivatives with each component of the four potential, this sum basically has to be zero, I'm just blowing up the Einstein notation:

$$0=\partial_{\mu} A^{\mu}=\partial_0 A^0+\partial_1 A^1+\partial_2 A^2+\partial_3 A^3 \tag{17}$$

The point is now you have contracted indices up and down you don't even have to go through this process of checking this arbitrary matrix against an arbitrary field and calculating the transformed field and then showing that this transformed field still has this property (17), in other words you don't have to check that:

$$\partial_{\mu'} A^{\mu'}=0 \tag{18}$$

Switching into the new reference frame, you don't have to check it because the Lorenz condition is all obviously relativistically invariant because if we assume that $\boldsymbol{A}$ is a four vector this expression $\partial_{\mu} A^{\mu}$ is completely contracted space-time indices which by definition that's how they're constructed, that's what it means to be completely contracted, this is a Lorentz invariant expression by its very nature and that's why we say it's manifestly Lorentz invariant. If you have these indices structured in a reasonable way that's all you need to know, if you can put your expressions in that form, you have created a manifest Lorentz invariant and if you really wanted to go in there, I could take A^{μ} and transform it with a Lorentz transformation and I could take this ∂_{μ} , that's a space time index, I can transform that with a Lorentz transformation and this becomes:

$$\Lambda_{\mu'}{}^{\mu}\ \partial_{\mu}\left[\Lambda_{\mu}{}^{\mu'}\ A^{\mu}\right]=0 \tag{19}$$

That's the Lorenz condition in the new frame of reference and this is a constant in flat space time so you can pull them out and you get these two Lorentz transformations that contract to one which again shows you that all these Lorenz transformations, done right, really are multiplying by the identity matrix in the end:

$$\Lambda_{\mu'}{}^{\mu}\ \Lambda_{\mu}{}^{\mu'} \partial_{\mu}\ A^{\mu}=0 \tag{20}$$

That's why the Lorenz condition is valued is because it's you can make all these boosts and nothing happens to it now you'll see when we quantize things that there is some flaws with this you introduce degrees of freedom that you have to get under control but it so does the Coulomb gauge in a way has its own issues. The point is, I just want to demonstrate what they mean by manifestly invariant. Now we tackle Maxwell's equations, we've got a new form for the fields, it's all buried in the Electromagnetic field tensor (11), (12), (13), we have a four potential now we need to make Maxwell's equations work and so why not go to Wikipedia to talk about this?

Four Current

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\varepsilon_0} \quad , \quad \nabla\times\boldsymbol{B}-\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}=\mu_0\boldsymbol{J} \tag{21}$$

Here we have the two in-homogeneous Maxwell's equations, we have the charge density and the current density and everything here is a regular vector, ρ is a scalar and the divergence of a vector field is a scalar so the first is a scalar equation, the second is a vector equation.

If we want to now turn this into something that just uses the Electromagnetic field tensor, we actually have to it turns out create yet another four vector called the four current and this is the construction of the four current just as stipulated right here in Wikipedia:

$$J^{\alpha}=\left(c\rho\,,\boldsymbol{J}\right) \tag{22}$$

The time component of the four current is the speed of light times the charge density ρ and the space component is the current density and combined together that gives you a four current and that four current what that means is you can transform this from one Lorentz frame to another using our Λ matrices all day long but once you've done that now the claim is that this is the two in homogeneous Maxwell's equations combined together into one very tidy statement:

$$\partial_{\alpha}F^{\alpha\beta}=\mu_0 J^{\beta} \tag{23}$$

Notice that this tidy statement they're still first order this is still a first order differential equation but instead of two you have one and the just the sheer notation density is much improved, the parlance of the business is if you can express complicated things in a simple way you're learning something you've improved your Physics you've improved your interpretation of how nature is organized mathematically so the idea is to say that this is the favored mathematical way of writing it down because it takes something that was complicated and makes it simple. There's a there's a lot to say about that because obviously we know that buried in here (23) is actually a big mess (11), (12), (13) so you can almost argue, I'm just hiding stuff under the rug, I'm hiding things and this is not actually any simpler it's just I've defined things in ways that make it look simpler. That's probably not true, it's probably actually is a a better way of looking at nature, it's manifestly Lorentz invariant first of all, because α is contracted and you're just left with space time four vectors on both sides so it's very clean, Lorenz invariance is a fundamental part of Physics but (21) are Lorentz invariant too, it's just not obvious that it's that way.

Now you can argue about this but the point is is that (23) is a very important form to understand, you can't go marching into QED without having a good grip on this form but notice we did have to create another four vector here but that just tells us that the current is actually a four vector which is important too so then we do have two other Maxwell's equations, the homogeneous ones:

$$\nabla\cdot\boldsymbol{B}=0 \quad , \quad \frac{\partial\boldsymbol{B}}{\partial t}+\nabla\times\boldsymbol{E}=0 \tag{24}$$

Those get squeezed into this new form related to what's famously called the Bianchi identity:

$$\partial_\gamma F_{\alpha\beta}+\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}=0 \tag{25}$$

and this can actually be compressed even further because you note that it's appropriately symmetrized you just shift the α , β and γ comes over, and that's what this is supposed to say. Now the only reason you can get from (25) to (26) is because $F_{\alpha\beta}=-F_{\beta\alpha}$

$$\partial_{[\gamma} F_{\alpha\beta]}=0 \tag{26}$$

The point being is that (26) really is six terms and (25) is three but you can combine these six terms together using the anti-symmetry of F so that's very important to understand. The point is that you've simplified everything down quite a bit and in fact this is the way to look at it:

$$\left.\begin{array}{c}\nabla\cdot\boldsymbol{E}=\dfrac{\rho}{\varepsilon_0}\\ \nabla\times\boldsymbol{B}-\dfrac{1}{c^2}\dfrac{\partial\boldsymbol{E}}{\partial t}=\mu_0\boldsymbol{J}\end{array}\right\}\rightarrow\ \partial_\mu F^{\mu\nu}=\mu_0 J^\nu \tag{27}$$

$$\left.\begin{array}{c}\nabla\cdot\boldsymbol{B}=0\\ \nabla\times\boldsymbol{E}+\dfrac{\partial\boldsymbol{B}}{\partial t}=0\end{array}\right\}\rightarrow\ \left\{\begin{array}{c}\partial_\alpha F_{\mu\nu}+\partial_\mu F_{\nu\alpha}+\partial_\nu F_{\alpha\mu}=0\\ \partial_{[\gamma} F_{\mu\nu]}=0\end{array}\right. \tag{28}$$

We've we had four Maxwell's equations, these two get compressed into one (27) and these two get compressed into one (28). This is a legitimate compression because you've now got first order differential equations but only two of them your mathematical objects are more general and have wider applicability and they are easier to interpret, (28) can be interpreted in terms of ideas of differential geometry, (27) this is a four divergence so now instead of having a divergent and curl you have one for divergence and this divergence and curl turns into this Bianchi identity which is really convenient. I think I need to do the demonstration that (27) and (28) are correct so let's check that out. What are we after here, we're after one particular Maxwell's equation:

$$\nabla\cdot\boldsymbol{E}=\partial_x E_x+\partial_y E_y+\partial_z E_z=\frac{\rho}{\varepsilon_0} \tag{29}$$

This divergence is defined through this combination of spatial derivatives of the various components of $\boldsymbol{E}$, it's a divergence of a vector field which is a scalar so somehow this has got to be buried inside our new form of expressing Maxwell's equations (27), which we're going to work on this for divergence of the electromagnetic field tensor. Somewhere buried in (27) has got to be this expression (29) and likewise all of the other expressions we're going to demonstrate this for two of the expressions. Let's blow up this sum (27) using the standard Einstein convention and we remember that this is a sum in Minkowski spacetime so we have a mostly minus convention so we end up with these three minus signs but the leading term is a plus and that is just executing the sum over μ which is a dummy index:

$$\mu_0 J^{\nu} = \partial_0 F^{0\nu} - \partial_1 F^{1\nu} - \partial_2 F^{2\nu} - \partial_3 F^{3\nu} \tag{30}$$

Now we see we have four equations one for every value of ν . Because we're dealing with this charge density ρ , let's speculate that the relevant equation takes out J^0 because we know that the current density's zeroth component, its time component is the charge density times the speed of light so let's substitute everywhere we see ν substitute for the index zero and when we do that we get:

$$\mu_0 J^0 = \mu_0 c\,\rho = \partial_0 F^{00} - \partial_1 F^{10} - \partial_2 F^{20} - \partial_3 F^{30} \tag{31}$$

We know that ε_0 is some combination of μ_0 and c , we remember that from electrodynamics so hopefully that'll flush out. There's definitely needs to be a c^2 in there so we got to keep an eye out for that but it's trending in the right direction and then we get this time derivative. We know that $F^{00}=0$ because we know that F is anti-symmetric and its diagonal elements are zero. What we realize is (13).

$$\mu_0 J^0 = \mu_0 c\,\rho = \partial_1 E_x / c + \partial_2 E_y / c + \partial_3 E_z / c \tag{32}$$

Recall $\varepsilon_0 \mu_0 c^2 = 1$. The zeroth index equation returns to us the first Maxwell equation (29). Now we just have to hunt down the other Maxwell equation that's tied to this expression (27). If setting index 0 is so helpful why don't we just set index equal to 1 and see what happens. We examine what we've got:

$$\mu_0 J^1 = \partial_0 F^{01} - \partial_1 F^{11} - \partial_2 F^{21} - \partial_3 F^{31} \tag{33}$$

We know that this term F^{11} is going to go away because it's the diagonal term in the electromagnetic field tensor which is anti-symmetric. We just have to substitute what we know these components are in terms of the fields and what do we discover:

$$\mu_0 J^1 = -\frac{1}{c} \partial_0 E_1 - \partial_2 B_3 + \partial_3 B_2 \tag{34}$$

$-\partial_2 B_3 + \partial_3 B_2$ are clearly looking like the piece of a curl, remember the components of a curl. This would be the x component of the curl of the magnetic field and so it looks like we are now finding the x component of the curl equation for the magnetic field.

Component Equations

Specifically if we are claiming that these two equations (27) are bound up in here the (second) equation is a vector equation, you have a vector on the left you have a vector on the right the time derivative of a vector is still a vector, this expression has x, y, z components meaning this vector on the left and this vector on the right each has three component equations so it looks like we're hunting down one of those components and specifically if we go back up to this example of what a curl looks like (10) we see the curl of $\boldsymbol{A}$ and we see the component of the $\nabla \times \boldsymbol{A}$, That pattern is exactly the pattern we have in our expression (34). This is going to be, if you rearrange it so this term goes here and this term goes here and you relabel 2 and 3 to z and y this is the x component of $\nabla \times \boldsymbol{B}$ and then the derivative of E_x .

$$\mu_0 J^1 = -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \left(\partial_z B_y - \partial_y B_z\right) = -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \left(\nabla \times \boldsymbol{B}\right)_x \tag{35}$$

This expression is the x component of the relevant Maxwell's equations and does it equal the right thing? It equals $\mu_0 J_x$. There you have it so we do this three times meaning we go back here and we replace ν with 2 and 3 and we'll get all three of the relevant components and now we have four component Maxwell's equations. You can do the same thing for the other two Maxwell's homogeneous equations (28), you can just expand the Bianchi identity and once you expand it out, you'll discover embedded in it are these very these two Maxwell homogeneous equations.

We have now shown that we can take these four Maxwell's equations and we can compress them into these two equations tensor equations, they're not vector equations, they're tensor equations so you're you're solving for the components of the electromagnetic field tensor presumably given a charge in current density that you can combine into a four current, you're solving for these components and you use a lot of the fact that this is asymmetrical, keeps the number correct on both sides. The number of equations and the number of unknowns is still all good.

This is what you need to know as a prerequisite, you definitely need to understand Maxwell's equations in terms of the Electromagnetic field tensor, this is something that I just can't imagine arriving at any reasonable form of QED book anywhere and not having a very good understanding the problem is QED books know when you get there that you may not know this very well and they do a little bit of review and it's good enough but if you're here you're worried about superficial knowledge or knowledge that requires a lot of evolution before you really feel comfortable with the subject so hopefully what we've done is we've given you the motivation to dig a little bit deeper and gather up your knowledge and exercise this matter. For example you can do this yourself, you can find these two equations by doing this just take note when you do solve this yourself meaning when you go through the exercise we just did μ and ν can't take the same value so that eliminates a lot of options. You pick μ, ν and that you don't have to go through every conceivable combination of these indices, in fact a lot of them are absorbed for you because this is all the cyclic combinations in front of your face.

Let me make that clear, we have gone far enough you can stop right here if you're interested in just the prerequisites for QED but I simply can't resist pointing out one other thing and in particular you can ask the question well I've taken four first order equations and turned it into two and if I count the number of pen strokes it takes to to write this stuff down I suspect I've improved the pen strokes I definitely I can improve pen strokes here too right I can write this as I think $\partial_{[\gamma} F_{\mu\nu]} = 0$. I think that we're compressing this notation using this brackets that symmetrize things so I've definitely reduced pen strokes I've increased information density at the expense of having to know the definition of this right so I'm always curious are we making progress in the sense that we are truly unifying things and the idea now is that F is now the basic thing of our study but we've reduced it from four to two I guess the obvious question that floats around my head at least is can we go from two to one. Is it possible that we can actually create just one equation? To be honest the answer as far as I understand is yes it is possible, the question now is it worth the time it would take to develop what is required to do it and I'm going to strike a compromise with the caveat that we are done as far as the prerequisite promises this is the prerequisite material I cannot help but simplify this yet again one more layer of simplification I think it's worth doing and it fits very beautifully in with our with the what is a tensor material that we did before so I simply can't resist it.

Unfortunately the simplification I offered doesn't really get us to the one point, what it does is it takes this equation (27) and this equation (28) and writes them even simpler than they are but you still are left with two equations as far as I know. Taking the step all the way down to one equation at that point I'm not really an expert but there is a field of study that I am interested in learning that does do it and so I may do a video on that as I learn it, that's a subject called Geometric algebra but using the language of differential forms and the machinery of the Exterior algebra we can write down these two equations in a much more logical format than just this purely tensor format. I guess I can't say it's more logical I mean there's nothing illogical in (27) and (28) but the Exterior algebra form is very beautiful and it's very telling and honestly I guess if you really understand the exterior algebra it's quite intuitive.

I'll do that in a supplementary lecture next time but the truth is and the irony is that no matter how far you push this, when you solve EM problems you always go back to the original Maxwell's equations. I've never seen anybody solve a wave guide using differential forms analysis and I'm and now that I've said that I'm sure a million people are going to point me to references where it's done and I'm going to be impressed but the point being is that just because you can go from this 19^{th} century version of things to this 21^{st} century thing over here. Doesn't mean that this is useless or old or obsolete in any way.

Outro

What have we done today? We have shown that Maxwell's equations can be expressed in this purely beautiful relativistic form we discussed a little bit about the relativistic and variance of the Lorenz Lorenz condition and we demonstrated how these vector expressions these classic vector expressions are buried inside these tensor component expressions and next time what we're going to divert ourselves a little bit away from the mainstream notion of prerequisites of QED and we're going to show how exterior algebra can arguably simplify these two equations even more. I'll see you next time