QED Prerequisites Geometric Algebra 13: Tensors

Introduction

Welcome back, we're going to continue this read of “Space-time algebra as a powerful tool for electromagnetism” and at roughly an hour lesson we're on our 13th hour, obviously I have no compunction about going slow and digging into this and producing this material, it is definitely an inefficient way to approach any subject but again, that's not my thing, inefficiency is my thing I guess is the way I should say it. Here's a reference for where you can get the paper (arxiv) and I want to emphasize of course that I have no relationship with the authorship of this paper full credit goes to these guys and thank you very much to all three of you for doing this, this is definitely in my opinion the best paper out there on the subject and any gaps that I fill are necessary just because it's a thick subject and I just want to help people who might not be at this paper's level to be elevated into that level and that's why you get all this extra time involved because I fill in a lot of blanks and I accept a ton of redundancy exactly the kind of thing you can't afford to put in a paper like this.

We have been attacking Section 3.3.2 “Reversion and inversion”, we finished that Section and then we covered reciprocal bases we got that done, we just finished last the Section on components, how to calculate components of things and this lesson we're really going to dig into the notion of tensors so let's begin. Here's Section 3.4, we covered this “Reciprocal Basis”. I'll remind you that this is incorrect, that is an error in the printing of the paper.

$$\cancel{\gamma^{\mu}\gamma_{\nu}}=\gamma^{\mu}\cdot\gamma_{\nu}=\delta^{\mu}_{\nu} \tag{1}$$

As a reminder that error is almost obvious because this space-time product would be written as $\gamma^{\mu}\cdot\gamma_{\nu}+\gamma^{\mu}\wedge\gamma_{\nu}$ and obviously that expansion does not equal $\gamma^{\mu}\cdot\gamma_{\nu}$ because you've got this piece $\gamma^{\mu}\wedge\gamma_{\nu}$ so clearly that's not correct so the 1st part goes away, however the 2nd part is correct by definition and then we discussed how components are done in the reciprocal basis versus the reference basis for an arbitrary vector v and they introduce the notion of the *contravariant* components of v and the *covariant* components of v which are basically if the index is in the up position for the components those are contravariant components if the index is in the down position those are covariant components and do not waste any time at this point wondering where these words come from, what's covariant what's contravariant that is definitely something that can be learned but now is not the time and now is it's also not relevant it's just relevant that we have a way of speaking of the contravariant and covariant components:

$$v=\sum_{\mu}v^{\mu}\gamma_{\mu}=\sum_{\mu}v_{\mu}\gamma^{\mu} \tag{2}$$

I guess the way to say it, the way this paper would want to say it is the contravariant components are the components in the reference basis, the covariant components are the components in the reciprocal basis, this is always the reference basis, this lowered index and the reciprocal basis is always this raised index and then they discussed well how would bi-vectors be handled component wise and they gave us these formulas for bi-vector components in the reference basis and the reciprocal basis:

$$\boldsymbol{F}=½\sum_{\mu,\nu}F^{\mu\nu}\gamma_{\mu}\wedge\gamma_{\nu}=½\sum_{\mu,\nu}F_{\mu\nu}\gamma^{\mu}\wedge\gamma^{\nu} \tag{3}$$

We could do without the summations here because the Einstein sum process is fine but the summations are just to make it absolutely clear what's going on but I won't be using those summations when I jot this stuff down and then they showed us how to calculate these components and we spent the last part of our lesson exemplifying this one here I think we exemplified that (4)

Section 3.5 Bi-vector components

$$F^{\mu\nu}=\gamma^{\mu}\cdot\boldsymbol{F}\cdot\gamma^{\nu}=\gamma^{\nu}\cdot\left(\gamma^{\mu}\cdot\boldsymbol{F}\right)=\left(\gamma^{\nu}\wedge\gamma^{\mu}\right)\cdot\boldsymbol{F}=\gamma^{\nu\mu}\cdot\boldsymbol{F}=\left(\gamma_{\mu\nu}\right)^{-1}\cdot\boldsymbol{F} \tag{4}$$

You should feel pretty comfortable, if you can calculate these components out from this these dot products here these various dot products, if you have the ability to interpret, that's the contraction part of a bi-vector bi-vector space-time product and then execute that full contraction, if you can see, well that's the contraction part of a bi-vector vector space-time product and then you execute that and then you say oh well then what's left is the contraction part of a vector vector space-time product because this contracts to a vector so this contracts to a scalar well that's when I'm after a scalar. Hopefully that was the point of the last lesson, was to develop some comfort with that and if you need a little exercise just repeat them for the covariant components of the tensor $\boldsymbol{F}$, we haven't really called it a tensor yet, we're still calling it as a bi-vector $\boldsymbol{F}$ so look for the covariant components of the bi-vector and this calculation runs exactly the same way, I think the down indices become up into season and but you should get that comfort and now we can begin our reading here.

The components, the contravariant components are defined this way (4), we did all that and they “can be extracted using the contractions defined in Section 3.3.1”, which we did, “they're called the rank $(2,0)$ (contravariant) components of $\boldsymbol{F}$, while these lower components $F_{\mu\nu}=\gamma_{\mu}\cdot\boldsymbol{F}\cdot\gamma_{\nu}$ are called the rank $(0,2)$ (covariant) components of $\boldsymbol{F}$.” This language rank $(2,0)$ and rank $(0,2)$ have to do with how tensor analysis describes tensors and that's really going to be the subject of today's lesson so we're going to come back to this material here in a minute. Moving on, “similarly the (mix) components $F^{\mu}{}_{\nu}=\gamma^{\mu}\cdot\boldsymbol{F}\cdot\gamma_{\nu}$ and $F_{\mu}{}^{\nu}=\gamma_{\mu}\cdot\boldsymbol{F}\cdot\gamma^{\nu}$ are rank $(1,1)$ (matrix) components for $\boldsymbol{F}$ ”, they're called the matrix components for $\boldsymbol{F}$, again, we'll catch up with that in a moment. “There are overtly 16 real components in each case, of which only $(16-4)/2=6$ are non-zero and independent due to the anti-symmetry; these six independent components correspond precisely to the components of (the bi-vector) $\boldsymbol{F}$ when expanded more naturally in a bi-vector basis.”

“When expanded more naturally in a bi-vector basis.”, well they actually did expand it in a bi-vector basis (3) so I'm not quite sure what the “more naturally” is, what I believe they're getting at here is, I mean, when they write “more naturally”, I think what they're saying here is this, “there are 16 real components in each case of which only six are non-zero and independent”, fair enough so what they're saying is if you treated these components as though they were each an independent number then you would end up with 16 independent numbers and then you study the anti-symmetry and you realize it looks like 16 but only six are truly independent but then they're saying but if you just took a bi-vector and expanded in the reference or the reciprocal basis you automatically only get six independent components right from the fact that you only have six basis vectors. Notice however, the summation is still going over 16 objects so this is not all that far away from just treating $F_{\mu\nu}$ as a 16 object, a collection of 16 numbers and then thinking, oh yeah, but half of them are the opposite of the other half and four of them are zero so if you really wanted to do this where it was truly expanding $\boldsymbol{F}$ in a basis of six vectors, you'd have to really use the other notation which would be:

$$\sum_{\mu<\nu} F^{\mu\nu}\gamma_{\mu}\wedge\gamma_{\nu} \tag{5}$$

Notice there's no one half there and you are only summing over six possibilities so that is if you really only summed over these guys right here where $\mu<\nu$, yeah then it would be a more natural thing but basically they've thrown away that natural advantage by using this particular form of the expansion, that's totally a quibble by the way because the point is still well taken, if you wrote $\boldsymbol{F}$ in terms as a tensor you would definitely have 16 components so let's have a look at that.

Section 3.5 Tensor components

Consider $\boldsymbol{F}$ as a tensor, the way this $\boldsymbol{F}$ would be written as just a pure straight up tensor it's an anti-symmetric tensor is it would have components $F^{\mu\nu}$ because this is going to be a rank $(2,0)$ tensor so in the work of tensor analysis on a manifold, in a tangent space we would write this tensor $\boldsymbol{F}$ as:

$$\boldsymbol{F}=F^{\mu\nu}\partial_{\mu}\otimes\partial_{\nu} \tag{6}$$

Those are the basis vectors in the tangent space, these partial derivative operators, these directional derivative operators, moreover $\otimes$ would be a tensor product. With tensor products it's completely fine to have $\partial_0\otimes\partial_0$. You can have the indices be the same, that's a legitimate basis vector in the tensor product space so is $\partial_1\otimes\partial_2$ and $\partial_2\otimes\partial_1$, these two are different in a true tensor product space so a tensor product space is also a vector space, a rank $(2,0)$ tensor product space has 16 basis vectors and it's all combinations of $\partial_{\mu}\otimes\partial_{\nu}$. Each of these is a legit basis vector and there's 16 of them now if $\boldsymbol{F}$ is anti-symmetric the first thing you realize is that all $\boldsymbol{F}$ all the components where $\mu=\nu$, those go to zero and that'll be four go to zero and if $\boldsymbol{F}$ is anti-symmetric, you'll end up with $F^{12}\partial_1\otimes\partial_2$ and then you'll have $-F^{12}\partial_2\otimes\partial_1$ as two of the components of this summation, this is an Einstein sum (6), it would have been $+F^{21}\partial_2\otimes\partial_1$ but because it's anti-symmetric I switch it with $-F^{12}\partial_2\otimes\partial_1$ like that and when I do that I can pull out the $F^{12}(\partial_1\otimes\partial_2-\partial_2\otimes\partial_1)$ and I can do this exact little maneuver for all pairs of μ and ν that are not equal and where $\mu<\mu$ and when I do that I end up with basis vectors combined in this way for all of these pairings and if I take that basis vector and I give it a new name, how about $\partial_1\wedge\partial_2$? That is literally defined to be $\partial_1\wedge\partial_2=\partial_1\otimes\partial_2-\partial_2\otimes\partial_1$ so I create this basis, and there's only six possible ways of doing this so this is the origin of the wedge product in tensor analysis, you don't think of it in terms of a little plane and geometric space though, we're thinking of it in terms of of of the basis of the tensor product space.

If you were to blow it up this way (6), then yes you have to maneuver your head around the fact that you've got to combine basis vectors together but you still can get to the 6 dimensionality, the difference is Geometric algebra starts right here it starts right here so this issue about dimension six is built into the thing, now the counter argument is, well so that's great as long as $F^{\mu\nu}$ is anti-symmetric, what if I have a tensor $A^{\mu\nu}$ that's not anti-symmetric? How does that tensor live in the Geometric algebra, and the answer is it doesn't, it doesn't so you are sacrificing this, this tensor product space is too big an architecture to really be necessary for the study of electromagnetism, I guess is our theory here, this big architecture certainly covers what we need but it's a little extra unwieldy plus it lacks this geometric interpretation and I feel like that's what the claim is.

Back to the paper, "these six independent components correspond precisely to the components of $\boldsymbol{F}$ when expanded more naturally in a bi-vector basis. The factor of one-half in (3.21) ensures that the redundant components (different only by a sign) are not double counted." That factor of one-half just shows they're not really expanding this in the bi-vector basis, it is with a full level of commitment that they could, I guess what I'm getting at is I really do prefer that other form of notation where $\mu<\nu$ and because they don't use it they're being a little hypocritical here in my opinion but they are moving on. "The various ranked tensor components corresponding to $\boldsymbol{F}$ all refer to the action of the same anti-symmetric tensor $\underline{F}$, which is a multi-linear *function:*

$$\begin{aligned}\underline{F}(a,b) &\equiv a\cdot\boldsymbol{F}\cdot b = b(a\cdot\boldsymbol{F}) = (b\wedge a)\cdot\boldsymbol{F}\\ &= \sum_{\mu\nu} a^{\mu} F_{\mu\nu} b^{\nu} = \sum_{\mu\nu} a_{\mu} F^{\mu\nu} b_{\nu} = \sum_{\mu\nu} a_{\mu} F^{\mu}_{\nu} b^{\nu} = \sum_{\mu\nu} a^{\mu} F^{\nu}_{\mu} b_{\nu}\end{aligned} \tag{7}$$

They're introducing new notation here, the classic anti-symmetric tensor $\underline{F}$ corresponds to the bi-vector $\boldsymbol{F}$, "the various ranked tensor components corresponding to $\boldsymbol{F}$", now notice "various rank tensor components", what they're saying there is that you can choose whether to express $F^{\mu\nu}$ in the reference basis in which case you get these contravariant components or you could express $F_{\mu\nu}$ in the reciprocal basis in which case you get these covariance components or you could express $F_{\mu}{}^{\nu}$ in this mixed basis and then you get the so-called "(matrix) components for $\boldsymbol{F}$" and that's where the word "various", that's how this word "various" appears in here, the various rank tensor components corresponding to $\boldsymbol{F}$, they all refer to the same $\boldsymbol{F}$ though, that's the beauty of this geometric interpretation and it's also true in the tensor analysis interpretation but in the geometric interpretation you immediately think of $\boldsymbol{F}$ as some little circulating plane element and that thing has an identity independent of what basis you choose and so all of these different components can correspond to, depending on how you decide to cast the basis, reciprocal or reference, in which reference basis you actually choose.

Anti-symmetric tensor

The anti-symmetric tensor, they're reminding us defined tensors are actually defined as multi-linear *functions*, that take either vectors or forms to give numbers, now if you look at the way the paper goes after it, it's a multi-linear function $\boldsymbol{F}$ that takes two vectors and returns a real number, that's how a tensor works, they don't literally say that they say it's a multi-linear function but they don't remind you that it's a multi-linear function that returns a real number when all of its arguments are provided so in this case $\boldsymbol{F}$ which is a $(0,2)$, in this case the way they're defining $\boldsymbol{F}$ so that means it looks for two vectors and returns a real number, that's that's what a $(0,2)$ rank tensor does, it eats two vectors and returns a real number. In the language of this paper, if you were a $(2,0)$ rank tensor you would eat 2 reciprocal vectors and return a real number and in the language of tensor analysis you would eat 2 one-forms and return a real number so there is a bunch of language issues here that well we probably should talk about it so let's go ahead and talk about it.

Tensor Analysis

In the regular tensor analysis process of a tensor $\boldsymbol{F}$ you think of the tensor $\boldsymbol{F}$ as a multi-linear function and you can think you can cast the tensor $\boldsymbol{F}$ in the basis of vectors in the tangent space of a manifold which is how we studied it in our QED electromagnetism course in which case you get these partial derivatives and it's going to eat two objects that are one-forms and produce a number so when you calculate this whole thing out you end up with a number but notice the function, this is a function here this is a multi-linear function and these are the arguments of the multi-linear function and they're both one-forms likewise if you cast the tensor $\boldsymbol{F}$ in its form that has covariant components this is the $(0,2)$ form and this is the $(2,0)$ form:

$$\begin{aligned}\boldsymbol{F} &= \left[F^{\mu\nu}\,\partial_{\mu}\otimes\partial_{\nu}\right]\left(a_{\alpha}\,dx^{\alpha},b_{\beta}\,dx^{\beta}\right) \;\rightarrow (2,0)\\ &= \left[F_{\mu\nu}\,dx^{\mu}\otimes dx^{\nu}\right]\left(a^{\alpha}\,\partial_{\alpha},b^{\beta}\,\partial_{\beta}\right) \;\rightarrow (0,2)\end{aligned} \tag{8}$$

This guy is looking to eat two vectors and return a real number argument, so this guy looks for two vectors and here we've cast the vector a and the vector b in their contravariant forms and that means the components by the way are the contravariant forms the basis vectors are covariant (2nd line of (8)) and here the basis vectors are contravariant and the components are covariant (1st line of (8)) but when we talk about the form of it, we talk about the form of the components so you have these two options for casting a tensor $\boldsymbol{F}$ and you move between one or another using the metric and there's some other rules here I just don't want to get into too much of this but ultimately these are functions and the way the functions are calculated is by understanding that a form dx^{α} takes a vector a basis element of, I guess the one-form basis element dx^{α} takes the one vector ∂_{β} and returns:

$$\left\langle dx^{\alpha},\partial_{\beta}\right\rangle=\delta_{\beta}^{\alpha} \tag{9}$$

When you use that fact you can immediately see how the entire result of this process is driven by these components so these components actually do the calculation and if you were actually to execute this calculation, if you were to to sit down and execute this calculation the way it would look is you would end up with pulling out all the linear terms because everything is linear in this business and then:

$$\boldsymbol{F}=F^{\mu\nu}\,a_{\alpha}\,b_{\beta}\left\langle dx^{\alpha},\partial_{\mu}\right\rangle\left\langle dx^{\beta},\partial_{\nu}\right\rangle=F^{\mu\nu}\,a_{\alpha}\,b_{\beta}\,\delta_{\mu}^{\alpha}\,\delta_{\nu}^{\beta}=F^{\mu\nu}\,a_{\mu}\,b_{\nu} \tag{10}$$

That is how you calculate the number that results from taking these two vectors and feeding it to this tensor and so this is the process you would normally use in tensor analysis and then by the way this process (2nd line of (8)) will give you the exact same answer and so they point that out right here (7), here is the tensor, a, b and here is how it's calculated, which is exactly the way I just showed you using the tensor analysis formula, I demonstrated this version (2nd term on 2nd line of (7)) just a split second ago I demonstrated this version right here but this is the second version (1st term on 2nd line of (7)) which is actually they favor and then there's this mixed version (3rd term on 2nd line of (7)), you can do all of these versions but what they want to say is, the way we want to understand it in this paper is we're going to define the way to calculate this through the bi-vector $\boldsymbol{F}$ and these dot products are going to be these, this is going to be the bi-vector vector contraction part and then what's left will be the vector vector contraction part and you'll get a real number and they show you there's three ways of interpreting this but they all lead to the same real number which is given by this (7) and notice it's calculated exactly the same way as you would have done in regular tensor analysis.

Here's what they're saying: " $\underline{F}$ is a multi-linear function that takes two vector arguments" so when it says it takes two vector arguments they're automatically saying that the tensor calculus part wouldn't say that, it would say the $(0,2)$ rank tensor that takes two vector arguments but the $(2,0)$ rank tensor takes two one-form arguments but in the work we're doing there are no one-forms there's just vectors in the reference basis and then vectors in the reciprocal basis so they've eliminated this one-form notion and replaced it with reciprocal basis vectors but reciprocal basis vectors and reference basis vectors are both literally vectors in the same vector space whereas these one-forms don't live in the same vector space as these basis vectors, these guys live in the vector space V and these guys live in the vector space V^* , those are not the same vector spaces but the language that we're working with now is different, that we do understand that it "takes two vector arguments" so it's OK for the paper to say this is a multi-linear function on two vectors even though sometimes in tensor analysis, no it's it's takes forms but because in this paper the forms are really replaced by these reciprocal vectors and everything in sight is a literal vector so anyway.

The point being is that it now takes two vector arguments a and b but you have to decide what basis to use, the reference basis or the reciprocal basis so that's getting at the connection between these things. "And produces a scalar through the total contraction with the bi-vector $\boldsymbol{F}$. We use the under-bar notation $\underline{F}$ for functions to disambiguate them from products of multi-vectors." I guess they're saying they're treating this as a function, this $\underline{F}$ is a function, well the prescription for how to calculate the number that pops out of this function is this double contraction $a\cdot\boldsymbol{F}\cdot b$ of the bi-vector $\boldsymbol{F}$ with the arguments of the function so that's their formula and for all practical purposes these are the ways of calculating that number of executing this formula (7). Now, when we practiced it in the last lesson we broke it down piece by piece but in the end this is what we ended up with, so that's good.

I guess in the end we ended up with with this but we we we're using basis vectors so we ended up with δ functions in various places but it's the equivalent formula. This is an illustration of how this calculation is done, this is supposed to be the tensor function $\underline{F}(a,b)$ that takes two vectors a and b and returns a real number and that function is calculated using this formula $a\cdot\boldsymbol{F}\cdot b$ as one of several options but here I'm blowing it up for you:

$$\begin{aligned}\underline{F}(a,b)=a\cdot\boldsymbol{F}\cdot b &= ½\left(a_\mu\gamma^\mu\right)\cdot F^{\alpha\beta}\gamma_\alpha\wedge\gamma_\beta\cdot\left(b_\nu\gamma^\nu\right)\\ &= ½F^{\alpha\beta}a_\mu b_\nu\left[\gamma^\mu\cdot(\gamma_\alpha\wedge\gamma_\beta)\cdot\gamma^\nu\right]\\ &= ½F^{\alpha\beta}a_\mu b_\nu\left[\gamma^\mu\cdot\left[(\gamma_\beta\cdot\gamma^\nu)\gamma_\alpha-(\gamma_\alpha\cdot\gamma^\nu)\gamma_\beta\right]\right]\\ &= ½F^{\alpha\beta}a_\mu b_\nu\left[(\gamma_\beta\cdot\gamma^\nu)(\gamma^\mu\cdot\gamma_\alpha)-(\gamma_\alpha\cdot\gamma^\nu)(\gamma^\mu\cdot\gamma_\beta)\right]\\ &= ½F^{\alpha\beta}a_\mu b_\nu\left[\delta^\nu_\beta\delta^\mu_\alpha-\delta^\nu_\alpha\delta^\mu_\beta\right]\\ &= ½\left(F^{\mu\nu}-F^{\nu\mu}\right)a_\mu b_\nu=F^{\mu\nu}a_\mu b_\nu\end{aligned}\tag{11}$$

In this procedure we basically did the same procedure we did in the last lesson but now I have to introduce the reciprocal basis form for a and the reciprocal basis form for b and I'm using the reference basis form for $\boldsymbol{F}$ and we just do the same thing, we pull out all the real numbers to go in front so this comes out this goes out this goes out they all land here at the front and then what's left is just the manipulations with basis vectors which are always simplified by the fact that the basis vectors are orthogonal and it's interesting because this is all about the space-time product of things but we really find a lot of our work is done with the wedge, the inflation and contraction operators, the wedge operator and the dot operator. The wedge operator usually only has one meaning but the dot operator can have several meaning and in this case you have the dot operator between a bi-vector and a vector and we've memorized this expansion by now (2nd line of (11)) to be (3rd line of (11)), it cleans up being a vector itself with components that are given by these Minkowski dot products, these Minkowski contractions and then you're left with this vector vector dot product so right here in these two lines you see all three of the versions of the dot product. One is bi-vector vector dot product, another is vector vector dot products which are Minkowski contractions and one is also vector vector, that's also a vector

vector dot product so you see both of them you see, in the 2nd line of (11), it's vector by-vector and here in the 3rd line of (11), it's a vector vector. You're left with the answer and that's exactly the answer that they want you to see here (7) that's this technique, where they use the summation and we just maintain the Einstein summation by implication

That is how they want to define the tensor is through this bi-vector exercise and ultimately it's entirely equivalent of course but "We use the under-bar notation $\underline{F}$ to disambiguate them from products of multi-vectors. Note that the different ranks of components for the tensor $\underline{F}$ correspond to different ways of expanding the arguments a and b into different bases.", exactly so if I choose a to be in the reference basis $a=a^{\mu}\gamma_{\mu}$ then I have to use this formula $a^{\mu}F_{\mu\nu}b^{\nu}$. If a is in the reciprocal basis $a=a_{\mu}\gamma^{\mu}$ I have to use this formula $a_{\mu}F^{\mu\nu}b_{\nu}$, it's that simple and that's it for that paragraph.

Moving on, "Importantly the electromagnetic field is intrinsically a bi-vector $\boldsymbol{F}$ and *not* its associated anti-symmetric tensor $\underline{F}$, which is the multi-linear function that performs contractions with $\boldsymbol{F}$ to produce a scalar; the confusion between these two distinct concepts arises because they have the same characteristic components $F^{\mu\nu}$." That goes back to this analysis I did earlier (8), where I described a $(2,0)$ and $(0,2)$ tensor $\boldsymbol{F}$ and I put it in the basically the tensor product space and I said look the components are exactly the same as the bi-vector components but this object here is distinct from this object here (11), this bi-vector, the question now is how distinct is it really? Remember how we can think of these things as, we combine these basis vectors together and we redefine these basis vectors this pair of basis vectors as a wedge product between the basis vectors, that is how it's done in tensor analysis and I mean that is a mathematical formal equivalence between that and these bi-vectors, this bi-vector form but they're emphasizing, I guess the distinction is this function $\underline{F}$ this is a literal function that eats vectors whereas $\boldsymbol{F}$ is a geometric object and I think that's the distinction, they both do have the same components when expressed in the basics in this case, it's the basis of geometric objects that are little planes and this is the basis of of functions that take forms or vectors to real numbers and they have the same component so it's easy to mix up the two, I guess that's what they're getting at there.

Component Based Analysis

"Component-based tensor analysis obscures the subtle conceptual distinction by neglecting the k -blades themselves in favor of functions that can be defined by contractions with these k -blades, all while emphasizing component descriptions that depend on a particular basis expansion." That I get so what they're talking about there is when you study tensor analysis or certainly when you do it in Physics, when you study it in the context of Physics you think of $F^{\mu\nu}$, this bunch of components, that you think of as the tensor itself, we interpret this literally as a tensor, we don't use these basis vectors that I keep talking about here, they never show up in almost all treaties on Physics that deal with tensors, they entirely deal with this component-based analysis and therefore you start thinking of these things as these contractions, these procedures to calculate real numbers and that really does hide it, not only hides the k -blades but it hides these basis vectors as well, that whole process hides everything and it's very confusing to learn for students which is why I made this whole lecture series called "What is a tensor?" It was to address this exact point but they're basically saying they're going even further by neglecting k -blades, you're neglecting a geometric interpretation of tensors and it's not like there's never been an attempt to do a geometric interpretation of tensors before Geometric algebra, the first time I saw it plain as day was in Misner, Thorne and Wheeler in their legendary General relativity book "Gravitation", they go out of their way to do geometric interpretations of things and it's not the same as what's in Geometric algebra because they're more interested in the geometric interpretation of forms as opposed to vectors and so shifting this to the other side of the picture from the form picture to the vector picture that's fun and it is interesting but it's not like it's never been done before.

Going on, “more distressingly, one cannot construct more general multi-vectors or the Clifford product using tensor notation. As a result, we regard component-based tensor analysis as a part of but not a replacement for the space-time algebra used in this report.”, so that's true, this this notion of multi-vectors makes no sense in standard tensor analysis because the $(0,2)$ tensor vector space, that's which has basis vectors that look like this (8), these tensor products between these these partial derivatives you can't add anything from this vector space to this vector space $(2,0)$, there is no addition operation defined between different tensor product spaces you certainly can never construct anything that looks like this $\partial_\mu+\partial_\nu\otimes\partial_\alpha$, you could never construct anything like that, you could never add vectors to tensors, you just can't do it in tensor analysis, there is some forms of exterior calculus or Exterior algebra where they do try to get that going but they don't quite get it to the extent that or I don't recall it being quite as robust as it is in Geometric algebra.

Summary

That's cool and so where does that leave us? Well that leaves us having made the connection with standard tensors and now we are in Section 3.5 *The pseudo-scalar I, Hodge duality and complex structure*. This is a really really important Section and we will attack it next time we're going to start talking about the duality between the different pieces of the Clifford algebra that we're using here or the space-time algebra that we're using here and this is fun now we have an understanding of how tensors work here if you didn't pay follow all of the stuff about tensors because you're coming from a place where you don't know much about tensors I do recommend you watch my playlist “What is a tensor?” The first 10 or 11 lessons do a pretty good job I think of getting at this subject but with that we will begin Section 3.5 in the next lesson so I'll see you then.