Maxwell's Equations via Differential Forms Part 3

Welcome back, today we are going to continue our hunt for Maxwell's equations via differential forms. Excellent exercise, we've already reviewed our notion of differential forms and now we're going to start exploiting that use to reconstruct Maxwell's equations using those differential forms so let's begin by taking a quick look at the electric field as we understand it in regular electricity and magnetism physics meaning our elementary physics expression, our elementary engineering expression we treat the electric field as a vector in the sense of a little pointy thing that has magnitude and direction:

$$\boldsymbol{E}=E_x\hat{x}+E_y\hat{y}+E_z\hat{z}=E^1\partial_1+E^2\partial_2+E^3\partial_3 \tag{1}$$

This notion of the word vector, we're changing a lot now right we've we've discussed vectors in various courses on this channel in many different ways and the least way we've ever discussed vectors is a little pointy object with magnitude and direction, that's the first thing you know when you come into this class but it's the last thing we use to actually discuss a vector. Ultimately, we're trying to get away from that idea but now we're going back to this notion of the vector as you learn in elementary physics. Often this is called in books that discuss this stuff meaning discuss differential forms they call this the vector in the sense of Gibbs which is basically vector analysis as you learned in college and high school even, the Gibbs vector analysis formalism is what we're taught.

That's a totally fine formalism, in fact it's the one that is by far dominant in the education in the field of this subject and that's what I mean by this guy right here (1). We have a component of the electric field in the x direction and this $\hat{x}$ is its unit vector in the x direction, component in the y direction, the unit vector in the $\hat{y}$ in the y direction, the unit vector $\hat{z}$ in the z direction and the component in the z direction. Now we take this notion of the vector and we're going to start moving it into the more sophisticated versions of things and what we realize is that our electric field exists on some manifold, we have some manifold that has some coordinate system and x, y, z Cartesian coordinate system and now this being a manifold, it's a very simple one, it's an x, y, z spatial manifold only, there's no fourth dimension there's no time in it yet so with this manifold it's very easy to say we can just take this unit vector and immediately translate it into the language of vectors in a tangent space on a manifold, meaning what we can now do is we can now say that these unit vectors are actually differential operators that work in the tangent space and we now presume that we have a tangent space at each point of space in the manifold and that's how we describe vectors on a manifold.

We've already gone from the elementary notion to this more sophisticated notion, now you can do it by direct substitution you directly take these unit vectors and you just say we're going to use unit vectors on a manifold so those are differential operators so you're almost just doing a symbolic rewrite and that's the first and easiest step but notice that I was very fond of these subscripts for the x, y, z in our standard practice but now I turn them into superscripts in order to be able to exploit ultimately the notion, the Einstein summation convention $E^\alpha\partial_\alpha$. Now we start going to this up down convention once we make this change into a vector field on a manifold so that's important to keep an eye on.

Since we're still talking about regular space and Newtonian physics really, we note that the inner product on all of these tangent spaces or the metric as you will on the manifold is just your standard Euclidean metric which means that the inner product $(\partial_\alpha, \partial_\beta)=\delta_{\alpha\beta}$ in each tangent space and because of this the values of E whether the subscript is up or the subscript is down is irrelevant, it's not affected by going to the dual space and because we've chosen this metric we have a one-to-one correspondence between all the vectors that live in the tangent space and all the forms that live in the cotangent space so for every electric field vector in our manifold we have an easy one-to-one correspondence with an electric field co-vector and we just simply replace the vector vector operators there, the unit vectors with unit differential forms:

$$E_1\, \mathrm{d}x^1 + E_2\, \mathrm{d}x^2 + E_3\, \mathrm{d}x^3 \tag{2}$$

This one-to-one correspondence, right here between these two is all driven by our choice of the metric and what's more important is this choice of the metric means that this particular choice of the metric this trivial choice where we have this δ delta function it means that these super-scripted versions and these subscript inversions are the same. Had we chosen the metric of Minkowski space-time which is a four-dimensional space time and we will do this we would have had an E_0 here but the E_0 would not have been the same it would have been $E^0 = -E_0$ because the metric there includes $(\partial_0, \partial_0) = -1$ or sometimes they depend on your convention sometimes these guys here are all equal to -1 and this guy equals $+1$ depending on the convention.

The point of this though is that the way we're going to look at our Maxwell's equations today in three dimensions or at least right now in three dimensions with a regular standard Euclidean metric, all of this stuff is basically equivalent right there's isomorphisms between this vector space V and this vector space V^* and then there's an isomorphism between this vector space (1) and this vector space (2) these isomorphisms are slightly different and what drives them this (1) is purely a notational thing, we're not exploiting the machinery of tangent spaces when we do your regular Gibbs vector algebra but as soon as we start talking in terms of manifolds which is the real mature grown up way to do this then we need these vector operators to tell us that we're dealing with a vector that lives in the tangent space at every point so that transition here is really more of just a formalism translation because in principle the Gibbs notation is living in the tangent space as well.

We're just not concerned, we don't use all the rest of the machinery of differential form so we just ignore it and we just give it these little letters, it's almost just a purely symbolic change between the first and second parts of (1) but from going from (1) to (2) that's a statement about the metric on the tangent spaces, that's the statement about this inner product for each tangent space and the metric on the manifold and because it's Euclidean we can make these equivalences and all of this up down index machinery is a bit hidden under the hood because the metric is so trivial and there are plenty of books that do in fact, I would say make things even more confusing by just going ahead and putting the subscripts down here right and they'll even put the subscripts on these forms down here. I've seen papers do that, they don't care because they know the metric is the regular Euclidean metric and none of this really matters but for what we're talking about right now I want to make sure that we understand that we can move very freely between these three objects in Euclidean space and actually the way I've done it you can even move freely between these two objects in Minkowski space as long as you keep track of the up or down indices because the metric will take care of the sign differences in this case. Here we have the electric field as a vector an electric field as a one form and this movement between them is completely free we are allowed to make those changes. What's next? I want to move away from this for a moment and go back to our quick review that we did in a previous lesson about differential forms and show you that we studied the exterior derivative of a function on a manifold and we came up with this expression:

$$\mathrm{d}\mathrm{f} = \partial_\eta \mathrm{f}\ \mathrm{d}x^\eta \tag{3}$$

Note this uses Einstein notation.

What's important to see is if we flush this out what do we get let's do it we get this very obvious I mean I've just blown up the Einstein sum and we get:

$$\mathrm{d}\,\mathrm{f}=\partial_1\mathrm{f}\ \mathrm{d}x^1+\partial_2\mathrm{f}\ \mathrm{d}x^2+\partial_3\mathrm{f}\ \mathrm{d}x^3=\frac{\partial\,\mathrm{f}}{\partial\,x^1}\ \mathrm{d}x^1+\frac{\partial\,\mathrm{f}}{\partial\,x^2}\ \mathrm{d}x^2+\frac{\partial\,\mathrm{f}}{\partial\,x^3}\ \mathrm{d}x^3 \tag{4}$$

When you look at this we have a one form, this is a one form you can see the one form unit co-vectors there and the derivative is a number because we are presuming that this is a one form given place and time as it's written this is a one form on the manifold because f is a function, remember f is a function of x^1, x^2, x^3 which is the equivalent of saying that f is a function of x, y, z in the Gibbs presentation but it's the same thing so the derivatives of f are also a function of x^1, x^2, x^3 on the manifold but multiplied by these unit co-vectors so (4) is a one form on the manifold but at any given place on the manifold each of these functions takes on a value and suddenly becomes a real number and so at each point you end up with a real number and ergo at each point you end up with a no kidding co-vector which is a scalar real number multiplied by the one form unit vector.

We've now created this one form, we've taken the exterior derivative of a function by this definition that we studied earlier gives us this expression right here (4) and now we execute our freedom to turn that into a vector so how do we do that well I'm going to take this I'm going to now simply erase these one forms right and I am going to replace them with vectors:

$$\nabla\,\mathrm{f}=\frac{\partial\,\mathrm{f}}{\partial\,x^1}\partial_1+\frac{\partial\,\mathrm{f}}{\partial\,x^2}\partial_2+\frac{\partial\,\mathrm{f}}{\partial\,x^3}\ \partial_3 \tag{5}$$

Now we have a vector living at every point in space time, we can do that because of this immediate correspondence between vectors and co-vectors and I could go as far as instead of writing those I could have written $\hat{x}, \hat{y}, \hat{z}$, the point being that if you write it with $\hat{x}, \hat{y}, \hat{z}$ your elementary brain should be going, wait a minute this now as written in a vector form is simply the gradient, that is literally the definition of the gradient of a function as we've learned from vector calculus so the point is that the gradient of a function is very tightly connected to the exterior derivative of a function $\nabla\,\mathrm{f}\sim\mathrm{d}\,\mathrm{f}$, in fact they're entirely equivalent and that exploitation is pretty important because we're going to now start doing that thing connecting Gibbs notions to differential form notions for everything we do.

Before I move on I do need to point out this is not an equality right I did not write gradient function equals the exterior derivative of a function because it literally isn't, you've got this issue of you end up with a form on this side and this is our vector notion so I went with I went with this symbol something like I guess an arrow like you can replace the gradient with the differential with the exterior derivative understanding that we have to always track between these isomorphisms between forms and vectors and this Gibbs unit vector idea so the gradient of a function is tightly bound to its being a differential operator and in fact you can see the coefficients are the same they all have unit vectors so you can make these replacements, this is an important replacement, we're going to probably be exploiting in order to write our our Maxwell's equations in the language of differential operators because you know there's a lot of gradients in Maxwell's equations but they're not gradients of scalar functions they're gradients of vector fields so we still have a little work to do but this is a start and so let's go to the next step here.

Now we are going to study this notion of the divergence of a vector field so the way this is written here of course is the standard electricity and magnetism first year course vector analysis expression for the divergence of a vector field $\boldsymbol{E}$:

$$\nabla\cdot\boldsymbol{E}=\partial_1 E_1+\partial_2 E_2+\partial_3 E_3 \tag{6}$$

When I say standard I mean those the Gibbs vector analysis vector field, that's what this is and this is the divergence which is always expressed in the basic Gibbs form as follows:

$$\nabla=\partial_x\hat{x}+\partial_y\hat{y}+\partial_z\hat{z} \tag{7}$$

If I wrote it in just the straight up way that everybody's used to seeing it this is this gradient operator, it's a gradient vector field but it's got these differential operators for components and so the divergence of a vector field is given by this simple expression when the vector field we're talking about is the electric vector field $\boldsymbol{E}$ expressed in this form here (6) so what we're going to do is we're going to see how we can translate this into the language of differential forms and we're going to do this step-wise It's motivated by how it ends so we'll just do it step by step and we'll see that we land in the right place.

For example, first thing I'm going to do is I'm going to translate this electric vector field of standard classical mechanics into the differential forms version right and that is a free movement so all of a sudden this object now is moved from the Gibbs form we're just immediately dropping it off into the one form, I shouldn't say Gibbs form I don't want to use that word so I want to say the language of Gibbs or the language of standard vector calculus to the language of one forms and this movement is absolutely free. That's the first step we do and then we are going to find its Hodge dual right and we know how to do the Hodge dual because we did that in the entire last lecture so we know the Hodge tool is linear over the coefficients of the one form so we pull out this E_1, E_2, E_3 and we're now looking at the Hodge dual of the basis vectors which we calculated last time:

$$\begin{aligned}*E&=E_1\left(*\,\mathrm{d}x^1\right)+E_2\left(*\,\mathrm{d}x^2\right)+E_3\left(*\,\mathrm{d}x^3\right)\\&=E_1\left(\mathrm{d}x^2\wedge\mathrm{d}x^3\right)-E_2\left(\mathrm{d}x^1\wedge\mathrm{d}x^3\right)+E_3\left(\mathrm{d}x^1\wedge\mathrm{d}x^2\right)\end{aligned} \tag{8}$$

With all that in place we now have calculated the Hodge dual of the electric field (8). Notice the Hodge dual of a one form, the electric field is a one form, the Hodge dual of a one form in three dimensional space is a two form and we did all those calculations to go from the two lines of (8) in the previous lecture so you should be comfortable with that. It was totally not trivial because we had to figure out what this sign was and sometimes the sign throws you for a loop and you end up with a minus sign there but where you don't develop a sign in three-dimensional Euclidean space is you don't develop a sign because of the inner product having a -1 in its signature, you don't have that in three dimensional space, you're do in four but not in three. We have calculated the Hodge dual of the electric field $\boldsymbol{E}$.

Our next step is to calculate the exterior derivative of the Hodge dual of the electric field so let's now do that. so to calculate this exterior derivative I'm going to use the formula and method that we talked about in the previous lectures it comes down to finding this one form sum of the components, you form a one form of the components and then you wedge it with the two form basis vectors.

$$\begin{aligned} \mathrm{d}(*\boldsymbol{E}) = & \left[\partial_1 E_1 \, \mathrm{d}x^1 + \partial_2 E_1 \, \mathrm{d}x^2 + \partial_3 E_1 \, \mathrm{d}x^3\right] \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ & - \left[\partial_1 E_2 \, \mathrm{d}x^1 + \partial_2 E_2 \, \mathrm{d}x^2 + \partial_3 E_2 \, \mathrm{d}x^3\right] \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 \\ & + \left[\partial_1 E_3 \, \mathrm{d}x^1 + \partial_2 E_3 \, \mathrm{d}x^2 + \partial_3 E_3 \, \mathrm{d}x^3\right] \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \end{aligned} \tag{9}$$

The exterior derivative is always one form higher than the form that you're taking so the exterior derivative of a two form has to be a three form so here we have two form basis factors that we're summing over because we're summing over a collective index and the collective index is $2,3$, $1,3$ and $1,2$ but the answer has to be a three form. You need to express what multiplication you're dealing with and we're dealing with wedge multiplication. The formula we're using is this formula:

$$\alpha = \sum_I \alpha_I \mathrm{d}x^I \text{ and } \mathrm{d}\alpha = \sum_I \partial_\mu \alpha_I \, \mathrm{d}x^\mu \wedge \mathrm{d}x^I \tag{10}$$

We exercised this a couple lessons ago but just as a reminder of where this came from and voila that's what we're doing here (9), we're actually blowing it up so now we have found the exterior derivative of the Hodge dual of the electric field one form and that's what this thing is. What you see here is you have $\mathrm{d}x^3 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ for example so that one is zero because you have a condition where you have the wedge of $\mathrm{d}x^3$ with itself so that term goes away but one term survives per line. The three form in three dimensional space is one dimension in other words there's only one three form, it's $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, we shuffle these around and every interchange introduces a minus sign but in the end what do you get?

$$\mathrm{d}(*\boldsymbol{E}) = \left(\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{11}$$

Now we start realizing well look the coefficient of this looks a lot like the Gibbs definition of the divergence so the Gibbs definition of the divergence lands as the coefficient of the three form that results by taking the exterior derivative of the Hodge dual of the electric field but that's not what we want we just if we're interested in just purely extracting the divergence what do we have to do now? We have to get rid of unit one forms, well that's easy, we just take the Hodge dual of this object (11) and then that takes the three form and gives you a zero form so we're now interested in the Hodge dual of the exterior derivative of the Hodge dual of the electric one form field and what do we end up with?

$$*\mathrm{d}(*\boldsymbol{E}) = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \nabla \cdot \boldsymbol{E} \tag{12}$$

We end up with something that exactly equals the divergence of the Gibbs vector field $\boldsymbol{E}$ and that is an equality. That's really interesting so now we know how to write down the divergence of a three form, the divergence of a three form is the Hodge dual of the exterior derivative of the Hodge dual of that three form and that ends up being the divergence and you can see what we're going to do with this I mean ultimately Maxwell's equations there's there's a divergence right this first one is the divergence of the electric field this is literally what we want and here's the divergence of the magnetic field which written in Gibbs's language is written this way and so you can immediately see that this equation is:

$$\nabla \cdot \boldsymbol{B} = 0 \text{ is going to be } *\mathrm{d}(*\boldsymbol{B}) = 0 \tag{13}$$

Now this equation at least is written in the language of differential forms but we'll get to that a little more thoroughly when we actually do this, right now we have to understand how do we interpret these Gibbs expressions in terms of differential forms and so far we figured out only the divergence actually all of these divergences here to be true, with the Gibbs language should be written down as vectors:

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \rho \\ \nabla \times \boldsymbol{E} + \dfrac{\partial \boldsymbol{B}}{\partial t} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{B} - \dfrac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j} \end{cases} \tag{14}$$

Now we know the divergence of a one form is given by this expression here (12) so now let's have a look at a different process, we're now going to look for the curl. Let's have a look at that. With the curl we start with the Gibbs form of the magnetic field in this case, we could do this we could ask for the curl the electric field I'm just choosing to exemplify this with the magnetic field because we've already just done the electric field what's an interesting thing to understand here is that this vector form of the magnetic field, which we'll immediately write in this form in the language of the manifold:

$$\boldsymbol{B} = B^1 \partial_1 + B^2 \partial_2 + B^3 \partial_3 \;\rightarrow\; B_1 \,\mathrm{d}x^1 + B_2 \,\mathrm{d}x^2 + B_3 \,\mathrm{d}x^3 \tag{15}$$

In other words if I'm going to skip the Gibbs form all together, I'm just going to write $\boldsymbol{B}$ is this vector field on the manifold, it's exactly analogous to what we did up here with $\boldsymbol{E}$. We're just going to start right here, we're skipping this step now so we're starting right away with $\boldsymbol{B}$ being a vector field on the manifold, the magnetic field on the manifold and remember you may remember that there's axial and polar vectors in electromagnetism so we're still stuck with that right now because this is just really a regular vector but we know the property of magnetic fields is such that it behaves as an axial vector.

There's two words, there's axial and a polar vector in the Gibbs world. In the Gibbs world you have axial vectors and polar vectors and the magnetic field $\boldsymbol{B}$ is an axial vector and the electric field $\boldsymbol{E}$ is a polar vector so you have these two different kinds of vectors in the Gibbs world and a lot of what we're going to be discussing here attacks that and tries to create just know there's only one vector and we want to understand what's under the hood with this axial and polar vector thing going on and a lot of what I've been reading about lately is a form of electromagnetism based in this notion of something called Geometric algebra which completely eliminates this notion of axial and polar vector which is fascinating but right now we're just dealing with the magnetic field as a vector field understanding that it is an axial vector field so this is also called a pseudo vector because of its behavior under reflections but with that aside let's not worry about it right now we're just going to treat it as a vector field and we're immediately going to look at it in its one form version. Remember what we did up here in this example (8), the first thing we did is we took the Hodge dual right away now instead of doing that let's take the exterior derivative right away and the exterior derivative of a one form is very easy to calculate it's this is the formula (10) for it basically it's this these partial derivatives of the component fields and it's just wedged right into each of the components of the field itself and it ends up being a two form which of course is correct if we treat the magnetic field as a one form field which is totally free the exterior derivative of a one form field is a two form field defined this way:

$$
\begin{aligned}
\mathrm{d}\boldsymbol{B} &= \partial_{\mu} B_{\nu}\ \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\mu} \\
&= \partial_1 B_1\ \mathrm{d}x^1 \wedge \mathrm{d}x^1 + \partial_2 B_1\ \mathrm{d}x^2 \wedge \mathrm{d}x^1 + \partial_3 B_1\ \mathrm{d}x^3 \wedge \mathrm{d}x^1 \\
&+ \partial_1 B_2\ \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \partial_2 B_2\ \mathrm{d}x^2 \wedge \mathrm{d}x^2 + \partial_3 B_2\ \mathrm{d}x^3 \wedge \mathrm{d}x^2 \\
&+ \partial_1 B_3\ \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \partial_2 B_3\ \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \partial_3 B_3\ \mathrm{d}x^3 \wedge \mathrm{d}x^3
\end{aligned}
\tag{16}
$$

$$
\mathrm{d}\boldsymbol{B} = \left(\partial_1 B_2 - \partial_2 B_1\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\partial_1 B_3 - \partial_3 B_1\right) \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \left(\partial_2 B_3 - \partial_3 B_2\right) \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{17}
$$

We blow this guy up, blowing it up meaning executing the two implied sums we end up with all of these nine terms. Now we're looking right here at this expression (17) and it doesn't take too much squinting to see that this is beginning to look like the components of the curl of a vector field. The curl of a vector field is another vector field so we have we have the components that match up the curl but the unit vectors are actually two forms so this (17) is a two form whose components actually exactly match the curl of a vector field $\boldsymbol{B}$ so in other words if this was $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \rightarrow \hat{x}^3$, $\mathrm{d}x^1 \wedge \mathrm{d}x^3 \rightarrow \hat{x}^2$ and $\mathrm{d}x^2 \wedge \mathrm{d}x^3 \rightarrow \hat{x}^1$, if that's the way it was then that would be exactly the curl of the vector field $\boldsymbol{B}$ written with these components but that's not what we have what we have is a two form whose components happen to match that of a curl so if we want to extract the curl we have to convert these two forms back into one forms and then the one forms we would convert back into vectors but we know how to do this. Well it's so we have a tool to convert a two form into a one form now we're not certain that that tool will do what we want but let's give it a try.

$$
*\,\mathrm{d}\boldsymbol{B} = \left(\partial_1 B_2 - \partial_2 B_1\right) \mathrm{d}x^3 - \left(\partial_1 B_3 - \partial_3 B_1\right) \mathrm{d}x^2 + \left(\partial_2 B_3 - \partial_3 B_2\right) \mathrm{d}x^1 \tag{18}
$$

$$
\nabla \times \boldsymbol{B} = \left(\partial_1 B_2 - \partial_2 B_1\right) \partial_3 - \left(\partial_1 B_3 - \partial_3 B_1\right) \partial_2 + \left(\partial_2 B_3 - \partial_3 B_2\right) \partial_1 \tag{19}
$$

Now we have an expression for the curl of a vector field in the Gibbs sense is the Hodge dual of the exterior derivative of the equivalent form field and so now we can express curls and we can express divergences for one forms. Let's take a look at how we might exploit this and I just wrote a few very quick little lines here but say we were interested in the curl of the gradient of a function. Well we now know that we can take the gradient of a function and substitute its exterior derivative that's a substitution we discovered we can make very early on and now we just learned that the curl of a function becomes this:

$$
\nabla \times (\nabla \mathrm{f}) = *\,\mathrm{d}(\mathrm{d}\,\mathrm{f}) = *\,\mathrm{d}\,\mathrm{d}\,\mathrm{f} = 0 \tag{20}
$$

$$
\nabla \cdot (\nabla \times \boldsymbol{A}) = *\,\mathrm{d} * (*\,\mathrm{d}\,\boldsymbol{A}) = *\,\mathrm{d}\,\mathrm{d}\,\boldsymbol{A} = 0 \tag{21}
$$

Now we want so we want the curl of a one form right that's what this whole expression is, it's the curl of a one form and we just learned that the curl of a one form is the Hodge dual of the exterior derivative of a one form so we're now going to take the Hodge dual of the exterior derivative of this one form which is the equivalent of the Hodge dual of the double exterior derivative of a function but the double exterior derivative of anything is zero so we immediately can see that the curl of the gradient equals zero right away, that proof becomes super trivial, it become a one liner. That's an important idea not that the curl of the gradient is zero that of course is an important idea, this theorem is an important idea that the curl of a gradient field is zero or I should say the curl of a vector field which is defined as the gradient of some scalar function that is zero, is very important, it's not trivial but this proof suddenly becomes trivial once you use the language of differential forms. The reason that's important is because if we discuss Geometric algebra you'll see me talking about this a lot probably. I'm wondering is this all just bookkeeping? Is this underlying stuff really simplifying things? Or is this all fancy bookkeeping?

I think one of the arguments against this being just fancy bookkeeping instead of a battle between the Gibbs notation and the differential forms notation and to say no no the differential form notation has an underlying fundamental quality to it that goes beyond just notation there's actually it's a better way of looking at the way that nature works. One of the hints that you're onto something is when complicated proofs become really simple and when complicated proofs become really simple you know that you're probably dealing with something that has some fundamental information embedded in it that means there is material in here that does not exist in this form so the Gibbs form simply doesn't have certain amount of information and that's why I talked about the axial and polar vectors. In the Gibbs world you have a vector and you have to give it a label is it a vector or is it a pseudo vector that you have to decide and every vector you have to say oh here's a vector in three-dimensional space oh this one is a polar vector but this one here is an axial vector and you have to track that, you have to somehow remember but what we're going to discover is that's not the case with differential forms with differential forms you can escape the notion of axial and polar vector what you really realize is that polar vectors are one forms and axial vectors are actually better expressed as two forms and that's where the minus sign when you flip or reflect in the mirror in two forms minus times start generating naturally when you do these flips and so you don't have to articulate that, oh don't forget the add the minus sign if you flip the coordinate system it's actually part of the design of a two form that it'll change sign.

We'll discuss that later but right now what we have learned is that the divergence of a one form, we've learned how to calculate the divergence of a one form and we've learned how to calculate the curl of a one form and now we need to learn how to do the divergence and curl of two forms. Before we go on though let me do another quick proof here how about the divergence of a curl (21)? We know that the divergence of a curl is always going to be zero well. We treat $\boldsymbol{A}$ as a vector field in Gibbs form so in the Gibbs language $\boldsymbol{A}$ is a vector field so we're going to turn $\boldsymbol{A}$ into a one form field and we know that the curl of one form field is the Hodge dual of the exterior derivative of the one form and we know that divergence is the Hodge dual of the exterior derivative of the Hodge dual of a one form.

This, (21), is going to be a one form and this is the Hodge dual level one form which will be a two form this will be the exterior derivative of a two form which will be a three form and this will be the Hodge dual of that three form so the Hodge dual of a Hodge dual cancel each other so that just goes away so you end up with the Hodge stool of the exterior derivative of the exterior derivative of $\boldsymbol{A}$ and for the same reason the exterior derivative of an exterior derivative is always zero. We've proved this with one little line so that really makes me feel like wow this is actually really big news and it's an important adjustment in our way of thinking getting out of the vector notation of Electromagnetism into this form notation that's got to be worth something. Next we need to do this process for two forms and what we're going to learn is that the divergence of a one form has the same structure as the curl of the two form, the curl of a one form has the same structure as the divergence of a two form and then we're going to see how we can translate Maxwell's equations into form equations and we will do that next time all right so I'll see you next time.