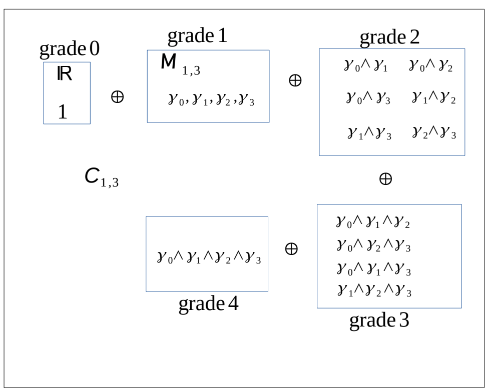
Geometric Algebra 6: Multivector Products

Welcome back, we're now going to proceed with our study of Geometric algebra focusing for now on the space-time algebra because that's our our guiding paper does and that actually is as far as I know the most interesting Geometric algebra as far as physics is concerned. The Geometric algebra of three dimensions is also very interesting and is frequently used for the introduction into the subject but we will proceed with our work, we had just begun, let's see where did we leave off, we just finished the notion of the space-time product and now it is time to move into this topic of multi-vectors and that's a big topic and we will begin it today so let's get started.

I'll begin today with a bit of an errata, somewhere I'll call it an errata but it's somewhere between an errata and a clarification. I spoke about bi-vectors, tri-vectors and quad-vectors and we put them in these different vector spaces and each of these vector spaces where he's given a grade, a way of identifying them so the real numbers are grade zero vector space, the grade one vectors are in $\mathsf{M}_{1,3}$, the grade two vectors are all the bi-vectors, the grade three vectors are all the tri-vectors and the grade grade four are all of the quad-vectors and I've labeled them here by the basis vectors and I've switched to this γ notation so that's the first thing I want to point out is I've been using e for our basis vectors and now I've gotten rid of e and we are going to use these γ and the reason we use γ is because those who are really interested in Geometric algebra frequently use γ and they use γ because they're very proud of the fact that these basis vectors in $\mathsf{M}_{1,3}$ turn out to have one-to-one direct and unambiguous correspondence to the Dirac matrices of the Dirac equation of relativistic quantum mechanics.



Now these guys are not 4×4 matrices like the Dirac matrices are but they're just basis vectors of $\mathsf{M}_{1,3}$ but it turns out that Dirac didn't know about Geometric algebra and he had to discover them through a particular representation a matrix representation of these matrix basis vectors which is interesting and that's one of the things that everybody's so proud about regarding Geometric algebra is that it captures the Dirac matrices so elegantly and without using matrices with just using them as basis vectors.

The basis vectors are now going to be labeled with these γ so all the bi-vectors, tri-vectors the quad-vector which we're going to call a pseudo scalar pretty soon, these are the pseudo-scalars, these are the scalars, these are the pseudo scalars, these are the vectors, these are going to be these pseudo-vectors and these are the bi-vectors. That's the first little change that's not really an errata that's just a change the paper does it most papers do it so we're going to jump into that too. The errata was the use the word k -blade and I used it synonymous with grade so a grade three object was a three blade, a grade one object was a one blade, the language I used in the last lesson and that's not really the way the language is used throughout most of this subject. Turns out in this paper they do actually use that language they use k -blade synonymous with grade one so you could be a grade one vector or a grade two vector and you're a one blade or a two blade but the reality is most papers reserve the word blade for something that's known as a simple k -vector so we still have all this weird language stuff. The bi-vectors are all two vectors which we give the word bi-vector because it's a little easier I guess to say than two vector, the tri-vectors are three vectors the quad-vectors are four vectors, we don't usually say one vector we just say vector and the grade zero we just call a scalar but they are zero vectors and one vectors.

Now so if you're a grade one object you're some linear combination of these basis vectors γ_i, if you're a grade two object, you're some linear combination of these bi-vector basis vectors $\gamma_i \wedge \gamma_j$, likewise for tri-vectors. However if you're a simple two vector that is something different than being just a general two vector so not all two vectors are simple but all simple two vectors are of course two vectors so let's make that clear and flush that out. To understand what I should have meant by a k-blade is if we took say this sum of two by vectors:

$$e_1 \wedge e_2 + e_2 \wedge e_3 = e_1 \wedge e_2 - e_3 \wedge e_2 = (e_1 - e_3) \wedge e_2 \tag{1}$$

The sum of two bi-vectors is a bi-vector because a bi-vector is a vector space, the bi-vector vector space allows you to add bi-vectors together so this is the sum of two bi-vectors (1). Now if I took these two bi-vectors and I flip these two elements, I'll generate a minus sign there so that's what I do right here and then if I take bi-linearity, I take e_2 and I move it out, what I've done is I've taken two bi-vector products and I've converted it into one by vector product and I've written this as still a vector wedge a vector, this is a vector wedge a vector but it's plus another vector wedge a vector this is just a vector wedge with a vector, well this is what we want to call, this construction where you just have nothing more than a vector wedged with another vector, if that's the whole expression or you can convert it into that type of expression that is what constitutes a two blade.

This this thing here is interpreted as a little slice of area inside our space well that looks like a blade it's a little blade sitting in space so we call this a two blade and then we expand the notion of blade to three blade and four dimensions and a four blade or a five blade but the point is it's just one little piece of area so that is a two blade there and that is a two blade here (1) but when you add them together it's not obvious that it's a two blade it turns out that it is because I can still take this sum and I can turn it into a single wedge product by just finding the right vector. Here's another example, Well can I turn that into something that looks like $a \wedge b$? The answer is yes. a and b are vectors and there's only one wedge so this constitutes a two blade that means this sum (2) is a two blade.

$$e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3 = \overbrace{(-e_1 - e_2)}^{a} \wedge \overbrace{(-e_1 - e_3)}^{b} \tag{2}$$

This can't always be done, there are certain times when it can always be done, for example in three dimensions you can always take any combination of two blades and the result is still a two blade but in four dimensions that's not always true. This is well known to the people who study differential forms because this is well known the idea of a simple form versus a general form but this is a simple two vector, this is a compound two vector but I don't want to get too far over that because (1) is not really compound because it really is a two blade but there are times where you cannot make this addition work and in four dimensions you definitely cannot always add two blades together to get another two blade but when you can do it then what you're dealing with are two blades otherwise you're just dealing with an element of the grade two vector space. Having said that we now understand that only these simple k -vectors really are eligible to be called blades however if we go to the paper, we haven't started reading this section on multi-vectors yet but when we do we will get to the following statement, this paragraph is in the section on multi-vectors, they write here “we will refer to the elements of the grade k sub-space as k -blades and what follows to disambiguate them from grade one vectors” so what they're dealing with here is this problem of language that I've hinted at before that exists in this subject because the old it's a four vector of four vectors problem.

They talk about bi-vectors, tri-vectors, quad-vectors but they say this word vector we want vectors to just mean the grade one object so all these other things we're going to call blades so we don't start getting this word vector leaking into our minds in the wrong way. I think that's what they're after here so anything in the grade k sub-space, they're going to call a k -blade, however that is not really super standard usage, I think the standard usage of k-blade is only the circumstance where you can literally write the element of the sub-space as $a \wedge b \wedge c$, that would be a three blade, but if you can't do that if you have to live with that plus $j \wedge h \wedge i$ and there's no way to combine these two into a single blade $w \wedge k \wedge s$, if you can't take these two and find a combination of w, k, s that gives you this, if you can't do that then you don't have this is not a three blade, this is just a grade one or a grade three vector or a member of the grade three or tri-vector it's a tri-vector.

Here in this paper they actually are using blades synonymous with an element of the algebra so because of that I am going to lean into it because this paper is what we're following and this is the language the author has chosen the authors made it very clear what they mean by k -blade. What they don't do is they don't alert you to the fact that that's not exactly standard usage of the word blade. Who am I to say what is standard and what isn't especially since this is clearly not my home-based subject, I'm learning this I'm probably just a little bit ahead of those of you who are watching this or most of those of you who are watching at least so going just straight up to Wikipedia you know how do they define the word blade and they say specifically a k -blade is a k -vector that can be expressed as the exterior product of one vectors and is of grade k . That's what I just said right? A k -blade, if you've got to be a k vector and you've got to be expressed as an exterior product of one vector so this would be a k -blade $a \wedge b \wedge v$ because I can express it as the exterior product of three one vectors.

All scalars are zero blades, all vectors are one blade every vector is simple, like I said, the other word for this is simple, when you can do this so simple and blades are in fact synonymous, as far as I know. A two blade is a simple bi-vector so now they say so a two blade is not just any bi-vector it's a simple bi-vector, sums of two blades are also bi-vectors but it's not always simple a two blade may be expressed as the wedge product of two vectors a and k so we just did that, what they don't say there is that in three dimensions that is always true every bi-vector is simple in three dimensions a three blade is a simple tri-vector it may be expressed as wedge product of three vectors. In a vector space of dimension n , a blade of grade $n-1$ is called a pseudo-vector or an anti-vector, well we'll get to that later that's something else. The highest grade element in space is called the pseudo-scalar, a little foreshadowing there, we'll talk about that later and well whatever. The point is is that this is the correct I shouldn't say correct because if you define your terms you can it's your paper right you can write it the way you want I just want to alert anybody who's watching this that the use of the word blade is a little bit more specific than this paper actually uses it, so with that we will now move on to some additional multi-vector multiplication.

When we do our multi-vector multiplication we have learned how to multiply scalars times vectors that's an easy one, space time multiplication of those two is simply real number multiplication and we can take grade one objects and multiply them together we know the rule for that and that rule is just remember the dot product part plus the wedge product part so and we've learned how to do in the last lesson we learned how to take a grade one object and multiply it by a grade two object which we're going to do that again in relative form what we don't know how to do is grade one times grade three yet which is very similar to grade one times grade two, we don't know how to do grade one times grade four so let's see if we can knock those two out in this lesson. Let me begin by just laying down some rules that apply to Geometric algebra in total generality and everybody who studies the subject has to learn these rules eventually and the basic idea for these rules that I put here is we're talking about this the space-time product of a vector times a k vector. K is a k -vector or as this paper would call it it's a k blade, it doesn't really matter if you call it a k -blade because any k -vector is the sum of k -blades so whatever you can prove for a k blade by linearity will extend to all of the k -vectors so it really doesn't matter what you call it.

$$v\,K = ½(v\,K + K\,v) + ½(v\,K - K\,v) \tag{3}$$

$$v\,K = v\cdot K + v\wedge K \tag{4}$$

We know that whenever you do this you can create a symmetric piece and an anti-symmetric piece, it's really that clear that you can do that, what's important to understand is that in the case where you're multiplying by a one vector by a regular vector if one of these two terms is a regular vector then it's always going to be the case that one of these two pieces will lower the grade of the product meaning if you have a k -blade here one of these two pieces (3) will be a $k-1$ -blade and the other piece will be a $k+1$ -blade but we don't know what which one is which until you study the subject but you do know that you're you're going to take the one that lowers the rank of the k -blade to be the dot product definition. The definition of the dot product will literally be which of these two pieces, whichever of these two pieces lowers the grade of k and the wedge product will be the opposite, the piece that raises the grade of k so the dot and wedge product here (3) aren't defined by this is always commutes or this always anti-commutes or something like that, in fact that's not the case it turns out there's times when this first term anti-commutes and this second term actually commutes.

$$v\cdot K = ½[v\,K + (-1)^{k+1}\,K\,v] = (-1)^{k+1}\,K\cdot v \tag{5}$$

We have to figure out which is which and they've already figured this out for us and we could go through these long proper proofs that show these facts but just to get it under our fingers or just to get it in our heads understanding right away that the dot product the part that lowers the value of the k -blade depends on the value of k and as you can see if the value of k is even then the dot product is the symmetric piece and if the value of k is even then the dot product commutes $v\cdot K = K\cdot v$ on the other hand if the if k is odd then the dot product is actually the anti-symmetric piece and likewise if it's odd then it actually commutes (5). I hope I said that right if k is odd then it is the symmetric part and it commutes, if k is even then it is the anti-symmetric part and it anti-commutes.

$$v\wedge K = ½[v\,K + (-1)^{k}\,K\,v] = (-1)^{k}\,K\wedge v \tag{6}$$

The formula is correct so I hope I spoke it correctly there then the Wedge product is really the same formula just you get rid of the one right and so the wedge product is the opposite of whatever the dot product is but the big point here is that the wedge product is always taken to be the thing that raises the rank and the dot product is always the thing that lowers the rank and when I say raises and lower I guess what I should say is that the space-time product of a vector times any multi-vector, well let's say any k -blade I just want to stick within one graded algebra because again you can extend all this by linearity so you just need to understand how to multiply vectors times k blades and then everything follows so whenever you take a vector and you take the space-time product with a k blade you're always going to get something one degree lower than the k -blade and one degree higher than the k -blade and the piece of these that lowers it is the dot product part and piece that raises it is the wedge product part. That is a formula that we need to know.

Another thing I want to remind you of from our last lesson is that any multi-vector M can be broken down into a grade zero part, grade one part, two part, three part and four part in the space-time algebra.

$$M=\overbrace{\langle M\rangle_0}^{\Lambda_0(\mathsf{M}_{1,3})}+\overbrace{\langle M\rangle_1}^{\Lambda_1(\mathsf{M}_{1,3})}+\overbrace{\langle M\rangle_2}^{\Lambda_2(\mathsf{M}_{1,3})}+\overbrace{\langle M\rangle_3}^{\Lambda_3(\mathsf{M}_{1,3})}+\overbrace{\langle M\rangle_4}^{\Lambda_4(\mathsf{M}_{1,3})} \tag{7}$$

In general algebra this continues on forever depending on how many dimensions you have in your base space and the base space is always this $\Lambda_1(\mathsf{M}_{1,3})$ but this guy's $\langle M\rangle_0$ always a member of $\Lambda_0(\mathsf{M}_{1,3})$ or the grade zero space and this is the bi-vector space the tri-vector space and the quad-vector space The way we separate M into these parts is we put these angle brackets around and say zero so this means the zero vector part of the multi-vector M with that caveat that we don't fully understand this plus yet but we're faking it until we make it. Hopefully we'll make it in one of the lessons coming up.

Now the formula that we should understand from this though that is an important aspect of Geometric algebra is the idea of multiplying taking the space-time product of two, in this case blades, a q -blade and a p -blade so A is going to be a q -blade for our discussion and B is going to be a p -blade:

$$A_q B_p=\langle A B\rangle_{|q-p|}+\langle A B\rangle_{|q-p|+2}+\cdots+\langle A B\rangle_{|q+p|} \tag{8}$$

$$\langle A\rangle_2\langle B\rangle_2=\langle A B\rangle_0+\langle A B\rangle_2+\langle A B\rangle_4 \tag{9}$$

It's pretty intricate proof using this proper form language but you can always show that this space-time product is going to be the sum of a wide variety of things where the lowest grade object is a $q-p$ -blade where I guess I should be a little bit more careful because p could be larger than q so these are all absolute values in here (8). A $|q-p|$ part, a $|q-p|+2$ part all the way up to the largest part being a $|q+p|$ part and so the product of two blades for arbitrary q and p can no longer really be thought of as just an anti-symmetric part and a symmetric part, it's actually spread a little more thoroughly and it covers more of the sub-spaces of the algebra. These products are a little bit more difficult to understand now in the context of the work we're doing the two blade times two blade is a thing we should we could definitely have to wrestle with and two blade times a three blade but if you'll notice a two blade times a three blade ultimately you would get a five blade but you can't have a five blade in a space-time algebra so it gets cut off once this series gets larger than n where n is the dimensionality of $\mathsf{M}_{1,3}$.

If the dimension of your one space whatever vector space you're dealing with, what the dimensionality of that thing is n then this this chain gets cut off at the top but the absolute value is the way we've done it prevent you from going to zero but it does that's an important thing to remember so for example a two blade times a two blade let me change the way that's written change it to:

$$A_2 B_2=\langle A B\rangle_0+\langle A B\rangle_2+\langle A B\rangle_4 \tag{10}$$

The two representing the k -ness of the object so A is a two blade, B is a two blade, that's not meant to be components, I don't know any other way to stick a value on there so I'm going to go with 2 and 2 for now but whatever that product is, really what it is, I should just write $A B$ and we understand these are two two blades so the space-time product of $A B$ I can suck out the zero part and put it there suck out the two part and put it there suck out the four part and put it there (10) and that won't be a problem because there is no three there's no one part and there is no three part if A and B are both two blades, you'll only have a $2-2=0$, $2-2+2=2$ and $2+2=4$ and that's it so this cannot really be broken into an anti-symmetric and symmetric piece that are the same as well you can always break into a symmetric and anti-symmetric piece you could always write $½(AB+BA)$, don't get me wrong, the problem is this is not uniquely some single blade this is some multi-vector that's a combination of zero blades, two blades and four blades so it becomes less useful to make that symmetric anti-symmetric combination so we have to learn how to do this a little bit more explicitly. A lot of papers don't bother because you don't often use these products but I can't stand it I have to know how it's done so we're going to give it a shot. That's the general stuff about multiplication inside the Geometric algebra notice this is all true for any dimensionality of Geometric algebra and so it applies to our space-time algebra just as well. Let's review our two vector times a regular vector the space-time product of a two vector B times a regular vector v .

$$\begin{aligned} Bv &= B\cdot v + B\wedge v \\ &= B^{ij}\gamma_i\wedge\gamma_j\cdot v^k\gamma_k + B^{ij}v^k\gamma_i\wedge\gamma_j\wedge\gamma_k \\ &= B^{ij}v^k\gamma_i\wedge\gamma_j\cdot\gamma_k + B^{ij}v^k\gamma_i\wedge\gamma_j\wedge\gamma_k \\ &= B^{ij}v^k\left[(\gamma_j\cdot\gamma_k)\gamma_i - (\gamma_i\cdot\gamma_k)\gamma_j\right] + B^{ij}v^k\gamma_i\wedge\gamma_j\wedge\gamma_k \\ &= B^{ij}v^k[\eta_{jk}\gamma_i - \eta_{ik}\gamma_j] + B^{ij}v^k\gamma_i\wedge\gamma_j\wedge\gamma_k \\ &= B^{ij}[v_j\gamma_i - v_i\gamma_j] + B^{ij}v^k\gamma_i\wedge\gamma_j\wedge\gamma_k \\ &= B^{ij}v_j\gamma_i - B^{ji}v_j\gamma_i + B^{ij}v^k\gamma_i\wedge\gamma_j\wedge\gamma_k \\ &= \left[B^{ij} - B^{ji}\right]v_j\gamma_i + B^{ij}v^k\gamma_i\wedge\gamma_j\wedge\gamma_k \end{aligned} \tag{11}$$

Let's review our two vector times a regular vector the space-time product of a two vector B times a regular vector v . Now in this case because one of the space-time products is a one vector we know we can break it up in this form because we basically have $2-1=1$ which is this is going to be a one blade and $2+1=3$, this is going to be a three blade and that's the only two terms and we've already shown that we can split this up into the symmetric and anti-symmetric parts using the logic that we learned early in our lessons so knowing that this is the case, we will directly make a substitution and we're going to do this in a relative form which means we're going to cast it in a set of basis vectors (second line of (11)) where $B^{ij}\gamma_i\wedge\gamma_j$ is the relative form of a bi-vector B , it's simply the component and the basis vector it's that simple it's the component times the basis vector. We could understand $B=u\wedge v$, there's B is written as a blade, this is a two blade now and we could say that well that's just:

$$B=u\wedge v=u^i\gamma_i\wedge v^j\gamma_j=u^iv^j(\gamma_i\wedge\gamma_j) \tag{12}$$

Then we just take this u^iv^j and say that's a number and that's the product of two numbers so why don't I just write it as $B^{ij}(\gamma_i\wedge\gamma_j)$. Just to be clear how we can go from this form (12) of a bi-vector call it B and then end up with a two component object, in case that's confusing to anybody. Back to (11), we have this two component object and we're going to dot it with a one component object and that's this term right this term and then we take the two component object and we wedge it with a one component object and we basically do the same thing. This second term (in second line) is actually done, that's about as simple as it gets. It's it's just going to follow us.

We're going to study this first term so the first term we're going to first thing I'm going to take the v^k and move it out front leaving the bi-vector basis dotted into the one vector basis or this γ_k but I know what this expression is because we did it in the last lesson, remember if we followed the formula of our last lesson we would see that:

$$e_i \wedge e_j \cdot e_k = +(e_j \cdot e_k) e_i - (e_i \cdot e_k) e_j \tag{13}$$

This has to do with that projecting e_k on the plane of $e_i \wedge e_j$ and then rotating it 90º, that's where that minus sign comes from so we finished that up in our last lesson now of course this is going to be equal to zero if e_k is orthogonal to e_i and e_j so the k is going to have to equal i or j in this particular case for this (13) to be non-zero but that's what we're facing here, the point is I can make that substitution for this expression and I get the fourth line is (11). That substitution is sweet and then this is where the natural existence of the metric inside Geometric algebra just pops out in 5th line is (11). Then $\gamma_i \cdot \gamma_k$ is the definition of the matrix, of the element, of the number that is the metric between j and k so I can replace that literally with the metric value in this coordinate system which in our case is the Minkowski expression given by η . If we were dealing with four-dimensional Euclidean algebra this would be different, instead of having this here I would have δ_k^j and with δ_k^j I wouldn't need these up down indices so much but I can still use the formalism if I'd like but we don't have a Euclidean system we have a Minkowski system so I want to keep this η_{jk} and then over here this becomes η_{ik} .

Then of course this part just falls down, the wedge product part follows us around. Now I'm just going to distribute this v^k inside and then lower the index using the metric from 5th to 6th lines in (11). To be clear when I said if we had a Euclidean metric it would have been δ but if we had an arbitrary metric it would have been g_{jk} so this lowering of this index is actually very important because this value of the lowered index, v^k lowers to v_j does not equal v^k in the Minkowski system necessarily, if this was the Euclidean system then these two would in fact be the same value all the time so we don't need to keep track of this up and down this but the fact that it's down means it's been processed by the metric so this v_j you cannot assume it equals v^k and that's why I'm preserving this up down index notation is to track all of that in the language of General relativity of course this is the covariant component and the contravariant component, anyway but the metric just lowers the index simple as that we don't need to think about it any more than that and then you have this expression plus the little piece that's carried around, you add to it the wedge product piece.

We were right here (6th line of(11)) where we keep bringing down the wedge part and we had just lowered the index on v^k to get v_j and v_i and then we just distribute the B^{ij} but when we distribute the B^{ij} to this piece the second part, I just change i to j and j to i arbitrarily because they're dummy indices so I can change the dummy indices but that'll allow me to pull out the $v_j \gamma_i$ from both terms and I get the 8th line of (11) and that is our expression for the grade lowering part of Bv so I can actually write in this last line if I get rid of last part, I can write:

$$B \cdot v = [B^{ij} - B^{ji}] v_j \gamma_i \tag{14}$$

$$B \wedge v = B^{ij} v^k \gamma_i \wedge \gamma_j \wedge \gamma_k \tag{15}$$

Now the wedge product is equal to this expression here (15), we got that right away so that was easy, the dot product (14), we did a little math and we ended up down there so now the question is can we do this for a tri-vector? Let's give that a shot, not too hard:

$$
\begin{aligned}
T\,v &= T^{ijl}\gamma_i\wedge\gamma_j\wedge\gamma_l\cdot v^k\gamma_k + T^{ijl}v^k\gamma_i\wedge\gamma_j\wedge\gamma_l\wedge\gamma_k \\
&= T^{ijl}v^k\gamma_i\wedge\gamma_j\wedge\gamma_l\cdot\gamma_k + T^{ijl}v^k\gamma_i\wedge\gamma_j\wedge\gamma_l\wedge\gamma_k \\
&= T^{ijl}v^k\left[\gamma_i\wedge\gamma_j(\gamma_l\cdot\gamma_k)-\gamma_i\wedge\gamma_l(\gamma_j\cdot\gamma_k)+\gamma_j\wedge\gamma_l(\gamma_i\cdot\gamma_k)\right] \\
&\quad + T^{ijl}v^k\gamma_i\wedge\gamma_j\wedge\gamma_l\wedge\gamma_k \\
&= T^{ijl}v^k\left[\eta_{lk}\gamma_i\wedge\gamma_j-\eta_{jk}\gamma_i\wedge\gamma_l+\eta_{ik}\gamma_j\wedge\gamma_l\right] \\
&\quad + T^{ijl}v^k\gamma_i\wedge\gamma_j\wedge\gamma_l\wedge\gamma_k \\
&= T^{ijl}\left[v_l\gamma_i\wedge\gamma_k - v_j\gamma_i\wedge\gamma_l + v_i\gamma_j\wedge\gamma_l\right] \\
&\quad + T^{ijl}v^k\gamma_i\wedge\gamma_j\wedge\gamma_l\wedge\gamma_k \\
&= \left[T^{ijl}v_l - T^{ilj}v_l - T^{lji}v_l\right]\gamma_i\wedge\gamma_j + T^{ijl}v^k\gamma_i\wedge\gamma_j\wedge\gamma_l\wedge\gamma_k
\end{aligned}
\tag{16}
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The space-time product of a tri-vector times a vector, well I make the same substitution, your tri-vector substituted in relative form using this coordinate basis and there is our vector $v^k\gamma_k$ and that's the dot product part (first line of (16)) and then the wedge product part is right over here with the full four vectors wedged together. Now this can only be a one-dimensional object over here at this point in our space-time algebra but if we were in ten dimensions there's some dimensionality I think for what I want to call it a four blade or I said this isn't a blade now because you have these summations you're adding many blades together here but this four vector is just one of a large pantheon of k vectors that live in the algebra but for space-time algebra this is the largest algebra in the sense that it's got the most number of wedge products in its basis vector, it's actually a smaller. I shouldn't say algebra a vector space, it's actually a smaller vector space because it's only one dimensional in a four-dimensional algebra anyway the point is it's very easy to calculate but this guy, the dot product, we will have to work on a little bit and what we do here actually is fun what you end up doing is once you pull out this v^k up front, you pull the v^k out to be up front and you're left with this structure right here that you've got to work on and the dot product of that structure follows the exact same logic we used for the bi-vector actually but we're going to take the scalar product of the last two of the basis vectors here.

In this case you'll have 3rd line of (16), we'll get a dot product with γ_k and these are two one vectors now, we're treating this scalar product as one vector so I can actually say this is the literal dot product as it's actually the Minkowski metric contraction of these two basis vectors and this process has to be unique so you can't just favor γ_l because it happens to be hanging off the end, what you need to do next is you need to flip j and l and then hit j with it but when you flip j and l you pick up a minus sign but now you get $\gamma_j\cdot\gamma_k$ and $\gamma_i\wedge\gamma_l$ and that's this term here and $\gamma_j\cdot\gamma_k$ becomes its version of the metric and then you have to get γ_i scalar product with γ_k so you have to flip this thing twice to do that so you don't lose any minus sign, here you got a minus sign, here you picked it up and then you put it back down and you got the plus back. You make these substitutions just like we did before exactly the same way in 4th line of (16). We expand it by bringing v^k inside and raising and lowering the index of v^k from this upper index form to the lower index form which means it's been modified by the metric and we retain these basis vectors so now we see we have what do we got, 5th line of (16).

Well the l contracts so this is now a two component object it looks like a three component object but you've contracted out the l and so you have a two component object over a two component basis vector in line 5th line of (16), this term here is a bi-vector now what did I say before? I said a k vector space-time product with a vector is going to give you a $k-1$ vector and a $k+1$ vector, well if this is a three vector then this is the two vectors so this is the grade lowering part and the same analysis happens for the other two terms, you just contract a different index away and after you've done all that contraction what you can do next is you rename in each term the indices of the basis vectors to i and j so you're just renaming dummy indices that's all you're doing and so you can pull out a $\gamma_i \wedge \gamma_j$ but you'll notice now you've jumbled up the indices in T but you've also turned all of the indices in v to l so you have:

$$T\,v=\left[T^{ijl}-T^{ilj}-T^{lji}\right]v_l\,\gamma_i\wedge\gamma_j+T^{ijl}\,v^k\,\gamma_i\wedge\gamma_j\wedge\gamma_l\wedge\gamma_k \tag{17}$$

$$T\cdot v=\left[T^{ijl}-T^{ilj}-T^{lji}\right]v_l\,\gamma_i\wedge\gamma_j \tag{18}$$

$$T\wedge v=T^{ijl}\,v^k\,\gamma_i\wedge\gamma_j\wedge\gamma_l\wedge\gamma_k \tag{19}$$

We introduce these two minus signs and then you pull out the v_l and you're left with this expression as the dot product of a tri-vector with a vector (18) and the wedge product of a tri-vector with a vector is really easy (19), it's just this thing here that tri-vector is never hard tri-vector just drops right out but the grade raising part, the quad-vector part is really easy, the grade lowering part takes a little bit of effort. We're going to use the exact same logic to do two-vectors times two-vectors so based on what I've talked about before we know that if we have two bi-vectors A and B and we're looking at the space-time product of them we're going to end up with three different degrees or grades inside that thing you can have a zero grade, a two grade and a four grade, two minus two, two minus two plus two and two minus two plus four so let's see if we can figure these out one by one.

$$A\,B=\langle A\,B\rangle_0+\langle A\,B\rangle_2+\langle A\,B\rangle_4 \tag{20}$$

Well the first one is this easy one at the right, let's look at that one first so I have two bi-vectors I'm calling it A^{ij} , I've broken it down into its relative form we're doing all of this in relative form for now because the paper ultimately does everything in relative form so we're just going to start getting used to it so how can I create a unique four vector out of this and what suggests itself is in fact the answer you just write $A^{ij}B^{kl}$ and you wedge everything together and that's going to give you the exact structure you need:

$$A^{ij}\gamma_i\wedge\gamma_j\quad,\quad B^{kl}\gamma_k\wedge\gamma_l\rightarrow A^{ij}B^{kl}\gamma_i\wedge\gamma_j\wedge\gamma_k\wedge\gamma_l \tag{21}$$

There's nothing arbitrary about it, you just throw the two together and you end up with this (21) and that is your four vector part of the space-time product of A and B . If you think about it carefully you could create a four vector part using some different rule where you swap i and j for some reason and you end up with a weird minus sign in the whole thing but it turns out the only way to do it that's not completely arbitrary is this way and if we did the extended proof that looked a lot like the proof we did I think early in the last episode or the episode before where you're adding and subtracting stuff and combining things and it just gets crazy the bigger the spaces you get and the more vectors you get but this so I'm going to do some special pleading here and say this is the answer, the answer is the obvious thing, you just wedge, you literally wedge A and B together and you get the largest component of this process so now we have to look at these other two so now I want to take these two bi-vectors and I want to create another bi-vector. In this example the space-time product actually has another bi-vector in it so the way we're going to do that, the mechanical method that's used to do that goes as follows:

$$A^{ij}\gamma_i\wedge\gamma_j \quad , \quad B^{kl}\gamma_k\wedge\gamma_l \rightarrow A\cdot B=A^{ij}B^{kl}\left(\left(\gamma_i\wedge\gamma_j\right)\cdot\left(\gamma_k\wedge\gamma_l\right)\right) \tag{22}$$

We pull those two out and what we're going to do is we're going to take $\gamma_i\wedge\gamma_j$ and we're going to create this dot notion (22) and we're going to call this the dot product of A and B . Now notice that our previous introduction of the dot product was very strongly linked to a symmetric and anti-symmetric part and the dot product was always the grade lowering thing. Clearly if we're going to define this then this dot product is the grade not changing things A and B are both grade two and $\langle AB\rangle_2$ is a grade two part of the product so here we're using the dot product in a slightly different way but it turns out that if you go through the full bodied proof of it, this expression here can only be calculated in one meaningful way and that is to literally take the vector dot product that's sitting right in front of you j and k and say we can do this in any order so we can do this dot product first and pull it out and as long as we do this in every conceivable order with all the possible patterns, we'll capture the unique and only way, the only way to get a unique grade two object so this is going to be:

$$\begin{aligned}A\cdot B=A^{ij}B^{kl}\Big(& \left(\gamma_j\cdot\gamma_k\right)\gamma_i\wedge\gamma_l-\gamma_j\wedge\gamma_i\cdot\gamma_k\wedge\gamma_l \\ & -\gamma_i\wedge\gamma_j\cdot\gamma_l\wedge\gamma_k+\gamma_j\wedge\gamma_i\cdot\gamma_l\wedge\gamma_k \Big)\end{aligned} \tag{23}$$

This dot product now is actually the regular Minkowski contraction. We cleared out this middle piece, we cleared it out, turned it into a number, pulled it out front and left a bi-vector behind. The problem is is that's not particularly unique because it really is depending on how you chose to order these things so we have to add to that every other possible combination that we could have so for example I could flip i and j and if I flip i and j I get a minus sign and then I end up with the 2nd term of (23) and then I could leave this side alone but flip the other side and I would get the 3rd term of (23) and then lastly I could flip them both and I would get 4th term of (23).

If I did this then now I just have to do all of these contractions and I leave behind a variety of bi-vectors so the definition here just to understand how this notation is working what I'm really saying is the bi-vector dot product between two bi-vectors can be broken down into this formulation of the Minkowski contraction of vectors and bi-vectors (23) so this is not the same dot product as you see here (22) but this is the only way to uniquely create a bi-vector out of two bi-vectors being multiplied together and it turns out it is the correct way if you do this long proof that I've never actually worked through so with that in mind we can simplify this by executing these contractions and pulling them out front just like we did with this first term:

$$A\cdot B=A^{ij}B^{kl}\left(\eta_{jk}\gamma_i\wedge\gamma_l-\eta_{ik}\gamma_j\wedge\gamma_l-\eta_{jl}\gamma_i\wedge\gamma_k+\eta_{il}\gamma_j\wedge\gamma_k\right) \tag{24}$$

I hope I wrote this down correctly, I kept the thing out front I I made all of these contractions so this should be $\gamma_i\cdot\gamma_k=\eta_{ik}$, $\gamma_j\cdot\gamma_l=\eta_{jl}$ and so on. That's all good and in our next line we just blow this up I'm going ahead and I'm using the metrics to lower these indices so I get these raised and lowered forms on these two component items so now you have mixed components there and then you have all of these basis vectors that are different.

$$A\cdot B = A^{ij}B_j{}^l\gamma_i\wedge\gamma_l - A^{ij}B_i{}^l\gamma_j\wedge\gamma_l - A^{ij}B^k{}_j\gamma_i\wedge\gamma_k + A^{ij}B^k{}_i\gamma_j\wedge\gamma_k \tag{25}$$

The next goal is to just make all these basis vectors the same by just renaming indices and pulling them out and adding things up. Notice before we do this that we started with $\left((\gamma_i\wedge\gamma_j)\cdot(\gamma_k\wedge\gamma_l)\right)$ but notice down here (25), i,j and k,l do not show up, it's all the other combinations i,l , j,l , i,k and j,k but no i,j and k,l , that's interesting so we are going to turn this into, let's turn everything into i,l and I hope I did all this right but if you take all of this stuff and you make all these note conversions so in this case I leave it alone but in this case I'm swapping, wait a minute, I want everything to be i,l so I have to switch j with i so I swap j with i there, I swap this i becomes a j , this j becomes an i and I've got i,l here so that works and then in this case I want the k to become an l . This is just one dummy index so I just switch the k to an l and I'm good so that one's easy so that switch is done here:

$$\begin{aligned} A\cdot B &= \left(A^{ij}B_j{}^l - A^{ji}B_j{}^l - A^{ij}B^l{}_j + A^{ji}B^l{}_j\right)\gamma_i\wedge\gamma_l \\ &= \left(A^{ij} - A^{ji}\right)\left(B_j{}^l - B^l{}_j\right)\gamma_i\wedge\gamma_l \end{aligned} \tag{26}$$

Switching k yo l is done in this 3rd term of (25) and then in the 4th term of (25), this one required two switches but one is just k to l which is just really easy then you switch i and j and that's done in this line so each of these gets adjusted so they're all $\gamma_i\wedge\gamma_l$ and then you pull out the $\gamma_i\wedge\gamma_l$ here and you left behind with the 1st line of (26) and then it turns out that if you look at that carefully you can factor it into the 2nd line of (26), if I got it all right. The point is that this is a real number, it's unambiguous and unique and it's a two vector. I claim this is an expression for this middle part so now we know how to do two parts of our space-time product of A and B , we can do grade four part and the grade two part so what's left? Well what's left is the grade zero part which we can do that very easily, well we can't do it very easily but can we do it. Here I have done that this is all in the pursuit of completeness, we want to know how to do all of the space-time products that we can reasonably figure out.

$$A^{ij}B^{kl}\gamma_i\wedge\gamma_j : \gamma_i\wedge\gamma_j \tag{27}$$

Here we've introduced this double dot notation to say we're going to take this thing and we got to distinguish it from the one we just did, we've got to distinguish it from the single dot, that gives us a two vector, we're looking for something now that is a scalar so I create this double dot notation which isn't my creation, there actually is a double dot product for tensors and that's basically what we're doing here we're doing a double product for tensors but with a little added twist that we've got these anti-symmetric two blades tensors but there's only really one way to proceed, you can't do too much, you can only take the dot product of all four of these things in some various combination and looking at it you can take the dot product of these two in the middle and then the dot product of the two that are on the outside so the two in the middle is $\gamma_j\cdot\gamma_k$ and now this is the Minkowski contraction and $\gamma_i\cdot\gamma_l$ is the Minkowski contraction the first and the last one and you didn't change any order so you don't change any signs but then you could flip i and j and then you get $\gamma_j\cdot\gamma_k$ which would be this piece and then j is now at the end, j has now moved over to here and i has moved over to here so then you're left with j and l so $\gamma_j \cdot \gamma_l$ but you did do a flip so you change the sign and then you could flip it two other ways and I did that but then I noticed that the two other ways give you the same thing.

$$\begin{aligned}&= A^{ij} B^{kl} (\gamma_j \cdot \gamma_k)(\gamma_i \cdot \gamma_l) - A^{ij} B^{kl} (\gamma_i \cdot \gamma_k)(\gamma_j \cdot \gamma_l) \\ &- A^{ij} B^{kl} (\gamma_j \cdot \gamma_l)(\gamma_i \cdot \gamma_k) + A^{ij} B^{kl} (\gamma_j \cdot \gamma_k)(\gamma_i \cdot \gamma_l)\end{aligned} \tag{28}$$

If you do another flip you'll get $\gamma_j \cdot \gamma_l$ and $\gamma_i \cdot \gamma_k$ again and $\gamma_j \cdot \gamma_k$ and $\gamma_i \cdot \gamma_l$ so these two really we already have accounted for them essentially, the signs are the same because it takes two flips or no flips but the point is that so these two don't even really belong. This is the Minkowski contractions, these are all Minkowski contractions so I can replace this with:

$$\begin{aligned}&= A^{ik} B^{kl} [\eta_{jk} \eta_{il} - \eta_{ik} \eta_{jl}] \\ &= A^{ij} B_{ji} - A^{ij} B_{ij} = A^{ij} (B_{ji} - B_{ij})\end{aligned} \tag{29}$$

All the indices are correct and when you do that now you just raise and lower the indices like you're supposed to and you end up with the 2nd line of (29) and you'll notice the A^{ij} is just factor out and that is a number so that is the method that is used to find the grade zero part of a bi-vector bi-vector space-time product. Now this may have seemed like a little bit of hand waving because the way I've done this here (28). I was trying to show you the mechanical ultimate result of how these inner products are calculated. The proofs of the fundamental principles of the algebra are actually pretty long and I don't think I've ever seen them completely written out and I do not have the patience to work through them but this is a good picture of what you're left with in the end and you can do this also for the bi-vector tri-vector product and there's really nothing different about that and I'll leave that to you because I've already burnt out on these products but that does show us how to do all the products between arbitrary elements of each of these graded spaces so that's great

With that in mind we will call it a day saying we're now going to claim we have a full understanding of how to do all the space-time products that matter in the graded algebra, everything that we've covered all of these things this this notion of a vector times bi-vector a bi-vector times tri-vector, a vector times another vector, well we did not do a vector times a quad-vector but it's the exact same process that we studied a vector times a tri-vector, a vector times a quad-vector is going to equal zero, well the wedge part is going to equal zero and the dot part is going to equal a tri-vector and you calculate that tri-vector exactly the way we calculated the bi-vector so that's a good exercise for you to do and then we also now have covered a bi-vector times a bi-vector which has three pieces, the grade zero, grade two and grade four pieces and I'll leave you to ponder a bi-vector times a tri-vector using the same technique that we did, this needs to end up being a grade one object and a grade three object and a grade five object. Now of course the grade five object is immediately zero but the grade one and the grade three object has got a vector object and a tri-vector object and you use the same logic we just used with the bi-vector bi-vector product so with that in mind we are pretty comfortable with where we're at and the next lesson will be finally we will attack these little plus signs that should be a quick lesson actually, see you next time.