What is General Relativity by XylyXylyX,

What is General Relativity Lesson 1: Prerequisites, Books, Units, and Syllabus

Elementary GR: 1-year, 1-semester

Prerequisite:
What is a tensor – 21 –
What is a manifold, up to – 11 –

Special Relativity (Flat space)

Books: What are the good GR books?

Advanced books:
- Misner, Thorne, Wheeler "Gravitation" (1973). (MTW)
- Wald "General Relativity".
- S. Chandrasekhar "The Mathematical Theory of Black Holes".

Elementary books: Introducing Einstein's Relativity by Ray d'Inverno

Units, a mix of natural units and "geometrized" units

Einstein field equations → $$\boldsymbol{R}_{\mu\nu} - \tfrac{1}{2}\boldsymbol{R}\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\boldsymbol{T}_{\mu\nu} \tag{1}$$

c is the Speed of light, units $\frac{[L]}{[T]}$, since we choose $c=1$ then $[L]=[T]$.

G is the Gravitational constant, units $\frac{[L]^3}{[M][T]^2}$, since we choose $G=1$ then $[L]=[M]=[T]=meter$.

$g_{\mu\nu}$ is a $(0,2)$ Tensor field on a manifold that is modeling space-time, it is called the Metric tensor.

Λ is the Cosmological constant, We will set $\Lambda=0$ at least at the beginning of the course.

In the Newtonian Limit we end up with Gravitational potential and Poisson's equation, integration over spherical volumes so the π starts to show up.

Charge of the electron e, in Coulomb $[C]$. Unit of Energy: eV (electron-volt)

Since $p=mv$ then $[P]=[M]=[E]=eV$ in the Special Relativity world. Those are Natural units.

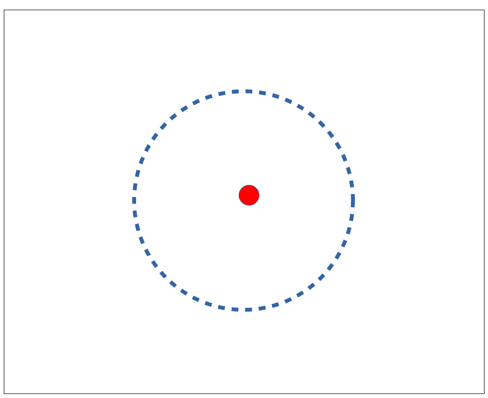
We are going to use a Geometrized unit system

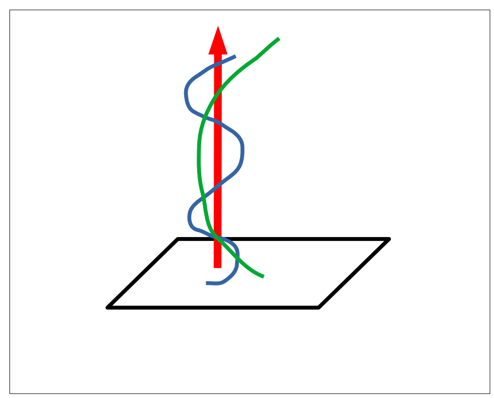
Syllabus: We can go either way
- $\boldsymbol{G}_{\mu\nu}=8\pi\boldsymbol{T}_{\mu\nu}$, then calculate the "structure" of space-time i.e. the metric $g_{\mu\nu}(x)$.
- Let's assume we know $g_{\mu\nu}(x)$, then calculate Geodesics of test particles.

What is General Relativity? Lesson 2: Worldlines

Let's begin this whole process by starting to think about Physics in a way that is conducive to understanding General Relativity and of course when I say that I mean this is how it worked for me or and work to the extent that I am familiar with this I mean I'm not certainly any more knowledgeable about this than many this is just the way I like to think about things in order to align with the nature of the subject very well and I think this is a great way of doing it and it has to do with thinking in terms of world lines I think world lines is really there's a lot of information to extract just by really getting a sense of how world lines work for example world lines are followed by test particles and there's a lot of interesting information about what is a test particle world lines exist in space-time and that's an important jump to make from the notion of space and time to the notion of space-time a lot of that should have been done in Special Relativity and ultimately the world line is sort of the foundational concept in my opinion to get to what I find it to be the most interesting thing well in the elementary form of relative General Relativity which is geodesics.

Let's see how much we can extract from the world line is sort of a concept to get us thinking mentally in terms of General Relativity in terms of Special Relativity and then the key issue is to get that time axis to get that x^0 axis there in the forefront of your thinking if you ever think of the motion of a particle and you don't think of it in terms of a time axis along with the space axis you're hopelessly stuck in the detached Newtonian view and you're never going to ever get a good conceptual grasp of General Relativity so this this implies for example that when you think of the orbit of a planet around the Sun here's the Sun or a star and you think of the orbit of a planet around it if your picture is like this then you're never going to get anywhere in the subject.





The picture that you need to have for a planet orbiting the Sun is a picture that looks like this you put that orbit in a plane you put that's the Sun at a point in the plane the Sun is just the Sun has a path through space-time that is a straight line because we are going to be this do this in the coordinate system where the Sun is motion-less and this planet orbits the Sun, let me just do a solid line, planet orbits the Sun like this and that is the way you need to view orbits around the Sun you can also do a flyby orbit, a flyby orbit would look something like this where some object is some test particle is moving towards the star and then moving away from the star but all pictures of motion from now on I want you never to have any motion that's only understood in terms of space from, now on everything has to work through this corkscrew through time or just flyby through time and time has to be part of the actual geometric picture. If you don't do that none of this is ever going to make real good sense if you do do it it's going to make sense pretty darn fast actually. One of the things that helps with this whole picture is understanding that you have to be moving in this picture you must must be moving you can't stand still in a geometric picture of space-time because standing still would be implying never leaving that spot there's a spot, there it's at location $\left(x^0, x^1, x^2, x^3\right)$ and if you're at that spot and you're never moving from that spot in space-time that basically means x^0 is frozen because it happens at an instant in time and time never stops the best you can do is apply all of that movement into the time light direction but you must be moving through time and that's a really interesting point say everything is moving through time and you're all standing still relative to each other so you've got all save got several guys that are there's no real relative motion but everybody's moving through time how fast how fast you moving through time how quickly is this point or is this particle moving through space time I mean the Sun is just sitting there moving through time you can imagine another Sun maybe this guy and it's moving through time well they should both be moving through time at exactly the same rate now if there's no relative motion if there's no relative motion between these 2 stars then Special Relativity tells us that yes their proper time is the same it's you can synchronize it and it will stay synchronized so what is that speed that 2 things who are not in relative motion move through time with and understanding that we are measuring distance in terms of time in terms of meters that speed is the speed of light there's only one possible speed what else could it be it has to be the speed of light that's the only universal constant available to us and so everything moves through time at the speed of light and so that's kind of a fascinating way of looking at the Universe especially just listening to this lecture you're sitting there listening this lecture you are moving through the speed of light c and if you're just picture yourself sitting still you're not doing what I'm telling you to do I'm telling you to picture everything moving through time at the speed of light things that are standing still relative to you, your book, maybe the mobile device you're watching this on, you know everything is moving at the speed of light through time and you should actually think of it that way, you should think of yourself as flying through time at the speed of light and you're not standing still at all.

The problem is is that everything is traveling at the speed of light not just you standing still but this particle that's on this flyby trajectory around the Sun that thing is also moving along at the speed of light it's moving along along this line at the speed of light and this planet that's moving around the Sun it's also moving at the speed of light it's moving along its world line at the speed of light every one of these particles at every point on its trajectory through space-time has a 4-velocity, we will call that 4-velocity and it's exactly the same 4-velocity you learned about in Special Relativity so there's a 4-velocity at every one of these locations there's a 4-velocity for the Sun that's standing still there's a 4-velocity for this flyby orbit, there's at every point in every instant there's this 4-velocity and if I call those 4-velocity say I call that v we know from Special Relativity we know how to write down v :

$$v=\left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) \tag{2}$$

We know that $d\tau$ is the proper time so the proper time is the time that's clicking off along on the clock attached to these test particles that are moving down these world lines now I haven't specified quite yet that these world lines are free-falling, I haven't said that I've said it specifically but it doesn't have to be what I'm saying is irrelevant of whether these lines are free falling on this could actually be a rocket ship that is accelerating toward the Sun not only because it's falling but it's got its engines on or it's pulling away from the Sun and it turns on its engines, it doesn't have to be a free-falling object.
this guy going around the Sun let's say it is the Sun of self of course is just free-falling in basically empty space the reason these guys are test particles moving around the Sun is because they don't influence the Sun they don't supply gravity all the gravity might be supplied by the Sun although I shouldn't even be invoking gravity yet I mean we haven't discussed it. I'm implying that this particles moving in an orbit around the Sun whatever keeps it in that orbit it doesn't matter all I can tell you is

that it's following this trajectory in space-time and as such this is its 4-velocity and all of these guys have a little clock on them, this is a test particle what is a test particle let's talk about our test particle for a second our test particle for the purposes of this course will be some kind of little contained object that has a clock which I'll draw as a little cesium atom, cesium clock, and an atom is an oscillator so that's the oscillator for the clock but a clock has to have an oscillator and a counter so here's a little digital register that counts ticks of the clock so that's the clock and then what else does it have so a test particle has a little clock on it it's got a certain mass test particle has a certain mass but that mass is very very very very small compared to everything else it's so small that it doesn't cause any gravitational effect relative to the problem that you're studying so that's it's mass is very tiny so you've got a mass but it's small you've got a clock on it oh and it's physical extent is very so very tiny that it doesn't experience any internal stresses or strains due to gravitational tidal forces which and we'll talk about that a little later you can put that in your memory back for now internally it has no stresses or strains measurable stresses or strains due to tidal forces and when we talk about tidal forces we do that in a lot more rigorous detail that's what a test particle is also test particles generally speaking should have the ability to just emit a burst of light so they should have some kind of lamp attached to it I guess there should be a lamp on every spot so it can it can emit a little sphere of light so it can emit a spherical light wavefront in flat space it would be a perfectly spherical wavefront going in all directions that's what a test particle is.

Anyway so these test particles have this 4-velocity and that τ in the denominator this τ is the proper time which is measured by the clock on the test particle so we've seen this before and we can break it down, we just take these derivatives so the lab frame time or the coordinate time with respect to the coordinate time times the coordinate time with respect to the proper time:

$$v=\left(\frac{dx^0}{dx^0}\frac{dx^0}{d\tau},\frac{dx^1}{dx^0}\frac{dx^0}{d\tau},\frac{dx^2}{dx^0}\frac{dx^0}{d\tau},\frac{dx^3}{dx^0}\frac{dx^0}{d\tau}\right) \tag{3}$$

I know that looks silly but that's because of our units actually, then the space time with respect to a coordinate time a coordinate time with respect to the proper time and the other space-time coordinates in a similar fashion. This is all Special Relativity, this is all our basic course where this guy here we call γ that's the γ factor:

$$\gamma=\frac{dx^0}{d\tau} \tag{4}$$

and of course in our units the is simply 1, in alternative units this would be ct, that would be a ct here. In an alternative units that would be dct/dt so that's how you may have seen it in a Special Relativity course where so you end up with a factor of c but in our case $c=1$ and we don't have to deal with that, one of the nice reasons of choosing the $c=1$ set of units. We end up with:

$$\vec{v}=\left(\gamma,\gamma v^1,\gamma v^2,\gamma v^3\right) \tag{5}$$

That is our 4-velocity. I'll put a give a little vector up there, that's our 4-velocity for basically any particle, you give me v^1,v^2,v^3 and I give you a 4-velocity so you go anywhere on this world line anywhere on this world lines specify v^1,v^2,v^3 and you end up with this 4-velocity and the velocity that goes into the γ because remember, I guess there's a bit of a review now γ is:

$$\gamma = \frac{1}{\sqrt{1-v^2}} \tag{6}$$

In many textbooks is when you use the units where $c=1$ you end up with this is a β instead of a v and β is defined as v/c so if your units are not $c=1$ then you just take the ratio of v to c you get %beta and you end up with:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{7}$$

But our unit is $c=1$ so we don't have to bother with that process. I'm just saying this because different people come from Special Relativity with different formats so we just go with this expression (6):

$$v = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \tag{8}$$

This is the definition of our 4-velocity but let's work this out a little bit, what is the magnitude of the 4-velocity? The magnitude of the 4-velocity is:

$$|\vec{v}| = |v_{\mu} v^{\mu}| = |\eta_{\alpha\beta} v^{\alpha} v^{\beta}| \tag{9}$$

This is our Special Relativity Minkowski metric $\eta_{\alpha\beta}$ which we're always going to use as:

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

That's our choice our choice of the Minkowski metric is (10), that's the flat space-time metric of Special Relativity and so when we do this calculation we will get:

$$\begin{aligned} |\vec{v}| &= -\gamma^2 + \gamma^2 (v^1)^2 + \gamma^2 (v^2)^2 + \gamma^2 (v^3)^2 \\ &= \gamma^2 \left[-1 + \left((v^1)^2 + (v^2)^2 + (v^3)^2\right)\right] \\ &= \gamma^2 (v^2 - 1) = 1 \end{aligned} \tag{11}$$

The point is that the magnitude of an arbitrary 4-velocity is always $c=1$. Which is what I was saying, you just sitting still are traveling at the speed of light. If you're sitting still you've got:

$$\vec{v} = (\gamma, 0, 0, 0) \tag{12}$$

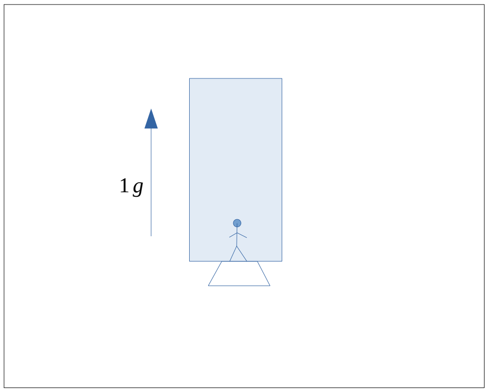
is your 4-velocity but the magnitude of your spatial velocity is 0 so $\gamma=1$ and the magnitude of this thing is going to be 1 so you're traveling at the speed of light all the time and that's the thing you got to think about, nothing is stationary in General Relativity, nothing sits still it's impossible to be motionless you can talk about being motionless relative to something I suppose but as far as any form of space-time measurement meaning you have any kind of coordinate system everything is moving in that coordinate system along a world line nothing is stationary and furthermore what's even more interesting is that no matter what world line you're on, you're always traveling at the speed of light you're always traveling at $c=1$ and so that's pretty profound idea that nothing is stationary in this business and so that's an important way of starting to think about General Relativity because we're going to be building these world line constructions and we're going to squeeze 3 space dimensions down into 2 so we can visualize it like this, that's not too hard because we'll have enough symmetry in our problems that that's okay, there are problems that I would love to get into involving curved black holes where you can't really do it, you can't really squeeze everything down into 2D so then you have to kind of imagine well how do you picture something in 3D like you know if we have our chunk of space, here we have a chunk of space how do we imagine this chunk of space moving through this chunk of space in all 3D some arbitrary movement in this chunk of space how do we imagine that world line because we can't add a time dimension.

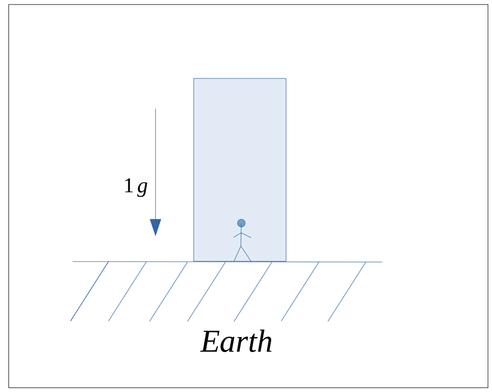
I've seen people try different things, I've seen people try coloring, they'll start with blue and then as it goes on and on the little turn as time marches on the line kind of changes color you know. I've seen people try to do it where shrinking you know. Imagine this is the cube at one time, and then you imagine at a later time the cube is this little thing and it's kind of shrinking and that shrinkage is the time dimension, this would actually looks like something that people would call a Tessaract you know this 4D cube it sort of got that 4D … this is really really not worth the trouble, I don't think it's worth your trouble trying to imagine that 4th dimension but there's a lot of times where something like that might be useful but for now we're okay because we're going to be able to squeeze our 4 time dimensions into into 2 space dimensions and then 1 time dimension so we're not going to have too much trouble with that.

This world line concept that's one big important thing I want you to think about, so the next thing is sort of one of these fun postulates ideas is that we're going to be talking about let's imagine we have this world line. I'll stick to my sort of orbiting I'm not even sure that's a really good orbit I think an orbit is better understood like this, you have the sort of orbiting object in our space-time, there it is that's its world line there's coordinates on this space-time remember how we drew and then what is a tensor series we drew coordinates like this and I said call that x^0 and x^1 and the very fact that you can do that, the very fact that this is possible, if I say this is symbolically representing space-time S , I can actually go into space-time and draw coordinates on space-time and the fact that that's possible, that I can put 4 coordinates on space-time that means that space-time is a manifold and that is kind of an assumption of our mathematical structure of the theory, we're assuming that we can lay down coordinates on space time and that space-time is actually a manifold and the 4 coordinates are $\left(x^0, x^1, x^2, x^3\right)$ and this process can actually be done. The implication of this of course is since it's a manifold we know that what we really have is we really have some $\mathbb{R}^4$ out there and if this region of space-time is some chart U_α , remember we were giving the charts U_α was their name then this mapping from $\mathbb{R}^4$ which has this coordinate grid, that mapping is the inverse of the chart mapping γ^{-1} laying down these coordinates into space-time so when I draw this thing and I say okay here's our x^0 and here's our x^1 and this here is our x^2 axis, this thing here is meant to be space-time with coordinate lines laid down through this inverse mapping from some $\mathbb{R}^4$ that's what this is space-time and these lines I've drawn here are supposed to be this thing I've been symbolically talking about in the manifold lectures. This is actually like a realization an instant of this and so there's some other $\mathbb{R}^4$ out here that's mapping into this region with some kind of inverse mapping and laying down these coordinate lines I'll put those coordinate lines in blue the coordinate lines are like this and then so the coordinate lines, the x^1 coordinate lines, the x^2 coordinate lines and it's chopping up and dividing up and slicing up space-time so space-time is a manifold and therefore if space-time is a manifold we have kind of all of those tools that we were talking about before.

If I'm now going to say this is a path through space-time well it's a path, it's a curve, well what's a curve? A curve in a manifold since space-time is a manifold I'm not even going to draw the coordinates or I'll do this I'll draw these little coordinates x^0, x^1, x^2 the time-like coordinate always be x^0 , we'll discuss what distinguishes the time-like coordinate from the space-like coordinates a little later but but if this is a curve and this is here as here's our space-time manifold and here's a path in space-time and there's a test particle that's following this path presumably in other words this path is modeling the motion of a test particle and General Relativity what does this mean from our manifold perspective? Well I have some interval on $\mathbb{R}^1$ and a parameter β and I have a map from that parameter into space-time and that map is going to be called $x^{\alpha}(\beta)$ it's a function of β so from every value of β I can create a point in the manifold whose coordinate is x^{α} and as β moves from 0 to 1 this test particle moves from $x^{\alpha}(0)$ ultimately to $x^{\alpha}(1)$ and every value of β in here, this mapping will now plot the location or the coordinates of this test particle at all of these different points so this might be $x^{\alpha}(0.5)$ for example and when I look at x^{α} of course I'm thinking of $\left(x^0(0.5), x^1(0.5), x^2(0.5), x^3(0.5)\right)$ and that is how we understand a path in space-time. It is parameterized as a curve on a manifold and that tells us quite a lot, I mean once we have a curve in space-time we can now talk about tangent vectors on that curve just as they we have in the previous lesson lessons about tangent vectors and we can also talk about the length of this curve and we're going to do that we're going to talk about the length of this curve in space-time but the point I'm trying to get now is before we go into those specifics is that space-time is a manifold and that's a basically an assumption of the theory, the theory leans on that mathematics of being a manifold and so now this there's a little potential for confusion here I think and I'm not sure I've ever really resolved it myself but understanding that if space-time is a manifold, here's our manifold of space-time, remember manifold requires that this region has a chart to $\mathbb{R}^4$ and every region has a chart to $\mathbb{R}^4$ if we are dealing with a 4D manifold, now notice that says nothing about the metric, being a manifold does not tell you anything about the metric sometimes you'll see people say that because space-time is a manifold every little region of space-time locally must look like a Lorentzian flat space-time and that's I don't think that's the right way to look at it, it is true that every local piece of space-time looks Lorentzian and that's going to be the last point we make in this first lesson but all this says is that every region of our manifold has to have coordinates in 4D now in addition to that it turns out that there's another assumption of the theory that does get you this far and we'll talk about that but just because it is a manifold doesn't mean anything about the metric all it means is that you can lay down coordinates and it's a Differentiable manifold in the sense that you can to transition functions are all differentiable functions just like we learned about in the what is a manifold lecture so the key assumption that we're dealing with though is that space-time is a 4D manifold. On top of that then there is the first principle that's actually really important and we'll cover that as our last subject for today.

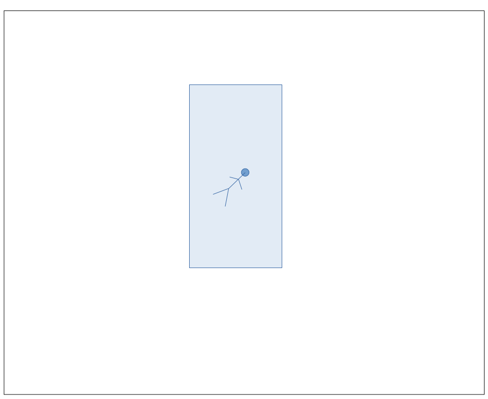
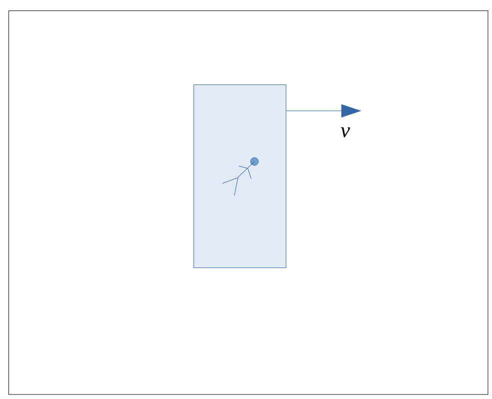
It's called the Einstein equivalence principle (EEP). The EEP is usually … I think I don't like the way it's presented in books when I learned about it because the presentation doesn't get you to sort of the mathematical modeling of it but the typical presentation is this guy in an elevator and I really I mean it makes a lot of sense I totally get it that you know you have a guy in an elevator and he's basically the elevator is in open space.





It shouldn't even be an elevator just say a guy in a box and he's just floating in open space there's no gravity there's no Earth there's nothing, he's just floating there and if you start accelerating the block very smoothly and supplied sometimes they'll put a little rocket engine on it saying that this rocket engine is providing a constant force metered very carefully so that the acceleration is constant, and so once he does that then you are a little guy is no longer floating he's now standing in the elevator.

The key is that as long as this acceleration is constant and let's actually say it's acceleration of gravity it's 1g of acceleration then that this is indistinguishable there's no experiment that can be performed inside this little box that distinguishes it from the situation of a guy standing in a box on the surface of the Earth, where this is the Earth, and in the Einstein equivalence principle basically is various versions of this moving rocket elevator situation that covers a variety of different situations and the key one of this so this one, this one here this particular thought experiment shows that uniform acceleration is indistinguishable from a gravitational field locally meaning if this box is small enough and we'll talk about it being big enough in a moment, but it also means that a person floating in free space, in absolutely empty space, the little guy in a box in free empty space, that is indistinguishable from a guy in a box moving with uniform velocity v .

Guy floating in a box with velocity v he can't be distinguished from a guy freely falling in a gravitational field so guy who's freely falling under the influence of gravity has no way of knowing if he's not in a box just moving along if he's not in a box where there is or isn't any gravity he can't detect, there's nothing that he can do to detect the existence of the local presence of the Earth accelerating him short of a non-local experiment for example if he looked out the window and looked at the Earth and sought there or shone a beam of a laser beam to the Earth and reflected it back and kept track of his relative motion with the Earth you know he could do that but that's non-local that's non-local locally that means inside this little box, there's really no experiment he could perform to decide if he was in a gravity field or if he was just floating out in space.

The question is what is local and this is a pretty important question because you can certainly imagine this circumstance where here's the Earth I'll just make a funny or here's a large massive object that's clearly gravitating very dense massive object and there's 2, this a very kind of standard picture by the way there's 2 test particles that are falling and we know that they're going to fall towards the center of the object and what's evident by this picture is that the distance between these 2 test particles it gets smaller because they're both going to the same spot so in principle if our box if the box that I was talking about that the lab that was covering this was this big you clearly will sit in the lab and as the lab falls and the balls fall you will see the balls get closer and closer together could be like sitting in a lab looking at 2 balls that just tended to come together there actually the distance between them again it's like they're attracting each other that's what it'll look like to you it'll look like to you as though those 2 balls are attracting each other that's what it'll look like to this guy here who's with the lab.

He'll think these balls are just attracting each other and clearly once he sees that he'll recognize wait a minute I must be in this circumstance I must be in this circumstance and therefore I am in a gravitational field so there's an experiment he could do to determine that he was in a gravitational field if he was not in a gravitational field if he was just floating way out in space then those 2 balls would just sit there in the lab far apart from each other and he wouldn't notice any change but because he sees these balls coming together he knows he's in a gravitational field so we can't have that, so what do we do well we shrink the lab but obviously that problem you know if you shrink the lab well that doesn't really change anything I mean the balls are still going to you know starting in 2 different places they're still going to the same place there's still going to get closer together so as long as you can detect it you can shrink the lab more and more and more until you can no longer detect it and it really has to do with the quality of your measuring devices.

If you have a measuring device that is so precise that he can take 2 balls one centimeter apart and an Earth that's a million miles away really far away if you're so good that you can detect the slight slight drift of these balls together due to the tidal forces in that situation well then you're not local, local is sort of this relative … it's always local relative to your ability to measure and then the higher your precision of measuring the smaller your lab has to be right so that this situation it's too big you've got to just get smaller so the point is, about this, is that nature will always beat your measuring equipment there's no level at which your measuring equipment is so precise that that nature you can't imagine a smaller lab it's a limiting process it's an $\epsilon - \delta$ type limiting process you set up how precise your ability to measure is and nature will find an ϵ laboratory size that will beat your measuring apparatus that's just the way it is.

MTW talks a lot about this and it's pretty interesting all you need to know is that local just means small enough that you can't detect this very process you cannot detect this process. Very well but how does this implicate the theory so we're going to go back to our notion of world lines and we're going to talk about you know we're talking about world lines but now imagine a freely following falling object in space-time and I like to draw like this because I always think of orbits so here's an object just orbiting a star and it's in free fall, it's in free fall and free fall just simply means there's no other forces there's no forces at all acting on it not even because gravity we're not calling a force gravity is going to be this following of a geodesic trail in space-time a little foreshadowing but a force would be an electromagnetic force an electromagnetic force of some kind or failing electromagnetic force perhaps the strong force of the weak force, so if it's none of those forces then if there's none of those forces are in play the particle is free falling and it still could be accelerating of course because of gravity but these forces are not acting on the particle at all.

The particle is free-falling and its path through space time is given by x^{α} of some parameter and I'll call that parameter τ for now in our next lecture we'll see why τ is going to be the proper time but this path that is what this path is, it's $x^{\alpha}(\tau)$ and τ being the parameter of course that's just $\mathbb{R}^1$, to finding a curve, so τ is the parameter and the point is that we're allowed to make coordinate transformations, we talked a lot about coordinate transformations when we talked about it tensors what is a tensor and how tensors change under coordinate transformations namely the essence of a tensor doesn't change but what the free-falling particle, the way this Einstein equivalence principle all of this elevator stuff, the way it translates to our world line understanding is it means no matter what
world line you're on in as long it's free falling it's free-falling no matter what free-falling world line you're on you could always do a coordinate transformation so that your coordinates at that point on the world line that coordinates is Minkowski coordinates, you could always find a coordinate transformation that will place the coordinate system of Special Relativity at your spot and that coordinate system will be free falling coordinates they will be coordinates that are associated with your falling motion and they're called geodesic coordinates and this is what we're going to learn to how to build the geodesic coordinates but the Einstein equivalence principle is this assumption, it's this statement that anywhere on a free-falling world line you can take this coordinate system and say screw it, I'm going to put up a coordinate system here at that particle or I can establish a coordinate system such that at this point we have Minkowski space-time and that's and what that means is that means not only do you have 4D of coordinates but you also have a metric at that point is equal to (10).

That's the metric tensor will exist everywhere on the manifold because the metric tensor is a tensor field but whatever that metric tensor field is, at this point, its value is equal to this so the way to think about that in terms of the work we've done in the previous lecture is we have our space-time coordinates we know we have a metric tensor field $g_{\alpha\beta}(x)$, it's a function of space-time so now we know that at every single point in space-time you can create a new set of coordinates, a new coordinate transformation that will take and at that particular point say that point P $g_{\alpha\beta}(P)$ is (10).
I could always find a coordinate transformation such that this is true at the point P and in fact we can even go a little bit farther and we can say we can always find a coordinate transformation such that this is true at the point P and furthermore $g_{\alpha\beta,\gamma}(P)=0$, all the first derivatives are 0 of the metric tensor components and the metric tensor components equals the Minkowski metric components at each point in space-time and that's what this means you're freely falling, this freely falling condition looks to you like absolutely flat space-time, there's nothing you can do to tell the difference therefore you believe you are in the space-time of Special Relativity and the metric of the space-time the Special Relativity is a $\eta_{\alpha\beta}$ (10), therefore whatever coordinate system you are using before or whatever coordinate systems say he's using, you can find a coordinate system where you believe that the metric at you, your calculation of the metric at your location is (10), his calculation will be different if he uses a different coordinate system it's as simple as that now the Physics of everything is always going to be the same because these coordinate changes don't mean anything all the laws of nature must comply or must be identical in every coordinate system therefore they must all be tensorial and that is yet another assumption is that the nature of general covariance, all the laws of nature must be the same but the Einstein equivalence principle that's this idea of this of the freely falling elevator being the same as the guy in regular straight-up inertial motion of Special Relativity, the uniformly accelerator being identical to the Earthbound elevator, all of these things, what they really mean is that we can make coordinate transformations that will take any point in the space-time, say you're on a free-falling path through space-time well at every point on that path you should be able to make a coordinate change so you feel

like you're in Minkowski space, regular space of Special Relativity but it's true for every point in space-time. At every point in space-time you can create a free-falling coordinate system that looks exactly like the Minkowski space and therefore a transformation exists from whatever coordinate system you're using to one that is absolutely flat in the medium vicinity of that point and exactly flat means $g_{\alpha\beta}(P)$ is (10) and also the first derivatives are equal to 0. You can't get the second derivatives to be 0 by the way you can only get the first derivatives to be 0 and of course it's always an approximation because the second derivatives aren't 0, so you know you are limited to being local for this coordinate system to have all this true, this metric will only equal these values at the point P literally so any deviation from P they're not quite the same so you have to shrink your lab to a point where you can't notice those differences in order to really sort of live through this kind of experiment if you're going to try, to experience that kind of experiment you have to really shrink your lab down.

Understand this is by no means obvious, that's the key here is that if I were to do this in our general terms I said okay here's our manifold and we have $g_{\alpha\beta}$ and say we use the full-on notation of our previous $dx^{\alpha}\otimes dx^{\beta}$ and this is a field so this is the coordinate basis and these are the components and it's a field so it's a function of the space-time where x is just shorthand for x^{μ}, which is in self shorthand for (x^0, x^1, x^2, x^3) so this has a value meaning there are16 numbers that have a value at this point P and what the Einstein equivalence principle says is that I can always find a coordinate transformation where I now switch to new coordinates and at the point P in the new coordinate system $g_{\alpha'\beta'}(x_{P'})$ so it's still at the same point but in the new coordinate system then $dx^{\alpha'}\otimes dx^{\beta'}$ where this is all the new coordinate system, the value of this at that point is going to be (10) and the rest 0. I don't like thinking of the metric as a matrix, that's a real bad idea so I'm just trying to say $g_{00}=-1$, $g_{11}=1$, $g_{22}=1$ and $g_{33}=1$. I could always find a coordinate transformation that will make that happen for this point and furthermore it can make the derivatives of these guys with respect to the spatial coordinates it can make: $g_{\alpha'\beta',\gamma'}=0$ for all those coordinates. The fact that that can be done that you know I mean think about this we're talking about in the black coordinates we're saying okay here's our manifold we've got a metric and I told you in the last lesson we know this metric can be arbitrary mathematically speaking you can create all kinds of metrics but the fact that I'm allowed to make coordinate transformation so that every point will always have a metric of this type that is a bit more of a constraining statement I have to make sure that for every point I can make this coordinate transformation and shove my metric into this form and there's only a limited number of possible forms that a metric can be, there's not that many, you can be Euclidean where they're all plus signs or something like that but that's not the metric of Special Relativity, the metric of Special Relativity is 1 minus sign and 3 plus signs or 1 plus sign and 3 minus signs. Maybe the way the Einstein equivalence principle is basically asserting that you can make that transformation for any point in space-time and it's kind of important.

Where this is all headed and we'll start beginning in next lecture when we talked about geodesics is ideally our goal is to find out, well our first goal is to find out this function $x^{\mu}(\tau)$, I want to know given a space-time and when I say give it a space-time mean I mean given a metric $g_{\alpha\beta}dx^{\alpha}\otimes dx^{\beta}$ if I'm given a metric and I have a particle, a test particle freely falling in space-time that has this metric I want to know what is its path and that is asking how do I take this parameter from $\mathbb{R}^1$ map it into the manifold through this function $x^{\mu}(\tau)$ and I will get coordinates for each of these points, I remember those coordinates were laid down on the space-time because space-time is a manifold so that allows me to know there's coordinates here I can find these coordinates or I can establish the existence of these coordinates and I know that there's a path because this is Physics, there's definitely a path of the particle and I can parameterize the path one way or another and I'm going to actually parameterize it with the proper time of the particle eventually you'll see that next time and my goal is to calculate this given this metric calculate this path, calculate these functions and we're going to do that for the metric of the Schwarzschild space-time we're going to do this for the metric surrounding a Black Hole or a very very dense star and we're going to calculate these paths and it's not it's going to start easy but I'm going to push it to as far as I can but that to me is the kind of the goal, I want to be able to calculate these functions I want to know you tell me what time it is according to this guy and I'll tell you where it is I'll tell you where he is now, it is interesting when I put about this $x^0(\tau)$ he tells me what time it is, the particle tells me what time it is but now I've got my own clock, because I've got a coordinate system that has a time coordinate an external coordinate I have to define what that means but this is the relationship between coordinate time and proper time, that's what this function is $x^0(\tau)$, it's the relationship between as proper time ticks how does coordinate time tick. this coordinate time ticks slower or faster than proper time and so that's the interesting one, the other ones are pretty easy $x^1(\tau)$ is just well where are you on the x^1 axis when the test particle zone clock clicks a certain number, of course I could also ask where am I on the x^1 axis when the lab clock x^0 clicks its own number, so this might be more apropos to a sort of a lab observer, a lab clock and a lab location, so but ultimately if I have this $x^{\mu}(\tau)$ then I get $x^0(\tau)$ and then I can calculate this $x^1(x^0)$ without much trouble so this is really it, if I can do all this I'm in really in great shape but in General Relativity what is important to understand and is that the proper time is always going to be the time as ticked off by the clock of somebody floating through space-time whether they're in free fall, whether they're in some kind of accelerated motion doesn't matter the clock built into the little test object that oscillator in that counter will always click the proper time.

The coordinate time however in Special Relativity, the coordinate time, was always the time of some sort of laboratory, you would have a laboratory that's sitting still and they would say it's time t and x and then there would be some moving thing and there's a moving lab or moving frame with a relative speed of v and you would go through all of these exercises with you know how what's the Lorentz contraction and the time dilation of such a thing and if you really know what you're doing you understand that the coordinate transformation is the sort of slanted coordinates $x^{0'}$ and $x^{1'}$ and these coordinate is bend inward but light is always going at a 45° angle but the point here is that in Special Relativity this lab time is always the proper time of a guy standing in a laboratory and the time and the proper time of the sky moving is always the proper time, the time ticked off by a clock guy moving with that frame the problem is that in General Relativity you can always establish a proper time of some test particle that never changes but this coordinate time may not actually be the proper time of any particular observer, we may choose time coordinates that don't tick off on anybody's clock, in fact there's this set of coordinates called Penrose coordinates, that clearly have a time coordinate I'll call it an x' coordinate there's nobody in the Universe whose clock ticks at some at the rate of Penrose time, likewise there's these coordinates associated with Black Holes, Eddington-Finkelstein coordinates, I got to remember exactly the call but again it's they're very useful coordinate system, it's a coordinate system where the event horizon of a black hole is essentially meaningless and all you have all of these coordinate lines it just cross the event horizon without any trouble whatsoever and they're beautiful coordinate system but the time axis on that thing, the time axis doesn't represent the ticking time of any particular observer in the Universe it is just a mathematical construction that provides a time axis that is indeed time-like which is something we'll discuss when we get to it but it doesn't necessarily match anybody's particular clock and that's an important distinction, it's an abstract distinction it is hard to make that jump from Special Relativity were every time you see is attached to somebody but in General Relativity that's not true there's nobody these cosmological diagrams you'll get in Special Relativity, these Penrose diagrams nobody actually has a clock that ticks that way.

What did we cover, I just went back to the beginning, we just covered this idea that I want you to think about I want you to think about motion in space-time as literally in space-time you should always have that time coordinate in there and you're always understanding this motion as as nothing is stationary everything is moving and everything is moving along their world line at the velocity c , and that can never change it can never change by the way because the 4-velocity the 4-acceleration you can calculate the 4-acceleration is the derivative of the 4-velocity with respect to proper time, this is all Special Relativity that 4-acceleration if you calculate $a^{\mu} v_{\mu}$ which would be the projection of the 4-acceleration on the 4-velocity that's always 0 it's kind of a tricky calculation actually because if you write down this in full generality which would be this (5) then you have to calculate this:

$$a^{\mu}=\frac{d v^{\mu}}{d\tau} \tag{13}$$

by taking derivatives of each of these terms with respect to τ , and you know you have to so you have to do like $d\gamma/d\tau$ and it can be done, it's not totally tricky it's just you've got to keep track of things but by the time you calculate this and you execute this you're going to get a 0 and so no acceleration in the Universe can change your 4-velocity because all acceleration is forever orthogonal to 4-velocity so that's the way I want you to think about movement in General Relativity always think about in the context of moving through space-time. Space-time is a 4 dimensional manifold we talked about that and in addition it has a metric structure such that at every point in the 4D manifold you can erect coordinates that are free fall coordinates and that's the Einstein equivalence principle another key assumption that Einstein realized was very very important here and with those points in mind, the mathematical manifestation of that is you can always get this transformation to transform away the metric to a very very simple metric and you can do it at every point where the metric becomes very simple and all of its first derivatives vanish not the second derivatives. I guess the other important point here is that if you can make a single transformation so that every point so that $g_{\alpha\beta}(x)$ is (10) for all x if you can do that then you have Special Relativity, then you have flat space, there's no curvature whatsoever so if you can make a single transformation that will turn everything into this metric at every point you have flat space-time and so keeping that in mind then our goal is to calculate these world lines and calculating the world lines means finding these functions $x^{\mu}(\tau)$ meaning as the proper time of an object moving of the world line ticks off how does the motion look in any given coordinate system okay so that's all for now little rambling but that's sort of to get us warmed up into how to think about what's going on.

Lesson 3: Geodesics and the Equivalence Principle

We are now going to talk about how the Einstein equivalence principle (EEP), how that mathematically actually translates into something very important namely Geodesics and a really interesting subject because we're going to go from this little story of elevators and people floating in elevators being floating in free space or standing on the surface of the earth or elevators that are free falling, or elevators that are being accelerated by some kind of rocket engine, all those little stories and we learned last time that all of these little stories translate into the fact that at any point P in space-time we have a metric tensor field $g_{\alpha\beta}(x)\, dx^{\alpha} \otimes dx^{\beta}$, we know how this thing works and in our classical notation we're just going to call this $g_{\alpha\beta}$, that means this whole thing kind of gets sucked into the classical notation of $g_{\alpha\beta}$ and we know that it's a function of space-time so we don't need this x and we're using the index gymnastic system and now that we understand tensors this isn't a big mystery to us anymore, we know that it's all the bookkeeping things so we don't need $dx^{\alpha} \otimes dx^{\beta}$, however during this lecture I will pop that part back up so this $g_{\alpha\beta}$ is what we're left with so we know that $g_{\alpha\beta}$ at the point P so I'll do this evaluate it at the point P we know that that's going to equal:

$$g_{\alpha\beta} = \begin{cases} -1 & g_{00} \\ +1 & g_{11} \\ +1 & g_{22} \\ +1 & g_{33} \end{cases} \text{and all the rest are going to be } 0 \tag{14}$$

That's what we understand from these little stories of elevators that's what it means, it means that the metric tensor add up to any given point can be put into this form and how do you put it into this form well there's some coordinate system remember these guys here, this is we're going back to our tensor stuff really matters we know that this stuff here is dependent on the coordinate system therefore obviously the components are dependent on the coordinate system that's sort of the whole point is you change coordinate systems you change $dx^{\alpha} \otimes dx^{\beta}$ but you're also kind of changing $g_{\alpha\beta}$ and you change the tensor components, if you change coordinate system so the point is, if you have a coordinate system and at the point P say the metric at the point P in the new coordinate system is:

$$g_{\alpha\beta}\Big|_P = \begin{pmatrix} w & x & y & z \\ a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix} \tag{15}$$

Of course the metric tensor is symmetric and it has values and some random collection of values that satisfy the possibility of being a metric, the point is that what these values are constrained in one particular way first they have to be symmetric of course but aside from being symmetric it has to be that there's a coordinate transformation that will change to the black coordinates where (15) will end up becoming (14) at the point P so at every given point you can make that happen now furthermore you can also make the derivatives equal to 0 , you can make:

$$g_{\alpha\beta,\gamma}\Big|_P = 0 \tag{16}$$

Remember all these things are functions of space-time I got to emphasize that part because people sometimes tend to forget because they just see this letter without this indication that it's a function on the space-time, if I had new coordinates where $g_{\alpha\beta}$ has some value here if that's given by (15) I can always find a transformation to the black coordinates where $g_{\alpha\beta}$ has this set of values (14), and that's the Einstein equivalence principle (EEP) in mathematical form. This transformation Λ^{α}_{β} exists, now it does not have to be a Lorentz transformation, it just has to be a general coordinate transformation, it turns out that once you've once you've found this set of coordinates, found the coordinates that give you this, this fruit we call it they're really free-falling coordinates are the coordinates that are falling free that's the idea, if you're in free-fall you experience no gravitational field that means that in free-fall the Physics of the your Universe is the Physics of Special Relativity, well the Physics of Special Relativity are all Lorentz invariant so once I have a coordinate system where (14) is true, I can do any Lorentz transformation and get another one, so there's an infinite number of these guys all related by Lorentz transformations however this relationship Λ^{α}_{β} is a general coordinate transformation there's some general coordinate transformation that will get you into that free-falling coordinate system but once you're in the free-falling coordinate system you can boost all day long and you'll still have the Physics of Special Relativity.

In pictures, if we were going to go back to our little pictures I've never really thought about this before but if you were going to go back to pictures so you have you know here's a big gravitating planet and here's an elevator that's freely falling and the freely falling elevator, of course, everything that's happening in there, shouldn't say elevator, I mean it's not an elevator, it's a freely falling little box with a person in it that's all Special Relativity but likewise if I suddenly boost the box so it has this relative velocity, it's still Special Relativity, you just now have a box that instead of just having a trajectory onto the planet it's now got some kind of trajectory and it's still experience it, it's still freely falling in Special Relativity so there's an infinite number of reference frames here all related by this velocity v which could be arbitrary so it's all related by a boost. Anyway the point that's how the Einstein equivalence principle is applied.

Now what next below now let's remember how path lengths are determined in Special Relativity we know that the differential of the proper time this is a very common Special Relativity equation the Minkowski metric:

$$d\tau^2 = -\eta_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} \tag{17}$$

Now in Special Relativity classes you interpreted this in a very standard way of calculus this is (left) an infinitesimal increment of proper time and this is (right) the path length along of some trajectory. You have to come to this in order to make sense out of this if there's 2 things we want to do first I want to rewrite this in the terms of the what is a tensor mathematics meaning I want to go back into the installing this with the full tensor notation with the basis vectors and see how this comes about, we should do that but also you need to be able to quickly understand how these 2 components here, this is how these 2 components here relate to this the path itself and implied in this of course is that there's a path in space-time x^1, x^0 and as we've always said before because it's a manifold and we're talking about a path which is a curve we can parameterize it, will parameterize it with τ and there's the function is $x^{\mu}(\tau)$ evaluated at $\tau=0$ is the beginning of the path and then the end of the path or at least the end that we're going to consider for the problem is $x^{\mu}(\tau)$ at $\tau=1$ and then maybe τ will go from 0 to 1 , really doesn't matter what τ goes to this is just some arbitrary space, arbitrary gap but τ will be the proper time so that's what x is and implied that's what these x in (17) are. These these x are the coordinate values $x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)$, those are the coordinates and these are these little differentials in the coordinates so we're thinking this in Special Relativity the way it's normally presented so we actually think of these as differentials, calculus style differentials now we'll have to make the connection between those and the true 1-forms and the tensor notation that we had before so we'll get to that but this expression (17) tells us the path length in space-time and this the space-time of Special Relativity so we know that η is (14) so clearly we have:

$$d\tau^2 = -\left[-dt^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right] \tag{18}$$

The minus sign here in the front that can be traced to the fact that in this metric using the mostly plus metric, time-like paths or the paths of material particles will always end up being negative and we're going to but we're defining the proper time which we want to count off in a positive way so we have that negative sign there to give us the proper time to be positive. That wouldn't be there if we were using the mostly minus convention so anyway so that tells us the proper time and the proper time is going to actually be this integral:

$$\int_0^1 d\tau \tag{19}$$

That's the proper time interval that it takes this particle to traverse this line which is what is really the length of the line because remember τ in our units where $c=1$ it's going to literally be a length it's going to be measured in meters and to convert back to seconds of course you would divide by whatever your favorite units of c are so this integral represents the amount of proper time or the path length in Special Relativity. Now let's try to broker that into something that's actually a bit more mathematical. I guess we have to spend a little bit more time in the abstract before we can actually jump to the math. Just so I can explain to you what's at stake here for that very first step. Imagine that there's some black hole maybe it's the black hole at the center of our galaxy, what is it Sagittarius A star or something, I can't remember the name, but there's this black hole the center of our galaxy say which is not an active galactic nuclei, it's an inactive one so there's these stars orbiting it so this would be say the plane of the orbit of one of the stars and this star starts here and it kind of orbits maybe it has a very elliptical orbit around that star and then there's another star that might be much closer say and its orbit is a lot faster around the center of the galaxy something like that and then maybe there's even another one that's really far away that kind of comes like this and goes around and presumably it presumably it bends back around and keeps looping up and it's just very far so here's a bunch of stars in a space-time diagram going around the galaxies now notice I'm trying to start by not thinking about this in terms of your regular picture with you have a Black Hole and here's you know one orbit maybe that would be that the orbit of the red star and then here's the orbit of the blue star and then maybe the orbit of the green star or something like this. Just thinking of it if all of those orbits were in the same plane so the first thing is this is not an accurate representation of what's going on the center of our galaxy except that there are star known to be orbiting a supermassive black hole but they're not all on the same they're not all doing the orbit in the same plane of course they're doing orbits and all kinds of different wacky planes but if they were doing it all in the same plane then we could compress the picture into a plane and then put our time axis here and this is the picture where we get the point is that our little guy in the elevator might start here, well let's say he starts here and he has some strange state of motion but let's just say he gets sort of captured by this first star so he's kind of in orbit around this first star but maybe he wanders a little too close to this star and ends up in orbit around that star and then kind of somehow swings through this whole thing and maybe even gets ejected from the whole process, that is possible I think, when you're dealing with multi bodies in orbits around each other you can actually get ejected from the system one way or another and I'm sure that's even extra true in the presence of a supermassive Black Hole.

But the point is that while this guy's doing it, while our little man in the lab the astronaut in the lab is free-falling this is a totally free-fall condition, the only thing that's happening is he's just following a space-time trajectory according to a geodesics, these are geodesics, which were which we're actually going to be beginning to touch on today, he's just if this astronaut is just freely falling there's nothing this astronaut can do to figure out that this is the path that he's on as long as he only relies on measurements inside his little laboratory. This is this whole process is totally invisible to him and when I first learned about this I was thinking: wait a minute what about inertial guidance systems for airplanes, inertial guidance systems and things of that nature and if you don't remember inertial guidance system would be equivalent to a device that's essentially a mass there's a very sort of solid mounted object that's fixed to your aircraft or your spaceship and then there's a spring and then there's a mass and of course if you move, if you accelerate in this direction the spring will get compressed so you would say: well look you know here you are here and you're falling into the star so clearly you're accelerating you look your space-time path is changing all over the place so there's all kinds of accelerations so this compression can be measured and we can track it we can actually figure out that you're accelerating in these weird ways and from that acceleration we can extrapolate your velocity we can calculate velocity we can have 3 of these one in each direction and you could actually reproduce this trajectory but the problem is that inertial guidance systems don't work in free-fall, inertial guidance systems only work when we're talking about non gravitational forces and that's because this ultimately this mass is subject to the exact same gravitational field conditions that the laboratory itself is and therefore it's not going to compress the string.

You should understand this by the way if you don't understand why an inertial guidance system will not work in free fall then a truly inertial guidance system meaning one that's designed something that can be modeled vaguely in the way I've described here with a mass in a spring and then you're a little bit behind for where you should be for this this is an important principle that, I mean it's basically a statement of the Einstein equivalence principle because if it's true that we can find a reference frame that is total Special Relativity then that means it's in uniform motion and these things will never make any registration of uniform motion so if that frame exists then of course an inertial guidance system wouldn't work. We have this free-falling observer in this gravitational field and now we know that regardless of what coordinate system we have this guy will always be able to establish a coordinate system or what they're experiencing because they're free-falling they're experiencing the coordinate system the empty coordinate system of Special Relativity and likewise the path length of this path is the proper time that is ticked off on the clock that's sitting with this observer so this guy he's got his clock his little cesium clock you know the atom with its register that little clock ticks and you can actually plot even amounts of proper time ticked off here and there will be times where he will be moving faster and slower like he may be moving a little slower out here away from the star but then as he gets to the star he's moving faster so the ticks you know would be slow and then speed up and here they could be really fast and here you can actually have some kind of time dilation where the ticks go but the amount of space in this point you would actually have time dilation the ticks and the amount of space are actually under the influence of the theory of General Relativity itself and it would be fun when we get to a chance, we'll be able to study this we'll be actually be able to see that the proper time ticks off at a very different rate at different places near this supermassive Black Hole.

The length of this line however is not subject to the rule that we just learned that the path length (17), that's for Special Relativity in this path this path is clearly subject to all kinds of gravitational forces and we're talking supermassive black hole we know the general theory is what applies here and now we have to think a little bit differently we still get the benefit of the proper time we're still going to calculate the time but now it's going to be:

$$d\tau^2 = -g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} \tag{20}$$

and this metric of the space-time is in to calculate this path length it's the whole darn space-time, it's this coordinate system, the coordinate system that is used to capture the coordinate time and the space and that is you know this guy is a function of space-time, this represents the path just as before $x^{\alpha}(\tau)$ as a function of τ that is this path but remember this metric can only be made to be inertial at any given point in space-time, at any individual point but along this whole path it won't be so you're going to have to do this full integral to capture the time of the path of this particle and furthermore to complicate this thing even more is this is for a material particle which means that $d\tau$ well it means that it's time-like path so at every point in its motion it's always a time-like interval and because of that, this integral makes sense (19) but if it was light we were talking about, now the trajectory of a light ray which would be moving very fast relative to all this so it would be just bent a little bit by the supermassive Black Hole, that particular trajectory that always will be 0, that's always going to be have a path length of 0 and calculated that using this method so that makes it very difficult first of all you can't calculate a proper time for light for that very reason but we'll try to split them split the situations up when we talk about geodesics time-like null geodesics in a moment but now what you have to remember is that in General Relativity the path length is calculated using the full metric and in Special Relativity is calculated using the Minkowski metric and that's because Minkowski metric is the full metric in Special Relativity. In Special Relativity everywhere in space has this metric but in General Relativity there is a coordinate system where every point could be given this metric $\eta_{\alpha\beta}$ but only at the point, if you could find a transformation of a coordinate system such that the entire space-time had this metric then you would be basically working in a Special Relativity condition, the space-time would be flat, that's actually what we mean by flat, flat space-time means that everywhere that's true $\eta_{\alpha\beta}$, for curved space-time that's impossible to make that true everywhere.

Now we're going to squeeze it into some very interesting mathematical modeling that will give us ultimately some really important information about how particles movements based on, just based on one thing only just based on the Einstein equivalence principle that at a given point we can make (14) true. Let me show you first, by the way, how (17), how we make contact between our understanding of here we have just the component of the metric tensor in this case the flat space metric tensor how do we make contact with:

$$\eta_{\alpha\beta}\, dx^{\alpha} \otimes dx^{\beta} \tag{21}$$

which is an object that takes 2 vectors and returns a real number with (17), Well first we notice (17) is a real number so that's good and here (17) we're interpreting dx^{α}, dx^{β} as differentials and here (21) we're interpreting dx^{α}, dx^{β} as co-vectors or alternatively 1-forms and this is a tensor product instead of just some regular real number multiplication so how does this work and the answer of course is we need 2 vectors to feed into (21) to get something that we can interpret like (17) because we feed into vectors we'll get a real number so that real number better be interpretable like (17), in this form here. Imagine one point in space-time and another point in space-time and then you can imagine a vector connecting the two, now this has to be really small but if it is really small we can call that vector say $(\Delta x)^{\alpha}\partial_{\alpha}$ because this is the vector basis in coordinate representation and $(\Delta x)^{\alpha}$ will be a component. In the past we would have written say $x^{\alpha}\partial_{\alpha}$ to be an arbitrary vector in the coordinate basis, this time I'm going to call it $(\Delta x)^{\alpha}$ to give it a suggestively so that would be like a little vector that would extend from one point in space-time to another. Now the problem of course is I've emphasized that these vectors live this thing lives in a tangent space here, if it goes from the point A to the point B there's a tangent space and this vector lives inside this tangent space but the units can still be right and it will still, the idea is that it will still give us if this is the coordinates of the point $x^{\mu}(A)$ and this is the coordinates of the point $x^{\mu}(B)$ you can definitely imagine that:

$$x^{\mu}(A)+(\Delta x)^{\mu}=x^{\mu}(B) \tag{22}$$

If B is close enough to A this logic is just fine (22), and even though this vector literally lives inside this tangent space when these things get so close together you can actually allow yourself to imagine this as a vector extending from A to B and if you did that you would put in (21):

$$\eta_{\alpha\beta}\, dx^{\alpha}\otimes dx^{\beta}\left((\Delta x)^{\gamma}\partial_{\gamma},(\Delta x)^{\varepsilon}\partial_{\varepsilon}\right) \tag{23}$$

You're putting in the vector twice and then you crunch the tensor product and you end up with:

$$(\Delta x)^{\gamma}(\Delta x)^{\varepsilon}\eta_{\alpha\beta}\langle dx^{\alpha},\partial_{\gamma}\rangle\langle dx^{\beta},\partial_{\varepsilon}\rangle \tag{24}$$

These are just $\langle dx^{\alpha},\partial_{\gamma}\rangle=\delta_{\gamma}^{\alpha}$ and $\langle dx^{\beta},\partial_{\varepsilon}\rangle=\delta_{\varepsilon}^{\beta}$ so what you end up with is:

$$\eta_{\alpha\beta}(\Delta x)^{\alpha}(\Delta x)^{\beta} \tag{25}$$

Then you continue to push the notation further with the Δx has become infinitesimally small and now you recover the differentials of calculus and you get:

$$\eta_{\alpha\beta}\, dx^{\alpha} dx^{\beta} \tag{26}$$

You understand this is still the same thing, you're talking about a little interval but here we create that little interval inside our tangent space that should be what it takes to create this coordinate addition in the appropriate way and then we just crunch that the tensor product machine on those 2 vectors and we basically get this object which in its infinite decimal form is completely and plausibly linked to this object (26) and everything's a real number the way it's supposed to be.

That works obviously for flat space, it works well for any $(0,2)$ tensor this exact analogy will work. That's the connection between our work in the classical notation so we won't deal with it this way (21) for the purposes of this because (17) is how most of the books you're going to be reading and following along with to deal with it.

We'll begin by understanding that in this little sealed lab that's just freely falling in some space-time that we know that we can establish a coordinate system where a test particle in this lab say the test particles this little red thing moves in a straight line and we know that because the Einstein equivalence principle says so that's a stipulation of the theory, it's an axiom, well I guess axioms a little bit more of a mathematical term, that's a postulate, it's a basic premise of the theory it only comes from insight there's no development of it. Einstein famously just said it was the happiest thought of his life because he wrote it down thought about it and the General Theory of Relativity in many ways flows forth from it but let's see how.

We're going to call the straight-line motion of this particle in the space-time, we're going to give those coordinates a name, we used to call them x in that previous thing but here we're going to call them ξ and that will be the coordinate system that we're dealing with that is inertial by its inertial it's a freely falling coordinate system that exists because it's the stipulation of the Einstein equivalent principle and in that freely falling coordinate system we have straight-line motion which implies that:

$$\frac{d^2 \xi^{\alpha}}{d\tau^2}=0 \tag{27}$$

The proper time of course is … remember this is a path that's parameterized by the proper time $\xi^{\mu}(\tau)$ so the second derivative of this path with respect to the proper time of particle is going to be 0 because this motion is straight line motion. In straight line motion of Special Relativity and of course again this is all inside this local frame that's as small as necessary for this to be true. The proper time is still the case because of its we're in the side frame that we're dealing with:

$$d\tau^2=-\eta_{\alpha\beta}\ d\xi^{\alpha}\ d\xi^{\beta} \tag{28}$$

This is still true, this is the Special Relativity condition so we've asserted that and all and we have now invoked the Einstein equivalence principle at this point so our next step is to say well let's make an arbitrary coordinate transformation, we're going to now introduce a coordinate system x^{μ} and it could be any coordinate system at all, we're really unrestricted it can be curved it can be rotating and spinning it can be accelerated, accelerated motion is fine it doesn't really matter all that has to matter is that the freely falling coordinates … so any given point so here's our ξ coordinates so there's ξ^0 and ξ^1 , it's supposed and then here is our other coordinates x , I'll call that x^0 and x^1 and obviously there's a 2 and 3 for both of these but one thing that is going to be clear or that has to be the case for what we're working with is that ξ^0 is a function of (x^0, x^1, x^2, x^3) which I can write as $\xi^0(x)$. I could also write this as I'm just trying to do this in every way you might see it $\xi^0(x^{\alpha})$, each of the α are in there but it's a function of the other coordinates, likewise $\xi^1(x^0, x^1, x^2, x^3)$ so this is perfectly sensible of course, this is our coordinate transformation, these are our transition functions from what is a manifold, that's what these are, these are the transition functions. Take this point here you know it's got coordinates in x and it's got coordinates in ξ and the coordinates in ξ are some function of the coordinates in x presumably if you know the coordinates of this point in the x system you'll be able to calculate it in the ξ system and likewise you have one of these for ξ^2 and ξ^3 .

Using the chain rule we start with our expression that we know to be true because of the Einstein equivalence principle, we start with (27) and we invoke all of the 4 functions this is a way to write in one fell swoop basically all of these, we just write it down as $\xi^{\alpha}(x)$ so knowing that this exists that we have this other coordinate system well how does that how can we work with this equation what do we do next well we of course invoke the chain rule and (27) becomes:

$$\frac{d}{d\tau}\left(\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial \tau}\right)=0 \tag{29}$$

We execute the next derivative and we are now using essentially the chain rule and that will give us:

$$\frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau} + \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial \tau^2} = 0 \tag{30}$$

What we've basically done is we've taken this derivative (27) by using this expression $\xi^{\alpha}(x)$. Each ξ^{α} is a function of x so this derivative makes perfect sense and of course $\partial x^{\mu} / \partial \tau$ is the path of this particle in the new coordinate system as a function of the proper time. As long as we can have that we're fine, we can take all of these derivatives and when we do, this simple expression (27) in the flat coordinates, a statement that's very trivial in the flat coordinates located at this particular point P or this particular point inside our spaceship which is in free fall coordinates, that equation of motion in the free fall coordinates becomes a bit more complicated in the more general coordinate system. The transition functions are $\xi^{\alpha}(x^{\mu})$ and likewise there should be an $x^{\mu}(\xi^{\alpha})$ and then these guys depend on us knowing the path $x^{\mu}(\tau)$ relative to the proper time otherwise you can't calculate those derivatives, so you end up with this expression (30).

To move on from this we want to simplify this a little bit using an interesting fact. We're going to multiply both sides of the equation by $\partial x^{\lambda} / \partial \xi^{\alpha}$

$$\frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau} + \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 x^{\mu}}{\partial \tau^2} = 0 \tag{31}$$

What's important about this is that I can look at this last term here. Looking at this one this expression:

$$\frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} = \delta^{\lambda}_{\mu} \tag{32}$$

To convince yourself of that you should spend a little time convincing yourself that that's the case. With that fact in mind now I can rewrite everything again:

$$\frac{\partial^2 x^{\lambda}}{\partial \tau^2} + \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau} = 0 \tag{33}$$

We could actually rewrite this thing in a very suggestive way:

$$\frac{\partial^2 x^{\lambda}}{\partial \tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau} = 0 \tag{34}$$

This is our final expression where I've made the obvious renaming of that complex set of derivatives:

$$\Gamma^{\lambda}_{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \tag{35}$$

Let's look at this complex set of derivatives (35), x^{μ} is a function of ξ well that's just an inversion of ξ^{α} as a function of x i.e. $\xi^{\alpha}(x^{\lambda}) \leftrightarrow x^{\lambda}(\xi^{\alpha})$ and then the second term is just the second derivative of the function we talked about in the beginning which is basically $\xi^{\alpha}(x^{\lambda})$, the inertial coordinates of the little lab in the arbitrary coordinates of the lab, you can take those second derivatives. This object here is based on the second derivative of these coordinate changes multiplied by a first derivative of the inverse of the change but either way these functions are pretty well known these derivatives are established you're assuming the coordinate systems aren't totally insane so that all these derivatives should be smooth and exist which by the way is a requirement for a Differentiable manifold which is what space-time is modeled on so now we're actually leaning on our second assumption of all of this which is that space-time is a Differentiable manifold so these derivatives are fine and now you end up with this equation (34) and this differential equation where the independent variable is τ , and the dependent variable is x^{λ} so the answer to this thing is $x^{\lambda}(\tau)$ so it is the path of your particle in space-time in the arbitrary coordinate system as a function of the proper time of the particle that's what this expression will tell you and this expression (34) is the Geodesic equation and this guy (35) from our previous lecture about what is a tensor that is the connection. This is a derivation of the Geodesic equation and where did we start we started with the presumption of the Einstein equivalence principle, without the incentive of Einstein equivalence principle we cannot say that this freely falling observer, this guy's freely falling and it is this freely falling frame, this particle path is completely straight.

Here's the freely falling laboratory inside this thing it's flat space-time according to ξ coordinate system and here's the particle now this space-time as I showed in the end this sequence, I showed this freely falling frame is doing all kinds of crazy stuff in the Universe because it's freely falling under all the influence of gravity so standing out there by that star and watching this thing fall we see this frame move all kinds of weird ways and that's where the frame's doing but if you're a guy inside the frame and you're falling and you see this little ball and it's moving like this, that's its motion with respect to you meaning you're in this frame it's like you're an astronaut in the Space Shuttle and you have a little ball and you're watching it move in front of you well when this frame moves to this position by that time this ball has moved to say that position here from the perspective of this guy in ξ frame, the balls moved in a perfectly straight line and that perfectly straight path is $\xi^{\alpha}(\tau)$ so when we started our equations, when we started our work back here, that's what this was $\xi^{\alpha}(\tau)$ that is the straight path as seen in the ξ coordinate system which is moving with the falling lab frame but now the question is in this frame x^{μ} who's maybe with the stars in some other coordinate. This person's got the x^{λ} coordinate system, what does he see this ball doing well he sees this ball doing this kind of thing so let's figure that out if it was totally motionless it would sort of follow this black line. That's how he would see the ball go if it was motionless like that's the path he sees the head of this observer on but because there is some motion this thing drags down just a little bit, so it ends up here. The point is that it sees a much more complicated path of that ball, that complicated path is $x^{\mu}(\tau)$ that's this path here and the relationship to figure out what this path is, that's the interesting path to figure out, what that path is I have to solve this equation (34) and notice all of the relationships between the 2 coordinate systems, the relationship between this coordinate system and this coordinate system it's all buried in this guy (35) which we're going to call the connection and we remember the connection from our previous work represented how to understand parallelism in a coordinate system so ultimately this whole thing leans on the equivalence principle because we stipulated that this ξ must exist and it exists it's freely falling so it follows that that little lab frame and x is an arbitrary coordinate system and the manifold assumption it gives us what these derivatives are tells us these derivatives exist the manifold assumption tells us we can lay down the coordinates and that we can create these derivatives and thus we end up with this equation (34) and so ultimately well what is that equation, how do you tackle that equation well we're certainly not going to do it in this lecture.

If I was to take this thing, this is a series of equations, each is a double sum, there's 4 of these guys. What do I do there? There's going to be 16 of these terms, there's going to be 4 for the first index and for the second. It turns out if you look closely at this definition back here (34) because these partial derivatives, this is a continuous function real value and because of the nature of calculus it's symmetric in μ and ν so 16 really gets cut back to 10, so this is actually 10 terms so you can imagine is quite a mess, it's a very complicated equation and of course they're coupled together in a really fascinating way so it's not easy to solve for this path in an arbitrary space-time but that is the assignment that is our job in Special Relativity to me that's the one of the most interesting questions and we're going to spend a lot of time on it but anyway that's the Geodesic equation directly from the equivalence principle and of course before I go it's obvious if you want to review this in more detail and much more elegantly the place you want to go is Steven Weinberg's Gravitation and Cosmology. I'm sure everybody understands I did not invent the Geodesic equation this is clearly a long old established law of General Relativity and it can be found in many many different books however the only place I've ever seen the derivation straight from the equivalence principle is in Weinberg's book and I think it's really important an elegant place to start.

The next several lectures are going to be about different derivations of the Geodesic equation, we're really going to sort of study that pretty thoroughly and because I think it's fun and interesting, so that's all for today thank you and I'll see you next time.

Lesson 4: Introduction to the Connection

In this lecture we are going to talk about the connection and we were introduced to the connection in our last lecture when we talked a little bit about the Einstein equivalence principle and geodesics and we're gonna see we're gonna now focus this time on the connection now what did we do last time, well last time we assume that we had an inertial coordinate system and we've been drawing this picture of a guy sort of in a free-falling lab but imagine that he's just in a lab moving in the space-time of Special Relativity so the whole space-time everything is all Special Relativity which ultimately means there's no gravity and space-time is flat and it means that the metric can always be written in the form everywhere in the space-time:

$$\eta_{\alpha\beta}=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{36}$$

It means that at every point in space-time this is the value of the metric it's completely constant and it's the same everywhere now what is important though is that the metric is in this form if you're dealing with what amounts to a Cartesian coordinate system in space (x, y, z) in space and then a simple time coordinate bound together by the fact that the interval of proper time is:

$$d\tau^2=-dt^2+(dx)^2+(dy)^2+(dz)^2 \tag{37}$$

If you've got to have this kind of coordinate system to have this kind of proper time element (37) to give you that specific metric (36) but this coordinate system (x, y, z) is arbitrary we could do Special Relativity. In Special Relativity flat space-time but without this coordinate system we can use a Polar coordinate system we could use a Cylindrical coordinate system and that wouldn't change the fact that space is flat just because we're sort of laying out the coordinates in a different way and if you did that, this metric would be different, it would be different but it would still be flat. Not only would it be different but it may not be the same at every point in this case it is this number (36) and every single point in space time but if you change the Spherical coordinates and we'll see later when we do this example in this lesson that this metric is actually now a function of space-time and yet the space is still entirely flat. From this idea we're gonna create the notion of a connection in completely flat space-time and then will invoke the Einstein equivalence principle (EEP) to talk about how this might be a good theory of gravity.

Going back to just the connection as it is, let's consider that we are using these ξ coordinates and I'll use the letter a for those coordinates and these coordinates are going to be yours straight up Cartesian spatial system with the Lorentzian time element so it's going to be your basic Minkowski coordinate system and then there's an alternative coordinate system which we'll call x^{σ}. In this coordinate system this could be Spherical coordinates so if it was circled the coordinates in Minkowski system $x^0=$time would still be sort of the time coordinate, $x^1=r$ might be the radial coordinate, $x^2=\theta$ might be the polar angle which comes second in Spherical coordinates and usually I think the third is usually the azimuthal angle $x^3=\varphi$. This might be your Spherical coordinate system but ξ is just going to be your regular (t, x, y, z) system so those are the 2 systems so I'm not going to fix this as the Spherical coordinate system, I'll do that as an example later but it could be any system for example the Spherical coordinate system but for it to be a system of any use at all it has to be the case that I can write the coordinates in one system in terms of the coordinates in another.

I know it that I have these kind of expressions $\xi^a(x^\sigma)$, I have an expression that will tell me the a coordinate in the Minkowski space if I know all of the coordinates in the alternative system and of course likewise we have the same situation going backwards, I can figure out any individual coordinate of a point if I know all the coordinates of the point in the Minkowski system $x^\sigma(\xi^a)$. Now we have an object that is moving in a straight line in the space-time of Special Relativity, we are in an inertial coordinate system which means that our coordinate system ξ^a is inertial. Last time when we looked at this I said because of the Einstein equivalence principle (EEP) we know that a coordinate system exists in the special space-time of General Relativity, we know that such a coordinate system exists at a given point, in fact it's a little better than that it's usually along a line but maybe you will talk about that a little more when we talked about geodesics but we definitely know that at any given point we can find this coordinate system. Right now we're not gonna do that, we're gonna say the whole thing is Special Relativity so we don't even have to look for a coordinate system that satisfies … we don't need the equivalence principle because we're not dealing with anything that's free-falling we're dealing with things that are just moving uniform velocities entirely inertially in space-time of Special Relativity and this coordinate system has it and we know that this coordinate system is rectilinear Minkowski so it's got $\eta_{\alpha\beta}$ as its metric and I don't even really need to write the metric as a function of space-time, it is but it's a constant function and it's always (36), that's what we're dealing with here.

Now we have this alternative coordinate system x^μ and we we know both of these functions we know $x^\sigma(\xi^a)$ as a function of the inertial system and we know the inertial system $\xi^a(x^\sigma)$ as a function of the arbitrary system and because we are dealing with an object that's just moving under no forces in this coordinate system $\xi^a(x^\sigma)$ I will now write down the path of that object and the path of that object I'm going to call $\xi^a(\tau)$ where τ is the proper time of that particle moving in this inertial coordinate system and I know because it's inertial motion and there are no forces applied I know that $d^2\xi^a/d\tau^2$ the second derivative of that path with respect to its proper time is going to be 0 it's not accelerating. That's another way of saying that in this coordinate system $\xi^a(\tau)$ this object is not accelerating in any way, it's just moving along on its steady path.

This whole equation remember here we're leaving out the fact that very frequently in General Relativity textbooks don't include this key variable τ in these kind of expressions but this is what they mean that's a τ right there and so that again it can get confusing pretty fast if you're not on top of it. We know that this is the case so now what did we do, we just took both derivatives, we took the first derivative with respect to τ and then the second derivative with respect to τ and we obtain (35).

If we think in terms of force equals mass times acceleration, if we just kind of go back to ξ coordinate system, this is telling us about the acceleration of things, this is the acceleration of the body that acceleration is 0 , so if I multiplied everything by mass here I'd get mass times acceleration equals 0 so the forces are 0 . If we use the Geodesic equation (34), here I would get mass times acceleration, the first term here represents force but then I also get a mass in the second term here times something else in order to get this 0 so this is mass times acceleration and then if I move this to the other side I get:

$$m\frac{\partial^2 x^\lambda}{\partial\tau^2}=-m\Gamma^\lambda_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau}=-m\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{38}$$

I get the velocities in this new coordinate system times some interesting number which is a connection multiplied by mass so this side has got to be a force because this side is mass times an acceleration so the right hand side is a force.

We have this force that appears just because we're in this arbitrarily new different coordinate system that's not straight up inertial Minkowski space and that term, normally the first exposure to anything like this is the Coriolis effect when we took when we were simply exchanging coordinate systems to the, I think it's the surface of the Earth is the common one, and we start seeing you know forces that move things to or away from the equator depending on how they're spinning and all of that stuff but this is a force that appears only because we've changed our coordinate system and that is typically called a Fictitious force. This is the important part, these fictitious forces appear and you'll notice this key thing the force is proportional to the mass it did not have to be that way we could imagine a force that was, well it did have to be that way if you believe that $F=ma$ there's nothing in our logic that mass is forced to be in there but it's a fictitious force and it's proportional to the mass.

That is the key observation that makes us think we'll wait a minute what other force is proportional to the mass the force of gravity is some constant times the mass of the gravitating body the body that's the source of the gravitation times the body that's the subject of the gravitation divided by the square of the distance between them so this means that force is proportional to the mass of the body that's subject to the gravitation now clearly there's an issue here if you're dealing with 2 bodies that are subject to each other's gravitation interestingly you could always kind of move to a frame where that sort of goes away In the approximation that we have a gravitating mass and a test particle gravity is a force that is proportional to the mass of the test particle but you notice how this fictitious force appeared entirely through considerations of nothing but the geometry of how we lay out our meter sticks in space-time and our clocks as well and this is the central idea upon which, in my opinion, General Relativity kind of rests, it rests on the idea that gravity is in fact an artifact of, not simply the coordinate system we choose but it's an artifact of the fact that the inertial nature of space-time changes under the influence of gravitational forces and it does so in a very specific way and these guys here (38) are what drives it.

Now in this case what this is, it is simply a bookkeeping tool to keep track of the difference between say spherical and rectilinear coordinates but in this case we're also able to say that all of space-time was flat everywhere in this coordinate system we had a metric like (36) and whatever the coordinate transformation is, this new coordinate system x^{μ} has its own metric which may in fact depend on space-time or in the position in space and time but it's still flat and so there is always the question of how do we know when we look at a metric whether or not it's flat and the answer is: well can you transform it back into (36) everywhere, if the metric is everywhere transformable simultaneously to (36) you know you have a flat space-time and another way of saying it is that the Riemann curvature tensor, which is a rank 4 tensor, is 0 in every coordinate system, if it's 0 i.e. all of the components have to be 0 in every coordinate system and they will be 0 in every coordinate system because that's what tensors do, tensors if they're all 0 and one coordinate system they'll all be 0 in all coordinate system. This notion of this Fictitious force is really important and the connection comes up in that context so maybe we can just whip up a quick example in flat space.

We'll illustrate this in flat space and we'll use Spherical coordinates and we're actually going to do this in terms of pure geometry just because it's really just illustrating the process so meaning when I say pure geometry we're just going to consider very easy to visualize (x, y, z) right hand Euclidean space and then the Spherical coordinates (r, θ, φ). It's pretty straightforward the relationship between the 2. Just to stick with the spirit of the notation of this subject this I'm going to call (ξ^1, ξ^2, ξ^3) and this I'm going to write (x^1, x^2, x^3). The way this should look is:

$$\begin{cases} \xi^1 = x^1 \sin x^2 \cos x^3 \\ \xi^2 = x^1 \sin x^2 \sin x^3 \\ \xi^3 = x^1 \cos x^2 \end{cases} \text{ these are } \xi^a(x^\mu) \tag{39}$$

This can be inverted I should be able to write:

$$\begin{cases} x^1 = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2} \\ x^2 = \arccos\left(\dfrac{\xi^3}{\sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2}}\right) \\ x^3 = \arctan\left(\dfrac{\xi^2}{\xi^1}\right) \end{cases} \text{ these are } x^\mu(\xi^a) \tag{40}$$

Everything is totally flat space-time here all we're doing is changing the coordinate systems so now once I have that now I can create the these important functions over here. We need the derivatives with respect to each other well what we need is we want to $\Gamma^\gamma_{\alpha\beta}$ according to (35). What are we doing in this circumstance we're taking a coordinate transformation in flat space-time in the space-time, well it's not even the space-time of Special Relativity, now because we're not including time, we're just this is just normal Euclidean geometry in 3D so we've got 3D of flat space in Spherical coordinates and in Cartesian coordinates and these connection coefficients are going to be introduced as a result of this in this coordinate system inertial motion which is going to generate these connection coefficients.

That basically means that we're going to see things forcing an acceleration in the r direction because remember as a particle moves linearly as a particle moves say in this direction r gets smaller and smaller and then bigger and bigger but not at a steady rate. r changes faster right around here (the shortest distance to the origin) and so the change in r is actually accelerating and then the farther it gets away r tends to just change less and less and appears to have great and greater similarity to what say the y coordinate would if it was moving in this direction and that acceleration is driven by a fictitious force that's captured through this connection coefficient and it's expressed in (34).

$$\Gamma^1_{22} = \frac{\partial x^1}{\partial \xi^1}\frac{\partial^2 \xi^1}{\partial x^2 \partial x^2} + \frac{\partial x^1}{\partial \xi^2}\frac{\partial^2 \xi^2}{\partial x^2 \partial x^2} + \frac{\partial x^1}{\partial \xi^3}\frac{\partial^2 \xi^3}{\partial x^2 \partial x^2} \tag{41}$$

$$\Gamma^2_{12} = \frac{1}{r} \quad \Gamma^3_{13} = \frac{1}{r} \quad \Gamma^1_{22} = -r \qquad \Gamma^3_{23} = \frac{\cos(\theta)}{\sin(\theta)} \qquad \Gamma^1_{33} = -r\sin(\theta)^2 \qquad \Gamma^2_{33} = -\cos(\theta)\sin(\theta)$$

It can be done for any system as long as you can write down the actual transition functions, once you have these transition functions you can calculate these connection coefficients, In this work we've it was all flat space so we were dealing with purely a coordinate transformation that was nothing else than a coordinate transformation and to understand this in the context of the Special Relativity that we've been talking about in the beginning we know that the metric the flat space-time metric for all of space is (36) so what is the metric in the new coordinate system? In the new coordinate system you know that this τ interval, this interval of proper time (28), that's invariant and we know that you can convert this by just using the chain rule:

$$d\tau^2 = \eta_{\alpha\beta}(\xi^\sigma)\, d\xi^\alpha\, d\xi^\beta = \eta_{\alpha\beta}(\xi^\sigma) \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}\, dx^\mu\, dx^\nu \tag{42}$$

This object here which is a function of space time is:

$$g_{\mu\nu}(x^\mu) = \eta_{\alpha\beta}(\xi^\sigma) \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \tag{43}$$

The flat space-time metric gets converted into a different metric using this prescription here (42) and that prescription should be entirely recognizable to you as the result of a change of basis of the coordinate system, it basically changed at the coordinate system and it goes right back to our what is a tensor lecture for how the components of a $(0,2)$ tensor transform and that's all it is, you can look at it from this perspective as simply the chain rule and in fact in most physics courses that's all they do to get you there they say okay this is now the definition something that transforms that way this is the definition of a $(0,2)$ tensor. That is what the metric is but this is still a flat space-time metric because I've stipulated that $\eta_{\alpha\beta}(\xi^\sigma)$ is a coordinate system where the metric is flat and constant everywhere $g_{\mu\nu}(x^\mu)$ is still a flat space-time it's just expressed in different coordinates.

When I relax that assumption when I say, no, this is only true at a single point, whatever the metric is in this space-time, I can only guarantee you that at a single point or truly along a single line, a free-falling observer but at any given point for sure I can take this coordinate system, I can take whatever coordinate system I'm running in and whatever that metric is in that coordinate system I can convert it to a different coordinate system where at that point the metric will look like $\eta_{\alpha\beta}(\xi^\sigma)$, but I can only do it by the point by point basis, that's the application of the Einstein equivalence principle to our Physics and that's not a trivial or arbitrary maneuver, that's actually very profound. What I'm basically saying is I'm not dealing with the entire flat space-time of Special Relativity, I'm not dealing with it where $\eta_{\alpha\beta}(\xi^\sigma)$ exists everywhere in this inertial coordinate system but I am saying that there is a coordinate system covering this general relativistic space-time which would include gravity, there is a coordinate system out here and that there's also a metric associated with that space-time $g_{\mu\nu}(x^\mu)$ and that metric is a function of these coordinates and there is that metric out there but I can never be sure to be able to take that metric and turn it into the flat metric everywhere, there's no conversion from x^μ back to some inertial system ξ^a where the metric goes from some value on the space-time and all different values all over the place to $\eta_{\alpha\beta}(\xi^\sigma)$ everywhere, that doesn't exist that coordinates transformation doesn't exist but if I go to any single point and I calculate $g_{\mu\nu}(x^\mu)$ at that single point say x^P , I'll get some collection of numbers and there is a coordinate transformation that will take that particular collection of numbers and turn it into $\eta_{\alpha\beta}(\xi^\sigma)$ at that point and once I stipulate that, that is what on what I'm now saying is that space-time is modeled by a manifold and that manifold gives me a coordinate system by definition of what a manifold is I get a coordinate system, space-time is somehow the model of our physical reality and I claim that it's a manifold so I can give that manifold a metric tensor I certainly can give that manifold tangent spaces and in every one of those tangent spaces there's a vector space at every little location in space-time those are the tangent spaces and I can apply to every one of those tangent spaces and inner product and therefore I can give a metric to every position in space time and the Einstein equivalence principle says: yes you can do that but you better make sure that every one of these points that has the value of the metric at every point in space-time which is another way of saying the inner product and every single point in space-time, for the tangent space at every point in space-time you better be able to just arbitrarily swap out these coordinates to something so that that metric at any given point is always going to be is $\eta_{\alpha\beta}(\xi^{\sigma})$.

That's an application of the Einstein equivalence principle to General Relativity and that's the stipulation that we use and we use that stipulation to generate that Geodesic equation. By the way I didn't reference this as the Geodesic equation this time but you see it is I think I may have actually because I'm talking about the path in a different coordinate system but the geodesics are the same because in this case because we're talking about entirely flat space-time back here in this situation the geodesics are always going to be straight lines it's just straight lines expressed in some random coordinate system versus straight lines expressed in the flat space coordinate system, the inertial coordinate system where they really do look like straight lines themselves, the coordinate system lines themselves are straight.

That's enough for now we will continue discussing the connection in the next lecture where we will talk about it in terms of the Covariant derivative and we're going to start getting into the Covariant derivative.

Lesson 5: The Catalogue of Spacetimes

For this lecture I've actually decided to lean into some kind of source material to help everybody go along through this course and the source material I've chosen is the catalog of space-times and the catalog of space-times can be found on the web here at this address.

It'll be sort of a textbook for us in a sense, I don't want to just take a textbook and work through because that's probably what everybody's already done. this is more of a reference material and what it purports to be is a list of all the different solutions to the Einstein equation:

$$\boldsymbol{G}_{\mu\nu} = 8\pi \boldsymbol{T}_{\mu\nu} \tag{44}$$

which we will talk about eventually and will actually execute a solution of that equation for the Schwarzschild geometry but this is a catalog of all of them, the Schwarzschild geometry is difficult as it is, is actually easy as far as these things go but there are some much more difficult ones. This book doesn't show you how the solutions were made, this doesn't derive any of these space-times, it only shows you the final results which is kind of interesting but we might hunt down some other solutions but we'll definitely go through the Schwarzschild one together and otherwise if the catalog of space-time is this nice reference material. Another piece of language you could say here is, it's a catalog of metrics because remember how what we're supposed to be understanding as the basic concept of General Relativity, we model space-time with a 4D manifold but we know that space-time is modeled by a manifold and that manifold will have a metric and that metric is going to be a solution of the Einstein equation (44) so whatever that metric is, it's big constraint as it has to solve the Einstein equation and that metric lays down an inner product at every point in space-time which is attached and if it's the inner product is on the associated tangent space at each of these points in space-time and it's a symmetric as a function of the space-time coordinates so we have a manifold we have a metric that satisfies the Einstein equation and also whatever this metric is, not only does it satisfy the Einstein equation but it's got to be such that if I pick any one point in space-time, any individual point, I can find a coordinate transformation $x^{\mu} \rightarrow x^{\mu'}$ and that coordinate transformation will guarantee that the value this point I picked, let's say I picked the point S that:

$$g_{\mu'\nu'}(S) = \eta_{\mu'\nu'} \tag{45}$$

meaning that I can always make the metric Lorentzian at any given point in space-time, that is assured by the Einstein equivalence principle, that is a statement of the Einstein equivalence principle. For flat space it's much stronger, if there is no curvature, one of the solutions of the Einstein equation is in fact the metric of flat space-time everywhere being (36). That means if I lay down a coordinate system a Cartesian coordinate system on a flat space-time the metric (36) will be a solution of the Einstein equation and will satisfy the Einstein equation and by default the Einstein equivalence principle is satisfied because every single point has this metric so it's a constant metric, you change coordinates on the space-time if it's flat the metric never changes as long as you're in this Cartesian coordinate system and that's the way we determine a flat space-time is if you can take your metric and you can find a coordinate transformation that will make that metric equal to the standard Cartesian coordinate system flat space-time Lorentzian metric which is (36), if you could find such a transformation where everywhere in space-time you get this, you have flat space if you can find a transformation where you can only do it at one point at a time or I should say actually one line of points at a time it turns out if you can do it for more than just one point at a time you can do it for a line of points.

At least at one point at a time but not the whole space at a time then you have a curved space-time and then the idea is that curved space-times model the real world. That's really what we're going on and the catalog of space-time has all of these wonderful space times in it so so the idea is that you guys get this you download it for free and I mean it's an academic publication that's freely available and thank these guys for putting it together because it's really great.

What will we see in this catalog? Well the first in the beginning of the catalog in the early sections they talked about the basic objects of a metric and this is gonna be a little bit of a guideposts to how our course is gonna work the basic objects of the metric means once you have a metric if you assume that they have a metric on space-time which is what we're doing remember we agreed that we were going to start our work assuming the existence of a metric and learning everything we need to learn and then at the end going back and solving the Einstein equation for a metric, but we're gonna start with an assumed metric these are very hard problems finding these metrics that work but solve this big big hard hard problem but when you have a metric understanding the properties of the space-time once you know the metric, not so hard and you can answer questions like how do free what are the paths of freely falling bodies in space-time you can figure that out like you know you have a star here and then you have some kind of orbit you know you can discover that things will orbit really funny around stars if you know the metric around a dense star and I say a funny orbit because look at the one I just drew it what kind of loops around and comes back and crosses itself you can't do that in Newtonian mechanics no matter how dense this star is Newtonian mechanics will never predict an orbit that loops and crosses itself but of course that's totally doable in General Relativity and in fact in General Relativity you can even have an orbit that doesn't hit the star but is still permanently sucked into the star that can't happen in Newtonian space-time, if you're freely falling unbound you will remain unbound and you will depart in some kind of hyperbolic orbit but in a Schwarzschild space-time. You can actually be captured but the point is that these paths all of these paths can be calculated once you know the metric for the space-time around the star so we're assuming that you've solved all that, you've got this metric and then you can learn a lot of stuff from it.

The way this catalog works is they start with the metric and then they say okay you got all these things you can build from the metric: the Christoffel symbol of the first kind and the Christoffel symbol of the second kind. We'll have to decide why we're gonna call it a Christoffel symbol but notice the Christoffel symbol the second kind looks just like the connection $\Gamma^{\mu}_{\nu\lambda}$, we call this the connection but they're calling it the Christoffel symbol of the second kind and there's a reason, it's definitely a connection, there's no doubt that Christoffel symbol of the second kind is a connection and it does exactly what we've already said it does in the first few lectures talking about fictitious forces and then in the lecture on manifolds discussing the covariant derivatives and the notion of parallelism or might we talked about how to take a vector and find a parallel vector at another point and how the Christoffel symbol is so critical for that or how the connection was so critical for that.

$$\Gamma^{\mu}_{\nu\lambda}=\frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda}+g_{\rho\lambda,\nu}-g_{\nu\lambda,\rho}\right) \tag{46}$$

The Christoffel symbol is a very specific type of connection and look, look at what this is, this is so cool, the Christoffel symbol the connection is a function of the metric, don't worry about how now but look, if I know the metric I can take its derivative with respect to λ :

$$g_{\rho\nu,\lambda} \stackrel{\text{def}}{=} \partial_\lambda g_{\rho\nu} = \frac{\partial g_{\rho\nu}}{\partial x^\lambda} \tag{47}$$

This is telling us that we should expect to learn somewhere in this course how the metric itself defines the connection and remember the connection told us about parallelism and it also told us about how paths in space-time are calculated through the geodesic equation, we've done those already before we've not done any examples yet but we've understood in principle how that should work but what's important now is you can see here that we better learn why this is the case but it's really really interesting and it shows you why the metric is so important, the metric actually defines the connection and when you do that when you use the metric to define the connection, the connection becomes what they call the Christoffel symbols and it's plural because of course there are many of them, all the Christoffel symbols gathered together make a connection.

That's a little enlightening piece and it's a guidepost to how we want to learn our material here, then they have the Christoffel symbols of the first kind which are very very similar, the difference of course is entirely this first term $g^{\mu\rho}$, these are related by:

$$\Gamma^\mu_{\nu\lambda} = g^{\mu\rho}\Gamma_{\nu\lambda\rho} \tag{48}$$

Now the problem here of course with the notation of these Christoffel symbols and the connection is I've always been very careful to write my connections $\Gamma^\mu_{\nu\lambda}$ where there's no space, I'm not writing it $\Gamma^{\mu}{}_{\nu\lambda}$, I'm not doing it that way because this implies that the ordering is critical because this is not a tensor, there is no $\Gamma^{\mu}{}_{\nu\lambda}\, dx^\mu \otimes dx^\nu \otimes dx^\lambda$, that is not what we mean, this is a non tenser object so having this space here I think is misleading because it implies that the order is critical but it's not a tensor object but then look at this (48), if you look at this you see that you're kind of raising an index you're taking this ρ here and you're raising it up there in some weird way, normally if you did this, if this was a tensor you would end up with $\Gamma_{\nu\lambda}{}^{\mu}$ like that but you're not in this case and also if you use the regular metric to lower that μ where would it drop? The point is this definition is not the same as raising and lowering indices that's very important to keep in mind that the Christoffel symbols of the first kind in the second time they have a very clear relationship that looks a heck of a lot like raising and lowering indices but it's not because this is not a tensor so you got to kind of memorize this.

Then there's these other tensors: there's the Riemann tensor, this is about the curvature of space we're gonna understand how, if you have to geodesic paths 2 paths of objects falling through space time they're going to tend to separate meaning the distance between these points, the spatial distance between these points or actually I should say the space-time interval distance between these 2 points along these paths it's gonna grow or alternatively could shrink: if it was going one way it would be shrinking, if it's going that other it would be growing, doesn't matter. If these are geodesic paths, when I say geodesic I mean free fall particles that are falling only under the influence of gravity and when they do that they follow these critical paths these geodesic paths that can be determined by the Geodesic equation which we learned about and the geodesic equation is completely driven by the Christoffel symbols but now we know the Christoffel symbols are completely driven by the metric so these geodesic paths are completely driven by the metric but we know that there's going to be this change, these paths are going to separate, in flat space parallel geodesic paths never separate but in curved space when they start parallel they tend to separate, somehow they tend to either separate or converge and a measurement of that is driven by the Riemann tensor and ultimately we'll see how that kind of connects to the idea of tidal forces and things of that nature.

$$\text{Riemann Tensor}\rightarrow\ R^{\mu}{}_{\nu\rho\sigma}=\Gamma^{\mu}_{\nu\sigma,\rho}-\Gamma^{\mu}_{\nu\rho,\sigma}+\Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma}-\Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\rho} \tag{49}$$

Notice the Riemann tensor depends on the Christoffel symbol of the second kind which is the connection and it depends on this connections derivative, that's what this comma is, that tells you're taking a derivative but regardless taking a derivative is easy, still depends on the connection it depends on the Christoffel symbols and second kind which depends on the metric so this thing depends on the metric, you can actually write this out as a function of the metric if you substitute it for each one of these things, this guy (46) and what a mess because just substituting for say one of them is this 3 term object 2 factor and 3 term object here multiplied by that so that's a 3 term object and then here you have to take derivatives of those guys and put them in there so by the time you get that full substitution done it's a lot of different derivatives of the metric and notice it'll be the second derivative of the metric because here you're taking the derivative of the Christoffel symbol, the first derivative the Christoffel symbol but buried in that is the Christoffel symbol which itself has the first derivative of the metric so ultimately you're gonna have a term that depends on $g_{\rho\nu,\lambda\sigma}$, that's a second derivative and I'm pretty sure the second derivative is written this way, it's not with 2 commas.

This guy (49) depends on the second derivative of the metric ultimately the first derivative of the Christoffel symbols but then second derivative the metric then you have other things that you can build out of the metric you can build the Ricci tensor which is an important part of the Einstein equation and that's an actual tensor so this is actually raising and lowering well this is a contraction:

$$\text{Ricci tensor}\rightarrow\ R_{\mu\nu}=g^{\rho\sigma}R_{\rho\mu\sigma\nu}=R^{\rho}{}_{\mu\rho\nu} \tag{50}$$

You're contracting on ρ and you're contracting on σ and you end up with a contraction and a raise, is what it is and you only ultimately end up with one contraction on ρ . You're basically taking this guy $R^{\mu}{}_{\nu\rho\sigma}$ and you're contracting μ with ρ when you do that you end up with something called the Ricci tensor and then ultimately the Ricci tensor that can be contracted on itself because it has 2 components:

$$\text{Scalar curvature}\rightarrow\ R=g^{\mu\nu}R_{\mu\nu}=R^{\mu}{}_{\mu} \tag{51}$$

This is the Scalar curvature. I've never heard this word until I saw that in the catalog the Kretschmann scalar. I've heard of the Ricci scalar, never heard of the Kretschmann scalar but regardless it's just the curvature, the scalar curvature. This isn't obvious I'm just going through this, this isn't meant to be literally a lesson in all this but this is what the catalog says: once you have the metric we're gonna talk about these important objects and this catalog is going to catalog all of these objects for each known metric, that's really cool.

There are a few others here, the Weyl tensor is a function of the curvature tensor with a lower first index and then there's some anti symmetrization going on here, you may remember that notation where we put things in brackets when we anti symmetrize them, it's a little bit of anti symmetrization going on with the metric tensor there and that's another special tensor. The Weyl sensor is very interesting object it's an advanced object we're not going to need that one for a while. The covariant derivative of the metric is 0 that's has to do with a metric that's called Metric compatibility we don't have to worried about that now but the covariant derivative of vector fields established here that's just a definition that you already know from the what is a tensor lecture and the covariant derivative of a tensor field, I think

I also put that equation down so that those are just reference equations. Ultimately the killing equation will be very important to us, we don't need to discuss it now but it's basically saying the covariant derivative of a vector field it's a statement of a property if the covariant derivative of a certain vector field in this case I guess this vector field here, the covariant derivative of that vector field satisfies this expression meaning the covariant derivative μ with respect to ν and ν with respect to μ , if you add it up and get 0 you end up with a Killing vector and that's a very useful thing to know, every metric has its own collection of killing vectors and this has to do with directions in which the metric is constant, we'll talk about that eventually.

Then let's see, natural local Tetrad and initial conditions for geodesic. The natural local Tetrad that has to do with our Einstein equivalence principle and that's finding a frame of reference where the metric at that point is our famous and ever useful Lorentz flat space metric (36), if we can find a collection of directions at any point in space where the metric is expressed in those directions is this that's the natural local Tetrad now we haven't used the language Tetrad yet but we will and we'll talk about this when we're ready it's not too long from then all

Every book has a beginning and the chapter 1 of that this this reference book was sort of a definition of terms and explanation of the things that they're going to talk about so chapter 2 is where the fun begins and it talks about the space times that solve the Einstein equations and we can actually dive right in because the first space time it talks about is one that we already know a lot about, it's the Minkowski space-time which is the flat space-time and it's the space-time of Special Relativity. Let's just see how the catalog presents this because we'll learn a lot about what we're after. You notice that the first thing it does is it says what coordinate system are we're dealing with the space-time is not the same as the coordinate system. The space-time is the natural structure that models the real world so I've always drawn the space-time like this and I've thrown down these coordinates in a very indiscriminate and arbitrary way because they could be any system the space-time doesn't care what system you use.

When we cite in a catalog the space-time what we really mean is we're citing the metric, the metric of the space-time is what defines the space-time so space time equals metrics and in order to write down the metric I have to write down an object $g_{\mu\nu}(x)$ where x is the space-time coordinates of the points. In order to write this down I have to have a basis, remember the way we've learned this $g_{\mu\nu}(x)\, dx^{\mu}\otimes dx^{\nu}$ this requires, these 2 one forms, these 2 covectors, they are written in the coordinate representation so in order to write down this component to give you a number for this component I have to have a coordinate system because the coordinate system will define the covector and those covector will have components that define the metric. Likewise $g^{\mu\nu}(x)\, \partial_{\mu}\otimes\partial_{\nu}$. Now we're kind of dealing with just components here but the point is you need a coordinate system otherwise this doesn't make sense and so it talks about the space-time which is referring to a metric then it says that metric for the Minkowski space-time which is flat, that flat space-time, if I have Cartesian coordinates meaning I have 4 real numbers, every point in space-time can now be written as (t, x, y, z) now they'd like to do that in this catalog, they like to sort of use these basic sort of elementary physics engineering notation for the coordinates so we would have written this as (x^0, x^1, x^2, x^3) and in fact in order to understand the way I'm going to teach it you're gonna have to be able to move between these two really smoothly because I want to be able to write $g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$ and I want you to see that there's an Einstein summation, you can't do that if you if every coordinate has its own name, there's no way of creating an Einstein sum so you have to understand that this is the pairing and this is going to happen for every coordinate system we talked about I'm going to move fluidly between this kind of notation and whatever convenient set of symbols the catalog uses, also the catalog does not set $c=1$ so from time to time I'm positive I'm gonna get tripped up and confused by that, you shouldn't be.

Perhaps I should put ct here for x^0 : (ct, x, y, z) and (x^0, x^1, x^2, x^3) . What I've just done is I've just added that c there and my x0 is still going to be in length units so ct would be in meters so at least all of them are the same units this way. Truth be told the catalog does not do that, the catalog does not put a c there, the catalog has time in different units from space as far as the point go so a point in space-time here, the catalog is going to write as a time on a clock and then a position in space (t, x, y, z) , I want to do it as a position in space and time (x^0, x^1, x^2, x^3) and then there's this intermediate place where we say (ct, x, y, z) which is a position in space and a position in time but I throw the c in to the coordinate. If you leave the c out of the coordinate you're gonna see that you need to put the c in somewhere and it'll be in the metric but let's get to it first.

What what are they trying to show us here? Well the first thing they jump on is this thing called the interval ds^2 is the space-time interval and this is how you calculate an infinitesimal piece of space-time interval but look what they say the Minkowski metric in Cartesian coordinates reads this, they don't say the Minkowski interval, they say the Minkowski metric so what gives? Well what gives is that they're leaning on $g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$, they think you know this by heart and of course you do, so you know that this thing really is:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \tag{52}$$

This thing equals $g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$. I left out all of the non-diagonal terms, I only have this diagonal terms because I know they're 0 in this interval, the point is the reason they say the Minkowski metric in Cartesian coordinates is this and then they present to you the interval is because they think you're so smart that you just understand the whole point of showing you the interval so you can read off the metric and with this case the metric is (36). That's how you use this thing, when you want to see what the metric is you've got to read it off the interval that's how they present it. they don't present it as a list of components of metric and some books do but not this one, this one leans on the interval.

All Christoffel symbols as well as the Riemann and Ricci tensor vanish identically so what's important to understand there is the Christoffel symbols are those connection coefficients $\Gamma^{\mu}_{\alpha\beta}$ that we calculated and in this coordinate system only in the Cartesian coordinate system of a flat space-time well these Christoffel symbols vanish and they will always vanish because if you go back to the equation we use to calculate them (35), remember there was the second derivative of a constant so this derivative is always 0 so every one of these terms is going to be 0 . The Christoffel symbols which is just another word for the connection which we calculated in our last lecture using this formula using (35). We're not using the Christoffel symbol definition in (46), that's something we're going to derive later this form it's going to be ultimately equivalent for these various space times but remember the connection itself has some more to do with the transition between the coordinates so it's no surprise that Christoffel symbols are 0 . The Riemann and Ricci tensors we haven't studied those yet but when we do we'll realize they're related to curvature and they will vanish identically not because of the special coordinate system we're in, the Cartesian system and that Cartesian system makes these Christoffel symbols vanish identically but the Riemann and Ricci tensor those are going to vanish identically because the space-time is flat so what that means is that when we change coordinate systems for example we go to cylindrical coordinates we're going to discover that the Christoffel symbols may not be 0 but the Riemann and Ricci tensor will always be 0 .

The Cylindrical coordinate system is Minkowski space expressed in cylindrical coordinates so what we have to appreciate now is that we're starting from the coordinates of a point in the Cartesian system ξ^{α} and we're moving to a point in the Cylindrical system x^{μ} so if this is the coordinates of the point in the Cartesian coordinates of a point in space-time in Minkowski space-time in the Cartesian system and the coordinate at the same point in some curvilinear system so we know that we should be able to write the points in the Cartesian system as a function of the coordinates in the curvilinear system $\xi^{\alpha}(x^{\mu})$ and likewise we should be able to take a point to the curvilinear system and find its coordinates if we know the point in the Cartesian system $x^{\mu}(\xi^{\alpha})$ and these are the transition functions. Now we can calculate the connection coefficients and for example we could calculate $\Gamma^{r}_{\varphi\varphi}$ but in the cylindrical coordinates you may remember z doesn't change so the transition function is going to be $z=z$ the others will be $r=\sqrt{x^2+y^2}$ and $\varphi=\arctan2(y,x)$. $\arctan2$ is a fancy $\arctan$, it's the inverse tangent function but it returns the correct quadrant depending on the sign of y and the sign of x , it eliminates the ambiguity, it's actually a computer science term but just think of it as the inverse tangent it's just as a computer term. The catalog actually uses it in an earlier section, the catalog uses this $\arctan2$ but that is just a function a special version of the inverse tangent that gets the quadrant.

These are your transition functions here's a transition function for r for φ and for z and likewise there's a transition functions going the other way and from those transition functions you can calculate Christoffel symbols but you'll notice that they're not all 0 , there's 2 of them that are nonzero:

$$\Gamma^{\varphi}_{r\varphi}=\frac{1}{r} \quad \Gamma^{r}_{\varphi\varphi}=-r$$

This metric is not not crazy different than the original metric but it's not constant in space-time, now you have to remember that $g_{\mu\nu}(x)$ is a function on space-time in particular it's a function of x , in this case is a function of r so if this is the origin of your space-time coordinate system (arbitrary point) the farther out you go the bigger r is the bigger this g_{22} term gets, so now you have a metric that's not constant on space-time, however, the beauty is you execute one coordinate transformation, you take this metric and convert it to (t,x,y,z) so one transformation will eliminate this r dependence of the metric and convert it into Minkowski in Cartesian coordinates. Likewise in Spherical coordinates same story. The time part doesn't change much, I'm going to change the coordinate on time in the next lesson.

The local Tetrad that'll be an interesting thing to go through but it is basically the Einstein equivalence principle frame that has a metric that doesn't look like this the metric in a different frame looks like the regular Cartesian metric, there's a frame of reference where this metric guaranteed looks like Minkowski in Cartesian coordinates and that's what this Tetrad thing is all about. This Tetrad thing is all about finding that frame of reference, that special frame where the metric is flat and Cartesian.

Let's look at one where we do it again it's just another coordinate transformation but we're going to start with the cylindrical coordinates so the new coordinate system will be called Rotating coordinates and instead of just changing the way we lay down things in space, we're gonna lay down change how we lay out some things in time so normally we're dealing with polar coordinate system, I guess in this case it's φ and we also have θ which I'm suppressing and of course we're suppressing time but now what we're going to do is we're going to make arbitrary coordinates. We can make any coordinate transformation we want and in this case the one we're gonna make is we're gonna say that in the new system, I guess we'll call it the prime system, $t'=t$, $r'=r$, $z'=z$ and $\varphi'=\varphi+\omega t$. This is a completely arbitrary coordinate transformation.

$$ds^2 = -\left(1 - \frac{\omega^2 r^2}{c^2}\right)\left[c\,dt - \Omega(r)\,d\varphi\right]^2 + dr^2 + \frac{r^2}{1 - \omega^2 r^2/c^2}\,d\varphi^2 + dz^2 \tag{53}$$

with $\Omega(r) = \dfrac{r^2\omega/c}{1-\omega^2 r^2/c^2}$ then $\quad g_{\mu\nu} = \begin{pmatrix} \omega^2 r^2 - c^2 & 0 & \omega r^2 & 0 \\ 0 & 1 & 0 & 0 \\ \omega r^2 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

First of all notice that we're starting the coordinate system with this polar, the un-primed coordinates is the polar system, the prime coordinates is very close to the polar system except for this last coordinate which in the transformation is $\varphi' = \varphi + \omega t$ so it's actually a function of 2 of the old coordinates which is fine because $\varphi'(t, r, z, \varphi)$. Now the way we really want to write this is $x^{\mu'}(x^{\mu})$, that's my favorite notation I like to put the new coordinates give it a prime but I put a prime on the index. An alternative way of writing it is $\hat{x}^{\mu}(x^{\mu})$, the new coordinates are the hatted ones the old coordinates are the non-hatted ones. The implication of this is that what we're doing is we're rotating the coordinate system. The axis that defines φ is actually rotating with an angular velocity ω so ω has to be in units of angle per time, now notice that in this case it's pretty clear time is measured in seconds so it will be angle per seconds because remember they always they have their c not set to be 1 but if $c=1$ it's no problem, it's just the angular velocity would now be instead of rad/sec it would be rad/m where meter is the measure of time because $c=1$, just keep that in mind, it's just the units are or all that matters and notice the way this is done by the way it doesn't matter because whether time is they're not that transformation is going to be just $t' = t$ whether t is measured in meters or seconds, here the significance is you got to get the right units for your angular velocity.

Anyway so you're going to make this transformation but so everything is the same: $dt' = dt$, $dr' = dr$, $dz' = dz$ and $d\varphi' = d\varphi + \omega\,dt$ so now we can make this substitution into our spherical coordinate system that means we're starting with this metric here:

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\varphi^2 + dz^2 \tag{54}$$

We're making a coordinate change by changing φ according to $\varphi' = \varphi + \omega t$ so that $d\varphi' = d\varphi + \omega\,dt$ I can solve this thing for $d\varphi$ and I get $d\varphi = d\varphi' - \omega\,dt'$. I can put that up in (54) and so I can now take (54) and just transform it straight out, meaning I make this substitution for the prime coordinates It's an identity transformation for everything except this last one. If I did that substitution I'm gonna end up with this metric (53) (without the primes) and notice how quickly these metrics get to be a mess, first of all when you look at this metric what are you gonna see? First they felt the need to combine the dt and the $d\varphi$ together because they could factor it together.

What is important and maybe this is why they did it, they're concealing a mixed term because this is going to have a term that looks like $-2c\,\Omega(r)\,dt'\,d\varphi'$ so there is actually an off diagonal element term in the metric so when you write down this metric you're now gonna have diagonal terms and $g_{t\varphi}$ and then the rest of these will be 0 so you now have off diagonal terms and I don't like to say off diagonal because it implies that the metric is some kind of matrix and it's not this is just a bookkeeping system just an organizational system so I can quickly find $g_{\alpha\beta}$ by just looking it up on a table.

This transformation that's what it's done $d\varphi \to d\varphi + \omega dt$ it's driven an off diagonal term so even if you have off diagonal terms this thing can still, through a simple transformation go back to $\eta_{\alpha\beta}$ which is (36), the Cartesian system flat space-time or one transformation away and it is not obvious, if you started with this thing don't tell me that you could look at (53) and say oh yeah this is obviously flat space this is still Minkowski flat space, it's just the reference frame is spinning that's really what it's all about, so how would you recognize that to be flat space you would have if you started from this and you had to come up with a coordinate transformation to get you back to this $\eta_{\alpha\beta}$ in order to prove it was flat space you'd be in big trouble so the way you actually do it is you actually take this, you extract the metric tensor and they realize that they for some reason they just love this line element form and therefore they realize well we better actually write out the components so people don't mess around. My point is in order to see that this is flat space what you're gonna have to do is you're going to have to calculate the Riemann curvature tensor $R^{\mu}_{\alpha\beta\gamma}$ and if you calculate this curvature tensor and all the components are 0 then you know that you're dealing with flat space in some kind of weird form.

That's pretty good I think we've made some real good progress today we've jumped on the catalog of space times I've sort of introduced it as like that point of departure for this course I've decided that that's the point of departure we'll use and then we've reviewed and talked about flat space in several different coordinate systems and these are familiar coordinate systems with the possible exception of the rotating system but if you did classical mechanics you've done some work with a rotating coordinate system but now we've talked about it in terms of the metric and transformation of the metric and remember we started with this metric $\eta_{\alpha\beta}$ and then we took the transformation equations and we got this metric (cylindrical) and then we took this metric again $\eta_{\alpha\beta}$ and took the transformation equations for spherical coordinates so we got this metric (spherical) then we took this metric and we decided to take another transformation on this metric to get this metric (53) and all these things are just one transformation after another and they're arbitrary, there's no restriction on how I can do that, I'm allowed to take any kind of transformation I want and it will always end up with a metric that is equivalent always a transformation away from the flat Cartesian Minkowski metric.

On my next lesson we're going to do this one more time in a very counter intuitive but very useful way and I'm gonna show you how we can take the Minkowski flat metric, this Minkowski $\eta_{\alpha\beta}$ space-time metric and what we're gonna do is we're gonna compactify it, we're gonna instead of making it all real numbers we're gonna put it in a finite region of the plane and that is gonna be a very helpful thing down the road but also just get you loose with the idea of radical coordinate transformations, coordinate transformations indiscriminate, radical, wild coordinate transformation then that's the way it's going to look to us of course because we haven't studied this subject all our lives but it took many years for people to figure out the best transformations to use and why, it's a lot like topology was around for decades before someone finally figured out the best way to define open sets and things of that nature.

Likewise these coordinate transformations came about after a lot of thinking but I want you to understand that they are quite arbitrary and they can be used very flexible and powerful way so we'll do one more lesson like this but we're going to get away from the standard coordinate transformations Cartesian to Cylindrical to Spherical and even these rotating coordinates are pretty standard and we're going to do something that's very very strange and flexible that's going to have the intent of taking all of these these variables of Minkowski coordinate all these coordinates from the entire real line down to a limited amount of space and I think that's a good exercise to do next even though it's a little out of order for the non-standard course because it really starts giving you flexibility and understanding how arbitrary and important these coordinate changes are it also talks a lot about the time coordinate which confuses a lot of people and it shows how coordinate time can be this just arbitrary wacky number because even here in all these this time is the actual time of somebody, you just lay down Cylindrical coordinates you have a lab clock and you measure measure time and in the Spherical coordinates all you've done is thrown the time on so that's not a problem at all and even in the rotating coordinates you're tracking time, your time coordinate doesn't really change, whereas the g_{tt} the time coordinate is still just the coordinate time on somebody's clock.

Lesson 6: Introduction to compact coordinates

We're going to continue and get deeper into this idea of coordinates and we are not done really stretching the limits of what can actually be done with coordinate transformations and we are going to continue to focus on the manifold of Special Relativity, all General Relativity courses seem to start with a review of Special Relativity, we're not actually doing a review of Special Relativity you better get your own review of Special Relativity but the Lorentz transformations and all of the paradoxes and to some extent even the Special Relativity dynamics in particular of subatomic particles and things like that, that I'm expecting you to already feel comfortable with I mean nobody's ever a full expert on everything about everything I don't expect that I mean it'd be crazy it'd be hypocritical, I'm not an expert but but we're still going to use the manifold of Special Relativity and we understand that that manifold I can draw it in many coordinate systems and we went through several of those coordinate systems in the last lecture, we went through the Spherical coordinate system, we went through the Cylindrical coordinate system, we went through the Rotating coordinate system and we went through the flat coordinate system. these well they're all flat so that's not the right way to put it, they're all flat and by flat we mean that the metric can't always be put into the form (36) where these are meant to be the diagonal elements of the metric.

We always know that this doesn't have to be the metric of your particular manifold, it depends on the coordinate system but you should always be one coordinate transformation away from (36), with this constant value at every point in space-time. Normally of course in the other coordinate system for example the Spherical or the Rotating one that we did you'll have a metric $g_{\alpha\beta}(x^{\mu})$ where the metric coefficients will be a function of space-time, sometimes we could write it like that probably best to not have the superscript there because what we intend to mean is all 4 points at once if I leave the superscript in there you may think we you have to substitute for it and it's either a function of x^1 , x^2 or something like that, so it's actually easier to leave it like this $g_{\alpha\beta}(x)$ where you know that by x what we intend to mean is all 4 coordinates of the point (x^0, x^1, x^2, x^3) so that's the argument of this.

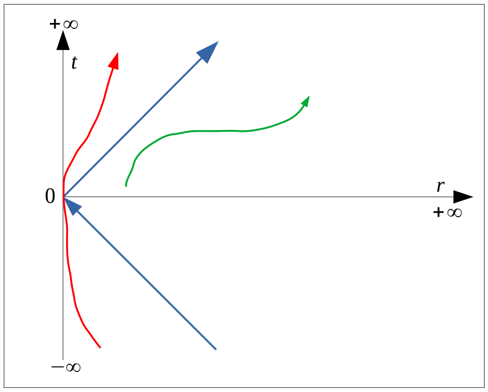
We know that for Spherical coordinates the metric is a function of space-time but 1 coordinate change later and boom you're in a new coordinate system: Cartesian rectilinear, Cartesian coordinates with a flat space-time metric so now we're going to push this into some really exotic territory and we're going to begin by taking our flat space-time and we're going to put it into Spherical coordinates with t for time r and then we're going to suppress θ and φ and that will be our Spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2[\text{Angular Part}] \qquad (55)$$

The angular part is meant to be this piece:

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \qquad (56)$$

We're going to talk a lot about the angular part eventually in this conversation so we're going to be using this metric here and this is now our starting point that's our first transformation, we can say that we've actually gone from $\eta_{\alpha\beta}$, we've done one set of coordinate transformations and we've gotten (55) and the new coordinates are no longer (t, x, y, z) , they're now the new coordinates are (t, r, θ, φ) and now we will plot this system in a very important and convenient way first of all time and space are still going to be measuring the same unit so I'm always going to have $c=1$ which is different than the catalog, they still retain a value of c but we're gonna have $c=1$.



t can go to $+\infty$ and it can extend all the way down to $-\infty$, r can go all the way to $+\infty$ from 0 and so the part of the plane that's relevant for this is clearly just this right half-plane. The θ and the φ angles are suppressed in this diagram and it's time to get used to that this is really important: when I pick out a point in space-time, if I plot it, I'm definitely plotting its time and I'm giving it a specific radial distance from the origin, I need to be a little caution with my word distance, I'm giving you a specific radial coordinate from the origin.

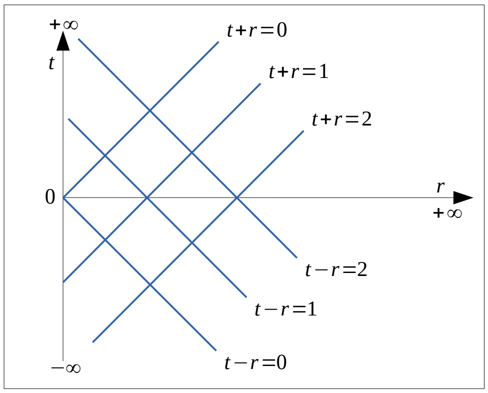
Remember distance is always driven by this metric (55) and I can easily have a coordinate system, for example without a metric for whatever reason, and all I'm doing is assigning labels to points at that point no metric and no structure whatsoever, there's some topological structure, that goes back to the what is a manifold thing, you don't need a metric to define a manifold so there is topological structure but there's no geometric structure, once you have a metric you've actually introduced some geometric structure you actually have a notion of true geometry without it you've got none of that.

I've established (t,r) but I can only plot 2D on this flat plane so (θ,φ) are suppressed which essentially means when I put a point on this plane I'm actually imagining an entire sphere and that sphere has radius r and that sphere exists at time t and the sphere itself has these 2 other coordinates, it has the polar angle coordinate θ and some azimuthal angle coordinate φ so there's a point on this sphere that has an a polar angle and an azimuthal angle that are unspecified so I'm thinking when I don't specify them I have to think of the entire sphere so when I look at a point in this diagram I need to think sphere and I need to think sphere of radius r existing at time t and it looks like a point but it's actually in a whole sphere and that's really important to remember and it'll be more important as we as we go on and start getting closer to our conformal geometry that we're interested in.

I could restore one of these, I could make it so that the only suppressed angle is θ for example I could take every point and swing it around into a circle, I can just rotate about this axis the t axis and I could create a disc, well a circle so then this point, I will need a 3D graph, but I could specify a φ coordinate between $[0,2\pi)$ so I would end up with what would amount to be a rotational symmetric graph at that point but I can never add θ, there's no way to add θ in also. I don't need to do a whole sphere if I wanted to, the price I pay is I have to kind of draw in 3D. I just want to have one point represent an entire sphere.

How does light move in this particular system well at this point it's really no different than how you might have seen it in Special Relativity drawings, light will move in a 45° line (blue) because we've still sets $c=1$ and our time axis and our space axis is identical and light will move, inbound light will come on a line from $c=1$ and outbound light will go out on a line from $c=1$ and if we did this little rotational trick of course this line would actually look like a cone and that is the light cone of the origin this would be the past light cone of the origin and there would be the future light cone of the origin and it would be rotated so this is one advantage of rotating it is you can really see a light cone so these 2 lines, this light cone line that's that's how things would move but you got to understand now that this point here on the line and this point here on the line these represents spheres so if I were to remove all of my light cone descriptions I would just talk about the light wavefront is at this point it's at this point and it's at this point and it's at this point in each time I look at those points I have to think of a bigger and bigger spheres of expanding sphere of light and if so it's really a light hyper cone because each of these points would represent a sphere if I was able to restore all the θ and φ .

Then of course from here we would understand that all time-light paths in this system would have to be in this area (red) something like this all time-light paths that go through the origin would have to be in this region in between the light cone and the t -axis and of course we know that since $r \geq 0$ and we're not doing this rotation trick we're only dealing with things in the right-half plane. Now that we understand this idea of the light cone and where these time-light paths would be, I guess I should indicate space-light paths would be in here (green) space-light pass would be in those 2 regions.



Now that we have that we can actually say well wait a minute maybe these light cone lines are are interesting to us what is that line and we said well that's the line where where $t=r$ or $t-r=0$ and this is the line where $t=-r$ or $t+r=0$ and I could then create a coordinate system where I have different values of $t-r$ and $t+r$ as the coordinates where these lines of $t-r$ equaling different amounts for example this one in which $t-r=0$ here $t-r=1$ or 2 or 3 or 4 or 5 . I want constant $t+r$ I need the lines that go this way.

I can lay down a coordinate system that is of course at this 45° using this simple coordinate transformation so let's formally do that let's formally introduce a coordinate and that coordinate will be:

$$\begin{cases} u=t-r \\ v=t+r \end{cases} \tag{57}$$

That is the coordinate transformation for the new coordinates as a function of the old coordinates so this would be $u(t,r,\theta,\varphi)$ likewise $v(t,r,\theta,\varphi)$ and then I can figure out:

$$\begin{cases} t=½(u+v) \\ r=½(v-u) \end{cases} \tag{58}$$

That's the other part of the coordinate transformation and from this I can calculate dt and dr and once I have dt and dr which should by the way the obvious it's going to be:

$$\begin{cases} dt=½(du+dv) \\ dr=½(dv-du) \end{cases} \tag{59}$$

With this I can very quickly make a substitution into (55) and when I do that then I've made a complete coordinate change I've made a coordinate transformation from t,r,θ,φ to u,v,θ,φ . Notice that I'm still keeping θ and φ so I guess the full transformation should include $\theta'=\theta$ and $\varphi'=\varphi$.

Things are going to cancel out and the only thing that's going to be left is:

$$ds^2 = -du\,dv + ¼(v-u)^2[\text{Angular Part}] \tag{60}$$

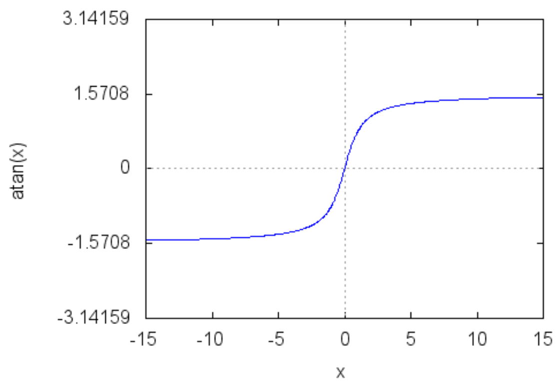
The angular part is unchanged because these two transformations are identity transformations so this is now the new metric in our new coordinate system and the only restriction comes from (57) since $r \geq 0$ that means necessarily that $v \geq u$. What's important to notice about this new metric is, if we were to example define $x^0 = u$, $x^1 = v$, $x^2 = \theta$ and $x^3 = \varphi$ just in order to put it into sort of a more familiar notation from the tensor lectures then we would say that $g_{01} = g_{10} = -½$ and that accounts for the symmetric metric coefficients connected to $du\,dv$ and then the metric coefficient $g_{11} = g_{00} = 0$ and then the metric coefficient $g_{22} = ¼(v-u)^2$ and $g_{33} = ¼(v-u)^2 \sin^2\theta'$:

$$ds^2 = -\mathrm{d}(u)\ \mathrm{d}(v) + \frac{(v-u)^2\,\mathrm{d}(\theta)^2}{4} + \frac{\sin(\theta)^2\,(v-u)^2\,\mathrm{d}(\varphi)^2}{4} \tag{61}$$

That is derived directly from the classic metric of the Spherical coordinates. This is not a diagonal metric, just because you start diagonal you don't have to stay diagonal, remember that's the power of arbitrary coordinate transformations: you can immediately make things non diagonal. That's one interesting thing and the other interesting thing is that there's no obvious time coordinate u and v , individually moving down a u and v axis, if we consider these two axes with the proviso that $v \geq u$, motion in the u direction we already know this is light-like motion, motion in the v direction is light-like motion. In our old system motion in the r direction was space-like which meant that if you evaluated this metric (55) where $dt = 0$ because you're not moving in the time direction and $d\varphi = d\theta = 0$ then integration of their motion in the purely r direction would give ds positive and positive intervals are space-like and likewise if you held r and θ and φ constant and only moved in the time direction you would end up with a negative interval and that's defined as time-like so you have a time-like axis and that's we can call this the coordinate time of our system because motion down that access is time-like that's the definition of what a time-like axis is but here you don't have a time-like access because motion on this axis u is null when motion on this axis v is null so you have to null axes and then you have to space-like directions because you still have θ and φ and it is still true that a point here is still an entire sphere and in principle now though it's a bit harder to talk about rotations around the u axis because v can go from $-\infty$ to $+\infty$ so this point still has an unspecified θ and φ so this is still an entire sphere but now light rays move along the axes and that's why this might be an attractive system is because light rays may move along the axes but obviously we've done nothing to change the fact that the metric of this space-time is flat despite the coordinate transformation which in this case really was kind of simple, this is not a complicated this is much simpler than the coordinate transformation that got us into this Spherical system to begin with, that was actually kind of a tricky coordinate transformation with sines and cosines and stuff this one is just this little additive thing and you got to kind of square a two term object and you end up with a metric that is not diagonal but it is still completely flat and it's flat because you can undo this transformation and get back to the Spherical system which you know is flat.

This starts getting a little bit more and more abstract but we're not done with our abstraction, we're actually going to go another layer of abstraction because now we're going to consider the uv system and one thing that's pretty clear is that it goes off infinitely, it goes off from $-\infty$ to $+\infty$ subject to the restriction that $v \geq u$, now the problem is that all of space-time is in this thing and I can't draw the whole space-time in here because I can't go to infinity on both sides.

What I would like to do is I'd like to bring this stuff at infinity into view with a mathematical transformation on the coordinate and this is actually not very hard to do, it seems like it'd be completely weird to take an infinite plane and squeeze it down into some sort of finite picture but we can actually do that, so we can you do that using the inverse tangent function. I can go from $-\infty$ to $+\infty$ and through the inverse tangent function I can go from $-\pi/2$ to $+\pi/2$ and I can get something that looks like this:



I have a monotonically increasing function that goes from $-\pi/2$ to $+\pi/2$ and the actual point at $-\infty$, I assign the value of $-\pi/2$ and at the point at $+\infty$, I assign the value of $+\pi/2$ so I can take the infinite line and I can squeeze it into a finite interval. The word we use for this is compactify which is kind of a loosely used term because it's not the same as compact in topology but it has the sort of the same feel: I've taken something infinite and created something finite. It turns out this final finite product is not really a compact set as defined in topology but it's not too far off, I mean the idea is I'm now going to take this infinite period of time, this infinite axis and I'm just gonna squeeze it into this little box and I can do that because I can just create a new coordinate transformation.

Specifically I will now write $U=\tan^{-1}(u)$. u has and infinite range U is now this compactified range likewise $V=\tan^{-1}(v)$. Once I've done this coordinate transformation, by the way as usual $\theta''=\theta'=\theta$ and $\varphi''=\varphi'=\varphi$ so we're not transforming those angular variables so this is yet another transformation we're going to execute and in this transformation pretty much everything kind of stays the same, the only difference now is if I write this whole problem with U and with V , now I have a square, my whole Universe fits in this square and it goes from $-\pi/2$ to $+\pi/2$ on both U and V , but we still have the restriction that $v\geq u$ which kind of puts us in the lower triangle, now we're actually pretty far afield from our original (t,x,y,z) coordinates with a metric of $\eta_{\mu\nu}$ constant, we're now in some in a new set of coordinates (U,V,θ,φ) , coordinates with a metric that we haven't quite calculated yet. Let's see if we can kind of get our heads around this whole thing, we now have:

$$\begin{cases} U=\tan^{-1}u \\ V=\tan^{-1}v \\ \theta''=\theta' \\ \varphi''=\varphi' \end{cases} \text{so} \begin{cases} u=\tan U \\ v=\tan V \\ \theta'=\theta'' \\ \varphi'=\varphi'' \end{cases} \text{and} \begin{cases} u=t-r \\ v=t+r \\ \theta'=\theta \\ \varphi'=\varphi \end{cases} \tag{62}$$

Now we want to have a look at the new metric in this final UV system here, so in order to do that I have to substitute out du and dv in our previous metric (61) so I need to know what those things are I need to know what is du and what is dv :

$$\begin{cases} du = sec^2 U\, dU \\ dv = sec^2 V\, dV \end{cases} \tag{63}$$

This is the key substitution and so the old metric was (60) so I have to take these two guys (63) and substitute it in (60):

$$ds^2 = -sec^2 U\, sec^2 V\, dV\, dU + ¼(\tan U - \tan V)^2 [\text{Angular Part}] \tag{64}$$

It doesn't take too much pushing to start working on some elementary trig stuff to simplify this I guess I should do it out in its long and ugly glory, this is often written:

$$ds^2 = -\frac{dU}{\cos^2 U}\frac{dV}{\cos^2 V} + \frac{1}{4}(\tan^2 U + \tan^2 V - 2\tan U \tan V)[\text{Angular Part}] \tag{65}$$

Now you can kind of guess, the straightforward way is just to replace tangent by sine over cosine:

$$ds^2 = -\frac{dU}{\cos^2 U}\frac{dV}{\cos^2 V} + \frac{1}{4}\left(\frac{\sin^2 U}{\cos^2 U} + \frac{\sin^2 V}{\cos^2 V} - 2\frac{\sin U}{\cos U}\frac{\sin V}{\cos V}\right)[\text{Angular Part} \tag{66}$$

Then what do you do? You basically multiply and get:

$$ds^2 = -\frac{dU}{\cos^2 U}\frac{dV}{\cos^2 V} + \frac{1}{4}\left(\frac{\sin^2 U \cos^2 V + \sin^2 V \cos^2 U - 2\sin U \sin V \cos U \cos V}{\cos^2 U \cos^2 V}\right)[\text{Angular Part}] \tag{67}$$

What does the numerator look like I'm smelling a couple of trig identities that's I think it's gonna go.

$$ds^2 = -\frac{dU}{\cos^2 U}\frac{dV}{\cos^2 V} + \frac{1}{4}\left(\frac{(\sin U \cos V - \sin V \cos U)^2}{\cos^2 U \cos^2 V}\right)[\text{Angular Part}] \tag{68}$$

Which means the whole metric ends up being:

$$ds^2 = -\frac{dU}{\cos^2 U}\frac{dV}{\cos^2 V} + \frac{1}{4}\frac{\sin^2(U-V)}{\cos^2 U \cos^2 V}[\text{Angular Part}] \tag{69}$$

Now that's the new metric where we can still extract the metric coefficients the same way we did before I won't bother this time but now we're in the UV system and of course the angle part doesn't change so we've done yet another transformation but this time what's really nice is everything exists inside a little square well I guess not even a square a little triangle in the plane. The entire Universe exists inside that little triangle because we've compactified it using this transformation (62) and because of that we've now got the entirety of space-time is jammed into this little triangle. What that means though? Maybe we should think about what exactly that means to have the whole Universe strand into a triangle. If I have U and V and I know that both U and V can be between $-\pi/2$ and $+\pi/2$ and I also know that $V>U$ and so this is that triangle of allowable coordinates so the point is when I say it's all jammed in any space-time coordinate you can imagine in the original system so we could actually draw the original system, which by original I'm actually going to t and r . A point here represents a complete sphere, the same is true in UV , by the way a point here represents a complete sphere because we're leaving unspecified θ and φ so we'll take all possible θ and φ and will get a sphere.

The point is any point in (t,r) has a mapping that will drag it into (U,V) , there is a series of transformations that'll get you from this point (t,r,θ,φ) and get you a (U,V,θ'',φ'') point and that point will always be inside this lower triangle and the important part is since $t\in(-\infty,+\infty)$, there are arbitrarily large values of t well off the (t,r) chart, we'll still find their way into this lower triangle somewhere and where will it be? You kind of have to work the transformations to sort of figure that out but ultimately as you get arbitrarily large in time you're going to end up arbitrarily close to this little point up here (upper right corner), if you're arbitrarily far in the past you're going to be jammed up into this little point here (lower left corner) so clearly it's not a nice uniform distribution of points like a patch of points way down here at some very large number it's gonna find themselves all jammed into a very small spot down here. Likewise the patch points at some arbitrarily large number is gonna find themselves way jammed up here but likewise if you go out an infinite radial distance or a large radial distance what's gonna happen then? It turns out you can kind of think about it but you'll end up getting closer and closer to this point here (lower right corner). Each of these things still represents spheres as you can still be anywhere on θ and φ so if you're very far away with radial infinity you can still be anywhere on a huge sphere that sort of encompasses an arbitrarily large swath of space-time and also this is all flat space-time so this is the infinite flat space-time of Special Relativity that's what this triangle represents.

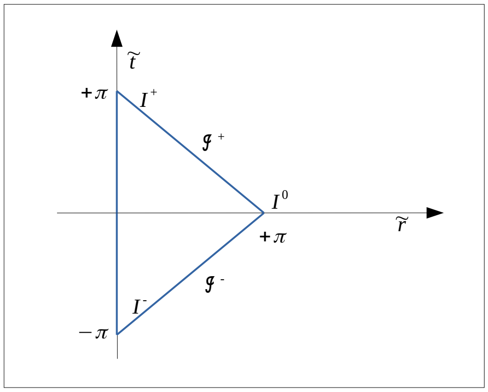
We're not done because I don't like this triangle it's on its side it's hard to understand I mean well it's not hard to understand it's a simple collection of transformations but I can still make it a little bit easier to pay attention to and understand also it still got this annoying problem with the metric here (69) and I shouldn't say problem but the metric doesn't have a time-like coordinate I kind of like a time-like coordinate so I'm gonna do yet another another coordinate changing, that's the point of this lesson I can do coordinate changes all day: I don't like what I got I'm gonna change it so I'm gonna create a new coordinate system:

$$\begin{cases}\tilde{t}=U+V\\ \tilde{r}=U-V\end{cases} \tag{70}$$

I can now of course do the reverse:

$$\begin{cases}U=½(\tilde{t}+\tilde{r})\\ V=½(\tilde{t}-\tilde{r})\end{cases} \tag{71}$$

Then I can make this transformation which essentially it's going to take this triangle here and it's going to rotate it into a vertical position which would then be like this the $\tilde{t}$ axis and the $\tilde{r}$ axis and this will the maximum value of $\tilde{t}$ is going to be $+\pi$, likewise the minimum value of $\tilde{t}$ would be $-\pi$ and for $\tilde{r}$ the situation is that it can be as big as $+\pi$ also so you'll end up with this as the triangle for $\tilde{t}$ and $\tilde{r}$ and now it's the same situation though it's it's now all of space-time is jammed into this region.



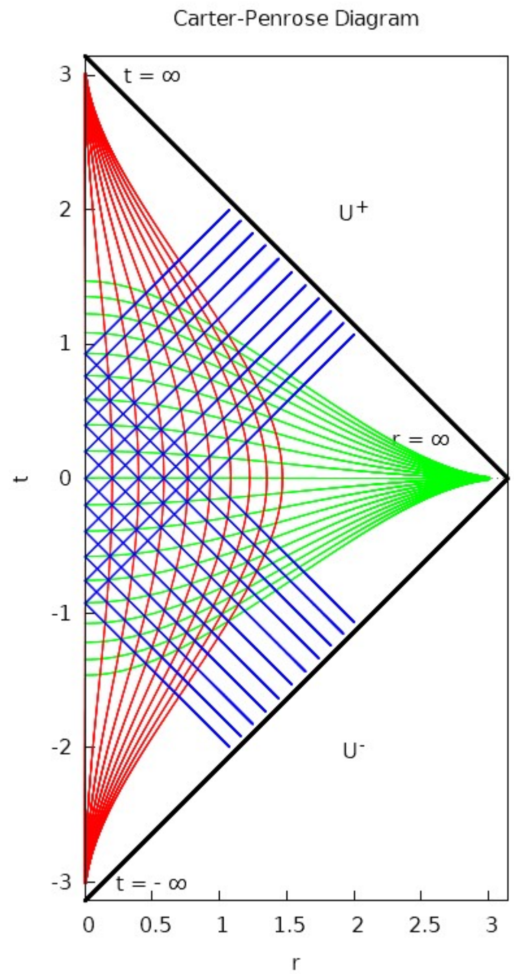
Each spot is still a whole sphere because we haven't included our $\tilde{\theta}$ and $\tilde{\varphi}$. Now we finally have the final picture this is where we're going to end up. What about the metric for this? Well the exercise is exactly the same, I'm going to do a quick short circuit on this one because I just I'm gonna write this down as in a slightly different form. What I want is let's go all the way back to t and r :

$$\begin{cases} t+r=u=\tan U=\tan ½(\tilde{t}+\tilde{r}) \\ t-r=v=\tan V=\tan ½(\tilde{t}-\tilde{r}) \end{cases} \tag{72}$$

What I've got is the original coordinates expressed in all the way to the end so I now have these $(\tilde{t},\tilde{r})$ coordinates that compactify the entire Universe in the same spot as I have (t,r) and this makes sense. I have these numbers here are always going to be between $(0,\pi)$ so I can take the tangent of that but I'll get numbers that can be infinitely large and (t,r) are unbounded because that's what tangent does. I have this formula here (72) for the final linkage between the original coordinates and the new coordinates and the reason that's important is because ultimately when we look at this metric in the catalog (2.1.4 Minkowski metric in conform-compactified coordinates) what we see is this metric here:

$$ct+r=\tan\frac{\psi+\xi}{2} \quad , \quad ct-r=\tan\frac{\psi-\xi}{2} \tag{73}$$

That's one of the reasons I did this is you can actually now go to this section of the catalog (2.1.4) and you can see how these coordinates came about and also a little bit of the motivation. This (2.1.17) is the ultimate metric using those coordinates where $\psi=\tilde{t}$ and $\xi=\tilde{r}$, they have the same θ and φ . What you see is again the denominator is the same in both of these terms and the numerator is this thing here and we're gonna have a lot more to say about that numerator in our next lesson or two. That numerator of course is this thing up here (2.1.15). You can calculate the Christoffel symbols for this coordinate system and this is now they go directly from (t,r) of regular Spherical system to the (ψ,ξ) of the compactified system so ultimately what you end up with this compactified system. There's a symbol for each of these points: I^+ , I^- and I^0 , the lines are called $\mathscr{I}^+$ and $\mathscr{I}^-$. The idea is if you actually did all the calculations and figured out how regular time and regular radial coordinate appear in this diagram you would realize that if you had a point at some fixed radial distance r that just sat there and just moved straight through time, this is Special Relativity so it's world line will be straight and it's been at that distance for all eternity and it's never gonna move and it just goes off and through all eternity, it ends up at I^+ and it turns out that a point such as that at a fixed radial distance will end up having a trajectory in the compactified coordinate system that starts at I^- , kind of moves off along a constant radial line and ends at I^+ . If it was much much farther away (bigger r) arbitrarily far away, it would have a trajectory that would look much closer to I^0 in these compactified coordinates. All of these purely time-like trajectories all start at the same place I^- and end up at the same place I^+ and they don't cross so that's all of these kinds of trajectories.



There are other kinds of trajectories too, there's this space-like trajectories at different slices of time they just go out to infinite distance and those trajectories will all start on the vertical line and all will end up at point I^0 it'll all start out at different places at the vertical line and end up at point I^0 , that is all the purely space-like trajectories but the other important thing about this is that at any point in this system a light ray, which moves at a 45º line in this system, emanating from any point in space-time, they will still be represented by a 45º line despite the fact that the whole Universe is captured inside this little triangle and that is the art of being a Conformal metric.

Ultimately what have we gained? Well we've gained a little bit of access to this particular section of flat space-time, now we understand, now we can look, the idea is like you're looking at this reference book this catalog of space times and you're seeing really weird things: conformal compactified coordinates well now we've explained what they are and how they work and why they are the way they are, this thing here should not be a mystery this part (2.1.15) is just the numerator of (2.1.17). We'll explain why this is really interesting too, this goes back to Einstein's original solution to the field equations, as original thing, it's actually coupled a little bit with the story of the cosmological constant. We're going to discuss that in fact I'm probably going to break from my promise to avoid the Einstein equation and just do a quick diversion into it because we're really well situated to study more but what you've seen is that we've been able to do a crazy amount of coordinate changes.

We end up with a legitimately interesting coordinate system with this metric (2.1.17) which by the way is just as flat just as flat as $\eta_{\mu\nu}$ and that is the story of conformal compactified coordinates so the idea here is that you should not be afraid of these coordinate changes you can just do them almost at will, I mean you've got to have some pretty good function, tangent is a pretty nice continuous function it's got a good big obviously well-defined domain and range, it's invertible and all of those things are important but if you have those things then you are in really good shape, his idea of the conformal transformation, we haven't discussed that so we only understand half of what this is means and this is by the way the definition of the conformal transformation $ds^2=\Omega^2 d\tilde{s}^2$ but we're not quite ready to talk about it yet.

Next time I think we're gonna actually go a bit deeper into this to discuss the conformal transformation and maybe a little bit of introduction to the an elementary solution of the Einstein equation.