[Geometric Algebra: Introduction and Motivation]

I'd like to begin the pursuit of a bright shiny intellectual object called Geometric algebra, the motivation for this pursuit is to get Maxwell's equations down to one simple equation. We want $\text{something} = \text{something}$ and we want to say that that embodies all of Maxwell's equations. The only way I know to do this is to construct Maxwell's equations in the context of this new thing called Geometric algebra. Now I'm not an expert in Geometric algebra this is absolutely not a prerequisite to QED you can learn all the QED in the world you want and never touch Geometric algebra but we have been compressing Maxwell's equations and we might as well compress it to the known end and then come up with an opinion about whether or not Geometric algebra is what its proponents claim it to be which is some universal language that really captures all known essential physical concepts is buried inside this mathematical structure called the Geometric algebra.

According to the proponents of it the modern understanding of Geometric algebra has never fully blossomed outside of a few fetishists who really love it so let's have a look, let's take a little diversion outside of the prerequisites of QED finish up this idea of how low can we go with Maxwell's equations and in the meantime let's learn Geometric algebra together and say together because I've only done one pass through this subject and so this is unlike the other topics here that I've gone through in this channel that I'm extremely familiar, I'm actually going to be learning this one about the same rate that you will if you follow along and actually I've done a full pass on the subject so I am ready to begin but so this will be my second pass through the subject but I do find it very interesting and I'd like to get your opinions on it so with that being said, let's begin.

The way I propose to study this material is I'm going to pick one particular paper, this paper is called “Spacetime algebra as a powerful tool for Electromagnetism” by Justin Dressel, Konstantin Y. Bliokh and Franco Nori and they're at nice good institutions. We have a nice mix of very qualified people I'm sure, I don't know any of them but I've gone through a lot of papers about the subject of Geometric algebra and they're all good in their own way this one I feel is the best. I'm very excited to review this paper again and that is the plan we are going to read this paper together for the good parts of the paper together or I shouldn't say it's all good I guess I should say the dense parts together. It's very ambitious because the paper wants to introduce space-time algebra which is just one form of Geometric algebra so the concept of Geometric algebra is actually quite broad it covers many different possible algebras but space-time algebra is only one and it's the one that is applicable to Special relativity and it's the one that's applicable to understanding the vast majority of space-time Physics in flat space time.

This paper has to teach Geometric algebra specialize it to space-time algebra and then apply it to the study of Electromagnetism so it's trying to do an awful lot and as such you know it refers to other papers about the algebra itself and we're going to make those hyperlinks together I'm going to try to fill in the gaps that are necessarily present for such an ambitious paper so if we look through the contents of the paper they talk about the motivation and we should go through that a little bit and then they talk about the insights this is the fetishist part meaning that anybody who's really excited about Geometric algebra really likes to talk about why it's so good and all of the special insights you gain by using this approach as opposed to any other approach and so I think actually we'll probably skip this right and what we'll do is we'll proceed directly to the approach which is section three of this paper and we'll let you decide how the insights surface and then in the end maybe we'll go back and have a look at the insights or perhaps as we go through this when we discover insights that are cited up here I'll make note of it but I think it's better just to dive right into it and let the insights percolate by themselves.

The second section “A Brief History of Electromagnetic formalisms” we probably will not discuss that this goes through some of the battles between the Quaternion formalism versus the Gibbs Heaviside vector formalism versus the formalism of differential forms and a few other formalisms that are in there suffice it to say all of these formalisms in a history of the study of Electromagnetic are derivable, buried in, a sub piece of or somehow contained within the space-time algebra which is one of the important motivations all of these formalisms from the day that Maxwell started the mathematical construction of the theory of Electromagnetism all the way to the present day, all those formalisms it turns out can be unified in one form and that's the claim, that's the big claim and that's why we're studying it.

When you see space-time algebra in this paper, that is the Geometric algebra of space-time and I'll have to spend a little time teasing apart the distinction between these special instance of Geometric algebra which is space-time algebra from the body of all Geometric algebra so I'll take care of that. Then they talk about the basic components of the algebra which are the multi-vectors then they go through a whole bunch of ways of understanding how multi-vectors work in the space-time algebra. We're going to immediately cover this section, I've actually brought this section of the paper which I guess is from 16 to 44. I've actually put them right here into our lesson plan and in order to finally get Maxwell's equations we're going to have to do the space-time calculus and Maxwell's equations in a vacuum and then notice they also do a lot of other stuff, I don't know if we'll go all the way through everything here but I'd like to see us get through section 5 so we'll begin with section 3, we'll shoot for section 5, I don't know how long this will take, I don't know how quickly it will go but it will be very interesting. By the way if you want this paper here you go this is where you'll find it, you'll find it in the Cornell archive database “Space time algebra as a powerful tool for Electromagnetism” and this is the identifier right here archive 1411.5002 so anybody can get this paper and read along with us.

Let's begin with section 3, let's begin with this quote by David Hestenes, I don't know exactly how to pronounce his name I should check hold on so I found a lecture by him and he pronounces it David Hestenes and let's have a quick look at what he says:

> Mathematics is taken for granted in the Physics curriculum – a body of immutable truths to be assimilated and applied. The profound in influence of Mathematics on our conceptions of the physical world is never analyzed. The possibility that mathematical tools used today were invented to solve problems in the past and might not be well suited for current problems is never considered.

That's his position now first of all Mr Hestenes is very acclaimed, this guy is he's no fly by night, he's a real physicist he's been thinking about this literally his entire career and he's unambiguously a very brilliant man and he's an evangelist for Geometric algebra and interestingly I've arrived at a pretty advanced state of my knowledge of Physics and I never heard of it, I never heard of Geometric algebra until stumbling through this material so unfortunately despite this evangelism I'm not sure that he's made a lot of real progress penetrating to regular students of the subject, of course that really depends on where you stand but where I stand it's a very new idea now what he's suggesting of course is that all of the Mathematics we're using today this is the part here the Mathematics we're using today is not adequate for the work we're trying to do and he says it's never considered, I tend to doubt that I've met a lot of theoretical physicists who are always chomping at the bit for some new Mathematics I think immediately about super-symmetry which is a radically new form of Mathematics interjected into the subject of Lee algebra's that really people jumped on understanding that the current conception of Lee algebra's is insufficient, certainly of the study of super-symmetry on the other hand I'm not completely sure that the idea is never considered is all correct.

On the other hand the basic point that he's trying to make here or that the paper is trying to set up here is that it's time for us to understand that the way we formulate at least Electromagnetism is not adequate for what Electromagnetism is and that's the excitement that those who are interested in this subject are trying to get us lay scientists to understand is like hey guys look at this, this is the way we need to approach and think about this. Let's give them their day in court and read through this paper.

"Physically speaking space-time algebra is a complete and natural algebraic language for compactly describing physical quantities that satisfy the postulates of Special relativity" so right away that first sentence is very optimistic, it's very exciting right we have a mathematical structure that compactly describes Special relativity now we've already had one that describes Newtonian relativity or I guess Galilean relativity but bonding everything together into something that naturally describes Special relativity is definitely exciting. "Mathematically speaking space-time algebra is the largest Associative algebra that can be constructed with the vector space of space-time equipped with the Minkowski metric", all right so this is the first place we are going to stop and make sure that we fully understand what's going on right here we're looking for we've got this the notion of an Associative algebra the vector space of space-time equipped with the Minkowski metric.

Let's break this down a little bit all right so we'll begin with the idea of an Algebraic structure and now an Algebraic structure is also called "an Algebra" these are synonymous, it has to do with a particular mathematical structure that exists in the subject of Abstract algebra and let's quickly review what an Algebraic structure has to contain? What is it? How do we create an Algebraic structure and we begin with a vector space and remember I always draw vectors spaces with a little box, I give it a name V for vector space, I define a basis in the vector space usually and I'll write the basis as e_{μ} those are the basis vectors of a vector space. The vector space has an addition $+$ defined on it and it has very importantly some bin of of scalars, this is a bin of scalars and that bin of scalars comes from some field F so the field F could be the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ traditionally it's either real or complex numbers but it could it could actually be any field, we for the purpose of our work in this material we're going to do real numbers because we're dealing with Special relativity deals with the real number manifolds on space-time or vector spaces with real scalars so our scalar field is going to be the real numbers. It's Quantum mechanics that uses the complex numbers for scalars.

Our vector space here and now what makes it an algebra though? A vector space is not an algebra this is just a vector space so to make it in algebra we have to add a bilinear operator which is often called a bilinear product and it's often referred to as just multiplication. The idea is any two vectors in the vector space, if we pull out one vector $\boldsymbol{v}$ and then we pulled out a vector $\boldsymbol{w}$, we can certainly add $\boldsymbol{w}+\boldsymbol{v}$ using the basic addition of vectors, this addition allows us to add any two vectors, we can certainly multiply by a scalar, if we pulled out a scalar b we can certainly write $b\boldsymbol{w}$ to give us a new vector which we call the vector $b\boldsymbol{w}$. Notice this is a multiplication, this is a multiplication just like the new one that we're going to create, we have to create a multiplication a bilinear operator, a bilinear product we have to create one but we already have one, we have what's called scalar multiplication that's what this is. We've got a scalar here and a vector here so scalar multiplication already exists.

Scalar multiplication is linear so we know that $b(\boldsymbol{w}+\boldsymbol{v})=b\boldsymbol{w}+b\boldsymbol{v}$ as long as we know that b is the scalar and $\boldsymbol{w},\boldsymbol{v}$ are vectors. We already have that and this is what makes it linear so we are now seeking a new multiplication and this is a multiplication that works for vectors it's the vector $\boldsymbol{w}$ multiplied by the vector and we can put them right next to each other just like that and this needs to return another vector which I guess I'll call $\boldsymbol{w}\boldsymbol{v}=\boldsymbol{y}$ so this is now a multiplication a bilinear product or bilinear operator, I guess there's no operator here if I wanted to do an operator I would have to maybe do something like $\boldsymbol{w}*\boldsymbol{v}$ this guy here becomes the so-called operator that we're seeking but if I just put it next to each other like this $\boldsymbol{w}\boldsymbol{v}$ the operator is implied, there's an implied star here between the and the $\boldsymbol{v}$ so now I need to create this new operation that takes two vectors and produces another vector and by vector remember what we just mean is an element of this vector space V , we're not talking about yet, we're not talking about little pointy things that go off in a direction and when I used to teach this or when I taught this in the “What is a tensor section”, I really emphasized that this thing, I didn't even want you to call it a vector, I wanted to call it a little pointy thing so you would erase this from your thinking. When you think of a vector in this context you're only thinking of it as an element of this vector space which obeys some rules, this addition rule, it's got a basis, the basis by the way has to have some dimension assigned to it so I should actually include that in my list of things here this has we have to assert some dimensionality of the vector space.

Vector space is quite abstract but it will have as long as it's finite dimensionality you have to assign it a finite dimension, presuming that you want a finite dimensional vector space. We can talk about infinite dimensional vector spaces but we're not for any of the work we're doing here we're always going to talk about finite dimensional vector spaces, in fact we can even go one step further and just say and just go ahead and say $n=4$. We're dealing with a four dimensional vector space and we're dealing with the unit vectors e_0, e_1, e_2, e_3 , that's what these e_μ are just those four starting at zero and going to three. We might as well because that's all we're going to use in this collection of material.

The point is that once we have a vector space in order to promote it to an algebra we have to create this bilinear product and in this case bilinear means something very straightforward, this new multiplication we're inventing is bilinear with respect to the addition operator so you get $\boldsymbol{w}*(\boldsymbol{v}+\boldsymbol{s})=\boldsymbol{w}*\boldsymbol{v}+\boldsymbol{w}*\boldsymbol{s}$. Notice that this $+$, this is the addition operation for the vector space which means that this guy $\boldsymbol{w}*\boldsymbol{v}$ must be an element of the vector space V and this guy $\boldsymbol{w}*\boldsymbol{s}$ must be an element of the vector space V so we can actually add them together otherwise addition doesn't make sense because addition is only defined between two vectors in the vector space.

That is what an algebra is but you'll also notice that it used the word Associative algebra in the sentence that we're deconstructing and that's pretty simple that just simply means $\boldsymbol{w}\boldsymbol{v}$ now I'm omitting the operator symbol, whenever you see two vectors next to each other like this, there's only one multiplication operation between them so there's no ambiguity so we can just drop it. The point is $\boldsymbol{w}(\boldsymbol{v}\boldsymbol{s})=(\boldsymbol{w}\boldsymbol{v})\boldsymbol{s}$. Some of the most important algebras in Physics, in particular the Lee algebra are non-associative, we're not going to worry about that now but just understand that for what we're doing we have an Associative algebra and so that is what Geometric algebra is. You we're going to somehow create vector spaces, we're going to define this multiplication on this vector space and somehow we're going to create one a vector space that's four dimensional and somehow this is going to completely reassemble our understanding of space time and the subject of Electromagnetism.

Let's go back here what have we done we've figured out this word Associative algebra that can be constructed with the vector space of space-time equipped with the Minkowski metric so obviously they're going to use what they're calling the vector space of space-time and the Minkowski metric on that vector space to create this Associative algebra so I guess we need to understand, what do they mean by the vector space of space time? What is this all about and equipped with the Minkowski metric, what is that all about? Those two are much easier because what I suspect that they're doing here is because space time is flat this vector space is going to be space time itself, positions in space time so a space-time coordinate of t, x, y, z , a space-time coordinate like that is going to be transferred into a vector that's going to be $t e_0 + x e_1 + y e_2 + z e_3$ and because space-time is flat you can do this you can treat the points in space as though they are vectors and by doing so you vectorize space time and then the last thing is pretty obvious they're just saying that we're going to use the Minkowski metric where $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ or $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$ is either one of these two diagonals, I can't quite remember which one this paper used we'll find out in a moment I'm sure and that allows us to find the distances between two events in space time so if I express an event in space time in this form its distance to another event in space time is a contraction of this vector against this metric, the Minkowski metric so that's all that means and we'll see exactly how they push that in there pretty soon.

Let's go on, the space-time algebra "is an orthogonal Clifford algebra which is a powerful tool that enables manifestly frame independent and coordinate free manipulations of geometrically significant objects". Here I'm interested in this word, I'm interested in this question of geometrically significant and frame independent and coordinate free manipulations so okay so first of all this notion of it being an orthogonal Clifford algebra well that's what the space-time algebra is so that's just really we can treat that right now as just another word for the space-time algebra. Manifestly frame independent well my understanding of frame independent has always been that you don't bust things out into coordinates right away which is another way of saying we don't select a basis, we don't select a specific basis necessarily right away that's what coordinate free is is when we create our vector space we have one special chosen basis and I think as we go through this we're going to actually see that we're going to do a lot of proofs using coordinate bases because it's a lot easier in some cases however you don't have to do that I'll try to show a few examples as we go on but the idea is that one of the problems with studying Special relativity is you have to say oh you know we've got the lab and we've got the moving guy and here's what it looks like from the lab and here's what it looks like from the moving guy and you have these coordinate transformations.

Presumably we can get away from that by putting everything in the context of the space-time algebra if we escape this need to refer everything to some coordinate system so we'll see how that claim pans out and what is the notion of a geometrically significant object well so to me a geometrically significant object is anything that can be described as a field over the space-time manifold, that to me is what geometrically significant means so we'll have to see exactly what they mean but vectors certainly are geometrically significant objects forms are certainly geometrically significant objects but notice object is now this is a mathematical construction we're talking about we're talking about a mathematical object that models something about physical reality so presumably we have physical reality and a bunch of mathematical objects that can be placed on a manifold our physical reality exists in some manifold of space-time, a four-dimensional manifold of space-time and Special relativity, anything that can exist inside this thing as expressed in the terms of a vector or tensor field that is a geometrically significant object in from from my understanding.

Moving on, unlike the dot and cross products using standard vector analysis the Clifford product between vectors is often invertible and not constrained to three dimensions so right away they've introduced this notion of Clifford product so that Clifford product so that's our bilinear multiplication of vectors, they're giving it the name the Clifford product so make no mistake when we build our space-time algebra the product between the elements of the vector space we're dealing with is the Clifford product, they call it the Clifford product so we'll talk we'll use that word from now on that is whenever I describe this bilinear operator and this multiplication that does all of this stuff we're now going to call that the Clifford product, all right, that's what that is. What about it?

Well "the Clifford product between vectors is often *invertible*", that's interesting, I didn't realize they made such a deal out of that but it's often invertible and not constrained to three dimensions. This one I'm real familiar with, they want to make a big point here is that the algebra that we're constructing for space time, you can construct an algebra for any dimensionality out there and it's not the space-time algebra at that point it's a different form of the geometric algebra but it immediately generalizes to any number of dimensions and this is not true of the Mathematics we have learned for Electromagnetism, this notion in particular of the cross product the cross product cannot be defined except in three dimensions but the cross product is, there is a vector cross another vector and it does equal another vector $\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{y}$ so the cross product looks a lot like an algebra and indeed in three dimensions the cross product can be understood as a bilinear operation that defines an algebra but as soon as you go to four dimensions this cross product idea breaks down, it only works in three dimensions I think it also works in seven dimensions or something, I've never really explored that but you can somehow for some reason you can define a cross product in seven dimensions of all things.

You certainly can't do it in four five six eight nine or other dimensions so having the idea that our Electromagnetism can be forced into a mathematical structure that has great generality otherwise that actually is compelling that's interesting, our universe has settled in on one particular mathematical structural form of many and you can speculate why that might be so, that's interesting and then the notion that it is invertible, that's interesting, that basically that's saying that if I have a vector $\boldsymbol{w}$ there is a vector $\boldsymbol{z}$ out there such that $\boldsymbol{w}\boldsymbol{z} = \mathbf{1}$ is essentially, well it's some kind of unity. Now there is a problem here, there's no such thing really as a unit, there are unit vectors but there's no such thing as $\mathbf{1}$ in the world of spatial vectors, so this notion here is actually quite important and interesting because what I want to write down is I want to write down that the Clifford product between $\boldsymbol{w}\boldsymbol{z} = \mathbf{1}$ ergo $\boldsymbol{z}$ is the inverse of $\boldsymbol{w}$ and $\boldsymbol{w}$ is the inverse of $\boldsymbol{z}$ but $\mathbf{1}$ is not a vector $\mathbf{1}$ is a scalar and so this notion of inversion is already strange if you think of these vectors as little pointy things so we're going to understand what this idea of invertible means as we go along but it is a very important point to take note of is that the idea of being invertible for two vectors being multiplied together to give you a third vector that is actually a puzzle. It turns out it's solved by broadening our notion of what a vector is, remember a vector is an element of a vector space so basically, a little foreshadowing here, if I add the scalars directly to the vector space so the vector space contains like a bunch of little pointy things and all the scalars well now one is part of the vector space and in principle this Clifford product between two little pointy things might actually be a scalar because the scalar is now inside the vector space.

Now these two little arrows up here they don't mean I'm a part of the vector space, they mean I'm a part of the vector space but I'm a little pointy thing also and you might have to put some other symbol up here or the absence of a symbol says oh I'm a part of the vector space but I'm a scalar so we'll see how that works as we as we move on in the early phases of discovering how this was all put together. Then we read “unlike the component manipulations used in tensor analysis”, which we've done already “Spacetime algebra permits the compact component-free derivations that make the intrinsic geometric significance of physical quantities transparent” so I guess the physical quantity that I'd like to see the geometric significance of is the Electromagnetic field tensor F , that's a tensor so perhaps there is something in Geometric algebra that takes that tensor from being this very abstract mathematical idea and lifts it into something that's obvious, likewise a differential form ω , we treat fields as a physical quantity described by say a two form, and the two form ω we already have some notion of how ω is physically significant maybe this brings it out even more.

Let's continue reading “when space-time algebra is augmented with calculus then it subsumes many disparate mathematical techniques into a single comprehensive formalism including multi linear algebra, vector analysis, complex analysis, Quaternion analysis, tensor analysis, Spinor analysis, group theory and differential form” so this is it, this is the evangelism right here, all of those things that you have to learn, you can pick a class in each of those subjects, well somehow Geometric algebra takes all those things and puts them in one comprehensive mathematical framework that's actually pretty exciting right so that's part of our motivation. Actually earlier I said that we're not going to spend a lot of time on this motivation but apparently they slipped in some of the motivation right here so we're we're having to deal with some of the motivation.

“Moreover for those who are unfamiliar with any of these mathematical techniques space-time algebra provides an encompassing framework that encourages seamless transitions from familiar techniques to unfamiliar ones as the need arises. As such space-time algebra is also a useful tool for pedagogy” so that's pretty hopeful, the idea that you would take this thing which is still obscure this idea of Geometric algebra and all of a sudden it's going to be used to teach people stuff you really want to move it to the mainstream so these the people interested in this subject really feel like this is ready for prime time and the final paragraph that we'll read today is “during our overview we'll make an effort to illustrate how space-time algebra contains and generalizes all the standard techniques for working with Electromagnetism. Hence one can appreciate space-time algebra not as an obscure mathematical curiosity but rather as a principled, practical and powerful *extension* to the traditional methods of analysis. As such all prior experience with Electromagnetism is applicable to space-time algebra approach, making the extension readily accessible and primed for immediate use”.

This is why we're doing what we're doing so the next time we will begin by studying exactly how they define and create space time and how we'll we'll actually construct the space-time product so we'll get through 3.6 section 3.1 and hopefully we'll get well underway into section 3.2. See you next time.