Maxwell's Equations via Differential Forms Part 4

Welcome back we are going to continue our story of Maxwell's equations via differential forms a little sidebar here that I'm more than happy to draw down and we've already made some pretty pretty significant progress in the last lecture for example we discovered that if we were interested in understanding the divergence of a vector $\boldsymbol{E}$ where now $\boldsymbol{E}$ is understood to be a vector in the Gibbs sense. Remember the Gibbs sense is your classic vector analysis, we call that Gibbs apparently it's been the history of how our mathematical notation has come down to us has become more of a topic of conversation in recent years and we know that Gibbs and I think Heaviside champion this notation over some Quaternion formalism of electromagnetism which evidently Maxwell was in favor of for whatever reason but ultimately if we start with your classic electric field or any classic vector field we can get what would be called the divergence of that three-dimensional straight up three-dimensional vector field which is obviously this from your classic vector analysis:

$$\nabla \cdot \boldsymbol{E} = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = * \, \mathrm{d}\left(* \boldsymbol{E} \right) \tag{1}$$

That divergence ends up being equal to, as we saw in the last lecture, the Hodge dual of the exterior derivative of the Hodge dual of the vector field understood as a vector field on a manifold where we now replace the Gibbs unit vectors x, y, z with our partial derivative unit vectors which is how manifolds work. We understood that for where $\boldsymbol{E}$ ends up basically being a one form so we took the Gibbs form, we converted $\boldsymbol{E}$ into its equivalent one form which we saw was very easy to do in Euclidean space because there's just one-to-one correspondences between all of these things when you're dealing with straight up Euclidean space or in Physics ensconced in Euclidean space so once we convert the electric field to a one form, notice I don't have an equal sign. These two really are equal in the sense that we're just rewriting the unit vectors but these two are not equal so I have these green arrows saying it's a correspondence that we make so we treat $\boldsymbol{E}$ as a one form and then we've get our divergence of $\boldsymbol{E}$ using this formula (1), the Hodge dual of the exterior derivative of the Hodge dual of $\boldsymbol{E}$ and the result is a real number.

Then we also did the same thing for the curl, that is, we took the vector $\boldsymbol{B}$, pushed it into its one form version and we discovered that if we take the exterior derivative of $\boldsymbol{B}$ then we end up with a two form because remember the exterior derivative raises forms by one and we discovered we had these terms that looked awfully a lot like curls and when we finally took the Hodge dual of that, we ended up with the Hodge dual of the exterior derivative of the one form version of $\boldsymbol{B}$ gives us exactly the components in front of these forms that match the curl and we can just make the correspondence:

$$* \, \mathrm{d}\boldsymbol{B} = \left(\partial_1 B_2 - \partial_2 B_1 \right) \mathrm{d}x^3 - \left(\partial_1 B_3 - \partial_3 B_1 \right) \mathrm{d}x^2 + \left(\partial_2 B_3 - \partial_3 B_2 \right) \mathrm{d}x^1 \tag{2}$$

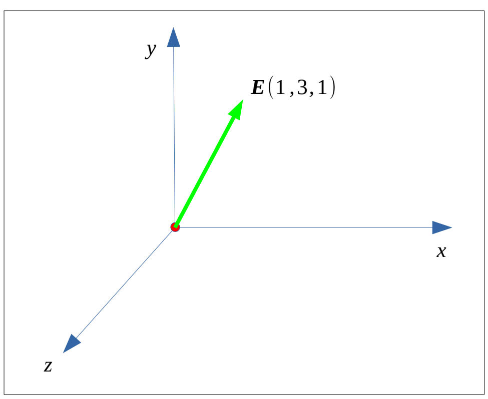
$$\nabla \times \boldsymbol{B} = \left(\partial_1 B_2 - \partial_2 B_1 \right) \partial_3 - \left(\partial_1 B_3 - \partial_3 B_1 \right) \partial_2 + \left(\partial_2 B_3 - \partial_3 B_2 \right) \partial_1 \tag{3}$$

We have found basically ways of taking the divergence and the curl of one forms. One question would we want to start by asking is why would we do this? Why would we want to know the curl or the divergence of a two form? One forms are easy to understand in the sense that if we start with vectors, we convert them to one forms in Euclidean space which is just a trivial correspondence then we understand we're basically finding the divergence and the curl of vectors which we understand because that is something we've learned to do in basic Physics, it's a process we need to understand but curl and divergence have always been taught in the context of three dimensions and Euclidean Physics and so in four vectors so why would we think that something like a curl and a divergence would even be relevant to a different mathematical object that's a two form which is not strictly speaking a vector in the sense of vectors from basic Physics. Why is that? This is a really important point to understand and we have to get through that. Why would we hunt for an equivalent of curl and divergence for a two form at all? This is very profound and I think quite confusing but extremely worth while point to understand so we're going to have a quick look at why we would want to do this before we do it. Because doing it is actually very easy but understanding the motivation is quite difficult.

To understand this we're going to begin with a very simple Cartesian coordinate system that we have erected inside the space where all of our Physics is happening and we've selected a point for the origin we've selected an x , y and z axis design so it's all right-handed, your classical Cartesian coordinate system and so this is the place where our electromagnetic Physics is going on so when we think of like electromagnetic Physics we want to think of things that are really straightforward and very obvious like directions that particles move, directions of fields of force things of that nature that we've discussed in just elementary electromagnetism and you can't get more fundamental than the Lorentz equation:

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B} \tag{4}$$

I've written down the Lorentz force law and this is the when I say the arena where our Physics is taking place this is essentially the Physics we're talking about, there's going to be a charged particle located at the origin, there's going and that charged particle will have the charge q and the force on a particle due to the electric field is $q\boldsymbol{E}$, the force on a particle due to the magnetic field is $q\boldsymbol{v} \times \boldsymbol{B}$ so this is what we're dealing with when we talk about force so this red dot here that is our particle and now we are going to calculate the force on this particle.



How do we do this? Well first let's calculate the electric part so I'm going to presume an electric field that exists at this point right here and that electric field will have components $1,3,1$ so the components of this electric field at this point is one unit in the x direction three in the y and one in the z which means it's coming out of the plane a little bit it's not illustrated well here but you can see the numbers and therefore we know that this vector is $\partial_1 + 3\partial_2 + \partial_3$. That's the vector formulation in the tangent space of this Euclidean classic electromagnetic universe.

I want to say Newtonian but Newton, of course, didn't formulate these things I guess it's Newtonian in the sense that you've got a force so forces mass times acceleration of the particles so in that sense it's very Newtonian but here it's all just straight up Gibbs classic mathematics. Gibbs vectors and classical simple electromagnetism. The point is that when we set up this coordinate system, that was an arbitrary choice. I had a bunch of arbitrary choices I could have oriented it any way I wanted to I chose Cartesian instead of say Polar or something else but I chose the names of the x, y, z so that things were right-handed, if x curls into y you end up with z . z curls into x you end up with y . If y curls into z you end up with x using your right hand, which is another arbitrary choice and so it should be of course that it stands just fundamental reason that the Physics, the real Physics of what's happening should be completely irrelevant to the choice of coordinate system. I could have done any coordinate system and nothing's going to change about how the particle actually moves. What we'll change is the description of how it moves but not its motion itself, there's something intrinsic about the motion itself that we want to preserve so let's imagine for a moment that we had chosen a different coordinate system right or that we make a transformation of the coordinate system and a very simple one. Instead of making the positive x axis go off to the right, we're going to make the positive x axis go off to the left. Likewise with the y axis, instead of going up it'll go down and the same with the z axis, it'll go into the page instead of out of the page.

We've literally flipped the coordinate system so now this is the $-x$ axis the $-y$ axis and the $-z$ -axis and so had we done that instead we would notice that the right hand rule gets a little bit messed up because now when you curl x into y with your right hand you get $-z$ and when you curl y into z you get $-x$ and when you curl z into x you get $-y$ so now you have a left-handed coordinate system you've got to use the left-handed rule but what's a more of note is that if you keep these components the same $\boldsymbol{E}(1,3,1)$ components is now going to end up pointing in this direction it's going to be coming out of the page slightly because it's got this positive one but it's going to be pointing down into the left right because the components are always in reference to the underlying coordinate system. Now in this case understand that this is all of space, we're dealing with the tangent space at this point so when we made this change that x goes to $-x$ and y goes to $-y$ and z goes to $-z$ we've also made the change that $\partial_x=\partial_1$ goes to $-\partial_1$, $\partial_y=\partial_2$ goes to $-\partial_2$ and $\partial_z=\partial_3$ goes to $-\partial_3$ so if I just maintain these labels like I do here and maintained these component numbers then $\boldsymbol{E}$ would have apparently flipped because ∂_y is going down and so $3\partial_y$ obviously has to go down as we look at this page instead of up.

What this tells us right away is that when we reflect our entire physical space this is a passive reflection this is called passive because all I'm doing is changing the coordinate system itself, Physics we usually use active transformations but I think to illustrate this passive is much better because it's so clear if all you're doing is changing the coordinate system and leaving the material aspects of it alone it's much more obvious that nothing really should change and what you need to do. The point is if we make this passive transformation, the one I've described right here then we have to flip the signs of all of the components in order to make this work and there's two ways to think about it, I just re-label x,y,z , and I keep the symbols $\partial_1,3\partial_2,\partial_3$ but I change the sign of the component so I get $-\partial_1,-3\partial_2,-\partial_3$ or I could say no no when I made the transformation I also ended up transforming these basis vectors and that's where the sign change came from but regardless, I do need to know that I've got to get those component sign flipped one way or another and the way we the way we remember this is we say that $\boldsymbol{E}$ is a Polar vector and it's a vector so obviously this is true why do I need to call it a Polar vector? We'll see in a second, of course, that I'll remind you hopefully because hopefully you've seen some of this before but there are some types of vectors where the sign doesn't change and that's the crux of why we want to understand how to take the curl of two forms.

Right now most vectors we deal with are polar vectors in fact let's look at the Lorentz force law. If the force is going off in this direction and I suddenly flip around the coordinate system the force on a particle is not going to change so force is polar so this whole side is polar so both sides of the equal sign have to transform the same way so all of this (4) has to be polar. Well that's not a problem because we already figured out that $\boldsymbol{E}$ is polar so what's left? Well we've got the velocity, well again if this is your origin and the particle is moving in this direction and you flip the coordinate system well, you still want the particle moving in the same direction so $\boldsymbol{v}$ has got to be polar meaning, you'll have to flip its components one way or another so that leaves the question of this magnetic field and you might want to say, well the magnetic field has to be polar too because this whole thing's got to be polar because this whole right side has to be polar. The problem is this cross product here, because the cross product has inherently a preference for right-handed systems the way the cross product is defined and we're flipping from a right hand into a left-handed system so it's not entirely obvious that the magnetic field needs to be polar so now we're going to examine the magnetic field and see how that has to work.

I'm now going to make an effort to explain the notion of a pseudo vector as I had to work through it when I was a student. I got to admit I took it a little level probably more than I needed to and this may be a level more than most of you care about this is an elementary idea the notion of a pseudo vector and if you already know it and you're comfortable with it fine, you're probably as comfortable with it as you think but you may be misjudging how well you understand it I don't know but I'm going to try to explain it to you the way I best finally felt comfortable understanding it so with that in mind let me just begin with a regular right-handed coordinate system x curled into y gives you z , z curled into x gives you y and y curled into z gives you x and these are the curl equations for the unit vectors in a typical Euclidean space right-handed coordinate system of classical electrodynamics.

$$\begin{aligned} \hat{x}\times\hat{y}&=\hat{z}\\ \hat{y}\times\hat{z}&=\hat{x}\\ \hat{z}\times\hat{x}&=\hat{y} \end{aligned} \tag{5}$$

I'm not even using the fancy manifold form of the unit vectors, I'm just using standard stuff from elementary electrodynamics and this is how we define this notion here is what embodies the right hand rule if you execute the right hand rule in this coordinate system then this is how your cross products are going to turn out. Now I want to make a coordinate transformation from the unprimed system to the prime system and the way I'm going to do it is I'm simply going to reflect each axis about the origin.

$$\begin{aligned} \hat{x}&\rightarrow\hat{x}'\\ \hat{y}&\rightarrow\hat{y}'\\ \hat{z}&\rightarrow\hat{z}' \end{aligned} \tag{6}$$

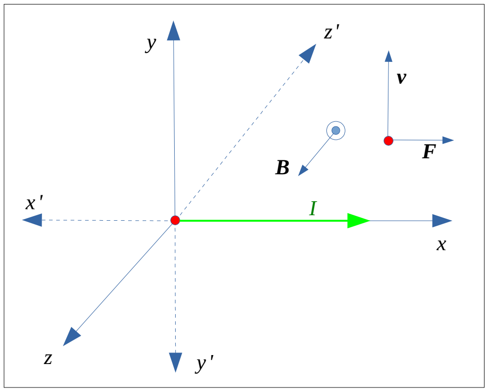
The positive coordinates of the primed system are going to be the to the left of the origin for the x axis it's going to be below the origin for the y' axis and it's going to be into the plane of the board for the z' axis so it's going to be the left for the x' axis and likewise positive for the unprimed coordinates are going to be as we just discussed so we learned from this that if you're going to write an electric field vector $\boldsymbol{E}$ in terms of the unprimed system, it's going to be:

$$\boldsymbol{E}=E_x\hat{x}+E_y\hat{y}+E_z\hat{z} \tag{7}$$

If you write the same electric field in terms of the primed system, you have to write:

$$\boldsymbol{E}=-E_x\hat{x}'-E_y\hat{y}'-E_z\hat{z}' \tag{8}$$

You have to change the sign of the components and that's because $\boldsymbol{E}$ is a polar vector, we just did that and that ensures that $\boldsymbol{E}$ will always be pointing in the correct direction because in the unprimed system these are positive component values in the prime system they're negative component values so you have to flip the sign because E_x, E_y, E_z represents the components in the original right-handed system or in the unprimed system I should say. Now let's consider the magnetic field which of course is going to be a pseudo vector, it's not going to behave this way, it's going to behave some other way. We begin by considering a current flowing in the positive x direction of the unprimed system or the negative x direction of the primed system but right now we'll say we're in the unprimed system and we've got this current flowing in the positive x direction so this current that's flowing in the positive x direction is going to produce a magnetic field and that magnetic field using the right hand rule of the unprimed system is going to come out of the board and when something when a vector may you may remember when it comes out of the board you give it a dot.



Now we consider a particle right here's a particle right here and we give that particle some velocity let's say a velocity $\boldsymbol{v}$ in the positive y direction so that's the velocity of the particle and we can now determine the force on the particle by calculating $q\boldsymbol{v}\times\boldsymbol{B}$ where $\boldsymbol{B}$ is this blue vector coming out of the field. In calculating $q\boldsymbol{v}\times\boldsymbol{B}$ in our right-handed coordinate system gives us the force $\boldsymbol{F}$ and that force is in this direction, this is pretty elementary stuff we've calculated the Lorentz force on the particle (4).

With that now we switch to the primed coordinate system, well in the primed coordinate system we still want I to flow to the right so I must be a polar vector so its components have change sign but we still have a vector where the where the current is flowing to the right but to greater negative values of x' where it was to greater positive values of x but now in order to calculate the magnetic field we are now in a left-handed coordinate system so we engage the left hand rule and the left hand rule tells us oh no we've got magnetic field going into the page, now we have our velocity vector which is still going to be a polar vector velocity is polar so now the velocity is going in the negative y direction but the components of the velocity of all change sign because it's a polar vector so now we implement the left hand rule between this velocity vector which is unchanged against this magnetic field vector which has changed and now that left-hand rule gives us a force vector which goes in this direction just as it should because the force shouldn't change just because we passively change the coordinate system.

We just have to remember that we're going from a left-handed to a right-handed coordinate system each time so now we see something, the magnetic field did in fact change direction simply because of a passive coordinate transformation it was coming out of the page and now it's going into the page and this is a byproduct of the change from a left-handed to a right-handed coordinate system. What's tricky about this is the direction of the vector is changed but we say that it is an axial vector and typically we say that the components don't change but of course the components don't change because here in the first case in the right-handed system the components of the vector were headed towards larger values of, positive z and when you flip it's still headed in positive values but now of z' so the components actually don't change while the basis vectors did in fact change so this is all together different behavior from the polar vectors and it all comes back down to this right-handed, left-handed arbitrary choice of of right hand rule and left hand rule. The magnetic field we normally think of as a vector:

$$\boldsymbol{B}=B_x\hat{x}+B_y\hat{y}+B_z\hat{z} \tag{9}$$

But it's not really a vector in the sense that vectors that we are familiar with, when you flip the coordinate system their components should flip sign, in this case the components don't change sign.

$$\boldsymbol{B} = B_x \hat{x}' + B_y \hat{y}' + B_z \hat{z}' \tag{10}$$

These components are now the same, there's no additional sign change and when those components are the same but these unit vectors have changed direction the vector itself appears to change direction despite the fact we've completely passively adjusted the coordinate system so the fact that this magnetic field vector behaves this way when you reflect it under parity, the fact that that's different from how these other vectors, the force, the velocity and the current in this particular example don't change things that transform differently under coordinate changes are different things and that's the part that we have to understand this guy we want it to be a vector but we have to realize you know what it doesn't transform the way we expect vectors to transform. Force is clearly a vector, velocity is clearly a vector and they have components that change sign under parity inversion, the way to say it is they're negative under parity but magnetic field is not it's positive under parity and therefore it's not really a vector at all it is something else it's called a pseudo vector and it's hard to tell because we just draw little lines and give it you know an origin point in space and because it's Euclidean we can slide these things around all over the place and we still they're arrows with magnitude and direction. We still want this magnetic field to look just like a vector but we need to understand if we change the coordinate system it behaves differently so fundamentally the magnetic field is not a vector it is a pseudo vector and the way we've concealed that fact is we said well there's two kinds of vectors in the world there's axial vectors like the magnetic field and there's polar vectors like the electric field right and these two kinds of vectors.

The answer is you can get away with that, first of all you have to memorize the fact that vectors come in two flavors but also it's very limited, it only really works in the arena where you've defined everything to work which is fine if all you're going to do is elementary electromagnetism or engineering Physics, this is perfectly good you just have to remember this the problem comes up is like we had the electric charge in there, well under the parity inversion we expect well that's not going to change the electric charge just because you flipped the sign of the coordinate system the electric charge isn't going to change in any way but there are quantities out there that are scalar quantities that do in fact change sign when you flip the coordinate system, these are called pseudo scalars and those that do not change are called scalars they don't change sign so we have scalars and we have pseudo scalars.

Now pseudo scalars are totally outside the purview of a classical electrodynamics but if we did have magnetic charges would be pseudo scalars and electric charges would be regular scalars so when we do the Lorenz Force law (4) this q , when we flip the coordinates system we don't change the sign of q , we don't change positive charge to negative charge but if we were doing this with magnetic forces and we had a Lorenz Force law associated with magnetic charges then yes, it would change sign. The point is in Physics we do have the notion of pseudo scalars floating out there but it's not relevant for what we're talking about right now but just keep that in mind. The point is is that we have these things called axial and polar vectors and the magnetic field is in fact an axial vector which is synonymous with pseudo vector, these are meant to be the same thing.

Now that we know that what does this have to do with the cross product of a two form so let's discuss that now. We begin by looking at this picture for three-dimensional space of the different vector spaces for the various forms, the vector space of all the one forms vector space of all the two forms, three forms and zero forms:

$$\begin{gathered}\mathrm{f}(x) \rightarrow \text{ 0-forms} \\ \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3 \rightarrow \text{ 1-forms} \\ \mathrm{d}x^1 \wedge \mathrm{d}x^2, \mathrm{d}x^2 \wedge \mathrm{d}x^3, \mathrm{d}x^3 \wedge \mathrm{d}x^1(x) \rightarrow \text{ 2-forms} \\ \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \rightarrow \text{ 3-forms}\end{gathered} \tag{11}$$

With three dimensions it's considerably smaller than the one we looked at for four dimensions which was this one you may remember from a previous lecture where we did one forms two forms three forms and four forms,

$$\begin{gathered}\mathrm{f}(x) \rightarrow \text{ 0-forms} \\ \mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3 \rightarrow \text{ 1-forms} \\ \mathrm{d}x^0 \wedge \mathrm{d}x^1, \mathrm{d}x^0 \wedge \mathrm{d}x^2, \mathrm{d}x^0 \wedge \mathrm{d}x^3, \mathrm{d}x^1 \wedge \mathrm{d}x^2, \mathrm{d}x^2 \wedge \mathrm{d}x^3, \mathrm{d}x^2 \wedge \mathrm{d}x^3 \rightarrow \text{ 2-forms} \\ \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3, \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \rightarrow \text{ 3-forms} \\ \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \rightarrow \text{ 4-forms}\end{gathered} \tag{12}$$

We lose the fourth one we cut down a dimension, I guess we get rid of this $\mathrm{d}x^0$ and we just live with the three dimensions so the more dimensions you have in your space the more one form basis vectors there are and the larger the dimensionality of the space but what's important to understand is that the dimensionality of the one forms is equal to the dimensionality of the $N-1$ forms where N is the total number of dimensions of the space so in our case here (12) N is four, there's four dimensions so the one forms is a four dimensional vector space and the $N-1$ forms or the three forms is also a four dimensional vector space. What we're doing is we're playing around in three dimensions so we have in our case $N=3$ so we have three dimensions for one forms and three minus ones are the two forms but there's also three dimensions for the two forms so these are two these are both three dimensional vector spaces so the question now is, well if they're both three dimensional vector spaces, why are we automatically going to this one for our electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ vector fields. We call them vector fields and so that's probably the one reason, we say they're vector fields, I know they're vector fields because I've studied them that way and they've always called them vector fields so what we naturally tend to do is we tend to say if $\boldsymbol{E}$ is a vector field then we have:

$$\boldsymbol{E} = E^1 \partial_1 + E^2 \partial_2 + E^3 \partial_3 \tag{13}$$

Then we immediately convert this into its one form version because it's free, there's a one-to-one correspondence for that and we're studying the nature of forms so we're going to convert this into the obvious dual one form this is the dual form so we go:

$$\underline{E} = E_1 \, \mathrm{d}x^1 + E_2 \mathrm{d}x^2 + E_3 \mathrm{d}x^3 \tag{14}$$

Well $\boldsymbol{E}$ is a vector, this is what we do and over here we have to maybe give e a slightly different symbol because now it's a form so maybe I will put a line underneath it or something like that. When we did that for $\boldsymbol{E}$ we said well $\boldsymbol{B}$ is a vector too so why can't I write $\boldsymbol{B}$ vector and then I just do the exact same thing, exactly like (13) I just write $\boldsymbol{B}$ the same way and I find $\boldsymbol{B}$ one form version the same way:

$$\boldsymbol{B}=B^1\partial_1+B^2\partial_2+B^3\partial_3 \tag{15}$$

$$\underline{B}=B_1\,\mathrm{d}x^1+B_2\,\mathrm{d}x^2+B_3\,\mathrm{d}x^3 \tag{16}$$

This is what we naturally tend to do because we want to think of $\boldsymbol{B}$ as a vector but now we know that it's not really a vector, it's a pseudo vector so what's interesting is because we have this rich structure of forms (11) we can actually choose whether we want our $\boldsymbol{E}$ and $\boldsymbol{B}$ fields to be represented by one forms or by two forms and we choose the one that is most appropriate and so how do we figure out which one is the most appropriate well we use our coordinate transformations and figure out which one of these actually transforms the right way do how do one forms transform under a parity inversion a full reflection about the origin and how do two forms transform about a full reflection about the origin.
It's not too hard to investigate how that works because what we're going to write is we're going to say:

$$x^1\rightarrow x^{1'}\quad,\quad x^2\rightarrow x^{2'}\quad,\quad x^3\rightarrow x^{3'} \tag{17}$$

That's the transformation and we also know that we're doing a parity inversion so I can say that:

$$x^1=-x^{1'}\quad,\quad x^2=-x^{2'}\quad,\quad x^3=-x^{3'} \tag{18}$$

Now I can also understand that the forms are going to transform exactly the same way:

$$\mathrm{d}x^1\rightarrow-\mathrm{d}x^{1'}\quad,\quad \mathrm{d}x^2\rightarrow-\mathrm{d}x^{2'}\quad,\quad \mathrm{d}x^3\rightarrow-\mathrm{d}x^{3'} \tag{19}$$

Now I execute this substitution (19) into the $\underline{E}$ field (14) and I'm going to get that the $\underline{E}$ field, the $\underline{E}$ one form field in the prime frame is going to be given by this expression:

$$\underline{E}'=E_1\left(-\mathrm{d}x^{1'}\right)+E_2\left(-\mathrm{d}x^{2'}\right)+E_3\left(-\mathrm{d}x^{3'}\right) \tag{20}$$

The minus signs all will come out and what we'll end up with is:

$$\underline{E}'=-E_1\mathrm{d}x^{1'}-E_2\mathrm{d}x^{2'}-E_3\mathrm{d}x^{3'} \tag{21}$$

Which is exactly what we want because we know that in the new basis system the components in the old basis system which E_1,E_2,E_3 have to change sing and there they do, they change sign straight up now the problem of course is that when we do this we'll get the same result for the magnetic field so we're going to end up with, where the $\underline{B}'$ also has a change of sign:

$$\underline{B}'=-B_1\mathrm{d}x^{1'}-B_2\mathrm{d}x^{2'}-B_3\mathrm{d}x^{3'} \tag{22}$$

But we know that $\underline{B}'$ does not change sing because $\boldsymbol{B}$ is a pseudo vector, this is fine for a polar vector so we can conclude that a polar vector should in fact be a one form so I can actually just write down $\boldsymbol{E}$ should be a one form because it transforms the right way, well look what would happen if instead of making $\boldsymbol{B}$ a one form we may $\boldsymbol{B}$ a two form, what would happen if we wrote $\underline{B}'$ down as a two form

$$\underline{\underline{B}}=B_1\ \mathrm{d}x^2\wedge\mathrm{d}x^3+B_2\ \mathrm{d}x^3\wedge\mathrm{d}x^1+B_3\ \mathrm{d}x^1\wedge\mathrm{d}x^2 \tag{23}$$

That's all the basis vectors in the space of two forms and the components are just B_1, B_2, B_3 . Now in principle this probably is best to write down as:

$$\underline{\underline{B}}=B_{23}\ \mathrm{d}x^2\wedge\mathrm{d}x^3+B_{31}\ \mathrm{d}x^3\wedge\mathrm{d}x^1+B_{12}\ \mathrm{d}x^1\wedge\mathrm{d}x^2 \tag{24}$$

Regardless, these are just still the components of these three basis vectors but now if I make my substitution (19).

$$\underline{\underline{B'}}=B_1\ (-\mathrm{d}x^{2'})\wedge(-\mathrm{d}x^{3'})+B_2\ (-\mathrm{d}x^{3'})\wedge(-\mathrm{d}x^{1'})+B_3\ (-\mathrm{d}x^{1'})\wedge(-\mathrm{d}x^{2'}) \tag{25}$$

Now look at this miracle that happens, the negative signs cancel out and what you end up with is the transformed vector right which is going to be:

$$\underline{\underline{B'}}=B_1\ \mathrm{d}x^{2'}\wedge\mathrm{d}x^{3'}+B_2\ \mathrm{d}x^{3'}\wedge\mathrm{d}x^{1'}+B_3\ \mathrm{d}x^{1'}\wedge\mathrm{d}x^{2'} \tag{26}$$

If you look at it we now are in the primed system but notice that the coefficients haven't changed, the components have not changed sign, which is exactly the behavior we would expect for a pseudo vector, this is a pseudo vector so now we have some understanding of this we have some insight, here we have this arbitrary distinction between two kinds of vectors this $\boldsymbol{B}$ type vector and the electric field vector $\boldsymbol{E}$ which was this (14), we notice that there's just two different types of vectors they transform different and I just gotta keep track of the fact that the components don't change sign for $\boldsymbol{B}$ and they do change sign for $\boldsymbol{E}$, for the electric field and polar and axial so I take note of it and I keep track of it that way but now what I realize is no no no, what's going on really is the electric field really lives in this world of one forms and the magnetic field lives in this world of two forms.

These things have all the transformation properties built in so the magnetic field is just a different type of mathematical object and now we are putting it really in its right place we're not doing some bookkeeping trick we're actually understanding this in the right way and the right way is as a two form so now that we're going to understand the magnetic field as a two form now we have to understand well heck I need to know how to calculate the curl of the magnetic field because I know that the curl of the magnetic field this is part of Maxwell's equations so in order to understand how to do the curl of a magnetic field, I have to understand what does it mean to take the curl of this object of a two form because I know how to take the curl of a vector, we've already demonstrated that the curl of a vector you convert to a one form now we know how to do curls of one forms but I need to know how to do the divergence of the vector $\boldsymbol{B}$ so I need to find the equivalent expressions for this in terms of what $\boldsymbol{B}$ really is which is a two form, I need to find what do I do in the world of forms to mimic these properties in Maxwell's equations because when I look at Maxwell's equations I'm thinking of $\boldsymbol{B}$ as a vector in the old sense, it is a pseudo vector not a vector but well I could say it's an axial vector but I usually say it's a vector and then I just have to remember oh it's an axial vector but now I'm going to say no $\boldsymbol{B}$ is a two form so I have to understand curls and divergences of two forms so that's the motivation, all that stuff I just set up to here which has been probably over half an hour of talking is just the motivation because we are now going to think of the magnetic field as a two form and now we have to understand what does it mean how do we transform these vector curls and vector divergences into two form language so that is our our final part of today's lesson.

The good news is this last part is actually very easy, the first thing is we realize we're dispensing with this idea that the magnetic field vector we're going to immediately write as a vector and as a one form instead we are going to express the magnetic field as a two form, we're going to choose a two form to represent the magnetic field $\boldsymbol{B}$. Now we still need to take the divergence as I said a moment ago so we know how to take the divergence of a one form. The divergence of a one form is:

$$\text{Divergence} \rightarrow \ * \mathrm{d} * [* \boldsymbol{B}] = * \mathrm{d} \boldsymbol{B} \tag{27}$$

But $\boldsymbol{B}$ is now a two form so what we're doing is I'm substituting in for this one form the Hodge dual of a two form and I'm hoping this will work because if it does work it's very easy because the Hodge dual of the Hodge dual is an inverted inverse of each other. If you take the Hodge dual of a Hodge dual you get back the original thing so this divergence expression for the two form $\boldsymbol{B}$ is just the Hodge dual of the exterior derivative of the two form. The question is I'm speculating because I've just written this down and I'm saying, well I know how to take the divergence of a one form so if I can turn this into a one form, take the divergence of that maybe this all works so let's check let's calculate the Hodge dual of the exterior derivative of the two form version of the magnetic field:

$$* \boldsymbol{B} = B_1 \ \mathrm{d}x^1 + B_2 \ \mathrm{d}x^2 + B_3 \ \mathrm{d}x^3 \tag{28}$$

First I need to calculate the exterior derivative, we just ignore (28) for right now, I need to calculate the exterior derivative of the two form and that's not hard because remember, the exterior derivative of a two form is a three form and that's already making me feel good because the Hodge dual of the three form when I take this guy and I Hodge dual the three form I'm going to get a zero form which is a scalar and the divergence is a scalar so I'm in the right zone:

$$\mathrm{d}\underline{\underline{B}} = \left[\partial_1 B_1 \ \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \partial_2 B_2 \ \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \partial_3 B_3 \ \mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\right] \tag{29}$$

We've already done several demonstrations of how to take the exterior derivative of something but it's not very tough, it's a sum and the only surviving terms are shown in (29) and if you shuffle this stuff around, $1,2,3$ is what we want, this is the basis vector of the three form vector space which is one dimensional and I end up with:

$$\mathrm{d}\underline{\underline{B}} = \left[\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3\right] \ \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{30}$$

Which is exactly the divergence, this is the divergence of the regular vector field from classical vector analysis and then this makes it a three form so it's not a scalar it's actually a three form in fact it's a pseudo scalar isn't it because if we did if you changed these coordinates if you did a parity inversion you would be left over with a sign change so it's interesting this is actually as it's written is a pseudo scalar with something I told you doesn't show up much but it is right here, you see it this is a genuine pseudo scalar so but we're going to take the Hodge dual of this thing right we're taking the Hodge Dual of the exterior derivative and when you do that this guy gets replaced with just the number 1 because the Hodge dual of the single one-dimensional three-four basis vector is just the number 1 .

$$*\,\mathrm{d}\underline{\underline{B}}=\partial_1 B_1+\partial_2 B_2+\partial_3 B_3 \tag{31}$$

There you have, it checks out that the divergence of a two form is given as the Hodge dual of the exterior derivative of the two form, likewise the curl of a one form was the Hodge dual of the exterior derivative of the one form so if I substitute in the Hodge dual of our magnetic field two form then what do I end up with? I end up with:

$$*\,\mathrm{d}\left[*\,\boldsymbol{B}\right]=*\,\mathrm{d}*\boldsymbol{B} \tag{32}$$

What we end up seeing is that the divergence of a one form ω and the curl of a two form β are actually given by the same expression: the Hodge dual of the exterior derivative of the Hodge dual of the one form gives you the divergence of the one form it corresponds with the divergence of the one form and the Hodge dual of the exterior derivative of the Hodge dual of the two form gives you the curl of vector:

$$\text{Divergence one form } *\,\mathrm{d}*\omega\rightarrow\nabla\cdot\boldsymbol{\omega}\text{ , Curl two form } *\,\mathrm{d}*\beta\rightarrow\nabla\cdot\boldsymbol{\beta} \tag{33}$$

I'm sorry this gives you the divergence of a vector and this gives you the equivalent of the curl of a vector and then the curl of a one form ω is the Hodge dual of the exterior derivative of the one form and that corresponds to the curl of some vector in our elementary system.

$$\text{Divergence two form } *\,\mathrm{d}\beta\rightarrow\nabla\cdot\boldsymbol{\beta}\text{ , Curl one form } *\,\mathrm{d}\omega\rightarrow\nabla\times\boldsymbol{\omega} \tag{34}$$

That has the same structure, I shouldn't say, has the divergence of a two form β , we just learned the divergence of a two form was the Hodge dual of the exterior derivative of the two form which corresponds to the divergence of some vector field. Now we can see that right away, we can write down this Maxwell's equations, the divergence of the electric field. We treat the electric field like a one form well what's the divergence of a one form? see it's the Hodge dual of the exterior derivative of the Hodge dual of the one form so this now is the differential forms version of this of the divergence of the source Maxwell equation:

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\varepsilon_0}\rightarrow*\,\mathrm{d}*E=\frac{\rho}{\varepsilon_0} \tag{35}$$

Likewise the divergence of the magnetic field $\boldsymbol{B}$. The magnetic field, we're now going to treat as a two form so now we know that this divergence expression is equivalent to in the form language as:

$$\nabla\cdot\boldsymbol{B}=0\rightarrow*\,\mathrm{d}B=0 \tag{36}$$

These two, (35) and (36) are very easy to see and as long as we're dealing with the static case with no sources, means there's no current and it's static so the $\boldsymbol{E}$ field isn't changing then I can write the curl of the magnetic field two form is:

$$* \mathrm{d} * B = 0 \tag{37}$$

That's the curl equation which is the same structure as a divergence equation of a one form and likewise in the static case where $\partial \boldsymbol{B}/\partial t = 0$. I now get the curl of a one form because $\boldsymbol{E}$ is a one form, I get:

$$* \mathrm{d} E = 0 \tag{38}$$

I've now converted Maxwell's equations at least in the static case to the language of forms and that's where we're going to stop for now and in the next lesson we're going to tidy this up even more using yet more conventions and more notation but this is the key point the key point now is we have organically discovered that the magnetic field vector as written in standard Maxwell notation language from your standard electromagnetic textbooks, that language of magnetic field as a vector is actually I'm going to go ahead and say it, it's just wrong in the sense that it doesn't capture the … there's a mathematical piece of machinery out there that does so much a better job capturing what it actually is, captures that transformation rule for example and it also allows us to tighten things up far more as we'll see in the next lesson but so now we are no longer talking about the magnetic field vector we're talking about the magnetic field two form, we're not talking about the electric field vector we're talking about the electric field one form and now we have these form equations as necessary.

There's no magnetic charges, if there were magnetic charges they'd be down here (36) but there aren't any but we do leave the source in up here, it's easy enough to leave the source for this Maxwell's equation (35) but down here if we said it was static we would get rid of $\partial \boldsymbol{E}/\partial t$ and we would leave the source term in and then we would say that this term (37), the Hodge dual the exterior derivative of the Hodge dual of the magnetic field two form equals zero $\mu_0 \boldsymbol{J}$ if we got rid of this time dependence or this a non-static term. In the next lesson we'll try to tighten this up a little further. I'll see you next time.