Geometric Algebra 12: Reciprocal basis, calculation of components

Welcome back, we are going to continue our examination of the Geometric algebra using this paper "Spacetime algebra as a powerful tool for Electromagnetism" and space-time algebra as our foundation and we have reached all the way to Section 3.4 so we have done all of the previous material and so now we will begin with Section 3.4 so let's begin. Section 3.4 "Reciprocal bases, components and tensors" begins here and we had just finished this idea of reversion and we left off with the definition of the inverse of any multi-vector. Of particular interest of inverse of multi-vectors is the inverse of this the regular vectors, I say regular vectors, I'm always referring to the four basis vectors from the Minkowski subspace of our Clifford algebra so I wrote that down over here this definition applied to a single vector

$$\gamma_{\mu}^{-1}=\frac{\gamma_{\mu}}{\tilde{\gamma}_{\mu}\gamma_{\mu}}=\eta_{\mu\mu}\gamma_{\mu} \tag{1}$$

A basis vector γ_{μ} if it's defined this way, well the reversion of a single vector is just the vector itself and the reversion square is just $\tilde{\gamma}_{\mu}$ space-time product with and we know what this space-time product in the denominator is going to be, $\gamma_{\mu}\cdot\gamma_{\mu}$ because the wedge part will be gone, remember every space time product of two vectors has the contraction and the inflation part and I'm not even going to write it down because this needs to be in your head at this point so the inflation part goes away leaving only the contraction part and that contraction part is defined as $\eta_{\mu\mu}$ so this can be broken down into:

$$\gamma_{0}^{-1}=\eta_{00}\gamma_{0}=\gamma_{0} \tag{2}$$

$$\gamma_{i}^{-1}=\eta_{ii}\gamma_{i}=-\gamma_{i} \text{ where } i=1,2,3 \tag{3}$$

We see that our basis vectors are their own inverse or the their own inverses times a factor of -1 and we'll use this in our next Section which we will begin now. Section 3.4 "reciprocal bases, components and tensors" so you know when you read this pair I mean he's literally telling you the things you better understand when you're done with this Section and that's the way you read a paper is you're assuming that the author is giving you a real clear road map into what they're doing and we are going to check those topics off as we learn them. "To connect with the standard tensor-analysis treatments of the electromagnetic field, where it", the electromagnetic field, "is considered to be an anti-symmetric rank 2 tensor $F^{\mu\nu}$, we now briefly consider the various ways of expanding proper multi-vector quantities in components". Now, I'm under the presumption that everybody here is following along at least in the sense that we're trying to compress Electromagnetism into a single equation right that's what triggered this rabbit hole of Geometric algebra that we're all enjoying together. I'm not going to review the tensor formalism of Electromagnetism, presumably you've done that but let's look at just a few facts and just to set the stage for what this all promises to deliver.

This is how the authors are talking about, a rank two tensor $F^{\mu\nu}$ that can be used to describe the electromagnetic field, now these are numbers right and we know that it is anti-symmetric so we know and the authors have already said as much that $F^{\mu\nu}=-F^{\nu\mu}$. Right now these real numbers are not part of a literal vector space because we're treating this as a collection of numbers, now it is completely legitimate and probably better and it's certainly the way I teach this material is to understand this as a set of components that belong to basis vectors that are in a tensor product space $F^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})$ where these basis vectors are functions that eat forms dx^{α} and dx^{β} and these forms go into slots and the slots get consumed by these functions that eat forms and they do so with this factor out in front and when you do this this becomes the basis vector of the vector space of these tensors. Now, what usually happens, however, as I've said many many times in these lessons is that we forego all of this stuff here and we just work with the components and the components themselves are real numbers and the anti-symmetry is captured in the anti-symmetry of these components $F^{\mu\nu}$.

$$F^{\mu\nu}(\partial_\mu \otimes \partial_\nu)(dx^\alpha, dx^\beta) \tag{4}$$

Now, we can also understand this as the components of the forms themselves $F^{\mu\nu}\ \partial_\mu \wedge \partial_\nu$. Don't confuse these, ∂_μ is a partial derivative operator which is acting as a basis vector, it's not to be confused with γ_μ , these are not equal, if the confusion exists, it's because I write like a child, a criticism which has been levied against me with complete fairness for the past few years I've been doing this. They are now going to talk about tensors in this paper, I'm quite sure of just in terms of the components themselves, they're not going to go to the tensor product space and introduce these basis factors and these functions but understand that we have to turn this now into geometric objects which means somehow $F^{\mu\nu}$ has to be either a scalar, it has to be a vector, a grade one object, it has to be a grade two object it has to be a grade three object or it has to be a pseudo-scalar like that γ_{0123} or some combination of them in principle but that's the goal here is we're going to take this tensor that we used $F^{\mu\nu}$ and we're going to discover its analogy in the Geometric algebra.

Now, they also use the word proper here, in this case they're actually saying we're going to take expanding a proper object into components so this time they are using the word proper to mean without reference to any particular reference frame because certainly when you introduce components components have to have a basis and a basis is a selected reference frame. “We shall see that we immediately recover and clarify the standard tensor analysis formulas, making the tensor analysis techniques available as a restricted consequence of the space-time algebra”. “Restricted consequence” is, I guess, a fancy way of saying, buried inside space-time algebra is all the machinery you need to execute the full rank two tensor theory of the electromagnetic field and then there's more on top of that.

Going on, “given a reference basis γ_μ for $\mathsf{M}_{1,3}$ ”, so now we've introduced a basis, “it is convenient to introduce a *reciprocal basis* of vectors” and now what we do is we define the reciprocal basis, as it's another set of vectors in $\mathsf{M}_{1,3}$ and it's super-scripted and $\mathsf{M}_{1,3}$ super-scripted is simply $\gamma^\mu = (\gamma_\mu)^{-1}$. Now this could be confusing for those of you who study General relativity and previous tensor analysis where we would know that the basis vectors with lower subscripts, if your basis vector had a lower subscript, that might be an element of your underlying vector space $\gamma_1 \in \mathsf{M}_{1,3}$ and we know that if the basis vector has a superscript it should be an element of the dual space $\gamma^1 \in \mathsf{M}^*_{1,3}$, these are not the same spaces that is how we've always learned about subscripts and superscripts, that is absolutely not what's happening here, they are both elements of the same space so in principle you could add without any trouble $\gamma_1 + \gamma^3 + \gamma_2$, you could add that and this is a legitimate vector, this sum is totally legit because everything in this chain lies inside the same vector space. You can add them together, you could also take space-time products of them you could take a space-time product $\gamma_1 \gamma^3$ which would be the same as $\gamma_1 \gamma_3^{-1}$.

That is a confusion that's potential there for new people who are new on the scene or old on this scene if you're new on the scene I guess you just you just buy the definition and there's the definition right here and by the way this actually is a definition, this is one that does legitimately deserve a three bar equal sign because you are defining the superscript version to be the inverse, the space time vector inverse of the subscript version and of course this only really applies to basis vectors because basis vectors are the ones that are given a subscript:

$$\gamma^{\mu} \equiv (\gamma_{\mu})^{-1} \tag{5}$$

Now, what's interesting is you can create a basis for $\mathsf{M}_{1,3}$ called the light-cone basis where the basis vectors are light-like and there is a good reason to do that, there's plenty of formalisms for Special relativity where being in the light-cone basis is actually a good idea and if you were in the light-cone basis then (5) wouldn't work because in the light-cone basis $M\widetilde{M}$ is zero. I'm curious to know how that would affect some of this analysis, it can't affect it too much but it certainly affects this definition but that aside as long as you choose a reasonable basis and the reasonable basis we generally will choose just so it's clear is γ_0 , the purely time-like basis vector is often going to be chosen to be the four velocity v of some material particle and that's the same thing as taking a reference frame and moving into the frame of some particle, you're moving into the reference frame attached to some material object and once you do that γ_0 points purely upward and the world line of that particle will always be a straight line in that reference frame so once you've done, that you definitely are not in light-cone coordinates or in any danger of light-cone coordinates and you drive the space-like parts to be very standard space-like parts, of course I'm losing a dimension in my little picture here so I wouldn't worry too much about that little fact if I'm sure if it comes up and it's relevant it will be called out later.

As far as the text goes, I feel like (5) probably should have landed right here at the tail end of that paragraph and (6) should have been front and center:

$$\cancel{\gamma^{\mu}\gamma_{\nu}} = \gamma^{\mu} \cdot \gamma_{\nu} = \delta^{\mu}_{\nu} \tag{6}$$

However, we've been warned that (6) is a textual error and we should be able to see that so this first part of the equality is just wrong and it slipped past editing, that is so obvious, we could have detected that error ourselves and it sure does look like something strange because we know that $\gamma^{\mu}\gamma_{\nu}$ has got to be $\gamma^{\mu}\cdot\gamma_{\nu}$ and now I have superscripts and subscripts plus $\gamma^{\mu}\wedge\gamma_{\nu}$, well I mean this looks pretty close if this part was zero then this equality (6) would be legit, the problem is this wedge part is not always zero, first of all we know that γ^{μ} is co-aligned with γ_{μ} , it's in the same direction as γ_{μ} and we know that because we we we've solved it, the reversion does not change the vector itself for a regular vector and the reversion squared is just a real number so whatever direction γ_{μ} is in, it's still going, it's going in the direction of its reversion which is the same direction so $\gamma^{\mu}\wedge\gamma_{\nu}$ will be zero when $\nu=\mu$ but when $\nu\neq\mu$ it will not be zero but if you look at the way (6) is written, this is written in total generality so it's obviously incorrect because it only equals this dot product in the case where $\nu=\mu$ which is exactly what is reflected by the second part of the equality, that is correct. We were warned by a viewer that this was incorrect and they're right so this is the definition (2nd part of (6)), of course, of the space time inverse breaks down to this, this δ function and of course that's referring to the contraction part.

Moving on, "this is an equally valid basis of vectors for $\mathsf{M}_{1,3}$ that satisfies the same metric relation $\gamma^{\mu}\cdot\gamma^{\nu}=\eta^{\mu\nu}$ ". In flat space the good news is $\eta^{\mu\nu}=\eta_{\mu\nu}$ so clearly each of the basis vectors that we work with $\gamma_0 \rightarrow \gamma^0$ that is its reversion,, you can clearly see that $-\gamma_1$ is obviously in the same direction as γ_1 or that is not linearly independent of γ_1 I guess it's in the opposite direction but it's certainly not linearly independent it's co-aligned which is sufficient to make sure that the wedge product between the two is definitely zero and I'll just make this $\gamma_i \rightarrow \gamma^i = -\gamma_i$ for $i=1,2,3$. Now the point being is that any basis vector that we've written before as say $a^i \gamma_i$ can equally be written as $a_i \gamma^i$

“Any multi-vector M is a geometric object that can be expanded in any basis, including the reciprocal basis.” That's what we just said. “For example, a vector v has the expansions”:

$$v = \sum_\mu v^\mu \gamma_\mu = \sum_\mu v_\mu \gamma^\mu \tag{7}$$

Which again is something what we just said of vector v , now here of course the Einstein sum is not implied anymore they broken out the Einstein sum but understand that we will use the Einstein sum all the time and I think they do too really I think they're just blowing it up just for exposition here but they show that you you can blow these things up in both base these vectors, there is a question of what is the relationship between the superscript and the subscript now the superscript stuff we normally call the contravariant components and the subscript we call the covariant components and usually in the old way of doing it γ^μ would be a basis vector for the dual space of the vector space but that's not happening here, I can't emphasize that enough, this notation is confusing in that regard because there's a good chance if you're studying Geometric algebra you know tensor algebra already or tensor analysis I should say, already and this is going to be something that you're going to have to wrestle with, is that (7) is not the dual vector, this isn't a duality relationship between γ^μ and v_μ , this is a straight up equality because everything lives in the same vector space here.

“For example a vector v has the expansions in terms of the reference basis and the reciprocal basis.” They're going to call this the reference basis, I'm going to try to use the language of the paper as often as I can, the reference basis γ_μ and the reciprocal basis γ_μ . “Traditionally, the components $v^\mu = \gamma^\mu \cdot v$ in the reference basis are called the *contravariant* components of v , while the components $v_\mu = \gamma_\mu \cdot v$ in the reciprocal basis are called *covariant* components”. We're going to see this thing a lot and it shouldn't be too hard to follow but if $v = v^\nu \gamma_\nu$ then $\gamma^\mu \cdot v$, notice we're we're using this dot as though it's some operation that lives in our vector algebra but remember the dot is always just a convenient way of breaking down $\gamma^\mu v$, the space-time product of two vectors, in this case you'll get:

$$\gamma^\mu v = \gamma^\mu \cdot v + \gamma^\mu \wedge v \tag{8}$$

Understand that we need to know how to do these calculations of the contracted part in the inflated part but it's not a literal operation of the Clifford algebra but anyway this is going to clearly be $\gamma^\mu \cdot v^\nu \gamma_\nu$ but v^ν is a real number so it's $v^\nu \gamma^\mu \cdot \gamma_\nu$ and by definition or by the way we've defined this, we use this equation right here (6) and we get $v^\nu \delta^\mu_\nu = v^\mu$ which is what they're after here, that's what they're saying We've exercised that to understand how that all came about so we understand that formula I just want to make sure you don't start assuming that this is some fundamental operation of the algebra, it definitely feels like it eventually because we do use this operation a lot. “Traditionally, the components $v^\mu = \gamma^\mu \cdot v$ ” and this is how you extract, it's a way of extracting the components “in the reference basis are called *contravariant* components of v , while the components $v_\mu = \gamma_\mu \cdot v$ in the reciprocal basis”, and you can repeat the same exercise we just did for this “are the *covariant* components of v ”.

When I studied tensor analysis and taught tensor analysis, I made it very clear why the words contravariant was used and why the word covariant was used I'm not going to go through that right now, those of you who are here wondering why do they use contravariant what is covariant, forget about it, in this case where we're at we we don't want to talk about the things that are varying and in fact there really isn't anything to it, it has to do with basis transformations and a few other things that we haven't reached yet, just take these as the names the contravariant components of v and the covariant components of v , what is important is that v is the geometric object itself v is the actual thing that lives in space-time and when you write down v^{μ} what you're actually writing down is (7), you're just expressing v in a basis, you're going from the proper form to the relative form and if the basis you use is the reference basis then these components are just called *contravariant* components, if the basis you use is the reciprocal basis, if the basis uses the reference basis you get the *contravariant* components, if the basis you use is the reciprocal basis, you get the *covariant* components and the relationship between the two is the same relationship is in our Minkowski Geometric algebra, the relationship between the two is the same as with the basis vectors meaning that the contravariant version of v is equal to the covariant version of v i.e. $v^0=v_0$ contravariant versions of the other ones differ by a minus sign $v^i=-v_i$.

"We see here that they can be interpreted as different ways of expressing the same geometric object". That's the key here, the geometric object is what it is, you have to choose a basis to get its relative form (7). "Evidently from (3.20) we have $v\cdot\gamma_{\nu}=v^{\nu}\eta_{\nu\nu}=v_{\nu}$ ", this will get you the contravariant component of v and it equals $\eta_{\nu\nu}$ times the covariant contribution v_{ν} , I get it reversed but it's still the same thing "which shows how the two component representations are related by the metric (which effectively raises or lowers the index)", the mechanics is actually very much the same as from tensor analysis but the interpretation of things is very different because again the reference and the reciprocal basis are in the same vector space. Now it is true that any vector space when you give it a basis it does have a reciprocal basis that's true for all vector spaces, it's just the notation we use in other branches of physics, (7) is this exact same notation but that reciprocal basis is literally a different vector space altogether and there's a lot of complications that are introduced but not here, in here it's all the same vector space.

"Since the graded basis elements of the space-time algebra are constructed with the anti-symmetric wedge product from the reference (or reciprocal) vector basis, then one can also expand a k-blade into redundant components that will be the same as the components of any rank-k anti-symmetric tensor. For example a bi-vector $\boldsymbol{F}$ ", notice the notation here, capital bold Roman letter, "may be expanded into components as follows:

$$\boldsymbol{F}=½\sum_{\mu,\nu}F^{\mu\nu}\,\gamma_{\mu}\wedge\gamma_{\nu}=½\sum_{\mu,\nu}F_{\mu\nu}\,\gamma^{\mu}\wedge\gamma^{\nu} \tag{9}$$

This is how they expand a bi-vector and you know here's the bi-vector basis in the reference basis and the reciprocal basis and the components are just given with superscripts or subscripts, the contravariant form or the covariant form, there's also presumably a mixed form out there but notice how this is anti-symmetric, $F^{\mu\nu}$ are anti-symmetric and you'll double count if you just were to sum this up and the reason you double count is you'll get $\gamma_1\wedge\gamma_2$ and then you'll get $\gamma_2\wedge\gamma_1$ and that'll have a negative sign relative to the $\gamma_1\wedge\gamma_2$ but $F^{\mu\nu}$ will have a negative sign because that's anti-symmetric so you'll end up with literally twice as much, the γ_1 and γ_2 basis vector will have a component that's twice as big if you just summed it up this way, because of the anti-symmetric symmetry of the basis vectors so that's where this ½ one half comes from. There's no genius amount of thinking that requires you to understand this is just a different basis and we have our convention that for this Einstein sum we have this up down convention and we're explicitly showing the sum here so the sum is a bit redundant. It's another important thing to note that the bi-vector itself, that's a geometric object, $\boldsymbol{F}$ is the proper form, these others forms in (9) are the relative forms. Let's attack this line here, “the components $F^{\mu\nu}$:

$$F^{\mu\nu}=\gamma^{\mu}\cdot\boldsymbol{F}\cdot\gamma^{\nu}=\gamma^{\nu}\cdot(\gamma^{\mu}\cdot\boldsymbol{F})=(\gamma^{\nu}\wedge\gamma^{\mu})\cdot\boldsymbol{F}=\gamma^{\nu\mu}\cdot\boldsymbol{F}=(\gamma_{\mu\nu})^{-1}\cdot\boldsymbol{F} \tag{10}$$

Notice that these are the reciprocal forms of the basis vectors so we have the contravariant components is the geometric object dotted into the reciprocal forms of the component so let's exercise this as best we can so I've written it down here this is what we're after:

$$\begin{aligned} F^{\mu\nu}=\gamma^{\mu}\cdot\boldsymbol{F}\cdot\gamma^{\nu} &= ½\gamma^{\mu}\cdot F^{\alpha\beta}\gamma_{\alpha}\wedge\gamma_{\beta}\cdot\gamma^{\nu} \\ &= ½F^{\alpha\beta}\gamma^{\mu}\cdot\gamma_{\alpha}\wedge\gamma_{\beta}\cdot\gamma^{\nu} \\ &= ½F^{\alpha\beta}\gamma^{\mu}\cdot\left[(\gamma_{\beta}\cdot\gamma^{\nu})\gamma_{\alpha}-(\gamma_{\alpha}\cdot\gamma^{\nu})\gamma_{\beta}\right] \\ &= ½F^{\alpha\beta}\gamma^{\mu}\cdot\left[\delta^{\nu}_{\beta}\gamma_{\alpha}-\delta^{\nu}_{\alpha}\gamma_{\beta}\right] \\ &= ½F^{\alpha\beta}\left[\delta^{\nu}_{\beta}\delta^{\mu}_{\alpha}-\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\right] \\ &= ½(F^{\mu\nu}-F^{\nu\mu})=F^{\mu\nu} \end{aligned} \tag{11}$$

The claim is that the contravariant components is this given by this formula (1st line of (11)), this formula here so I immediately replace $\boldsymbol{F}$, the proper form with its relative form. Notice this is in the reference frame. Then I pull out the real numbers and I'm left with just the basis vectors in all of these dot product forms. You still have the implied Einstein sum and you take the dot products this way. This is an example we've memorized how to do this, we know that when we have $(a\wedge b)\cdot c$, this dot product is the contraction part of the space-time product $(a\wedge b)c$, this is the contracted part, the vector part of that and we know what this formula is we know that this is:

$$(a\wedge b)c=(b\cdot c)a-(a\cdot c)b \tag{12}$$

Knowing that formula, we can now blow up this (2nd line of (11)) because this is the contraction part of $(\gamma_{\alpha}\wedge\gamma_{\beta})\gamma^{\nu}$ and make sure you understand this makes total sense because this is a vector in the vector space you can take its space-time product with any other vector in the vector space which includes this bi-vector and this so this space times product is totally legit but this dot here is telling us oh we only want the contraction part of it which is why we can apply this formula (12), what we're ignoring right now is the $\gamma_{\alpha}\wedge\gamma_{\beta}\wedge\gamma^{\nu}$ piece which would be zero whenever $\nu=\alpha$ or $\nu=\beta$ because despite the fact that ν is a superscript it will be linearly dependent on the subscript version γ_{μ} . It'll be either off by a sign or it'll be exactly the same thing depending on whether $\nu=0$ or $1,2,3$ so we're ignoring this for now because that's we have this formula that's popped up in the text (10) and we're following our nose with that formula so now we've got this structure here $\gamma^{\mu}\cdot\gamma_{\alpha}\wedge\gamma_{\beta}\cdot\gamma^{\nu}$ which we simplify to this thing here $(\gamma_{\beta}\cdot\gamma^{\nu})\gamma_{\alpha}-(\gamma_{\alpha}\cdot\gamma^{\nu})\gamma_{\beta}$, I guess simplifies a bad word for it because the latter is obviously messier than the former, that we expand it and then we have these dot products which we've learned by definition that $\gamma_{\mu}\cdot\gamma_{\nu}=\delta^{\nu}_{\mu}$, that defines these reciprocal basis vectors as this expression here which we learned a few minutes ago and so because of this we make the substitution for these two parts in parentheses with these δ functions (4th line in (11)) and then the process repeats itself but from the left side and you get another δ function (5th line in (11)) and then you can expand

$\boldsymbol{F}$ and then in the end we end up with $F^{\mu\nu}$ and it's the anti-symmetry of $\boldsymbol{F}$ by definition we're dealing with an anti-symmetric, we know that this has to be anti-symmetric and the anti-symmetry of $\boldsymbol{F}$ is what allows us to combine these two together to get $F^{\mu\nu}$. We have verified this formula.

The other formula (10) is really proceed exactly the same way, in this case you just do this process twice from the left and in this case you're actually doing a bi-vector bi-vector product, that should be interesting, maybe we should test that one out and this is just notation here (4^{th} equality in (10)), this bi-vector bi-vector product is exactly the same as this because we remember this rule $\gamma^{\mu\nu}=\gamma^{\mu}\wedge\gamma^{\nu}$, that rule, that expansion applies for the superscript or reciprocal basis vectors just as much as it does for the reference basis vectors but $(\gamma^{\nu}\wedge\gamma^{\mu})\cdot\boldsymbol{F}$ might be an interesting one to work out, just to make sure we're on top of our game here:

$$\begin{aligned}(\gamma^{\mu}\wedge\gamma^{\nu})\cdot\boldsymbol{F}&=½(\gamma^{\mu}\wedge\gamma^{\nu})\cdot F^{\alpha\beta}\gamma_{\alpha}\wedge\gamma_{\beta}\\&=½F^{\alpha\beta}(\gamma^{\mu}\wedge\gamma^{\nu})\cdot(\gamma_{\alpha}\wedge\gamma_{\beta})\\&=½F^{\alpha\beta}\left[(\gamma^{\mu}\cdot\gamma_{\beta})(\gamma^{\nu}\cdot\gamma_{\alpha})-(\gamma^{\mu}\cdot\gamma_{\alpha})(\gamma^{\nu}\cdot\gamma_{\beta})\right]\\&=½F^{\alpha\beta}\left[\delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}-\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\right]=½(F^{\nu\mu}-F^{\mu\nu})=F^{\nu\mu}\end{aligned}\tag{13}$$

Here I've actually written down this dot product between this bi-vector $(\gamma^{\nu}\wedge\gamma^{\mu})$ and I'm in the reciprocal basis dotted into the bi-vector $\boldsymbol{F}$. The first thing to understand is this dot product of course is full of meaning, it's the dot product between two bi-vectors and that's different than the dot product between a bi-vector and a vector which we understand we know that that is the contraction part of the space-time product of $\boldsymbol{F}$ times γ^{ν} . The only time we know for sure that this dot product and how to execute it just very naturally is when there's two vectors involved in which case $a\cdot b$ is always going to be the Minkowski contraction $\eta(a,b)$ but now we have a dot product between two bi-vectors and the paper hasn't really introduced that very obviously, it has actually introduced it in this table:

Description	Notation & Definition
Clifford product	ab
vector dot product	$a\cdot b=(ab+ba)/2=b\cdot a$
vector wedge product	$a\wedge b=(ab-ba)/2=-b\wedge a$
projection to grade-k	$\langle M\rangle_k$
algebraic trace	$\langle M\rangle_0$
Hodge duality	$M\mapsto MI^{-1}$
relative frame duality	$M\mapsto M\gamma_0$
bivector contraction	$a\cdot\mathbf{F}=(a\mathbf{F}-\mathbf{F}a)/2=-\mathbf{F}\cdot a$
bivector inflation	$a\wedge\mathbf{F}=(a\mathbf{F}+\mathbf{F}a)/2=\mathbf{F}\wedge a$
bivector dot product	$\mathbf{F}\cdot\mathbf{G}=(\mathbf{FG}+\mathbf{GF})/2=\mathbf{G}\cdot\mathbf{F}$
commutator bracket	$[\mathbf{F},\mathbf{G}]=(\mathbf{FG}-\mathbf{GF})/2=-[\mathbf{G},\mathbf{F}]$
bivector cross product	$\mathbf{F}\times\mathbf{G}=[\mathbf{F},\mathbf{G}]I^{-1}=-\mathbf{G}\times\mathbf{F}$
reversion (transpose)	$(ab)^{\sim}=ba$
	$(MN)^{\sim}=\widetilde{N}\widetilde{M}$

In the table you see bi-vector dot product, $\boldsymbol{F}\cdot\boldsymbol{G}=½(\boldsymbol{F}\,\boldsymbol{G}+\boldsymbol{G}\,\boldsymbol{F})=\boldsymbol{G}\cdot\boldsymbol{F}$ and it's defined as the symmetric part of the space-time product of the bi-vector $\boldsymbol{F}$ and the bi-vector $\boldsymbol{G}$. We know that this commutator bracket is defined as $[\boldsymbol{F},\boldsymbol{G}]=½(\boldsymbol{F}\,\boldsymbol{G}-\boldsymbol{G}\,\boldsymbol{F})=-[\boldsymbol{G},\boldsymbol{F}]$ and we've worked through that already, in fact I think I defined it with an additional factor of 2 than this paper did, so I defined the commutator bracket as simply what it what everybody knows the commutator to be $[\boldsymbol{F},\boldsymbol{G}]=\boldsymbol{F}\,\boldsymbol{G}-\boldsymbol{G}\,\boldsymbol{F}$. They actually throw in this additional factor of 2 in the definition of the commutator but that's a little different than the definition of my previous lesson, not that it's really that relevant because with my definition and theirs it's all the same and the point is, it is the grade two part of the space-time product of $\boldsymbol{F}$ and $\boldsymbol{G}$ and clearly this is the grade four part which means $\boldsymbol{F}\cdot\boldsymbol{G}$ is supposed to be the grade zero part. In my demonstration of how to take the space-time product of two bi-vectors, I would have called this $\boldsymbol{F}:\boldsymbol{G}$ and then the commutator bracket would have been $\boldsymbol{F}\cdot\boldsymbol{G}$ and then the bi-vector cross product would have been $\boldsymbol{F}\wedge\boldsymbol{G}$. Actually take away that last one, this would have been the bi-vector inflation.

The point being though is that I just want to take away that they're clearly using the dot product to mean the scalar part which of course makes perfect sense because what we're after in this formula is a scalar so we definitely want what I would have called the double dot product but in this paper they just call it the single dot product and I have no compunction about not being absolutely locked solid on my notation from the beginning of this lesson and on obviously I don't plan these things out that intensely and messing around with the notation is how you have to live when you read different papers and if you really want to study Geometric algebra this one paper isn't going to cut it and when you go you're going to have to be real flexible with notation so anyway we understand that we're looking for the scalar part of the space-time product of this bi-vector blade with this bi-vector $\boldsymbol{F}$ (13).

First thing I'm going to do is I'm going to take $\boldsymbol{F}$ and I'm going to expand it in the same basis as these reciprocal γ^{μ} matrices so I'll use the reference basis as the paper calls it with $\boldsymbol{F}$ and these are the scalars I'm after, I pull the scalars out and now I have a bi-vector product between these two blades blades in the reciprocal basis and blades in the reference bases but I demonstrated how to do this before I'm going to take the Minkowski product of the two that are touching and then the Minkowski product of the two that are on the outside here then I'm going to flip one of them and repeat that process picking up a $-$ for the flip and so I get these Minkowski products and we know that these Minkowski products are these δ functions by the definition of the reciprocal basis so these δ functions start to appear and I end up with $F^{\nu\mu}$. Notice it's $F^{\nu\mu}$ and I was expecting $F^{\nu\mu}$ and in fact if you go up here (10) that is what they're calculating but what I noticed is they calculated it with these Greek indices reversed and I didn't notice that when I did my work so notice that this dot product to get these components you have to actually take this double dot product or this contraction relationship in the reverse order so of course I end up with $F^{\nu\mu}$ because I used $\gamma^{\mu}\wedge\gamma^{\nu}$, if I had switched it to $\gamma^{\nu}\wedge\gamma^{\mu}$ this would have come out to be $F^{\mu\nu}$ and that ends up down here so this should equal $F^{\nu\mu}$.

The trick to getting this right though of course is to understand how to do this bi-vector contraction (2nd line in (13)), and that goes back to the earlier lesson so we're pretty comfortable I think with this notion of extracting the components, the goal is to extract the components from the bi-vector itself using these different formulas and notice there are all these different formulas to work with and this formula here (5th equality in (10)), by the way, is just a restatement of the previous one, just because $\gamma^{\mu\nu}$ actually means $\gamma^{\nu}\wedge\gamma^{\mu}$ and what's also interesting is that this inverse notion applies to these bi-vectors just as they do to the regular vectors so the fully covariant version of these basis vectors is also the inverse of the basis vectors so that's interesting.

Let's go back to the paper. The paper goes on to discuss in detail how to make this connection between tensors and the architecture of Geometric algebra, by the way, I spoke this incorrectly, the way (10) should read is, the inverse of the reference basis for bi-vectors is equal to the contravariant second rank form of reciprocal basis so the reciprocal version of the basis vectors is still equal to just this up down indices even if there's two of them that's what I'm trying to get at. I'm trying to say this:

$$\gamma^{\mu}\wedge\gamma^{\nu}=(\gamma_{\mu}\wedge\gamma_{\nu})^{-1} \tag{14}$$

I'm trying to speak this somehow and of course this ends up being:

$$\gamma^{\mu\nu}=(\gamma_{\mu\nu})^{-1} \tag{15}$$

That's what I'm trying to get at. Next time we'll actually begin the connection with tensors as we learned it previously with the formalism of the geometric algebra so I'll see you next time.