Geometric Algebra 11: Reversion and Inversion

Welcome back, we are going to continue our exploration of the space-time algebra and Geometric algebra in general using this paper "Spacetime algebra as a powerful tool for electromagnetism" by Dressel, Bliokh and Nori, I'll remind you that you can grab the paper right here off of arXiv and where are we? Well let's see, we have done 3.3.1 "Bi-vectors. Product with vectors" we have done this section, we've done all these previous sections and now we are on Section 3.3.2 "Reversion and inversion" which are operations that you could apply to multi-vectors and just to be clear they seem like they're their own operations but there's really only two operations there's only space-time multiplication ab and I should write that as multi-vector M and multi-vector N can be multiplied together using the space-time product and then there's addition where you can take a multi-vector M and you can add a multi-vector N and we've talked about both of those things. Reversion and inversion is an operation and it does apply to multi-vectors but ultimately these are the two real operations and everything else is introduced for convenience and it's very significant because it's important and convenient but understand that point. With that in mind let's review and begin.

I'd like to begin with one quick errata from lesson nine so this errata goes back to lesson 9 and a viewer pointed out that I was a little bit quick when I specified that the pseudo-scalar basis vector which is this object here which we shorten all the way down to:

$$I = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_{0123} \tag{1}$$

We've really shorten all the way down to I when I said there is only one four volume that is the pseudo-scalar basis and to be clear let me make sure I that was imprecise because there's many ways of defining a four volume in any four-dimensional space but what I'm trying to get at is if you have two four basis vectors that are are orthogonal so you have some basis set $\{\gamma_\mu\}$ then that basis set can be used to create a pseudo-scalar and that pseudo-scalar as we've seen is always going to look like γ_{0123} and the way we write things down, however if you had a different basis set say $\{\delta_\mu\}$, that would give you δ_{0123} and that would certainly be a different four volume depending on the relationship between δ and γ which in principle there would be a transformation between the two so they would be different so the four volume definitely depends on the basis that you choose but notice the structure that we write down is actually the same. Now having said that this is the same it is still true that we are assuming a certain handedness here and that I could also have chosen my four volume to be $0,2,1,3$ which would have been a right-handed or a left-handed system, well a system whose-handedness is opposite, these would have opposite handedness and so that is a choice we have to make and so in some sense there are two because the handedness is real, handedness is a very important thing it's arbitrary for us to choose how we want to establish our basis handedness which in this case would be opposite to each other but although we always will go with the first handedness in our work but things that have circulation, that circulation is not arbitrary $a \wedge b \neq b \wedge a$ likewise γ_{0123} is not the same as γ_{0213} so I guess what I was trying to say when I said there's only one is certainly there is only one dimensionality in the space-time algebra, there's only one dimensionality for $\Lambda_4(\mathsf{M}_{1,3})$, that's for sure that's absolutely true, it's a one-dimensional structure the question is, is there only one way to choose the basis meaning is there only one way to choose this four volume and of course no there's many ways of choosing the four volume there's as many ways as you can choose basis and then there's times two of those because you could choose the handedness but when I said there's only one what I mean is that you're only going to see us write it as γ_{0123} for this class so thank you for the observation and let's move on.

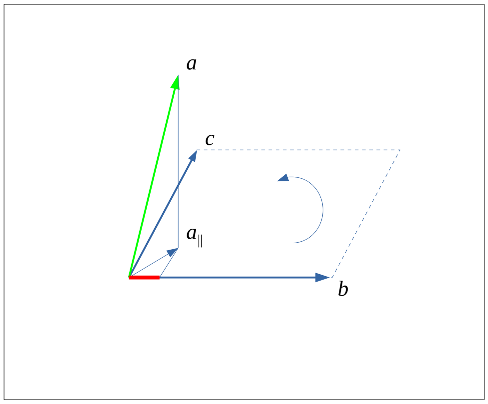
There's not too much need to review because actually the last whole lesson was almost a review, the last reading of the paper was this section about, what do they call it? They called it the bi-vector product, we talked about the bi-vector product and we went through this calculation in good detail where what we learned was when you do not have a coordinate system, you can still break things up into the projection and rejection parts, a vector can be broken up into projection and rejection and relative to a bi-vector and then once you've done that we've actually worked through these two calculations and that was fun:

$$\begin{aligned} a\boldsymbol{F} &= a\,b\,c = \tfrac{1}{2}\,a_{\parallel}(bc - cb) + a_{\perp}\wedge b\wedge c\,, \\ \boldsymbol{F}\,a &= b\,c\,a = \tfrac{1}{2}(bc - cb)\,a_{\parallel} + a_{\perp}\wedge b\wedge c\,, \end{aligned} \tag{2}$$

This is the thing that we need to memorize a bit, the contraction of the space-time product $a\boldsymbol{F}$ so remember $a\boldsymbol{F}$ is the space-time product and that equals this contraction plus this inflation:

$$\begin{aligned} a\cdot\boldsymbol{F} &= a_{\parallel}\cdot\boldsymbol{F} = (a_{\parallel}\cdot b)c - (a_{\parallel}\cdot c)b\,, \\ a\wedge\boldsymbol{F} &= a_{\perp}\wedge\boldsymbol{F} = a_{\perp}\wedge b\wedge c\,. \end{aligned} \tag{3}$$

We've basically learned that this contraction piece is this dot product and that's just really interesting how that works. By the way in case it's confusing at all, I think it's worth pointing out that the 1st line of (3) can be replaced by $(a\cdot b)c-(a\cdot c)b$ because $a\cdot b$ which is the projection of a on b , well if I just write $a_{\parallel}$ that is by definition the projection of a on b so these two have got to be the same $a_{\parallel}\cdot b = a\cdot b$, that's why, when we did this whole thing about you take the projection and rotate it by 90° degrees in the direction of $\boldsymbol{F}$ circulation, we didn't have these parallels there because you don't really need them the only reason they show up here is because they made this intentional decomposition earlier so once you've done that $a=a_{\perp}+a_{\parallel}$ why waste the effort but it does make it appear, if you're not careful in your reading here, makes it appear like I need to figure out what the projection of a is before I can do that and you don't have to, I guess the way to say it is, let me do a little drawing:



If you look at it this way you have the vector b , the vector c and the vector a and $b\wedge c$ so the circulation of this goes in the direction shown and the idea is that I take a and I project it onto the plane and I get $a_{\parallel}$ and then $a_{\parallel}\cdot b$ is the projection here, it's that little piece in red but that's the same as just taking a and projecting it right on b to begin with, that's my point, is that it's not that $a_{\parallel}\cdot b$ is something you have to calculate but knowing $a_{\parallel}$ is an intermediate step, you don't necessarily have to do if you know a and you know b and especially if you're in a coordinate system but this is a way of doing it without coordinates, I guess anyway, so we finished all that.

Now we are going to go on to this section called reversion and inversion so let's begin. "Before we continue, we make a brief diversion to define another useful operation on the space-time algebra known as *reversion*", with my caveat it's it is another operation but remember it's all dependent on the space-time product, “which reverses the order of all vector products:”, whenever you see vector products in this language you're talking about the space-time product, there's really no other product $(ab)^{\sim}=ba$ You literally just reverse the order of everything in these parentheses. “The reverse distributes across general multi-vector products recursively $(AB)^{\sim}=\tilde{B}\tilde{A}$ ” so if I have a general multi-vector A and a general multi-vector B and I reverse it, I flip B and A but I also have to reverse B and A individually I love the way they use this word recursively because that is a very useful word here. Here for example I might write out one multi-vector A and I'm writing it out now in its relative form, the form that has basis vectors and you see I have a blade zero part a blade one part a blade two part and a blade three part and B is also a general multi-vector and it has a blade zero, blade one, blade two, blade three part:

$$\begin{aligned} A &= 1+3\gamma_1+2\gamma_2+4\gamma_0\wedge\gamma_1+6\gamma_0\wedge\gamma_3+8\gamma_1\wedge\gamma_2\wedge\gamma_3 \\ B &= 5+3\gamma_2+2\gamma_{02}+4\gamma_{31}+8\gamma_{120} \end{aligned} \tag{4}$$

Now notice I use different notations here, I wrote out the space-time product completely and here I wrote out in terms of wedges, here I didn't use the wedges I well I probably should I probably should put the wedges in here all right so let me throw the wedges just to be just so this multi-vector is written one way and this is written another in this one I use the notation, just the subscript notation so just as a reminder $\gamma_{02}=\gamma_0\gamma_2$ and because of the orthogonality of a γ_0 and γ_2 , we know that the contraction part is zero so we only write the inflation part which is the wedge product $\gamma_0\wedge\gamma_2$ so all of this is a simplification and I could have done, this could be written $\gamma_0\wedge\gamma_1=\gamma_{01}$, this could be written $\gamma_0\wedge\gamma_3=\gamma_{03}$ and this could be written $\gamma_1\wedge\gamma_2\wedge\gamma_3=\gamma_{123}$.

The point is if I take A and multiply it by B and then I take the reversion which they like to throw the little symbol on the upper right of this for some reason I'm not sure if they do that for typographical reasons, when there's parentheses they throw it up here $(AB)^{\sim}$ but when you're talking about the reversion of a single multi-vector it seems like they put it on top $\tilde{A}$, at least in this paper but the point is A and B are multi-vectors and this says we'll reverse them, well you might be think well it's BA but it's $\tilde{B}\tilde{A}$ reverse B and reverse A so now I have to say well if this is B what's reverse B ? Well it proceeds linearly through B reversion so to reverse B we reverse each of the pieces and that's the level of recursion so reversing 5 doesn't change it, reversing a regular vector doesn't change because there's nothing to reverse, reversing a bi-vector does in fact change it so this bi-vector γ_{02} reverses to γ_{20} and reversing this bi-vector γ_{31} changes it to γ_{13} reversing this one γ_{120} changes it to γ_{021} , reversing this bi-vector γ_{01} changes it to γ_{10} , reversing this one γ_{03} is γ_{30} and reversing this one γ_{123} is γ_{321}

That's one way to look at reversion, well it's the only way, you reverse the two multi-vectors products but you have to recursively reverse each one individually as well so I guess the reversion of a product is the product of the reversions in the reversed order, it's a way to say it, but another way to look at it and the way we eventually will look at it is instead of doing this we realize, well when a bi-vector is reverse all you do is you change the sign of it because $\gamma_{10}=-\gamma_{01}$, likewise with this one so you would reverse that sign and the reversion of a tri-vector does that change the sign? Well, I need to turn it into γ_{321} so I have to move three over one and two so there's no sign change but then I need to move the one over once that is a sign change so the tri-vector also changes sign under reversion so I could just scratch the $+$ sign out and I put a $-$ sign there so I can treat reversions as literally reversing these basis vectors but I tend not to like that because when we set up our basis vectors we choose a basis so when we write A in a certain basis we want to stay in that basis all the time, when we do the reversion of A and B, once there's some basis vectors out there, we usually just change the signs and we leave the basis vectors alone.

The way to calculate it is, well reverse the basis vectors see what the relationship is between the reverse basis vectors and the basis vectors that were un-reversed and if there's a sign swap, if that's all there is, put it up there and that's all there could be there can only be a sign swap, that's how this all this stuff works so you do that recursively meaning this is the highest level of recursion, is switching A and B but then this is the next layer of recursion, reversing all of the parts of B and all of the parts of A and then it terminates the recursion bottoms out right there, I love recursion and so anyway but that is how you would do a recursion for two vectors that are expressed in the basis of the Clifford algebra or the space-time algebra.

There is an approach to studying Geometric algebra totally in terms of matrices I've seen that approach and it is interesting it's actually fun because things are a little bit more familiar in some ways but at the same time I'm pretty good with Abstract algebra so I'm not too worried about it but if you're not good at Abstract algebra that may be a better approach for you we're not going to indulge in this set of lectures though, moving on, “the reverse inverts itself $(\widetilde{M})^{\sim}=M$ ”, meaning the reversion of a reversion is the identity “which makes it an *involution* on the algebra”. This is the sentence that, if you know what an involution is, the sentence makes total sense but if you had no idea what an involution was you would have to infer the definition, an involution is designed as an operation which when applied twice does nothing, it undoes itself, that's an involution so clearly this is an involution right and any operation tilde that behave like this would be an involution. “The reverse is also summarized in Table 4 for reference.”

Let's look at Table 4 and I guess this is what they mean right here. Notice that they split up into two parts, if you have the vector product the product of two vectors because remember a little Roman lowercase a and b are vectors so if you have the product of two vectors and you do a reversion you just swap the order of the vectors, the reason is you can't reverse an individual vector so there's no recursion here this is the bottom level of any recursion but for general multi-vectors you have to remember this recursive step so you reverse the order of the multi vectors but each multi-vector must go through its own reversion so this is the recursive thing when you deal with general multi-vectors this is the bottom of the recursion or no recursion of it all if that's what you're dealing with so this is what they refer to in Table 4.

$$\begin{aligned}(ab)^{\sim}&=b\,a\\(M\,N)^{\sim}&=\widetilde{N}\,\widetilde{M}\end{aligned}\tag{5}$$

Reading on, “the reverse of a bi-vector $\boldsymbol{F}$ is its negation $\widetilde{\boldsymbol{F}}=-\boldsymbol{F}$ ”, that's obvious from $\gamma_{\mu\nu}=-\gamma_{\nu\mu}$ that's pretty straightforward, “which can also be seen by splitting each basis element of $\boldsymbol{F}$ into its orthogonal factors $\gamma_{\mu\nu}=\gamma_{\mu}\gamma_{\nu}=\gamma_{\mu}\wedge\gamma_{\nu}$ ”. I want to emphasize this is so important you understand this notation, this has got to be clear as a bell, the first is a space-time product the second is a wedge product, that's not true in general that a space-time product equals a wedge product, it's true when you've chosen an orthonormal basis or I guess an orthogonal basis, it doesn't necessarily have to be normal but these are orthonormal, it happens in our case, of course, but it's because that contraction part is zero when $\mu\neq\nu$, when μ and ν are in fact, the same and they don't stipulate that here but they probably, for complete clarity, they probably should, at this point in the paper you should know this, if $\mu=\nu$ then $\gamma_{\mu}\wedge\gamma_{\nu}=0$ and the only thing that exists is the contraction.

“The reversion of the factors then flips the wedge product $\tilde{\gamma}_{\mu\nu}=\gamma_\nu\gamma_\mu=\gamma_\nu\wedge\gamma_\mu=-\gamma_\mu\gamma_\nu=-\gamma_{\mu\nu}$ ”, then the notation of this last step is notation compression where you're compressing this wedge product back into this nice tight notation, “it flips the wedge product which results in a negation”. “Each grade of the general multi-vector can be reversed in an analogous way according to:

$$\begin{aligned} M&=\alpha+v+\boldsymbol{F}+\mathscr{T}+\beta I \\ \tilde{M}&=\alpha+v-\boldsymbol{F}-\mathscr{T}+\beta I \end{aligned} \tag{6}$$

If you do the reversion of M , α doesn't change, the vector stays the vector the bi-vector changes sign the tri-vector changes sign but the quad-vector, the pseudo-scalars they don't change sign so the only things that change sign when you do a reversion on M is it's grade two and grade three components which is interesting, grade two, grade three change but the others don't so there's these little non-symmetrical behaviors inside the Geometric algebra that really do all of the crazy bookkeeping for us which is really fun to flush out. “Notably the pseudo-scalar reverses to itself $I=\gamma_{3210}=\gamma_{0123}=I$.” The reversion of I is actually I . The reason this is important to see is because γ_{0123} by just exchanging the positions and turning it into γ_{3210} , you realize there's an even number of exchanges to do this, the permutation that goes from γ_{0123} to γ_{3210} , it's an even permutation and so the way that's written is $\operatorname{sign}(\sigma)=1$, the sign of the permutation is 1 therefore it's the same thing. The order doesn't matter up to this sign, it only has to do with the sign of the permutation between the subscripts so $\tilde{I}=I$.

We read on, “as a useful application of the reversion the product of a pure k -blade $M=\langle M\rangle_k$ ” which is the kth portion of the multi-vector M , the kth grade component of the multi-vector M , “the product of a pure k -blade M with its reverse $\tilde{M}$ produces a scalar”. This sentence here, I just want to emphasize, notice they drop in this word *pure* all of a sudden, I don't think I saw the word pure appear in this paper earlier so I am going to presume that by pure they mean simple which is probably the more standard language which is interesting because that's what a k -blade is supposed to be in the broader language, so saying a pure k -blade, if pure means simple which I'm pretty sure it does, that's redundant but because they're using this non-standard use of the word k -blade, they're using k -blade to refer to the k -grade which may or may not be simple, they're now saying, the simple k -blade so where it should be just k -blade. I'm just pointing that out only to emphasize that if you read other papers out there with this language in it just to be a little careful that this language is still not firmly established out there in the academic field of of Geometric algebra quite yet, I think it's pretty close.

From the papers I've read, there's there's more notation issues than language issues, let's put it that way. “As an example, for $M=\gamma_{123}$ ”, notice that's a simple multi-vector, “we have”:

$$\tilde{M}N=\gamma_{321}\gamma_{123}=\gamma_{32}(\gamma_1)^2\gamma_{23}=-\gamma_3(\gamma_2)^2\gamma_3=(\gamma_3)^2=-1 \tag{7}$$

Now, if you look how convenient this is, when you do this reversal the γ are right next to each other and that's what they're trying to show here but the notation makes it a little bit obscure in this case because it's $\gamma_3\gamma_2\gamma_1$ and then the reverse of that is $\gamma_1\gamma_2\gamma_3$ so you've already done the exchange and now you just pair things up and every time you do a pairing you get a square of something so you're going to get $(\gamma_1)^2$ then you're going to get $(\gamma_2)^2$ and then you're going to get $(\gamma_3)^2$ and if you're in the euclidean metric this is always going to be $+1$ but in our metric it could sometimes be -1 because well in this case it would be -1 because each of these squares to -1 in the convention we're using, the other convention would be γ_0 would square to -1 and these three would square to $+1$. Anyway in our case this actually ends up being -1 which is what they show here $(\gamma_1)^2$, $(\gamma_2)^2$ and $(\gamma_3)^2$. This $-$ comes from $(\gamma_1)^2$, the fact that the $-$ is missing here comes from $(\gamma_2)^2$ canceling with that $-$ and then what's left is $(\gamma_3)^2$ which actually equals -1 .

This reverse square, a square would be just $M\,M$ but the reversed square is $M\,\widetilde{M}$ or the other way around $\widetilde{M}\,M$. They did the example (7). “The resulting positive magnitude $|M|^2 \equiv |\widetilde{M}\,M| = 1$ ”, The magnitude of M squared is defined. If I'm going to ask, well what's the magnitude of a multi-vector? The magnitude on the multi-vector we have now defining as equaling the absolute value of the reversed square, that's the magnitude so even in our case where the magnitude of (7), the reverse square of that equal -1 , the magnitude still equals $+1$ so the definition of the squared magnitude, let me make sure I get that right, the definition of the squared magnitude is defined as the absolute value of the reversion square so it “is a product of the magnitudes of the factors of M , while the sign”:

$$\epsilon_M \equiv |\widetilde{M}\,M| / |M|^2 = -1 \tag{8}$$

The sign of a multi-vector M is equal to the reversed square divided by the magnitude of the reversed squared or the absolute value of the reversed square and that just gives you a sign and in this case it's either $+1$ or -1 . Now, eventually we'll see that this guy here doesn't have to be stuck at $+1$ or -1 , it turns out that this is called the signature, the *signature*. We're talking about just a single multi-vector here. Every multi-vector can have a signature but it turns out the signature does not have to be $+1$ or -1 , turns out the signature can actually have other values and we'll talk about that I think a little bit later. This is a very convenient and useful thing and “hence, the reverse square of a pure k blade”:

$$\widetilde{M}\,M = M\,\widetilde{M} = \epsilon_M |M|^2 \tag{9}$$

This produces the notion of a pseudo-norm for M so the idea now is a norm should always be positive, pseudo-norm can clearly be negative and the net signature of all of the sign is buried in this ϵ_M term and the norm itself is always positive “in exact analogy with the definition for vectors. It follows that if $|M|^2 \neq 0$ then” you can create the inverse of a multi-vector:

$$M^{-1} \equiv \frac{\widetilde{M}}{\widetilde{M}\,M} \tag{10}$$

Let me give you an example, by the way, of a situation where the magnitude of M actually is zero:

$$M = \gamma_0 + \gamma_3 \tag{11}$$

$$\begin{aligned} |M|^2 = |\widetilde{M}\,M| &= (\gamma_0+\gamma_3)^{\sim}(\gamma_0+\gamma_3) \\ &= (\gamma_0)^2 + \gamma_0\gamma_3 + \gamma_3\gamma_0 + (\gamma_3)^2 \\ &= (+1) + (-1) = 0 \end{aligned} \tag{12}$$

Here's an example of a vector whose magnitude is zero. We already know that the reversions of a vector is just the same vector so the $\sim$ actually goes away because it's a vector and then you just multiply this out linearly and you get these space-time products (12), all of these are space-time products but γ_{03} which is a known bi-vector, because of the orthogonality of the γ vectors, the γ basis vectors is obviously γ_{03} is the opposite of γ_{30} , these two cancel, these two clearly have to cancel because $\gamma_{03}=-\gamma_{03}$. $(\gamma_0)^2=1$ but $(\gamma_3)^2=-1$ so you add those together and you get zero so $|M|^2=0$ and you can't even define a signature because to define a signature, remember, we have to divide by this number and you can't divide by zero so there's just an example of a vector with a magnitude of zero, obviously this could have been a vector with a negative magnitude, if this had been $M=\gamma_0+\gamma_3+\gamma_2$ there would have been some leftover terms well let's say it was just like this, then it would have been $|-2|$ so there's an example just to clarify that part of the paragraph.

This last part (10), is interesting, this basically says when does a multi-vector have an inverse well as long as the denominator isn't zero then clearly if I want to write $M M^{-1}=1$, I better get the real number one out of this so now this is going to be:

$$\frac{M\,\widetilde{M}}{\widetilde{M}\,M}=\frac{\epsilon_M|M|^2}{\epsilon_M|M|^2}=1 \tag{13}$$

I'm using this formula here (9). Multi-vectors have inverses and that is something special about the Geometric algebra, vector spaces, generally algebras don't necessarily have inverses unless real numbers are somehow built into the algebra and in this case we've expanded our algebra to include real numbers so we can actually create this notion of an inverse which I guess is it's very important or so I'm told.

We've finished Section 3.3 and I think we'll stop there for today because I would like to try to keep the lessons definitely under an hour but an hour is actually asking a lot from the viewership but it's the way, it's my method but this is a little bit shorter this section was titled reversion and inversion but clearly reversion was the hard concept once you have reversion down this inversion is just a trivial definition, by the way it is a definition, the invert is literally defined to be (10) and clearly you combine (9) and (10) and it's the inversion is literally obvious as I just showed. Our next lesson is actually a fun section what will be Section 3.4 “Reciprocal bases, components and tensors”. Here we actually make a little bit of contact with the notion of tensors which don't really live in the Geometric algebra setting. Obviously Geometric algebra can do everything that the tensor mathematics we learned can do that's an important point but tensors don't strictly speaking live inside the Geometric algebra they're acquired through some other natural process that's in terms of geometric objects tensors are hard to visualize out there in the world when we study tensor analysis especially for General relativity, I did a whole lecture series on what is a tensor and it was a purely abstract pain in the butt notion of tensors are you know functions that eat vectors and return real numbers unless they don't and in which case they return other tensors if they only eat a few vectors and you know tensors of different rank and all of that stuff this material should teach us that tensors are to be understood in space-time algebra in a totally different way, in a geometric way that captures all of that and provides some additional insight that's what I'm hoping we see here so we'll begin that study next time.