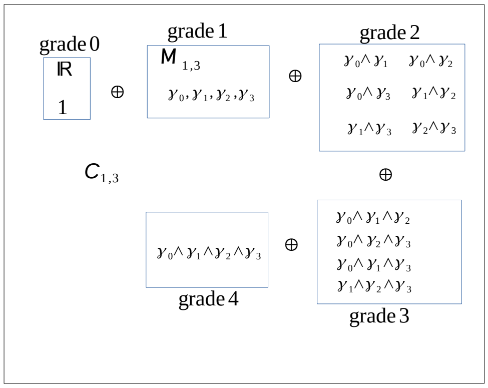
[Geometric Algebra 7: Multivector Addition](#)

We are going to proceed with our study of Geometric algebra, we're getting ready to review this section on multi-vectors, we've actually jumped ahead with a deep dive into the multi-vectors by constructing this chart:



We now understand we have five vector sub-spaces that all are part of the Clifford algebra the C_{13} and our key sub-space is our Minkowski sub-space $M_{1,3}$ this is also I should mention here that this is endowed with this metric the Minkowski metric which is so critical to the whole structure of things now we could have inserted any metric here and then we wouldn't be dealing with the Clifford algebra we'd be dealing with a different algebra. The Clifford algebra does have a Minkowski metric and that's all that it takes to bury space-time Special relativity into this mess.

I shouldn't even call it a mess that's very disrespectful, into this super-structure, this algebraic structure that we're creating here and we've discussed how to do all of the multiplications between the individual blades and each other so we can multiply any two blades or blades of any two grades, grade zero blades, grade one blades, grade two blades, three and four and now once we can multiply blades together we can multiply entire multi-vectors together so I've been promising to talk about this addition concept $\oplus$ with a little bit more detail. It's probably going to be a bit of an anti-climax but let's just dispense with this issue right away so let's begin there so we have to understand the notion of the Direct sum which is actually very easy to understand. We're going to take two vector spaces, I will call it vector space V and vector space W and those are any vector spaces real, complex, doesn't really matter, this is extremely generic process with this notion of the Direct sum of vector spaces. The idea is that we have of two vector spaces and we want to create a third vector space I guess we could call our third vector space Z so somehow Z is built out of V and W and in fact we're going to just go ahead straight up and write:

$$Z = W \oplus V \tag{1}$$

Z being the Direct sum of W and V means Z is a vector space, V is a vector space and W is a vector space so I take two vector spaces and I make a new vector space that's the idea here and for our work the Clifford algebra right as a vector space is supposed to equal the Direct sum of all of these vector sub-spaces:

$$C_{13} = \Lambda_0(M_{1,3}) \oplus \Lambda_1(M_{1,3}) \oplus \Lambda_2(M_{1,3}) \oplus \Lambda_3(M_{1,3}) \oplus \Lambda_4(M_{1,3}) \tag{2}$$

That's understanding C_{13} as an algebra and writing it out this way you see the significance of $M_{1,3}$ here. $M_{1,3}$ is what brings along the metric so I'm going to keep reminding you why Geometric algebra works the way it does, the metric of space time $M_{1,3}$, this is our vector space of four vectors and only four vectors are what Special relativity is built on but the only reason there are four vectors is because we have this Minkowski metric $\eta(,)$ that is relevant only to four vectors so $M_{1,3}$ really brings along this metric, it's an inner product space $M_{1,3}$ is actually an inner product space, it's a metric space, it's a vector space that comes with a metric. Now the significance of that is this particular sub-space that's part of this this sum actually this is $M_{1,3} = \Lambda(M_{1,3})$ as a vector space but this Direct sum it doesn't really care about the metric it only cares about the fact that these guys are vector spaces. The fact that this is an algebra that is equal to this sum is really only a statement that this set as a vector space is the Direct sum of all of these vector spaces. The algebraic properties of C are not literally derived from this Direct sum you still have to deliver to C it's space-time multiplication. This idea of a space-time multiplication does not flow from this Direct sum at all, this Direct sum provides the vectors that live in the algebra C_{13} but then we have to create the space-time algebra and we have to create the space-time multiplication and add that to C_{13} and finally when you do all of those things you really get C_{13} .

This equality is really a vector space equality so that's one thing to understand but we still let's unpeel how we understand the Direct sum so let's begin there. I'll start by giving you a little picture of the vector space V so the vector space is named V it has a bin of scalars that is some field F_1 and the dimensionality of V is some number $\dim(V)=N$ and we'll just talk about it as well it has a basis, it's a finite dimension and it's not infinity so it has some basis e_μ , the same exact thing can be said for W so it has its own dimensionality $\dim(W)=M$, it has a different a field F_2 that may be different from F_1 . W may have a different field of scalars and different basis vectors f_ν . Now we are going to talk about creating a new vector space which is the Direct sum and that new vector space is defined the following way, it is a set and it's a set of ordered pairs right so every element every vector in the set Z is an ordered pair and the first slot of the ordered pair is an element $v\in V$ and the second slot is an element $\omega\in W$ which I'll put with a Greek letter in this case I'll put ω and every ordered pair that's possible meaning every vector from W and every vector from V , you create all the possible order Pairs and you get the vectors that live in the vector space Z .

Now, I didn't put it in here but V , being a vector space has an addition property and that's an addition property in V so in order to specify that this addition is an addition just for the V vector space, I'm going to give it a little $+_v$ there and likewise W has its addition property and I'll give it a little $+_w$ and Z is a vector space also so Z just like V and W it has a field of scalars and it has a dimensionality of Z and it has an addition property and its addition property is the addition property $+_z$ so we know right away that that addition property for Z means you take a vector in Z such as (v,ω) and you add to it using the $+_z$ property another vector in Z which I would call, let's say (s,α) and we should get another ordered pair which is a vector in Z also and it's pretty obvious what we're going to do we're going to say this new ordered pair is:

$$(v,\omega)+_z(s,\alpha)=(v+_v s,\omega+_w \alpha) \tag{3}$$

That is how we define the sum of two vectors in Z which is the Direct sum of the vector space V and W . It's this ordered pair rule and the key is that everything in Z is an ordered pair. Z is a completely legit vector space now. What about the basis of Z ? Well the basis of Z is pretty obvious if you look at this ordered pair idea you have $(e_0,0),(e_1,0),\ldots,(e_{N-1},0)$, it should be all zeros and then you also have the basis vectors $(0,f_0),(0,f_1),\ldots(0,f_{M-1})$ and these are the basis vectors for the vector space Z Obviously any vector in Z that's expressible as (v,ω) can be expressed as:

$$(v,\omega)=(v^i e_i,\omega^j f_j)=(v^i e_i,0)+_Z(0,\omega^j f_j) \tag{4}$$

We can immediately split that up and it's not too hard to see that this is just going to be the N basis vectors of the vector space V and this is going to be the M basis vectors of the vector space W and so the dimensionality of Z is going to be $\dim(W)=N+M$ and the basis vectors themselves are just exactly as I described so this is actually all pretty easy stuff and it also is very obvious to extend to additional vector products meaning if a vector of sums meaning if I add another one up on the front here, if I added a vector space Q with all of its stuff you would end up with an ordered triplet where you have an element of Q here, an element of V here and an element of W here, the ordering does matter otherwise you can't define this addition very well, the ordering is relevant now it doesn't really matter what order you do it in every any ordering will be isometric to any other ordering isometric as a vector space so the ordering doesn't matter in that sense but once you've decided the ordering you know to really zero in on what Z is, you better remember what order we're doing.

Now we can look at this and we can immediately understand what C_{13} here so when we look at C_{13} in (2) we see wow so C_{13} is a vector space of order quintets so every element of C_{13} looks like this it has five slots, the first slot that's where you have to put in a real number, the second slot that's where you have to put in a vector, the third slot that's where you have to put in a bi-vector, the fourth slot that's where you put in a tri-vector $v\wedge w\wedge z$, you put that here and then the fifth slot you put a quad-vector which we're going to call pseudo-scalar so that would that last slot is $v\wedge w\wedge z\wedge q$ and it's in that order so everything in C_{13} ultimately if we take seriously, the formal definition of this Direct sum is this product, actually if you got really super formal, the second sum would be once you add two together and you get an ordered pair, if you add three together in principle if you really were a stickler you would have an ordered pair where the first slot is an ordered pair from the first sum and then the second one is an ordered pair from the third and then the fourth inner product would be an object that takes an ordered pair from the first pairing and combines it with a second and you start building stuff up like that but unpacking that is isomorphic to this idea of just having an unordered quintet of objects. Now we understand how to add things together because whenever we add things we're actually using the addition property of the Direct sum space so the addition property for C_{13} actually looks like this:

$$a,b\in C_{13} \quad \begin{matrix} a=(7,2\ \gamma_2,3\ \gamma_1\wedge\gamma_3,5\ \gamma_0\wedge\gamma_1\wedge\gamma_2,7\ \gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3) \\ b=(1,3\ \gamma_2,-\ \gamma_2\wedge\gamma_3,2\ \gamma_0\wedge\gamma_1\wedge\gamma_3,4\ \gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3) \end{matrix} \tag{5}$$

C_{13} is this long Direct sum (2) so if I have two elements of the C_{13} they look like this, element a well the 1st slot comes out of the 0th exterior power $\Lambda_0(M_{1,3})$ which we know is just real scalars, the 2nd slot that's the slot that draws from the 1st exterior power $\Lambda_1(M_{1,3})$ which is basically just our underlying vector space which is $M_{1,3}$ so this slot here is vectors in $M_{1,3}$. This 3rd slot are bi-vectors, the 4th slot are tri-vectors and the 5th slot is pseudo-scalars which we'll introduce the word, pseudo-scalar for this last one soon but it's a quad-vector so that's what a member of C_{13} looks like and if I have another member of C_{13} which may look like b , it's the same idea, they're different I have them written as blades but they could be elaborate sums right you could have the sum of blades in fact you're going to see that in a moment because if the sum we were now interested in what do we mean by $a+b$, well the first thing we mean is we're talking about the plus, a is a member of the vector space C_{13} and C_{13} addition property is symbolized by $+_Z$ so whenever I write a plus I mean that, now we can also use the computer science method of looking at this which I actually like which is, look we know a and b are members of C_{13} we know that so since we know since that fact was given, you don't need to waste your energy with a Z , it can only be that, there's no other you're not making a mistake by leaving it off because we're informed of what addition we're using by what's on both sides of the addition symbol so it's implied and it can only be one thing, given that what is this? Well it's now going to be:

$$a+b=\Big(7+1,2\,\gamma_2+3\,\gamma_2,3\,\gamma_1\wedge\gamma_3-\gamma_2\wedge\gamma_3,\\ 5\,\gamma_0\wedge\gamma_1\wedge\gamma_2+2\,\gamma_0\wedge\gamma_1\wedge\gamma_3,\\ 7\,\gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3+4\,\gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3\Big) \tag{6}$$

$7+1$ on the 1st slot but what is this $+$? Well this $+$ has to be the $+$ from Λ_0 , that's the Λ_0 $+$ and what about the next $+$? Well, that's going to be $2\,\gamma_2+3\,\gamma_2$ where this $+$ comes from the Λ_1 sub-space. Then the next one is $3\,\gamma_1\wedge\gamma_3-\gamma_2\wedge\gamma_3$, it's a bi-vector. What about this $-$ sign? Well the $-$ sign is a bi-vector $-$ sign and if you're unhappy, you just make it a $+$ sign and you keep the $-$ sign inside parenthesis, that's fine. Then the next two are pretty obvious. Take note of how painful it was to write that down because one thing we're going to do pretty quickly is create notation to simplify all of that but you can see we have all the slots and that's scalars, vectors, bi-vectors, tri-vectors, quad-vectors (pseudo-scalars) and each addition comes from the underlying vector space. We could write that like this every time but this is what it means to add a vector to a bi-vector, literally the addition of a vector to a bi-vector is going to be:

$$\left(0,A^{\mu}\gamma_{\mu},0,0,0\right)+_Z\left(0,0,\tfrac{1}{2}F^{\mu\nu}\gamma_{\mu}\wedge\gamma_{\nu},0,0\right) \tag{7}$$

That is the technical way of adding a vector to a bi vector, by the way, when I write down this notation for a bi-vector I need to be a little careful, this is anti-symmetric and there's usually a one-half that is a part of it because you double count. The number of independent vectors is half of what you would get if you you did all this and of course the μ equals ν bi-vector is zero so you end up double counting so there's usually a one-half here, I didn't use that in the last lesson but I don't think it mattered in the last lesson, I think it would have all flowed okay without it but say you're adding those two together well the result is the obvious thing it's:

$$\left(0,A^{\mu}\gamma_{\mu},\tfrac{1}{2}F^{\alpha\beta}\gamma_{\alpha}\wedge\gamma_{\beta},0,0\right) \tag{8}$$

Just to be clear let me change this so it's α , β because just so we're not reusing μ . I guess I'm now introducing this one half. This ordered quintet is now that is what it means to add a vector to a bi-vector you literally create this ordered quintet, that's the mathematical machinery behind all of this. Obviously this applies to tri-vectors and quad-vectors, the pseudo-scalar, bi-vector and that's the machinery that's under the hood so when we are writing so now you can understand that if we write a vector say a vector plus a bi-vector which might be $v+u\wedge w$ what we're really doing is we are creating this entire operation here we're taking the v , we're stuffing it in the second slot of an ordered quintet because that's the only way to make it a member of the Clifford algebra, we're taking the wedge, we're stuffing it in the third slot of the ordered quintet of C_{13} and then we're adding it together and we're producing a ordered quintet that has them both in there, that's the literal Machinery behind all of this.

However, once we appreciate that we can take radical shortcuts and we can just say, you know what? That means (8), you know what else means? That $v+u\wedge w$, that means and this $+$ sign that's always means $+_{C_{13}}$, it's always the Clifford algebra addition, what we do is we introduce this addition symbol realizing I'm not going to write down these parentheses and all these zeros all the time, a vector is just this thing in the 2nd slot, do I need really need to include all of those zeros? Probably not because I don't lose anything by just creating this $+$ sign and so whenever we see a $+$ between different graded objects, if you really want to go back to the fundamentals this is what the fundamentals mean that $+$ is literally this ordered quintet, let's see what else do I want to say about this? Let's see.

I guess the only other point is there's a strong desire to interpret this, I mean we've interpreted $u \wedge v$ as a little slice of oriented area with a circulation and we've interpreted $\gamma_0 \wedge \gamma_1 \wedge \gamma_3$ as an oriented volume a three-dimensional volume and we've interpreted $\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$ as a four-dimensional volume element and that's fine but interpreting them as a sum, you know I'll keep an eye out but I think we're just going to have to live with the idea that, no all we mean is that we're carrying these things along together, now it's easier to interpret the space-time product portions like the space-time product portion of v so we have a space-time product of v with some blade F , this is always going to be some projection of v onto F with a rotation of some kind or I should say one of the resulting components, one of the resulting graded objects is going to have that interpretation and each graded object has some interpretation but the sum of them together that might be a little bit much to ask for but there's always somebody who comes up with an interpretation and people jump on it very quickly because it really does, if it's a good interpretation, it really does make things a little easier for us human brains to understand but I think what's really going on here is all of this architecture just does an amazing job of keeping track of pretty much everything you ever wanted to know about space-time, at least that's the promise of it all, so moving on.

We're now going to return to the paper, let's do a little foreshadowing of our next subject, we've talked about multi-vectors quite a bit so now we're going to return to the paper and if you have any questions left about multi-vectors, give this paper a chance. What you already know the authors here probably, they're cutting to the core of it very quickly and I bet the paper itself will clear up any remaining questions so we will be we will begin this section on multi-vectors, section 3.3. As you can see it starts with a very clever notation tightening of how to understand multi-vectors they give the different blade types names and interpretation so we're going to talk about that you've already seen a lot of this but you haven't seen their compression, I've given you the uncompressed version where everything has a bunch of wedge products but why draw all those wedge products if you don't need them and there are important circumstances where you don't need to draw in wedge products and we'll talk about how they how they do that we've already looked at this paragraph a little bit when we talked about k -blades, these paragraphs introduce that special notation you can see a little bit of foreshadowing right here of how this is going to work. This is space-time product on the left, this is wedge product on the right so you might wonder you know this is a space-time product usually there's other pieces but where would those other pieces go? That'll be fun to see, they rename a space-time product you'll see how we do that and then ultimately they start giving this compressed notation for any multi-vector in C_{13} .

It going to look like that ((3.7) in paper) so they're going to introduce different fonts or different alphabets and fonts for each different part of a multi-vector so we can write it in a much more compressed form and you'll see how that works also then they introduce this notation ((3.8) in paper) which we've already introduced so it'll be the second time you're seeing it and this section actually goes on and on and on quite a bit. They have another chart that breaks everything down and then they talk about some of the stuff we talked about multiplying or about space-time multiplication of multi-vectors so there's actually a lot of reading ahead of us, This project of going through this paper is not going to be short and I don't know when it will finish it when, we'll finish it I hope I don't run out of steam if I do run out of steam though I promise to run out of steam at a place where if you've been following you can just press forward yourself so thank you for paying attention and I'll see you next time