The Dirac Equation

Let's continue with our introduction to field equations and let's see if we can make any sense of the Dirac equation so we're going to try to give a somewhat of a logical reason to want to have something like the Dirac equation. Remember all of this Physics at this point is poking and prodding through our axiomatic systems looking for things that actually model nature and so there's no direct line as I said in the last lecture right there's no direct line from Classical Mechanics right into Quantum Mechanics right into Relativistic Quantum Mechanics right into Quantum Field Theory. I mean there's a straight line in retrospect as we look back the history of the subject and the development of the subject but if you're standing here (Classical Mechanics) there's no obvious clue about how to get to here (Quantum Mechanics) and there's no obvious clue about how to get to here (Relativistic Quantum Mechanics) and there's no obvious clue about how to get to here (Quantum Field Theory), you have to do a little bit of guesswork and so guesswork is kind of the way this process works on the other hand if you have this (Quantum Field Theory) then you can definitely get to this definitely get to this and I mean inside here all of this is contained in limiting cases but going this way is insight and the activity of geniuses and the accumulation of knowledge in well-regulated institutions that examine each other that the whole process of science and philosophy gets you here, coming back we're kind of basically what we're doing is we are going backwards even though it looks like we're going this way and I'm claiming we started from Classical Mechanics I'm doing prerequisite stuff, I assume you know Quantum Mechanics now we're going to talk a little bit about that Relativistic Quantum Mechanics it looks like we're going this way but the only reason we have a path to walk this way is because people have already created this thing and I'm actually going backwards I'm just telling a backward story backwards.

Enough about that let's begin and let's see if we can now introduce the idea of the Dirac equation, so we had this mind map and the first thing I'd like to point out is a very astute observer noticed I had a sign error, I can't believe this error too, I had a sign error in my classical Klein-Gordon field analysis and the problem that I had was, it's a tricky little problem it's worth seeing, well let me put it this way, it's not a tricky little problem, it's a stupid problem but it's the kind of stupid problem that we all go through.

$$\text{Classical Klein-Gordon field} \rightarrow \ \mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}M^2\phi^2 \qquad (1)$$

The issue is taking the derivative of the Lagrangian with respect to the field variable requires taking the derivative of this with respect to ϕ not $\dot{\phi}$ and the 2 comes out, it cancels this ½ and you end up with $-M^2\phi$ and that's what I was carrying through the whole problem, the issue is though that I'm looking for in the Euler-Lagrange equations I want to take this, I want the negative of this derivative and that's what I failed to do and so this sign in the previous lecture was sitting as a minus sign and the ultimate equation ended up with a minus sign which is absurd, absurd in the sense that if you know what the Klein-Gordon equation is you know that there's no minus sign there, this is a positive number.

$$\text{Recall} \rightarrow \ \frac{1}{2}(\partial_\mu\phi)^2 \equiv \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi \qquad (2)$$

$$\text{Euler-Lagrange equation} \rightarrow \ \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathscr{L}}{\partial\phi} = 0 \qquad (3)$$

$$\partial_{\mu}\left[\frac{\partial}{\partial(\partial_{\mu}\phi)}\left[\frac{1}{2}\partial^{\beta}\phi\,\partial_{\beta}\phi\right]\right]+M^2\phi=0 \tag{4}$$

$$\text{Recall}\rightarrow\ \frac{1}{2}\partial^{\beta}\phi\,\partial_{\beta}\phi\equiv\frac{1}{2}\eta^{\alpha\beta}\,\partial_{\alpha}\phi\,\partial_{\beta}\phi \tag{5}$$

$$\frac{1}{2}\eta^{\alpha\beta}\overbrace{\frac{\partial[\partial_{\alpha}\phi]}{\partial(\partial_{\mu}\phi)}}^{\delta_{\alpha}^{\mu}}\partial_{\beta}\phi+\frac{1}{2}\eta^{\alpha\beta}\,\partial_{\alpha}\phi\overbrace{\frac{\partial[\partial_{\beta}\phi]}{\partial(\partial_{\mu}\phi)}}^{\delta_{\mu}^{\beta}} \tag{6}$$

$$\frac{1}{2}\eta^{\mu\beta}\,\partial_{\beta}\phi+\frac{1}{2}\eta^{\alpha\mu}\,\partial_{\alpha}\phi=\partial^{\mu}\phi \tag{7}$$

$$\partial_{\mu}[\partial^{\mu}\phi]+M^2\phi=\left[\partial_{\mu}\partial^{\mu}+M^2\right]=0 \tag{8}$$

Also the other thing that made it truly absurd was that I had it correct over here when we did the sort of the relativity version of it where we first quantize the energy momentum relation and this was correct so I had two wrong two different equations for the Klein-Gordon equation at the end of the last lecture so this is the correct version.

$$\text{Klein-Gordon equation from Relativity}\rightarrow\ E^2=p^2c^2+M^2c^4 \tag{9}$$

$$\hat{E}^2-\hat{p}^2c^2-M^2c^4=0 \tag{10}$$

$$\left[i^2\hbar^2\partial_0^2-\left((-i)^2\hbar^2\nabla^2\right)c^2-M^2c^4\right]\Phi=0 \tag{11}$$

$$\left[-\partial_0^2+\nabla^2c^2-\left(\frac{Mc^2}{\hbar}\right)^2\right]\Phi=0 \tag{12}$$

$$\left[\partial_0^2-\nabla^2+M^2\right]\Phi=0 \tag{13}$$

$$\left[\partial^{\mu}\partial_{\mu}+M^2\right]\Phi=0 \tag{14}$$

Before we discuss the Dirac equation, let's remind ourselves this guy here (9), from Special Relativity it's the law, it's the rule, there's there's no reason why you would expect any kind of relativistic energy momentum relation to be different than this, this is the one from Special Relativity and it must be obeyed by all particles. When we create this equation (14), we're definitely saying that this field Φ, this field variable which is interpreted because remember this is derived from the First quantization rule right we take the energy momentum relation we replace in (11) the energy with $i\hbar\partial_0$ and then we replace it with $(-i)\hbar\nabla$ so when we do that we're hoping that this guy Φ is going to come out and be a Wave function so we're thinking about this in terms of a Wave function but a Wave function's still a field it still is a scalar field, it has a value at each point in space, it's the interpretation of that value that is now different it's now not a classical quantity that can be measured it is now part of a quantum mechanical Wave function that has its own interpretation in principle now the famous problem with this is that interpreting this as a literal Wave function doesn't work very well.

In fact it fails and my understanding is that was recognized pretty quickly I think I think the story goes that Schrödinger actually did this. This is sort of obvious, this is a legendary equation that's staring everybody in the face so this notion of First quantization, if you're going to try this idea of First quantization of substituting these operators (15) into something, if that's your plan then this is where you would start because everybody knew this was true.

$$\text{First quantization} \rightarrow \ \hat{E}=i\hbar\partial_0 \ , \ \hat{\boldsymbol{p}}=-i\hbar\nabla \ , \ \hat{p}_i=-i\hbar\partial_i \ , \ \hat{p}^\mu=i\hbar\partial^\mu \tag{15}$$

Unfortunately it didn't work and so he unfortunately he created the regular Schrödinger equation which did seem to work but is not relativistic so they kind of put relativistic principles on the shelf and developed regular Quantum Mechanics and they fully understood it wasn't relativistic so I guess they decided well we'll have to come back and figure that problem out later and just work in a non-relativistic regime but the point is that the fact that (10) has to be satisfied is an important piece of understanding how the Dirac equation surfaces. One other thing I'll say before we begin the Dirac equation is here I write down the Pauli matrices:

$$\sigma^0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \ , \ \sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \ , \ \sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \ , \ \sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{16}$$

I'm trying to be consistent with Peskin and Schroeder, “An introduction to QFT”, it's a legendary textbook, really important textbook in field theory, not necessarily the best for everything but it's a great textbook to start with and I'm trying to use all of its conventions and I captured all the conventions here and that is more correct.

$$\sigma^\mu \equiv (1,\boldsymbol{\sigma})=(\sigma^0,\sigma^1,\sigma^2,\sigma^3) \tag{17}$$

$$\tilde{\sigma}^\mu=(1,-\boldsymbol{\sigma})=(\sigma^0,-\sigma^1,-\sigma^2,-\sigma^3) \tag{18}$$

Having reminded ourselves of all these facts now let's turn to the idea behind the Dirac equation. We are going to do, this is the most common approach I think to explaining why the Dirac equation is a thing. The idea is here I'm writing down the Klein-Gordon equation in full, I've blown up the $\partial^\mu\partial_\mu$ part and when you do that you end up with this expression here and I've collapsed these units $M^2c^2/\hbar^2$ into just μ^2 so we have this guy. What I'm going to try to ask myself is well first of all this thing is a second order differential equation and this is one of the problems with the Klein-Gordon equation, this is the Klein-Gordon equation:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+\mu^2\right]=0 \text{ where } \mu=\frac{Mc}{\hbar} \tag{19}$$

Right here we have the Klein-Gordon equation and the point is if you were to solve this you would solve for some function Φ of space-time and in order to uniquely solve it you're going to need an initial condition which is a value of $\Phi(x)$ at some particular value $x=x'$ establishing its value at t, x_1, x_2, x_3 and you also would need to know it's first derivative you would need to know say the value of the first derivative $\partial_\mu \Phi(x)$ because you need two conditions really to establish the value of of the Wave function uniquely, you're trying to solve a second order differential equation so you need two pieces of information.

Dirac equation via "factoring" is what this sort of section of the mind map is all about.

$$\left[A\partial_1+B\partial_2+C\partial_3+iD\partial_0\right]^2\Phi=\mu^2\Phi \tag{20}$$

$$\begin{array}{c} A^2\partial_1^2+AB\partial_1\partial_2+AC\partial_1\partial_3+iAD\partial_1\partial_0 \\ BA\partial_2\partial_1+B^2\partial_2^2+BC\partial_2\partial_3+iBD\partial_2\partial_0 \\ CA\partial_3\partial_1+CB\partial_3\partial_2+C^2\partial_3^2+iCD\partial_3\partial_0 \\ iDA\partial_0\partial_1+iDB\partial_0\partial_2+iDC\partial_0\partial_3-D^2\partial_0^2 \end{array} \tag{21}$$

Notice in Quantum Mechanics you don't need two, you only need one because it's first order in time, the Schrödinger equation is first order in time and the Dirac equation is second order in time actually so I guess the conditions you need are these spatial value of the Wave function and it's and the spatial derivative the time derivative of the Wave function at two different times but the point being that at two different conditions in order to get it. Right away the interpretation of the Wave function becomes a problem because the understanding of a Wave function is when I know the Wave function at a certain time, the Schrödinger equation tells me how that Wave function will evolve straight, I don't need to know the Wave function and how the Wave function is changing, I don't need to know that extra piece of information so that's one problem: is this extra amount of information that you need to to nail down Φ if the whole equation is second order.

The way the again the story goes and I don't mean to demean the history by saying the story goes but I'm not a historian of Physics and I haven't read the letters and messages and all the original papers or anything like that I just read like any other person who's learned the subject in the modern era I read distillations of this stuff so the story as I've heard it is that Dirac says let's try to make the whole thing first order, the non-relativistic one is first and second order first order in time second order in space, let's see and Klein-Gordon doesn't work very well and it's second order in time and space let me find one that is first order in time and space and find an equation for the Wave function of an electron or a particle that is first order in time and space and make it relativistic and so I'll solve this problem of relativity and so what he sets out to do is he knows that the Klein-Gordon equation has to have something to do with this because it's a direct expression of relativity, it falls right out of this basic principle (9) and the only thing people know how to do is to first quantize (15) so he's saying what I'm going to do is I'm going to say that there are these numbers A, B, C, D and if I take a linear operator to straight up a linear operator first order meaning and I have these arbitrary constants A, B, C, D and I square it then I'm going to get the Klein-Gordon equation and then what I'll do is I will take the square root of it meaning, I'll eliminate these little squares and what's left over is going to be a first order equation where if this function Φ , satisfies that first order equation it's definitely going to also satisfy the Klein-Gordon equation because this operator that drives the first order equation is just sort of the square root of this thing.

Notice this is the Klein-Gordon equation (19) but I write it down as A, B, C, D and I add this i here (20) but if you check it all out, this is the same thing as this (19). If you what would you do you would take you take this thing here move it over there you have a minus sign multiply everything by -1 you get a plus sign there those guys become plus you get a minus there but then I'm going to turn it into a square so I take that minus I drive it to be an i and you see now we have the structure that I've written out right here (20). This is equivalent so now we gotta look for these numbers A, B, C, D and I emphasize numbers because of course they're not going to end up being numbers so we think about this we say here's this thing I'm going to square it, it's going to equal this mass term Φ , let's blow this up and see what we got and when we blow it up we just do the obvious thing I mean we're just squaring this quad nominal, making sure we keep track of this i a little bit and we have this squared term and these cross terms. The square terms are here on this diagonal and then you have all these cross terms and then the problem is: this guy here (21) has got to end up equaling this guy here (20) and that's a bit of a problem, I mean we've already can see right away that $A^2 \partial_1^2$ is going to have to correspond to $-\partial^2/\partial x^2$, that $B^2 \partial_2^2$ is going to have to correspond to $-\partial^2/\partial y^2$ and $C^2 \partial_3^2$ is going to end up having to correspond to $-\partial^2/\partial z^2$ and then the t^2 part $-D^2 \partial_0^2$ is going to have to correspond to $1/c^2 \partial^2/\partial t^2$

That's all the parts, which means everything else has to cancel out, now what you'll notice is that again just like I said before I'm working backwards from knowing the answer because when I blew up this quad nominal as I'm calling it I guess I'm keeping track of the order of things A^2 is fine but when I do AB but when I do BA , I leave it as BA I don't say it's AB and I get $2AB$, I don't assume they commute and that's because I know that if I assume they commute the problem doesn't solve, you end up with $2AB$ here, the AC becomes $2AC$ and the CA is gone and the AD becomes a $2AD$ and DA is gone and you're stuck, you don't have a square root whose square is actually the Klein-Gordon equation and so looking ahead I know that you and me we see this and we go like well, this is a failure we just move on and maybe try something else or we become like biologists or something.

Not Dirac, Dirac said I think the problem is I need AB to cancel with BA and I do know something that could do that and it's not a number it's not a real number it's not a complex number, it's a matrix so A, B, C, D have to be matrices which is what he decided was the case. The reason that this was so significant is because sure matrices will get all these cancellations to work if you find the right ones. Matrices can make these things cancel but can you find four matrices whose squares will give you (19) and whose commutators will cancel well, which will cancel when they commute and there are four of them can we find matrices to do this? There's that question but even if you did then you end up with a matrix equation and if you have a matrix equation you're now going to be looking for an object that is a column vector that has the same dimensionality as your matrices, Φ ceases to be a Wave function like we've always seen in Quantum Mechanics, it now becomes some kind of column vector.

Now indeed by this time it was already known that you could ad hoc add into Quantum Mechanics things like two by two matrices and you could add into the Wave function a column vector of two parts with a function attached to it and you would call this a Spinor, that was already known but it was totally ad hoc meaning they just realized that if you did this mathematical thing and you now created this notion of a Spinor field as a Wave function you can solve problems involving angular momentum spin ½ angular momentum but without that it was it couldn't be done but they just noticed it and they added it in there as an assumption to the theory.

The idea now is that this is different, if you can find these four matrices and now Φ becomes a column vector by its very nature and remember those four matrices are forced on you by the principle that you need to satisfy this ultimately this relativistic principle (9) so this relativistic principle drives you to ask this question and the answer forces you to accept matrices which forces this Wave function to now be some kind of column vector and I'm sure at this point he's like well I'm on the right track because that might explain this ad hoc crap we've had from Quantum Mechanics so let's go, this is this is going to be great so right now we're kind of optimistic about all this so what do we do next? Well, what are the requirements for these four matrices and as I said before they have to cancel all of these cross terms:

$$AB+BA=AC+CA=BC+CB=AD+DA=CD+DC=BD+DB=0 \tag{22}$$

You have all of these requirements here and then when you square them you have to end up with one:

$$A^2=B^2=C^2=D^2=1 \tag{23}$$

Finding these matrices well that's the next thing we do and immediately you might get excited about this but then you realize what there are no two by two matrices that do this right, there are no two by two matrices that will satisfy all of these requirements so whatever this is it's not going to immediately reproduce this ad hoc invention that saved Quantum Mechanics and made Quantum Mechanics really relevant, it's something else. It turns out the smallest size matrices that can do this it turns out are four by four matrices which means this guy Φ has to be a column vector of four elements which means now we're talking about some kind of field that at every point in space-time boom, there's some kind of structure that has four elements in it, four degrees of freedom I guess you could say and every point in space-time is assigned some value, some numbers and these will be complex numbers and they'll be four of them for every point of space-time so you have this field we'll still call it a Spinor field, even though the Spinors from Quantum Mechanics only had two dimensions, we're still going to call this a Spinor field and now the mission is to find these matrices all right? Let's have a look at that question.

Dirac

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix} \tag{24}$$

Majorana

$$\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} \tag{25}$$

Weyl

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} , \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix} , \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix} , \begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix} \tag{26}$$

The good news is that there's a lot of matrices that satisfy this and there are three sets that are typically used in the literature and these four matrices have common names $\gamma^0,\gamma^1,\gamma^2,\gamma^3$ in the same order. The difference is that they're different representations of these, what we're going to call these γ matrices but they all satisfy the requirements that we need to make all of this work. Now there is a little bit of an issue between what I've defined as A,B,C,D and these γ matrices but we'll clear that up in a moment. The point is these γ matrices are the matrices that go into the Dirac equation and there are four different, these are the three most common representations in fact the Dirac representation and the Weyl representation are by far the more common and the Majorana representation is far less commonly used however it's a really interesting representation I think I want to explore it a little bit in our prerequisites series just to really get a good feeling on how the Dirac equation works.

$$\textbf{Dirac}\rightarrow\ \gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix} , \gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix} \tag{27}$$

Majorana

$$\gamma^0=\begin{pmatrix}0&\sigma^2\\\sigma^2&0\end{pmatrix} , \gamma^1=\begin{pmatrix}i\sigma^1&0\\0&i\sigma^1\end{pmatrix} , \gamma^2=\begin{pmatrix}0&\sigma^2\\-\sigma^2&0\end{pmatrix} , \gamma^3=\begin{pmatrix}i\sigma^3&0\\0&i\sigma^3\end{pmatrix} \tag{28}$$

$$\textbf{Weyl}\rightarrow\ \gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix} , \gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix} \tag{29}$$

There is a simplified notation for these three representations, the Dirac representation (27), these are meant to be four by four matrices because each of these are two by two matrices so there's two by two so this 0 clearly means the zero two by two matrix there that's what that means. Likewise this 1 here means the two by two identity matrix and this -1 is the opposite of the two by two identity matrix so this is a bit of a shorthand for how to write down these γ matrices. Now you'll notice for the Dirac representation (27) these γ matrices are very simple, you just have the poly matrices for $1,2,3$ versions and the 0th version is just a diagonal matrix.

For the Majorana representation (28), there's no super duper shorthand, you've got to express each one separately but notice there's a certain amount of favoritism to the second poly matrix and there's a reason for that, if you look at these matrices carefully what you discover is that if you go back to the conventions (16), σ^2 is the only complex poly matrix so if you go down here back to the Majorana representation and you notice that σ^2 is there. Now I have σ^3 is real, σ^1 is real, σ^2 is complex but notice I'm taking these real ones and I'm multiplying them by i so now all four of these γ matrices are complex. Here (27) that's not the case σ^2 is complex but the others are not. The same is true for the Weyl representation (29). Notice the only distinction between the Weyl representation and the Dirac representation is in γ^0 , these guys γ^i are the same actually so these different representations work and when you actually make contact with this material right here (20) and (21) you end up with the Dirac equation and here it is:

$$\left[i\gamma^{\mu}\partial_{\mu}-M\right]\Psi(x)=0 \tag{30}$$

This is the Dirac equation and if you square this Dirac equation you will get the Klein-Gordon equation. There is a good important point there, you get the Klein-Gordon equation because the Dirac matrices obey very specific rules and these rules these anti-commutation rules are what are so important to allow this squaring to give you the Klein-Gordon equation.

$$\begin{cases}\{\gamma^{i},\gamma^{j}\}=-2\delta^{ij}\\ \{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}\end{cases} \tag{31}$$

What's interesting now, these are typically called Dirac matrices however that's different, these are all then Dirac matrices (24), (25) and (26) but these are Dirac matrices in the Dirac representation (24) so I want to make sure the language is clear, these are Dirac matrices in the Dirac representation, these are Dirac matrices in the Majorana representation (25) and these are Dirac matrices in the Weyl representation (26). The Weyl representation by the way is sometimes called the Chiral representation and the Weyl representation is the one that's used by Peskin and Schroeder so we'll probably be leaning into this one a little bit more than the others, although I have although like I said we're not doing field theory we're just sort of doing the prerequisites so we'll spend some time comparing some of these I think instead of just going through the solution of the direct equation which we eventually will.

I want to compare with the Majorana representation and so I'm not sure which representation we'll use to solve it, it becomes kind of unimportant at that point it's just understanding the nature of the solution really doesn't depend that much on the representation. The Majorana representation can be a bit tricky though so in my opinion and mostly because we're unfamiliar with it. Like I said, my first textbook that I ever learned from use the Dirac representation, Peskin and Schroeder uses the Weyl representation.

This is now the Dirac equation (30) and now it's talking about a four component Spinor field called a Dirac Spinor and this Dirac Spinor object, each of these values has four components $[\psi_1,\psi_2,\psi_3,\psi_4]^T$ and each of these things for example $\psi_1\in\mathbb{C}$ is a complex number, it's ultimately a complex number so it's a four component object of four complex numbers applied to each point in space-time so every point in space-time because that's what a field is, it's a Dirac field so at every point in space-time it's given one of these things and the whole Dirac field is written usually a $\psi(x)$ and its evolution is, its equation of motion is going to be this guy right here, the Dirac equation of motion (30).

Now a really annoying confusion is there's four components, there's four components space-time has four components and so you've got things that have four components that are unrelated as far as you should be worried about now Spinors have four components because, I guess the way we want to think about it in this line of thinking is because the representations we're using for these γ matrices are four by four it's the smallest one that's available three by three doesn't work, two by two doesn't work, and scalars don't work four by four works, it's the smallest one so we're going with it. This forces the Wave function that we're seeking to have four components but notice that why are there four Dirac matrices? Why are there four of these guys? Well you go back up here (20), (21) and you realize, well we threw a coefficient in front of each of these partial derivatives and there's one of these for each direction and space-time, one for the time direction and three for the space direction so because space-time has four, should I say degrees of freedom? Four coordinates this equation (20) has four coordinates and therefore four operators energy and momentum, we end up with what we were looking to be numbers and we discovered that they're matrices so we end up with four matrices, one for each direction in space-time and so we give them an index to reflect that and we write down these Dirac matrices with an index μ , this index μ is in fact a space-time index and you can take it seriously as such, you can raise and lower it and you sum over it you sum over it with other space-time indexes, so you write:

$$\gamma^{\mu}\partial_{\mu}=\gamma^{0}\partial_{0}+\gamma^{1}\partial_{1}+\gamma^{2}\partial_{2}+\gamma^{3}\partial_{3} \tag{32}$$

You treat it like a space-time index, however the Dirac Spinor, these components $[\psi_1,\psi_2,\psi_3,\psi_4]^T$ they are not associated with space-time indices, these are a totally different index, it's coincidental that it has the number four, well from this point of view, it's coincidental that it has the number four, at some deeper level I'm not going to comment but the fact is that it has the number four because this four by four matrix pops out as the smallest matrix to satisfy our requirements but these are not space-time indices so you will never see anything like some vector in this case a contravariant vector $p^{\mu}\psi_{\mu}$, you'll never see that because you don't index the Wave function with a space-time coordinate in fact I'm going to erase that fast because I don't want anybody to think otherwise.

Usually when you see ψ you very rarely break it up into $[\psi_1,\psi_2,\psi_3,\psi_4]^T$ and talk about them independently, very very rarely and whenever you see ψ you immediately think of a four component object and you can never if you do see ψ with some sort of index here, that index can never be summed against a space-time index like that, you'll never see something like that because it's just a different index and this is a major problem in QED, a major problem for new students in QED is you'll see indices all over the place and they represent indices of totally different natures and you have to keep track what is the space-time index where are the space-time indexes in your problem and where are the internal indices that represent the components of the Spinors and they show up and the more complicated you get in the field theory the more different indices show up and it becomes quite a challenge to keep track of them but if you do keep track of them you never make a mistake.

There we have our Dirac equation (30) and the derivation is as I've expressed it if you actually go to some of Dirac's original work or some older treatments of this what you end up seeing is let's just start with a linear equation let's just invent a linear equation that one squared gives you the Klein-Gordon equation so you end up seeing these α matrices and β matrices and it's the same analysis you just square this thing and you're still looking for matrices that end up satisfying the exact same rules we just found but there is this thing where you talk about Dirac calling these the α matrix and the time one ends up being well then there's this β matrix:

$$\begin{aligned} i\partial_t\Phi &= \left[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m\right]\Phi \\ &= \left(\alpha_1 p_1+\alpha_2 p_2+\alpha_3 p_3+\beta m\right)\Phi \\ &= \left(-i\alpha_1\partial_1-i\alpha_2\partial_2-i\alpha_3\partial_3+\beta m\right)\Phi \end{aligned} \tag{33}$$

$$\partial_t^2\Phi=\left(-i\alpha_1\partial_1-i\alpha_2\partial_2-i\alpha_3\partial_3+\beta m\right)\left(-i\alpha_1\partial_1-i\alpha_2\partial_2-i\alpha_3\partial_3+\beta m\right)\Phi \tag{34}$$

Just be aware that there's this other sort of notation out there when you try to find your own version of this in whatever textbook you're favoring. That's it for the Dirac equations introduction and so now we've studied the Klein-Gordon equation and the Dirac equation so the next thing we're going to do is to look at Maxwell's equations which are also field equations and we're going to understand how Maxwell's field equations have to be massaged into a form that you're very familiar with, well you have to become very familiar with a different form of Maxwell equations to make any sense of field theory and so before we go solving the Dirac equation, the Klein-Gordon equation I'm going to revisit Maxwell's equations and we're going to talk about Maxwell's equations in reciprocal space and that will be our next lecture. I'll see you next time.