Geometric Algebra 10: Bivector-vector products

We will proceed with our study of Geometric algebra through the space-time algebra and we left off last time deep in this Section 3.3 on multi-vectors and so how this paper is actually organized is it talks about multi-vectors and it leans on by-vectors because for the space-time algebra bi-vectors and their product with vectors is something that's very important how we understand that product to work is critical, I've already made the case that we need to understand how all of the different blades can be space-time multiplied with all of the other blades so they start with bi-vectors and vectors which we've already studied and then you see later on, here's bi-vectors with vectors then they study bi-vectors products with other bi-vectors so they delay that for a while and you never see anything like bi-vectors with tri-vectors and all these other things and this is because they're creating a structure that's going to make all that a little bit simpler but is very particularized to the space-time algebra so the approach that I had earlier was a little bit more general in principle, I was talking about how this would work in multiple dimensions and or higher dimensions and here they're going to focus on keeping it simple for the purposes of space-time algebra and that's fine so we're going to mimic that.

The way they do this is they introduce bi-vectors then they start talking about reversion and inversion which are operations related to multi-vectors, however keep in mind that all multi-vector operations, there really is only two there's space-time product and there's vector addition you can add multi-vectors together and you can take their space-time product everything else is somehow derivative from those two things in particular derivative from the space-time product so when I write $a \cdot b$ or even $a \wedge b$, I'm talking about the symmetric and anti-symmetric parts of the space-time product of two multi-vectors and it's not sure which is symmetric and which is anti-symmetric that does change depending on the multi-vectors you're dealing with but these are derivative from the space-time product, likewise they're going to describe things they call reversion and inversion and those are operations that you can execute on a multi-vector but ultimately that all refers to space-time product in some way.

Then what do they do next? They're going to talk, it's worth our while to look at how now we are prepared, we understand enough that we can now look ahead and see how this paper is organized so let's spend a quick moment doing that. Reciprocal basis components and tensors, they're going to make contact with your standard tensor calculus or tensor analysis or tensor algebra, well the theory of tensors and components, we've actually discussed components a little bit so they're going to repeat that the reciprocal basis is an important idea that's going to end up looking a lot like the dual basis in our previous work our previous work on tensor spaces and their dual spaces but then after that they really dig into the pseudo-scalar and the whole value of the pseudo-scalars to execute Hodge duality and complex structure and we'll see a little hints of that in this lecture when we look at one of their tables that reveals all this stuff but the pseudo-scalar big function in a lot of the work in this paper is to execute Hodge duality and create a notion of complex structure in a couple different places.

Then they dig into bi-vectors again so the bi-vectors they introduce it here and then they really dig into this some so-called canonical form of a bi-vector so that'll be interesting to see when I use the word canonical what I mean is something that's forced upon us, we don't have to make any choices and they're going to introduce a form of the bi-vector that is very reasonable and we don't have to add anything to the theory to to create this particular form of the bi-vectors but this word form does not have anything to do with one forms, two forms and three forms, the dual one vectors, two vectors and three vectors from Exterior calculus, this is just the cut the canonical way of writing a bi-vector then they finally attacked by vector products with bi-vectors and then once they do that everything in the space-time algebra is under control because they're going to create a system where tri-vectors are actually vectors through this Hodge duality so you never need to worry about multiplying things with tri-vectors per se in the structure that they're offering up here, well we talked about it when we spoke in full generality of things, well if you have complex structure then you have complex conjugation so they'll talk about that.

Then they'll talk about relative frames and para-vectors, this is a really important formalism here's the Hodge duality formalism this is going to be a duality that's based on γ_0 which has a unique role in all of this stuff because it is the only time-like basis vector but it also really defines the observer's inertial reference frame so this whole notion of para-vectors, relative reversion and the space time split big deal and it's a form of duality so it's the second form of the duality, Hodge duality is one form of duality and this space-time split depends on a second duality that we'll talk about. Then there's a whole sector called the bi-vectors: commutator bracket and the Lorenz group and this is probably where I will pause to talk about the notion of rotations in the Geometric algebra, how do we rotate vectors how do we rotate, if we rotate the coordinate system, how do we rotate these vectors or in this case I think we'll do it actively, we'll be rotating the vectors and the bi-vectors and the tri-vectors inside our Geometric algebra and how we execute those rotations is very important and then Spinor representation in the Pauli and Dirac matrices. I guess this is where they're going to make the connection between the γ unit vectors and the γ matrices of Dirac, which will be very interesting.

With that review, let us continue just reading the paper. The last time we left off with this paragraph where they talked about the algebraic trace operation which is really collecting these scalar parts of multi-vector products:

$$\langle M N\rangle_0 = \langle N M\rangle_0 \tag{1}$$

This algebraic trace was, I think that's the last place we stopped and then they mentioned the Dirac matrix representation which we'll talk again in Section 3.8 then there's in the stream of the text a few tables appear and you're tempted to look at these tables like you're tempted to really look at table two and what you'll see is things that you've been introduced to and things that you haven't like we now know that their notation, this is a notation table and I love notation so I'm happy to look at it but you'll know some of these things we've seen right so now we know that all scalars will be Greek letters great we know all four vectors will be Roman lowercase letters both the Greek letters for scalars are lowercase too, bi-vectors will be bold uppercase Roman letters with straight structures, they're block Old Roman and they're the bi-vectors. Now pseudo-vectors, look how they write the pseudo vector, they're giving it two parts the first part is a lowercase Roman letter and then the second part is I and we know what I is, $I=\gamma_{0123}$ so we haven't learned this part yet, we haven't learned about how we're going to write pseudo-vectors as though they are vectors multiplied by the pseudo-scalar and ultimately so but we know pseudo-vectors are are base our grade three objects so these are all of the grade three parts of any multi-vector is the pseudo vector part e.g. $\langle M\rangle_3$ so with that we haven't learned yet but we have learned about the pseudo-scalars $v I$, a real part times this pseudo-scalar basis vector so we know about pseudo-scalars. Multi-vector, we know the symbol for multi-vector is just a Roman uppercase letter e.g. M then there's this notion of a complex scalar and a complex vector so we have not studied those two yet. The complex scalar though we've been foreshadowing into it, we already know about the real number and we know about the pseudo-scalars so obviously just by looking at this I can see that a complex scalar, e.g. $\zeta=\alpha+\beta I$, is the addition of those two, I already know that I does square to -1 so it certainly works like a complex, the complex imaginary number I but it's not, it's this quad-vector, it's this unit hyper volume so clearly we've got some learning to do there but it's already not that hard because I can almost figure out what that has to be two one-dimensional objects added together.

Likewise a complex vector, well I don't know yet how these pseudo-vectors work but I see I'm adding a vector and a pseudo-vector $z=v+wI$ I know it's a pseudo-vector because it's got the same form as this side entering the table the pseudo-vector entry so apparently there's some complex structure to make there but it's a complex vector structure so that's interesting I can't wait to figure that out and then they talk about the gradient and the tensor and the tensor adjoint which we have not seen yet at all so then the next section of this table, what else what here have we seen already? Well we've seen the orthonormal basis vectors we have not seen the reciprocal basis but it's pretty clear this definition is pretty straightforward $\gamma^{\mu}=\gamma_{\mu}^{-1}$, we are going to write it with an upper subscript, it's very reminiscent of covariant and contravariant basis vectors so that's cool but we have not been introduced to that yet so I'll leave that in a little green mark. Relative three vectors, don't know what that means because three vectors in our business is a pseudo-vector, in other words this is a language problem pseudo-vectors are all of these objects that are constructed out of a grade three thing, that's a pseudo-vector now that is a three-vector, we call this a three-vector when we have a construction like that, the wedge product of three-vectors together is a three-vector, however this notion of relative three-vectors something's up there because notice the notation goes back to our classic notation from vector algebra with a little arrow on top e.g. $\vec{E},\vec{B}$ so these relative three-vectors are not the same as $a\wedge b\wedge c$ so this is a language issue and they've already prepped us because they know that they're talking about when they talk about the tri-vectors they're talking about grade three objects and these three vectors are your classic three vectors but somehow embedded in the Clifford algebra so we'll learn about those later.

Then they have the orthonormal basis for three vectors so obviously if these are vectors they must have a basis so evidently they will introduce us a basis, relative three gradient relative time derivative we don't know about any of that stuff, scalar components well that we do know about, we've I've already mentioned and I think I've even screwed up $F^{\mu\nu}\gamma_{\mu}\wedge\gamma_{\nu}$, these are components and they're scalar components but what they're saying here is when we deal with scalar components, Greek letters mean it goes from zero to three, Roman letters it's just the spatial parts, one to three. Minkowski metric tensor, we got that the Euclidean metric tensor is also the Kronecker Delta and the anti-symmetric Levi-Civita symbols are here so this is just a notation thing, the point is even though we're in the middle of this paper it's okay to dive into these tables and see well what do I know and what do I not know and it's a good way to track how you're proceeding with understanding the paper, eventually you should understand everything on the paper if you want to be really comprehensive or you just select the parts you're interested in like we might stop after we understand the space-time algebra and not worry about the space-time calculus for example. We will worry about the space-time calculus because ultimately our goal is to understand Maxwell's equations but once we do enough space-time calculus to see Maxwell's equations, I'm not convinced that we'll go to the end of this paper together at least.

The next table (table 3), they are basically showing various physical quantities that are modeled by blades of different degree so charge, mass and proper time, that seems logical that they're modeled by scalars, what's modeled by a pseudo-scalar though, magnetic charge $q_m I$ and something they're calling phase φI so there are real things out there called pseudo-scalar, now in the physical world we don't know if there are magnetic charges but if there were, they'd have to be pseudo-scalars. Now vectors are anything that's a four-vector that you learn from classic Special relativity is still a vector in the Clifford algebra. The bi-vectors, electromagnetic fields $\boldsymbol{F}=\vec{E}+\vec{B}I$, well we understood the electromagnetic field to be a bi-vector even in our study of Maxwell's equations via Differential forms so that's not a big surprise, Angular momentum, Torque and Vorticity, not completely sure that those are going to fit super solidly into the electromagnetism formulas story that we're talking about but certainly classical Physics as long as it happens in the space time of Special relativity this Clifford algebra should be able to handle it and certainly all of these things are, they're part of space-time Physics so they should be in there and they get linked to bi-vectors. Pseudo-vector is the magnetic potential and magnetic pseudo-current and the Helicity pseudo-current, not sure what Helicity pseudo-current is but these other things they are pseudo vectors not surprisingly they involve magnetism because we know the magnetic field has always been a pseudo-vector so it's not a surprise that these guys are too.

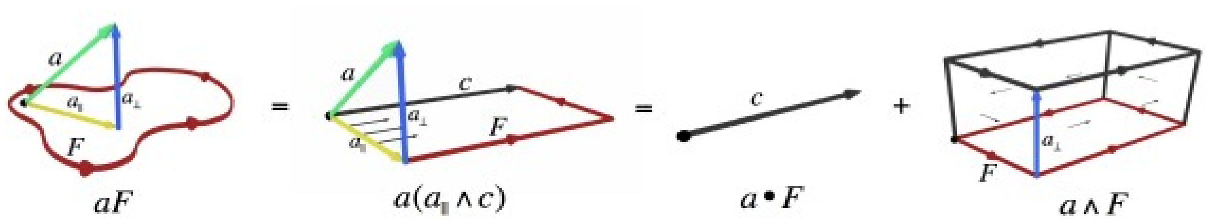

Figure 3: The product $a\mathbf{F}$ between a vector a (green) and bivector $\mathbf{F}$ (red) decomposes into a vector $a \cdot \mathbf{F}$ and a trivector $a \wedge \mathbf{F}$ part. A spacelike $\mathbf{F}$ is an oriented plane segment that may be deformed into a parallelogram $\mathbf{F} = a_{\parallel} \wedge c$ that shares a side with the part of the spacelike vector $a = a_{\parallel} + a_{\perp}$ that is parallel to $\mathbf{F}$ (yellow). The contraction extracts the orthogonal vector $c = a \cdot \mathbf{F}$ (black), while the inflation drags the bivector $\mathbf{F}$ along the perpendicular vector $a_{\perp}$ (blue) to produce an oriented trivector volume $a_{\perp} \wedge \mathbf{F}$.

This is just to show you that Physics does map on to this structure, the next figure (figure 3) that they show, this is a really fun figure because it really shows you how to take the space-time product of a vector with a bi-vector, we need to know how to do that and I think it's worth going through this table even though we haven't reached this part of the text just because hey here it is and it's in front of us and it's really glorious but the idea is that $\boldsymbol{F}$ is your bi-vector and you notice that on this side of the equal sign they don't give $\boldsymbol{F}$ any defined shape because it doesn't have a defined shape it only has an area and a circulation and it also has an orientation, by orientation I mean it's got components in the basis of bi-vectors so for example $\boldsymbol{F}$ has got to equal in this case I'll just say one example $F^{12}\gamma_1 \wedge \gamma_2$ and that's it, that's so $\gamma_1 \wedge \gamma_2$ is the orientation of the plane. Now it could have more complicated orientation or it could actually be right now it's depicted as a simple bi-vector, if it was a not simple bi-vector and I don't know if there's a word for that, there would be another little area segment with an orientation or essentially in an orthogonal space relative to $a\boldsymbol{F}$ and it would be floating out there separately. It turns out, two is the most you need, every bi-vector in the Clifford algebra can be made into the sum of two simple ones, if you write down $F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$, no matter how many $F^{\mu\nu}$ components are not zero ultimately you can rewrite this as the sum of just two bi-vectors.

Anyway, that aside this question is what is the space-time product? How do we understand the space-time product of a vector and a bi-vector and here's your vector a and here's your bi-vector $\boldsymbol{F}$ and looking at it from a purely geometric point of view without any reference to coordinates, meaning I'm not going to introduce any γ_μ into this analysis, I knew that the vector a certainly can be broken down into a part that is parallel to the bi-vector and a part that's perpendicular, the parallel part is called the *projection* and the perpendicular part is called the *rejection* and the way we want to understand this is that well look there is a there is some vector in the world c where if I just take the projection of a and I sweep it along c , the area that I sweep out will be the area of $\boldsymbol{F}$ whatever $\boldsymbol{F}$ is. I have to find this vector c such that c , well $a_{\|} \wedge c = \boldsymbol{F}$. If I can do that, well I can do that and when I do do that, that vector c is uniquely defined and you'll notice the way they've done this in the picture they say this (leftmost diagram) equals this (middle diagram) because this area has no definition, we can define the shape any way we want, the area is well defined but the shape is indeterminate so why don't we just establish that the shape is going to be a parallelogram and one side of the parallelogram is going to be the projection of a , that is part of this space-time product that's really easy to do.

Now I'm talking about the space-time product of a with the projection of a wedged with this new vector c that's defined such that the area of a parallel wedge c is equal to the area of $\boldsymbol{F}$ and then I say well that's based on product is going to have a vector part and a tri-vector part because it that's the way space-time products work with multi-vectors as they have parts that lower the grade and parts that raise the grade, sometimes they have parts that leave the grade alone but not always and in this case they don't. Whenever you multiply by a vector a times any multi-vector M where if this multi-vector, well let's say a multi-vector of grade seven, you're always going to have a part that has grade six and you're going to add to that a part that has grade eight right you're going to have two parts:

$$a M_7 = \langle a M_7 \rangle_6 + \langle a M_7 \rangle_8 \tag{2}$$

Whenever you have this a vector multiplying some blade, you can have a blade of one lower degree and blade of one greater degree so in our case we have a blade of grade one and a blade of grade three so this vector c , well that's the answer to what is the lower grade blade this is the contraction they call this the contraction because you're contracting the blade to a lower number $a \cdot \boldsymbol{F}$, they call this the inflation because you're inflating the blade $a \wedge \boldsymbol{F}$ but the inflation is really easy, you just take $\boldsymbol{F}$ in whatever shape it is and you just drag it along the rejection of a and you end up with this little three-dimensional volume so that's a really a nice way of looking at the picture. I don't think I need to read this caption except for one thing, they say "a space-like $\boldsymbol{F}$ is an oriented plane" and they talk about the "space-like vector" and I'm not quite sure why they do that, except there's nothing about this analysis that demands that anything here either the bi-vector or the vector itself be entirely composed of $\gamma_1, \gamma_2, \gamma_3$ and have no component in the time light direction γ_0 so space-like usually implies that we're dealing with just the space-like basis vectors.

I think the reason they're saying space-like is because they want to draw it in space so they're saying we're just going to look at the space parts so we can draw it because they don't know how to draw something in time although I gotta say in Special relativity we always write space-time diagrams like this (x, t) and we have no trouble talking about paths in space-time with time as one of the axes and nothing about this analysis will really change if you have a time-like component to the vector a or a time-like orientation to the bi-vector $\boldsymbol{F}$ but they just want to emphasize in this case space-like but that's not a requirement for anything, it's not a requirement for any of this, that anything be space-like, this is the general way of interpreting this, interpreting the space-time product of a vector a and a bi-vector $\boldsymbol{F}$, I guess what they're saying is it's hard to do this spatial interpretation if you're dragging something through time because what's a Parallelepiped in time?

I'm used to picturing those things from General relativity, I have no trouble picturing three volumes where one of the sides of the volume extends in time, anyway so that's this beautiful picture I really love this picture I think it's a great way of understanding the space-time product of a vector and a bi-vector, well it's not a great way, it's the only way but it's very well drawn and well explained so to continue, "all proper physical quantities in space-time is either correspond to pure k -blades or multi-vectors that combine k-blades of different grades". That goes back to that chart we did before where all the physical quantities were attached to different blades that would model them. "These are the only frame-independent objects permitted by the constraints of Special relativity. Hence, the expansion of a general multi-vector" into its blade parts, "indicates the full mathematical arena in which relativistic physics must occur. The algebraic structure allows us to manipulate each of these proper objects with equal ease and finesse".

That's the promise of Geometric algebra, in space-time algebra it promises that all of Special relativity and all Physics done in the arena of Special relativity is handled by the space-time product of two multi-vectors and addition between multi-vectors, that's what it's saying. There's also space-time calculus that needs to be introduced that was those gradient things but ultimately those operations with Calculus are still firmly attached to the space-time algebra, we'll get there eventually. "The added complication of multi-vectors $M \in C_{13}$ over the original four-vectors of $v \in M_{1,3}$ ", remember we all learned Special relativity right in here, everything we did was inside $M_{1,3}$, now this paper is saying not good enough, you should have been doing it here C_{13} and there's some "added complications" and that "may seem excessive and it may be intimidating at first glance, however, as we have been suggesting, the added structure contained in the full multi-vector is in fact necessary for a proper description of space-time quantities".

What they're getting at is that you can sit around and say $\vec{B}$, the magnetic field that's a space-time quantity it's a vector but what it's a pseudo vector so don't forget, it's a pseudo vector well if you understand how pseudo vectors are constructed now in the space-time algebra you don't even really have to remember that it's a pseudo vector you just have to understand that $\vec{B}$ is attached to say tri-vectors although this paper will formulate it differently, I guess what I should I should back this up a little bit. Pseudo-vectors have a home in the Clifford algebra that is not ad hoc, everything has a place and there's a place for everything that matters so "indeed as we will see in more detail in Sections 3.4 and 3.6, each distinct blade of a general multi-vector is in fact a familiar type of object that appears in the standard treatments of Relativistic Physics. We summarize several examples in table 3 for reference". We've already looked at table 3, "the multi-vector construction unifies all these quantities into a comprehensive whole in a principled way". Well the notion of a principled way is a little bit vague but basically I believe what they're after here is look we built C_{13} in a purely mathematical way from the ground up using nothing more than the logic of basically of Abstract algebra really and we interjected it the minimum amount of Physics that we know about namely the Minkowski metric and then from the Abstract algebra plus the Minkowski metric, those are the principles and yet this thing actually does a great job modeling reality.

It's not like we literally built it and kludged it together to model reality like the Epicycles of Ptolemy in astronomy where everything went around the Earth and they had to really, they make a great model of things going around the Earth but that great model was just really kludged together, this is not kludged together, this is actually built from basic vector space principles and the Minkowski metric and voila, it all falls out so it's a comprehensive whole because it covers everything but it's not just a bunch of stuff thrown into a box it's well organized from first principles that don't obviously, shouldn't necessarily obviously, tell us about the real world but it does. "Moreover, the Clifford product provides them with a wealth of additional structure which we will exploit to make manipulations of multi-vectors simple in practice" and that's the I think what they're referring to there is the beautiful simplifications that they're going to do to keep us away from having to think perpetually in terms of this mess and getting much closer to Figure 2 and we're getting even simpler, we've already gotten simpler because $I = \gamma_{0123}$ and these tri-vectors $\gamma_{123}, \gamma_{230}, \gamma_{310}, \gamma_{120}$ we're going to see are going to become things like $\gamma_0 I$ and things of that of that ilk, I think γ_{123} is probably $\gamma_0 I$. You can already see it $\gamma_0 I = \overbrace{\gamma_0 \gamma_0}^{1} \gamma_1 \gamma_2 \gamma_3$ and you're left with $\gamma_0 I = \gamma_1 \gamma_2 \gamma_3$ which is γ_{123} so we're even going to get rid of $\gamma_1 \gamma_2 \gamma_3$ and we're going to probably end up writing just $\gamma_0 I$ for that but stay tuned but the beauty is that they're saying is that space-time algebra has some real efficiencies in it where we can just grind this down to a beautiful simple notation and I'm looking forward to seeing it and just sharing it with you.

We finished that so now they do their Section on bi-vectors, let me see if we have enough time to do some more, yeah sure so we've seen all this so this should go quite smoothly for us, this is our second approach, we've already worked hard we know how to take bi-vector products with vectors I've already told you that this formula:

$$a(b\wedge c)=(a\cdot b)c-(a\cdot c)b+a\wedge b\wedge c \tag{3}$$

That's worth memorizing and this is a bi-vector product with a vector so we already know, straight up grade two parts always easy, the blue part is the grade one part, the green part is the grade three part so let's see how they presented here, it should look a lot like what we did because I learned it from this stuff and I'm repeating back to you what I've already read in advance and then we're going to read it together and that's the way we're going to go through this whole paper. “We will see in the following sections that several dualities to space-time algebra allow a complete understanding in terms of scalars, vectors and bi-vectors”. That's what I was getting to a moment ago, the tri-vectors can all be written as vectors times the single pseudo-scalar basis vector so we lose tri-vectors as their own entity eventually but we haven't done it yet there are several dualities and we have to study those still. I'm only aware of two dualities so I don't know about several maybe, I've forgotten some, “out of these three fundamental objects bi-vectors are the least familiar, so we'll make an effort to clarify their properties as we progress through this introduction”. Well by now I'm hoping that you all are not unfamiliar with bi-vectors bi-vectors are interpreted as an oriented area, simply that an area with the circulation and bi-vectors are written as wedge products $\boldsymbol{F}=a\wedge b$, the area of this wedge product is derived from the components of a and the components of b in a given basis and it's a parallelogram a swept along the direction b so I have a vector b and a vector a and I sweep a along b and I get a parallelogram and that area is the area of the bi-vector and the circulation from a to b is the circulation and it just sits there in space as appropriate, sitting in space by being oriented with b here and a here.

Those two vectors are sitting in space, we have no trouble with that so orienting the plane is pretty easy because it's the plane spanned by the two vectors, totally simple so it is least familiar but we've been talking about it so much that it's not least familiar and if you came here from Exterior calculus actually if you came here from our study of Differential forms there is a problem because Differential forms are not interpreted this way, Differential forms $\boldsymbol{d}x\wedge\boldsymbol{d}y$, that is not interpreted as a little element of plane in fact it's interpreted as something that would be cut by little elements of plane, it's understood as essentially a honeycomb in space and $\boldsymbol{d}x$ is not interpreted as a little pointy thing in space even though $\boldsymbol{d}x$ lives in a vector space called the co-vector space, you cannot interpret it as a pointy thing and in fact what you have to interpret $\boldsymbol{d}x$ as is as a series of little planes that have a density but no extent meaning the distance these planes have no extent but they do have a density in this direction and a vector as I've drawn here can cut a bunch of planes and some vectors cut more planes than others but that's how a single form works and two forms would be a bunch of planes in one direction and a bunch of planes in another direction forming a honeycomb and a honeycomb can be cut by a little piece of plane how many honeycomb segments does it cover. Differential forms and exterior products of vectors are interpreted very differently so if you're coming from that place then three two vectors and three vectors are in fact a little bit unfamiliar because you don't use them as much in other studies. Anyway, moving on “we now consider how bi-vectors and vectors relate to one another in more detail, defining two useful operations (*contraction* and *inflation*) in the process. It is worth emphasizing that all operations in space-time algebra are derived from the fundamental vector product and sum, so are introduced for calculation convenience, conceptual clarity, and for contact with existing formalism.

That's what we were getting at with, you have a vector bi-vector space-time product, this is the fundamental product, this is considered the contraction and inflation piece and they're introduced just to help us understand this product but but they have to be understood, you have to be able to calculate this and calculate this to build this but it's a way of getting us to ultimately building the space-time product That's what they're saying there but “a product between a vector a and a bi-vector $\boldsymbol{F}$ can be split into two terms (vector and tri-vector) in an analogous way to the product of vectors”.

$$a\boldsymbol{F}=a\cdot\boldsymbol{F}+a\wedge\boldsymbol{F} \tag{4}$$

Because this looks like our vector expression but $a\cdot\boldsymbol{F}$ is not a dot product, this is not a Minkowski contraction between two vectors such as it would be if this was $ab=a\cdot b+a\wedge b$, $a\cdot b$ would be a Minkowski contraction, in (4) it's not, it's something that is a little more artful that they define but we've already gone through it, I've already shown you how to do that calculation but then they point out that in this case the contraction part is the anti-symmetric piece and the inflation part is the symmetric piece and I showed you that several lessons ago.

$$\begin{aligned} a\cdot\boldsymbol{F}&\equiv\langle a\boldsymbol{F}\rangle_1=½(a\boldsymbol{F}-\boldsymbol{F}a)=-\boldsymbol{F}\cdot a, \\ a\wedge\boldsymbol{F}&\equiv\langle a\boldsymbol{F}\rangle_3=½(a\boldsymbol{F}+\boldsymbol{F}a)=\boldsymbol{F}\wedge a. \end{aligned} \tag{5}$$

“Notably, the grade-lowering dot product, or *contraction*, is anti-symmetric while the grade raising wedge product, or *inflation*, is symmetric, which is opposite to the vector product case”. Exactly, now $\boldsymbol{F}\wedge a$ is a tri-vector which is totally anti-symmetric but it is the symmetric part of the space-time product and I showed you why a couple lessons ago that actually makes sense because basically if you're going to flip $a\boldsymbol{F}$, if you're going to flip that to $\boldsymbol{F}a$, it takes two flips to do it and if $\boldsymbol{F}$ is $b\wedge c$ then $\boldsymbol{F}\wedge a=a\wedge b\wedge c$ and to flip $b\wedge c$ to the front takes two flips and you lose your sign so you need total anti-symmetry in $\boldsymbol{F}\wedge a$ to capture the symmetry of the vector bi-vector product.

“The geometric meaning of these products can be ascertained for space-like $\boldsymbol{F}$ by decomposing the vector $a=a_{\perp}+a_{\parallel}$ ” (rejection and a projection) “into parts perpendicular $a_{\perp}$ and parallel $a_{\parallel}$ to the plane of $\boldsymbol{F}$. After factoring $\boldsymbol{F}$ into constituent orthogonal vectors (i.e., $b\cdot c=0$),” into any constituent orthogonal vectors “ $\boldsymbol{F}=b\wedge c=bc$ ”, we can write out this expression:

$$\begin{aligned} a\boldsymbol{F}=abc=½a_{\parallel}(bc-cb)+a_{\perp}\wedge b\wedge c, \\ \boldsymbol{F}a=bca=½(bc-cb)a_{\parallel}+a_{\perp}\wedge b\wedge c, \end{aligned} \tag{6}$$

This is a nice algebraic expression of that picture we saw, notice the first thing they do “after factoring $\boldsymbol{F}$ into constituent orthogonal vectors”, you can always do this, $\boldsymbol{F}$ is a bi-vector it equals some vector $b\wedge c$ but it could equal literally any two vectors $b\wedge c$ as long as they give you the same area so let's just assume that if all we know is that we're dealing with a bi-vector, let's just assume that b and c are orthogonal and that $b\cdot c=0$, look how nice that is, now $\boldsymbol{F}=b\wedge c=bc$ but $b\cdot c=0$ and let's see how we use that. Well if $b\cdot c=0$ then, well what is the space-time product of b and c , that's $bc=b\cdot c+b\wedge c$ but as I just said $b\cdot c=0$ because we're constructing $\boldsymbol{F}$ out of orthogonal pieces so everywhere I see $b\wedge c$ or everywhere I see $\boldsymbol{F}$ I can replace it by the space-time product of b and c so $a\boldsymbol{F}$ now becomes

abc and $\boldsymbol{F}a$ becomes bca , really beautiful stuff so now I break up a into the projection and rejection so $a=a_{\|}+a_{\perp}$, a projection plus a rejection, we do this a lot in Electromagnetism, you always do the parallel electric field and the perpendicular electric field and they transform differently relativistically but you can always do this, break this up and so everything being linear we are going to say is that this is going to actually, looking at (6) they've left out quite a few steps here because they're not very difficult steps but I think it's worth filling in the gaps as our last part of today's lecture so let's look at this formula very closely:

$$
\begin{aligned}
a\boldsymbol{F}&=abc=\left(a_{\|}+a_{\perp}\right)bc=a_{\|}bc+a_{\perp}bc\\
&=a_{\|}\left[\tfrac{1}{2}(bc-cb)+\cancel{\tfrac{1}{2}(bc+cb)}\right]+a_{\perp}\left[\tfrac{1}{2}(bc-cb)+\cancel{\tfrac{1}{2}(bc+cb)}\right]\\
&=a_{\|}\left[\tfrac{1}{2}(bc-cb)\right]+a_{\perp}\left[\tfrac{1}{2}(bc-cb)\right]\\
&=a_{\|}\left[\tfrac{1}{2}(bc-cb)\right]+\tfrac{1}{2}\left[a_{\perp}(b\wedge c)-a_{\perp}(c\wedge b)\right]\\
&=a_{\|}\left[\tfrac{1}{2}(bc-cb)\right]+\tfrac{1}{2}\left[(a_{\perp}\cdot b)c-(a_{\perp}\cdot c)b-(a_{\perp}\cdot c)b+(a_{\perp}\cdot b)c\right]\\
&\quad+\tfrac{1}{2}\left[a_{\perp}\wedge b\wedge c-a_{\perp}\wedge c\wedge b\right]
\end{aligned}
\tag{7}
$$

It's pretty easy each step. The first step is a here is replaced by $a_{\|}$ plus $a_{\perp}$ guaranteed easy step. bc we've already understood how $\boldsymbol{F}$ is replaced by bc and that's goes right here (1st line in (7)) so that's all easy so I just blow that up by linearity so I'm going to do this totally step by step, we blow this up by linearity. $a_{\|}$ plus $a_{\perp}$ are just vectors, two vectors that add together to form a so then I take the space-time product of bc and I'm going to bust it out into its anti-symmetric and symmetric parts. b and c are vectors and they're orthogonal remember we've decided that we're making $\boldsymbol{F}$ out of two orthogonal vectors so I can add that Minkowski contraction between these two vectors is zero.

This step is purely formal in the sense that it's basically adding and subtracting zero to bc and getting this symmetric and anti-symmetric split and we do that for this bc on the perpendicular side and the bc on the parallel side so that and that expression are the same. We know that $b\cdot c=0$ but we also know that $b\cdot c=0=\tfrac{1}{2}(bc+cb)$, it's the symmetric part of the space-time product of the vectors b and c because that's that was part of our assumption in this process so now this reduces to a parallel times the anti-symmetric part of the space-time product of b and c and a perpendicular times the anti-symmetric part so the final form that they leave here is you can see there's the anti-symmetric part multiplied by $a_{\|}$ (1st line of (6)) so we've actually got this piece down we're done with that piece then this piece, it's not that hard to understand, we just realized that b and c being orthogonal, I know that the space-time product of b and c is going to equal $b\wedge c$ because $b\cdot c$, which would have been the scalar part, well I know that is zero by assumption so I've replaced bc with $b\wedge c$ and likewise cb gets replaced with $c\wedge b$ and that's in this perpendicular part but now I have the space-time product of a vector with a bi-vector and I know how that works because I've taught you how that works so this is going to be the 5th line of (7).

Now these two tri-vector parts will add together and double because this part here is about to double because you're going to flip c and b and then you're going to change that sign to a plus so you're going to get a two out front so what you're going to get left with is $+a_{\perp}\wedge b\wedge c$ with a two canceled, The 2nd term is all zero because $a_{\perp}$ is the rejection of the bc plane, by definition it has no projection in the b direction, if it did the it wouldn't be the rejection from the plane so likewise it has no projection on the c direction either so every one of these dot products is zero so this entire 2nd term goes away and what you're left with is this, the parallel part of a times the anti-symmetric piece of the space-time product of bc and the perpendicular part of a times the wedge product of b and c which is what they came up with in the 1st line of (6) so with that in place you can see that if you switch this 2nd line of (6) nothing different happens, you just end up with the stuff on the right hand side.

$$a\boldsymbol{F}=½\,a_{\|}[bc-cb]+a_{\perp}\wedge b\wedge c \tag{8}$$

Now notice that this is exactly a description of what those boxes were all about, this is the tri-vector part (2nd term of (8)) and this here is a vector (1st term of (8)), maybe it's not so obvious that this part is a vector because this is the anti-symmetric part of the product of two vectors so being anti-symmetric this is going to be $a_{\|}(b\wedge c)$, this will be a parallel space-time product $b\wedge c$ but that's going to break up into $a_{\|}\wedge b\wedge c$ but $a_{\|}\wedge b\wedge c$ has got to be zero because $a_{\|}$ is in the plane of b and c and therefore it's not linearly independent and you can't make a volume out of it, it's a flat plane so that whole thing is going to be zero so all that's left is $(a_{\|}\cdot b)c-(a_{\|}\cdot c)b$, which is going to equal this vector so that's a vector so this guy here (1st term of (8)) is a vector and it reflects that picture we saw earlier.

It was worth the exercise of digging into the mechanics of how this calculation was made and so their approach is, well they did this for $a\boldsymbol{F}$ and they came up with the 1st line of (6) and then they switched it and they did it for $\boldsymbol{F}a$ and they came up with the 2nd line of (6), then they just said well, we're going to take the $a\cdot\boldsymbol{F}$ to be $a_{\|}\cdot\boldsymbol{F}$ and equaling this expression right here (1st line of (9)) which we already know, we just used it and $a\wedge\boldsymbol{F}$ is going to be $a_{\perp}\wedge\boldsymbol{F}$ which is this expression (2nd line of (9)):

$$\begin{aligned} a\cdot\boldsymbol{F}&=a_{\|}\cdot\boldsymbol{F}=(a_{\|}\cdot b)c-(a_{\|}\cdot c)b\,,\\ a\wedge\boldsymbol{F}&=a_{\perp}\wedge\boldsymbol{F}=a_{\perp}\wedge b\wedge c\,. \end{aligned} \tag{9}$$

When you see the dot product of a with a bi-vector basically that equals the projection part of the vector dotted into the bi-vector and it equals this expression that we've learned long ago and if you see a wedged into a bi-vector you can forget the projection part and only look at the rejection part and that forms this beautiful little tri-vector and this is just another demonstration of how all of this works.

What do they say? They say “for space-like a,b,c , these expressions have intuitive geometric interpretations. The contraction $a\cdot\boldsymbol{F}$ produces a vector in the plane of $\boldsymbol{F}$ that is perpendicular to $a_{\|}$. Indeed, choosing $b=a_{\|}$ so $\boldsymbol{F}=a_{\|}\wedge c$ ”, which is that c vector we were talking about, “yields $a\cdot\boldsymbol{F}=c$ ” That's all from this picture we studied before, $a\cdot\boldsymbol{F}=c$ and here now we choose $\boldsymbol{F}=a_{\|}\wedge c$ and this vector c is one of the pieces of our space-time product of a and $\boldsymbol{F}$ and the other piece of course is $\boldsymbol{F}$ in any shape just drag through the rejection part of a and that's the last thing they say here, “the wedge product $a\wedge\boldsymbol{F}$ produces a tri-vector with a magnitude equal to the volume of the Parallelepiped constructed by dragging the plane segment $\boldsymbol{F}$ along a perpendicular vector $a_{\perp}$. These constructions are all Illustrated in Figure 3. We again got a little bit ahead of ourselves there, doesn't matter whether you learn this from the figure or learn this from the text, they're both very comprehensive.

That takes us through their Section on multiplying vectors and bi-vectors and you should be good at that now, that's a thing we have learned. Notice they didn't use any basis vectors here so this is all a proper demonstration but when you do a proper demonstration you are allowed to break it up into this parallel and perpendicular part but that is always relative to some arbitrary bi-vector $\boldsymbol{F}$ so you don't even have to be too specific if you gave $\boldsymbol{F}$ components then $a_{\|}$ and $a_{\perp}$ you'd have to figure out the parallel and perpendicular components of a so here it's actually simpler to avoid all that so the next Section we're going to introduce some operations called reversion and inversion and we'll just proceed down all of this paper, where are we now? We've done 3.3.1 so now we are at 3.3.2 so I'll see you next time.