Maxwell's Equations Via Differential Forms 5 (Final)

Welcome back, in this lesson we have laid the groundwork already to finally finish our complete description of Maxwell's equations via differential forms. We understand all the differential forms when we understand all the relevant operations and we understand how those operations tie to the curl and the divergence which appear in the familiar form of Maxwell's equations in their standard vector analysis system have come down to us so now we're ready to actually finish this up, this might be a little bit of a long lecture tying it all together because I just want this to be the last lecture on this subject so let's begin and see if we can get through the entirety of the material.

Where we left it? We had come to the understanding of how to calculate the divergence of a vector using a method of differential forms first of all we end up treating the vector as a one form so we have to convert your standard vector to a one form and then we understood that if you have a one form in three dimensional space when you take the Hodge dual of that one form you end up with a two form and you take the exterior derivative of that two form and you get a three form which is one dimensional and then you take the Hodge dual of that three form and you get a zero form which is also one dimensional and so this divergence of a vector is translates to the Hodge dual of the exterior derivative of the Hodge dual of the equivalent one form and so we can replace the left hand side of this Maxwell's equation with that term. the right side has a source on it so we're just going to leave that alone for now.

$$\nabla \cdot \boldsymbol{F} = * \mathrm{d} * \boldsymbol{F}^{\flat} \tag{1}$$

Likewise we came to understand that the curl of a vector can be converted into the language of forms by turning the vector again into a one form that the electric field $\boldsymbol{E}$ is a one form and by taking the exterior derivative of that one form and then the Hodge dual of that one form we get the equivalent of the curl it works out mathematically to have the same information.

$$\nabla \times \boldsymbol{F} = \left(* \left(\mathrm{d} \boldsymbol{F}^{\flat} \right) \right)^{\#} \tag{2}$$

If we operate in an environment where everything is static and I'm not going to struggle with the difference between static and stationary here, I'm just going to just say magnetic field is not a function of time so $\partial \boldsymbol{B} / \partial t = 0$. Then we have this Maxwell's equation:

$$* \mathrm{d} \boldsymbol{E}^{\flat} = 0 \tag{3}$$

That is conditional on $\partial \boldsymbol{B} / \partial t = 0$. Then what we learned is that all right so we're good with the electric field $\boldsymbol{E}$ being transformed into a one form and we understand that because the electric field has been classically called a vector field in particular a polar vector field but we understand that the magnetic field $\boldsymbol{B}$ is an axial vector field a pseudo-vector field so we had to repeat the analysis for the pseudo-vector field and we understood that that has to be modeled by a two form so where I see $\boldsymbol{E}$ I see a one form but where I see $\boldsymbol{B}$, I see a two form and we now realized when we marched through this we realized that the curl of a two form has the same structure as the divergence of a one form:

$$* \mathrm{d} * \underline{\underline{B}} = 0 \tag{4}$$

In other words the curl of a two form is the Hodge dual of the exterior derivative of the Hodge dual of a two form and again in the context where there is no sources and where there is no time dependence on the electric field $\partial \boldsymbol{E}/\partial t=0$, we can write this Maxwell's equation in this tight form using differential forms (4) the have the same structure and then the famous divergence of the magnetic field equaling zero we understand that the magnetic field converted to a two form, the divergence of the two form is, it gas the same structure as the curl of a one form and so we get this famous equation:

$$* \,\mathrm{d}\,\underline{\underline{B}} \tag{5}$$

I will say that these last two in particular we can take the Hodge dual of both sides and we end up with something a little more plain:

$$\mathrm{d}\,\underline{\underline{B}}=0 \quad , \quad \mathrm{d}\,\boldsymbol{E}^{\flat}=0 \tag{6}$$

It's a little simpler, not that it matters much but it saves a step if you were to calculate it that way so now the problem we're going to deal with is this is very there's all these caveats there's this caveat of no sources although I left it in here but the big caveat is that the time dependence was ignored so what we really need to understand is that that when we are doing this work. $\boldsymbol{E}$ which is a one form is a function of x^0, x^1, x^2, x^3 where our x^0 is our time and I call it time because in the standard the standard notation the standard I don't want to call it notation because it's more than notation, the standard mathematical architecture used for electromagnetism usually has time labeled as t whereas the standard architecture used for differential forms we we use this x^0, x^1, x^2, x^3 where x^0 is the time variable and likewise the same is true of the magnetic field sometimes we can just write it as $B(x^{\mu})$ something like that, that's another way of it's a shorthand this is shorthand for each other sometimes it would just be $B(x)$.

You do have to remember that we're now talking about four variables and that's what we're going to have to do to make all this work, we're going to have to go from three-dimensional space which is so standard and we have been using three-dimensional space up to up to this point but we now have to go to four dimensional space because we have to add this time variable so we now have four dimensions and furthermore our metric is no longer going to be Euclidean, we're now going to have to use a Lorentz metric and not a big problem but we have to make sure we keep track of all that so that's what we're going to do now we're going to tie everything together into a four-dimensional structure so we can include time and then we're going to determine how to create Maxwell's equations that way. Let's begin so our first process will be with to deal with these two equations:

$$\begin{gathered}\nabla \cdot \boldsymbol{B}=0 \\ \nabla \times \boldsymbol{E}+\frac{\partial \boldsymbol{B}}{\partial t}=0\end{gathered} \tag{7}$$

These are the source free equations that's what they're called as opposed to the source full equations the one that contain sources, current $\boldsymbol{J}$ and the charge density ρ are the sources. The source free equations are the ones that automatically have nothing on the right hand side so we are going to work with those equations first so our first step is exactly the same step we took previously, we are going to begin by modeling the magnetic field as a two form so when we see $\boldsymbol{B}$, I'm not going to put two lines under it like I did in the previous lesson just to emphasize that it was a two form or one line to emphasize it was a one form, now we just know right we just know that the magnetic field $\boldsymbol{B}$ is a two form.

$$\underline{\underline{B}} = B_{23}\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_{31}\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_{12}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \tag{8}$$

Here is the two form structure, it's the same two form structure as two dimensions, that's important because the member the magnetic field we still understand as a vector in the world so it has to be a an object that has three dimensions to it. Now we do have access to a $\mathrm{d}x^0$ but we're still modeling the magnetic field $\boldsymbol{B}$ exactly the same way, likewise the electric field $\boldsymbol{E}$. The electric field we're going to model as one form just like we did before so this step is actually the same even though we now have access to a $\mathrm{d}x^0$ term even though we have access to this term we're not adding it into the model of $\boldsymbol{B}$ and $\boldsymbol{E}$ because they are still three-dimensional objects.

$$\underline{E} = E_1\, \mathrm{d}x^1 + E_2\, \mathrm{d}x^2 + E_3\, \mathrm{d}x^3 \tag{9}$$

What we will do is, we'll take these three-dimensional objects or we'll take these two objects we've created and we're going to bind them together we're actually going to mix them into a single object that we're going to call F and F is going to be the electromagnetic field. We can call it the electromagnetic field two form if we like because it is going to be a two form, you can see that on the left hand side is F and it's defined through $\underline{\underline{B}}$ which is a two form so if you're going to add something to a two form it better be a two form well this is a one form wedged with $\mathrm{d}x^0$:

$$F = E \wedge \mathrm{d}x^0 + B \tag{10}$$

You can add two forms together but notice now we're using all four of the different dimensions the time dimension gets attached right here to the electric field so they're now the electromagnetic field really is a bunch of two forms but our electric field has somehow been augmented in time and that's an interesting thing it's the magnetic field is living entirely in space but the electric field part of the electromagnetic field has this time element attached to it and that is what makes this so interesting so with that in mind we will start operating on this object this integrated object contains the electric and magnetic pieces of the electromagnetic field all bound together in one mathematical object and our ability to turn Maxwell's equations into differential forms structure is going to be completely operating on this object F and I'm just going to demonstrate how that works.

$$\begin{aligned} F = {} & E_1\, \mathrm{d}x^1 \wedge \mathrm{d}x^0 + E_2\, \mathrm{d}x^2 \wedge \mathrm{d}x^0 + E_3\, \mathrm{d}x^3 \wedge \mathrm{d}x^0 \\ & + B_{23}\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_{31}\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_{12}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \end{aligned} \tag{11}$$

Our those two Maxwell's equations (7) are going to be the exterior derivative of this electromagnetic field to form the exterior derivative of that equaling zero is going to be the equivalent of these two of the four Maxwell equations so that's the end here so the way we're going to do this is just through straight up demonstration we're just going to demonstrate that that's the fact. In this demonstration we're going to calculate the exterior derivative of the electromagnetic field:

$$\mathrm{d}F = \mathrm{d}\left[E \wedge \mathrm{d}x^0 + B\right] = \mathrm{d}\left[E \wedge \mathrm{d}x^0\right] + \mathrm{d}B \tag{12}$$

That the exterior derivative of F will distribute right so I replace F with its definition and then I show that I can take the exterior derivative of the first piece and then the exterior derivative of the second piece. Now we're fine because, this is a three form and this is a three form so then this is what we want to calculate now as part of this calculation what so a lot of books do is they separate the exterior derivative into a spatial part and a time part and I'm not too big a fan of that because you can just calculate this you can just use the rule of exterior derivatives and calculate this straight out and that rule remember was if your form α is this sum over this multi-index:

$$\alpha=\sum_I \alpha_I \, \mathrm{d}x^I \rightarrow \mathrm{d}\alpha=\sum_I \partial_\mu \alpha_I \, \mathrm{d}x^\mu \wedge \mathrm{d}x^I \tag{13}$$

Remember α is a function of space time, it's a function of time and x, y, z so the exterior derivative of α is the sum over these indices but then there's this internal sum also and this internal sum goes you know goes over the index $0, 1, 2, 3$, the zero part, that's the time part and so you're always wedging whatever piece of the sum you're doing you're wedging that into this multi-index so this exterior sum is the sum over these multi indices and this interior sum is the sum over the space time indices so this allows you to break this down right if I blow up this interior sum at this this contraction, if I blow that up I get these four terms, there's this time term where we have the time derivative of the coefficient and then we wedge the time one form with the multi-index so I'm still summing over the multi-indices.

$$\begin{aligned} \mathrm{d}\alpha=\sum_I \partial_0 \alpha_I \, \mathrm{d}x^0 \wedge \mathrm{d}x^I + \partial_1 \alpha_I \, \mathrm{d}x^1 \wedge \mathrm{d}x^I \\ + \partial_2 \alpha_I \, \mathrm{d}x^2 \wedge \mathrm{d}x^I + \partial_3 \alpha_I \, \mathrm{d}x^3 \wedge \mathrm{d}x^I \end{aligned} \tag{14}$$

Now I've broken up that inner sum so this part here is the space part of the exterior derivative (last three terms) and this part here is the time part of the exterior derivative (first term). It's still the same this is still literally the definition of the exterior derivative. I could break this up into the timeline piece and the space-like piece.

$$\mathrm{d}\alpha=\overbrace{\sum_I \partial_0 \alpha_I \, \mathrm{d}x^0 \wedge \mathrm{d}x^I}^{\text{Timelike}} + \overbrace{\sum_I \partial_i \alpha_I \, \mathrm{d}x^i \wedge \mathrm{d}x^I}^{\text{Spacelike}} \tag{15}$$

I don't know if I like the word that's my language this adding the like to it is probably wrong because that implies there's some causality issues here, I'm just literally breaking the sum into the time chunk and the space junk and then I can say the exterior derivative of any form is simply the time chunk exterior derivative plus the space chunk exterior derivative you have to have them both to get the full exterior derivative so the thing that's important to see here is that the time chunk exterior derivative will always wedge a $\mathrm{d}x^0$ into the multi-index and the space chunk will always wedge some space piece into the multi-index and that simplifies a little bit of our thinking but it's not too much really because now if I take the exterior derivative, I break into its space piece and it's time piece.

$$\mathrm{d}F=\mathrm{d}_s F+\mathrm{d}_t F=\mathrm{d}_s\left[E \wedge \mathrm{d}x^0\right]+\mathrm{d}_t\left[E \wedge \mathrm{d}x^0\right]+\mathrm{d}_s[B]+\mathrm{d}_t[B] \tag{16}$$

Likewise once I distribute this exterior derivative over the parts of the electromagnetic field tensor I find myself taking the space exterior derivative of $\boldsymbol{E}$ wedge $\mathrm{d}x^0$ and the time exterior derivative $\boldsymbol{E}$ of $\mathrm{d}x^0$ and then the space exterior derivative of $\boldsymbol{E}$ and the time exterior derivative of $\boldsymbol{E}$ and I get a little simplification here because I know that the space exterior derivative of something is always wedged with $\mathrm{d}x^0$ but this already has a $\mathrm{d}x^0$ in it so I know that's going to be zero. I know $\mathrm{d}_t[E\wedge \mathrm{d}x^0]=0$ so this is going to go away but that's all I really know because the space part will not necessarily go away there is one space piece in $\boldsymbol{E}$ because remember $\boldsymbol{E}$ is defined straight up as a one form so it has space pieces only but the space exterior derivative has all three of the spatial one forms in it so some will survive and some won't, It simplifies the calculation because I don't waste my time with the space part and for $\boldsymbol{B}$ it's the same thing the space part some will survive and some won't and the time part they'll all survive actually because $\boldsymbol{B}$ has no time piece in its definition the magnetic field two form it's all purely spatial so when I take the exterior derivative time part and I wedge it with $\mathrm{d}x^0$ in front every where so they're all survive so it's not a huge simplification.

$$\begin{aligned}\mathrm{d}F=&\mathrm{d}_s\left[E_1\,\mathrm{d}x^1\wedge\mathrm{d}x^0+E_2\,\mathrm{d}x^2\wedge\mathrm{d}x^0+E_3\,\mathrm{d}x^3\wedge\mathrm{d}x^0\right]\\&+\mathrm{d}_s\left[B_{23}\,\mathrm{d}x^2\wedge\mathrm{d}x^3+B_{31}\,\mathrm{d}x^3\wedge\mathrm{d}x^1+B_{12}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\right]\\&+\mathrm{d}_t\left[B_{23}\,\mathrm{d}x^2\wedge\mathrm{d}x^3+B_{31}\,\mathrm{d}x^3\wedge\mathrm{d}x^1+B_{12}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\right]\end{aligned}\tag{17}$$

There is a more elegant way to proceed onward from here but I decided you know what I'm going to do I'm just going to calculate it I'm just going to go ahead and straight up calculate it so I'm just using this basic formula (14), these basic formulas here to do the calculation and ultimately this guy is just this guy here right which we learned about in earlier lessons so you can think of it this way just take the exterior derivative of F and drag everything to zero that deserves to be zero anyway so let's do this.

$$\begin{aligned}\mathrm{d}F=&\partial_1E_1\,\cancel{\mathrm{d}x^1\wedge\mathrm{d}x^1\wedge\mathrm{d}x^0}+\partial_2E_1\,\mathrm{d}x^2\wedge\mathrm{d}x^1\wedge\mathrm{d}x^0+\partial_3E_1\,\mathrm{d}x^3\wedge\mathrm{d}x^1\wedge\mathrm{d}x^0\\&+\partial_1E_2\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^0+\partial_2E_2\,\cancel{\mathrm{d}x^2\wedge\mathrm{d}x^2\wedge\mathrm{d}x^0}+\partial_3E_2\,\mathrm{d}x^3\wedge\mathrm{d}x^2\wedge\mathrm{d}x^0\\&+\partial_1E_3\,\mathrm{d}x^1\wedge\mathrm{d}x^3\wedge\mathrm{d}x^0+\partial_2E_3\,\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^0+\partial_3E_3\,\cancel{\mathrm{d}x^3\wedge\mathrm{d}x^3\wedge\mathrm{d}x^0}\\&+\partial_1B_{23}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3+\partial_2B_{23}\,\cancel{\mathrm{d}x^2\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3}+\partial_3B_{23}\,\cancel{\mathrm{d}x^3\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3}\\&+\partial_1B_{31}\,\cancel{\mathrm{d}x^1\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1}+\partial_2B_{31}\,\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1+\partial_3B_{31}\,\cancel{\mathrm{d}x^3\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1}\\&+\partial_1B_{12}\,\cancel{\mathrm{d}x^1\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2}+\partial_2B_{12}\,\cancel{\mathrm{d}x^2\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2}+\partial_3B_{12}\,\mathrm{d}x^3\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\\&+\partial_0B_{23}\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3+\partial_0B_{31}\,\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1+\partial_0B_{12}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\end{aligned}\tag{18}$$

For the magnetic field $\boldsymbol{B}$ we lose two of the three which makes perfect sense we're taking the spatial exterior derivative of something that's already fully spatial so there's only one more spatial variable we can add in there. A lot of terms are going to go away. Then lastly we take the time piece of the magnetic field and what we discover this time, there's only one derivative to take and that's ∂_0 , this is the only one that has a time piece left in it and it's pretty clear that all three terms are going to survive because you're adding a $\mathrm{d}x^0$ and none of these have a $\mathrm{d}x^0$ in it so all three of these terms survive. Now we can just get rid of the ones that don't survive and then we just keep the ones that do.

Let's look at the time derivatives of $\boldsymbol{B}$ that survive, remember all three of those survive well if we look at it we can shuffle this $\mathrm{d}x^0$ to the back in each case, now it's in the front we can shuffle it to the back and when we do that shuffle we move it one, two so the two sign changes cancel out so you have no sign change in the end and so you get for this first piece:

$$\begin{aligned}&\partial_0 B_{23}\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3+\partial_0 B_{31}\,\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1+\partial_0 B_{12}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\\&=\partial_0\left[B_{23}\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3+B_{31}\,\mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1+B_{12}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\right]\\&=\left[\partial_0 B\right]\wedge\mathrm{d}x^0\end{aligned}\tag{19}$$

Likewise we take the magnetic field parts that survived from the spatial exterior derivative and we combine those together, only three of them survive, Those three pieces can be combined to give us

$$\begin{aligned}&\partial_1 B_{23}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3+\partial_2 B_{31}\,\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1+\partial_3 B_{12}\,\mathrm{d}x^3\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\\&=\left(\partial_1 B_{23}+\partial_2 B_{31}+\partial_3 B_{12}\right)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\end{aligned}\tag{20}$$

This is a purely spatial three form. now remember we're dealing with four dimensions so three forms aren't one dimensional anymore, three forms are four dimensional and four forms are one-dimensional so this is just the purely spatial three form but notice over here (19), you don't have a purely spatial three form, you've got three forms that have space parts and a time part tacked on at the end and then we take what's left over from the electric field $\boldsymbol{E}$ and when we take what's left over from the electric field we're combining these guys together and by reshuffling the order of these various three forms we end up with, again mixed time and space three forms.

$$\left[\left(\partial_1 E_2-\partial_2 E_1\right)\mathrm{d}x^1\wedge\mathrm{d}x^2+\left(\partial_3 E_1-\partial_1 E_3\right)\mathrm{d}x^3\wedge\mathrm{d}x^1+\left(\partial_2 E_3-\partial_3 E_2\right)\mathrm{d}x^2\wedge\mathrm{d}x^3\right]\wedge\mathrm{d}x^0\tag{21}$$

We can pull out the time one form and leave behind this two form and the coefficients of these two forms end up being the curl of the electric field. You can see where this is going here's something that looks like a curl, $\partial_0 B$ looks like the time derivative of a magnetic field $\boldsymbol{B}$ two form and (20) looks like a divergence. Ultimately that's correct, that's the way this is going to work because (19) and (21) both share $\mathrm{d}x^0$ but (20) does not have any $\mathrm{d}x^0$ in it at all.

Now we know from Maxwell's equations, we know that the divergence of the magnetic field $\boldsymbol{B}$ is zero and that's exactly what this is these are the components of the magnetic field vector right inside this thing here and therefore we know that this has to be equal to zero and also from Maxwell's equations we know that the curl of the electric field $\boldsymbol{E}$ and the time derivative of the magnetic field combined that equals zero as well so our next step is to translate this into vector language well to do that we just combine everything we can combine. First of all we recognize that (21) is the exterior derivative of the electric field, the electric field is a one form, the exterior derivative is a two form and that's exactly what this is and we know that this is what's left over when you take the exterior derivative of a one form you get a two form and that's what this is in here.

What we would do normally next is if we were really interested in this being a curl is we would take its Hodge dual to convert these two form basis vectors into the one form basis vector so you have what looks like a curl but right now we just leave it this way and (19) of course is just the time derivative of a two form so this is a two form this is a two form you add it together you can wedge it with $\mathrm{d}x^0$ to get a three form if you want and well that's required so $\mathrm{d}F$ is a three form and then (20) is a three form but the coefficient is straight up the divergence of the magnetic field two form which we know how to write down so we can definitely take this piece and we can write it down as the Hodge dual of the exterior derivative of the two forms. The exterior derivative of a two form is a three form the Hodge dual the three form is a zero form which is what this is a scalar and then you have of course the three form basis vector over here.

$$\mathrm{d}F = \left[\mathrm{d}E + \partial_0 B\right] \wedge \mathrm{d}x^0 + \left[*\,\mathrm{d}B\right]\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = 0 \tag{22}$$

When we look at this, now we have to apply our Physics we know that this guy has to be zero, we know it has to be zero because we know Maxwell's equations, we understand Physics, there's nothing about this that forces $\mathrm{d}B$ to be zero, we're just showing that the exterior derivative of the electromagnetic field tells us something or contains this information or contains this structure and sure enough it contains something that looks like the divergence of $\boldsymbol{B}$ but we know from our vast knowledge of Physics that $\mathrm{d}B = 0$. We also know (7), these Maxwell's equations are true and we understand that in the concept of differential forms:

$$\nabla \times \boldsymbol{E} \rightarrow *\,\mathrm{d}E \tag{23}$$

We know that it is the Hodge dual of the exterior derivative of the electric field one form, electric field one form the exterior derivative is a two form and the Hodge dual the two form is back to a one form and $\nabla \times \boldsymbol{E}$ is a vector so this is the one form associated with the vector that is the curl of the electric field. This isn't quite a match though because we only have $\mathrm{d}E$ without the Hodge dual in (22) but imagine if we took the Hodge dual of this whole thing, we would end up with:

$$*\,\mathrm{d}E + \partial_0\left[*\,B\right] \tag{24}$$

Remember now $\boldsymbol{B}$ is, we've chosen to make it a two form so the Hodge dual of $\boldsymbol{B}$ is going to be the $\boldsymbol{B}$ one form which is associated with a vector so what I have in $\mathrm{d}E + \partial_0 B$ is the Hodge dual of Maxwell's equations which would be:

$$*\,\mathrm{d}E + \frac{\partial *B}{\partial t} = 0 \tag{25}$$

I know this guy's zero like the Hodge dual of both sides will also be zero so I know that $\mathrm{d}E + \partial_0 B = 0$ and therefore I can say that hey Maxwell's equations those two are embodied in the idea that the exterior derivative of F is equal to zero. That takes care of just two of the four Maxwell's equations so you could ask the question right now as well how profound is this? Is this just some bookkeeping exercise? We always have that question, is this just a bookkeeping exercise and it really comes down to how capable are we of, this is my opinion now, how capable are we of interpreting the electromagnetic field two form? Is that interpretation something that gives us insight where now we can say what is what do we mean by the exterior derivative of a two form equaling zero and if that's the case does that help us understand something about the nature of electric and magnetic fields when combined together in a two form and specifically in this the two form of the electromagnetic field? I think the answer there is generally yes we're not going to go into that in this lecture but this is more than just bookkeeping.

The fact that this can be done and that you can end up with this nice simple equation for these two Maxwell's equations where the fact that we're taking these two basically messy complicated and difficult to understand (7), why would this be and break it down into something so combine them together into something so nice I think there is some profundity there I mean there obviously is some profundity there the question is how actually useful is it be when you're faced with a problem in electromagnetism certainly as an engineer you're going to go right to this (7), you're not going to go start from here (22) so now the question is what about the other two. I think I can move relatively quickly through these second two Maxwell's equations:

$$\begin{gathered}\nabla \cdot \boldsymbol{E} = \rho \\ \nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{J}\end{gathered} \tag{26}$$

It's very similar, if $\rho=0$ and $\boldsymbol{J}=0$ it would be the exact process that we did for (7), it would just be flipping $\boldsymbol{E}$ and $\boldsymbol{B}$ so the whole problem comes down to understanding what to do with the charge density ρ and the current density $\boldsymbol{J}$. We begin the same way we're going to model the magnetic field $\boldsymbol{B}$ as a two form which is the correct model because as we've said earlier it captures the parity inversion behavior and we're going to model the electric field $\boldsymbol{E}$ as a one form which also captures its parity inverting nature and we are going to combine them into the same electromagnetic field F :

$$F = E \wedge \mathrm{d}x^0 + B \tag{27}$$

We're gonna go and wedge with $\mathrm{d}x^0$ to turn the electric field one form into a two form and we're gonna add it to the two form of the magnetic field and you'll notice that this addition is I'll just remind you that this addition will be basically disjoint in the sense that the two form components of the $\boldsymbol{B}$ field are these spatial ones and the two form components of the $\boldsymbol{E}$ field wedge $\mathrm{d}x^0$ will all be a space and time two form. All the electric field components will live in these time these time wedged two forms and the space components will be in the magnetic field two forms.

That's the same one we did before but now the difference is we're going to create the four current. The four current is going to be the current vector, the current in the x direction y direction z direction with the x, y, z basis vectors where I'm now using the manifold notation of the tangent space basis vectors in the coordinate basis:

$$\boldsymbol{J} = \rho\, \partial_0 + J^1 \partial_1 + J^2 \partial_2 + J^3 \partial_3 = J^0 \partial_0 + J^1 \partial_1 + J^2 \partial_2 + J^3 \partial_3 \tag{28}$$

Which is a direct substitution for E_x, E_y, E_z or $\hat{i}, \hat{k}, \hat{j}$ depending on what fields you're arriving here from but we take those three and we give them the spatial components but we give the charge to the time component so this is now the four vector that will represent the current but notice that the charge is considered to be the current that's in the direction of purely with time, it doesn't have any spatial direction it's the directions associated entirely with time which I find fascinating I don't know why I think that's really an interesting way of blending these things together this is not a big mystery nor is it unique to this particular branch of study having a four current is totally normal in relativity theory.

Now I'm going to take ρ and instead of calling it the charge density, I'm going to make it the 0^{th} component of the current four vector so now we have it where everything has an up and down index for every component so ρ now becomes J^0 and now we convert this vector to the four current one form and we do this by doing the one to one correspondence between the vectors and the co-vectors and we take these up indices to down indices but now we do have to remember now we're dealing with a four vector and we're dealing with the Lorentzian metric so this is fine you can just simply lower the indices that's completely fine but we do have to remember that the super scripting is no longer equals the subscript index for the 0^{th} component in the convention we're using it actually equals the negative of that that's a very standard relativistic thing that we've covered in previous lessons.

$$J^0=\rho=-J_0 \quad , \quad J^i=J_i \tag{29}$$

Understand now it's no longer the straight up Euclidean metric anymore, it's Euclidean for the space parts but this time part introduces this negative sign so now this $\boldsymbol{J}$ one form if you really were to compare it to the four current as we originally wrote it we'll have minus ρ here:

$$J=J_0\,\mathrm{d}x^0+J_1\,\mathrm{d}x^1+J_2\,\mathrm{d}x^2+J_3\,\mathrm{d}x^3=-\rho\,\mathrm{d}x^0+J_1\,\mathrm{d}x^1+J_2\,\mathrm{d}x^2+J_3\,\mathrm{d}x^3 \tag{30}$$

We could have also written it wouldn't have been very reasonable to write $-J^0\,\mathrm{d}x^0$, that's not totally very conventional to have these two superscripts together but remember J^0 is a number it equals $-J_0$ so you could do that, the beauty of the notation is that minus sign is tracked for us just by lowering the subscript and knowing that these things are not necessarily equal however remember these three spatial are equal. Anyway the upshot of all this is that you end up with the four current one form (30).

Now we can start tying these things back together again so now this time we are going to show that this expression the Hodge dual of the exterior derivative of the Hodge dual of the electromagnetic field two form equals the current one form that we just assembled:

$$*\,\mathrm{d}*F=J \tag{31}$$

We're going to show that this is equivalent to those two Maxwell's equations (26). I'm just going to demonstrate that this is the case. First we do some logic testing, F is a two form in four dimensional space so we have a four dimensional space and F is a two form so the Hodge dual of a two form in four dimensional space is another two form so this object right here $*F$ is a two form even though F is a two form it doesn't change because the two forms are Hodge dual to themselves and then just to be clear about that remember that if the underlying vector space V is a four dimensional then the co-vector space is four dimensional, the zero forms are still the functions and the four forms are one dimensional space with this single quadruple wedge product then the one forms are Hodge dual to the three forms and the two forms are Hodge dual to themselves. In three dimensional space it was different, in three dimensional space there would only be one three form it would just be a single three form and then you would lose all the two forms with a zero on it so you'd end up with these three two forms, these three one forms, this zero form, the four form goes away and you just are left with the three form and so now one forms are dual with two forms, that's the way it was for three dimensions.

With that in mind, now that we understand that this actually does make sense (31), that means the left hand side is a one form because the as I said this was a this the Hodge dual of the electromagnetic two form is another two form the exterior derivative of a two form is a three form and in four dimensional space the Hodge dual of a three form is a one form and J is a one form so both sides are one forms. Now we have to calculate these things piece by piece and see what we get and it's not going to be too difficult we're it's all familiar the Hodge dual of the electromagnetic two form is:

$$\begin{aligned}
*F &= *\left[E\wedge \mathrm{d}x^0 + B\right] = *\left(E\wedge \mathrm{d}x^0\right) + *B \\
&= *\left[E_1\ \mathrm{d}x^1\wedge \mathrm{d}x^0 + E_2\ \mathrm{d}x^2\wedge \mathrm{d}x^0 + E_3\ \mathrm{d}x^3\wedge \mathrm{d}x^0\right] \\
&\quad + *\left[B_{23}\ \mathrm{d}x^2\wedge \mathrm{d}x^3 + B_{31}\ \mathrm{d}x^3\wedge \mathrm{d}x^1 + B_{12}\ \mathrm{d}x^1\wedge \mathrm{d}x^2\right] \\
&= E_1\ \mathrm{d}x^2\wedge \mathrm{d}x^3 + E_2\ \mathrm{d}x^3\wedge \mathrm{d}x^1 + E_3\ \mathrm{d}x^1\wedge \mathrm{d}x^2 \\
&\quad + \left[B_{23}\ \mathrm{d}x^0\wedge \mathrm{d}x^1 + B_{31}\ \mathrm{d}x^0\wedge \mathrm{d}x^2 + B_{12}\ \mathrm{d}x^0\wedge \mathrm{d}x^3\right]
\end{aligned} \tag{32}$$

I've rewritten them here:

$$\begin{aligned}
*\left(\mathrm{d}x^2\wedge \mathrm{d}x^0\right) &= \mathrm{sign}(2\,0\,3\,1)\,\varepsilon(2)\,\varepsilon(0)\ \mathrm{d}x^3\wedge \mathrm{d}x^1 \\
&= (-1)(-1)(-1)\ \mathrm{d}x^3\wedge \mathrm{d}x^1 = \mathrm{d}x^3\wedge \mathrm{d}x^1
\end{aligned} \tag{33}$$

$$\begin{aligned}
*\left(\mathrm{d}x^1\wedge \mathrm{d}x^0\right) &= \mathrm{sign}(1\,0\,2\,3)\,\varepsilon(1)\,\varepsilon(0)\ \mathrm{d}x^2\wedge \mathrm{d}x^3 \\
&= (-1)(-1)(-1)\ \mathrm{d}x^2\wedge \mathrm{d}x^3 = \mathrm{d}x^2\wedge \mathrm{d}x^3
\end{aligned} \tag{34}$$

$$\begin{aligned}
*\left(\mathrm{d}x^3\wedge \mathrm{d}x^0\right) &= \mathrm{sign}(3\,0\,1\,2)\,\varepsilon(3)\,\varepsilon(0)\ \mathrm{d}x^1\wedge \mathrm{d}x^2 \\
&= (-1)(-1)(-1)\ \mathrm{d}x^1\wedge \mathrm{d}x^2 = \mathrm{d}x^1\wedge \mathrm{d}x^2
\end{aligned} \tag{35}$$

$$\begin{aligned}
*\left(\mathrm{d}x^1\wedge \mathrm{d}x^2\right) &= \mathrm{sign}(1\,2\,0\,3)\,\varepsilon(1)\,\varepsilon(2)\ \mathrm{d}x^0\wedge \mathrm{d}x^3 \\
&= (1)(1)(1)\ \mathrm{d}x^0\wedge \mathrm{d}x^3 = \mathrm{d}x^0\wedge \mathrm{d}x^3
\end{aligned} \tag{36}$$

$$\begin{aligned}
*\left(\mathrm{d}x^2\wedge \mathrm{d}x^3\right) &= \mathrm{sign}(2\,3\,0\,1)\,\varepsilon(2)\,\varepsilon(3)\ \mathrm{d}x^0\wedge \mathrm{d}x^1 \\
&= (1)(1)(1)\ \mathrm{d}x^0\wedge \mathrm{d}x^1 = \mathrm{d}x^0\wedge \mathrm{d}x^1
\end{aligned} \tag{37}$$

$$\begin{aligned}
*\left(\mathrm{d}x^3\wedge \mathrm{d}x^1\right) &= \mathrm{sign}(3\,1\,0\,2)\,\varepsilon(3)\,\varepsilon(1)\ \mathrm{d}x^0\wedge \mathrm{d}x^2 \\
&= (1)(1)(1)\ \mathrm{d}x^0\wedge \mathrm{d}x^2 = \mathrm{d}x^0\wedge \mathrm{d}x^2
\end{aligned} \tag{38}$$

At this point you should be familiar with how to do these Hodge duels we've done it several times so where did I go so the Hodge dual of F is given by this expression (32) which is another two form but notice now that $\boldsymbol{E}$ is attached to purely spatial parts and $\boldsymbol{B}$ is now attached to purely time parts which is interesting so this is the dual of the electromagnetic two form if we were to put this in a matrix form you remember the electromagnetic four tensor when you write it in matrix form the electric fields are on the edges and the magnetic field is in the middle where everything's diagonal but the dual version it swaps, the magnetic fields are on the edges because they have now the time component and the electric fields are in the middle so and that's reflected here right now you have the electric fields are associated with purely spatial components and the exterior derivative of the Hodge dual of the electromagnetic field tensor that's a little bit more of a hassle to calculate but we've done it several times and we're expecting a three form which is exactly what we get:

$$\begin{aligned}
\mathrm{d}*F &= \partial_0 E_1\ \mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + \partial_1 E_1\ \mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\\
&= \partial_0 E_2\ \mathrm{d}x^0\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1 + \partial_2 E_2\ \mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^1\\
&= \partial_0 E_3\ \mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + \partial_3 E_3\ \mathrm{d}x^3\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\\
&= \partial_2 B_{23}\ \mathrm{d}x^2\wedge\mathrm{d}x^0\wedge\mathrm{d}x^1 + \partial_3 B_{23}\ \mathrm{d}x^3\wedge\mathrm{d}x^0\wedge\mathrm{d}x^1\\
&= \partial_1 B_{31}\ \mathrm{d}x^1\wedge\mathrm{d}x^0\wedge\mathrm{d}x^2 + \partial_3 B_{31}\ \mathrm{d}x^3\wedge\mathrm{d}x^0\wedge\mathrm{d}x^2\\
&= \partial_1 B_{12}\ \mathrm{d}x^1\wedge\mathrm{d}x^0\wedge\mathrm{d}x^3 + \partial_2 B_{12}\ \mathrm{d}x^2\wedge\mathrm{d}x^0\wedge\mathrm{d}x^3
\end{aligned} \tag{39}$$

You can definitely see that now we're going to assemble these partial derivatives in something like Maxwell's equations better show up. You can work it out yourself, I'll leave it up here in case you want to see how I did it. We can look in advance and we're seeking this differential equation (26) here so we're expecting derivatives of the electromagnetic field with respect to time and we're expecting cross products of $\boldsymbol{B}$ so we we should see different in components of $\boldsymbol{B}$ and we should see sums of some components of $\boldsymbol{E}$ and then somehow we should see $\boldsymbol{J}_\alpha$ in there. We shouldn't see those in this part because we're only calculating the left-hand side. Here's clearly time derivatives of the electric field, we see time derivatives of $\boldsymbol{E}$ and we see some space derivatives of $\boldsymbol{E}$, we see three and they look like divergence and clearly here we see derivatives of $\boldsymbol{B}$ and we know that we can subtract them. We still have to take the Hodge dual of it down here:

$$\begin{aligned}
*\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 &= \mathrm{sign}(0,1,2,3)\,\varepsilon(0)\,\varepsilon(1)\,\varepsilon(2)\ \mathrm{d}x^3 = -\mathrm{d}x^3\\
*\,\mathrm{d}x^0\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 &= \mathrm{sign}(0,2,3,1)\,\varepsilon(0)\,\varepsilon(2)\,\varepsilon(3)\ \mathrm{d}x^1 = -\mathrm{d}x^1\\
*\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^3 &= \mathrm{sign}(0,1,3,2)\,\varepsilon(0)\,\varepsilon(1)\,\varepsilon(3)\ \mathrm{d}x^2 = +\mathrm{d}x^2\\
*\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 &= \mathrm{sign}(1,2,3,0)\,\varepsilon(1)\,\varepsilon(2)\,\varepsilon(3)\ \mathrm{d}x^0 = -\mathrm{d}x^0
\end{aligned} \tag{40}$$

When we do that Hodge dual calculation we end up with these, it looks a little shorter because now all these three forms have become one forms because the Hodge dual of the three form in four dimensional space is one form:

$$\begin{aligned}
*\,\mathrm{d}*F &= -\partial_0 E_1\ \mathrm{d}x^1 - \partial_0 E_2\ \mathrm{d}x^2 - \partial_0 E_3\ \mathrm{d}x^3\\
&= -\partial_1 E_1\ \mathrm{d}x^0 - \partial_2 E_2\ \mathrm{d}x^0 - \partial_3 E_3\ \mathrm{d}x^0\\
&= -\partial_2 B_{23}\ \mathrm{d}x^3 + \partial_1 B_{31}\ \mathrm{d}x^3\\
&= -\partial_1 B_{12}\ \mathrm{d}x^2 + \partial_3 B_{23}\ \mathrm{d}x^2\\
&= -\partial_3 B_{31}\ \mathrm{d}x^1 + \partial_2 B_{12}\ \mathrm{d}x^1
\end{aligned} \tag{41}$$

Now we can just look at it straight up, I think I've organized this correctly, the first line is clearly the opposite of the time derivative of $-\partial_0\boldsymbol{E}$, that's what this is, we just pull ∂_0 out where $\boldsymbol{E}$ one form and then the second line is $(-\nabla\cdot\boldsymbol{E})\,\mathrm{d}x^0$, this is a number remember this is a real number and this makes it a one form but it's a one form entirely attached to $\mathrm{d}x^0$ and that's important. That association is very easy and then the last one here if you look at this three last lines, is $*\mathrm{d}*B$. We can write down this final version, because all this stuff is added together to give us:

$$*\mathrm{d}*F=*\mathrm{d}*B-\partial_0 E-(\nabla\cdot\boldsymbol{E})\ \mathrm{d}x^0=-\rho\ \mathrm{d}x^0+J \tag{42}$$

Actually that that's not good because this is in vector form and we really want it to be just the one form of $\boldsymbol{J}$ so we just want this to be $J_1\ \mathrm{d}x^1+J_2\ \mathrm{d}x^2+J_3\ \mathrm{d}x^3$, it's supposed to be the spatial part of $\boldsymbol{J}$. Now when you look at this $*\mathrm{d}*B-\partial_0 E$, these are purely spatial, $\partial_0 E$ is the time derivative of E but E is totally a spatial one form (9) and $*\mathrm{d}*B$ is also purely spatial, we can look at it (41), you can see all of these co-vector parts are spatial co-vectors so this is purely spatial. $(-\nabla\cdot\boldsymbol{E})$ is a number and $\mathrm{d}x^0$ so this is purely time so you compare to the right hand side and $-\rho\ \mathrm{d}x^0$ is purely time and J is purely space so we conclude that $(\nabla\cdot\boldsymbol{E})\ \mathrm{d}x^0=\rho\ \mathrm{d}x^0$ and $*\mathrm{d}*B-\partial_0 E=J$ for this expression to equal J where J is the full two form that we described in here (30). This part here $(\nabla\cdot\boldsymbol{E})\ \mathrm{d}x^0=\rho\ \mathrm{d}x^0$ is just an expression of Maxwell's equation the first of these Maxwell's equations (26), that's as simple as that so we know Maxwell's equations are true so we know that this coefficient and that coefficient equal and we get that condition and $*\mathrm{d}*B$ represents the curl of a magnetic field and $\partial_0 E$ is the time derivative of E and it equals J , well that's exactly the second expression here (26).

Therefore we know that the way we modeled the electromagnetic field tensor and the construction of this exterior algebra actually takes those two Maxwell's equations and burns them into a single one. Now we have demonstrated that these two Maxwell's equations can ultimately be expressed by the Hodge dual of the exterior derivative of the Hodge dual of the electromagnetic field two form (31).

$$\begin{cases} \mathrm{d}F=0 \\ *\mathrm{d}*F=J \end{cases} \tag{43}$$

These two very simple expressions give us the same information that's in Maxwell's equations. Look how tidy this is this is very and this tidiness is what should impress you because it means that modeling the electromagnetic field as a two form the way we have is somehow profound because by doing that modeling we get much simpler expressions for the underlying Physics and so interpreting this F is actually worth our time to see what is it that it's trying to tell us about the electromagnetic field. First of all I think it is telling us that we live in a four-dimensional world because when you add that 4th dimension things tighten up very much and for Physics this is a really beautiful thing, for engineering though you're just going to do what we've always done I don't think there's any advantage in for engineers to immediately jump into this necessarily.

I'm going to check on that, I'm going to find some engineering problems that do in fact start from this posture, I feel like even if they did start from this posture they would actually start from the three-dimensional version of this right you can do this in three dimensions as well and these equations become a bit messier, you end up having to insert time derivatives by hand where the three space derivatives are all taken care of in the vector formalism but the point of this lesson is to show that we can get these four equations down to two but you'll remember we've already gotten them down to two we did it a few less several lessons ago where we got them down to these two potential equations for the scalar potential and the vector potential. Notice these these equations are relatively messy, you have your space blended up in here by the way, you see time and space are definitely the actors here but they're not merged into a single unit, well a singular formalism, they're still separated and because of that separation you have to work a lot more a lot harder with special derivatives of things and rules and everything like that.

$$\begin{cases} -\nabla^2 V - \frac{\partial}{\partial t}[\nabla \cdot \mathrm{A}] = \frac{1}{\varepsilon_0}\rho \\ \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\boldsymbol{A} = \frac{1}{\varepsilon_0 c^2}\boldsymbol{J} - \nabla\left[\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\dot{V}\right] \end{cases} \tag{44}$$

This is obviously in different units, this is in units where $c \neq 0$ and the other material we did just a moment ago, these units were different but you still see the charge densities here the current is here and now it's all the vector form but still using the vector form we got Maxwell's equations jammed down into two things. Our final question that I'm wrestling with playing with and I just don't think I'm going to be able to resist the temptation is to ask the question, is there a formalism that can take these two guys these here (43) and squeeze it down into a single equation a single expression is that possible?

My understanding is that it is but now I'm going into a field that I have yet to fully familiarize myself with a field called Geometric algebra and it is very interesting and I think I am going to not be able to resist the shiny object of going and studying Geometric algebra and it will be more of an adventure where we learn it together because well I know quite a bit about it already but not enough to in order to get to the level of teaching it. It will be a bit of an adventure, I'm not sure if I should continue it on as a tangent in the QED prerequisites because it's definitely not a QED prerequisite, even (43) is not a QED prerequisite. We did the QED prerequisites earlier, this is all a shiny object tangent but to go into Geometric algebra is really shiny object territory and you know I know myself well enough to know that I'm not going to be able to resist so well sooner than later we'll probably begin a sequence in Geometric algebra and hopefully finish it so there we have it, this is the how differential forms are applied to the theory of electromagnetism and I'll see you next time.