Lesson 13 Some important CFREE relations

In this lesson we're going to talk about Parallel transport of a (p,q) rank tensors and we'll be focusing on leaning on the CFREE notation to do this although as you've already seen from previous lectures the CFREE and the comp notations work together, they almost always slide from one to the other to take out these proofs down so we want to do parallel transfer for (p,q) rank tensors notice that we've done this already for vectors, we've got this for a vector:

$$\nabla_{\dot{\gamma}} Y = 0 \tag{1}$$

Likewise we could write this as:

$$\frac{d}{ds}\left[\Omega_{t\,s} Y(s)\right]\Big|_{s=t} = 0 \tag{2}$$

These are the two Parallel transport equations but you'll notice they're both defined for vectors, a vector field and of course we also have the comp notation:

$$\frac{dY^{\mu}}{d\tau} + \Gamma^{\mu}_{\alpha\beta}\,\dot{\gamma}^{\alpha}\,Y^{\beta} = 0 \tag{3}$$

These are the 3 forms of Parallel transport but they're all defined for vectors, you solve this equation you get an auto parallel vector field along the curve γ likewise same with these two or just alternative expressions of the same thing and all of these things are ultimately the same and that last lecture we did was a very long analysis of this guy (2) which I think was worth doing although you really don't see this too much in the books. The book that I studied it from was a book by a guy named Norbert Straumann (General Relativity with Applications to Astrophysics). It's a pretty good book it's one of the hardest ones, it's up there with Wald's and Chandrasekhar's books.

Now we have to think about this in terms of tensors, just general tensors, because everything should be able to do a Parallel transport. If I have some curve in my space-time with some parameterization τ of some curve γ that takes each of these points onto the curve at this point I've been showing a vector space but I could have any tensor product space I could have $Q^{\alpha\beta}{}_{\gamma\delta}$ and that's a tensor that would exist at this point t and then over here there's a tensor product space for $Q^{\alpha\beta}{}_{\gamma\delta}$ and this guy should be parallel to the other guy in a similar idea that vectors are parallel, the thing about vectors is we understand how it works in flat space-time because we visualize vectors in flat space-time so we understand the idea, we can understand what a parallel vectors are and with the notion of curved space with a connection $\Gamma^{\alpha}_{\mu\nu}$ then we understand the basis vectors of these parallel vectors are changing and we need a rule to penetrate through these vectors spaces, these separate vector spaces to somehow relate together this manifold. The idea of this happening for tensors of arbitrary rank it's a little bit tougher now you could go and lean on the idea, well first you could just accept it, you could accept the fact that we need to come up with a way of Parallel transporting tensors and tensors should be parallel there should be a way of doing that. In flat space you would say the same thing, if you had a tensor q with all the components out in flat space you might say well when you move from t to s the components should stay the same. That would be like Parallel transporting a vector in flat space-time you just keep the components the same.

It's another way of saying that the connection $\Gamma^{\alpha}_{\mu\nu}=0$ everywhere in the space-time. It's easy enough to understand how you might use that components don't change rule for flat space-time but it's not so easy to understand how that might change for a space-time that's curved or space-time where the metric connection $\Gamma^{\alpha}_{\mu\nu}$ is not equal to zero but ultimately the basic problem, the basic idea still is that now you have a tensor product space $T^{\alpha}{}_{\beta}\ \partial_{\alpha}\times dx^{\beta}$ and you have one of those tensor products is still a vector space, we all know that we have it, these are bi-linear maps, in this case and I should be able to pick a value a particular tensor a t and find its Parallel transported version at s using just the same process and I should have the same operator Ω_{ts} moving from s to t . This operator acting on a tensor should give me the parallel tensor at a different place and this is different than an arbitrary tensor field there needs to be, what I need is I need an equation to tell me what is the auto parallel tensor field given a tensor at one point and a curve and a space-time with a connection (S,∇) , I have an initial tensor at some point I should be able to calculate the parallel tensors at every other point on the curve and so we can do that for vectors now we have to do that for tensors.

By the way when I mentioned that it could go with the idea that the components remain the same that for example $T^{0}{}_{1}$ in this point in flat space would be $T^{0}{}_{1}$ over here that's flat space in a Cartesian system where the connection is actually $\Gamma^{\alpha}_{\mu\nu}=0$ everywhere. If I have a flat space-time where the connection is not 0 everywhere then I'm still gonna have these these components changing even in flat space that needs to be clear flat space in the Cartesian system where the connection coefficients vanish everywhere then literally the components should in principle just move from one point to another regardless of the path, in fact. If you have flat space but you do have a connection that's nonzero, then the components will change as you move Parallel transport from place to place. The other thing is that we could there is a visualization you could lean on for 1-forms, if you go back to our 1-form lecture and this is a part of the prerequisites. If you go back to the “What is a tensor?” series and you considered the cotangent space $T^{*}_{s}(s)$ and then say $T^{*}_{t}(s)$, now we're talking about the space of all 1-forms or just co-vectors so every co-vector would look like $A_{\mu}\ dx^{\mu}$, now we have a way of visualizing those things as these plane densities, basically a 1-form has this appearance of a bunch of stacks, we called it a stack and the density of that stack was relevant so you could imagine Parallel transporting those stacks and whatever the curvature of the manifold $\Gamma^{\alpha}_{\mu\nu}$, that is gonna somehow play in making the stacks here parallel to the stacks there and this curvature is going to make it so that it's not literally just the same direction, if you associate with the stacks of vector in the direction of the stacks which is really not a good way of looking at it because we're trying to get away from the arrows when we think about these stacks but the stacks are going off in a direction, there's no doubt about that and they have a certain density so that's another visualization that could work, if you've gotten good with this stack visualization of 1-forms then you could use that to bootstrap the logic that you would have of Parallel transporting vectors which we're pretty comfortable with.

We still got to attack this creature here $T^{\alpha}{}_{\beta}\ \partial_{\alpha}\times dx^{\beta}$ we've got to figure out how to Parallel transport an arbitrary tensor even if you were good with these stacks that's still only co-vectors and vectors that you're down with, we still have these things to figure out. By the way you don't need to stop at the stack visualization, if you're really pushed on to the end of the “What is a tensor?” series and you can do any form at all, you could do 2-forms and you're dealing with these honeycomb structures, you could do 3-forms and you're dealing with these, well I guess the 3-forms is the network or the cells, all the cells, the density of cells, there is some room to play in the lower dimensions, if you stick to say 3D or 4D or you stick to 2-forms I guess is what it is, you can imagine shifting those around parallel to each other so there is a way of visualizing the notion of what is becoming parallel to what with some tensors in particular the forms.

The point is we got to get away from visualizing things that's not the point if you stick to what you can actually visualize you're never gonna make it very far in this business. Now we're gonna dive into that abstract idea of how to create parallelism for arbitrary rank tensors and we're going to start with co-vectors. With co-vectors we are going to have to come up with some foundation that's going to drive us all forward through this. The first one then he established the notation:

$$\Omega_{ts}\alpha(s)\in T^*_{\gamma(t)}(s) \tag{4}$$

With co-vector α where α is at s , the parameter value s this guy is an element of the cotangent space of $T^*_{\gamma(t)}(s)$. This is just a reminder, this is an operator that's going to take any tensor, the ones we've been doing so far are vectors but we're gonna upgrade this thing so it can take any tensor, it acts on the right, in this case is taking 1-form that 1-form is a co-vector and it's a co-vector field along a curve so again we're always dealing with some curve γ that's parameterized by τ and at a certain point that value is $\tau=s$ and that point is $\gamma(s)$, that's what this point is in the manifold and if you want, the coordinates of that point are going to be $(X^0(\gamma(s)),X^1(\gamma(s)),X^2(\gamma(s)),X^3(\gamma(s)))$ that's when you really break it down, that's the point in $\mathbb{R}^4$ from the manifold so this is the manifold S and this is the point in $\mathbb{R}^4$ that is the coordinates of that spot.

We're referring to a vector located on this curve here at the parameter value $\tau=s$ and (4) is $\alpha(t)$ which is essentially some auto parallel field that is auto parallel along this line γ and this operator takes the vector that's at s and turns it into the one that's at t . We would say $\alpha(t)$ equals (4). This is just a review of the same notation the big step here is that we're promoting this so it doesn't just act on vectors it acts on any tensor and in this case we're just throwing in a $(0,1)$ rank tensor. It's already been demonstrated and used for a $(0,1)$ rank tensor so we've already done that. With all that in mind, going through this whole thing, repeating it in your head, getting it straight each time that's worth it especially when you pick up a new book you're gonna have to look at it and say what is all this notation actually mean we got a curve, we got a parameter for the curve, you got a name of the curve all that stuff you have to work out each and every time, I have to work it out each and every time.

What is our guiding principle? Our guiding principle is, well what is a 1-form? A 1-form $\alpha(s)$ is something that eats a vector, say some arbitrary vector $X(s)$ and it gives you a real number:

$$\langle\alpha(s),X(s)\rangle\in\mathbb{R} \tag{5}$$

We know that from our earliest days so this is true (5) for any 1-form and any vector where now X is an element of the tangent space $X(s)\in T_{\gamma(s)}(s)$. Notice (4) is at $\gamma(t)$ because although α is at s the operator turns it into something at t so this is an element of $\gamma(t)$ but $\alpha(s)\in T^*_{\gamma(s)}(S)$. We know that (5) is true, this is the dual space mapping of $\alpha(s)$ with some vector $X(s)$ will be a real number so the notion of Parallel transport axiomatically is going to mean that we want this value, whatever this number is we want that to be preserved, we want the dual space mapping to be preserved under Parallel transport which means that:

$$\langle\Omega_{ts}\alpha(s),\Omega_{ts}X(s)\rangle=\langle\alpha(s),X(s)\rangle \tag{6}$$

What this is telling me is that if I take a vector at s and I Parallel transport it to t and I take a co-vector at s and I Parallel transport it to t and I take the inner product of those two Parallel transported objects, I'd better get the same number as I would have gotten had I just taken the inner product in the tangent space at s and that is our fundamental assumption about what parallelism should mean and it certainly makes sense in flat space-time with regular vectors, I take a vector here and I take a co-vector here, that dual space mapping should, if I Parallel transport it, that dual space mapping should give me the same answer, the number of planes pierced by this arrow should be the same if we truly Parallel transported the stack along with the vector.

I should be a little careful because the prerequisites for this course don't include the stack visualization of 1-forms that we did in “What is a tensor?” so you either have to just accept this with or without the ability to visualize the stacks for 1-forms but it's really not that critical. The principle is pretty straightforward the dual space mapping should be preserved under Parallel transport. Notice though that's not the same as saying the inner product has to be preserved. If I took the inner product of two vectors $(X(s), Y(s))$ so that would be g , the metric tensor acting on $X(s)$ and $Y(s)$, if I really did it out the full way I would say:

$$g_{\alpha\beta}\, dx^{\alpha} \otimes dy^{\beta}\left(X^{\gamma}\partial_{\gamma}, Y^{\zeta}\partial_{\zeta}\right) \tag{7}$$

Something like that, so this is the metric tensor acting on a two vectors both located at the point s , I guess that would be $X^{\gamma}\partial_{\gamma} = X^{\gamma}(s)\partial_{\gamma}\big|_{s}$ that's what would go here. That demand, the demand that the inner product be the same under Parallel transport is totally different because this Parallel transport (6) is driven by the connection on the space-time which in the abstract notation it is ∇ in the comp notation it is $\Gamma^{\alpha}_{\beta\gamma}$ something like that, in two different notations but that connection is a structure of the manifold itself, this metric tensor $g_{\alpha\beta}$ is some other structure. The demand that the metric tensors inner product of two vectors or equivalently if I wrote it in this form $g^{\alpha\beta}$ tensor product of two co-vectors:

$$g^{\alpha\beta}\ \partial_{\alpha} \otimes \partial_{\beta}(\omega, \delta) \tag{8}$$

Likewise those inner products could also, in principle, be preserved under Parallel transport but if they were it's a different requirement than the one I'm setting up, this requirement (6) is Parallel transport that preserves dual space mappings and (8) is Parallel transport that preserves the inner product and these are not the same requirements. However, we're gonna discover is that if we demand (7), that's going to severely constrain our choices for the connection on the manifold, in other words you can't have both, you can't arbitrarily choose a connection and then demand (8). If you arbitrarily choose a connection you probably won't get (8). If you demand (8), you're forced into a connection, that's the trick, that's what we're headed for. We're trying to prove that because we're gonna demand (8) ultimately, we're gonna discover that it forces us to have a specific connection on the manifold and that's where we're headed with all this but for now we don't have any of this, we have no demand that the metric or that the inner product of vectors or co-vectors be preserved under Parallel transport, we're only demanding that the dual space mapping be preserved (6).

How would this demand (6) look for full-blown tensors, we just showed it for dual vectors but I could imagine I had a tensor at the point $Q(s)$ now this is a tensor of any rank so in the comp notation this thing would be like $Q^{i_1 i_2 \cdots i_p}{}_{j_1 j_2 \cdots j_q}$ and then that would be a (p, q) rank tensor, that's what the component would look like and then in the component form this thing would be:

$$Q^{i_1 i_2 \cdots i_p}{}_{j_1 j_2 \cdots j_q}\ \partial_{i_1} \otimes \partial_{i_2} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_q} \tag{9}$$

It's a mapping so we know that it's gonna take, its gonna take p co-vectors and q vectors as arguments so we write all that down like this:

$$Q(s)(\sigma_1, \sigma_2, \ldots, \sigma_p, Z_1, Z_2, \ldots, Z_q) \in \mathbb{R} \tag{10}$$

That is a real number because that's what these things are so (10) is in the abstract notation and (9) is a comp notation and when I say this is the comp notation, this is the full-on comp notation including the basis vectors and as I keep pointing out in your textbooks all you're gonna see is (9) without the basis vectors very very often because they don't usually drag around these basis vectors. It's all implied because the way these the indices are ordered, the ordering of the indices tells us how these basis vectors are arranged so it's redundant notation and that's the whole purpose of the "What is a tensor series", the first half of it at least. There is a textbook notation form by the way just be aware where you would write:

$$Q^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} \tag{11}$$

On top of each other so the indices are above each other when you do that you've got to think of it in terms of (9). Usually the author will tell you what he thinks is coming first in the basis vectors, the point is that (11) is a shorthand because typographically compressed form of (9) and I can appreciate why you might want to do it. When I use a lot of stuff in latex (9) actually becomes pretty extended sometimes and (11) is a lot tighter but strictly speaking I don't like to deviate from (9) this literally captures the entire structure of the tensor because you have the index ordering. With (11) you do have the index ordering but the moment you would want to raise one of these indices, for example one of the $j's$ or lower one of the $i's$ you immediately have to go back to (9) anyway so although admittedly you don't typically do that.

The point is we're gonna work with (10) so what does it mean to Parallel transport this thing? Actually before I begin that have to make it clear that these forms in (10) are all at s . $Q(s)$ is a tensor field and it's at s and remember we're still dealing with a curve so when I say it's a tensor field, it's got to be defined at least along this curve and it lives inside a tensor product space I guess that would be $T^p_q(s)$, that's the tensor product space and then these guys σ_i and Z_j of (10), live in the tangent space at the cotangent space $T^*_s(s)$ and tangent space $T_s(s)$ respectively. I've already slipped, this has got to be $T^*_{\gamma(s)}(s)$ because this subscript on these spaces should be a point in the manifold and remember s is not a point literally in the manifold, s is a value of some parameter that gets mapped to the curve through γ onto the manifold so actually the parameter is τ . The parameter is τ and s is the value of. I'm using this shorthand again and it typically happens a lot in this field, you've got to keep track of what this shorthand means so I repeat it each time it happens or each time I notice that I'm doing it.

These guys here (co-vectors) are all in the cotangent space at s these guys here (vectors) are all in the tangent space at s and $Q(s)$ is from the tensor product space at s so what does it mean to Parallel transport a tensor? What it means is the following simple principle: Ω_{ts} is allowed to Parallel transport things from s to t so Q has to start at s so $\Omega_{ts}Q(s)$ is the Parallel transport of Q and we're going to call that $Q(t)$. This is perpetually grounds for confusion between tensor fields and auto parallel tensors. I'm saying that I have a tensor act in the tensor product space at the point identified by s and I Parallel transport that and $Q(t)$ is what I'm gonna call that parallel version of Q . Q is an auto parallel tensor field, it's a field where each value of Q at every point along this curve is parallel to the other value. Typically when we talked about, if I write $Q(x)$, I'm talking about a tensor field on the manifold where this tensor field has some value at every point x and it can be an arbitrary tensor field that I'm talking about and such a field is not typically auto parallel, it's just a tensor field. This notation $Q(t)$ looks very similar to $Q(x)$ and indeed we are talking about Q as a tensor field but we're talking about Q that's constructed specifically to be auto parallel for the purposes of this demonstration. We could abandon this and just say that $\Omega_{ts}Q(s)$ is the auto parallel version of $Q(s)$ at t and that would be our symbol. That's probably the safer way to go but the key is this: the parallel of $Q(s)$ that's going to be a tensor at t and it's going to operate on a bunch of objects at t :

$$\Omega_{ts}Q(s)\left[\sigma_1(t)\sigma_2(t)\cdots\sigma_p(t),Z_1(t)Z_2(t)\cdots Z_q(t)\right]\in\mathbb{R} \tag{12}$$

This is now a tensor at t that's auto parallel or that's parallel to $Q(s)$ and in brackets these are a bunch of co-vectors and vectors at t and this guy here (12) is going to be a real number so that has to equal the following thing: the original tensor field at s i.e. $Q(s)$ operating on the inverse of this motion where each of these guys is Parallel transported to s :

$$Q(s)\left[\Omega_{ts}^{-1}\sigma_1(t),\Omega_{ts}^{-1}\sigma_2(t),\cdots,\Omega_{ts}^{-1}\sigma_p(t),\Omega_{ts}^{-1}Z_1(t),\Omega_{ts}^{-1}Z_2(t),\cdots,\Omega_{ts}^{-1}Z_q(t)\right] \tag{13}$$

This relationship (12)=(13) has to hold and that's very very similar to this relationship (6). In fact it's the idea that if you go back to one forms and you write down (6) you could also interpret that as:

$$\langle\Omega_{ts}\alpha(s),X(t)\rangle=\langle\alpha(s),\Omega_{ts}^{-1}X(t)\rangle \tag{14}$$

This second one (14) is essentially going to be:

$$\Omega_{ts}\alpha(s)\big(X(t)\big)=\alpha(s)\big(\Omega_{ts}^{-1}X(t)\big) \tag{15}$$

which is a co-vector at t operating on $X(t)$ which is this expression (12) where $p=0$ and $q=1$ then the right hand side of (14) is (13) again where $p=0$ and $q=1$ so that we get the equation (15). This expression (15) says I start with them of a vector $X(t)$ and I Parallel transported it back to s . Notice this Ω_{ts} operator is an operator that takes vectors from s and brings them to t so the inverse takes them from t to s , I could just switch the s and the t , by the way but I like to keep the inverse this way Ω_{ts}^{-1} . Now this guy $\Omega_{ts}^{-1}X(t)$, is a vector at s and I operate it with $\alpha(s)$, that should be the same number as if I take $\alpha(s)$, I'll transport it to t and then operate on this thing $X(t)$ that never moved. This is what we've already done is the $p=0$ and $q=1$ version of the general case.

We're going to simplify our notation a little bit so here's our curve, here's our parameter τ . What I'm going to say is that when $\tau=0$ we are dealing with the point we're gonna call $P=\gamma(0)$ which is the point P in the manifold and then I'm going to say that if we were to go to another point in the manifold parameterized by s , if i want to do Parallel transport from 0 to s , I'm now gonna just write $\Omega_s=\Omega_{s0}$. I'm taking my Ω_{ts} and I'm changing it up, I'm calling it say Ω_{st} and then I'm fixing $t=0$ so I'm saying this is always going to be Ω_{s0} and then I'm saying since it's always going to be 0 I'm just gonna write it like that Ω_s so this is always going to be interpreted as $\Omega_s=\Omega_{s0}$ and so Ω_s^{-1} can be interpreted as Ω_{0s} and where s can be any parameter value of τ , so Ω_s is the Parallel transport from 0 to s and $\Omega_s^{-1}=\Omega_{0s}$. Now I'm gonna lean on my notion of what I expect for the Parallel transport equation. The Parallel transport equation is going to be something like this:

$$\nabla_{\dot{\gamma}} Q=0 \tag{16}$$

I would normally write the connection, the derivative of the curve and the tensor that we're taking the Parallel transporting that normally equals zero (16), and I'm going to say well I'm going to use this operator form:

$$\left.\frac{d}{ds}\right|_{s=0} \Omega_s^{-1} Q(\gamma(s))=0 \tag{17}$$

That would have been $s=t$ in our previous notation, the inverse function of s meaning moving from s to 0 and I'm moving from s to 0 based on this notation here is from s to 0 of Q evaluated at $\gamma(s)$, that equals 0 . Both of these equations, (16) and (17), are defining equations for Parallel transport of Q and we're going to show that this this basically satisfies all of the demands that we had made over here with this stuff (12), (13), (14) and (15) with these expressions about Parallel transporting the arguments or Parallel transporting the tensor. Another thing we always have to keep in mind is that for $(0,0)$ tensors this guy:

$$\nabla_{\dot{\gamma}(t)} \mathrm{f}=\dot{\gamma}(t) \mathrm{f} \tag{18}$$

Remember that rule always exists for when f is a $(0,0)$ tensor or a function on the space-time the covariant derivative in the direction of the curve is the tangent vector to the curve acting on the function that's just a flat-out rule that we talked about a couple lessons ago that is just understood to all, that's the only way it could work, just flat-out functions that are $(0,0)$ tensor don't respond to the connection on the manifold $\Gamma^{\alpha}_{\mu\nu}$. With that we need to demonstrate a few key things before we can proceed, some algebraic things because what we're gonna do is, with that notation, we're going to go with this:

$$\Omega_s Q(s)\left[\sigma_1(0)\sigma_2(0)\cdots\sigma_p(0), Z_1(0)Z_2(0)\cdots Z_q(0)\right]= \tag{19}$$

$$Q(s)\left[\Omega_s^{-1}\sigma_1(0)\Omega_s^{-1}\sigma_2(0)\cdots\Omega_s^{-1}\sigma_p(0), \Omega_s^{-1}Z_1(0)\Omega_s^{-1}Z_2(0)\cdots\Omega_s^{-1}Z_q(0)\right. \tag{20}$$

That's the expression and we're using this abbreviated notation for the Ω operator so now consider this expression just for one particular case, let's say that $Q(s)$ is, I'm using Z as my argument tensor so I need let's say:

$$Q(s)=X(s)\otimes\alpha(s) \tag{21}$$

That's a $(1,1)$ rank tensor so I'm just constructing an arbitrary $(1,1)$ ranked tensor and I'm breaking it into its vector piece and its co-vector piece so that's Q . If I write:

$$\Omega_s Q(0) = \Omega_s [X(0) \otimes \alpha(0)] (\sigma(s), Z(s)) \tag{22}$$

What this is telling me is that I'm taking a tensor at 0 and I'm moving it to s. Given that you can imagine feeding this thing which is now a tensor living at s you want to feed it one co-vector at s. I'll call the vector at s $Z(s)$, that's what I've been using. If we do this, according to our definition of Parallel transport, that's going to be:

$$X(0) \otimes \alpha(0) \left(\Omega_s^{-1} \sigma(s), \Omega_s^{-1} Z(s) \right) \tag{23}$$

This, (23), we can calculate, that that's going to equal:

$$\left\langle \Omega_s^{-1} \sigma(s), X(0) \right\rangle \left\langle \alpha(0), \Omega_s^{-1} Z(s) \right\rangle \tag{24}$$

Once we have that expression (24), we can compare it with:

$$[\Omega_s X(0)] \otimes [\Omega_s \alpha(0)] (\sigma(s), Z(s)) = \langle \sigma(s), \Omega_s X(0) \rangle \langle \Omega_s \alpha(0), X(s) \rangle \tag{25}$$

Remember $\Omega_s X(0)$ is a vector located at s, tensor product with $\Omega_s \alpha(0)$ which is a co-vector and if we fed that $(\sigma(s), X(s))$ that thing is going to equal the above in (25). Now we compare the right hand side of (25) with the product in (24), but we've already know how this works for co-vectors we, demonstrated it before, in (6), back where we're demanding this parallel transport doesn't affect this inner product and I think I wrote it down in (14), that this Parallel transport, no matter how you do the Parallel transport, whether you transport the vector to where the co-vector is and do it whether you transport the co-vector to where the vector is and do the dual space mapping or you take a vector and co-vector there at one place and transport them both to a new place and do, it doesn't really matter, it's all going to be equal and so when you look at this and you compare (25), this expression here has got to equal that expression there, in (24) because this one is just taking the co-vector and Parallel transport it to where the vector is and taking the dual space map and this is taking the vector and Parallel transport it or the co-vector is so this product equals that product and likewise that product equals that product and the point of that is that this operation (left hand side of (25)) is equal to this operation (right hand side of (22)) and the lesson there is I guess I should write it:

$$\Omega_s [X(0) \otimes \alpha(0)] = [\Omega_s X(0)] \otimes [\Omega_s \alpha(0)] \tag{26}$$

The Parallel transport of a tensor equals the tensor of the Parallel transported components of the tensor or pieces of the tensor, in other words this operator Ω_s commutes with the tensor product that's another way of seeing it formally. This is important fact (26), that's the algebraic fact and remember we're dealing with CFREE notation so in CFREE notation we can't lean on the rules of multiplying real numbers like the components of tensors $X^{\mu}(x)$ or functions on the space-time and we can multiply these things together using our regular algebraic rules, we have to prove every algebraic rule from scratch so we need to show that this parallel transport operator Ω_s commutes with the tensor product and to do that we used our basic principle that and that proof, we did no component notation here but we did have to go back to our presumption (22), (23), (24) that Parallel transport preserves dual space mappings not necessarily inner products so this dual space mapping is equal to that dual space mapping therefore this expression (25) is equal to this expression (24) which allows us to write down (26) and say that parallel transport commutes with the tensor product operator. Now that we've shown that, let's demonstrate that the partial derivative in the direction of the vector X of tensor Q over the tensor product all at a point P :

$$\nabla_X[Q \otimes S]_P \tag{27}$$

Let's show that this obeys the Leibniz rule, let's show that this guy ∇_X operates on this tensor product (27) in a Leibnizian fashion, that's an important thing to know and it's true and we'll have to use our abstract definitions to show it. Now why not tighten up the notation again just because it's done so many different ways. Here what I'm saying is, I've picking a vector X , Q and T obvious all these things have to be associated with, this has to be some tensor field because we're taking the Covariant derivative of a tensor field ultimately we have to understand what that means, we're working on Parallel transport of a tensor field but we can already show some facts about this now which we will do but it's happening at a point P . Now a point P is an element of the manifold and that element has got to be nailed down to a say a parameter value s because it's on a curve γ so we always have this curve we always have τ we always have a point s we always have $\gamma(s)$ and that gives us what we're gonna call P which is in the manifold so I can now express Q and S in different ways of course I can say:

$$Q(P) \equiv Q(\gamma(s)) \equiv Q(s) \equiv Q_P \equiv Q_s \tag{28}$$

In the last set there was a point where we were dealing with:

$$Q(0) \equiv Q(\gamma(0)) \equiv Q_0 \equiv Q_P \text{ where } \gamma(0) = P \tag{29}$$

We can work with this notation also where we use this subscript now the problem of course is when you go to comp notation, you're gonna have $(Q_s)^{\mu}$ the μ component of Q_s this becomes the name of something and that subscript is not a space-time index and you're now having space-time index is mixed with named indexes and it's really not that much of a mess in Latex because this Q would be boldface and the s would be a different script or a different type of font or whatever so it's not that big a problem in modern production at all and any of these things were really working.

The one I don't like the most is this one $Q(\gamma(s))$ just because it gets so horizontally bulky but at the same time it tells you everything, gives you the name of the curve, it gives you the value of the parameter because it's in the argument it reminds you that Q is on this manifold, I mean, all that stuff should be at the forefront of your mind anyway but if you really have to revert to everything back to its basics then this is probably where you would end up and just have big heavy formulas with all the details when you do that by the way be careful to alternate your, especially if you do any latex work, be careful to alternate stuff parenthesis with brackets I would actually write this as: $Q[\gamma(s)]$, like that. You don't want to deal with this situation $(((())))$, you much rather deal with this $\{[()]\}$ and hopefully you don't have to go much farther than braces but I would definitely go parentheses brackets braces when you handle this stuff but enough of the diversion, let's get back to the proof

The proof, this one is actually pretty easy, we write any tensor:

$$\nabla_X [Q \otimes S]_P = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Omega_s^{-1} [Q_s \otimes S_s] \tag{30}$$

This is why we went to all the lengths to prove that other thing this fact. This is the definition, we're talking about the covariant derivative of a tensor product of two tensors and this (right hand side of (30)) is our old definition that we did in the last lesson, Ω_s^{-1} takes us from 0 to s so the inverse takes us from s to 0 and $Q_s \otimes S_s$ is a tensor product. In this example the curve $\gamma(\tau)$ is not referenced at all, it's only referenced through its parameter. What's interesting is that the parameter of γ is τ and s is a value of τ so we've actually lost all reference to the actual curve itself except perhaps this X which is presumably tangent to that curve. We're presumably doing this derivative this vector X , that lies in the directional part of the covariant derivative that's tangent to some curve that you're in principal dragging these points together along.

The curve is pretty much gone, I suppose the only way to really get it back here would be to do this $\nabla_{\dot{\gamma}}$ and that's fair, we can do that, but for the reason that we don't need to is you don't really need to think of a curve too much here because X is tangent to some curve, the very definition of X being a vector in the tangent space at the point P is that there's some curve out there where X is tangent to it that's what it means that's the very definition of vectors in a tangent space so we always know there's some curve out there that'll suffice for this and it doesn't really matter which one you choose there's actually a lot of curves that have X as the tangent vector and any one of those curves will suffice for the calculation of the derivative, you don't really need to know because everything's done in limits and with with these limits the way these curves differ away from the point P becomes less and less relevant, in fact it limits to not being relevant at all.

This is our expression (30), what we just proved a moment ago is that this is going to be the same as:

$$\nabla_X [Q \otimes S]_P = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} [\Omega_s^{-1} Q_s] \otimes [\Omega_s^{-1} S_s] \tag{31}$$

That's why we proved that last statement, is because we knew we would need it for this and now we're dealing with an ordinary derivative, here we're wondering what's the behavior of this covariant derivative in the direction of X vis-à-vis the tensor product operator but this is a regular derivative so we now write this down as:

$$\nabla_X [Q \otimes S]_P = \left\{ \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} [\Omega_s^{-1} Q_s] \right\} \otimes [\Omega_s^{-1} S_s]_{s=0} + [\Omega_s^{-1} Q_s]_{s=0} \otimes \left\{ \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} [\Omega_s^{-1} S_s] \right\} \tag{32}$$

We're done because this (1st brace in (32)) is the definition of the covariant derivative of Q_s along this curve, the curve comes here because s is parameterizing the curve of which the vector X is a tangent vector and this (2nd brace in (32)) is the covariant derivative of S_s so that means that this whole thing is, remember this is done it when $s=0$ and applies to the whole thing, which of course means all of these guys are at P which is what we were after:

$$\nabla_X [Q \otimes S]_P = \nabla_X Q \otimes S\big|_{s=0} + Q \otimes \nabla_X S\big|_{s=0} = \nabla_X Q_P \otimes S_P + Q_P \otimes \nabla_X S_P \tag{33}$$

This shows that the covariant derivative in the direction of some tangent vector X is Leibniz with respect to the tensor product so we've got the fact that it's Leibniz with respect to the tensor product and in order to prove that we use the fact that the Parallel transport operator commutes with the tensor product. This has all been in CFREE notation so far, we haven't dove into comp notation at all so in this lesson we covered the following points we defined using the operator form of this notation we defined the Parallel transport equation (17) in an operator CFREE form of an arbitrary tensor Q and we showed that the Parallel transport operator commutes with the tensor product operator.

We showed that the covariant derivative in the direction of X is Leibniz with respect to the tensor product operation which is a way of saying that this operator this covariant derivative in the direction of operator ∇_X is a derivation in the tensor algebra, that's the word we use if your Leibniz, you're a derivation, you could be an anti-derivation which would have an extra term in (17) and anti-derivations are big deals in the study of forms but this is just a straight up derivation and it's essentially the Leibniz rule where the product is the tensor product then this is our definition of how to Parallel transport a tensor we say that a tensor parallel transported from one point to another a distance s away along the curve acts on a bunch of vectors and co-vectors at that new point it's the same as leading the tensors alone and taking all those vectors and co-vectors and shifting Parallel transport back to the location of the tensor you better get the same value and taken to its simple conclusion for a $(0,1)$ tensor, you can derive the dual space mapping of a vector and a co-vector at a different point, likewise you could Parallel transport one to the other or the other back to the first and all of the numbers will be the same.

Now we have to prove something about contractions and once we've done that we'll start being able to do some conversion into a comp notation and we're getting closer and closer to our ability to define the metric connection where this idea of the preservation of the dual space mapping is converted to the preservation of inner products and once we do that we're going to bind the connection with the metric, that's our goal.