Geometric Algebra 8: Better notation for basis vectors

We're now going to continue our dive into this subject of Geometric algebra this diversion away from QED prerequisites but it's certainly entertaining and I'm really enjoying my time going through this material. The space-time algebra is a powerful tool for electromagnetism, is the vehicle by which we're learning this subject and this paper gets better and better every time I read it I really really enjoy the paper and the space-time algebra by the way is abbreviated often as STA which is one particular Geometric algebra and we have made a lot of progress and we are about to begin reading this paper's discussion of multi-vectors, we have done some preparatory work regarding multi-vectors so this section should be quite accessible to us now and after a little bit of review and errata let's begin section 3.3. I think I need to get used to doing errata in this material because it is not nearly as familiar to me as other mathematical Physics material, more classic mathematical Physics material and I'm just going to have an errata section in front of every lecture and thankfully there are those watching this series who understand this material pretty well and they are happy to point out my errors obvious and less so in this case it was actually an obvious mistake so I did discuss the notion of a Direct sum. The structure I used was we're going to take two vector spaces combine them to create a third vector space and you can do that however, if you study most linear algebra texts it usually posits everything the other way. It says given a vector space we want to ask is it the Direct sum of two of its sub-spaces or given two sub-spaces of a vector space is the vector space the Direct sum of these two sub-spaces?

Essentially it's equivalent to say if this original larger vector space can be constructed the way I have described it in the last lesson then the answer is yes but what's very different about those two approaches is if you think of V and W already as sub-spaces of Z then one thing very important, the scalars of V and the scalars of W are in fact the same as the scalars for Z and if you look at the way I wrote it down when I spoke about it I said V has some field F_1 and W have some field F_2 and then I glossed over the fact that Z obviously has a field but it is important to understand this bottom-up approach doesn't demand the way I wrote it that all these fields be the same field right and that is important. For all of this work the two vector spaces that you direct some together for our work need to have the same field I don't know if there's some exotic way of taking two vector spaces and creating a Direct sum if they have different scalar fields but obviously if you're going to scale a vector in Z and it's presumably this sum that scaling factor has to flow through both sums.

These subscripts that I wrote here really shouldn't be here, they should all be the same field and what I should have even been simpler because all of our work all of our work is there's only one field that matters that's the real numbers. This process of a Direct sum is truly more mathematically general but for everything that we do in the space-time algebra and indeed everything we do in the Geometric algebra it's real numbers, the fact that real numbers are the scalars for all the relevant vector spaces is a fact that's true for Geometric algebra. That's an errata, there was one other errata or an errata-like thing regarding this:

$$M=\overbrace{\langle M\rangle_0}^{\Lambda_0(\mathsf{M}_{1,3})}+\overbrace{\langle M\rangle_1}^{\Lambda_1(\mathsf{M}_{1,3})}+\overbrace{\langle M\rangle_2}^{\Lambda_2(\mathsf{M}_{1,3})}+\overbrace{\langle M\rangle_3}^{\Lambda_3(\mathsf{M}_{1,3})}+\overbrace{\langle M\rangle_4}^{\Lambda_4(\mathsf{M}_{1,3})} \tag{1}$$

We have what we described as multi-vector and I wrote down this notation for each of the grades of the multi-vector and this multi-vector can be considered the sum of grades and I won't bother any more talking about this sum we've covered that everything we said was correct you know this is some in a formal way this is some ordered quintuplet and the zero grade is always the scalar grade, the vector grade, the bi-vector grade the tri-vector grade the quad-vector grade and in principle for any arbitrary

Geometric algebra this can go up much larger and it terminates all depending on the dimensionality of of the grade one vector space so as far as I know this is universal language if I take a multi-vector M and I want to talk about the grade two part I throw it in brackets and drop down a little bit of a two so that's universal, what's not universal is this and I'm using A and B as bi-vectors for this demonstration so a bi-vector product will have a scalar part a bi-vector part and a quad vector part.

$$\begin{aligned} AB &= A:B + A\cdot B + A\wedge B \\ &= A.B + \langle AB\rangle_2 + A\wedge B \\ &= A\cdot B + [A,B] + A\wedge B \end{aligned} \tag{2}$$

The question is how do you write down using just the two bi-vectors, how do you make a sensible notation for the scalar part and there's universal agreement that is the largest part, the quad-vector part is just given by $A\wedge B$, as far as I've seen that in every single paper that's the case. The problem is with this scalar part and I've picked up whispering of a debate, I haven't read all of Hestenes work but apparently he made the claim that you can't really create a simple object just using the bi-vector nature of A and B and create a scalar, you have to break A and B down into their vector pieces and then I think somebody out there contradict that and said no no, you can actually do this you just have to use the double dot product of tensor analysis and it does seem to me like that double dot product does make sense but on the other hand it also seems like when you actually execute the double dot product, you are going into the guts of the vector structure of A and B so I'm not going to really worry about it but those that like the double dot product do use this notation but I think that might only be one person.

I like it so I'm going to stick with it and I offer it up the mechanism we showed in the last lesson about how to do it, is more important obviously than the notation but most papers do in fact use just this single dot product with with the dot either in the bottom depending on the type setting or the middle and this is leaning on the fact that the dot product between two objects, the mechanism of calculating it and the things on both sides might vary but we expect it to be a scalar and this is always a scalar but I have seen even more fluctuation in what goes in here, for example in this notation the dot product suddenly is just the thing that returns this the grade two's part, here's the literal language of the grade two part and this I think actually showed up in one of Hestenes papers although I can't find it right away and he prefers this notion of this commutator bracket, we're not going to deal with this notation that much, we're pretty much going to stick with (1) because this is nice and universal so although we might use this part here for the highest grade piece.

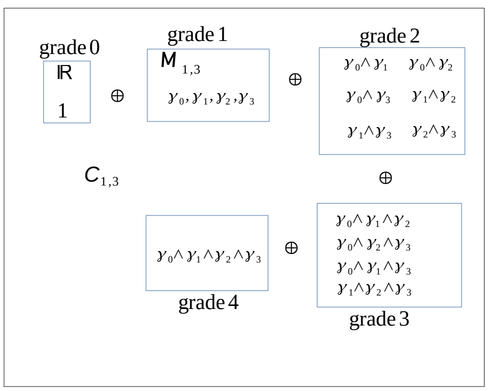
With those two errata in place, let's do a little review. We have made our first pass at the complete picture of C_{13} and we understand it to be the Direct sum of four vector spaces and we've given all the vector spaces a name grade zero, grade one, grade two, grade three and grade four and then by language we call these the scalars, vectors, bi-vectors, pseudo-vectors (tri-vectors) and the pseudo-scalars and dimensionality one, four, six, four and one and we've given them, I've leaned into the names I'm familiar with from Exterior algebra, these Λ and we understand that this of course is a Minkowski, this is an inner product space $M_{1,3}$, a four-dimensional inner products base with a Minkowski inner product and there you have it so we are and we understand how to do this space-time multiplication, we can take any multi-vector M and we can multiply it by any multi-vector N and we know how to do that now because we've learned how to do in the space-time algebra at least all of the multiplications of the different blades the different simple members of each of these grades and by the way when you can do this I just to be clear this algebra, when the algebra has this structure it's called a Graded algebra and the Graded algebra means you can break it up into these grades exactly as I've described here. That is what the multi-vector world is and now we will turn to the paper and read how they present it.

Section 3.3 Multi-vectors let's begin. “By iteratively appending all objects generated by the wedge product to the initial vector space $M_{1,3}$, we construct the full *space-time algebra* C_{13} ”. That's exactly what we've done, that sentence needs no more explanation because we have spent two lessons on explaining it or more than that, we have appended, that's the way this worked, they created this product and they said well to be an algebra it's going to have to be bigger, it's going to have to be bigger and the question is, there is some issue like if you do that and you create this mathematical product the space-time product, I mean in principle it could have been that, you take A and you take B and you produce something of a different nature D and then you take A then you multiply by D and then you get something completely different again and then this process never ends right and you can never close the algebra I mean that could have happened that appending things wouldn't have generated anything at all but in this case it does and that's the way math as a natural subject constrains itself, you find these things that actually close, well if it closed it's a thing it must be interesting and if it doesn't close and it's some infinite thing well you have to ask yourself can you get your head around this infinite thing and it's just this really interesting way that mathematicians explore the universe they just look for this stuff and they noticed, look that we appended and we created an algebra which was closed”.

This notation indicates that the space-time algebra is a *Clifford* algebra generated from the metric signature $(+,-,-,-)$ ”. What's important about that is what I said before is once you build this algebra this guy here is a center stage in my opinion, it's center stage. I'm sure that in some abstract view anything here could be considered as the center stage in fact usually they talk about once you decide on the pseudo-scalar that's what you need because the pseudo-scalar decides the handedness for the whole algebra but to me this metric is the most important thing and they're saying this particular grade one vector space comes with the metric that tells us this is a space-time algebra it has to be that metric could have been other metrics here and you would have got another mathematical objects that would have been interesting for other reasons but for our reasoning, you need this metric signature and the fact that it's a Clifford algebra is, I think this particular product, the product of the algebra is what makes this thing a Clifford algebra but regardless they're calling it a Clifford algebra. “Importantly, all components in this Clifford algebra are purely *real* – we will not need any *ad hoc* addition of the complex scalar field in what follows”.

This is a bit of a dig so first of all when they say “all components in the Clifford algebra are purely *real”*, it's important to understand what they mean there because components if you read this loosely you might think, all the parts that make up the Clifford algebra but I'm quite sure that's not what they mean, what they mean of course are the literal components, if you have a multi-vector M it's got some scalar S and then it's got some vector $A^{\mu}\gamma_{\mu}$ and then it's got some bi-vector $½F^{\alpha\beta}\gamma_{\alpha}\wedge\gamma_{\beta}$. I think I'm going to start throwing these over twos here until I know differently. What they're saying is that these numbers $S, A^{\mu}, F^{\alpha\beta}$, those are all elements of the reals, all of the components of each of the grades that make up any multi-vector those components are real numbers. Now the reason this is a dig is because you certainly can have vector spaces with complex components, Quantum mechanics lives and breeds on vector spaces where these numbers here would be complex and there's a whole different architecture and how it's done but the point is complex vector spaces are completely legit, completely fine and they're used to great power in branches of Physics but what they say is, no, no, no complex scalar fields those are ad hoc and when you see ad hoc when you see someone calling something ad hoc, that is a dig, that is an unambiguous dig which is fine, I mean, that's sort of the whole point but what they're saying is all of a sudden you have a theory that suddenly has complex numbers in it and in order for your theory to work you've got to have complex numbers and complex numbers are bad for what reason? Not completely clear but what is clear and I'm going to restructure this I'm now I'm now putting words in the mouth of the advocates for Geometric algebra so I need to be cautious there is a certain notion of let's keep it simple stupid and if we keep it simple, well certainly we think of reals is simpler than the complex numbers if for no other reason that the dimensionality of the reals is one and the dimensionality of the complex numbers is two.

If I can build up an entire algebra out of one-dimensional objects and have all the benefits of having the thing constructed out of two-dimensional objects well shouldn't I do that and I think that's probably what they're getting at here is you don't need complex numbers if you use Geometric algebra so any use of complex numbers is ad hoc relative to something more fundamental that and that's a completely real algebra and when I say a real algebra we're talking about real components so that takes care of this little paragraph and moving on. “The repeated wedge products produce five linearly independent sub-spaces of the total algebra known as *grades* which are Illustrated in figure two”. “Each grade is a distinct type of *directed number*”. I'm not going to expose figure two quite yet because I want you to see how this paragraph already matches our own but I'll call version of figure two, which is this thing here.



One, two, three, four, five grades just like the same. The real scalars, the pure numbers are grade zero while the four vectors (line segments) are grade one so notice that they're throwing in the name of the grade, they're identifying the grade, they're giving it a name and then they're giving it an interpretation so four vectors, the name, line segments, the interpretation in $\mathsf{M}_{1,3}$ are grade one, bi-vectors, plane segments, here's the symbolism are grade two $a \wedge b$.

Successive wedge products produce tri-vectors which are pseudo-vectors or three volume segments they're little volumes of grade three $a \wedge b \wedge c$ and quad-vectors which are to be known as pseudo-scalars which are four volume segments and they're expressed this way $a \wedge b \wedge c \wedge d$ and they're grade four and that completes the algebra we refer to the elements of grade k -sub-space as k -blades I talked about that in the last class and what follows to disambiguate them from grade one vectors, they don't want to use the word vector all over the place although they probably will continue to use bi-vector but they might be using two blades in this paper. I can't really remember but the point is using the word vector all over the place becomes a bit of a mess, we want the vector to just represent these guys here four vector, line segments, members of $\mathsf{M}_{1,3}$ so we're going to get rid of the word bi-vector, tri-vector, quad-vector and we're gonna call them all two blades, three blades, four blades, even though they probably should be two vectors, three vectors and four vectors. The more I think about it the more I like they're used to the word blade here. Moving on.

“For concreteness we systematically generate a complete graded basis for C_{13} as all independent products of the vectors γ_μ from $\mu = 0 \ldots 3$ in a chosen basis of $\mathsf{M}_{1,3}$ ”. Very rich and important concept right here, “for concreteness”, I think that is a euphemism for we're going to use basis vectors we're not going to do our entire work writing things like $u \wedge w$, we're going to do things using a chosen basis set and I suspect that's what they mean by concreteness so whether or not doing it in a certain basis is concrete or not is, I think it's fair to say that if you do things with the basis vectors you're fixing yourself to some particular frame of reference but what's interesting is when you do this we're not specifying much particularity what this basis actually is, where in the world it is, what it's modeling. Obviously if we switch bases we're going to switch to some prime basis from primed to unprimed but the point is if we do that the only thing that's meaningful is the relationship between the primed and the unprimed basis so like this relationship becomes what's important (3) but the point is we're going to do everything in basis vectors is what they're saying here.

$$\gamma_0, \gamma_1, \gamma_2, \gamma_3 \rightarrow \gamma'_0, \gamma'_1, \gamma'_2, \gamma'_3 \tag{3}$$

We're going to generate a complete graded basis, notice they use the word graded basis and what they're trying to say there using our own charts is well these (grade 1) are the basis vectors for $M_{1,3}$ but look, the basis vectors for this section (grade 2) are these bi-vectors so you take these basis vectors form all the bi vectors or all the linearly independent bi-vectors out of them and that becomes the basis for grade two, you form the tri-vectors so this is in a different grade, you have these grade one basis vectors grade two basis vectors grade three and grade four basis factors that's what they mean by the notion of a graded basis so that's pretty simple and they're all the independent products of the vectors γ so that in a chosen basis of $M_{1,3}$, again showing that this is our fundamental space here this $M_{1,3}$.

What's important is this notion "independent products of the vectors $\{\gamma_\mu\}_{\mu=0}^{3}$ " so when they say products the only thing they can mean is space-time product so how are we to understand this? Let's let's take a moment and step aside well actually, let's finish the paragraph and then we'll come back to this idea of the independent product. "We choose the starting vector basis to be orthonormal $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$ " so we've already done this so orthonormal "in the sense of Minkowski metric, so $\gamma_0^2 = 1$ and $\gamma_j^2 = -1$ for $j = 1, 2, 3$ ". Let's take a look at that because when we write γ_0^2 we're talking about the space-time product, not the Minkowski dot product, there's a good chance that if you read this and you just jumped in the middle of this paper you would somehow think that this statement here is meant to mean $\gamma_0 \cdot \gamma_0$ but it's not, it's meant to mean $\gamma_0 \gamma_0$ where this is a space-time product and $\gamma_0 \cdot \gamma_0$ is just the grade zero part of a space-time product or the fully symmetric piece of the space-time product so we have to clear up exactly what they mean there, these independent products and then they finally say at the end "the choice of notation of the basis is motivated by a deep connection to the Dirac γ matrices that will clarify in Section 3.8". As I said that they're choosing γ because of this Dirac γ matrix connection which is so interesting. Let's unpack this a little bit so we start with the space-time product of two of our basis vectors of $M_{1,3}$.

$$\gamma_\mu \gamma_\nu = \gamma_\mu \cdot \gamma_\nu + \gamma_\mu \wedge \gamma_\nu \tag{4}$$

We know the rule for that, when you're multiplying space-time product two vectors, two members of $M_{1,3}$, two four vectors, when we do that space-time product this is the rule we have this dot product part and this wedge product part (4), we know the first is a scalar then the second is a bi-vector now γ_μ and γ_ν are normal vectors, they're basis vectors, sure but they are elements of $M_{1,3}$ and they obey this rule (4) like any two other vectors in $M_{1,3}$, just like when we write:

$$a b = a \cdot b + a \wedge b \tag{5}$$

That's the first thing to understand is, just because their basis vectors doesn't mean they don't apply the rules, the point is is that when they are basis vectors the rules get a little bit simpler because we have already chosen the fact that the dot product between γ_{μ} and γ_{ν} is already known to be:

$$\gamma_{\mu}\cdot\gamma_{\nu}=\eta_{\mu\nu} \tag{6}$$

Which means:

$$\gamma_0\cdot\gamma_0=+1 \;\;,\;\; \gamma_1\cdot\gamma_1=-1 \;\;,\;\; \gamma_2\cdot\gamma_2=-1 \;\;,\;\; \gamma_3\cdot\gamma_3=-1 \tag{7}$$

We know what these dot products are, we also know that the space-time product of γ_0 with itself has got to be a real number so we know that $\gamma_{\mu}\wedge\gamma_{\nu}=0$ when $\mu=\nu$, we've already demonstrated this and that's true for all of these $\gamma_0^2=\gamma_0\gamma_0=+1$, $\gamma_1^2=\gamma_1\gamma_1=-1$, $\gamma_2^2=\gamma_2\gamma_2=-1$ and $\gamma_3^2=\gamma_3\gamma_3=-1$, where now we're talking about these squares as the space-time product squares and this affords us the ability to simplify things. The first thing we can say when we're dealing with these orthogonal basis vectors, the space-time product of well this is the space time square of any one of these basis vectors is either 1 or -1 depending on whether it's a space-like basis vector or if it's time-like spaces vector, zero, I'll try to call it the time-like basis vector and the others are space-like basis vectors so right away we know that this rule simplifies greatly for the case where $\mu=\nu$ but it also simplifies for the case when $\mu\neq\nu$ because in that case when when we have $\gamma_{\mu}\gamma_{\nu}$ where $\mu\neq\nu$, that is, well the dot product is zero which means $\gamma_{\mu}\cdot\gamma_{\nu}$ goes away and $\gamma_{\mu}\wedge\gamma_{\nu}$ is the only surviving term so we know we can write $\gamma_0\gamma_1$, we know that that's just a shorthand way now of writing $\gamma_0\wedge\gamma_1$:

$$\begin{array}{c}\gamma_0\gamma_1=\gamma_0\wedge\gamma_1 \;\;,\;\; \gamma_0\gamma_2=\gamma_0\wedge\gamma_2 \;\;,\;\; \gamma_0\gamma_3=\gamma_0\wedge\gamma_3\\ \gamma_1\gamma_2=\gamma_1\wedge\gamma_2 \;\;,\;\; \gamma_1\gamma_3=\gamma_1\wedge\gamma_3 \;\;,\;\; \gamma_2\gamma_3=\gamma_2\wedge\gamma_3\end{array} \tag{8}$$

We get all of these bi-vectors just by multiplying the basis vectors as a space-time product so this becomes a shorthand when we want to talk about the bi vector $\{0,1\}$, we just write the space-time product of γ_0 and γ_1 . We want to write the bi-vector basis vector $\gamma_1\wedge\gamma_3$, we just write $\gamma_1\gamma_3$ because we know that we have a particularly simple circumstance where this dot product term goes away because of the metric we have chosen because of this metric $\eta_{\mu\nu}$, this simplification is possible so the two simplifications are these bi-vectors are all written as space-time products and these squares immediately reduce to real numbers based on the metric, based on whether it's space-like or time-like and all of the basis vectors can be formulated this way: $\gamma_0\gamma_1\gamma_2=\gamma_0\wedge\gamma_1\wedge\gamma_2$, this triple space-time product and that can be easily seen because we know this is associative so I write:

$$\gamma_0\gamma_1\gamma_2=\gamma_0(\gamma_1\gamma_2)=\gamma_0(\gamma_1\wedge\gamma_2) \tag{9}$$

This is a vector space-time product with a bi-vector but we know how to take the space-time product of a vector with a bi-vector, remember the rule? This is the one where this vector gets projected into the plane of this bi-vector and then rotated 90º, that's the vector part because this whole thing has to break into a vector part and a tri-vector part, well the vector part is γ_0 projection into the plane of γ_1 , γ_2 in two different ways but γ_0 doesn't have a projection in the γ_1 , γ_2 plane, it's orthogonal to that little piece of plane, it's a different basis vector, in other words:

$$\gamma_0\gamma_1\gamma_2=(\gamma_0\cdot\gamma_2)\gamma_1-(\gamma_0\cdot\gamma_1)\gamma_2 \tag{10}$$

Well $\gamma_0\cdot\gamma_1=0$ and $\gamma_0\cdot\gamma_2=0$ so the vector part of this product is zero which just leaves the tri-vector part $\gamma_0\wedge\gamma_1\wedge\gamma_2$ so this is now shorthand for $\gamma_0\gamma_1\gamma_2$ and likewise for the pseudo-scalar $\gamma_0\gamma_1\gamma_2\gamma_3$ is the shorthand for the pseudo-scalar $\gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3$. Now we've made a choice about the pseudo-scalar because there's only one pseudo-scalar out there and so by listing it time-like on the left and then one, two, three, we've chosen essentially the handedness because remember if you interchange any two the sign of this should change so we'll talk about that a little bit more when we talk about the pseudo-scalar but the point is these space-time products are now going to be these blades, these two blades, three blades or four blades or one blades or zero blades, these space-time products is a very simple way of writing them down as long as you're using the basis vectors of the Clifford algebra.

Continuing with their writing, "the basis of zero blades is the real number 1. The basis of one blades is the chosen set of four orthonormal vectors γ_μ themselves. The $(16-4)=12$ possible products of those vectors that produce bi-vectors (i.e., with $\mu\neq\nu$) $\gamma_\mu\gamma_\nu=\gamma_\mu\wedge\gamma_\nu\equiv\gamma_{\mu\nu}=-\gamma_{\nu\mu}$ ", then the bi-vector basis is $\gamma_\mu\gamma_\nu$ when $\mu\neq\nu$ and they point out that there's only six of those and then notice how they do the notation here now they just introduce a notation well I guess they introduce it right here with this triple equal sign. The triple equal sign usually reads, in case you're unfamiliar with it, and this is my own interpretation, triple equal means *is defined to be* so what we're saying here is that the space-time product $\gamma_\mu\gamma_\nu$ well that equals $\gamma_\mu\wedge\gamma_\nu$ and that equality is a real equality in the sense that it's a calculation, you have to know that $\gamma_\mu\cdot\gamma_\nu=0$ if $\mu\neq\nu$, with that fact, that Orthonormality fact in place you can write this down but this is too much writing. The paper says, let's just go right down to the brass tags $\gamma_{\mu\nu}$, let's just give it two sub-scripts and declare it to be anti-symmetric in those indices and call it good so now when we go back to the work that we had done over here this could be written, well let's start with the bi-vectors, this could just be written as:

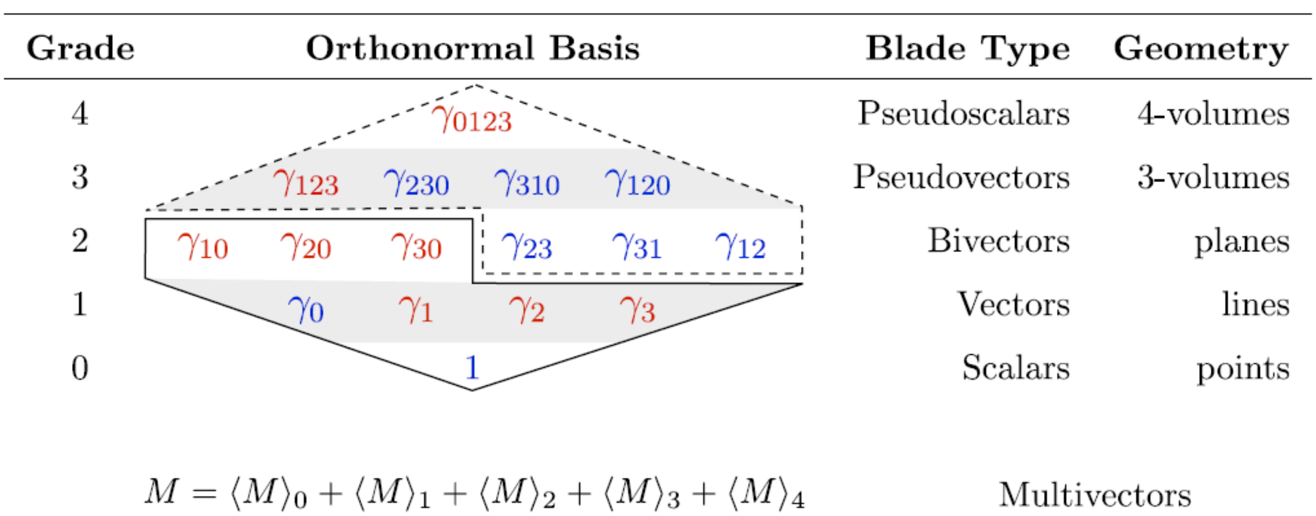
$$\begin{aligned}&\gamma_0\gamma_1=\gamma_0\wedge\gamma_1=\gamma_{01}\quad,\quad\gamma_0\gamma_2=\gamma_0\wedge\gamma_2=\gamma_{02}\quad,\quad\gamma_0\gamma_3=\gamma_0\wedge\gamma_3=\gamma_{03}\\&\gamma_1\gamma_2=\gamma_1\wedge\gamma_2=\gamma_{12}\quad,\quad\gamma_1\gamma_3=\gamma_1\wedge\gamma_3=\gamma_{13}\quad,\quad\gamma_2\gamma_3=\gamma_2\wedge\gamma_3=\gamma_{23}\end{aligned} \tag{11}$$

This guy here could be $\gamma_0\gamma_1\gamma_2=\gamma_{012}$ and $\gamma_0\gamma_1\gamma_2\gamma_3=\gamma_{0123}$, that's what that is so we're really compressing the notation but in order to do this we have to lean on the fact that these guys are basis vectors in $M_{1,3}$ in our Minkowski space and they satisfy the Minkowski metric otherwise you can't do this so this is one of the strengths of choosing a basis and doing everything with basis vectors is you can write everything in terms of space-time products and you're still going to be correct because $\gamma_1\gamma_3$ is a two blade, it is not a multi-vector with a scalar part because of the nature of γ_0 , γ_1 , γ_2 and γ_3 .

Continuing on, they give us this notation right this excellent notation which we're now going to use due to the anti-symmetry of the wedge product, well that's just this last piece "these independent elements form the oriented basis of two blades". It's oriented of course, because if we chose since we're choosing this to have, we're choosing γ well, I guess what they've done here is these bi-vectors are independent. Notice how those are backwards from the ones I chose, I just noticed that so I went with γ_{01} . They're saying nope, our basis vector is γ_{10} so I went with γ_{23} and they also went with γ_{23} and I went with γ_{13} but they went with γ_{31} so their bi-vectors basis is not the same as the one that I've been laying down and the reason I lay it down my way is I'm used to this notion of increasing indices where the indices must go up for the basis vectors but they don't and we'll flesh out the reason for that a little bit later but it doesn't really matter as long as you choose it you're good to go so they've chosen these to be the basis vector.

“The product of orthogonal vectors with bi-vectors $\gamma_\mu \gamma_\nu \gamma_\delta = \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\delta = \gamma_{\mu\nu\delta}$ ”, meaning this triple product, well I've already done that and they went with γ_{123} , γ_{120} , γ_{230} , γ_{310} . Notice these are not the same that I would have chosen, I would have gone with, I would have agreed with γ_{123} and I would have disagreed with γ_{120} and I would have gone with γ_{012} which by the way isn't even a sign flip difference, that's just yeah that's because zero moves over twice so that so it's really the same, it's the same, it doesn't even have a sign flip difference, γ_{230} does not have a sign flip difference either but I think γ_{310} does have a sign flip difference. The paper is choosing the basis vectors differently but as I said inevitably there's going to be a good reason for that but they form the oriented basis of three blades and here actually it is truly a blade, the basis vectors are blades under every definition of the word blade because every basis vector is clearly written as $a \wedge b \wedge c$ so this would be $\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\delta$ and that's clearly how blades are defined.

“Finally the product of all four orthogonal vector produces only a single independent element and that's this guy and it's only a single independent element $\gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \equiv \gamma_{0123}$ that serves as the basis for the four blades”, meaning it's the basis of the quad-vectors or four blades as the paper calls it it's defined this way it has no scalar or well I guess in this case it would have, if you multiply it by a bi-vector you would expect a bi-vector and quad-vector, if you multiplied it by a tri-vector you would expect a vector and another tri-vector and if you multiplied it by a single vector you would expect a tri-vector and if you multiplied it by itself you would expect a scalar so I'm not sure why I said that, it is irrelevant, the point is it's one-dimensional is what they're trying to say and they choose the classic γ_{0123} . You're gonna see why they made these other choices in a moment well eventually, “that serves as the basis for the four blades. Hence, there are $2^4 = 16$ independent basis elements for the space-time algebra partitioned into five grades of *4 choose k*”. this is just a way of counting how many are in each grade independent basis k -blades.



Now let's go to their figure and now we know all of the things we need to know to understand their figure 2. Now I want you to appreciate how much beautifully simple figure 2 is compared to my thing and the reason I've done it this way is, this is the under the hood stuff and I want you to appreciate the value of the notation, the notation they've chosen really compresses everything that I've written before makes it much sweeter and more convenient to see and they're not even done, this is their first step it's even going to get simpler than this but my sense is understanding Geometric algebra one of the problems which is typical for all studying of new mathematics is the notation itself can become a little bit bizarre if it's made too convenient too quick because you lose connectivity to the core nature of things that's why I kind of harped on the plus now this may be a totally a me problem this may be my hang up and I'm now forcing it on you but the truth is if you get too simple too fast while you're learning something new I guess in some way it gets you to the mechanics of things quicker and some place people would argue well you're actually concealing all the complexity which is good for the new learner, I just don't work that way but of course on the other hand I'm very slow to learn so I don't know anyway, here we are but I hopefully if everything's gone according to plan this now looks like a very beautiful simplification of things each grade is now described in this vertical axis the bottom has one dimension, it's the scalars of grade one the first rung of our grade ladder is the key basis vectors they're basis vectors so we got a name, we got a geometric interpretation.

Then we see we have all of the bi-vectors, now you'll notice they've split the bi-vectors into two pieces here which will they didn't mention yet but they will in a second so we'll talk about that and then you have the tri-vectors are here. Now notice they have committed to this ordering as their basis vectors so their basis factors are $\gamma_1 \wedge \gamma_2 \wedge \gamma_3$, $\gamma_2 \wedge \gamma_3 \wedge \gamma_0$, $\gamma_3 \wedge \gamma_1 \wedge \gamma_0$, $\gamma_1 \wedge \gamma_2 \wedge \gamma_0$. You'll notice the zero is on the end when they can do it, when there is a zero so that is something to take note of and we'll see why notice here (grade two) the γ_0 is on the right also so they've committed to the basis for the grade two part, they've committed to the basis $\gamma_1 \wedge \gamma_0$ and $\gamma_3 \wedge \gamma_0$ for example so they've committed to that which is very different than my picture where I committed to a more natural basis if you're coming from the subject of Exterior algebra where this is very typical, you always have this increasing index but they've chosen to give that up here and they have a reason for it and you'll see that reason later.

Again it doesn't really matter, I mean it's well it matters in the sense that it creates a better notation architecture an Arithmetic architecture, Algebraic architecture I suppose is the way to say it and you can see where this is coming from. You've already seen right here that they split the basis vectors of the basis bi-vectors into two parts, γ_{23} , γ_{31} , γ_{12} is purely space-like, meaning all the indices here are greater than zero, these guys γ_{10} , γ_{20} , γ_{30} have a time component so I'll just call them time-like until the paper changes the language if it does and the distinction, they colored the time-like red and they colored the space-like blue and you can even see that here in the basis vectors as well the time-like one is blue and the space-like one are red, they've added this color dimension to things and they've given this one a blue appearance so understanding that color dimension of things is actually pretty interesting also the tri-vectors are colored blue for time-like, if they have the zero in it so what's interesting is the time-like tri-vectors are the majority of the basis vectors for grade three but the time-like vectors are the minority for the basis vectors of grade one.

There's duality between these things there's a duality we're going to learn about between vectors and tri-vectors the bi-vectors are self-dual so the duality look like this and there's a duality between the scalars and the pseudo-scalars that we'll all learn about and they take advantage of all of this duality in this study and then of course they use this notation which is universal a general multi-vector is the sum of parts of each of the different grades and now we understand how sums work so those are the multi-vectors. Let's read this text (of the figure). I'm trying to make the text as big as I can because I know it's going to be really hard to read so I'll just read it out loud though the “graded basis for the space-time algebra C_{13} ”, so they're talking about this as the graded basis right because each basis vector lives in its own grade but as a whole these guys here are the basis vectors of the vector space of C_{13} . “Each multi-vector decomposes into a sum of distinct and independent grades”, we understand that, that's what this line here says, this square right here “which can be extracted as grade projections $\langle M \rangle_k$ ” so each of these pieces, they're calling it a grade projection, a grade projection is the word they're using. You can almost think of those angle brackets with the sub-scripted number as an operator that projects from M onto the grade two basis for example.

“The oriented basis elements of this of grade one $\{\gamma_\mu\}_{\mu=0}^{3}$ ” that's these guys γ_0 , γ_1 , γ_2 , γ_3 these basis vectors “are an orthonormal basis $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$ ” and that's so they emphasize the Minkowski nature of those things “for the Minkowski four vectors of $\mathsf{M}_{1,3}$ ”. We've talked about that a lot. “An oriented basis element of grade k such as $\gamma_{\mu\nu} \equiv \gamma_\mu \gamma_\nu = \gamma_\mu \wedge \gamma_\nu = -\gamma_\nu \gamma_\mu$ with $\mu \neq \nu$ and they show this is defined as γ_μ space-time product γ_ν which because of the Minkowski nature is just this bi-vector, so this is synonymous with this bi-vector but it was only synonymous because the space-time product due to the orthogonality reduces to this bi-vector it's a nice convenience here. They talked about the anti-symmetry “is constructed as the product of k of these orthonormal four vectors”.

“Interchanging indices permutes the wedge products which only changes the sign of the basis element; hence, only the independent basis elements of each grade are shown”. Fair enough we understand that obviously for example γ_{32} is not independent of γ_{23} so it's not on this list. “The color coding indicates the signature of each element with blue being a $+1$ and red being -1 . Remember the idea is if you take a space-time product of a vector with itself γ_0^2 , you end up with $\varepsilon|\eta(\gamma_0, \gamma_0)|$, a signature times the absolute value, like that. This ε is the signature. The magnitudes of these guys … that's a good point, I don't think they mentioned that yet, how to calculate the magnitude of all of these other things, like what's the magnitude of $a \wedge b$, I don't think they've defined that principle yet but there is such a thing as the magnitude of $a \wedge b$ and that magnitude there will be a signature in front of that and so you can anticipate that these signatures are going to be blue is positive and red is negative.

It looks pretty clear to me that the this figure is a little bit ahead of … or the caption of the figure is a little bit ahead of the text so where the text refers to the figure but right here if you read the caption the caption is ahead of where these references are which is fine I mean I like captions that have a lot of explanation in it because I know I can come back to it and it's a little bit of foreshadowing “the color coding indicates the signature the basis elements” so I know gee I better understand how magnitudes are calculated, “the boxes and shading indicate useful duality of the algebra”, so they're saying the boxes and the shading so I didn't notice the boxes so what they're saying is there's actually a variety of duality here and one set of the duality is the box duality so all of these guys have a duality with all of those guys but in addition to that there's also duality between the shaded stuff and that's what I was saying before the two shaded things are dual and this one in the middle is self-dual so there's two kinds of duality depicted in that picture, this dotted line, the solid line duality and the shaded to shaded and the non-shaded to non-shaded.

Now the notion of being dual is really an interesting mathematical idea there's a lot of different places where this word duality comes in Math in Exterior algebra it's Hodge duality so they say “the solid and dashed boxes are (Hodge) dual under right multiplication of the pseudo-scalar” and they're going to talk about that in section 3.5 which is well ahead of us, well within each box the shaded region is dual to the non-shaded region. “Within each box the shaded region is dual to the non-shaded region under right multiplication by the time-like basis vector γ_0 ”, so basically in this caption they've really defined all of the duality and they've also pointed out that I mislabeled the duality when I made it simpler that is when they say these guys (upper part of figure) and these guys (lower part of figure) they're related by Hodge duality where if I take an element out of this and I call it a and I do right multiplication by the pseudo-scalar I guess that would be with γ_{0123} then I turn it into something in this box down here and that's duality by right multiplication by the pseudo-scalar and then they're talking about another form of duality within each box, it's by multiplication by γ_0 so if I take these guys γ_1 , γ_2 γ_3 , multiply on the right by γ_0 I get these guys γ_{10} , γ_{20} , γ_{30} so that's the duality, it goes like that. Obviously if I take γ_0 and multiply on the right by γ_0 I get 1 so I have that duality there. Likewise up here if I multiply these guys on the right γ_{23} , γ_{31} , γ_{12} by γ_0 , I get these guys γ_{230} , γ_{310} , γ_{120} and If I multiply this guy γ_{123} on the right by γ_0 then wouldn't I get $-\gamma_{0123}$?

We'll flush this out more when we get to the appropriate section but there's a little foreshadowing to be done here and that's fine by me we're not we're not going to explain these duality right now but we're going to allude to them to get your mind primed for when we actually study them. Anyway great picture and the point is what you should be taking away right now is understanding how these this notation is constructed, how it relates to the wedge product and how it relates to the basis structure that we outlined in our chart and then you forget this and you live entirely in their world in the world of the paper which is a much simpler and more pleasant place and it's designed so that these dualities are easy to see and you'll see how important they are as we go through this paper.

Where are we? We're at this paragraph 3.3 and they're about to introduce the importance of the pseudo-scalar and a little more notation and I think that can wait for next time so we've made some progress in our paper and we're just going to keep plugging forward and try to capture what these dualities are all about and how their notation works and we'll carry on next time.