[Geometric Algebra 5: Multivectors]

Welcome back we are going to continue our study of the space-time algebra and in particular I want to say we're really going to continue our study of the Geometric algebra and we're using space-time algebra as our favorite exemplar first of all because it's probably the most physically relevant although the Geometric algebra of regular three dimensions is also pretty straightforward and interesting, it's a lot easier and from time to time we'll probably use some examples in the Euclidean three-dimensional space algebra I guess we would call it but we're following this paper and this paper is all about the space-time algebra and so we have gotten through most of the space-time product and let us review where we're at and then push into multi-vectors today.

$$
\begin{aligned}
a(bc)&=(ab)c \qquad (\text{Associativity})\\
a(b+c)&=ab+ac \qquad (\text{Left Distributivity})\\
(b+c)a&=ba+ca \qquad (\text{Right Distributivity})\\
a^2&=\eta(a,a)=\epsilon_a|a|^2 \qquad (\text{Contraction})
\end{aligned} \tag{1}
$$

We definitely did the axioms of the space-time algebra which apply just to members of $M_{1,3}$, these are the vectors so all of this stuff just applies to the notion of vectors which of course in space-time algebra are four vectors if you speak about Geometric algebra in general then these axioms apply just to the vectors and the these are like the base space and when we think of our four-dimensional space-time we're thinking of the four basis vectors in this base space and these axioms only apply to those four and then we immediately saw that this last axiom made us expand our algebra to include at least the real numbers because the space-time product of any vector with itself was the Minkowski contraction and we realized that right here this is where we force special relativity into our problem by saying that the space-time product of any vector with itself is equal to the Minkowski contraction of a vector with itself that's right in there that's the metric is literally built into the algebra right there in that statement and if we were doing Euclidean algebra if we were using the just the Euclidean 3D then this wouldn't be wouldn't be present it would be a different statement it would be let's see it would be much simpler it would be just this Pythagorean multiplication $a^2=a_1^2+a_2^2+a_3^2$ it's just the sum of the squares of the components of each of the three dimensions would be what the magnitude squared is and you wouldn't have to worry about ϵ_a because this would always be positive, you have a positive definite metric.

Our algebra works fine with a pseudo metric or a non-positive, an indefinite metric so we understood that and then we moved on and then we actually started beginning some definitions we said the space-time product of any two vectors it has a symmetric part and an anti-symmetric part which is just an algebraic statement, we only think we're not assuming is commutativity, of course:

$$ab=a\cdot b+a\wedge b \tag{2}$$

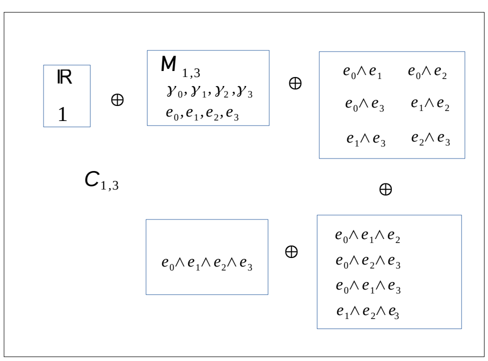
$$a\cdot b=½(ab+ba)=b\cdot a=\eta(a,b) \tag{3}$$

$$a\wedge b=½(ab-ba)=-b\wedge a \tag{4}$$

We define the symmetric part to be $a\cdot b$ and we define the anti-symmetric part to be $a\wedge b$.

Now those are just symbols that attach themselves to the symmetric and anti-symmetric part and then we demonstrated that this $a \cdot b$ necessarily because of this earlier axiom (last line of (1)) must end up being the Minkowski contraction of a and b which is why we chose the dot there because it's like the dot product of these two vectors and then down here it wasn't quite as clear we had a symbol to represent the anti-symmetric part but we need something that is truly anti-symmetric and involves two vectors and we basically went back to our experience with other branches of mathematics namely the theory of the Exterior calculus and we said we there's something out there that is a geometrically interpretable thing and it satisfies this rule beautifully and when we examined that we decided to interpret this anti-symmetric part of the space-time product as this circulating area that has no particular shape but is defined as the vector a swept across the vector b and that would define a little region in the plane but if you take this too literally you end up with this parallelogram shape but in principle, as the authors point out in this picture, the shape is arbitrary in fact the shape is not even arbitrary it's not it's just there is no shape it's just an area and with that interpretation we have a geometric interpretation of the symmetric part which is the magnitude of these two vectors.

In other words if you believe in the geometric interpretation of the vector as a little pointy thing in space in our case four-dimensional Minkowski space then the symmetric part gets interpreted as the Minkowski contraction, the anti-symmetric card gets interpreted as an element of oriented plane with a circulation, the orientation of this thing there's two parts of it a and b have components so in principle you would understand that this is sitting out in space and has certain alignment but it also has a circulation so there's two kinds of orientation usually when we talk about the orientation though you're talking about the circulation direction and that's the wedge product so now we can break up the space-time product between two vectors and that's really critical the only thing we know is how to take the space-time product between two vectors and then we added real numbers to our algebra and now we need to add these plane pieces because remember the space-time product of any two things in the algebra this (2) has to be part of the algebra, that's a real number we've already added that in there but these bi-vectors these plane pieces called bi-vectors we had to add that in as well so now our Clifford algebra C_{13} we've added the real numbers, we have our vectors which in the paper we're going to call these basis vectors for $M_{1,3}$, $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ and we understand them to be four vectors of Special relativity but we'll mix that between e_0, e_1, e_2, e_3 and I'll make the transition soon enough.



Then we add our bi-vector space and we call this the bi-vector space and it is written as $\Lambda_2(M_{1,3})$, that's all of the bi-vectors where the vectors come from $M_{1,3}$ this is notation that comes from the Exterior calculus so that's this next vector space that we add, by the way, I might have mentioned in a previous lesson that each of these was a sub-algebra of C_{13} and that's not correct, C_{13} is a vector space as all algebras are and these are definitely sub spaces of that vector space.

This is a vector space which is a sub space of the vector space C_{13} but is not a sub-algebra because there's no product to find that stays closed inside this obviously the space-time product between this vector space completely leaves the vector space so if I said any of these are sub-algebras that's incorrect I think a viewer called me out on that in the right, I knew that but it's so tempting to say oh this is the algebra the Geometric algebra these are sub-algebras that's incorrect, these are vector spaces in their own right the only algebra in sight is this complete Clifford algebra, thank you to whoever pointed that out and I've done a little bit of foreshadowing here, just as we've added the bi-vectors in this lesson we're going to understand that we add the tri-vectors and the quad-vector and there will be four tri-vectors and one quad-vector and those will complete the entire algebra of C_{13} and each of these are closed vector spaces in and of themselves. This is a little foreshadowing and that will complete all of these spaces.

This quad-vector we're going to learn is also called the pseudo scalar of the algebra and that is very important, the idea that there's a pseudo scalar because it's one dimensional so it's very similar to $\mathbb{R}$, this is one dimensional too so this is the regular scalars and this will be the pseudo scalars these will end up being the regular vectors these will end up being the pseudo-vectors remember how we talked about the distinction between vectors and pseudo-vectors is slapstick and arbitrary, what's the right word? The correct word for that in science is ad hoc, the notion of vectors and pseudo-vectors in the previous formalism it's an ad hoc addition to the formalism but this is natural in the formalism you don't have to track the distinction between two things that have geometrically the same idea, two vectors and say oh remember this one though is a pseudo-vector and this one is a regular vector now you have a completely different geometric object which will be an oriented three-dimensional volume and this will be an oriented four-dimensional volume and it is our pseudo scalar so now you have two different types of scalars that are completely distinguished geometrically and they behave exactly like the ad hoc versions of the of the scalars and pseudo-scalars in the previous formalism so this is one of the reasons why people love this formalism so much.

Let's see and notice of course there are six dimensions to this bi-vector space and any oriented plane area can be broken down in these components and the way that would look in component form is $F^{\alpha\beta} e_\alpha \wedge e_\beta$. Now there is an issue here when you look at this, you're double summing but if F is anti-symmetric, that's okay so F will end up having to be anti-symmetric because you're double summing over things that are not linearly independent because for example $e_0 \wedge e_2 = -e_2 \wedge e_0$ so if you took this sum literally you are double counting every basis vector but if this is anti-symmetric then you'll compensate for that negative sign so that's why these components inherit anti-symmetry from the anti-symmetric nature of the wedge product. That's a little bit of review to this point and a little bit of foreshadowing so let's move on.

“The vector product combines the non-associative dot $*$ and wedge $\wedge$ products into a single *associative* product” so we haven't done these two paragraphs yet now that paragraph to me that statement is a little bit weird because the dot product between two vectors $a \cdot b$. This dot product is a scalar so I can't take another dot product because you can't take a dot product with with a scalar and so you can't really even write down, you can't really create an association $(a \cdot b) \cdot c$ so it is truly non-associative in the sense that it's not logical to talk about associativity when you have an operator that only works in a binary way so sure I guess it's the dot product is non-associative, the problem really is with the wedge product because it's also true that $a \wedge b$ if you restrict it to just two vectors then $a \wedge b$ is associative but you can do $a \wedge b \wedge c$, that is a thing that exists and but that is in fact associative $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ which is exactly why you never see parentheses around these things because it's totally associative so the wedge product is actually associative but so I think what they're getting at is if you look at this expression (2) they say well this guy is associative which means $(ab)c = a(bc)$ you can split that up, you can put those parentheses either places but individually $*$ and $\wedge$ products are not associative but when you combine them in this way (2) you get something that's associative and the authors found that interesting and pointed that out. “The result of the space-time product thus decomposes into the sum of a scalar and bi-vector parts which should be understood as analogous to expressing a complex number as the sum of distinct real and imaginary parts”.

This is the hand waving argument of why you can add a scalar to a bi-vector so that because that's what's going on here, we are adding a scalar to a bi-vector this is a scalar and that's a bi-vector so this is a real number like 7 and this is you know $3e_0\wedge e_1+2e_2\wedge e_3$, that's a bi-vector right there so this part is bi-vector and is understood as some oriented area in the plane and this this is just a straight up real number and yet we're adding them together which we're adding apples and oranges and we always learn you can't add apples and oranges but they're saying I know you've learned that but you've ignored it when you write a complex number and if you write $7+3i$ you'd have no trouble with that because you understand this complex number is best understood as one thing even though it's split into two parts and you never combine it into one part and for example $Ae^{i\theta}$, you can convert this rectangular form into this polar form and it looks like one thing but understand that's not one thing that's $A(\cos\theta+i\sin\theta)$. This is really is two things and the definition of this is still the sum of real and complex parts through the Taylor expansion so what they're trying to say is don't worry about adding these things together because you've already jumped into that well so just keep jumping.

We don't have to study this notion of this addition because you've already bought into it so let's just move on which is completely appropriate for a review paper that's saying look you know we're not here to answer every conceivable question you could ever ask in the world we're just here to to get you moving on this subject so don't worry about this addition, on the other hand, I'm very worried about the addition so I am going to do a section talking about that addition but this is a fine way to proceed just say hey look you can add these things together just treat it the same way you do a regular complex number and then they go and say “just as with the study of complex numbers, it will be advantageous to consider these distinct parts as composing a unified whole rather than separating them prematurely”.

Imagine you had a complex number $7+3i$ but all of your math demanded you work on just this part or just that part, you have the universal math for this and universal math for that, you wouldn't even be reproducing it, you would lose the ability to do complex analysis so we'll explore the similarity more thoroughly in section 3.5 so what they're anticipating there and it's an important fact thing that people who love Geometric algebra love to talk about is, we don't have, there's no imaginary number, this entire algebra is real, all the scalars are real, let me make that clear, all of these vector spaces, you can add vectors together and you can multiply any vector in a vector space by its appropriate scalar and what's important is that in all cases the bin of scalars that they're looking at are real numbers, they're just real numbers, you never see a complex vector space and this one is there's an extra bit of real numbers right there, you never see complex numbers show up anywhere in this study and what we're going to do is we're going to show that there are pieces of this algebra which act as the imaginary number and the pseudo scalar is probably the most important of all of them because this thing squared is real important but there are other pseudo-scalars buried in here so Quaternions and complex numbers are actually sub-algebras, they're literal in that case, they literally are sub-algebras of C_{13} and eventually we'll get to seeing why that is.

To finish off our last paragraph “a significant benefit of combining both the dot and wedge products into a single associative product in this fashion is that an *inverse* may be defined”:

$$a^{-1}=\frac{a}{a^2} \tag{5}$$

Now first about this inverse, notice it's an inverse for just the vectors so the only things that they're talking about here are the members of $\mathsf{M}_{1,3}$ those guys get an inverse and not even all of them get an inverse because this is a scalar down here this a^2 so if that's zero and in Minkowski space if a is literally a light-like vector, which they actually say here, then (5) isn't going to make any sense but other than that, every vector will have an inverse and it's pretty easy to see, if aa is the space-time product of a with itself is the number $\varepsilon|a|^2$ with this ε here then clearly if you define a new vector a^{-1} inverse in this way (5) this a^2 will be in the denominator and it'll all cancel and you'll get a 1 . This number a^2 will be in the denominator, this all cancel and you'll get a 1 . Clearly every vector has an inverse in the space-time product, it doesn't apply to the bi-vectors or the tri-vectors or other combinations yet this definition as they've described it shows that it just applies to the vectors of $\mathsf{M}_{1,3}$ and in fact overall it is not true, well we already showed it's not true but it's not true for the a general element of the Clifford algebra it has an inverse, that's not correct or a unique inverse at least.

“Importantly, neither the dot product nor the wedge product alone may be inverted; only their combination as the sum retains enough information to define an inverse”, and what they're saying there is if I have $a\cdot b$, I end up with a real number $\eta(a,b)$. Say that number was 7 from 7 I can never really go back and figure out what a and b were and that's also true of $a\wedge b$. We show that,

$$a\wedge(b+\varepsilon a)=a\wedge b+\varepsilon a\wedge a \tag{6}$$

I add to b some value of some other vector that's in the direction of a co-linear with a , I break that up by linearity and I get the right and side but $a\wedge a$ of course is zero because any vector wedged with itself is zero so if I'm sitting on $a\wedge b$, if I'm sitting on some area, some circulated oriented area I can't get back uniquely to the two vectors that produced it, there are many vectors that could have produced that that area so it's not invertible either and that's what they're getting at there.

With all of that, we're now ready to talk about multi-vectors so I thought a bit about how to approach this and I'm typically doing is I'm reading the text and I'm expounding on the text and filling in what I think are since we have infinite amount of time in these lessons, I'm filling in all of the gaps that I find in there the back story, if this was if this was some in some fictional world where there's a lot of backstory that's unexpressed I'm filling in the back story but in this case with multi-vectors what I'm going to do is I'm actually going to start with some backstory and then we're going to attack the text and the text will actually look very simple because these authors do quite a bit to simplify the space-time algebra and if you just read through their paper and you attacked it as they present it you would end up with this beautiful picture and I can't wait to get to it but it's a very simplified version of how things work in the space-time algebra and it's efficient and it's smooth, I want to demonstrate it the hard way a little bit so you can appreciate the simplicity that they give you I'm not going to dive too deeply into it but we'll start their section a little bit after we do some of our own work.

Let's pause the paper and begin a study of multi-vectors so first our global picture although with the caveat that we have yet to dive into these things although we're probably not going to dive in too deep I'm just going to lay it out that this is this the space-time algebra the C_{13} algebra and it's made up of the graded sum of these four vector spaces, we still need to talk about how this sum is done but for the moment you're going to fake it till you make it using the prescription that they offered earlier that's it's just like complex numbers so this is a real number this is like a complex number this is like another complex thing like Quaternions have three different complex square roots of -1 they're all different so we have four so we push this under the rug for a minute and we're going to understand that we can now create a vector in this space a vector in this space a vector here a vector here a vector here and somehow we can add them together and anything that we do by creating vectors in any combination of these spaces and adding them together those things are elements of the vector space C_{13} and they are part of the Clifford algebra, they're closed under space-time multiplication which makes them part of the Clifford algebra and that universe of things are called multi-vectors.

A multi-vector would be a real number plus some bi-vector $a \wedge b$ plus some tri-vector $c \wedge d \wedge e$, plus some quad-vector $f \wedge g \wedge h \wedge j$. If I add these four things together I get some multi-vector any sum like this is a multi-factor now those, multi-vectors don't have to have each part right I could have a multi-vector with just a tri-vector and a real number I could have a multi-vector with just a bi-vector and a quad-vector, I could have a multi-vector which is just a bi-vector. Now if I had a multi-vector that was just a bi-vector that's actually given the name a “blade” and in fact this would be called a two blade. Now I know we've already said this we're going to call it a bi-vector and a two blade is a bi-vector the reason we want to go with blades is we don't want to say you know bi-vector tri-vector, quad-vector quinta-vector, if they were a larger dimensional space, you can't have more than a quad vector of course because if you wedge another vector to it you're going to get zero but $a \wedge b$ would be a two blade and then $c \wedge d \wedge e$ would be a three blade and $f \wedge g \wedge h \wedge j$ is a four blade and a scalar is a zero blade.

Every multi-vector is some combination of zero blades, two blades, three blades and four blades now importantly there is notation for this and I'm going to fast forward into this notation a bit and if we have a multi-vector M which has a zero blade part that we describe by these angle brackets around M :

$$M = \langle M \rangle_0 + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3 + \langle M \rangle_4 \tag{7}$$

This is the notation that we use so the vector M has these five parts to it, now some of them may be zero and some of them not and it may only have one in which case M is a pure blade. The real number is the zero blade so that is the notation of what a multi-vector is and we can take the space-time product of any multi-vector with another multi-vector so if I had another multi-vector which I'll call say N , I can create the space-time product of M and N just by putting them next to each other and by linearity this is very easy to take apart so the idea is if we have M as a multi-vector and it's some sum of these four different types of blades and H is a multi-vector that's some sum of four different type of blades

$$H = \langle H \rangle_0 + \langle H \rangle_1 + \langle H \rangle_2 + \langle H \rangle_3 + \langle H \rangle_4 \tag{8}$$

The MH is the space-time product of M multiplied by H which is a pretty big expansion:

$$MH = \langle M \rangle_0 \langle H \rangle_0 + \cdots + \langle M \rangle_i \langle H \rangle_i + \cdots + \langle M \rangle_4 \langle H \rangle_4 \tag{9}$$

The first term is going to be the zero blade of M times the zero blade of H and the last term is going to be the four blade of M times the four blade of H and you're going to have every combination of blades buried inside there that are going to be part of this sum so the point of this slide is to show that we can do the space-time product of two arbitrary multi-vectors as described here but we have to know how to do the space-time product of every one of the blades with every one of the other blades so our mission now is to understand the space-time product of vectors with bi-vectors of vectors with tri-vectors of vectors with quad-vectors or the pseudo scalar bi-vectors with bi-vectors, we already know vectors with vectors so we've solved that one but now we need to know vectors and bi-vectors and we need to know by vectors and tri-vectors and bi-vectors and quad-vectors and we need to know tri-vectors and quad-vectors and all of that stuff so we need to understand all of these different blade products in order to fully understand how this algebra works.

Here's a little secret, very few studies of Geometric algebra really do that in full generality and even our work here we're not going to do it in full generality but we are going to pursue it substantially and then focus on just what we need to know for the space-time algebra so we're going to take a bit of a digression into vectors times bi-vectors and then we're going to describe bi-vectors times bi-vectors and bi-vectors times tri-vectors and we're going to describe all of these things and then move into the way the paper wants you to understand the structure of Geometric algebra so we'll just attack one very thoroughly and then we'll wave our hands and assume some of the others but the one we're going to attack is the vector times a bi-vector so $A \in \Lambda_2(\mathsf{M}_{1,3})$ is the bi-vector so it's an element of the bi-vector sub space of our Clifford algebra and we're going to say that bi-vector is $u \wedge v$ so you can already see that we're going to use a proper formalism here, we're not breaking this down into components.

We're going to use a proper formalism and w, u, v are all part of the sub space of the Clifford algebra that is $\mathsf{M}_{1,3}$, it's an element of $\mathsf{M}_{1,3}$, $\Lambda_1(\mathsf{M}_{1,3})$ by the way is in fact $\mathsf{M}_{1,3}$, $\Lambda_0(\mathsf{M}_{1,3})$ is just real numbers $\mathbb{R}$ so it's a little excessive but this is to show that we're considering $\mathsf{M}_{1,3}$ as part of the Clifford algebra and then we're going to form the space-time product of the vector w with the bi-vector A and we're going to write that as w space-time product of $u \wedge v$ so we're going to proceed and analyze this and see if we can understand what the space-time product of w and A looks like.

$$\begin{aligned} wA &= w(u \wedge v) = ½\, w(uv - vu) = ½[wuv - wvu] \\ &= ½[wuv + uwv - uwv - wvu - vwu + vwu] \end{aligned} \tag{10}$$

We know u and v are vectors from $\mathsf{M}_{1,3}$ so we know that this is the anti-symmetric part of the space-time product of u and v which is exactly what this term here so that is by definition a true statement and then we just use linearity and we bring in the w , then this is a triple space-time product of vectors, no problem there, you can always do that because remember space-time product of u and v is in the Clifford algebra so that's just a member of the Clifford algebra and so is w and we know actually uv is going to be $u \cdot v + u \wedge v$ but if you made that substitution you would just go back. We're blowing this up into this triple vector product and then to show how annoying these proper proofs can be or these proper analyzes can be I don't know if I would call this a proof as much as an analysis is I take this wuv and minus wvu and I make all these very clever observations about how it's going to end up:

$$\begin{aligned} wA &= ½[wuv + uwv] - ½[wvu + vwu] - ½[uwv - vwu] \\ &= ½[wuv - uwv] - ½[wvu + vwu] - ½[uwv - vwu] \\ &= ½[wu + uw]v - ½[wv + vw]u - ½[uwv - vwu] \\ &= (w \cdot u)v - (w \cdot v)u - ½[uwv - vwu] \end{aligned} \tag{11}$$

Basically when I did this work I actually started from the end and went backwards but as long as you see how each step is logical you can appreciate the final result so we move the wuv and we're going to add to it uwv and then subtract uwv so we're adding zero and then we have the $-vwu$ part from our previous line and we subtract vwu , we bring down wvu and we subtract vwu and we add vwu so that's adding zero so we've taken this and added zero in this unique form and then of course we're going to cleverly regroup.

What's nice about this first piece that we've regrouped is that we have v at the back so I can pull v out and then the second one I have u at the back so I can pull u out and this third one I don't have v or u at the back or u or v at the front or anything so I leave it alone and what's important now as I look at this I say well that's the symmetric part of the space-time product of $w \cdot u$ and this is the symmetric part of the space-time product of $w \cdot v$ so I know what those things are, those are the Minkowski contraction of w and u and this is the Minkowski contraction of w and v so I replace it with these dot products which indicates that. Remember $w \cdot u = \eta(w,u)$ and $w \cdot v = \eta(w,v)$ this equals Minkowski contraction. I'm left with this piece over here (11) so I'm going to rewrite this with these Minkowski contractions:

$$\begin{aligned} wA = &\overbrace{(w\cdot u)}^{\eta(w,u)} v - \overbrace{(w\cdot v)}^{\eta(w,v)} u + (u\cdot w)v - u(v\cdot w) \\ &+ ½[-uwv + vwu + uwv - vwu + uvw - vuw] \end{aligned} \tag{12}$$

This part here (last line of (11)) is just a vector this part here is some triple space-time product it almost seems like we've made the problem worse but I bring down these two the vector part into this line and I'm going to cleverly work on this part by the same process I'm adding and subtracting stuff and in this case what I'm going to do is I'm going to add I take this minus sign and I turned it to a plus sign so I brought the minus signs inside these brackets.

It's very clever but very hard to see and it's a bunch of algebra so obviously somehow I have to compensate for all of these four things but what I notice is that uwv and uvw can be confined into this term $½u(wv+vw)$ which is u times the symmetric part of the space-time product of w and v and these two can be combined into this term which is a vector v times the symmetric part of the space-time product of w and u and so this is actually what I've added to these two terms to make the changes that I think are are relevant, canceling this and leaving behind these two terms so obviously if I'm going to add this to this stuff inside the brackets I need to understand these two is the same as this one half. These divisions by two come from here so I'm not there's no division by four anywhere but anyway the point is I have to subtract it somewhere else so I'm going to subtract it right here, what's convenient is this is a vector and this is a vector these two things are vectors, I now subtract this part $½u(vw+wv)$ and add this part $½(uw+wu)v$ and now this is just $(u\cdot w)v - u(v\cdot w)$, a real number times two more vectors just like these two guys.

It turns out it's exactly the same as these first two guys because $(w\cdot u)v = (u\cdot w)v$ because u and w this dot product commutes and $(w\cdot v)u = u(v\cdot w)$ because this commutes and the real number commute so this can be combined with this and this term can be combined with this term so the vector the vector v part can be added together and the vector u part can be added together and you just get:

$$w(u\wedge v) = 2(w\cdot u)v - 2(w\cdot v)u + ½[uvw - vuw] \tag{13}$$

$$½[uvw - vuw] = ½[uv - vu]w = (u\wedge v)w \tag{14}$$

Which is nice and then what's left behind is this thing here that is what's ultimately left of this mess is this just these two terms this term here and this term here when you combine them what you see is that you can pull out the W from the back and you get the anti-symmetric part of the space-time product of u and v times the vector w but that anti-symmetric part is by definition $(u\wedge v)w$ so you can write:

$$w(u\wedge v)-(u\wedge v)w=2(w\cdot u)v-2(w\cdot v)u \tag{15}$$

Now this there's no reason to think this commutes at all right this stuff does because we know that that's just a scalar real number but this wedge product the space-time product of a wedge product and w we can't it doesn't commute but if I take this and I move it over to the other side I end up with (15) equals this vector and if I clear the two I get:

$$½(w(u\wedge v)-(u\wedge v)w)=(w\cdot u)v-(w\cdot v)u \tag{16}$$

If you look at this, this is the anti-symmetric part of the space-time product of w with the bi-vector A , (16) is the anti-symmetric part of this product so we are going to define the anti-symmetric part as $w\cdot A$, now notice we used to define $w\cdot A$ as the symmetric part of something but not this time, this time we are going to define it as the anti-symmetric part so when we're talking about a vector space-time product with a bi-vector the anti-symmetric part is going to be associated with the dot product.

$$\text{Antisym}[wA]=w\cdot A \tag{17}$$

Now notice we can't say that this dot product equals $\eta(w,A)$ because that's not how Minkowski metric works, the Minkowski metric only works with vectors it doesn't work with bi-vectors that doesn't even make sense, there's no such thing so we're actually creating this new notion of a dot product and we're associating it with the anti-symmetric part of this product and it's a vector, it's not a scalar so this dot product doesn't even give you a scalar, it gives you a vector. This is a mathematical notation convenience we're introducing here and I'll explain why in a moment, well I'll explain why right now. Let's look at our notion we have a vector times a bi-vector and we are now saying that the anti-symmetric part we're going to call $w\cdot A$ the symmetric part is going to be $w\wedge A$ and we're going to do this by definition, it has an anti-symmetric part, it has a symmetric part so we can just rename these things now we just discovered w is a grade one object and A is a grade two objects, bi-vectors have grade two, they're two blades, if I didn't say that before two blades have grade two, one blades have grade one, zero blades have grade zero. That is this number down here in (7) represents the grade of that part so the grade two part of M is $\langle M\rangle_2$ whatever this two blade is the grade four part of H is $\langle H\rangle_4$ whatever this object is so over here we have:

$$wA=w\cdot A+w\wedge A \tag{18}$$

$$w\cdot A=½(wA-Aw)\ ,\ \ w\wedge A=½(wA+Aw) \tag{19}$$

w is a grade one object and A is a grade two object and we know that it devolves into a grade one object the symmetric part $w\cdot A$ is a grade one object so that just leaves this anti-symmetric part $w\wedge A$ which is a grade three object which we write down as $w\wedge u\wedge v$ because $A=u\wedge v$ so this piece $½(wA+Aw)$ is now a three blade so this object breaks into a one blade and a three blade now there were three grades on the left side and it drops to a one grade and a three grade on the right side so what we're seeing now is that the dot product is the piece of this space-time product that lowers the grade by one and the wedge product is this piece of the space-time product that raises the grade by one and this can be done whenever you have a vector as part of the space-time product.

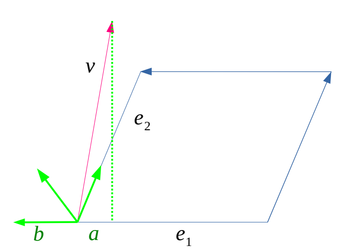
What the vector will do is it'll take the grade of whatever it's multiplying by in this case a grade two object but it could also work with a grade one object, a grade two object three or four but it will give you two pieces one which is one grade lower than what it's multiplying, what the other factor than the vector so the vector factor is grade one but this other factor is grade two, this first piece is one grade lower than the grade two object, the second piece is one grade higher, if this was a tri-vector then the grade one object would be a bi-vector, the dot product piece would be a grade two object a bi-vector and the wedge piece would be a grade four object a quad-vector but in this particular case this dot product ends up getting associated with the anti-symmetric part and this grade three part gets associated with the symmetric piece of the product.

This always confused me because you're looking at the symmetric part of this product is something which is the ultimate anti-symmetric object, how can these deeply anti-symmetric object be associated with the symmetric part of anything? But of course, the answer is actually pretty obvious, which is if I switched and wrote $A w$, I would have to switch A with w and the u would move over one bringing a minus sign and then the v would move over one bringing another minus sign and those two would cancel and you would end up with $u \wedge v \wedge w$ so the symmetry of this deeply anti-symmetric object actually is exactly what you need to define this symmetric part of $w A$ because the two minus signs cancel when you make this movement in other words you start with:

$$w \wedge u \wedge v = -u \wedge w \wedge v = u \wedge v \wedge w \tag{20}$$

You've switched A which is to the right and now A is on the left side and obviously you switched A and w and the sign didn't change so that is symmetric, that's the symmetric part of the product and we define it to be this completely anti-symmetric object which is interesting. Then we learned that we just figured out that the grade one part is equal to this vector $(w \cdot u) v - (w \cdot v) u$. That is the mechanics of proving how you do multiplication of a vector and a bi-vector so we could spend a little bit of time interpreting what this object is. This object $w \wedge u \wedge v$ is easy to interpret, it's a fully anti-symmetric thing we consider this a little unit of three volume so it's a three-dimensional Parallelepiped living in four-dimensional space and that's how we choose to interpret this thing.

Now that interpretation is a bit of a risk because all it is, is an algebraic object but we've learned that there's this geometric thing that is very aligned with this algebraic construction and so we just go ahead and we make that interpretation just like we did for a bi-vector. This guy $(w \cdot u) v - (w \cdot v) u$ is not so easy to interpret, it's the projection of w on part of A in the direction of v and the projection of w on part of A in the direction of u and there's that minus sign between them so how do we interpret this?



$$v = a e_1 + b e_2 + c e_3$$

Let's talk about that for a moment so let's do the setup here. Here's the plane element $e_1 \wedge e_2$, this is the bi-vector, I've even got to have its circulation, it's a unit bi-vector, it's a basis bi-vector and we want to take the space-time product of $e_1 \wedge e_2$ with this vector v so we're looking for $v(e_1 \wedge e_2)$ that's equivalent to $v A$ in our previous analysis so this is geometric objects so you want to think well what could this be? Well you got a vector and a plane so what are the interesting things that can happen to a vector in a plane well the vector could be in the plane or it could be out of the plane and the way I've depicted it here it's out of the plane and you know it's out of the plane because I have a e_3 component but this first part is actually in the plane so there is that question of its projection so we know that our formula is:

$$v\,A=v(e_1\wedge e_2)=(v\cdot e_1)e_2-(v\cdot e_2)e_1 \tag{21}$$

What is this first term? It's the projection of v on e_1 which I guess we could draw as the projection of v on e_1 would be something like that (in green) and that distance should presumably be a but that's just a number, that's the number a but it's in the direction of e_2 , it's a vector now in the direction of e_2 so I take a and I push it in the e_2 direction and then it's minus $(v\cdot e_2)e_1$ and $v\cdot e_2$ is some other number but it's whatever number it is, it's some projection on this axis here e_2 but we projected actually minus on e_1 so we project it onto e_1 and then we get this minus sign so we switch the sign so we end up with an amount b over this way and we add those things together and we get something that's in the plane, you'll notice that this result (21) is in the plane so this space-time product is, if nothing else, projected this vector v into the plane but notice that the minus sign really throws us for a loop because if it was simply a projection in the plane it would have just dropped into the plane and we would have gotten some vector in the plane but we didn't, we got a vector like that but then rotated by 90º.

This space-time product captures several things it captures the projection of the vector into the plane that is identified by the bi-vector but it also captures the orientation the circulation of the bi-vector by rotating it so all the essence of the bi-vector is captured and preserved in this space-time product you lose this component, this orthogonal component in this case but you still capture the circulation now in principle we're in four-dimensional space so we could have had a $d\,e_0$ here but that would have been lost also because you'll notice that this equation (21) doesn't change at all, it's just the projection into the bi-vector parts so it doesn't matter how many other dimensions you have you could have e_4 ,, e_6 forever and ever, they'd all be gone, you're still just projecting into the plane element so that's a way of interpreting this space-time product through this geometry and in fact when you study Geometric algebra in simply three dimensions you know this is a big deal, it's introduced very early but it's really no different, it's the same in all dimensions you'll notice that the metric doesn't appear here so the fact that this is Minkowski doesn't seem to matter much for this interpretation.

That's an interpretation of this space-time product, it's a projection of the vector into the plane and then rotated 90º degrees in the direction of the circulation of the plane that's the resulting vector that you get really sweet. Wait though, I should be a little bit more careful this is literally $v\cdot A$ the grade one part of that space-time product, the full space-time product if I had written it down correctly would have been (21) plus $v\wedge A$ which would be, the way I've written it, $v\wedge e_1\wedge e_2$ something like that if $A=e_1\wedge e_2$ which is it, there is also that part so the space-time product part has the grade one part and the grade three part so I'm talking about how to interpret just the grade one part so this whole thing is asking the question, how do we understand the grade one part of the space-time product? We already know how to understand the grade three part this is just going to be the well while we're here we might as well draw it out right this is just going to be the full Parallelepiped that you get between all of these vectors.

My previous story would have been completely correct if I had a dot there $v\cdot A$ but now I'm blowing it up to include the full space-time product $v\,A$ which includes this wedge product part, so we're going to leave it there for now. We still all we've done though is, just to be clear, all we now know how to do is we can multiply vectors with each other, we can multiply vectors and scalars with each other we can multiply bi-vectors and scalars and now we can multiply vectors and bi-vectors. There's still a few other things to do but we're not going to go into these demonstrations quite as deeply we're just going to talk about them in generality and we'll do that in our next lesson so I'll see you next time.