Black Hole Theory

Based on The Polyphysics Project Playlist: Black Hole Theory

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(%i2) `info:build_info()$info@version;`

(%o2)

5.38.1

(%i2) `reset()$kill(all)$`

(%i1) `derivabbrev:true$`

(%i2) `ratprint:false$`

(%i3) `fpprintprec:5$`

(%i4) `if get('draw,'version)=false then load(draw)$`

(%i5) `wxplot_size:[1024,768]$`

(%i6) `if get('optvar,'version)=false then load(optvar)$`

(%i7) `if get('rkf45,'version)=false then load(rkf45)$`

(%i8) `declare(s,mainvar)$`

(%i9) `declare(trigsimp,evfun)$`

1 Settings

2 Using optvar

3 Using ctensor

4 Using cartan

(%i66) `kill(labels,t,r,θ,ϕ)$`

(%i1) `if get('cartan,'version)=false then load(cartan)$`

(%i2) `init_cartan(ξ)$`

(%i3) `cartan_basis;`

$$[dt, dr, d\theta, d\phi] \tag{\%o3}$$

(%i4) `cartan_coords;`

$$[t, r, \theta, \phi] \tag{\%o4}$$

(%i5)
```
cartan_dim;
```

$$4 \tag{\%o5}$$

(%i6)
```
extdim;
```

$$4 \tag{\%o6}$$

Coframe

(%i7)
```
depends([V,W],[r,t])$
```

(%i15)
```
Eq_t:e_t=V*dt$ sol_t:solve(Eq_t,dt)$ Eq_r:e_r=W*dr$ sol_r:solve(Eq_r,dr)$ Eq_θ:e_θ=r*dθ$
sol_θ:solve(Eq_θ,dθ)$ Eq_ϕ:e_ϕ=r*sin(θ)*dϕ$sol_ϕ:solve(Eq_ϕ,dϕ)$
```

(%i16)
```
ldisplay(e:ev([e_t,e_r,e_θ,e_ϕ],Eq_t,Eq_r,Eq_θ,Eq_ϕ))$
```

$$e = [V\ dt, W\ dr, r\ d\theta, r\ d\phi\ \sin(\theta)] \tag{\%t16}$$

Exterior derivative of the coframe

(%i20)
```
de_t:ext_diff(e_t),Eq_t$ de_r:ext_diff(e_r),Eq_r$ de_θ:ext_diff(e_θ),Eq_θ$
de_ϕ:ext_diff(e_ϕ),Eq_ϕ$
```

(%i21)
```
ldisplay(de:[de_t,de_r,de_θ,de_ϕ])$
```

$$de = [-(V_r)\ dr\ dt, (W_t)\ dr\ dt, dr\ d\theta, dr\ d\phi\ \sin(\theta) + r\ d\theta\ d\phi\ \cos(\theta)] \tag{\%t21}$$

Cartan's first equation of structure: $\mathrm{d}e^{\hat{\alpha}} + \omega^{\hat{\alpha}}_{\hat{\beta}} \wedge e^{\hat{\beta}} = 0$

Cartan's second equation of structure: $R^{\hat{\alpha}}_{\hat{\beta}} = \mathrm{d}\omega^{\hat{\alpha}}_{\hat{\beta}} + \omega^{\hat{\alpha}}_{\hat{\gamma}} \wedge \omega^{\hat{\gamma}}_{\hat{\beta}}$

Relation between the Ricci tensor to the Riemann tensor: $R^{\hat{\alpha}}_{\hat{\beta}} = R^{\hat{\alpha}}_{\hat{\beta}\gamma\delta}\,\mathrm{d}\gamma \wedge \mathrm{d}\delta$

To convert the Riemann tensor in orthonormal basis to coordinate basis we use the fact that $R^{\hat{\alpha}}_{\hat{\beta}\gamma\delta}\, e^{\alpha}_{\hat{\alpha}}\, e^{\hat{\beta}}_{\beta} = R^{\alpha}_{\beta\gamma\delta}$, where $e^{\alpha}_{\beta}\,\mathrm{d}x^{\beta} = e^{\alpha}$ is the connection 2-form

ldisplay(de_t:ev(de_t,sol_t,sol_r,sol_θ,sol_ϕ))$ ldisplay(de_r:ev(de_r,sol_t,sol_r,sol_θ,sol_ϕ))$ ldisplay(de_θ:ev(de_θ,sol_t,sol_r,sol_
ldisplay(de_ϕ:ev(de_ϕ,sol_t,sol_r,sol_θ,sol_ϕ))$

(%i22)
```
Ω:genmatrix(lambda([i,j],sum(f[i,j,k]*cartan_basis[k],k,1,dim)),dim,dim)$
```

(%i23)
```
for i thru dim do for j thru dim do ldisplay(Ω[i,j])$
```

$$\Omega_{1,1} = f_{1,1,4}\ d\phi + f_{1,1,3}\ d\theta + f_{1,1,1}\ dt + f_{1,1,2}\ dr \tag{\%t23}$$

$$\Omega_{1,2} = f_{1,2,4}\ d\phi + f_{1,2,3}\ d\theta + f_{1,2,1}\ dt + f_{1,2,2}\ dr \tag{\%t24}$$

$$\Omega_{1,3} = f_{1,3,4}\ d\phi + f_{1,3,3}\ d\theta + f_{1,3,1}\ dt + f_{1,3,2}\ dr \tag{\%t25}$$

$$\Omega_{1,4} = f_{1,4,4}\ d\phi + f_{1,4,3}\ d\theta + f_{1,4,1}\ dt + f_{1,4,2}\ dr \tag{\%t26}$$

$$\Omega_{2,1} = f_{2,1,4}\, d\phi + f_{2,1,3}\, d\theta + f_{2,1,1}\, dt + f_{2,1,2}\, dr \tag{\%t27}$$

$$\Omega_{2,2} = f_{2,2,4}\, d\phi + f_{2,2,3}\, d\theta + f_{2,2,1}\, dt + f_{2,2,2}\, dr \tag{\%t28}$$

$$\Omega_{2,3} = f_{2,3,4}\, d\phi + f_{2,3,3}\, d\theta + f_{2,3,1}\, dt + f_{2,3,2}\, dr \tag{\%t29}$$

$$\Omega_{2,4} = f_{2,4,4}\, d\phi + f_{2,4,3}\, d\theta + f_{2,4,1}\, dt + f_{2,4,2}\, dr \tag{\%t30}$$

$$\Omega_{3,1} = f_{3,1,4}\, d\phi + f_{3,1,3}\, d\theta + f_{3,1,1}\, dt + f_{3,1,2}\, dr \tag{\%t31}$$

$$\Omega_{3,2} = f_{3,2,4}\, d\phi + f_{3,2,3}\, d\theta + f_{3,2,1}\, dt + f_{3,2,2}\, dr \tag{\%t32}$$

$$\Omega_{3,3} = f_{3,3,4}\, d\phi + f_{3,3,3}\, d\theta + f_{3,3,1}\, dt + f_{3,3,2}\, dr \tag{\%t33}$$

$$\Omega_{3,4} = f_{3,4,4}\, d\phi + f_{3,4,3}\, d\theta + f_{3,4,1}\, dt + f_{3,4,2}\, dr \tag{\%t34}$$

$$\Omega_{4,1} = f_{4,1,4}\, d\phi + f_{4,1,3}\, d\theta + f_{4,1,1}\, dt + f_{4,1,2}\, dr \tag{\%t35}$$

$$\Omega_{4,2} = f_{4,2,4}\, d\phi + f_{4,2,3}\, d\theta + f_{4,2,1}\, dt + f_{4,2,2}\, dr \tag{\%t36}$$

$$\Omega_{4,3} = f_{4,3,4}\, d\phi + f_{4,3,3}\, d\theta + f_{4,3,1}\, dt + f_{4,3,2}\, dr \tag{\%t37}$$

$$\Omega_{4,4} = f_{4,4,4}\, d\phi + f_{4,4,3}\, d\theta + f_{4,4,1}\, dt + f_{4,4,2}\, dr \tag{\%t38}$$

Change matrix multiplication operator

(%i39) `matrix_element_mult:"~"$`

(%i40) `Θ:expand(Ω.e)$`

(%i41) `map(ldisp,Θ)$`

$$[f_{1,4,3} r\, d\theta\, d\phi\, \sin(\theta) + f_{1,4,1} r\, dt\, d\phi\, \sin(\theta) + f_{1,4,2} r\, dr\, d\phi\, \sin(\theta) - f_{1,3,4} r\, d\theta\, d\phi - f_{1,1,4} V\, dt\, d\phi - f_{1,2,4} W\, dr\, d\phi + f_{1,3,1} r\, dt\, d\theta - f_{1,1,} \tag{\%t41}$$

$$[f_{2,4,3} r\, d\theta\, d\phi\, \sin(\theta) + f_{2,4,1} r\, dt\, d\phi\, \sin(\theta) + f_{2,4,2} r\, dr\, d\phi\, \sin(\theta) - f_{2,3,4} r\, d\theta\, d\phi - f_{2,1,4} V\, dt\, d\phi - f_{2,2,4} W\, dr\, d\phi + f_{2,3,1} r\, dt\, d\theta - f_{2,1,} \tag{\%t42}$$

$$[f_{3,4,3} r\, d\theta\, d\phi\, \sin(\theta) + f_{3,4,1} r\, dt\, d\phi\, \sin(\theta) + f_{3,4,2} r\, dr\, d\phi\, \sin(\theta) - f_{3,3,4} r\, d\theta\, d\phi - f_{3,1,4} V\, dt\, d\phi - f_{3,2,4} W\, dr\, d\phi + f_{3,3,1} r\, dt\, d\theta - f_{3,1,} \tag{\%t43}$$

$[f_{4,4,3} r\ d\theta\ d\phi\ \sin(\theta) + f_{4,4,1} r\ dt\ d\phi\ \sin(\theta) + f_{4,4,2} r\ dr\ d\phi\ \sin(\theta) - f_{4,3,4} r\ d\theta\ d\phi - f_{4,1,4} V\ dt\ d\phi - f_{4,2,4} W\ dr\ d\phi + f_{4,3,1} r\ dt\ d\theta - f_{4,1,}$ (%t44)

(%i45) `Eq:coeff(coeff(Θ[1,1],dt),dθ)=coeff(coeff(de[1],dt),dθ);`

$$f_{1,3,1} r - f_{1,1,3} V = 0 \tag{Eq}$$

Restore matrix multiplication operator

(%i46) `matrix_element_mult:"*"$`

(%i56)
```
ω:zeromatrix(dim,dim)$ ω[1,2]:-diff(V,r)/V/W*e_t+diff(W,t)/V/W*e_r$
ω[2,1]:+diff(V,r)/V/W*e_t-diff(W,t)/V/W*e_r$ ω[3,2]:-1/r/W*e_θ$ ω[2,3]:+1/r/W*e_θ$
ω[4,3]:-cos(θ)/sin(θ)/r*e_φ$ ω[3,4]:+cos(θ)/sin(θ)/r*e_φ$ ω[2,4]:+1/r/W*e_φ$
ω[4,2]:-1/r/W*e_φ$ ldisplay(ω)$
```

$$\omega = \begin{pmatrix} 0 & \frac{(W_t)\, e_r}{V W} - \frac{(V_r)\, e_t}{V W} & 0 & 0 \\ \frac{(V_r)\, e_t}{V W} - \frac{(W_t)\, e_r}{V W} & 0 & \frac{e_\theta}{W r} & \frac{e_\phi}{W r} \\ 0 & -\frac{e_\theta}{W r} & 0 & \frac{e_\phi \cos(\theta)}{r \sin(\theta)} \\ 0 & -\frac{e_\phi}{W r} & -\frac{e_\phi \cos(\theta)}{r \sin(\theta)} & 0 \end{pmatrix} \tag{\%t56}$$

(%i57) `ldisplay(ω:ev(ω,Eq_t,Eq_r,Eq_θ,Eq_φ))$`

$$\omega = \begin{pmatrix} 0 & \frac{(W_t)\, dr}{V} - \frac{(V_r)\, dt}{W} & 0 & 0 \\ \frac{(V_r)\, dt}{W} - \frac{(W_t)\, dr}{V} & 0 & \frac{d\theta}{W} & \frac{d\phi \sin(\theta)}{W} \\ 0 & -\frac{d\theta}{W} & 0 & d\phi \cos(\theta) \\ 0 & -\frac{d\phi \sin(\theta)}{W} & -d\phi \cos(\theta) & 0 \end{pmatrix} \tag{\%t57}$$

Change matrix multiplication operator

(%i58) `matrix_element_mult:"~"$`

(%i59) `list_matrix_entries(ω.e);`

$$[-(V_r)\ dr\ dt, (W_t)\ dr\ dt, dr\ d\theta, dr\ d\phi\ \sin(\theta) + r\ d\theta\ d\phi\ \cos(\theta)] \tag{\%o59}$$

(%i60) `is(%=de);`

$$\text{true} \tag{\%o60}$$

Restore matrix multiplication operator

(%i61) `matrix_element_mult:"*"$`