Lesson 7: The Glome (OPTIONAL)

I want to explore a little bit more about this interesting conformal compactified coordinate system (2.1.4 in the catalog) only because I just love this as an exercise in understanding how flexible coordinate systems are and how we can choose them to solve our problems. The problem now is that we don't have problems to solve, we're just talking about methods and how the coordinate systems work and last time we kind of walked from the beginning to the end starting with (t, r) , remember in our world $c=1$ so we don't have that c and ending with what we called $(\tilde{t}=\psi, \tilde{r}=\xi)$. I just like the idea that you have this reference book, this catalog that we can slowly figure out by just studying and I guess ultimately my plan here is to kind of take a nonlinear walk through this catalog step by step unpeeling everything that it means and that will be my sort of non systematic presentation of General Relativity which is always intended to be a compliment to some real systematic presentation there's a lot of good ones out there, there's a lot of good books but you learn something and then it leads you to ask another question which leads you ask another question and you know you can follow that flow it's not totally logical but if you've started with some foundation if you've done or prerequisites you can kind of go that way so ultimately we showed that this is a legitimate coordinate transformation (2.1.16) and it leads to the transformation of the metric to be this thing (2.1.17) which equals Minkowski space so we kind of did all that.

What I told you to ignore last time was this thing up here (2.1.15). That is not Minkowski space, that is the Minkowski metric in conformal compactified coordinates which means it is something called conformal to Minkowski space but it's not literally Minkowski space. The relationship is $ds^2=\Omega^2 d\tilde{s}^2$. The metric of the conformal compactified metric is the metric that we found which is the flat metric of Minkowski space-time this this thing here which is this expression that we found in the denominator: meaning we take the metric we just discovered which is flats base metric that's (2.1.17) and you multiply it by this:

$$\Omega^2=4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2} \tag{1}$$

You're left with (2.1.15), that is different than a coordinate transformation you just are multiplying the metric by some function of the coordinates, some function of space-time Ω is a function in this case of what they're calling ψ and ξ , where we would call $\tilde{t}$ and $\tilde{r}$, it's just not a function of the other coordinates θ and φ . It is a function of space-time but you're taking the entire metric and you're multiplying it by some function of space-time and what you're kind of doing is you're clearing this denominator and you end up with (2.1.15). It is not a straight up coordinate transformation, it is a transformation is a conformal transformation of the Minkowski metric, this Minkowski metric is the compactified coordinates so you could call this the Minkowski metric in compactified coordinates and then (2.1.15) is the Minkowski metric in compactified coordinates after a conformal transformation which is just a multiplication by Ω^2 . This Ω^2 could literally be any decent function of the coordinates if you had any decent function of the coordinates you will end up with a conformal transformed metric.

There's a reason they've done this and it has to do with the fact that (2.1.15) is a metric that solves something called the Einstein Static Universe and we're not quite ready to talk about that because it's a solution of the Einstein equation and that means it's when I say a solution of the Einstein equation what I mean is that is a metric $g_{\mu\nu}$ that solves $\boldsymbol{G}_{\mu\nu}=8\pi \boldsymbol{T}_{\mu\nu}$. This Einstein equation has solutions, one of the solutions and the solution means a metric, that's what we mean by a solution, the solution of this is a space-time a metric.

I put the matter in $\boldsymbol{T}_{\mu\nu}$ (the right hand side) and then this tells me on the left hand side, the space-time structure induced in the Universe by the matter that's placed there, that result pops out as a metric and in this case this metric here (2.1.15) is one of those possible solutions which means there's a certain distribution of matter that I can postulate and in this case it's an ideal Perfect fluid, if I put it in there throughout the entire Universe I end up with this solution (2.1.15) for the metric of the Universe. It's actually a cosmological metric as opposed to metrics that just reflect like Black Holes like you can get the Schwarzschild metric for a Black Hole or the Kerr metric for a spinning Black Hole and there's a variety of different situations in Physics that can give you metrics for different things but if you talk about the cosmology of the entire Universe this is a metric for the cosmology of the entire Universe under a certain assumption about the distribution of matter. It's interesting that that solution is related to completely flat space, it's not the same as completely flat space which means there's no coordinate transformation that will take these coordinates (2.1.15) in this cosmological metric which we have to talk about what they might mean, we don't even know what they quite mean, this kind of looks like a sphere, ψ looks like time so what's going on with all that but this cosmological solution is related to the flat space solution through this conformal relationship $ds^2=\Omega^2 d\tilde{s}^2$.

There's a reason why we like to use the conformal relationship I kind of alluded to it last time but I want to take a small diversion today just to talk a little bit about this metric (2.1.15) and because eventually we want to discuss the connection between these two metrics and we're on the subject now so I thought now would be a good time and the immediate problem is to understand something called the “Glome”. The “Glome” is also known as the 3-sphere. The 3-sphere is a surface you could think of it as a surface in 4D space and if the 4D space coordinates are (x, y, z, w) a 3-sphere is the set of points where $x^2+y^2+z^2+w^2=r^2$ where $r\in\mathbb{R}$ is the radius. The 3-sphere as a manifold has 3D to it.

A 2-sphere in the geometric sense is a 2D manifold because when you take the 2-sphere which would be $x^2+y^2+z^2=r^2$ you only need, with each of the coordinate patches on the 2-sphere, only go to 2D space, it'll go to θ and φ for example because we're going to consider r to be a constant for the 2-sphere so that's why it's called the 3-sphere so it's embedded in 4D space but it's a 3D manifold so a 3-sphere is just like the 2-sphere but it's obviously going to have an additional coordinate here because it's gonna map into a 3D coordinate system and it's a manifold and that's called the “Glome”.

It's obviously not something we can visualize but it's really I mean there's always a time in General Relativity where you kind of wrestle with visualizing 4D and everybody approaches it differently because there's a lot of different ways of thinking about 4D, think of it purely mathematically like this you can try to suppress a dimension, you can try to use stereo-graphic projection, there's a lot of great stuff on Wikipedia about stereo-graphic projections of the “Glome”, you can use different kinds of art it's a lot of ways of approaching it but in the end you know you've got to lean on just the math but the reason the “Glome” is important is because this metric here (2.1.15) (the spatial part) is the metric of the euclidean “Glome” meaning that if we have embedded this “Glome” in this space with a Euclidean metric, this is normal geometry, a regular Euclidean metric in 4D space, the metric induced on the surface of the “Glome” is this metric here (2.1.15) (the spatial part) and that's exactly like saying if you take a regular sphere embedded in 3D dimensional space the metric there is:

$$d\sigma^2=r^2\left[d\theta^2+\sin^2\theta\, d\varphi^2\right] \tag{2}$$

This is the metric induced on the surface of a sphere embedded in 3D space and the metric of 3D space induces this metric on the surface of the sphere and likewise the metric of 4D space induces this metric (2.1.15) (the spatial part) on a “Glome”. Now if you look at the “Glome”, it seems to have the regular 3-sphere metric kind of buried in there and then there's this new angular coordinate ξ and it kind of stands to reason if you have a 3D manifold you need 3 coordinates, well in the 2D manifold we had θ and we had φ , well now we need something else and trying to visualize how these angles work is kind of tough but I found a really great article that I thought was just totally wonderful and it talks about the “Glome” and I thought I would share the derivation and the article is “A derivation of of n-dimensional Spherical coordinates” by L. E. Blumenson University of Columbia, he wrote this and it was published in “The American Mathematical Monthly”. The reason i want to go through this derivation is because I want you to understand the coordinates of a “Glome” because ultimately we're going to use that to talk about the Einstein static Universe. Admittedly this is a bit of an aside I don't want you to think that this is critical to understanding General Relativity and you can skip this lecture and move on to the next.

This process is surprisingly simple and somewhat clever, we're going to use a bit a little bit of notation first of all I want you to understand that everything we're talking about is now going to be n dimensional geometry so let's say it's wearing n dimensions and we're gonna deal with vectors in n dimensions but it's gonna be all Euclidean. Notice if you go back to the to 2.1.4 in the catalog, this part of the metric (2.1.15) (the spatial part) is Euclidean. The introduction of ψ is what makes the metric have the potential to be less than zero so because of that, this becomes a Lorentzian metric but the spatial part is Euclidean and that's the part I'm going to talk about. Ultimately this piece $-d\psi^2$ is time and time is going to be another coordinate that doesn't show up anywhere in this spatial part and that's not totally uncommon I mean that happened in the regular Spherical coordinate system to remember in the in Minkowski in Spherical coordinates you had time hanging out by itself and then you had a big Spherical piece that was all spatial and even in the regular metric which is $\eta_{\mu\nu}$ the spatial part is Euclidean and time hangs out in front so we are now going to just study this part because I want to get you some intuition in these angles and why this formula is the way well I want you to understand these angles and understand how to make coordinate transformation into regular Cartesian coordinates.

For example in n-dimensional Euclidean space we'll have regular Cartesian coordinates (x^0, x^1, x^2, x^3) and it will also have some kind of Spherical coordinates $(r, \theta^1, \theta^2, \theta^3)$ because I don't really know much about them I do know that in 3D space it's (r, θ, φ) and I know that in 3D space $\varphi \in [0, 2\pi)$ but $\theta \in [0, \pi)$ so I kind of want to know what is the transformation from (x^0, x^1, x^2, x^3) to $(r, \theta^1, \theta^2, \theta^3)$ and what do these angles are kind of mean that's what I'm thinking and I want to be able to do this for any dimensions arbitrarily and that's what this little paper shows.

The good news about all this is that we can go back to our basic understanding of vectors that we learned in our most elementary work. I'll try to use the notation we've been using in “What is a tensor” stuff but we start with a vector space and in this vector space will chart out all of Euclidean space and we will create a basis for that and it will be $\vec{e}_\mu$

$$\text{Cartesian basis} \rightarrow \ \vec{e}_\mu \cdot \vec{e}_\nu = \delta_{\mu\nu} \tag{3}$$

We're in the Cartesian basis, this is our set of basis vectors $\vec{e}_\mu$ and we're going to adopt the classic notation from your standard Linear algebra material where the dot product of two basis vectors is going to be this δ function, orthogonal. Now notice we have this δ function unlike the “What is a tensor” series, the indices are both down and that just is to accommodate the fact that this is an inner product.

This isn't double space mapping obviously you're dealing with two vectors it's just an inner product and it's defined this way with these two indices down it's the old system, it's the system before we get all of our fancy tensor analysis but I am still going to retain the Einstein summation convention and I'll say:

$$\vec{X}=X^{i}\vec{e}_{i}\Rightarrow X^{i}=\vec{X}\cdot\vec{e}_{i} \tag{4}$$

The magnitude of $\vec{X}$ is:

$$|\vec{X}|^{2}=\vec{X}\cdot\vec{X}=\sum_{i=1}^{n}\left(X^{i}\right)^{2}=X^{i}X_{i}=X^{i}X^{j}g_{ji} \tag{5}$$

This is a result of the fact that we're dealing with a Euclidean metric which means the inner product is strictly Euclidean which is what (4) is saying. That metric is the Euclidean metric which is the identity. The other part is the angle between two vectors say $\vec{X}$ and $\vec{Y}$, that angle is defined through:

$$\cos\alpha=\frac{\vec{X}\cdot\vec{Y}}{|\vec{X}||\vec{Y}|} \tag{6}$$

That angle is important for our analysis and this is all from old classic linear algebra in classic vectors Now let's begin to construct our model of n-sphere so we start with a vector $\vec{X}$ whose magnitude is r and r is going to be sort of a fixed number, it's the magnitude of the vector $|\vec{X}|=r$ and from this fact we can now consider the angle between $\vec{X}$ and any of the basis vectors $\vec{e}_{i}$. I want to know what's the angle between $\vec{X}$ and $\vec{e}_{i}$, I'm gonna call that angle φ_{i} , this is the angle between $\vec{X}$, which is a vector that points to one spot on the n-sphere of radius r and the basis vector $\vec{e}_{i}$. These basis vectors are aligned with the coordinate axes so we know they're orthogonal to each other while $\vec{X}$ lives out there

$$X^{i}=\vec{X}\cdot\vec{e}_{i}=r\cos\left(\varphi_{i}\right)\text{ were }0\leq\varphi_{i}\leq\pi \tag{7}$$

We know that because that's the way the elementary dot product works, we know that the dot product is the magnitude of the vectors with the cosine of the angle between the two vectors, in this case $|\vec{e}_{i}|=1$. This is pretty easy stuff so far but from this what we do is we are going to write:

$$\vec{X}=r\cos\left(\varphi_{1}\right)\vec{e}_{1}+\sum_{k=2}^{n}X^{k}\vec{e}_{k} \tag{8}$$

Now that we have that, we can use the fact that we know how long this vector is:

$$|\vec{X}|^{2}=r^{2}=\sum_{i=1}^{n}\left(X^{i}\right)^{2}=r^{2}\cos^{2}\varphi_{1}+\sum_{k=2}^{n}X^{k}\vec{e}_{k} \tag{9}$$

Now we manipulate this a little bit in a clever but not particularly ingenious way:

$$r^2 - r^2\cos^2\varphi_1 = r^2\sin^2\varphi_1 = \sum_{k=2}^{n} X^k \vec{e}_k \tag{10}$$

Now comes the slightly clever part I'm going to define another number β^k like this:

$$X^k \equiv \beta^k r^2 \sin^2\varphi_1 \tag{11}$$

I can now write:

$$\vec{X} = r^2\cos^2\varphi_1 \vec{e}_1 + r^2\sin^2\varphi_1 \sum_{k=2}^{n} \beta^k \vec{e}_k \tag{12}$$

Notice that:

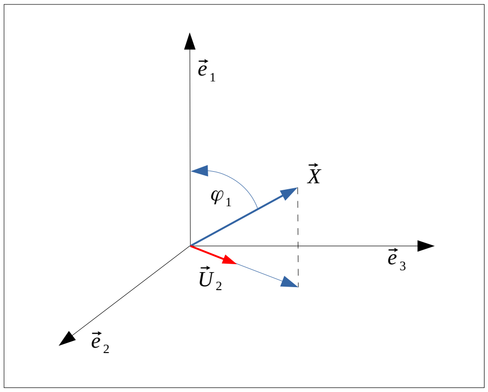
$$\sum_{k=2}^{n} \left(\beta^k\right)^2 = 1 \tag{13}$$

The reason is, in order for $|\vec{X}|^2 = r^2$ I can factor out r^2 , I end up with:

$$|\vec{X}|^2 = r^2 = r^2\left[\cos^2\varphi_1 + \sin^2\varphi_1 \sum_{k=2}^{n} \left(\beta^k\right)^2\right] \tag{14}$$

Let's look just at this part, this vector part (12) (the last part), I can actually define a new vector which I will call $\vec{U}_2$:

$$\vec{U}_2 = \sum_{k=2}^{n} \beta^k \vec{e}_k \tag{15}$$



First of all it doesn't fill the full vector space, the full vector space has $(\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n)$ but we're actually starting with $\vec{e}_2$ through $\vec{e}_n$ so whatever $\vec{U}_2$ is, it lives in a subspace of the full vector space and that subspace is spanned by $(\vec{e}_2, \cdots, \vec{e}_n)$ and therefore it is an orthogonal subspace to $\vec{e}_1$ because it's a Euclidean metric, its $n-1$ -dimensional, we know that $\vec{U}_2$ is orthogonal to $\vec{e}_1$ and it lives in this subspace, that's really redundant, if it lives in a subspace spanned by $(\vec{e}_2, \cdots, \vec{e}_n)$

it must be orthogonal to $\vec{e}_1$ so that's the one thing we know about it, it's also a unit vector because of (13). It's a projection of $\vec{X}$ into that subspace, I should say it's the direction of the projection so if we were to think about this in terms of just may be a situation in 3D we can visualize, we can imagine $\vec{X}$. $\vec{X}$ has an orthogonal projection and the unit vector of that orthogonal projection would be $\vec{U}_2$. Well now we just break down $\vec{U}_2$ the same way we broke down $\vec{X}$:

$$\vec{U}_2=\cos\varphi_2\vec{e}_2+\sum_{\ell=3}^{n}\beta^{\ell}\vec{e}_{\ell} \tag{16}$$

It has no component in $\vec{e}_1$ by definition because $\vec{U}_2$ lives in this separate subspace and then. This time there's no r because we choose a unit vector so that does an r , it's just a 1 . For the next sort we just repeat for $\vec{U}_2$ what did I before for $\vec{X}$. Now this is a subspace that starts with $\vec{e}_2$ and the sum goes from 3 to n and now we just make the same statement we did before:

$$\left|\vec{U}_2\right|^2=1=\cos^2\varphi_2+\sum_{\ell=3}^{N}\left(\beta^{\ell}\right)^2\Rightarrow 1-\cos^2\varphi_2=\sum_{\ell=3}^{N}\left(\beta^{\ell}\right)^2=\sin^2\varphi_2 \tag{17}$$

Now I'm going to do this again, I guess it's getting recursive, I'm going to write:

$$\beta^{\ell}=\sin^2\varphi_2\gamma^{\ell} \tag{18}$$

I'm going to factor out a $\sin^2\varphi_2$ from every β^{ℓ} . Remember φ_2 is the angle between $\vec{U}_2$ and $\vec{e}_2$ Once I've done that I can now rewrite the equation one more time:

$$\vec{U}_2=\cos\varphi_2\vec{e}_2+\sin\varphi_2\sum_{\ell=3}^{n}\gamma^{\ell}\vec{e}_{\ell} \tag{19}$$

Now you can almost see what's gonna happen I'm gonna substitute this back in to our definition of $\vec{X}$ and that is going to set me up to do it all over again, so what is $\vec{X}$? Well we got to go back to $\vec{X}$:

$$\vec{X}=r\cos\varphi_1\vec{e}_1+r\sin\varphi_1\left[\cos\varphi_2\vec{e}_2+\sin\varphi_2\sum_{\ell=3}^{n}\gamma^{\ell}\vec{e}_{\ell}\right] \tag{20}$$

I can expand that out without too much trouble to get:

$$\vec{X}=r\cos\varphi_1\vec{e}_1+r\sin\varphi_1\cos\varphi_2\vec{e}_2+r\sin\varphi_1\sin\varphi_2\vec{U}_3 \text{ where } \vec{U}_3\stackrel{\text{def}}{=}\sum_{\ell=3}^{n}\gamma^{\ell}\vec{e}_{\ell} \tag{21}$$

Now we repeat the process for $\vec{U}_3$. We know that it is a unit vector and is going to also exist in the subspace orthogonal to $\vec{U}_1$ and $\vec{U}_2$ meaning $\vec{U}_3$ is orthogonal to them. We repeat this exercise again and we then start with breaking down $\vec{U}_3$ just like before:

$$\vec{U}_3 = \cos\varphi_3 \vec{e}_3 + \sum_{m=4}^{n} \alpha^m \vec{e}_m \qquad (22)$$

Just like before:

$$\left|\vec{U}_3\right|^2 = 1 \Rightarrow \sum_{m=4}^{n} \left(\alpha^m\right)^2 = \sin^2\varphi_3 \qquad (23)$$

What am I building up? Well what I've built up $\vec{X}$ that looks like (21), just gets longer and longer and end up with:

$$\vec{X} = \cdots \vec{e}_1 + \cdots \vec{e}_2 + \cdots \vec{e}_3 + \cdots + \cdots \vec{e}_n \qquad (24)$$

Ultimately the point is that these something's in front of each of these unit vectors those are the coordinate transformations because these something's are always going to be functions of r , φ_1 , φ_2 all the way up to all of the angular coordinates and r in each of these boxes and that is going to be the transformation from these angular coordinates into these Cartesian coordinates and that's going to show you the Spherical coordinate transformation for some higher dimension. There is now there's a problem with what we've done so far in that it doesn't actually work in the very last case, you end up doing this process over and over again and you don't quite get all the way to the end. Let me show you where you do get, you get to this point here I will show you the general term after you've done all the recursive work, you're gonna find out that:

$$\vec{X} = \sum_{i=1}^{n-2} r \left[\prod_{k=1}^{i-1} \sin\varphi_k\right] \cos\varphi_i \vec{e}_i + r\left[\prod_{k=1}^{n-2} \sin\varphi_k\right] \vec{U}_{n-1} \qquad (25)$$

You can't get rid of $\vec{U}_n$ term using this process, when you're in $\vec{U}_{n-1}$ that means you are in a 2D space, there's two dimensions left and so $\vec{U}_{n-1}$ we have to substitute that out in terms of these basis vectors and we can't. This first part of (25) captures this process for $n-1$ iterations and you end up with the first part of (25), this guy and it shows you exactly: it's a sum of all the basis vectors up to $n-2$ so you still have $n-1$ and n to deal with and the second part of (25) tells you that you have one more unit vector $\vec{U}_{n-1}$ but $\vec{U}_{n-1}$ is composed of $\vec{e}_{n-1}$ and $\vec{e}_n$, in other words:

$$\vec{U}_{n-1} = \kappa^{n-1} \vec{e}_{n-1} + \kappa^n \vec{e}_n \qquad (26)$$

That's because $\vec{U}_{n-1}$ is a 2D vector that's built out of the last two of our basis vectors so it's some coefficient times $\vec{e}_{n-1}$ plus some coefficient times $\vec{e}_n$, that's sort of stating the obvious but if I knew what $\vec{U}_n$ was exactly, notice I'd have $\vec{X}$ defined completely in terms of the basis vectors and in terms of the angular functions of the angle between $\vec{X}$ and each of the basis vectors $\vec{e}_i$ because φ_k is defined to be the angle between $\vec{X}$ and each of the basis vectors $\vec{e}_i$. The idea is that these are the Spherical angular components that go from 0 to π and that defines the coefficient of each of these basis vectors which is the coordinate transformation.

I'll show you how that actually flushes out in a moment but we're still left with this last one and the substitution doesn't work for the last one but that's okay because the last one has got to be in this form so it turns out that it's still true that $\vec{U}_{n-1}$ is a unit vector:

$$\left|\vec{U}_{n-1}\right|^2=1=\left(\kappa^{n-1}\right)^2+\left(\kappa^n\right)^2 \tag{27}$$

We know that there's some angle λ such that these two these two κ components can be written as the $\cos^2\lambda$ and $\sin^2\lambda$ therefore I now know that $\vec{U}_{n-1}$ is got to be equal:

$$\vec{U}_{n-1}=\cos\lambda\,\vec{e}_{n-1}+\sin\lambda\,\vec{e}_n \tag{28}$$

There's one more angle out there, we've gotten all of these angles up to $n-2$ in the form of these φ_k and then when we get to the last one we are in this 2D subspace but we still dealing with a unit vector so we have one more angle λ . Now the problem is that now there's really no restriction on λ because this component κ^{n-1} i.e. $\cos\lambda$ that could be negative or positive so now λ this last one has to be $0\leq\lambda\leq 2\pi$. The last angle gets a full range. Thinking of this in terms of our Spherical coordinates we have θ and then we have φ , well we know that φ is allowed to go all the way around the circle in principle $\varphi\in(0,2\pi)$ but $\theta\in(0,\pi)$ so what we're now seeing is that the Spherical coordinates for n dimensions has a lot of them that look like θ but only one that looks like φ so no matter how high your dimension is there's only one full $\varphi\in(0,2\pi)$ coordinate in any spherical coordinate system in n dimensions but there's a lot of $\theta\in(0,\pi)$ coordinates well $n-2$ of these of these θ style coordinates.

We're almost finished, now we can write down sort of the final answer when you put all this stuff together what does it ultimately end up looking like and I'll write it in a way where you can see each of the coordinate transformations it starts with X^i :

$$\begin{cases} X^1=r\cos\varphi_1 \\ X^i=r\cos\varphi_i\prod_{k=1}^{i-1}\sin\varphi_k \text{ for } 1<i<n-1 \\ X^{n-1}=r\sin\lambda\prod_{k=1}^{n-2}\sin\varphi_k \\ X^n=r\cos\lambda\prod_{k=1}^{n-2}\sin\varphi_k \end{cases} \tag{29}$$

That's exactly what we see in the regular Spherical coordinates we see that the projection of a vector onto this plane, you had the last part this φ part you have the cosine part and the sine part. Let's actually break that down so let's look at the familiar case where $n=3$ (Spherical coordinates). For $n=3$ we don't even have the second line in (29) we have X^1, X^2, X^3 . How does that look like?

$$\begin{cases} X^1 = r\cos\varphi_1 \\ X^2 = r\cos\lambda\sin\varphi_1 \\ X^3 = r\cos\lambda\sin\varphi_1 \end{cases} \tag{30}$$

To make this our Spherical coordinates we just write this as $\varphi_1 = \theta$ and $\lambda = \varphi$. Now the reason we did that is to get to the “Glome” which is the 4D case so what does that look like?

$$\begin{cases} X^1 = r\cos\varphi_1 \\ X^2 = r\cos\varphi_2\sin\varphi_1 \\ X^3 = r\sin\lambda\sin\varphi_1\sin\varphi_2 \\ X^4 = r\cos\lambda\sin\varphi_1\sin\varphi_2 \end{cases} \tag{31}$$

That is the coordinates that is Spherical coordinates for the “Glome” for the 3-sphere We noticed we basically have θ_1 and θ_2 and you still have a φ and an r and that “Glome” is the 3-sphere and the reason I just find this interesting is because it's generalizing the higher dimensions is something to kind of get used to doing and I've always kind of wondered about this until I found this paper which made it really clear that you have to keep introducing these things the reason these two are similar is because $\theta_1, \theta_2 \in (0, \pi)$ but this last one always goes $\varphi \in (0, 2\pi)$ and we kind of saw how that worked in the development here.

Now you have a “Glome” and so the “Glome” has a metric and this is the metric, the spatial part of (2.1.15), where this last coordinate φ , that's the one that goes from $(0, 2\pi)$ while $\xi, \theta \in (0, \pi)$. The first part ψ is the time but remember we're in compactified coordinates so the time looks like an angle but it goes from $(-\pi, \pi)$ but remember we're talking about a time that's compactified but it really goes, the real time is going from $(-\infty, \infty)$ but it is compactified but you can now imagine a different kind of metric where instead of the spatial part of (2.1.15) forget the compactify just go $-dt^2$ plus the spatial part of (2.1.15), this is now the metric on the surface of the “Glome” where I've now defined what these angles are. You see now the whole point of the exercise I just did, to give you some understanding of what these angles could be these ξ, θ, φ , it's 4D that's where ξ comes from and you can see there's the $\sin^2\xi$, so the point is now if I just put a regular time up there I'm gonna end up with something that has this infinite time $t \in (-\infty, \infty)$ but in each instance in time we have a “Glome” and it's going to be a manifold that's going to be typically called S^3 which is $S^3 \times \mathbb{R}$ where $\mathbb{R}$ is time and this metric here (2.1.15) is in fact a solution of a certain version of the Einstein equation on this manifold where these S^3 coordinates are “Glome” coordinates and that's kind of why I did this whole exercise so you can see the “Glome” coordinates.

That's all for now like I said this was a bit of a diversion and we will get back to this ultimately when we talk about cosmological metrics but now I think I need to turn back to the catalog and we start need to go through the fundamental tensors of general relativity so I'll start that next time, thanks.