QED Prerequisites Geometric Algebra 15: Complex Structure

Introduction

Thank you, welcome back to our examination of this paper "Spacetime algebra as a powerful tool for electromagnetism". I'm very much enjoying reading through this paper in great detail except I'm noticing that it's taking quite a long time, we are working at basically like a full course for each of these sections and that's running a little less than an hour but some of them are very close to an hour so if I roughly set an hour for each of these you know already there's like 17 hours just in Section 3 so if I had planned this out I probably would have tightened it up to a point where I wouldn't have even done it at all but lucky for everybody or lucky for me I didn't actually plan it out and ergo I'm going to persist and I have the time to finish it so if you have the time to listen, let's just keep going.

Where are we now? We are at complex structure, we have done the pseudo-scalar, Hodge duality we've done all of this material and we are going to do complex structure, let's begin. We left off at this paragraph right here, "The duality transformation induced by I is equivalent to the *Hodge-star* transformation in differential forms (though arguably simpler to work with)". Now, I talked about that last time, there is a lot of differences with the Hodge-star transformation in its overall structure in differential forms of course, having to do with you're actually switching between unrelated well, vector spaces that are not part of the same vector space necessarily and things like that but it is the literal equivalent but what is important about this is that this I "splits the space time into two halves that are geometric complements of each other." and this is the secret that's going to allow us to introduce something that they call complex structure into this into our organization of the multi-vectors of the space-time algebra.

"Exploiting this duality, we can write any multi-vector M in an intrinsically complex form (in the sense of I)", meaning that the pseudo-scalar I replaces the notion of the imaginary number i . We get that's not what we're talking about we're going to talk about complexification relative to this pseudo-scalar and this pseudo-scalar is going to pair all quantities with their dual and when you do this our multi-vector which previously was M previously the multi-vector, let's see if I write it down, it was:

$$M=\alpha+v+\boldsymbol{F}+\mathscr{F}+\beta I \tag{1}$$

What we're going to do is we're going to take all of this stuff (1) and we're going to actually compress it into just three terms, now it's a bookkeeping matter in some sense right because this first term this piece right here is actually going to be just the sum of the scalar pieces meaning the actual scalar piece and the pseudo-scalar piece, when I say the scalar pieces I guess I mean the dimension one sub-spaces so you take the two dimension one sub-spaces and we're going to talk about their sum which is actually a dimension two sub-space $\zeta\equiv\alpha+\beta I$ so this piece here and this piece here get combined together into a complex scalar so we're going to call that the complex scalar ζ and that's going to be defined this way.

Complex Vectors

That doesn't seem so weird, the part that is a little bit stranger is this other part the complex vector so let me rearrange this a little bit so here we have this complex vector piece is what's interesting because now z which would have been a lowercase Roman letter which is now going to be representing a complex vector has a vector part which is grade one and a tri-vector part which is grade three, now we don't write it as a literal tri-vector, we write it as the dual of a vector so any tri-vector that's of interest in our analysis, if we have some tri-vector T , we have to turn that tri-vector into the dual of a vector and you can always do this, every tri-vector there is there exists a vector whose dual is that tri-vector so the way we write a complex vector is with a regular vector summed with a tri-vector but that tri-vector must be expressed as the dual of a vector and this is complicated a little bit also by the fact that over here I commutes with β , the pseudo-scalar commutes with all real numbers $\beta I = I \beta$ but I does not commute with vectors, I anti-commutes with vectors, these two statements are actually equal because the pseudo-scalar anti-commutes with vectors:

$$z \equiv v + w I = v - I w \tag{2}$$

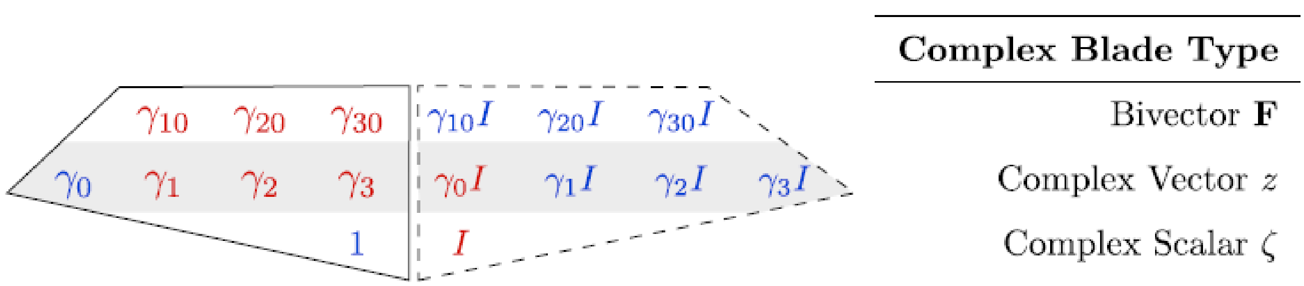
This doesn't act strictly like a regular complex number would, ζ does, this is literally like a complex number from complex analysis, (2) does not act like a complex number and so now we can ask the question, well, what are space-time products between complex scalars and complex vectors but that's not hard because that's just by linearity going to be is the space-time product that looks like:

$$\zeta z = (\alpha + \beta I)(v + w I) = \alpha v + \alpha w I + \beta I v + \beta I w I \tag{3}$$

You just blow this up by linearity and you get the space-time product of a scalar with a vector plus a space-time product of a scalar with a tri-vector plus the space-time product of a pseudo-vector with a vector plus the space-time product of a pseudo-scalar and a tri-vector. All of these things, we can calculate any one of these things with what we know now so that's not tough but it's also true that this does not commute $\zeta z \neq z \zeta$. I think they actually talk about that in a moment so using this duality we can write every tri-vector as the dual of a vector and then we can take the vector and dual tri-vectors, squeeze them together into so-called complex vector and likewise we can squeeze the scalar and pseudo-scalar together and create a complex scalar.

Illustration

They say “We illustrate this decomposition in Figure 5”. I've already discussed this part but let's look at Figure 5 and see how their illustration goes so here's Figure 5 and it says “Complex structure of the space-time algebra”.



The complex blade type, there's bi-vector blades, there's complex vectors and complex scalars so buried in the complex scalar now is the regular scalar and the pseudo-scalar and buried in the complex vector is the regular vector and the tri-vector and the bi-vector remains uncomplexified. “The complementary Hodge-dual halves of the algebra illustrated in Figure 4 become intrinsically complex pairings by factoring out the pseudo-scalar $I = \gamma_{0123}$. That satisfies $I^2 = -1$. The complex scalars and vectors have parts that are independent geometric objects. However the bi-vectors are self-dual and irreducibly complex objects that cannot be further decomposed into frame-independent parts”.

What's going on here? The complex decomposition includes one I on this side (left) is basically the side that is not being multiplied by I and this side (right) is the side being multiplied by I so that's the complex decomposition of the space now notice that although these could be basis vectors these are not the same basis vectors that we were mentioning before because there's going to be a sign issue that we went through before but this is a literal Hodge-duel, by definition, those two are Hodge-dual.

The Hodge-dual stuff is completely obvious in this decomposition and the part that is ζ is any factor times one like $\alpha+\beta I$ so this section is a basis for ζ . This whole section here is a basis, that's this complex vector space for scalars ζ , z is the complex vector space defined by these objects (middle line in the figure above) where the vector part is made of the left part and the tri-vector part is made of the dual of the vector part (right) and then the bi-vectors they're arguably divided up into a complex part but what it really is, is the part that includes the time-like piece and the part that does not, that's orthogonal to the time-like piece so this is a bit of a different kind of decomposition and we will talk about how this splits up later on but this complex structure is usually, well, so far it's referring to vector plus tri-vector and scalar plus pseudo-scalar but what's really nice is using this methodology you no longer see tri-vectors written down, you no longer see γ_{123} or γ_{023} or anything like that all you see are the dual of the tri-vectors written down so you've compressed the notation even further and you've basically gotten rid of the explicit tri-vector form.

Orthogonal complement

To carry on with the reading, “ α , β are real scalars, and v , w are four-vectors. As anticipated, the tri-vector $\mathscr{T} = wI = -Iw$ ”, which is the dual of a vector w , “has been expressed as the orthogonal complement of a vector, (i.e., a *pseudo-vector*)”, that's another way of saying this here in the figure above, the right part contains the orthogonal complements of γ_0 , γ_1 , γ_2 and in a four-dimensional space the orthogonal complement of a vector is a tri-vector is a whole volume, in three-dimensional space the orthogonal complement of a vector is a plane constructed of two vectors that are both independently orthogonal of the vector and if you have two vectors they define a plane and that plane has an orthogonal vector to it and so two vectors you can actually identify a vector as being orthogonal to two vectors and you can create this cross product notion but in four-dimensional space two vectors define a plane but any given plane has another plane that it's orthogonal to so you don't have this possibility of a simple cross product in four dimensions which is the reason for that, we talked about that in a previous lesson and those are pseudo vectors.

Quasi-commute

“Some care has to be taken with manipulation of complex vectors z ”, that's the vector $a+bI$, “since I anti-commutes with vectors. As a result, complex scalars and vectors only quasi-commute”.

$$\zeta z=(\alpha+\beta I)z=az+\beta Iz=za-\beta zI=(\alpha-\beta I)\neq z\zeta \tag{4}$$

Here you have an example, you have a complex scalar ζ times z a vector. ζ and α commute because we know that I anti-commutes with ζ so that's not the same thing as $z\zeta$ so because of that, there's this half of it commutes but the other half doesn't so they call it quasi-commute, I would just say they don't commute but I guess there's some value in understanding that there's this quasi commutation but they don't commute “while complex scalars and bi-vectors commute normally.”

Complex scalars

Complex scalars and bi-vectors don't have this problem if z was a bi-vector it would commute with I because remember I commutes with anything in the even sub-space which includes the scalars the bi-vectors and the quad-vectors, the scalars the pseudo-scalars and the bi-vectors so “Thus the pseudo-scalar I is only algebraically equivalent to the usual notation of the scalar imaginary i when restricted to the *even* graded sub-algebra of complex scalars and bi-vectors.” and that's of course because in real complex numbers the imaginary number i commutes with everything, complex number i commutes with, well there isn't much to commute with but the point is if we're going to substitute I and we're going to say that it corresponds with i , we have to be very careful to have no tri-vectors floating around which we now write as vectors as the orthogonal complement of vectors which is the dual of a vector and this still is a pseudo-vector, that's another way of saying it, if you have any pseudo-vectors floating around I does not commute with pseudo-vectors.

Bi-vectors

Let's go on to section 3.5.1 *Bi-vectors: canonical formulas.* “A bi-vector like the electromagnetic field, $\boldsymbol{F}$ ”, now notice they're anticipating that the electromagnetic field will be a bi-vector, this paper hasn't described why, we should know because if we're reading this we probably know about the differential form of electromagnetism but understand that ultimately the electromagnetic field will be described by a bi-vector so “A bi-vector like the electromagnetic field, $\boldsymbol{F}$, is an interesting special case for the complex decomposition of a general multi-vector, since the full bi-vector basis is already *self-dual*.”, that means $\boldsymbol{F}$ is going to be some combination, we've already written this down $F^{\mu\nu}\gamma_{\mu\nu}$ and this thing here covers all possibilities of $\gamma_{\mu\nu}$ but we know that they're all dual to one another from our relationships, in other words from the definition of duality and in fact they calculate it all right here:

$$\begin{aligned} \gamma_{10}I=-\gamma_{23}, \quad \gamma_{20}I=-\gamma_{31}, \quad \gamma_{30}I=-\gamma_{12}, \\ \gamma_{23}I=\gamma_{10}, \quad \gamma_{31}I=\gamma_{20}, \quad \gamma_{12}I=\gamma_{30}. \end{aligned} \tag{5}$$

All of the bi-vectors their dual is another bi-vector so if you're a general bi-vector you're constructed out of bi-vectors and their dual right which is obvious if you're dealing with something that is self-dual so that's the important part. “Since the full bi-vector basis is *self-dual*”. It's an interesting case for the complex decomposition so how do you do a complex decomposition when the basis itself is self-dual so that's what they're asking so how do you do this? What are they talking about? Now, “geometrically, these duality relations express the fact that in four dimensions the orthogonal complement to any plane is another plane (not a normal vector as in three dimensions). We can express these relations more compactly by collecting cyclic permutations and writing $\gamma_{i0}I=-\epsilon_{ijk}\gamma_{jk}$ and $\gamma_{jk}I=\epsilon_{ijk}\gamma_{i0}$, where $i,j,k=1,2,3$ and ϵ_{ijk} the completely anti-symmetric Levi-Civita symbol (no summation implied).”

I don't find this very enlightening, I've never found these definitions that have these Levy-Civita symbols in them as really be time savers or anything like that, in this case it's much easier just to calculate these things straight up (5), it's not hard but in principle you could figure out, you could work in all the subscripts and get these guys to match this formula and I don't find these formulas very insightful but you have to do it if it's possible, you've got to express it this way. You have these formulas that capture the obvious.

Let's see, “This self-duality of the bi-vector basis makes the signature of the general bi-vector *mixed*”. The signature of the general bi-vector mixed, well that's really interesting so first of all this a signature we've been so far when we've been did our signature calculations we came up with the interesting $+1$ or -1 so they're saying it's mixed so let's figure out exactly what that means. “That is, although each basis bi-vector has a well-defined signature $\gamma_{i0}\tilde{\gamma}_{i0}=(1)(-1)=-1$ or $\gamma_{jk}\tilde{\gamma}_{jk}=(-1)(-1)=1$ ”, that is, each basis vector of the bi-vector sub-space has a well-defined signature and then they throw down this little example of γ_{i0} multiplied by its reverse is -1 but of course that's not exactly the formula, the formula is:

$$\varepsilon_{\gamma_{i0}}=\frac{\tilde{\gamma}_{i0}\gamma_{i0}}{|\tilde{\gamma}_{i0}\gamma_{i0}|}=\frac{\gamma_{0i}\gamma_{i0}}{|\gamma_{0i}\gamma_{i0}|}=\frac{-1}{|-1|}=-1 \tag{6}$$

They've ignored the denominator here and they've ignored it because, come on, these are unit vectors we know this is going to be 1 , fair enough, but understand that the real definition of signature is not just a multi-vector times its reverse you've got to divide by this factor and it just so happens if you're dealing with a unit vector it's pretty easy but I just want to point that out, if you're a fully spatial bi-vector γ_{jk} because, remember, these Roman indices are $1,2,3$ so these are complete these are spatial plane elements well that times its reverse is going to be -1 times -1 divided by 1 and you get 1 so you have a positive signature for the purely spatial parts and a negative signature for the time-like parts which if we go back to the figure above, don't forget that the time-like planes that have a time-like element to it, that time-like basis vector as part of its plane, the plane that's extending in the time-like direction those have signature minus one and these guys, the dual of those planes, the dual of these time-like planes are all of the space-like planes which we no longer really write as space-like planes, we write them as the dual of time-like planes, to capture this complex structure and those guys have signatures of $+1$, very well.

General bi-vector

Now they say, “although each basis bi-vector has a well-defined signature” and they give these two examples, “a general bi-vector $\boldsymbol{F}$ will be a mixture of these two different signatures”, sure because a general bi-vector, being some linear combination of basis vectors that have negative signatures and basis vectors that have positive signatures, what is the signature of a general bi-vector? “Consider the simple mixed bi-vector $\boldsymbol{F}=\gamma_{10}+\gamma_{23}$ as an example”. All right before we move on let's just do this computation ourselves. Actually they demonstrate it right here, in the very next paragraph:

$$\begin{aligned}\tilde{\boldsymbol{F}}\boldsymbol{F}&=[\gamma_{01}+\gamma_{32}][\gamma_{10}+\gamma_{23}]\\&=\gamma_{01}\gamma_{10}+\gamma_{01}\gamma_{23}+\gamma_{32}\gamma_{10}+\gamma_{32}\gamma_{23}\\&=\gamma_1^{\,2}\gamma_0^{\,2}+\gamma_{0123}+\gamma_{3210}+\gamma_2^{\,2}\gamma_3^{\,2}\\&=(-1)(1)+I+I+(-1)(-1)=2I\end{aligned} \tag{7}$$

Then you blow this up by linearity to get this structure here (2nd line in (7)). Now what's interesting is when you get this structure you're going to find that you're going to get pieces that multiply all the way down to a real number and this is pretty obvious this is going to be $\gamma_1^{\,2}\gamma_0^{\,2}$ which is going to be -1 , well that's what you see right here (3rd line in (7)). Then you're going to get a part that is a quad-vector but notice how they designed the beginning, they designed $\boldsymbol{F}$ to be completely orthogonal, γ_{10} is completely orthogonal to γ_{23} , in other words you don't have say a γ_{12} in there, which would blend together with γ_{01} to give you something more like $\gamma_{01}\gamma_{12}$, in which case you would end up with a bi-vector because the γ_1 parts would go away and you would end up with a γ_{02} piece. Now to be clear if you did have that, if you did have a piece in here that was not orthogonal to one of these two factors all of these bi-vectors would end up canceling out interestingly enough maybe we can demonstrate that later so the result is that this structure $\tilde{\boldsymbol{F}}\boldsymbol{F}$ for a bi-vector $\boldsymbol{F}$ a general bi-vector $\boldsymbol{F}$ is always going to end up with just pseudo-scalars and scalars which is why we can use this example to be fully general.

When you continue to multiply this out you get $\gamma_{0123}+\gamma_{3210}$ and you can determine quickly γ_{3210} is the same as γ_{0123} which is just $I+I$ and then this final part $\gamma_{32}\gamma_{23}$, becomes $(-1)(-1)=1$ which cancels with that -1 so the real parts actually cancel and you're left with a purely imaginary part for this particular bi-vector $\boldsymbol{F}=\gamma_{10}+\gamma_{23}$. You end up with a pseudo-scalar result but take note that you could have ended up and you more likely would have ended up with a mixed result, you if this number were $\boldsymbol{F}=\gamma_{10}+2\gamma_{23}$ imagine that, well then you would have:

$$
\begin{aligned}
\tilde{\boldsymbol{F}}\boldsymbol{F} &= [\gamma_{01}+2\gamma_{32}][\gamma_{10}+2\gamma_{23}] \\
&= \gamma_{01}\gamma_{10}+2\gamma_{01}\gamma_{23}+2\gamma_{32}\gamma_{10}+4\gamma_{32}\gamma_{23} \\
&= \gamma_1^2\gamma_0^2+2\gamma_{0123}+2\gamma_{3210}+4\gamma_2^2\gamma_3^2 \\
&= (-1)(1)+2I+2I+4(-1)(-1)=3+4I
\end{aligned}
\tag{8}
$$

A general bi-vector reversion product or the reversion square of a bi-vector can end up with what we're now going to call a complex number $\zeta=3+4I$, it could be purely imaginary which is in our language purely pseudo-scalar or purely scalar.

Complex Polar Form

Now the rest of this section talks about how you can take this final result of the reversion square and put it into a polar form a complex polar form so for example $\zeta=3+4I$ if I was to just naturally left to my own devices to put it in a complex polar form I would take the magnitude of it which is $\sqrt{3^2+4^2}$ and I would multiply it by a phase angle which would be the inverse tangent of $4/3$ and this number here in complex analysis is completely equivalent to ζ and we understand how do I raise, I need an I there, a pseudo-scalar I . Now here we are using e raised to basically a real number times the pseudo-scalar I i.e.,, but we're comfortable treating that exactly like the imaginary complex number, it's a different object for sure than these basic imaginary complex number but it is still true that they both square to -1 and that's what's critical about the Taylor expansion of this thing so the Taylor expansion of this thing will work out exactly the same way as it would with a complex number so this is always going to be:

$$
e^{\beta I}=\cos\beta+I\sin\beta \tag{9}
$$

We can always use this equivalently just like we would in complex analysis and we're going to do this a lot and we're also going to do this with hyperbolic numbers meaning where I squares not to -1 but I squares to 1 so that's the next step, that's what this next set of paragraphs does but what I'm going to do is I'm actually going to go off menu and we're going to do this analysis instead of from this paper, we're going to do it from a paper written by David Hestenes himself because I think it's a little bit more comprehensive. This paper says hey we can obviously proceed this way and we can, it's true, you just think of this in terms of of magnitudes and phase shifts and you can do all of this, they do throw in a factor of one half here that I don't fully understand and it does not show up in Hestenes paper in the same way. There is a factor of a half in Hestenes paper but it's not put into this phase angle so I want to figure out what that's all about so we're going to go off script and study this important concept of creating a canonical form for an arbitrary bi-vector $\boldsymbol{F}$. We create this canonical form where any bi-vector $\boldsymbol{F}$ can be turned into a canonical bi-vector multiplied by a phase factor and we're going to do that from a different paper and then we'll come back to this paper so I'll see you next time.