QED Prerequisites Geometric Algebra 14: The Pseudoscalar

Thank you, welcome back, we are going to continue our study of this paper “Space-time algebra as a powerful tool for Electromagnetism” and this is intended to be part of a general introduction to Geometric algebra as best I can but I really do want to focus on the space-time algebra because that's what's most applicable to, at least elementary Physics and we have made huge progress, we've gotten all the way through Section 3.4 and now we are going to start on Section 3.5 *The Pseudo-scalar I, Hodge duality and complex structure,* so let's begin. “The expansion (3.7) of a multi-vector in terms of elements of differing grade is similar to the expansion of a complex number into its real imaginary parts as we emphasized in Section 3.3.” As a reminder this is the expansion 3.7, a multi-vector has a scalar part, a vector part, a bi-vector part, a tri-vector part and a pseudo-scalar part and already the pseudo-scalar part has been broken down into a real number times I where we have defined $I\equiv\gamma_{0123}$ which we can immediately translate to $\gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3$ because it's orthogonal, these vectors are all orthogonal and all of the lower grades vanish but the highest grade remains:

$$M=\alpha+v+\boldsymbol{F}+\mathscr{T}\ +\beta I \tag{1}$$

Every one of these others, you'll notice, is just given its own symbol but this last one is a real number β times I because there's only one dimension in the pseudo-vector space, this is all important for what we're going to talk about today so I always represents this thing, this quad-vector and it always gets slapped on at the back as the pseudo-vector. Turns out we're going to do this a couple more times, the simplification you're going to see a few more times and in fact is a little bit of foreshadowing while we're here, we could rewrite this as, let's see how could we rewrite this, hold on, we could rewrite this:

$$M=[\alpha+\beta I]+v+\boldsymbol{F}+\mathscr{T} \tag{2}$$

We are making this $\alpha+\beta I$ taking the pseudo-scalar, the one-dimensional pseudo-scalar and the one-dimensional real number, putting them together and then adding the vector, bi-vector and tri-vector and then this kind of looks like a complex number right and that's the idea that we're pursuing now in this Section is, this naturally has the feeling of a complex number especially when we remember that $I^2=-1$ in our particular algebra, our space-time algebra, so that's a little bit of foreshadowing.

Then they write, “In fact, this similarity”, that is, the similarity of real and imaginary parts, “is more than an analogy because the pseudo-scalar I also satisfies $I^2=-1$, which makes the sub algebra of the form $\zeta=\alpha+\beta I$ completely equivalent in practice to the complex scalar numbers. Hence we do not need to additionally complexify the algebra in order to reap the benefits of complex analysis; the complex algebraic structure automatically appears within the space-time algebra itself.” That just means we have access to complex numbers inside the space-time algebra if we think of the scalar and the pseudo-scalar part, the sum of them with this addition, is, of course the same kind of addition I talked about in my lesson on vector addition here but if we freely think of this as a complex number this will behave and produce all the results of complex analysis. Now, one thing to be a little bit careful of, is this notion of I^2, now as a review I^2 is literally the space time product of I with itself and you have to be very facile using this rule that for basis vectors that are orthogonal we get the anti-commute, so you can switch a and b as long as you introduce a minus sign:

$$\gamma_\alpha\gamma_\beta=-\gamma_\beta\gamma_\alpha \tag{3}$$

If you write this and the fact that the space-time product itself is associative these two facts together allow you to start moving these things around and shuffling these things around until everything is next to each other and you end up with:

$$I^2 = I\,I = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_0{}^2\gamma_1{}^2\gamma_2{}^2\gamma_3{}^2 \quad (4)$$

When you rearrange them so that each of these is squared, it turns out these $-$ end up landing on no $-$ at all and you just take this product. Now, normally if this was Euclidean space, that would be $(1)(1)(1)(1)=1$ because in Euclidean space $\gamma_\mu \cdot \gamma_\nu = \delta_{\mu\nu}$ so the space-time contraction of these basis vectors is this δ function and it would always be 1 but for the space-time algebra where we have the Minkowski contraction we get $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$ which induces a -1 right here for the $\gamma_0{}^2$ part so (4) becomes -1 . If you were doing Geometric algebra that was just four-dimensional Euclidean $I^2 \neq -1$, but $I^2 = 1$ so it is not true that for every Geometric algebra this pseudo-scalar squared, squares to -1 , that's not always true but in fact, it's usually not true for regular four-dimensional Geometric algebra, it's entirely this Minkowski metric that I don't know if it saves the day or ruins the day but it forces this pseudo-scalar to actually square to -1 so that's kind of an interesting point to take, well what matters is the total number of basis vectors you have which depends on in our case we're using $\mathsf{M}_{1,3}$ but the base it depends on the underlying vector space that establishes your grade one section $\Lambda_1(V)$, this vector space, whatever the dimensionality of that vector space is, that's the number of basis vectors you're going to be playing with and that number of basis vectors determines how many flips you need and whether this final sign will end up positive or negative. If everything's Euclidean it really depends on the dimensionality of your fundamental vector space but if everything's not Euclidean it depends on the dimensionality of the fundamental vector space and the nature of the metric so just keep that in mind, it's that it's not always true that the pseudo-scalar squared is -1 for every Geometric algebra.

To go on, “The pseudo-scalar I has intrinsic geometric significance that goes well beyond the theory of scalar complex numbers, however, and has several additional interesting and useful properties. It commutes with the elements of even grade (i.e., scalars, pseudo-scalars and bi-vectors), but anti-commutes with elements of odd grade (i.e., vectors and tri-vectors).” All right, let's check that out let's study how a basis vector has a space-time product form with a pseudo-scalar I :

$$\gamma_\mu I = \gamma_\mu\gamma_0\gamma_1\gamma_2\gamma_3 \overset{?}{=} -I\gamma_\mu \quad (5)$$

A nice space-time product form, we can go ahead and start commuting these things so I want to move γ_μ from the front to the back so if I did that, if I was able to do that that would be $I\gamma_\mu$ and that would be the commutation, I guess, of the beginning so the question really is, is that an equal sign or is there going to be some sort of sign difference between the two? Well, I can take γ_μ and pass it through γ_0 and I'll get a sign change if $\mu \neq 0$ but if $\mu = 0$ then I'll get a sign change because if $\mu \neq 0$ you don't even have to commute you just start commuting the γ_0 over so then I can commute it past γ_1 and again I will get a sign change if $\mu \neq 1$ then I can pass it through γ_2 and I'll get a sign change as long as $\mu \neq 2$ and then I finally pass it through γ_3 and I'll get a sign change if $\mu \neq 3$ but here's the deal, μ has to equal one of these four but if it equals one of those four, it will not pick up a sign change when it passes by that particular indexed basis vector, but well, it'll only not pick up a sign once in this process and every other time it'll pick up a sign so it has to make one, two, three, four flips, one of which will not pick up a sign so it will make three sing changes of which ultimately that means it's an odd number of sign changes which means we are left with a sign change a few so that's a long way of saying that the pseudo-scalar anti-commutes with each of the basis vectors and if it anti-commutes with each of the basis vectors you can do your own analysis and see that it will also anti-commute with a general vector a you know expressed in the reference basis and this is all true for the reciprocal basis as well so this is this this demonstrates that it anti-commutes with vectors. Now likewise tri-vectors are going to have the same issue, with tri-vectors you would have something like this:

$$\gamma_\mu \gamma_\nu \gamma_\beta I = \gamma_\mu \gamma_\nu \gamma_\beta \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -I \gamma_\mu \gamma_\nu \gamma_\beta \tag{6}$$

You would have three basis factors up front as each part of your tri-vector basis and γ_β will go to the back and get a sign change γ_ν will go to the back pick up a sign change which will cancel the sign change that the γ_β got but then γ_μ is waiting out there to go and pick the sign change back up so clearly I will anti-commute with tri-vectors as well. However, it will not with scalars, scalars don't matter at all and it clearly commutes with itself $II = II$, that's an easy one so we've accounted for every case, what about bi-vectors? Well bi-vectors would be without γ_μ :

$$\gamma_\nu \gamma_\beta I = \gamma_\nu \gamma_\beta \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -I \gamma_\nu \gamma_\beta \tag{7}$$

γ_β goes through and picks up a negative sign, fine but then γ_ν goes through and gets rid of that minus sign so it commutes with bi-vectors so that's the explanation of those different commutation rules so when they say: “It *commutes* with elements of even grade (i.e., scalars, pseudo-scalar and bi-vectors), but *anti-commutes* with elements of even grade (i.e., vectors and tri-vectors).” All right, we showed that. “More surprisingly, a right product with I is a *duality transformation* from grade k to its orthogonal complement of grade $4-k$ ”. Let's focus here on this really important concept is I a duality, let's just go with transformation from grade k to its orthogonal complement of grade $4-k$. Let's see exactly how that works, we start with a grade zero object which we'll call the real number α and then I'm going to take that real number α and I'm going to right multiply by the pseudo-scalar so I'm going to get:

$$\alpha I = \alpha \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \alpha \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \tag{8}$$

Space-time product of all the basis vectors which because of the orthogonality of the basis vectors equals this quad-vector so I've taken something from grade zero and multiplying by I and now have something in grade four right so this multiplying by I on the right has converted something from grade zero to grade four. All right, let's do it for vectors, we start with an object v and now I'm going to multiply v by I :

$$\begin{aligned} vI &= v^\alpha \gamma_\alpha \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \left(v^0 \gamma_0 + v^1 \gamma_1 + v^2 \gamma_2 + v^3 \gamma_3\right) \gamma_0 \gamma_1 \gamma_2 \gamma_3 \\ &= v^0 \left(\gamma_0{}^2\right) \gamma_1 \gamma_2 \gamma_3 + v^1 \gamma_0 \gamma_2 \gamma_3 - v^2 \gamma_0 \gamma_1 \gamma_3 + v^3 \gamma_0 \gamma_1 \gamma_2 \end{aligned} \tag{9}$$

By linearity we now are dealing with these four products with I so the first one is really easy $\gamma_0{}^2 = 1$. In order to simplify the second term I need to move this γ_1 from the left of γ_0 to the right so I can combine it with this γ_1 that's already there and get $\gamma_1{}^2 = -1$ so if I make that switch I introduce a $-$ sign but then I'm left with $\gamma_1{}^2 = -1$ in the convention that we are committed to so that $-$ sign with the swap cancels with the $-$ sign of $\gamma_1{}^2 = -1$. This final object is a tri-vector so a vector space-time product with the pseudo-scalar gives you a tri-vector.

That's the point that we are trying to drive home earlier that a vector times I gives you a pseudo-vector, you move from one subspace to another and if you did this process for a bi-vector what you would discover is that the bi-vector would simply meaning, well I guess we should do it:

$$F^{\mu\nu}\gamma_{\mu}\gamma_{\nu}\gamma_{0}\gamma_{1}\gamma_{2}\gamma_{3} \quad (10)$$

First I write the bi-vector in this form using our semi-compressed notation I guess the fully compressed notation would use would go like that $\gamma_{\mu\nu}$, we're going to use the semi-compressed notation because I want to demonstrate these space-time products so we write it this way and then I'm going to multiply by I so I'm going to multiply by $\gamma_{0}\gamma_{1}\gamma_{2}\gamma_{3}$. Now I could blow up all just like I did before I could blow up all six of these guys but I'm not, I'm going to just exemplify it with say $F^{12}\gamma_{1}\gamma_{2}$ and you'll see how this can generalized to all the others

$$F^{12}\gamma_{1}\gamma_{2}\gamma_{0}\gamma_{1}\gamma_{2}\gamma_{3} \quad (11)$$

The goal is to demonstrate what do we get when we do this product. Well, I'm going to take γ_{2} , I'm going to move it over here, past γ_{0} which gives me a sign change and I'm going to move it over here past γ_{1} which gives me another sign change which cancels the sign change and then I have γ_{2}^{2} which leaves a sign change and so that would take me to:

$$-F^{12}\gamma_{1}\gamma_{0}\gamma_{1}\gamma_{3} \quad (12)$$

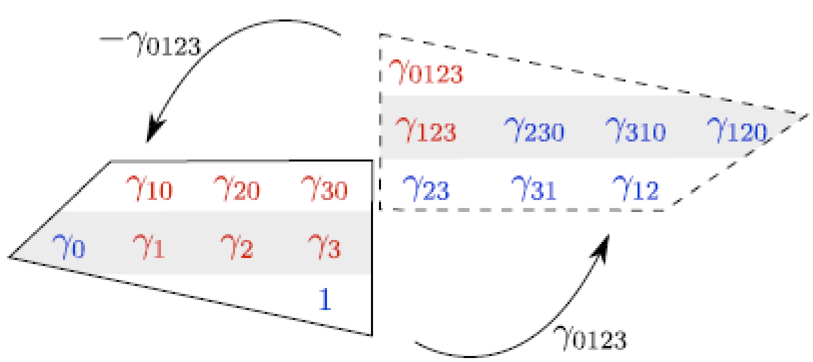
Then once I've done that then what's left? Well, γ_{1} moves over past γ_{0} which introduces the sign change and then γ_{1}^{2} is another sign change so we get no sign change. I get:

$$-F^{12}\gamma_{0}\gamma_{3}=-F^{12}\gamma_{0}\wedge\gamma_{3} \quad (13)$$

That is a bi-vector so bi-vectors times I go back into the bi-vectors. I'm not going to do three and four, four is really easy, I'll do four, you take a pseudo-scalar βI and you multiply it by I and you get $-\beta$ because $I^{2}=-1$ in the space-time algebra so objects of grade four obviously go back into objects of grade zero, likewise you'll see objects of grade three go back into objects of grade one so this process is a duality process that means I can take any object that's grade zero and I can find its dual in grade four likewise any vector has a dual vector in grade three where the duality is related to this multiplication by the pseudo-scalar I . Two is self-dual, there are vectors in two that are dual to other vectors of grade two so that's what we're getting at. Now the other word that they used here that was important was orthogonal complement and if you look at what we've done here in this case for example we were curious about $F^{12}\gamma_{1}\gamma_{2}$, $\gamma_{1}\gamma_{2}$ that's a little blade out there, that's a little blade that it's $F^{12}\gamma_{1}\wedge\gamma_{0}$ some little blade object but the result is $\gamma_{0}\gamma_{3}$, notice that neither γ_{1} nor γ_{2} appear in this blade so this blade here is actually orthogonal to original blade in the sense that it's not composed of any of the same basis vectors which would be different if we had started with, well in contrast this was $F^{13}\gamma_{1}\gamma_{3}$ which is $F^{13}\gamma_{1}\wedge\gamma_{3}$ this γ_{1} is shared with this γ_{1} so they do have a basis vector in common so that's what they mean by orthogonal complement and this was also true when we did the when we did the vector, when we looked at the vector we started with say $v^0 \gamma_0$ but when we did that calculation we ended up with the tri-vector that had $\gamma_1 \gamma_2 \gamma_1$ in it and no γ_0 and of course that makes sense because we squared out the γ_0 because ${\gamma_0}^2=1$. Likewise ${\gamma_1}^2=-1$ and it had no inflation part because of the orthogonality so the only part that existed was the contraction part and that's the Minkowski contraction which turned it into a real number.

Each of these components of the vector v returned through this duality transformation by multiplying by I an orthogonal tri-vector. Normally in three-dimensional space if you had a vector everything in the space orthogonal to the vector would live in a plane but in a four-dimensional space these orthogonality are volumes which is kind of fun and this is also true so this is how it always works this Hodge duality process of multiplying on the right by I by the pseudo-scalar produces an orthogonal vector that lives in the complementary or the dual sub-space so if you start with a grade k , the dual is grade $4-k$ so if I started in grade three, the dual is four minus three which is a vector so the dual of a tri-vector is a vector, the dual of a bi-vector is another bi-vector, the dual of a scalar is a pseudo-scalar so that is how all of this language here works, the grade $4-k$ part, that's what this I just went through that and they say, "a right product with I is a *duality transformation* from gate k to its orthogonal complement of grade $4-k$.", so this notion of this duality, this word duality it's trying to say that there's something about vectors and tri-vectors which the duality means these guys each one of these has an associated buddy in this dual space and there's a strong relationship between them and we're going to take advantage of that relationship in the rest of this paper.

Just reading on, maybe a little redundantly we see, "its unique oriented orthogonal tri-vector", which is given by $\gamma_0 I=\gamma_0 \gamma_0 \gamma_1 \gamma_2 \gamma_3=\gamma_{123}$, this is the dual of the vector γ_0 and you know because you're right multiplying by I , that's the definition of the duality transformation, a right product with I so I can read this as the dual of the vector γ_0 and then they work it out, they put here's γ_0 and here's I and they know that this is going to go away to be one, what's left behind is γ_{123} , that's really easy to show and they also do this again, the dual of the vector γ_{123} , the dual of that is $\gamma_{123} I=(\gamma_0 I) I=-\gamma_0$ and the way they show this is that they already know that γ_{123} is the dual of $\gamma_0 I$ so they just make that substitution $\gamma_0 I$ and then you have I times I because everything's associated, these are space-time products so everything's associative so $I^2=-1$ in our space-time algebra so you get that negative there and you have $-\gamma_0$. Now take note of this, they make it pretty clear, "the dual of a tri-vector is its unique normal vector" and here's an example, get the dual of γ_{123} is $-\gamma_0$, remember that particular one because I'm going to illustrate a confusion in a moment. "This duality is illustrated for the entire graded basis in Figure 4" so let's look at Figure 4:



“Figure 4: Hodge duality is illustrated using the same basis as Figure 2”. Well that's these basis vectors as described here in other words it's γ_{230} instead of γ_{023} as I would have done although that is the same basis vector isn't it? These are in fact the same because you flip zero twice so the sign doesn't change, however, γ_{310} I would have done γ_{013} so that would have been two flips and then another flip so that would have been the opposite so I would have had the opposite in my system it would have been γ_{013} but I've already spoken about that it's not when I say my system it's this thing I would have done differently, I have yet to figure out exactly why they've chosen these basis vectors, there is a reason and we're going to find the reason but now is not the time to to worry about it because this is a perfectly fine basis.

What this picture must be saying is that if you take the segment of the algebra spanned by these basis vectors, the solid basis vectors, the base vector surrounded in solid and you take its Hodge dual, you will land in the subspace spanned by these basis vectors surrounded by a fracture line and what's interesting about this is when I looked at it I was assuming somehow that the dual of each of these basis vectors, if you if you executed the Hodge dual and multiplied each one by I you would end up exactly with each of the basis vectors described on the right hand side so the dual of γ_0 we know is γ_{123} , we just did that calculation, the dual of γ_1 , which I calculated before, and notice I'm using that language because you see the I on the right, you see a multi-vector on the left so I can call this the dual of this multi-vector, whenever you see that I that you can start saying that, I did this calculation and I came up with $\gamma_1 I=\gamma_{023}=\gamma_{230}$. Those two are the same by exchange, you just move the γ_0 over two spots right and two spots cancel, two shifts cancel so you do get γ_1 going to γ_{230} , then I did γ_2 , I checked γ_2 and the dual of γ_2 is γ_{310} and then I did γ_3 so the dual of γ_3 is γ_{120} .

It is true that the dual of the basis vectors are in fact the basis tri-vectors in this system so now what about the bi-vectors? What it happens with the dual of γ_{10} ? Let's look at γ_{10} . Well, the dual of γ_{10} is according to my calculations $-\gamma_{23}$. This dual goes to here but there's a $-$ sign introduced. Likewise, the dual of γ_{20} gives you $-\gamma_{31}$ and γ_{30} gives you $-\gamma_{12}$. What about the dual of the pseudo-vector $I=\gamma_{0123}$? Well, the dual of the pseudo vector is obviously -1 so the dual of a pseudo-vector actually goes here with a minus sign. It's not true that every basis vector on the right side is the dual of every basis vector on the left side. I was surprised, I thought for sure that that was the way this was going to be organized but it probably can't be organized that way because if you think about how this whole process works the duality is not an Involution so you couldn't get back from this side to this side, in other words the dual of these guys, if you calculated them would not necessarily come back to it's pre-dual, I suppose is the right way of saying it and in fact I did that calculation too so let me erase this and let's look at these other dual. We did this one, we did the dual of γ_{0123} and we know that that comes back with -1 so we've already done that one but what about these other guys?

Well, we've done some of them already, the dual of the tri-vector $\gamma_{123} I=\gamma_0 I\, I=-\gamma_0$, Well, that's really easy because we know that the tri-vector γ_{123} is the dual of γ_0 so that times I so you get $-\gamma_0$ and by the way that's going to be true for all of the basis vectors, the tri-vector space basis vectors they're all going to come back to the opposite of the basis of the of the vector space and likewise if you look at the dual of the purely spatial bi-vectors and by purely spatial, I always mean there's no γ_0 which is the time like vector so all of these are planes in absolute space $\gamma_{23} I=(-\gamma_{10} I)I=\gamma_{10}$ so the two I cancel out and you actually get back γ_{10} itself so the dual of the bi-vector γ_{23} is actually the bi-vector γ_{10} . Likewise for $\gamma_{31} I=(-\gamma_{20} I)I=\gamma_{20}$ and $\gamma_{12} I=(-\gamma_{30} I)I=\gamma_{30}$, that's all straight up but these guys, the tri-vector basis vectors, they go to their opposites.

A $-$ appears there γ_{230} , if you do its calculation ends up with $-\gamma_1$, γ_{310} goes to $-\gamma_2$, γ_{120} goes to $-\gamma_3$ on duality transformation by I and I did those calculations here and they're all written down here $\gamma_{123} I=\gamma_0 I\, I=-\gamma_0$, $\gamma_{230} I=\gamma_1 I\, I=-\gamma_1$, $\gamma_{310} I=\gamma_2 I\, I=-\gamma_2$, $\gamma_{120} I=\gamma_3 I\, I=-\gamma_3$ so these guys come in, their dual has a $-$ sign over here so it's not true that the dual of every basis vector in both of the sides of the house here is another basis vector, it's this $-$ sign might sometimes appear, we've already seen, of course, that the dual of the quad-vector itself, the dual of the pseudo-scalar itself comes back with a $-$ sign for example so I was expecting these things to match up perfectly but they can't and I'm sure that the reason is traceable to the fact that I duality is not an involution which means multi-vector $(M\, I) I \neq M$ (not necessarily), sometimes it will but it doesn't necessarily equal M so that was something that I had to pull from this picture.

The part that I don't understand is the introduction of this $-$ sign here $-\gamma_{0123}$ so he introduces this $-$ sign, that's not a duality transformation that's literally the opposite, well the -1 times a duality transformation so if I do a right multiplication by that thing I end up with the opposite of the Hodge dual of something so I'm not sure why they put that in there but what kind of worries me is when I read the text of Figure 4, it says, “Right multiplication by the pseudo-scalar $I=\gamma_{0123}$ converts an algebraic element to its dual”. Now first understand that an algebraic element is a general algebraic element, that's an arbitrary multi-vector M . Don't let me lead you to think that duality transformations must be from the space span by these vectors into the space span by those vectors it almost makes it look as though that's the case this figure does but that's not true because I could take any multi-vector and I could even have this multi-vector $M=\gamma_0+\gamma_{31}$ that clearly is a multi-vector that has a vector part in this space and a bi-vector part in this space so this multi-vector is in both of those spaces but that multi-vector has a dual and that dual is $M\, I$ and that dual is going to clearly be $\gamma_{123}+\gamma_{20}$ so that's the dual of the multi-vector M , that's the dual of it.

Every algebraic element has a dual and that dual is the orthogonal complement in the geometric sense the entire space complementary to γ_0 is going to be spanned by γ_{123} , any other vector in the entire space is going to be inside, let's see how do I say? The largest part of the piece that's complementary to γ_0 is this this 3D volume element γ_{123} likewise γ_{31} it's missing the γ_{20} parts so it's “orthogonal complement in the geometric sense.” and then they give you this example “For example, the dual of a tri-vector γ_{123} is its unique normal vector $\gamma_{123}(-I)=\gamma_0$.” What gives it that $-$ sign? I see it here and I still don't understand it but that's not the way we defined a dual, we defined a dual by right multiplication by the pseudo-scalar I not the pseudo-scalar $-I$ and so now they have the dual of a tri-vector is its unique normal vector γ_0 but that exactly contradicts what they wrote down here $\gamma_{123} I=(\gamma_0 I) I=-\gamma_0$. They made it really clear that the dual of γ_{123} which is the right multiplication by I is $-\gamma_0$ which is what I would have calculated so I'm not going to pretend to understand exactly what they're getting at here. “Dual elements have opposite signatures indicated here by a flip in color coding.” so we'll talk about signatures in a moment but I just want to point out that this Figure 4 is a little bit awkward to my taste because I don't fully understand the introduction of this $-$ sign and I don't like the fact that this here contradicts what's in the text and I haven't found a satisfactory way through all of this so we're going to stick with the notion that the dual is always right multiplication by I and the part that my expectation that these guys had to literally be the dual of these guys that's just not true, it's the space spanned by these guys is totally dual to the space spanned by these guys and then on top of that the implication that the duality transformation only goes from the space spanned by these guys to the space spanned by these basis vectors that's sort of implied by this diagram but it isn't true and I've just given you an example, any multi-vector has a dual and that dual is right multiplication by I so that means we've covered the pseudo-scalar I and we've covered Hodge duality.

I want to cover this "Dual elements have opposite signatures" let's have a look at that so we begin with our favorite one $\gamma_0 I=\gamma_{123}$, the one that we've been scratching our heads about the dual of γ_0 is γ_{123} and you can see here is $\gamma_0\gamma_0\gamma_1\gamma_2\gamma_3$. This one's particularly well suited because or particularly easy because we just have $\gamma_0{}^2$ which through associativity we combine to $\gamma_0{}^2=1$ because it's the time like part so we know right away that this equals γ_{123} . Now what is the signature of γ_0 ? Let's remember how these things are defined so we have these three formulas to work with and this will give us a little practice to remind ourselves about reversions, the reversion square and the straight up square.

$$\varepsilon_M=\frac{\widetilde{M}\,M}{|M|^2}\quad,\quad |M|^2=|\widetilde{M}\,M|\quad,\quad \widetilde{M}\,M=\varepsilon_M|M|^2 \tag{14}$$

Let's calculate the signature for γ_0 . Well, that's not hard, we're going to first take the γ_0 reversion times γ_0 divided by the magnitude of $\gamma_0{}^2$ which is the absolute value of γ_0 reversion γ_0 :

$$\varepsilon_{\gamma_0}=\frac{\widetilde{\gamma}_0\gamma_0}{|\widetilde{\gamma}_0\gamma_0|}=\frac{\gamma_0{}^2}{|\gamma_0{}^2|}=\frac{1}{1}=1 \tag{15}$$

At the top the γ_0 of a vector has its own reversion, so we get to review that fact so this is just $\gamma_0{}^2$ on top and this is $|\gamma_0{}^2|$ on the bottom which is one over one which equals one so the signature of γ_0 is one. What about the signature of γ_{123} ? Well, it's going to be:

$$\varepsilon_{\gamma_{123}}=\frac{\widetilde{\gamma}_{123}\gamma_{123}}{|\widetilde{\gamma}_{123}\gamma_{123}|}=\frac{\gamma_{321}\gamma_{123}}{|\gamma_{321}\gamma_{123}|}=\frac{-1}{1}=-1 \tag{16}$$

This is beautiful because it's obviously we expand that into $\gamma_3\gamma_2\gamma_1\gamma_1\gamma_2\gamma_3$ and then we just start squaring things that squares to -1 then this squares to another -1 and then this last one also squares to -1 , that's three -1 so the upper product equals -1 the lower product the same thing but it's squeezed in these absolute value sign so the lower product is 1 so the signature of γ_{123} is equal to -1 and the statement was that the signatures of the dual part are opposite, well that has a signature of 1 ant the dual part has the signature of -1 . On Figure 4, the way you see it is γ_0 is blue for 1 and γ_{123} is red for -1 and this applies, that same methodology applies for all of them and none of them are any more or less interesting, they're all pretty straightforward as soon as you take the reversion you don't have to worry about shifting things left to right and how many shifts do you have and that all goes away with reversion, that's why reversion is so convenient so dual elements have opposite signatures is a general principle that they're offering and we just did an example of that.

Let's read on "this duality is illustrated for the entire graded basis in Figure 4". I spent a lot of time talking about that, "We will see in Section 5.2 that this duality is intimately connected with field-exchange *dual symmetry* of the electromagnetic field". Section 5.2 seems awfully far away at the rate we're going but we'll get there, so "This duality transformation induced by I is equivalent to the

Hodge-star transformation in differential forms (though is arguably simpler to work with)", I do agree with that, it is definitely arguably it is not even arguable to me, I think it's clearly simpler to work with but it is equivalent to the Hodge duality, I'm not going to review Hodge duality but we did talk about it quite a bit in the previous QED prerequisite lectures when we talked about of the electromagnetic field using differential forms, "splits the space-time algebra into two halves that are geometric complements of each other." That's the two halves we saw above but don't think that you can't be a multi-vector that dips into both halves, now "Exploiting this duality we can write any multi-vector M in an intrinsically complex form (in the sense of I)". This is a good place to stop we will talk about the complex structure, we'll talk about complex structure in our next lesson the next lesson we'll talk about complex structure. We've made some pretty good progress and I'll see you next time.