4616 Time Series Analysis

Eric Hillebrand
K140, Fuglesangs Allé
ehillebrand@econ.au.dk

Homework 1

due Wed, Sep 27, 2017 in class
Total: 60 points, Pass for 40 points or above.

1. (10 points) Consider the AR(1) model

$$x_t = c + \phi x_{t-1} + \varepsilon_t,$$
$$\varepsilon_t = WN(0, \sigma^2),$$

where $|\phi| < 1$, $\phi \neq 0$; c and $\sigma^2 > 0$ are constants. Derive the expected value $\mathbb{E}x_t$, the variance $\mathbb{E}(x_t - \mathbb{E}x_t)^2$, and the autocovariance function $\gamma(h) = \mathbb{E}[(x_t - \mathbb{E}x_t)(x_{t-h} - \mathbb{E}x_{t-h})]$.

2. (5 points) Consider the MA(1) model

$$x_t = c + \theta\varepsilon_{t-1} + \varepsilon_t,$$
$$\varepsilon_t = WN(0, \sigma^2).$$

Derive the expected value $\mathbb{E}x_t$, the variance $\mathbb{E}(x_t - \mathbb{E}x_t)^2$, and the autocovariance function $\gamma(h) = \mathbb{E}[(x_t - \mathbb{E}x_t)(x_{t-h} - \mathbb{E}x_{t-h})]$.

3. (15 points) Let X and Y be random variables with means and variances $\mu_X = \mathbb{E}X$, $\mu_Y = \mathbb{E}Y$, $\sigma_X^2 = \operatorname{Var} X$, $\sigma_Y^2 = \operatorname{Var} Y$ and correlation

$$\rho_{XY} = \frac{\operatorname{cov}(X, Y)}{\sigma_X \sigma_Y} \neq 0.$$

Show that the best linear predictor of Y given X, i.e.

$$\hat{Y} = aX + b$$

in the sense of minimizing the prediction error variance $\mathbb{E}(\hat{Y} - Y)^2$ is given by

$$a = \frac{\operatorname{cov}(X, Y)}{\sigma_x^2},$$
$$b = \mu_Y - \frac{\operatorname{cov}(X, Y)}{\sigma_X^2}\mu_X.$$

As a corollary of the above result, show the following. Let $\{x_1, x_2, \ldots\}$ be a time series with constant mean $\mu = \mathbb{E}x_t$ for all t and constant variance $\sigma^2 = \operatorname{Var} x_t$ for all t and

$$\gamma(h) = \operatorname{cov}(x_t, x_{t-h}),$$
$$\rho(h) = \frac{\gamma(h)}{\operatorname{Var} x_t}$$

the autocovariance and autocorrelation at lag h, respectively. Then, the best linear h-periods-ahead predictor

$$\hat{x}_{t+h} = ax_t + b$$

is given by

$$\begin{aligned} a &= \rho(h), \\ b &= \mu(1 - \rho(h)). \end{aligned}$$

4. (10 points) Show that

$$\begin{bmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{bmatrix}^t = \begin{bmatrix} \cos(\lambda t) & \sin(\lambda t) \\ -\sin(\lambda t) & \cos(\lambda t) \end{bmatrix}, \; t \in \mathbb{N}.$$

5. (20 points) Load the U.S. Housing Starts data 1959:1 to 2017:5 in `HOUSTNSA.xlsx` into Ox (or your favorite software) and find a basic structural time series model

$$\begin{aligned} y_t &= \log(\mathrm{HOUSTNSA}_t) \\ y_t &= \mu_t + \gamma_t + \varepsilon_t, \end{aligned}$$

where ε_t is the estimation residual, γ_t a seasonal dummy or trigonometric seasonal component, and μ_t is a deterministic trend function. For the trend, consider the alternative functions

$$\begin{aligned} \mu_t &= c + \alpha_1 t \\ \mu_t &= c + \alpha_1 t + \alpha_2 t^2 \\ \mu_t &= c + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 \\ \mu_t &= c + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4. \end{aligned}$$

Compare the alternatives using Akaike's Information Criterion (AIC) and Schwarz's Information Criterion (SIC/BIC). Discuss the fit and residual diagnostics. Submit a graph of the actual data together with the fitted values and a graph of the residuals.