3/4

Daniel Winkler

30. Oktober 2017

1 Construction of two random variables with given covariance

Let $Z_{1,t},\ Z_{2,t}$ be i.i.d. $N(0,1)$. We want to construct two variables $X_t,\ Y_t$ s.t.

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{x,y} \\ \sigma_{x,y} & \sigma_y^2 \end{bmatrix} \right) \tag{1}$$

Let $X \sim N(\mu_x, \sigma_x^2)$. Then:

$$\frac{X - \mu_x}{\sigma_x} \sim N(0,1) \equiv Z_{1,t} \tag{2}$$

It follows that:

$$\mu_x + \sigma_x Z_{1,t} \sim N(\mu_x, \sigma_x^2) \tag{3}$$

Now let $Z_{3,t} = \rho Z_{1,t} + \sqrt{1-\rho^2} Z_{2,t}$. We can show that this is also a $N(0,1)$:

$$\begin{aligned} &\rho Z_{1,t} + \sqrt{1-\rho^2} Z_{2,t} \\ &= \rho N(0,1) + \sqrt{1-\rho^2} N(0,1) \\ &= N(0,\rho^2) + N(0, 1-\rho^2) \end{aligned} \tag{4}$$

As $Z_{1,t}$ and $Z_{2,t}$ are independent:

$$N(0,\rho^2) + N(0,1-\rho^2) = N(0, 1-\rho^2+\rho^2) = N(0,1) \tag{5}$$

Thus:

$$\mu_y + \sigma_y Z_{3,t} \sim N(\mu_y, \sigma_y^2) \tag{6}$$

We still need to show that $cov(X,Y) = \sigma_{x,y} = \sigma_x \sigma_y \rho$.

$$\begin{aligned} cov(X,Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \mathbb{E}[(X-\mu_x)(Y-\mu_y)] \\ &= \mathbb{E}[(\sigma_x Z_{1,t} + \mu_x - \mu_x)(\sigma_y(\rho Z_{1,t} + \sqrt{1-\rho^2} Z_{2,t}) + \mu_y - \mu_y)] \\ &= \sigma_x \sigma_y \mathbb{E}[\rho Z_{1,t}^2 + \sqrt{1-\rho^2} Z_{1,t} Z_{2,t}] \end{aligned} \tag{7}$$

Lets look at the second part of the sum. $\sqrt{1-\rho^2}$ is a constant and can thus be taken out of the expectation. As $Z_{1,t}$ and $Z_{2,t}$ are independent:

$$\mathbb{E}[Z_{1,t} Z_{2,t}] = \mathbb{E}[Z_{1,t}]\mathbb{E}[Z_{2,t}] = 0 \tag{8}$$

where the last equality follows from the definition of the variables. Thus,

$$\begin{aligned} &\sigma_x \sigma_y \mathbb{E}[\rho Z_{1,t}^2 + \sqrt{1-\rho^2} Z_{1,t} Z_{2,t}] \\ &= \sigma_x \sigma_y \rho \mathbb{E}[Z_{1,t}^2] \end{aligned} \tag{9}$$

As $Z_{1,t}^2$ is the square of one normally distributed variable, we know that $Z_{1,t}^2 \sim \chi_1^2$. Since the expected value of a χ^2 distributed variable is just its degrees of freedom $\mathbb{E}[Z_{1,t}^2] = 1$. Thus:

$$cov(X,Y) = \sigma_x \sigma_y \rho \tag{10}$$

2 Conditional Expectation and Variance

2.1 Conditional Expectation of Y|X

$$\begin{aligned} \mathbb{E}[Y|X] &= \mathbb{E}[\mu_y + \sigma_y(\rho Z_{1,t} + \sqrt{1-\rho^2} Z_{2,t})|X] \\ &= \mathbb{E}[\mu_y + \sigma_y(\rho \frac{X-\mu_x}{\sigma_x} + \sqrt{1-\rho^2} Z_{2,t})|X] \end{aligned} \tag{11}$$

$\mathbb{E}[\frac{X-\mu_x}{\sigma_x}|X] = \frac{X-\mu_x}{\sigma_x}$. Note that in this case the realization of X is given, thus the fraction is a constant. μ_y, σ_y and ρ are chosen constants and thus independent of X. Thus we get:

$$\begin{aligned} &\mathbb{E}[\mu_y + \sigma_y(\rho \frac{X-\mu_x}{\sigma_x} + \sqrt{1-\rho^2} Z_{2,t})|X] \\ &= \mu_y + \sigma_y(\rho \frac{X-\mu_x}{\sigma_x} + \sqrt{1-\rho^2} \mathbb{E}[Z_{2,t}|X]) \end{aligned} \tag{12}$$

Since $Z_{1,t},\ Z_{2,t}$ are independent and the parameters $\mu_x,\ \sigma_x$ are chosen independently of $Z_{2,t}$:

$$\mathbb{E}[Z_{2,t}|X] = \mathbb{E}[Z_{2,t}] = 0 \tag{13}$$

Thus,

$$\mathbb{E}[Y|X] = \mu_y + \sigma_y(\rho\frac{X-\mu_x}{\sigma_x}) = \mu_y + \sigma_y\sigma_x\rho\frac{X-\mu_x}{\sigma_x^2} = \mu_y + \sigma_{xy}\frac{X-\mu_x}{\sigma_x^2} \tag{14}$$

2.2 Conditional Variance of Y|X

$$Var(Y|X) = Var(\mu_y + \sigma_y(\rho\frac{X-\mu_x}{\sigma_x} + \sqrt{1-\rho^2}Z_{2,t})|X) \tag{15}$$

As $\mu_y,\ \sigma_y,\ \rho$ and $\frac{X-\mu_x}{\sigma_x}|X$ are constants and thus have no variance:

$$= Var(\sigma_y\sqrt{1-\rho^2}Z_{2,t}|X) = Var(\sigma_y\sqrt{1-\rho^2}Z_{2,t}) \tag{16}$$

because of independence of X and the constants as well as $Z_{2,t}$.

$$Var(\sigma_y\sqrt{1-\rho^2}Z_{2,t}) = \sigma_y^2(1-\rho^2)Var(Z_{2,t}) = \sigma_y^2 - \sigma_y^2\rho^2 \tag{17}$$

Multiplying and dividing the second part by σ_x^2 yields:

$$Var(Y|X) = \sigma_y^2 - \frac{\sigma_x^2\sigma_y^2\rho^2}{\sigma_x^2} = \sigma_y^2 - \frac{\sigma_{xy}^2}{\sigma_x^2} \tag{18}$$

3 Problem 5

$$\begin{aligned}\gamma(h) = \mathbb{E}(x_t x_{t+h}) &= \mathbb{E}[(U_1 cos(\lambda t) + U_2 sin(\lambda t))(U_1 cos(\lambda[t+h]) + U_2 sin(\lambda[t+h])] \\ &= \mathbb{E}[U_1^2 cos(\lambda t)cos(\lambda(t+h)) + U_2^2 sin(\lambda t)sin(\lambda(t+h)) + \\ &U_1U_2 cos(\lambda t)sin(\lambda(t+h)) + U_2U_1 sin(\lambda t)cos(\lambda(t+h))]\end{aligned} \tag{19}$$

Because $U_1,\ U_2$ are random variables they are independent from each other and from the other terms. Thus,

$$
\begin{aligned}
&= \mathbb{E}[U_1^2]\mathbb{E}[cos(\lambda t)cos(\lambda(t+h))] + \mathbb{E}[U_2^2]\mathbb{E}[sin(\lambda t)sin(\lambda(t+h))] + \\
&\mathbb{E}[U_1U_2]\mathbb{E}[cos(\lambda t)sin(\lambda(t+h))] + \mathbb{E}[U_2U_1]\mathbb{E}[sin(\lambda t)cos(\lambda(t+h))] \\
&= \mathbb{E}[U_1^2]\mathbb{E}[cos(\lambda t)cos(\lambda(t+h))] + \mathbb{E}[U_2^2]\mathbb{E}[sin(\lambda t)sin(\lambda(t+h))]
\end{aligned} \tag{20}
$$

By definition of $U_1,\ U_2$

$$
= \sigma^2(\mathbb{E}[cos(\lambda t)cos(\lambda(t+h))] + \mathbb{E}[sin(\lambda t)sin(\lambda(t+h))]) \tag{21}
$$

We can drop the expectation because non of the terms are random anymore. Using the product identities this can be rewritten as:

$$
\begin{aligned}
&\sigma^2(\frac{1}{2}(cos(\lambda t - \lambda[t+h]) + cos(\lambda[t+h] + \lambda t)) + \\
&\quad \frac{1}{2}(cos(\lambda t - \lambda[t+h]) - cos(\lambda[t+h] + \lambda t))) \\
=&\frac{1}{2}\sigma^2(cos(\lambda t - \lambda t + \lambda h) + cos(\lambda t - \lambda t + \lambda h) \\
=&\sigma^2 cos(\lambda h)
\end{aligned} \tag{22}
$$

Problem 6

Show that:

$$
\int_{-\pi}^{\pi} e^{i(k-h)\lambda} d\lambda = \begin{cases} 2\pi, \text{ if } k = h, \\ 0, \text{ otherwise} \end{cases}
$$

For $k \neq h$ we evaluate the indefinite integral:

$$
\begin{aligned}
&\int e^{i(k-h)\lambda} d\lambda \\
&= \int cos((k-h)\lambda) + isin((k-h)\lambda) d\lambda \\
&= \int cos((k-h)\lambda) d\lambda + i \int sin((k-h)\lambda) d\lambda \\
&= \frac{sin((k-h)\lambda)}{k-h} - i\frac{cos((k-h)\lambda)}{k-h}
\end{aligned} \tag{23}
$$

Now we evaluate from $-\pi$ to π:

$$\begin{aligned}
&\frac{sin((k-h)\lambda)}{k-h} - i\frac{cos((k-h)\lambda)}{k-h}\Bigg|_{-\pi}^{\pi} \\
&= \frac{sin((k-h)\pi)}{k-h} - i\frac{cos((k-h)\pi)}{k-h} - \left(\frac{sin((k-h)(-\pi))}{k-h} - i\frac{cos((k-h)(-\pi))}{k-h}\right)
\end{aligned} \tag{24}$$

Since $k,\ h \in \mathbb{N}$ it follows that $sin((k-h)\pi) = sin((k-h)(-\pi)) = 0$.

$$i\frac{cos((k-h)(-\pi))}{k-h} - i\frac{cos((k-h)\pi)}{k-h} = 0 \tag{25}$$

as $cos((k-h)(-\pi)) = cos((k-h)\pi) = -1$.