1 Question 1

Let $X = (X_1, X_2, ..., X_K)'$ be a random vector with mean zero and covariance matrix

$$\Gamma = \mathbf{E}XX' \tag{1}$$

Assume Γ is singular. Then there exists an eigenvalue $\lambda_1 = 0$ of Γ with corresponding eigenvector $v_1 = (v_{11}, v_{12}, ..., v_{1K})'$. We know that

$$\Gamma v_1 = \lambda_1 v_1 = \mathbf{0_K} \tag{2}$$

where $\mathbf{0_K}$ denotes the zero vector of length K. Henceforth it will be denoted by $\mathbf{0}$. It is equivalent to say that

$$\mathbf{E}XX'v_1 = \mathbf{0}. \tag{3}$$

We can multiply v_1' from the left to get the variance of $v_1'X$.

$$\begin{aligned} \mathbf{E}v_1'XX'v_1 &= \mathbf{E}(v_1'X)(v_1'X)' = Var(v_1'X) \\ &= v_1'\mathbf{0} = 0 \end{aligned} \tag{4}$$

Since the variance is zero, we conclude that $v_1'X$ is deterministic and thus equal to a constant d. Thus we find

$$v_1'X = v_{11}X_1 + v_{12}X_2 + \cdots + v_{1K}X_K = d \tag{5}$$

We can rearrange (5) to

$$d - v_{1j}X_j = v_{11}X_1 + \cdots + v_{1j-1}X_{j-1} + v_{1j+1}X_{j+1} + \cdots + v_{1K}X_K, \ j \in (2, K-1) \tag{6}$$

Without loss of generality j can be equal to 1 or K as well by deducting the appropriate v and X instead when moving from (5) to (6). Now divide by v_{1j} and define $\alpha_i := -\frac{v_{1i}}{v_{1j}}$ for $i = 1, ..., K$. It follows that $\alpha_j = -1$. Thus

$$X_j + \frac{d}{v_{1j}} = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_K X_K \tag{7}$$

Now let $c := -\frac{d}{v_{1j}}$. Then

$$X_j = c + \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_{j-1} X_{j-1} + \alpha_{j+1} X_{j+1} + \cdots + \alpha_K X_K \tag{8}$$

$\gamma\gamma$