Analysis of Minimum Spanning Trees in Random Graphs

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CS124 Programming Assignment 1 Writeup
https://github.com/gzhang01/cs124prog/tree/master/prog1

Abstract

Our task was to find appropriate functions $f(n)$ to model the weight of the minimum spanning tree (MST) of a random, complete graph of n edges (in the 0-dimensional case) or n vertices (in the 2, 3, and 4-dimensional cases). Using C, we implemented a random graph generator and used Kruskal's algorithm to produce the MST of the graph. Our results showed that for the $d = 0$ case, we have the best fit function $f(n) = 1.2$, and for $d = 2, 3$, and 4, we have a best fit function of the form $f(n) = a \cdot n^{(d-1)/d} + c$, with constants a and c that are dependent on the value of d.

Implementation

Our first task was to construct the random graph generator (RGG) (see creategraph.c). We started with the 0-dimensional case, where we just assign each edge a random floating point value as a weight. To produce this random value, we seeded C's random number generator (RNG) with the current microsecond time, and used C's built-in rand() function. Of course, rand() produces a random integer between 0 and RAND_MAX, and so we converted them to floats and normalized so our values would fall between 0 and 1.

We then needed some way of representing an edge. We defined a new "edge" struct that contained two integers representing the vertices of the edge and a float representing the weight. Given this, finishing the 0-dimensional RGG was simple: loop over all possible combinations of edges and create a struct with the appropriate endpoints and random weight for each one.

The 2, 3, and 4-dimensional cases were simple as well. We created a two dimensional (n by d) array, where each row represents a d-dimensional point. For each of the coordinates, we produced a random floating point value, and for all possible combination of two vertices, we produced an edge struct. We now had two separate functions for generating graphs, so we wrote a wrapper function that seeds the RNG and then decides with generator to call based on the input value d. To finish off the graph generators, we wrote a simple print function that would allow us to test and debug our graphs.

Now that we had a way to generate random graphs, we could move on to producing the MST of a given graph. We decided to go with Kruskal's algorithm, as we thought the real-world run-time of Prim's algorithm would be much longer than Kruskal's. We noticed that if we were to use Prim's algorithm, we would have to look at all vertices connected to our current MST, and we decided that repeated looping through all those edges would be very time-inefficient. With Kruskal's algorithm, the main time bottleneck would be sorting all the edges, but after that, we would simply take the shortest of those remaining. We decided the one-time cost of sorting was preferable to the repeated searching we would have to do with Prim's.

Our task then was to implement a disjoint set data structure (see node.c). We created a node struct containing a value, a pointer to its parent node, and a rank, and implemented the functions makeSet, find, and join. Their implementations are very similar to the pseudocode we were given in class, so we will not go into specifics here. We again created a print function to help test and debug our code.

With the disjoint set data structure in place, we began the implementation of Kruskal's algorithm (see mst.c). We wrote a merge sort function to sort our edges. We then took the edge with the lowest weight, determined if the two vertices are already in our MST, and if not, added the edge to our MST. To save time when running the algorithm, we kept track of how many edges we had already inserted into our MST, and stopped searching when we reached the desired number (number of vertices minus one).

We now had a working algorithm to find the MST of a randomly generated complete graph. Our tests for small n passed easily, but unfortunately, when we tried for larger values of n, we ran into some memory errors. Our implementation had put certain data structures (mostly arrays) on the stack, and as n increased, the amount of memory needed to store the data structures increased as well. Ultimately, we began touching memory we didn't have access to when trying to access our large arrays, triggering segmentation faults. To resolve this issue, we ended up putting all data structures on the heap and passing around pointers instead.

The next issue we faced was run time. As n increased, the number of edges increased on the order of n^2, and so it became more and more time and memory costly to generate and sort the graph. To resolve this, we decided to prune our graph as we were generating it. We knew that once we had found a MST for a given graph, the rest of the edges that had not been searched yet had weights that were simply too high to possibly be included in the graph. We wanted to find a general estimate for this threshold, so that we could simply ignore the edge if we generate one that's too long and save time and space by not having to store and sort it.

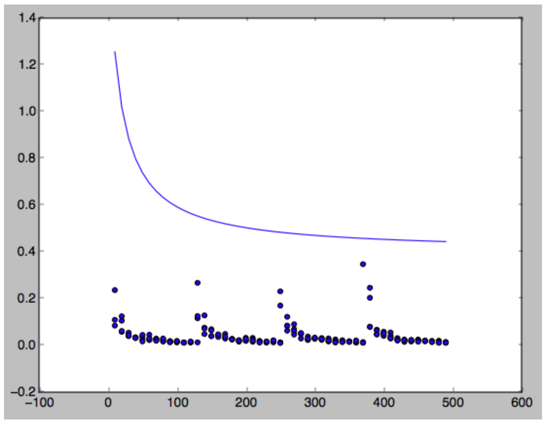


Figure 1: Actual vs. Predicted Max Edge Length in MST (n ∈ (0, 500), d = 0)

Thus we wanted some function $k(n)$ that could predict, given n, or the number of vertices in the graph, a value e such that any edge length greater than e could not possibly end up in the MST. We tweaked our program to record the maximum edge length in an MST and started collecting data on it. Using some Python scripts we wrote, we plotted the maximum edge length in a MST with n vertices (sample graphs seen at

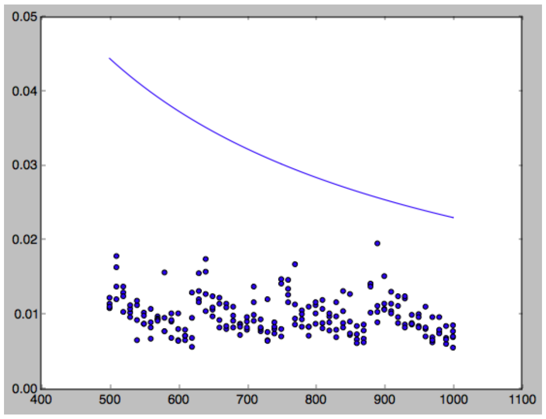


Figure 2: Actual vs. Predicted Max Edge Length in MST (n ∈ (500, 1000), d = 0)

right).

Using these scripts, we found that this threshold not only depended on the number of vertices, n, but also on the number of dimensions, d. We decided to not find one single function that would work for any given n, d, but instead found 4 functions that were dependent on d. In addition, we found that it would have been difficult to have a good fit for the entire range we were dealing with in a single expression, and so we essentially created piecewise functions. Each of these pieces (0-500, 500-1000, 1000+) were fitted to best approximate the data. Once we implemented pruning based on these threshold functions (see threshold(n, d) in creategraph.c), we saw a mild improvement in performance for our RGGs and a significant improvement in performance for our MST generators.

A side note on why we believe our pruning algorithm is correct: if it were not, then the MST produced without pruning would have a smaller total weight than the MST produced with pruning. This would mean that there exists an edge in the graph that should have been included in the MST that was accidentally pruned. However, since we did not trigger our assert statement (which makes sure that the MST has $n-1$ edges), we must have found $n-1$ edges that satisfy the properties of the MST. Thus the edge we pruned was replaced by some other edge that must have a larger weight. But if this edge has a larger weight, then it should also have been pruned, and thus it would be impossible for us to produce an MST if we accidentally pruned an edge that should be in the correct MST. Since we do not trigger any assert statements in our running of the program, we are confident that our pruning function adequately overestimates the threshold.

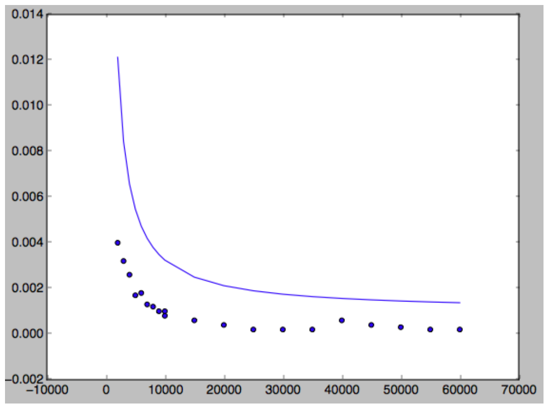


Figure 3: Actual vs. Predicted Max Edge Length in MST (n ∈ (1000, 60000), d = 0)

The final major issue we came across occurred when we attempted to run our program for $n = 65536$. Our MST generator up to this point had simply allocated more than enough memory to store the complete graph (namely, enough space for nC2 edges). However, this much memory was too big for even the heap. As a result, we changed our RGGs to increase the size of the graph array as needed. In this way, we could allocate a small amount of memory to begin and add more if necessary. This also necessitated we return the number of edges we found so we could know how long the array actually was, so we could manipulate it later.

Ultimately, our implementation of RGGs and Kruskal's algorithm was fairly straightforward, with the exception being the optimization for large n. Fortunately, with our optimization, we could now run our program and have results for large n in reasonable time.

Results
We ran our program a series of times on varying values of n and d. Since the problem statement required testing the powers of two, we ran those first. The results are shown in the table below:

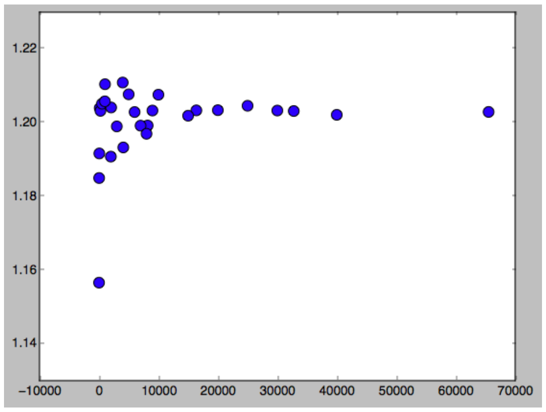
Table 1: Average Weight of MST with n Vertices and Dimension d

n	trials	d = 0	d = 2	d = 3	d = 4
16	10000	1.156588	2.706947	4.511162	6.137181
32	10000	1.184992	3.861921	7.166862	10.319945
64	5000	1.191658	5.433568	11.255388	17.146194
128	1000	1.203978	7.612868	17.600801	28.460093
256	300	1.203222	10.680196	27.607473	47.136620
512	300	1.205161	14.982530	43.397678	78.147415
1024	100	1.210444	21.080511	68.121613	130.106644
2048	30	1.204132	29.639332	107.188408	216.480087
4096	10	1.193288	41.676666	169.101974	360.339966
8192	5	1.199238	58.819538	267.806183	603.253052
16384	5	1.203346	83.127167	423.052734	1008.771790
32768	5	1.203170	117.585678	667.955505	1688.650024
65536	5	1.202927	166.020844	1058.974243	2829.117676

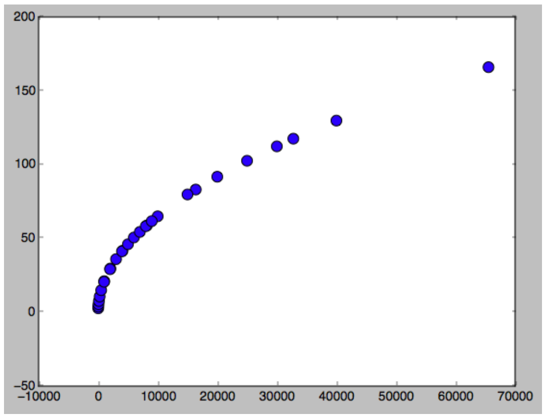
However, we decided to get a good model, we wanted a few more data points to work with. We ended up collecting the following additional data:

Table 2: Average Weight of MST with n Vertices and Dimension d (con't)

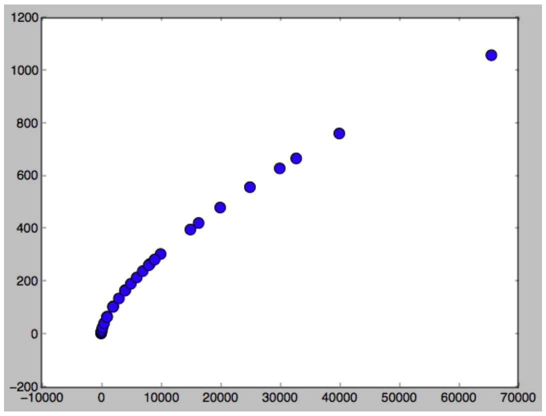
n	trials	d = 0	d = 2	d = 3	d = 4
1000	5	1.2058	20.6530	67.2194	127.9314
2000	5	1.1908	29.1936	105.1252	212.8607
3000	5	1.1990	35.9798	137.0924	287.7014
4000	5	1.2109	41.2662	166.7150	355.9292
5000	5	1.2077	46.0850	192.8012	418.7740
6000	5	1.2029	50.6606	216.8631	479.7808
7000	5	1.1992	54.4482	240.7518	537.3342
8000	5	1.1970	58.2977	263.0330	593.2632
9000	5	1.2033	61.7239	284.6641	647.4437
10000	5	1.2076	65.0109	305.5179	700.4604
15000	5	1.2019	79.8070	398.3995	944.5295
20000	5	1.2034	91.8058	481.6500	1168.7787
25000	5	1.2046	102.6266	558.9552	1379.5463
30000	5	1.2033	112.3914	630.0703	1580.5377
40000	5	1.2022	129.8327	762.3553	1958.2277



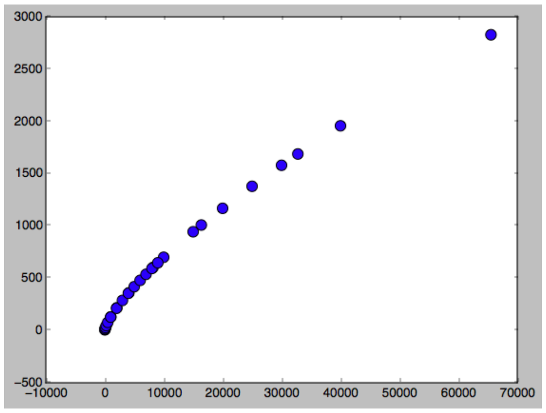
(a) Average Weight vs. n ($d = 0$)



(b) Average Weight vs. n ($d = 2$)



(c) Average Weight vs. n ($d = 3$)



(d) Average Weight vs. n ($d = 4$)

Figure 4: Plots of Results

We wanted to find a function $f(n)$ that could model the average weight of the MST given n. We notice immediately from the data that the function must depend on both n and d. In order to find what this function should be, we graphed the data (shown above). For the $d = 0$ case, if we ignore some initial noise for small values of n, it appears that the average weight of the MST is around 1.2. For the other cases, the relationship is not as clear. The curves look roughly like polynomial functions with fractional powers, so we attempted to fit the data to a function of the form $a \cdot x^b + c$. Using Python scripts we wrote, we got the following results:

Table 3: Coefficients of Best Fit Function

d	a	b	c
2	0.6543	0.4991	0.2054
3	0.6780	0.6629	0.7354
4	0.7378	0.7439	1.6431

We notice that the exponent b roughly takes the form $(d-1)/d$. Using this value of b, we recalculated our function to find better fits for a and c. The table below summarizes the best fit functions we found for the four values of d:

Table 4: Best Fit Functions

d	$f(n)$
0	$f(n) = 1.2$
2	$f(n) = 0.6477n^{1/2} + 0.2858$
3	$f(n) = 0.6510n^{2/3} + 1.9652$
4	$f(n) = 0.6907n^{3/4} + 5.9482$

Discussion

The purpose of the problem was twofold: to experience some of the problems involved with implementing an algorithm and exploring how minimum spanning trees behave in random graphs. We discussed much of the former in our implementation section, and so we dedicate this discussion section to talking about our findings from the latter.