Taking One Small Step Forward: Finding Low-Frequency Items in Data Streams

Abstract—**With the explosive growth of the amount of data and the rapid development of information technology, low-frequency items in data streams have attracted much attention due to their large amount and the rich information contained in them. However, as far as we know, there is no method finding low-frequency items in data streams. To address this problem, we propose an one-pass algorithm, called *BFSS*, which can find the items in a data stream whose frequency is no more than a user-specified support approximately. *BFSS* is simple and has small memory footprints. Although the output is approximate, we can guarantee no false negatives (*FNs*) and only a few false positives (*FPs*) exist in the result. Given a little modification, *BFSS* can be extended to *SBFSS*, which can handle data streams in limited and small space and guarantee only a few *FPs* and theoretically bounded number of *FNs* exist in the result. Experimental results on real-world dataset show *BFSS* achieves high performance using much less space compared to *nCount* and *rCount*, which are two naive methods finding low-frequency items in data streams. *SBFSS* outperforms *lCount*, a method finding low-frequency items in data streams using limited space with a simple replacement strategy, in both precision and recall.**

I. Introduction

In many real-world applications, information such as web click data [1], stock ticker data [2], [3], sensor network data [4], phone call records [5], and network packet traces [6] appears in the form of data streams. Motivated by the above applications, researchers started working on novel algorithms for analyzing data streams. Problems studied in this context include approximate frequency moments [7], distinct values estimation [8], [9], bit counting [10], duplicate detection [11], [12], approximate quantiles [13], wavelet based aggregate queries [14], correlated aggregate queries [15], frequent elements [16]–[19] and top-k queries [20], [21]. However, to the best of our knowledge, there is no algorithm finding low-frequency items in data streams. In fact, low-frequency items in data streams have attracted much attention because of the rich information they contain which can be seen through the formulas of entropy of data streams [22]. We take one small step forward to find low-frequency items in data streams approximately in this paper.

A. Motivating Examples

1) Individualized demands mining: With the rapid development of internet, we can shop and search whatever we are interested in online. Our demands, for example, buying a regular water glass online or searching the information about a tourist attraction etc, are popular most of the time and these demands can be easily satisfied. However, we are no longer satisfied with just popular demands nowadays. For example, it is not so easy for us to buy embroidery stitches or find the information about a nameless village in China online, because they are individualized demands.

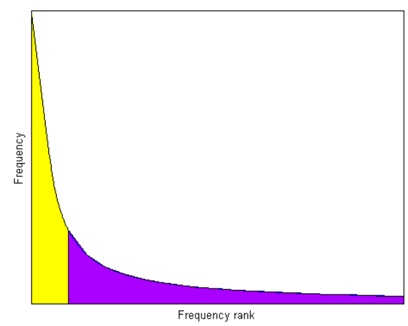


Fig. 1: Rank-frequency distribution

Microsoft lags so far behind Google in the search engine market because it focused a lot on the head of the queries and didn't acknowledge the long tail, said Yusuf Mehdi, senior vice president of the Online Audience Group for Microsoft Bing, at Search Engine Strategies[1]. For Microsoft, focusing on the head instead of the "long tail" meant that it returned popular queries but failed to satisfy less common queries. The long tail of queries, or the individualized demands, ended up yielding more sizable traffic and therefore more money for Google over the last 11-plus years.

From above, we can see that individualized demands, in some sense, are more important than popular demands, and finding individualized demands requires the algorithms finding low-frequency items in data streams.

2) Distribution estimation: Distribution is a basic property of a data stream, therefore it is important to estimate the distribution of a data stream efficiently. Sampling is a simple and fast method to estimate distributions, however, it may cause significant errors sometimes. Fig.1 is a power-law graph, being used to demonstrate the ranking of frequencies of items in data streams under power-law distributions[2]. From Fig.1, we observe that low-frequency items have a great influence on distributions, and combining the information of both frequent and low-frequency items, we can have a relatively more comprehensive estimation of the distribution of a data stream.

In fact, *BFSS* can approximately find both frequent and low-frequency items in data streams, given a little modification, *BFSS* can be extended to estimate the percentage of the total frequencies of a certain proportion of items in data streams, for example, we can approximately estimate the

[1]http://www.eweek.com/c/a/Search-Engines/Microsoft-Ignored-the-Long-Tail-in-Search-Bing-Boss-Says-396023

[2]the distributions of a wide variety of physical, biological, and man-made phenomena are approximately power-law distributions

percentage of the total frequencies of the most frequent twenty[3] percent of items in a data stream, which is similar to the expression of the famous Pareto principle, also known as the 80-20 rule.

B. Our Contributions

We are the first to pose and formally define the problem *s-Bounded Low-Frequency Elements (s-BLFE)*, the formal definition of which will be given later, and to the best of our knowledge, there is no method solving the problem till now.

We propose *BFSS*, which extends the classic algorithm *Space Saving* to maintain both frequent and low-frequency items in a data stream approximately. The basic idea of our solution is as follows: each item in a data stream is either a frequent one or a low-frequency one once the threshold $s[\in (0,1)]$ is confirmed, so we can maintain low-frequency items by filtering the frequent items out. A major problem we have to deal with is to maintain an itemset items in which appear in the data stream, and this can be done approximately using a Bloom filter. *BFSS* gurantees no false negatives and provably few false positives using small memory footprints.

However, the size of a Bloom filter must increase with the alphabet's size in order to keep low false positive rate, and here comes a problem: In many embedded devices, such as sensors and routers etc., the storage space is limited and small, in which case *BFSS* will produce many *FPs*. Inspired by the method presented in [11], we propose *SBFSS* which extends *BFSS* to deal with data streams in limited and small space, and *SBFSS* gurantees a few false positives and theoretically bounded number of false negatives.

C. Roadmap

In Section 2, we present problem statement and some backgrounds on the existing approches which deal with the problem *ϵ-Deficient Frequent Elements*. Our solutions are presented and discussed in Section 3. In Section 4, we experimentally evaluate our methods. Conclusions are given in Section 5.

II. PRELIMINARIES

This section presents problem statement and some representative algorithms solvimg *ϵ-Deficient Frequent Elements* [23] which will be formally defined below. Table I summarizes the major notations in this paper.

TABLE I: Major Notations Used in the Paper.

Notation	Meaning
BFSS	Our first algorithm
SBFSS	Our second algorithm
S	The input data stream
A	The alphabet of S
M	The size of A
n	The number of distinct items in S
s	The user-specified support parameter
ϵ	The user-specified error parameter
D	The synopsis used in *BFSS* and *SBFSS*
e	The item monitored in D
$f(e)$	The estimated frequency of e
$\Delta(e)$	The estimated error of $f(e)$
E	The item set monitored in D
min	The minimum value of $f(e)$ in D
C	The maximum number of counters in D
m	The number of counters used in D
N	The length of S
H	The number of hash functions used in *BFSS* and *SBFSS*
$f_S(e)$	The Frequency of item e in S
FPs	The items in S with frequency more than $\lfloor sN \rfloor$ wrongly output
FNs	The items in S with frequency no more than $\lfloor sN \rfloor$ wrongly neglected
BF	The Bloom filter used in *BFSS*
K	The size of *BF*
p_1	The percentage of the number of the items whose frequency is above $\lfloor sN \rfloor$
p_2	The percentage of the total frequencies of the items whose frequency is above $\lfloor sN \rfloor$
SBF	The Stable Bloom filter used in *SBFSS*
K'	The size of *SBF*
k	Number of cells used in *SBF*
d	Number of bits allocated per cell
max	The value a cell is set to
P	Number of cells we pick to decrement by 1 in each iteration
p	The probability that a cell is picked to be decremented by 1 in each iteration
p'	The probability that a cell is set in each iteration
W	The real data set we used in our experiments

A. Problem Statement

Consider an input stream $S = e_1, e_2, ..., e_N$ of current length N, which arrives item by item. Let each item e_i belong to a universe set $A = \{a_1, a_2, ..., a_M\}$ of size M.

The problem *s-BLFE* can be stated as follows: given a data stream S along with two user-specified parameters: a support parameter $s[\in (0,1)]$ and an error parameter $\epsilon[\in (0,1)]$ such that $\epsilon \leq s$.

At any point of time, with a small bounded memory, output a list of items with the following guarantees:

1. all items whose true frequency is no more than $\lfloor sN \rfloor$ are output.
2. no item whose true frequency is no less than $\lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ is output.

Imagine a user who is interested in identifying all items whose frequency is no more than $0.1\%N$. Then $s = 0.1\%$. The user is free to set ϵ to whatever she feels is a comfortable margin of error. As a rule of thumb, she could set ϵ to one-tenth or one-twentieth the value of s and use our algorithm. Let us assume she chooses $\epsilon = 0.01\%$ (one-tenth of s). As per Property 1, all items with frequency no more than $0.1\%N$ will be output, there will be no false negatives. As per Property 2, no element with frequency no less than $\lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ will be output.

B. Related Work

As far as we know there is no related algorithm addressing *s-BLFE*, however, the methods we propose are based on the algorithm solving *ϵ-Deficient Frequent Elements* which can be described as follows: given a input stream S of current length N and a support threshold $s \in (0,1)$, return the items whose frequency is guaranteed to be no smaller than $\lfloor (s-\epsilon)N \rfloor$ deterministically or with a probability of at least $1-\delta$, where $\epsilon \in (0,1)$ is a user-defined error and $\delta \in (0,1)$ is a probability of failure, so we examine several algorithms solving *ϵ-Deficient Frequent Elements*.

[3]"twenty" is just an example, and the specific value varies and will be explained later

Research can be divided into two groups: *counter-based* techniques and *sketch-based* techniques.

Counter-Based Techniques keep an individual counter for each item in the monitored set, a subset of A. The counter of a monitored item, e_i, is updated when e_i occurs in the stream. If there is no counter kept for the observed ID, it is either disregarded, or some algorithm-dependent action is taken.

Two representative algorithms *Sticky Sampling* and *Lossy Counting* were proposed in [23]. The algorithms cut the stream into rounds, and they prune some potential low-requency items at the edge of each round. Though simple and intuitive, they suffer from zeroing too many counters at rounds boundaries, and thus, they free space before it is really needed. In addition, answering a frequent elements query entails scanning all counters.

The algorithm *Space Saving* , the one we are based at, was proposed in [20]. The algorithm maintains a synopsis which keeps all counters in an order according to the value of each counter's monitoring frequency plus maximum possible error. For a non-monitored item, the counter with the smallest counts, *min*, is assigned to monitor it, with the items monitoring frequency $f(e)$ set to 1 and its maximal possible error $\Delta(e)$ set to *min*. Since $min \leq \epsilon N$ (this follows because of the choice of the number of counters), the operation amounts to replacing an old, potentially infrequent item with a new, hopefully frequent item. This strategy keeps the item information until the very end when space is absolutely needed, and it leads to the high accuracy of *Space-Saving*. Experiments done in [24], [25] showed *Space-Saving* outperformed other Counter-Based techniques in recall and precision tests.

Sketch-Based Techniques do not monitor a subset of items, rather provide, with less stringent guarantees, frequency estimation for all items using bitmaps of counters. Usually, each item is hashed into the space of counters using a family of hash functions, and the hashed-to counters are updated for every hit of this item. Those representative counters are then queried for the item frequency with less accuracy, due to hashing collisions.

The *Count-Min Sketch* algorithm of Cormode and Muthukrishnan [26] maintains an array of $d \times w$ counters, and pairwise independent hash functions h_j map items onto $[w]$ for each row. Each update is mapped onto d entries in the array, each of which is incremented. The Markov inequality is used to show that the estimate for each j overestimates by less than n/w, and repeating d times reduces the probability of error exponentially.

The *hCount* algorithm was proposed in [27]. The data structure and algorithms used in *Count-Min Sketch* and *hCount* shared the similarity, but were simultaneously and independently investigated with different focuses.

III. OUR ALGORITHMS

In this section, we will discuss our approaches *BFSS* and *SBFSS* in detail.

A. Challenges of s-BLFE

s-BLFE has two main challenges due to the different features between frequent items and low-frequency items over data streams.

The Long Tails in data streams. It can be easily proved that there are at most $\lceil 1/s \rceil$ frequent items whose frequency is more than $\lfloor sN \rfloor$ in any data stream, however, there is no upper bound of the number of the low-frequency items whose frequency is no more than $\lfloor sN \rfloor$. In fact, our experiments show that the low-frequency items occupy most of the distinct items in data streams, and it is almost impossible to maintain all of them in memory. Another observation is their frequencies are very low and close as well, and it may consume much space to separate low-frequency items from frequent items especially when s is very small.

Unpredictability. A basic and common idea of *Counter-Based Techniques* is to discard potential infrequent items dynamically, and it is based on the fact that potential infrequent items will never become frequent items if they don't appear afterwards, however, this fact no longer applies to low-frequency items in data streams because frequent items will possibly become low-frequency items if they don't appear afterwards. The unpredictability of low-frequency items makes it difficult to maintain them directly.

B. The BFSS Algorithm

In consideration of the challenges in *s-BLFE*, we tried to solve the problem indirectly by filtering frequent items out which is the underlying idea of *BFSS*.

Two algorithms are proposed for updating and outputting results separately. Algorithm 1 maintains a Bloom filter BF of size K with H uniformly independent hash functions $\{h_1(x), ..., h_H(x)\}$ and a synopsis D with C counters. Each of these H hash function maps an item from A to $[0, ..., K-1]$. Initially each bit of BF is set to 0 and D has C empty counters. Each newly arrived item in the stream is mapped to H bits in BF by the H hash functions and we set the H bits to 1. Then if we observe an item that is monitored in D, we just increment $f(e)$. If we observe an item, e_{new}, that is not monitored in D, handle it depending on whether there is an empty counter in D. If there is one, we just allocate it to e_{new} and set $f(e_{new})$ to 1 and set $\Delta(e_{new})$ to 0. Otherwise, we just replace the item that currently has the least hits, min, with e_{new}. Assign $f(e_{new})$ the value $min + 1$ and assign $\Delta(e_{new})$ the value min.

Algorithm 2 checks and outputs the items whose frequency is no more than sN. For each item in A, we first check whether it is in BF. If the item is not in BF, it must not be a low-frequency item. If the item is in BF but not in D, we output it as a low-frequency item. If the item appears both in BF and D, we identify it as a low-frequency item if $f(e) \leq \lfloor sN \rfloor + \Delta(e)$ with high probability. The time requirement of Algorithm 2 is linear to the range of universe. It is acceptable when the frequency of the requests is not high.

C. Analysis of BFSS

In this section, we present a theoretical analysis of *BFSS* described in Section III-B. We analyze *FNs*, *FPs*, space complexity, and time complexity. At last, we will identify the challenges to *BFSS*.

Algorithm 1 BFSS Update Algorithm

Input: Stream S, support threshold s

$N = 0, m = 0, C = \lceil 1/\epsilon \rceil, K = \lambda', H = \mu'$; {$N$: the length of the input stream; m: the number of counters used in D; C: the maximum number of counters in D; K: size of Bloom filter; H: the number of hash functions; The value of λ' and μ' will be discussed in detail later.}
The form of the counters in D is $(e, f(e), \Delta(e))$
for $i = 0$ to $K - 1$ **do**
  $BF[i] = 0$
end for
for each item e of stream S **do**
  for $i = 1$ to H **do**
    $BF[h_i(e)] = 1$
  end for
  if e is monitored in D **then**
    $f(e) = f(e) + 1$;
  else if $m < C$ **then**
    Assign a new counter $(e, 1, 0)$ to it
    $m = m + 1$
  else
    Let e_m be the item with least hits, min
    Replace e_m with e in D
    $f(e) = min + 1, \Delta(e) = min$
  end if
  $N = N + 1$;
end for

Algorithm 2 BFSS Query Algorithm

Input: *BF*, D, s, A, N, M
Output: low-frequency items with threshold s

$flag = true$; { *flag*: indicate whether an item is in *BF*}
for $i = 0$ to $M - 1$ **do**
  $flag = true$
  for $j = 1$ to H **do**
    if $BF[h_j(A[i])] == 0$ **then**
      $flag = false$
      break;
    end if
  end for
  if $flag == true$ **then**
    if $A[i]$ is monitored in D **then**
      if $f(A[i]) \leq \lfloor sN \rfloor + \Delta(A[i])$ **then**
        output $A[i]$ as a low-frequency item
      end if
    else
      output $A[i]$ as a low-frequency item
    end if
  end if
end for

*1) **Analysis of FNs:*** In this section, we will prove that there are no *FNs* in the output of *BFSS*. The proof is based on Lemma 1 to Lemma 5, and the detailed proof of Lemma 1 to Lemma 3 can be found in [20].

Lemma 1: $N = \sum_{\forall i | e_i \in E}(f(e_i))$

Proof: Every hit in S increments only one counter by 1 among the M counters which can be easily proved when D has empty counters. It is true even when a replacement happens, i.e., the observed item e was not monitored and D has no empty counters, and it replaces another item e_m. This is because we add $f(e_m)$ to $f(e)$ and increment $f(e)$ by 1. Therefore, at any time, the sum of all counters is equal to the length of the stream observed so far. ■

Lemma 2: Among all counters in D, the minimum counter value, min, is no greater than $\lfloor \frac{N}{m} \rfloor$.

Proof: Lemma 1 can be written as:

$$min = \frac{N - \sum_{\forall i | e_i \in E}(f(e_i) - min)}{m} \tag{1}$$

All the items in the summation of Equation 1 are non negative because all counters are no smaller than min, hence $min \leq \lfloor \frac{N}{m} \rfloor$. ■

Lemma 3: For any item $e \in E$, $0 \leq \Delta(e) \leq \lfloor \epsilon N \rfloor$.

Proof: From Algorithm 1, $\Delta(e)$ is non-negative because any observed item is always given the benefit of doubt. $\Delta(e)$ is always assigned the value of the minimum counter at the time e started being observed. Since the value of the minimum counter monotonically increases over time until it reaches the current min, then for all monitored items $\Delta(e) \leq min$.

Consider thses two cases: i) if $m = \lceil \frac{1}{\epsilon} \rceil$, $min \leq \lfloor \epsilon N \rfloor$; ii) if $m < \lceil \frac{1}{\epsilon} \rceil$, $\Delta(e) = 0$. For any case, we have $0 \leq \Delta(e) \leq \lfloor \epsilon N \rfloor$. ■

Lemma 4: For any item $e \notin E$ but appearing in S, $f_S(e) \leq min \leq \lfloor \epsilon N \rfloor$.

Proof: From Lemma 2, we can easily get $min \leq \lfloor \epsilon N \rfloor$ because there must be no empty counter in D and $m = \lceil \frac{1}{\epsilon} \rceil$. We only have to porve that any item satisfying $f_S(e) > min$ must be monitored in D, i.e. $e \in E$. The proof is by contradiction. Assume $e \notin E$. Then it was evicted previously, and we assume that it had been monitored in $i(> 0)$ time slots, and e appeared $n_j (0 < j \leq i)$ times in the jth time slot, therefore n_j satisfies:

$$\sum_{j=1}^{i} n_j = f_S(e) \tag{2}$$

We assume that $\Delta(e_j)(\geq 0)$ denotes the error estimation assigned to e at the start of the jth time slot, and we have the following inequality because the minimum counter value increases monotonically:

$$\begin{gathered} \Delta(e_1) + n_1 \leq \Delta(e_2) \\ \Delta(e_2) + n_2 \leq \Delta(e_3) \\ \dots \\ \Delta(e_{i-1}) + n_{i-1} \leq \Delta(e_i) \\ \Delta(e_i) + n_i \leq min \end{gathered} \tag{3}$$

After adding up the left and right sides of inequality group 3, we can get:

$$\Delta(e_1) + \sum_{j=1}^{i} n_j \leq min \tag{4}$$

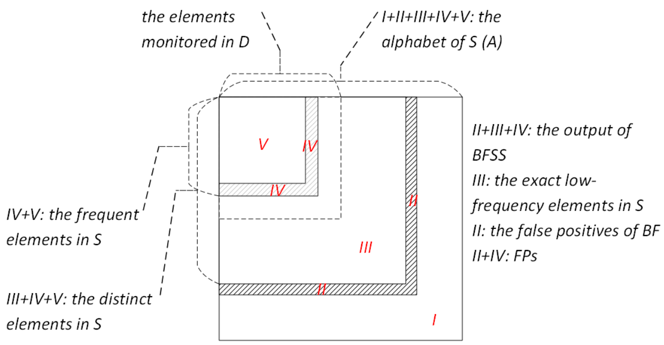


Fig. 2: The schematic diagram of the analysis of *BFSS*

Therefore $f_S(e) \leq min$, which contradicts the condition $f_S(e) > min$. ■

Lemma 5: For any item $e \in E$, $f(e) - \Delta(e) \leq f_S(e) \leq f(e) \leq f(e) - \Delta(e) + min$

Proof: Since the value of the minimum counter monotonically increases, we have $\Delta(e) \leq min$, which indicates $f(e) \leq f(e) - \Delta(e) + min$. $f(e) - \Delta(e)$ is the true frequency of e since it was lastly observed, so $f(e) - \Delta(e) \leq f_S(e)$. From Lemma 4, we can find that $\Delta(e)$ over estimated the frequency of e before it was observed, and it clearly indicates $f_S(e) \leq f(e)$. ■

Theorem 1: There are no *FNs* in the output of *BFSS*.

Proof: We only hava to prove that the items we don't output contain no low-frequency items. Algorithm 2 shows that two kinds of items are not ouput: i) the items filtered out by *BF*. ii) the items monitored in D with $f(e) - \Delta(e) > \lfloor sN \rfloor$. The first kind of items are obviously not low-frequency items because they never appeared in S. From Lemma 5, we know that the items monitored in D with $f(e) - \Delta(e) > \lfloor sN \rfloor$ must satisfy $f_S(e) > \lfloor sN \rfloor$, which means the second kind of items must be frequent items. ■

*2) **Analysis of FPs**:* In this section, we will give a theoretically bound of the expectation of the number of *FPs* in the output of *BFSS* regardless of the distribution of S, and a tighter bound can be derived for data streams under Pareto distribution [28].

Lemma 6: Any item e with $f_S(e) > \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ must not be output.

Proof: From Lemma 4, we know that any item e with $f_S(e) > \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ must be monitored in D, i.e. $e \in E$. Then from Lemma 5, we can get that the items monitored in D must satisfy $f_S(e) \leq f(e)$, and that is to say any item e monitored in D with $f_S(e) > \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ must satisfy $f(e) > \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$. At last from Lemma 3, we can prove that any item e with $f(e) > \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ must satisfy $f(e) > \lfloor sN \rfloor + \Delta(e)$, which means e will not be output. ■

Lemma 7: The probability of a false positive of *BF* is no more than:

$$(1 - (1 - \frac{1}{K})^{HM})^H \tag{5}$$

Proof: A false positive of *BF* means an item in A not appearing in S but not filtered out by *BF*. Observe that after inserting M keys into a table of size K, the probability that a particular bit is still 0 is exactly

$$(1 - \frac{1}{K})^{HM} \tag{6}$$

Hence the probability of a false positive in this situation is $(1 - (1 - \frac{1}{K})^{HM})^H$. However, we know that M denotes the size of A which is the alphabet of S, so the number of the distinct items in S must be no more than M, i.e. $n \leq M$, and further we have $(1-(1-\frac{1}{K})^{Hn})^H \leq (1-(1-\frac{1}{K})^{HM})^H$. ■

Theorem 2: Assuming no specific data distribution, the expectation of the number of *FPs* in the output of *BFSS*, denoted as $E(\#FPs)$, satisfies:

$$E(\#FPs) < M(1 - (1 - \frac{1}{K})^{HM})^H + \lfloor \frac{1}{s} \rfloor \tag{7}$$

Proof: From Algorithm 1, we can observe that two kinds of items contribute to *FPs*: i) the false positives of *BF*; ii) the items with $f_S(e) > \lfloor sN \rfloor$ wrongly output. The two cases correspond to the gray areas in Fig. 2 which is abstracted out from the analysis of *BFSS*. Concretely speaking, the ligth gray area represents the items with $f_S(e) > \lfloor sN \rfloor$ wrongly output, i.e. the second case, and the dark gray area represents the items not filtered out by *BF*, i.e. the first case. Furthermore, the output of *BFSS* is represented by $II + III + IV$, and the exact low-frequency items in S is represented by III.

For the first case, we define the independent 0-1 random variables $x_i (1 \leq i \leq M)$ for each item in A, and the value of x_i depends on whether a_i appeared in S or not. If a_i appeared in S, then $x_i = 0$; If not, then with a probability of $(1 - (1 - \frac{1}{K})^{Hn})^H$, $x_i = 1$; From the defination of x_i, we can find that the expectation of the number of the false positives of *BF* equals $E(\sum_{i=1}^{M} x_i)$, i.e. $(M - n)(1 - (1 - \frac{1}{K})^{Hn})^H$.

For the second case, we know from Lemma 6 that items with $f_S(e) > \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ must not be not output, so only items with $\lfloor sN \rfloor < f_S(e) \leq \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ are likely to be output, and the maximum number of these items is $\lfloor \frac{1}{s} \rfloor$ because there are at most $\lfloor \frac{1}{s} \rfloor$ items with $f_S(e) > \lfloor sN \rfloor$ in S. From above, due to the linear properties of expectation, we can get:

$$E(\#FPs) \leq (M - n)(1 - (1 - \frac{1}{K})^{Hn})^H + \lfloor \frac{1}{s} \rfloor \tag{8}$$

In addition, $n \leq M$, so inequation 7 can be easily derived from inequation 8. ■

However, from above, we can see that $\frac{1}{s}$ is a very loose bound of the number of the items with $\lfloor sN \rfloor < f_S(e) \leq \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$. We will get a tighter bound if S .

Theorem 3: Assuming noiseless Pareto data with parameter α and $f_S(e_m)$, where $\alpha (> 0)$ and e_m denotes the item with the lowest frequency, we have:

$$E(\#FPs) < M(1 - (1 - \frac{1}{K})^{HM})^H + T \tag{9}$$

$$T = min\{((\frac{f_S(e_m)}{\lfloor sN \rfloor})^\alpha - (\frac{f_S(e_m)}{\lfloor sN \rfloor + \lfloor \epsilon N \rfloor})^\alpha)M, \lfloor \frac{1}{s} \rfloor\} \tag{10}$$

Proof: The Pareto distribution[4] has the following property: If X is a random variable with a Pareto distribution, then the probability that X is greater than some number x, i.e. the tail function, is given by:

$$Pr(X > x) = \begin{cases} (\frac{x_m}{x})^{\alpha} & x \geq x_m \\ 1 & x < x_m \end{cases}$$

where x_m is the minimum possible value of X, and α is a positive parameter. In such case, the expected value of the number of the items with $\lfloor sN \rfloor < f_S(e) \leq \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ is $((\frac{f_S(e_m)}{\lfloor sN \rfloor})^{\alpha} - (\frac{f_S(e_m)}{\lfloor sN \rfloor + \lfloor \epsilon N \rfloor})^{\alpha})n$. ∎

From Inequation 7, we observe that the upper bound of $E(\#FPs)$ is related with the values of H and K, in addtion, the value of K directly influence the space consumption of *BFSS*, and the value of H directly influence the update time of *BFSS*, so it is of great significance to choose the appropriate values of K and H to keep a good balance of the precision, the space consumption and the update time of *BFSS* .

*3) **Space complexity**:* In this section, we will analyze the space complexity of *BFSS* including how to choose the appropriate values of H and K.

Choosing the appropriate values of H and K. Our goal is to choose the appropriate values of H and K to keep $\#FPs$ as small as possible, meanwhile, we have to take the space consumption and update time into consideration as well. However, we can only get the upper bound of the value of $E(\#FPs)$, which is not equivalent to $\#FPs$. In fact, from the Chernoff bound, we know that there is a high probability that an independent random variable hovers near its expected value, which means we can approximately treat $E(\#FPs)$ as $\#FPs$.

Obviously, the upper bound of $E(\#FPs)$ given in Inequation 7 is not so tight, and the upper bound of $E(\#FPs)$ given in Inequation 8 is much tighter. However, it is difficult for us to choose the appropriate values of H and K to minimize the upper bound given in Inequation 8 because the value of n is variable, and the task will be much easier for the bound given in Inequation 7, besides, the values of M and s are confirmed at the very beginning, therefore our task is to minimize the value of $(1 - (1 - \frac{1}{K})^{HM})^H$, denoted as F, and we have:

$$F \approx (1 - e^{-\frac{HM}{K}})^H = e^{H \ln(1 - e^{-\frac{HM}{K}})} \tag{11}$$

Let $P = e^{-\frac{HM}{K}}$ and $G = H \ln(1 - e^{-\frac{HM}{K}})$, and in order to minimize F, we only have to minimize G:

$$G = (-\frac{K}{M}) \ln(P) \ln(1 - P) \tag{12}$$

The first derivative of G with respect to P, denoted as G', is:

$$G' = \frac{K}{M}(\frac{1}{1-P} \ln(P) - \frac{1}{P} \ln(1 - P)) \tag{13}$$

From Equation 13, we can easily observe that when $P = \frac{1}{2}$, G reaches its minimum value. In this case, we have:

$$H = \frac{K}{M} \times \ln 2 \tag{14}$$

[4] https://en.wikipedia.org/wiki/Pareto_distribution

TABLE II: The value of F under various $\frac{K}{M}$ and H combinations.

K/M	H	$H = 8$	$H = 9$	$H = 10$	$H = 11$	$H = 12$
13	9.01	0.00199	0.00194	0.00198	0.0021	0.0023
14	9.7	0.00129	0.00121	0.0012	0.00124	0.00132
15	10.4	0.000852	0.000775	0.000744	0.000747	0.000778
16	11.1	0.000574	0.000505	0.00047	0.000459	0.000466
17	11.8	0.000394	0.000335	0.000302	0.000287	0.000284

Combined with Equation 11 and Equation 14, we have:

$$F \approx (1 - e^{-\frac{HM}{K}})^H = (\frac{1}{2})^H \approx (0.6185)^{\frac{K}{M}} \tag{15}$$

From the above derivation, we know that once the value of $\frac{K}{M}$ is confirmed, we can get the appropriate value of H to minimize the value of F. For example, if the value of $\frac{K}{M}$ is set to 13, the appropriate value of H is $\frac{K}{M} \times \ln 2 \approx 9.1$, because the value of H is an integer, we set it to 9. Considering the length limit, we give a small fraction of the value of F under various $\frac{K}{M}$ and H combinations in Table II.

From Equation 15, we notice that the larger the value of H is, the smaller the value of F is. However, the value of K increases monotonically with the increasing value of H once the value of M is confirmed, which can be observed from Equation 14, in other words, in order to decrease the value of $E(\#FPs)$, we have to increase the size of *BF*, i.e. K.

For example, if $M = 1M$ and $s = 0.001$, then from Table II, we see that if $K = 16M$, the value of H should be 11, therefore $F \approx 0.000459$, and the upper bound of $E(\#FPs)$,calculated by Inequation 7, is $1M \times 0.000459 + 1/0.001 = 1459$. If $K = 17M$, the corresponding values of H and F are 12 and 0.000284, therefore $E(\#FPs) = 1284$. In fact, a much tighter bound of $E(\#FPs)$ can be derived from Inequation 8 once the value of K and H are confirmed.

Analysis of $E(\#FPs)$. Our goal is to caculate the maximum value of $E(\#FPs)$ for different values of n when the values of H and K are confirmed. Let $F' = (M - n)(1 - (1 - \frac{1}{K})^{Hn})^H$, and from Inequation 8, we have $E(\#FPs) \leq F' + \lfloor \frac{1}{s} \rfloor$, in which the value of $\lfloor \frac{1}{s} \rfloor$ is confirmed once the value of s is confirmed, therefore our task is to calculate the maximum value of F'.

Theorem 4: The space complexity of *BFSS* is $O(\lambda M + min(\lceil \frac{1}{\epsilon} \rceil, M))$, where $\lambda \in N^*$, the value of which is discussed detaily above.

Proof: The space complexity of *BFSS* includes two part: i) the size of *BF*, i.e. K. ii) the number of counters used in D, i.e. m. In order to get the minimum value of $E(\#FPs)$, the value of K should be multiple times of the value of M, i.e. $K = \lambda M (\lambda \in N^*)$. From Algorithm 2, we have $m \leq min(\lceil \frac{1}{\epsilon} \rceil, M)$. So the space complexity of *BFSS* is $O(\lambda M + min(\lceil \frac{1}{\epsilon} \rceil, M))$. ∎

In conclusion, there exists a trade off between the number of *FPs* and the space consumption of *BFSS*. Theoretically, once M is confirmed, the larger the value of λ is, the larger the value of K will be, and the smaller the value of $E(\#FPs)$ will be.

Furthermore, consider a naive method to solve *s-BLFE*: we simply allocate each item in A a counter, and update the corresponding counter for each item in S, when a query comes, we just output the items with $f_S(e) \leq \lfloor sN \rfloor$. Obviously the method can maintain the low-frequency items precisely, and the space complexity of the method is $O(M)$. *BFSS* has no advantage over the naive method in space complexity considering big O though, *BFSS* needs not to store the exact value or the fingerprint of each item which may consume much space especially when the value is long string, like URL. The experimental results indicate *BFSS* is much more space efficient.

*4) **Time complexity**:* In this section, we will discuss the time complexity of *BFSS*.

Theorem 5: Processing each item needs $O(1)$ time, independent of N.

Proof: From Algorithm 1, we know the processing time for each item has two parts: i) the time spent hasing each item into *BF*. ii) the time spent updating the corresponding counter in D. Obviously, the time spent hashing each item into *BF* is constant because H is constant. Consider two cases in updating the corresponding counter in D: i) item e is monitored in D, and we only have to update the corresponding counter. ii) item e is not monitored in D, and we have to locate the counter with the minimum $f(e)$ first, then update it. Obviously, the time spent for any case is constant because $\lceil \frac{1}{\epsilon} \rceil$ is constant. Therefore, processing each item needs $O(1)$ time. ■

*5) **Challenges to BFSS**:* In this section, we will identify the existing problems in *BFSS*.

From the analysis in section III-C3, we observe that K should increase monotonically with M in order to keep a low rate of *FPs*, however, what if the space available is limited and small? The precision of *BFSS* will be very low in this case.

Inspired by the algorithm proposed in [11], we present the *SBFSS* algorithm, which can find low-frequency items over data streams in limited and small space with a few *FPs* together with theoretically bounded number of *FNs*.

D. The BFSS Algorithm*

In this section, we will introduce *BFSS**, which approximately estimates the distribution of a data stream.

The goal of *BFSS** is to estimate the percentage of the number of the items whose frequency is above a user-specified support, denoted as p_1, and the percentage of the total frequencies of these items, denoted as p_2.

*BFSS** and *BFSS* share the same update process, and the query process of *BFSS** is described in Algorithm 3. For each item e in A, we first check whether it is in *BF*, if the item is in *BF*, we increment the counter n, and if the item appears in D and $f(e) > \lfloor sN \rfloor + \Delta(A[i])$, we increment n' and add $f(e) - \Delta(e)$ to f. At last, we set $p'_1 = n'_s/n'$ as the approximate value of p_1 and $p'_2 = f'/N$ as the approximate value of p_2.

*E. Analysis of BFSS**

In this section, we will give theoretical analysis of p_1 and p_2. Obviously, the time and space complexity of *BFSS** are the same as *BFSS*, and we don't discuss here.

Algorithm 3 BFSS* Query Algorithm

Input: *BF*, D, s, A, N, M
Output: The percentage of the items whose frequency is above $\lfloor sN \rfloor$ and the percentage of the total frequencies of these items

$flag = true, n'_s = 0, n' = 0, f'_s = 0, p'_1 = 0, p'_2 = 0$;{*flag*: indicate whether an item is in *BF*, n: the number of distinct items in S, n'_s: the estimated number of the items whose frequency is above $\lfloor sN \rfloor$, f'_s: the estimated total frequencies of the items whose frequency is above $\lfloor sN \rfloor$, p'_1: the estimated percentage of the items whose frequency is above $\lfloor sN \rfloor$, p'_2: the estimated percentage of the total frequencies of these items.}
for $i = 0$ to $M - 1$ **do**
  $flag = true$;
  for $j = 1$ to H **do**
    if $BF[h_j(A[i])] == 0$ **then**
      $flag = false$;
      break;
    end if
  end for
  if $flag == true$ **then**
    $n' = n' + 1$;
    if $A[i]$ is monitored in D **then**
      if $f(A[i]) > \lfloor sN \rfloor + \Delta(A[i])$ **then**
        $n'_s = n'_s + 1$;
        $f'_s = f'_s + (f(A[i]) - \Delta(A[i]))$;
      end if
    end if
  end if
end for
$p'_1 = n'_s/n', p'_2 = f'_s/N$;
Output p'_1, p'_2;

*1) **Analysis of** p_1:* From the defination of p_1, we know that $p_1 = \frac{n_s}{n}$, in which n_s denotes the exact number of the items whose frequency is above s. From Lemma 3 and Lemma 5, we observe two facts: i) any item with $f_s(e) > \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ must satisfy $f(e) > \lfloor sN \rfloor + \Delta(e)$. ii) any item with $f_s(e) \leq \lfloor sN \rfloor$ must satisfy $f(e) \leq \lfloor sN \rfloor + \Delta(e)$. From the two facts, we know that only items with $\lfloor sN \rfloor < f_s(e) \leq \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ are likely to be neglected, which means $n_s \geq n'_s$. Let $\Delta(n_s) = n_s - n'_s$, and the maximum value of $\Delta(n_s)$ is the number of the items with $\lfloor sN \rfloor < f_s(e) \leq \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$. Obviously, for any data stream, $\Delta(n_s) \leq \lfloor \frac{1}{s} \rfloor$, which is a very loose bound, and for data streams under Pareto distributions, we have $E(\Delta(n_s)) \leq ((\frac{f_S(e_m)}{\lfloor sN \rfloor})^\alpha - (\frac{f_S(e_m)}{\lfloor sN \rfloor + \lfloor \epsilon N \rfloor})^\alpha)n$, the proof of which can be found in the proof of Theorem 3.

From Algorithm 3, we can get $n' \geq n$ because *BF* causes no false negatives but only false positives. Let $\Delta(n) = n' - n$, and the value of $\Delta(n)$ equals to the number of the false positives caused by *BF*. Therefore, we have $E(\Delta(n)) = (M - n)(1 - (1 - \frac{1}{K})^{Hn})^H < M(1 - (1 - \frac{1}{K})^{HM})^H$, the proof of which can be found in the proof of Theorem 2.

From above, $p'_1 = \frac{n_s - \Delta(n_s)}{n + \Delta(n)} \leq p_1$, and we can theoretically get the upper bound of $E(\Delta(n_s))$ for data streams under Pareto distributions and the upper bound of $E(\Delta(n))$ for any data streams. Furthermore, we can minimum $E(\Delta(n))$ by selecting appropriate parameters, which has been discussed before. Experimental results in [24], [25] show that *Space Saving* has a very high precision (almost 1), which means the number of the items with $\lfloor sN \rfloor - \lfloor \epsilon N \rfloor < f_s(e) \leq \lfloor sN \rfloor$ wrongly output is very small, and if we set $s = s + \epsilon$, then the number of the items with $\lfloor sN \rfloor < f_s(e) \leq \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ wrongly output is vary small too, in other words, the value of $\Delta(n_s)$ is very small. In fact, the experimental results also show that p_1' is nearly the same as p_1.

*2) **Analysis of** p_2:* From the defination of p_2, we can get $p_2 = \frac{f_s}{N}$, in which f_s denotes the exact total frequencies of the items whose frequency is above $\lfloor sN \rfloor$. Obviously, $f_s \geq f_s'$ because we may neglect some items with $\lfloor sN \rfloor < f_s(e) \leq \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ just like the analysis of p_1, besides, for the items with $f(e) > \lfloor sN \rfloor + \Delta(e)$, we just add $(f(e) - \Delta(e))(\leq f_s(e))$ to f_s'. Let $\Delta(f_s) = f_s - f_s'$, from above, $\Delta(f_s)$ consists of two parts: i) the total frequencies of the items with $\lfloor sN \rfloor < f_s(e) \leq \lfloor sN \rfloor + \lfloor \epsilon N \rfloor$ wrongly neglected, denoted as $\Delta_1(f_s)$. ii) the total missing frequencies of the items with $f(e) > \lfloor sN \rfloor + \Delta(e)$, denoted as $\Delta_2(f_s)$. For $\Delta_1(f_s)$, obviously, we have $\Delta_1(f_s) \leq \Delta(n_s) \times (\lfloor sN \rfloor + \lfloor \epsilon N \rfloor)$, in which $\Delta(n_s)$, for *Space Saving*, is very small (nearly 0). For $\Delta_2(f_s)$, we have $\Delta_2(f_s) \leq \lceil \frac{1}{s} \rceil \times \lfloor \epsilon N \rfloor$ because there are at most $\lceil \frac{1}{s} \rceil$ items with $f(e) > \lfloor sN \rfloor + \Delta(e)$, then from Lemma 5 and Lemma 3, we have $f_s(e) - (f(e) - \Delta(e)) \leq \Delta(e) \leq \lfloor \epsilon N \rfloor$.

From above, $p_2' = \frac{f_s - (\Delta_1(f_s) + \Delta_2(f_s))}{N} \leq p_2$. Let $\Delta(p_2) = p_2 - p_2'$, and we have $\Delta(p_2) \leq \frac{\Delta(n_s) \times (\lfloor sN \rfloor + \lfloor \epsilon N \rfloor) + \lceil \frac{1}{s} \rceil \times \lfloor \epsilon N \rfloor}{N}$. In fact, the experimental results show that p_2' is a little smaller than p_2.

F. The SBFSS Algorithm

In this section, we will introduce *SBFSS*, which modifies *BFSS* and can find low-frequency items in data streams using limited and small space.

The major difference between *BFSS* and *SBFSS* is the Bloom filter we use. Concretely speaking, *BF* is a regular Bloom filter, while *SBF* change bits in *BF* into cells, each consisting of one or more bits. In addition, from the explaination above, the size of *BF*, i.e. K, should increase monotonically with M, however, there can be no linear relationship between the size of *SBF*, i.e. K', and M, and K' can be any integer predefined by users. Defination 1 is the detailed defination of *SBF*.

Definition 1: SBF is defined as an array of integer $SBF[1], SBF[2], ..., SBF[k]$ whose minimum value is 0 and maximum value is max. Each item of the array is allocated d bits. The relation between max and d is then $max = 2^d - 1$. Compared to bits in a regular Bloom lter, each item of the *SBF* is called a cell.

Like *BFSS*, *SBFSS* consists of two phases: updating and quering. Algorithm 4 and Algorithm 5 give the detailed descriptions of them. Due to the length limit, no more tautology here.

From Algorithm 4, we observe that there are two kinds of operation on *SBF*. First randomly decrement P cells by 1 so as to make room for fresh items; then set the same H cells as in the update process to max.

Algorithm 4 SBFSS Update Algorithm

Input: Stream S, support threshold s
The form of the counters in D is $(e, f(e), \Delta(e))$
for $i = 0$ to $k-1$ **do**
    $SBF[i] = 0$
end for
for each item e of stream S **do**
    Select P different cells uniformly at random $SBF[j_1]...SBF[j_P]$, $P \in \{1, ..., k\}$
    for each cell $SBF[j] \in \{SBF[j_1]...SBF[j_P]\}$ **do**
        if $SBF[j] \geq 1$ **then**
            $SBF[j] = SBF[j] - 1$
        end if
    end for
    for $i = 1$ to H **do**
        $SBF[h_i(e)] = max$
    end for
    if e is monitored in D **then**
        $f(e) = f(e) + 1;$
    else if $m < C$ **then**
        Assign a new counter $(e, 1, 0)$ to it
        $m = m + 1$
    else
        Let e_m be the item with least hits, min
        Replace e_m with e in D
        $f(e) = min + 1, \Delta(e) = min$
    end if
    $N = N + 1;$
end for

Intuitively, due to the random deletion operation, *SBF* does not exceed its capacity in a data stream scenario, however, false negatives may come up as well.

G. Analysis of SBFSS

Like *BFSS*, we will give theoretical analysis of *SBFSS* in this section, including the equivalence of *SBF'* (the *SBF* used in [11]) and *SBF*, *FNs*, *FPs*, parameters setting, and time complexity.

*1) **Analysis of the equivalence of SBF' and SBF**:* *SBF'* is used to detect duplicates over data streams, in fact, from Algorithm 4 and Algorithm 5, we can observe that *SBF* is used to filter out the items that never appeared in S, and we check whether the item in A is duplicated one by one, if the item is a duplicate, we just claim it appeared in S, in this sense, *SBF'* and *SBF* share the same function. So the properties of *SBF'*, including the probability of a false positive and a false negative of *SBF'*, are fully applicable to *SBF*.

*2) **Analysis of FPs**:* In this section, we will give a theoretically upper bound of the expectation of the number of *FPs* in the ouput of *SBFSS*.

The detailed proof of Lemma 8 can be found in [11]. Because of the limitation of length, no more tautology here.

Lemma 8: The probability of a false positive (FP) of *SBF* is no greater than :

$$(1 - (\frac{1}{1 + \frac{1}{P(1/H - 1/m)}})^{max})^H \tag{16}$$

Algorithm 5 SBFSS Query Algorithm

Input: *SBF*, D, s, A, N, M
Output: low-frequency items with threshold s

```
flag = true; { flag: indicate whether an item is in SBF or not}
for i = 0 to M − 1 do
  flag = true
  for j = 1 to H do
    if SBF[h_j(A[i])] == 0 then
      flag = false
      break;
    end if
  end for
  if flag == true then
    if A[i] is monitored in D then
      if f(A[i]) ≤ ⌊sN⌋ + Δ(A[i]) then
        output A[i] as a low-frequency item
      end if
    else
      output A[i] as a low-frequency item
    end if
  end if
end for
```

Theorem 6: Assuming no specific data distribution, the expectation of the number of *FPs* in the output of *SBFSS*, denoted as $E'(\#FPs)$, satisfies:

$$E'(\#FPs) < M\min\{(1-(\frac{1}{1+\frac{1}{P(1/H-1/m)}})^{max})^H, \quad (1-(1-\frac{1}{k})^{HM})^H\} + \lceil\frac{1}{s}\rceil \tag{17}$$

Proof: From Fig. 3, we notice that the *FPs* of *SBFSS* has two parts: i) the false positives of *SBF*, i.e. II; ii) the items with $f_S(e) > \lfloor sN \rfloor$ wrongly output, i.e. VI. In fact, V is the only difference between the *FPs* of *SBFSS* and the *FPs* of *BFSS*, for *BFSS*, the items in V will be output, however, due to the false negatives of *SBF* in *SBFSS*, i.e. $IV + V + VIII$, the items in V will be neglected by *SBFSS*, in this sense, the false negatives of *SBF* inversely reduce the number of *FPs* of *SBFSS*.

In fact, the posibility of a false positive of *SBF* must be smaller than that of *BF* under the same conditions because *SBF* brings in decrement operation. For the first case, we can get the upper bound of the expectation of the number of the false positives of *SBF* through Lemma 7 and Lemma 8. For the second case, it is clearly to see from Fig. 3 that the maximum of the area of VI is $\lceil\frac{1}{s}\rceil$, which is the area of the square surrounded by dotted lines, and obviously it is a loose bound. The residual derivation is obvious and exactly the same as Theorem 2, and no more tautology here. ∎

Like *BFSS*, a tighter bound of the area of VI can be obtained if we assume the distribution of S, and the bound is $(\frac{f_S(e_m)}{\lfloor sN \rfloor})^\alpha(1-\frac{1}{2^\alpha})M$, the derivation of which is exactly the same as E.q. 10, if the distribution is Pareto. In addition, due to the exsitence of V, the number of the elemens with $f_S(e) > \lfloor sN \rfloor$ wrongly output is smaller than that of *BFSS* regardless of the data distribution of S.

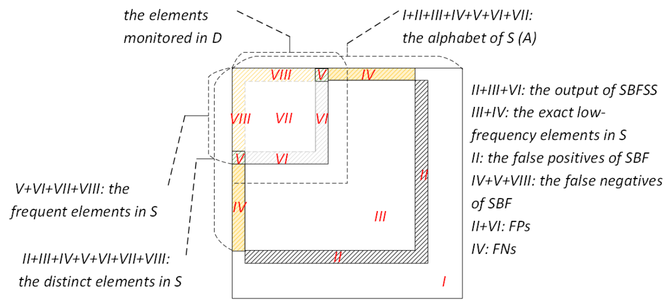


Fig. 3: The schematic diagram of the analysis of *SBFSS*

*3) **Analysis of FNs**:* In this section, unlike *BFSS*, we will give a theoretically upper bound of the expectation of the number of *FNs* in the ouput of *SBFSS*.

From Fig. 4, we can see that the *FNs* of *SBFSS*, IV, is a portion of the false negatives of *SBF*, $IV + V + VIII$. Therefore the upper bound of the number of *FNs* in the output of *SBFSS* is the number of the false negatives of *SBF*.

First, let us consider the probability of a false negative of *SBF* which is an error when an item in S wrongly filtered out by *SBF*. Therefore only the items in S can generate false negatives, obviously the number of false negatives is related to the input data distribution.

Suppose an item $A[i]$ appearing in S whose last appearence in S before a query comes is $e_{i-\delta_i}$, where δ_i represents the number of items in S during the time between the last appearance of $A[i]$ and a query, is hashed into H cells, $SBF[C_{i1}], ..., SBF[C_{iH}]$. A false negative happens if any of those H cells is decremented to 0 during the δ_i iterations when a query comes. Let $PR_0(\delta_i, p'_{ij})$ be the probability that cell $C_{ij}(j = 1...H)$ is decremented to 0 within the δ_i iterations, and A_l denotes the event that within the N iterations the most recent setting operation applied to the cell occurs at iteration $N - l$, and A'_N denotes the event that cell has never been set within the whole N iterations. $PR_0(\delta_i, p'_{ij})$ can be computed as:

$$PR_0(\delta_i, p'_{ij}) = \sum_{l=max}^{\delta_i-1}[Pr(SBF_{\delta_i}=0|A_l)Pr(A_l)] + Pr(SBF_{\delta_i}=0|A_{\delta_i})Pr(A'_{\delta_i}) \tag{18}$$

where

$$Pr(SBF_{\delta_i}=0|A_l) = \sum_{j=max}^{l}\binom{l}{j}p^j(1-p)^{l-j} \tag{19}$$

$$Pr(A_l) = (1-p'_{ij})^l p'_{ij} \tag{20}$$

$$Pr(SBF_{\delta_i}=0|A'_{\delta_i}) = \sum_{j=max}^{\delta_i}\binom{\delta_i}{j}p^j(1-p)^{\delta_i-j} \tag{21}$$

$$Pr(A'_{\delta_i}) = (1-p'_{ij})^{\delta_i} \tag{22}$$

Based on E.q 18, the expected value of the false negatives of *SBF*, $E(\#FN)$, can be easily derived, and the tailed derivation of E.q 18 and Lemma 9 can be found in [11], therefore no more tautology here.

Lemma 9: There must be an "average" $\widehat{\delta}$ and an "average" $\widehat{p'}$ such that:

$$E(\#FN) = n[1 - (1 - PR_0(\widehat{\delta}, \widehat{p'}))^H] \tag{23}$$

Theorem 7: Assuming no specific data distribution, the expectation of the number of *FNs* in the output of *SBFSS*, denoted as $E'(\#FNs)$, satisfies:

$$E'(\#FNs) \le M[1 - (1 - PR_0(\widehat{\delta}, \widehat{p'}))^H] \tag{24}$$

Proof: Like *FPs*, two kinds of items contribute to *FNs*: i) the elemtents with $0 < f_S(e) \le \lfloor sN \rfloor$ wrongly filtered out by *SBF*; ii) the elemtents with $f_S(e) \le \lfloor sN \rfloor$ monitored in D wrongly neglected. In fact, the items with $f_S(e) \le \lfloor sN \rfloor$ must satisfy $f(e) \le \lfloor sN \rfloor + \Delta(e)$, which means no items with $f_S(e) \le \lfloor sN \rfloor$ monitored in D wrongly neglected. For the first case, we assume that the false negatives are all items with $0 < f_S(e) \le \lfloor sN \rfloor$, i.e. neglecting $V + VIII$ in Fig. 3, which is the worst case, and in this way we have $E(\#FN) = E(\#FNs) = n[1 - (1 - PR_0(\widehat{\delta}, \widehat{p'}))^H] \le M[1 - (1 - PR_0(\widehat{\delta}, \widehat{p'}))^H]$. ■

From E.q 17, we observe that the upper bound of $E'(\#FPs)$ is completely related to *SBF* once the parameter s is confirmed, and the upper bound of $E'(\#FNs)$ is totally related to *SBF* which can be observed from E.q 24. In this way, we will give some analysis of how to choose appropriate parameters of *SBF* next.

4) ***Parameters setting****:* In this section, we will discuss how to choose appropriate parameters of *SBF* in *SBFSS*.

For *SBFSS*, users will specify s, $\#FPs$ and memory limit L. The memory consumption of *SBFSS* mainly contains two parts: *SBF* and D. From Algorithm 4, we know that there are at most $\lceil \frac{1}{s} \rceil$ counters in D, and in this way we can estimate the memory available for *SBF*, and through E.q 17, we can estimate the FP rate, in fact, the FP rate can be a little larger than the caculated one beacause only a part of items cause false positives. Therefore our goal is to choose a combination of max, H and P that minimizes the number of FNs under the condition that the FP rate is within a confirmed threshold and a fix amount of space.

In fact, a detailed discussion of how to choose these parameters has been given in [11], which can concluded as: A formula can be obtained to compute the value of P from other parameters provided that max and H have been chosen already; then find the optimal values of H for each case of $max(1, 3, 7)$ by trying limited number(≤ 10) of values of H on the FN rate formulas; Last, max can be set empirically, and specially in the case that no prior knowledge of the input data is available, max was suggested to be 1;

5) ***Time complexity****:* In this section, we will analyze the time complexity of *SBFSS*.

For *SBFSS*, the amount of space has been confirmed, so we just focus on time complexity.

Theorem 8: For *SBFSS*, Processing each data stream item needs $O(1)$ time, independent of the size of space and the stream.

Proof: Like *BFSS*, the processing time for each item also consists two parts: i) the time spent updating *SBF*; ii) the time spent updating D. In fact, from Algorithm 1 and Algorithm 4, we can see that *BFSS* and *SBFSS* share the same operations on D, so for *SBFSS*, the time spent updating D is constant once s is confirmed, which means the size of space has no impact on it. For the first part, a detailed explaination has been given in [11] to prove that the time spent updating *SBF* is O(1), independent of the size of space and the stream. Therefore the time complexity of *SBFSS* is O(1). ■

IV. Experiments

In this section, we first describe our data sets and the implementation details of *BFSS* and *SBFSS*, then we tested the performance of *BFSS* under different parameter settings on both real and synthetic data sets, then we compared the results against the method *nCount* and the method *rCount*. We were insterested in the *recall*, the number of correct items found as a percentage of the number of correct items, and the *precision*, the number of correct items found as a percentage of the entire output, then we calculated F_1. We also measured the space used by *BFSS*, *nCount* and *rCount*, which is essential to handle data streams, however, due to the page limit, we can not show the results of synthetic data here. For *SBFSS*, we compared the recall and precision against the method *lCount* on real data in different size of limited space. The detailed implementations of *nCount*, *rCount* and *lCount* will be given later. Last, we give a summarization of the experimental results. All the algorithms were compiled using the same compiler, and were run on a AMD Opteron(tm) 2.20GHz PC, with 64GB RAM, and 1.8TB Hard disk.

A. Data Sets

Real Data. For real data experiments, we used the dataset[5] which consists of all the requests made to the 1998 World Cup Web site between April 30, 1998 and July 26, 1998. We extracted 7×10^7 URLs in total.

Synthetic Data. We generated several synthetic Zipan data sets with the Zipf parameter varying from 0.5, which is very slightly uniform, to 3.0, which is highly skewed, with a fixed increment of 0.5. The size of each data set, N, is 10^7, and the alphabet was of size 10^6.

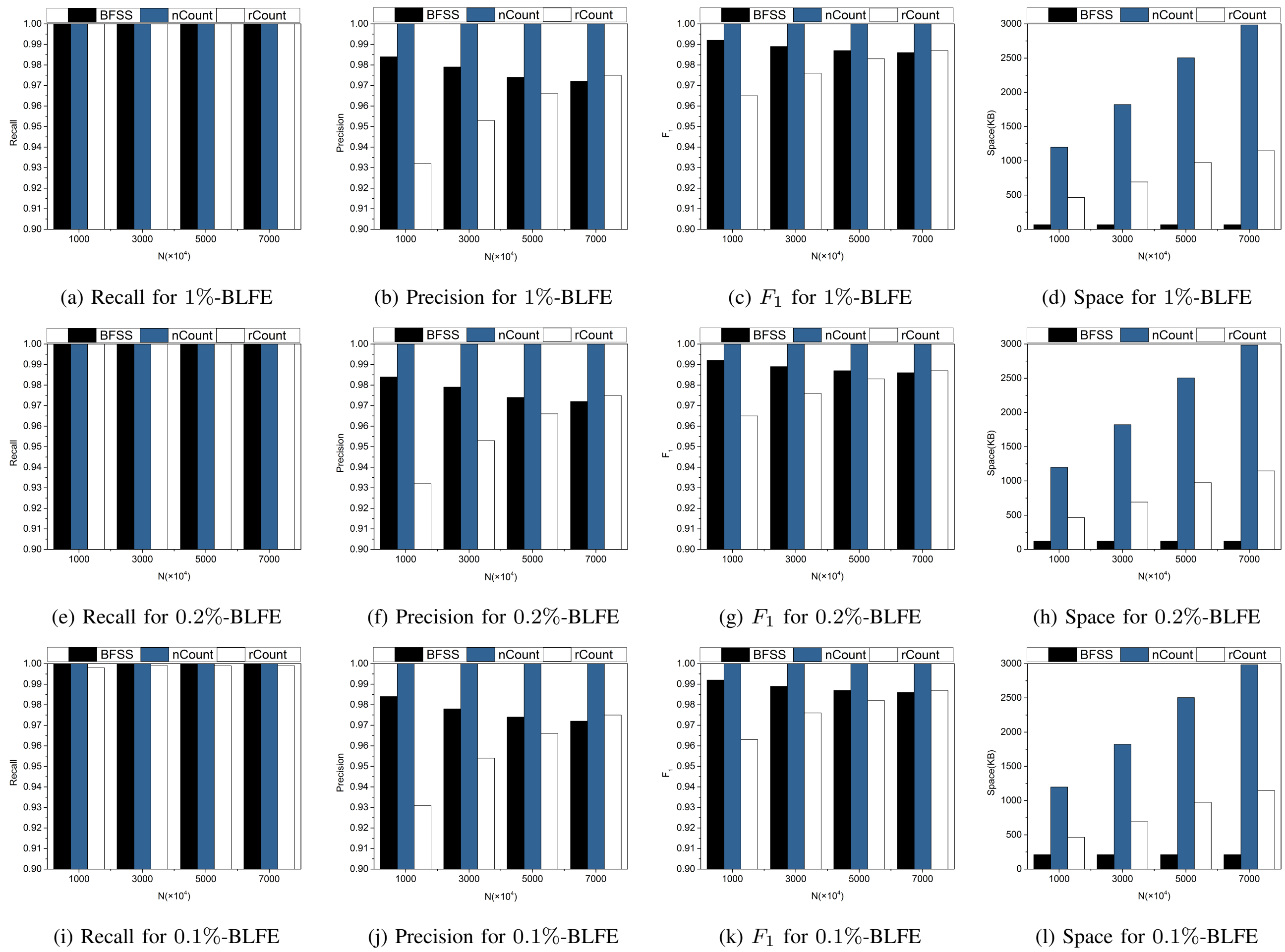
B. Implementation Issues

***BFSS* Implementation.** It is simple and straight forward to implement *BFSS*: 1) hash each incoming stream item into H numbers, and set the corresponding H bits to 1. 2) we maintain a min heap to index each counter in D, if there exists an empty counter, we just insert a new counter into the heap, otherwise we replace the top counter of the heap, at last we readjust the heap. 3) when a query comes, we just do as Algorithm 2 indicates.

Now the key point is how to choose the appropriate values of H and K. The

***SBFSS* Implementation.** The only difference between *BFSS* implementation and *SBFSS* implementation is the implementation of *BF* and *SBF*. For *SBF*, we generate a random number in each iteration, decrement the corresponding cell

[5]http://ita.ee.lbl.gov/html/contrib/WorldCup.html



(a) Recall for 1%-BLFE (b) Precision for 1%-BLFE (c) F_1 for 1%-BLFE (d) Space for 1%-BLFE

(e) Recall for 0.2%-BLFE (f) Precision for 0.2%-BLFE (g) F_1 for 0.2%-BLFE (h) Space for 0.2%-BLFE

(i) Recall for 0.1%-BLFE (j) Precision for 0.1%-BLFE (k) F_1 for 0.1%-BLFE (l) Space for 0.1%-BLFE

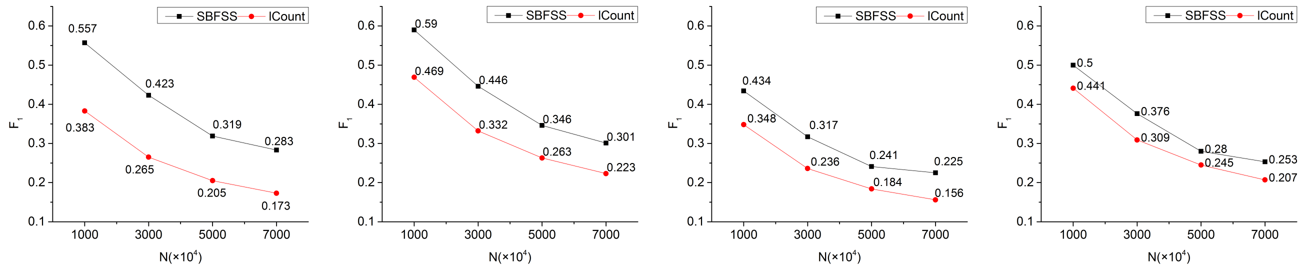
Fig. 4: Performance Comparison between *BFSS*, *nCount* and *rCount* Using Real Data - Varying s, N

TABLE III: Performance of *BFSS* Using Real Data - Varying s, K and N

s	K	Space	Precision				Recall			
			$N=10^7$	$N=3\times10^7$	$N=5\times10^7$	$N=7\times10^7$	$N=10^7$	$N=3\times10^7$	$N=5\times10^7$	$N=7\times10^7$
1%	270000	$49KB$	0.970	0.959	0.949	0.944	1	1	1	1
	400000	$66KB$	0.984	0.979	0.974	0.972	1	1	1	1
	800000	$116KB$	0.996	0.994	0.992	0.992	1	1	1	1
0.2%	270000	$103KB$	0.969	0.959	0.949	0.943	1	1	1	1
	400000	$120KB$	0.984	0.979	0.974	0.972	1	1	1	1
	800000	$170KB$	0.996	0.994	0.993	0.992	1	1	1	1
0.1%	270000	$193KB$	0.969	0.958	0.948	0.943	1	1	1	1
	400000	$210KB$	0.984	0.978	0.974	0.972	1	1	1	1
	800000	$260KB$	0.996	0.994	0.993	0.992	1	1	1	1

TABLE IV: Performance Comparison between *SBFSS* and *lCount* Using Real Data in Limited Space - Varying s, N

s	Space	Precision (SBFSS / lCount)				Recall (SBFSS / lCount)			
		$N=10^7$	$N=3\times10^7$	$N=5\times10^7$	$N=7\times10^7$	$N=10^7$	$N=3\times10^7$	$N=5\times10^7$	$N=7\times10^7$
1%	$54KB$	0.999/0.916	1/0.919	0.999/0.926	0.999/0.949	0.386/0.242	0.268/0.155	0.190/0.115	0.165/0.095
	$72KB$	1/0.920	1/0.930	0.999/0.940	0.999/0.959	0.419/0.315	0.287/0.202	0.209/0.153	0.177/0.126
0.2%	$54KB$	0.981/0.828	0.982/0.842	0.984/0.855	0.983/0.859	0.279/0.220	0.189/0.137	0.137/0.103	0.127/0.086
	$72KB$	0.997/0.886	0.996/0.906	0.999/0.922	0.998/0.907	0.334/0.294	0.232/0.186	0.163/0.141	0.145/0.117
0.1%	$54KB$	0.981/0.828	0.982/0.842	0.984/0.855	0.983/0.859	0.279/0.220	0.189/0.137	0.137/0.103	0.127/0.086
	$72KB$	0.997/0.886	0.996/0.906	0.999/0.922	0.998/0.907	0.334/0.294	0.232/0.186	0.163/0.141	0.145/0.117



(a) s=0.01, Space=54KB (b) s=0.01, Space=72KB (c) s=0.001, Space=54KB (d) s=0.001, Space=72KB

Fig. 5: Performance Comparison of F_1 between *SBFSS* and *lCount* - Varying s and Space

and $(P-1)$ cells adjacent to it by 1; this process is faster than generating P random numbers for each item; although the processes of picking the P cells are not independent, each cell has a probability of P/k for being picked at each iteration. Our analysis still holds. Then we hash each incoming stream item to H numbers, and set the corresponding H cells to max. Taking the number of *FNs*, the number of *FPs*, update time and limited space we can use into consideration, we set $max = 3, H = 4$ and $P = 1$.

***nCount* Implementation.** The basic idea of *nCount* is to allocate each distinct item in S a counter, we use a hash table to index these counters to speedup updating, and it is time consuming if we use a linked list or an array. When an item comes, we increment the corresponding counter using hash table. When a query comes, we traverse the hash table and output the low-frequency items.

***rCount* Implementation.** *rCount* shares a similar idea with *nCount*, however, unlike *nCount*, *rCount* uses Rabin's method [29] to hash each item in S to a 64-bit fingerprint, or it will consume too much space especially when the items are URLs. With this fingerprinting technique, there is a small chance that two dierent URLs are mapped to the same fingerprint. When an item comes, we first calculate its fingerprint and then increment the corresponding counter using hash table. When a query comes, we traverse the alphabet A and calculate the fingerprint of each item in A, then we check the corresponding counter and output it if it is a low-frequency item.

***lCount* Implementation.** *lCount* maintains a limited and fixed space, like *rCount*, *lCount* uses Rabin's method in order to adjust the space consumption through the maximum number of counters *lCount* can maintain. In addition, we use a max heap to index the counters in *lCount*. When an item comes, if there exists an empty counter, we insert a new counter into the heap, otherwise we replace the top counter of the heap, i.e. the counter with the maximum value, and set the value of the new counter to 1, at last we readjust the heap. When a query comes, we traverse the alphabet A and calculate the fingerprint of each item in A, then we check if the corresponding counter exists in the heap and output it if it is a low-frequency item.

C. Performance of BFSS

The query issued for *BFSS*, *nCount* and *rCount* was to find all items whose frequency is no more than $\frac{N}{s}(s = 1\%, 0.2\%, 0.1\%)$. The results comparing the recall, precision, F_1 and space used by the algorithms are summarized in Fig. 4. N varied from 10^7 to 7×10^7.

The recalls achieved by *BFSS* and *nCount* were constant at 1 for all values of N, as is clear from Fig. 4i, however, due to the existence of hash collisions, *rCount* may neglect a few low-frequency items, and the recall of *rCount* remained above 0.95. From Fig. 4j, with the increasing of N, the precision of *BFSS* decreased ranging from 0.969 to 0.943 since the false positive rate of *BF* increased with the increasing of N, and the precision of *rCount* increased ranging from 0.931 to 0.978 because only the items not appearing in S might cause false positives which can be observed from the detailed implementation of *rCount*, and the number of these items decreased with the increasing of N, in this way the precision of *rCount* would increase with the increasing of N, in addition, the precision of *nCount* remained unchanged at 1 because *nCount* just kept a counter for each item in S. Being concerned with both precision and recall, we calculated the values of F_1 of these algorithms based on the precisions and recalls as Fig. 4k shows, and we can find that *BFSS* outperformed *rCount* for all values of N. The advantage of *BFSS* is evident in Fig. 4l, which shows that *BFSS* achieved a reduction in the space used by a factor ranging from 6 when N was 10^7 up to 16 when N was 7×10^7, and it is interesting to note that with the increasing of N, the space used by *nCount* and *rCount* increased rapidly since more counters were needed with the increasing of N, however, the space used by *BFSS* was stable and even decreased a little with the increasing of N because the space complexity is not related to N which can be seen from the analysis of the space complexity of *BFSS*, and the minor difference between the space used by *BFSS* of different N existed in the space used by D.

Table III shows the performance of *BFSS* with different parameter settings. We neglect the minor difference of the space between the different N with the same s and K. We can see that the precision increased with the increasing of K, and once K and N were confirmed the precision was almost confirmed too, which means s had little affection on the precison of *BFSS* and the most *FPs* were caused by *BF*.

D. Distribution estimation

Table V shows the estimation of the distribution of the real data when N was 7×10^7, and we have two observations. First, the estimated results given by our method is precise

TABLE V: Distribution Estimation of Real Data.

s	$> 1\%$	$> 0.55\%$	$> 0.1\%$
estimated percentage of the items	0.027%	0.22%	1.1%
estimated percentage of the frequencies	6.0%	36%	72%
true percentage of the items	0.028%	0.23%	1.1%
true percentage of the frequencies	6.0%	36%	72%

enough, and nearly no difference exists between the estimated results and the real results, as is clear from the table. Another interesting obervation is that the most frequent one percent of the items in W occupied more than seventy percent of the number of the items in W, which means most people were interested in a few webpages and most webpages were browsed only a few times, and from table V, we can see that ninety-nine percent of the items in W occupied no more than thirty percent of the number of the items in W, in fact, thirty-two percent of the items in W were browsed only once, which is realy a "long tail".

E. Performance of SBFSS

As mentioned before, when the total space is limited, the space left for the Bloom filter is limited as well, and it may cause many *FPs* when the space is small. Table IV shows the performance comparison between *SBFSS* and *lCount* when the space was limited. When s was 0.01, *SBFSS* outperformed *lCount* in both precision and recall, and we can see that the precision of *SBFSS* was stable and constant at 1 for most values of N, and the precision of *lCount* increased with the increasing of N the reason of which is much like *rCount*. In addition, the recalls of *SBFSS* and *lCount* decreased with N since the space was limited and the information we could maintain was limited too, which directly caused the decreasing of the recalls. When s was 0.001, the rules of how the precisons and recalls changed was similar to the case when s was 0.01, and we can find that *SBFSS* still maintained high accuracy, however, the general performance of *SBFSS* was not as good as the case when s was 0.01 because much more space could be allocated to *SBF* when s was smaller. Another interesting observation is that *SBFSS* performed better when the limited space is small, which can be clearly seen from Fig. 5, which shows the comparison of the F_1 between *SBFSS* and *lCount*. We can observe that when s was confirmed, the smaller the space available was, the larger the gap between the F_1 was, and this obervation proves the efficiency of *SBFSS*.

V. Conclusion

Low-frequency items in data streams contain much useful information which can be widely used in commercial decision-making, data analysis, etc., and locating them is the first step. We present *BFSS*, an effective and space efficient algorithm that aims to solve the problem *s-BLFE* approximately. *BFSS* can output all low-frequency items and a few frequent items, the number of which can be theoretically bounded. The experimental results show the recall of *BFSS* was constant at 1 and the precision was above 0.95 using a small space. When the memory is limited and small, in which case *BFSS* can not guarantee high precision, we propose *SBFSS* which can guarantee .

The future directions of our work can be, but are not limited to: 1) output low-frequency items along with their approximate frequencies. 2) handle sliding window queries. 3) support both insertion and deletion of items. 4) find low-frequency item sets. In fact, like frequent items, there are many work can be done towards low-frequency items, however, due to the special features of them, we still have a long and tough way to go and we just take a small step forward.

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