

希腊字母

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1 Pre

1.1 Normal Distribution

$N(x)$ 为标准正态分布 (Standard Normal) 的累积分布函数 (CDF, Cumulative Distribution Function) :

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz$$

同时由于正态分布的对称性, 易知

$$N(-x) = 1 - N(x)$$

$N'(x)$ 为标准正态分布的概率密度函数 (PDF, Probability Density Function):

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

同样由于正态分布的对称性, 易知:

$$N'(x) = N'(-x)$$

1.2 Greeks

Greeks	
Delta	$\delta = \frac{\partial V}{\partial S}$
Gamma	$\gamma = \frac{\partial^2 V}{\partial S^2}$
Theta	$\theta = \frac{\partial V}{\partial t}$
Rho	$\rho = \frac{\partial V}{\partial r}$
Vega	$\nu = \frac{\partial V}{\partial \sigma}$

2 Black formula

$$c = e^{-r\tau} [FN(d_1) - KN(d_2)]$$

$$p = e^{-r\tau} [KN(-d_2) - FN(-d_1)]$$

其中有：

$$\begin{aligned}d_1 &= \frac{\ln(F/K) + \sigma^2/2\tau}{\sigma\sqrt{\tau}} \\d_2 &= \frac{\ln(F/K) + \sigma^2/2\tau}{\sigma\sqrt{\tau}} \\d_2 &= d_1 - \sigma\sqrt{\tau}\end{aligned}$$

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F} = \frac{\frac{\partial \ln(F/K)}{\partial F} \sigma\sqrt{\tau}}{(\sigma\sqrt{\tau})^2} = \frac{\partial(\ln F - \ln K)/\partial F}{\sigma\sqrt{\tau}} = \frac{1}{F\sigma\sqrt{\tau}}$$

2.1 Delta

$$\begin{aligned}FN'(d_1) &= \frac{F}{\sqrt{2\pi}} e^{-d_1^2/2} \\KN'(d_2) &= \frac{K}{\sqrt{2\pi}} e^{-d_2^2/2} = \frac{K}{\sqrt{2\pi}} e^{-d_1^2/2 + d_1\sigma\sqrt{\tau} - \sigma^2\tau/2} \\&= \frac{K}{\sqrt{2\pi}} e^{-d_1^2/2 + \ln(F/K)} \quad (d_1\sigma\sqrt{\tau} = \ln(F/K) + \sigma^2\tau) \\&= \frac{F}{\sqrt{2\pi}} e^{-d_1^2/2} \\&= FN'(d_1) \\\text{Delta}_c &= e^{-r\tau} [N(d_1) + FN'(d_1) \frac{\partial d_1}{\partial F} - KN'(d_2) \frac{\partial d_2}{\partial F}] \\&= e^{-r\tau} N(d_1) \\\text{Delta}_p &= e^{-r\tau} [KN'(-d_2) \frac{\partial d_2}{\partial F} - N(-d_1) - FN'(-d_1) \frac{\partial d_1}{\partial F}] \\&= e^{-r\tau} (-N(-d_1)) \quad (KN'(-d_2) = KN'(d_2), FN'(-d_1) = FN'(d_1)) \\&= e^{-r\tau} (N(d_1) - 1)\end{aligned}$$

3 BSM formula

$$\begin{aligned}C_t &= S_t e^{-q\tau} N(d_1) - K e^{-r\tau} N(d_2) \\P_t &= -S_t e^{-q\tau} N(-d_1) + K e^{-r\tau} N(-d_2)\end{aligned}$$

其中有：

$$\begin{aligned}d_1 &= \frac{\ln(S/K) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \\d_2 &= \frac{\ln(S/K) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \\d_2 &= d_1 - \sigma\sqrt{\tau}\end{aligned}$$

Lemma 3.1.

$$\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{\frac{\partial \ln(S/K)}{\partial S} \sigma\sqrt{\tau}}{(\sigma\sqrt{\tau})^2} = \frac{\partial(\ln S - \ln K)/\partial S}{\sigma\sqrt{\tau}} = \frac{1}{S\sigma\sqrt{\tau}}$$

$$\begin{aligned}\frac{\partial d_2}{\partial \tau} &= \frac{\partial d_1}{\partial \tau} - \frac{1}{2} \frac{\sigma}{\tau} \\ \frac{\partial d_2}{\partial \sigma} &= \frac{\partial d_1}{\partial \sigma} - \sqrt{\tau} \\ \frac{\partial d_2}{\partial r} &= \frac{\partial d_1}{\partial r}\end{aligned}$$

Lemma 3.2. 对于 BSM 公式, 有:

$$SN'(d_1) = Ke^{-r\tau}N'(d_2)$$

证明. 已知:

$$\begin{aligned}d_2^2 - d_1^2 &= (d_2 - d_1)(d_2 + d_1) \\ &= (-\sigma\sqrt{\tau})(2d_1 - \sigma\sqrt{\tau}) \\ &= (-\sigma\sqrt{\tau}) \left(\frac{2\ln(S/K) + 2(r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau} \right) \\ &= -2 \left[\ln \frac{S}{K} + r\tau \right]\end{aligned}$$

则有:

$$\ln \left(\frac{N'(d_1)}{N'(d_2)} \right) = -\frac{d_1^2}{2} + \frac{d_2^2}{2} = \frac{1}{2}(d_2^2 - d_1^2) = - \left[\ln \frac{S}{K} + r\tau \right]$$

对等式两边取指数:

$$\begin{aligned}\frac{N'(d_1)}{N'(d_2)} &= \exp \left(- \left[\ln \frac{S}{K} + r\tau \right] \right) \\ &= \exp \left(\ln \frac{K}{S} - r\tau \right) \\ &= \frac{K}{S} e^{-r\tau} \\ SN'(d_1) &= Ke^{-r\tau}N'(d_2)\end{aligned}$$

□

3.1 Delta

$$\begin{aligned}\text{Delta}_c &= \frac{\partial C}{\partial S} = N(d_1) + \frac{1}{S\sigma\sqrt{\tau}} [SN'(d_1) - Ke^{-r\tau}N'(d_2)] = N(d_1) \\ \text{Delta}_p &= \frac{\partial P}{\partial S} = -N(-d_1) + \frac{1}{S\sigma\sqrt{\tau}} [SN'(d_1) - Ke^{-r\tau}N'(d_2)] = N(d_1) - 1\end{aligned}$$

对于已知 $\partial C/\partial S$, 可对 PCP 求导:

3.2 Gamma

对于 gamma 而言, calls 和 puts 相同

$$\text{Gamma} = \frac{\partial^2 V}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{\tau}}$$

3.3 Vega

对于 vega 而言, calls 和 puts 相同

$$\text{Vega} = \frac{\partial V}{\partial \sigma} = S N'(d_1) \sqrt{\tau}$$

$$\begin{aligned} \text{Vega} &= \frac{\partial C}{\partial \sigma} = \frac{\partial}{\partial \sigma} [S_t N(d_1) - K e^{-r\tau} N(d_2)] \\ &= S_t N'(d_1) \frac{\partial d_1}{\partial \sigma} - K e^{-r\tau} N'(d_2) \frac{\partial d_2}{\partial \sigma} \\ &= S_t N'(d_1) \frac{\partial d_1}{\partial \sigma} - K e^{-r\tau} N'(d_2) \left[\frac{\partial d_1}{\partial \sigma} - \sqrt{\tau} \right] \quad (\text{对 } d_2 = d_1 - \sigma \sqrt{\tau} \text{ 求导}) \\ &= [S_t N'(d_1) - K e^{-r\tau} N'(d_2)] \frac{\partial d_1}{\partial \sigma} + K e^{-r\tau} N'(d_2) \sqrt{\tau} \\ &= S_t N'(d_1) \sqrt{\tau} \quad (\text{已知 } S_t N'(d_1) = K e^{-r\tau} N'(d_2)) \end{aligned}$$