希腊字母

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1 Pre

1.1 Normal Distribution

$N(x)$ 为标准正态分布（Standard Normal）的累积分布函数（CDF，Cumulative Distribution Function）:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$$

同时由于正态分布的对称性，易知

$$N(-x) = 1 - N(x)$$

$N'(x)$ 为标准正态分布的概率密度函数（PDF，Probability Density Function）:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

同样由于正态分布的对称性，易知：

$$N'(x) = N'(-x)$$

1.2 Greeks

$$\text{Delta} = \frac{\partial V}{\partial S} \qquad \text{Gamma} = \frac{\partial^2 V}{\partial S^2}$$
$$\text{Theta} = \frac{\partial V}{\partial t} \qquad \text{Rho} = \frac{\partial V}{\partial r} \qquad \text{Vega} = \frac{\partial V}{\partial \sigma}$$

2 Black formula

$$C = e^{-r\tau}[FN(d_1) - KN(d_2)]$$
$$P = e^{-r\tau}[KN(-d_2) - FN(-d_1)]$$

其中有：

$$d_1 = \frac{\ln(F/K) + \sigma^2/2\tau}{\sigma\sqrt{\tau}}$$
$$d_2 = \frac{\ln(F/K) + \sigma^2/2\tau}{\sigma\sqrt{\tau}}$$
$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F} = \frac{\frac{\partial \ln(F/K)}{\partial F}\sigma\sqrt{\tau}}{(\sigma\sqrt{\tau})^2} = \frac{\partial(\ln F - \ln K)/\partial F}{\sigma\sqrt{\tau}} = \frac{1}{F\sigma\sqrt{\tau}}$$

2.1 Delta

$$
\begin{aligned}
FN'(d_1) &= \frac{F}{\sqrt{2\pi}} e^{-d_1^2/2} \\
KN'(d_2) &= \frac{K}{\sqrt{2\pi}} e^{-d_2^2/2} = \frac{K}{\sqrt{2\pi}} e^{-d_1^2/2 + d_1\sigma\sqrt{\tau} - \sigma^2\tau/2} \\
&= \frac{K}{\sqrt{2\pi}} e^{-d_1^2/2 + \ln(F/K)} \qquad \big(d_1\sigma\sqrt{\tau} = \ln(F/K) + \sigma^2\tau\big) \\
&= \frac{F}{\sqrt{2\pi}} e^{-d_1^2/2} \\
&= FN'(d_1) \\
\text{Delta}_c &= e^{-r\tau}[N(d_1) + FN'(d_1)\frac{\partial d_1}{\partial F} - KN'(d_2)\frac{\partial d_2}{\partial F}] \\
&= e^{-r\tau} N(d_1) \\
\text{Delta}_p &= e^{-r\tau}[KN'(-d_2)\frac{\partial d_2}{\partial F} - N(-d_1) - FN'(-d_1)\frac{\partial d_1}{\partial F}] \\
&= e^{-r\tau}(-N(-d_1)) \qquad \big(KN'(-d_2) = KN'(d_2),\, FN'(-d_1) = FN'(d_1)\big) \\
&= e^{-r\tau}(N(d_1) - 1)
\end{aligned}
$$

3 BSM formula

$$
\begin{aligned}
C_t &= S_t e^{-q\tau} N(d_1) - K e^{-r\tau} N(d_2) \\
P_t &= -S_t e^{-q\tau} N(-d_1) + K e^{-r\tau} N(-d_2)
\end{aligned}
$$

其中有：

$$
\begin{aligned}
d_1 &= \frac{\ln(S/K) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \\
d_2 &= \frac{\ln(S/K) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \\
d_2 &= d_1 - \sigma\sqrt{\tau}
\end{aligned}
$$

Lemma 3.1.

$$
\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{\frac{\partial \ln(S/K)}{\partial S}\sigma\sqrt{\tau}}{(\sigma\sqrt{\tau})^2} = \frac{\partial(\ln S - \ln K)/\partial S}{\sigma\sqrt{\tau}} = \frac{1}{S\sigma\sqrt{\tau}}
$$

$$
\frac{\partial d_2}{\partial \tau} = \frac{\partial d_1}{\partial \tau} - \frac{1}{2}\frac{\sigma}{\tau}
$$

$$
\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \tau} - \sqrt{\tau}
$$

$$
\frac{\partial d_2}{\partial r} = \frac{\partial d_1}{\partial r}
$$

Lemma 3.2. 对于 *BSM* 公式，有：

$$
SN'(d_1) = Ke^{-r\tau}N'(d_2)
$$

证明. 已知：

$$\begin{aligned}
d_2^2 - d_1^2 &= (d_2 - d_1)(d_2 + d_1) \\
&= (-\sigma\sqrt{\tau})(2d_1 - \sigma\sqrt{\tau}) \\
&= (-\sigma\sqrt{\tau})\left(\frac{2\ln(S/K) + 2(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau}\right) \\
&= -2\left[\ln\frac{S}{K} + r\tau\right]
\end{aligned}$$

则有：

$$\ln\left(\frac{N'(d_1)}{N'(d_2)}\right) = -\frac{d_1^2}{2} + \frac{d_2^2}{2} = \frac{1}{2}(d_2^2 - d_1^2) = -\left[\ln\frac{S}{K} + r\tau\right]$$

对等式两边取指数：

$$\begin{aligned}
\frac{N'(d_1)}{N'(d_2)} &= \exp\left(-\left[\ln\frac{S}{K} + r\tau\right]\right) \\
&= \exp\left(\ln\frac{K}{S} - r\tau\right) \\
&= \frac{K}{S}e^{-r\tau} \\
SN'(d_1) &= Ke^{-r\tau}N'(d_2)
\end{aligned}$$

□

3.1 Delta

$$\begin{aligned}
\text{Delta}_c &= \frac{\partial C}{\partial S} = N(d_1) + \frac{1}{S\sigma\sqrt{\tau}}\left[SN'(d_1) - Ke^{-r\tau}N'(d_2)\right] = N(d_1) \\
\text{Delta}_p &= \frac{\partial P}{\partial S} = -N(-d_1) + \frac{1}{S\sigma\sqrt{\tau}}\left[SN'(d_1) - Ke^{-r\tau}N'(d_2)\right] = N(d_1) - 1
\end{aligned}$$

对于已知 $\partial C/\partial S$，可对 PCP 求导：

3.2 Gamma

对于 gamma 而言，calls 和 puts 相同

$$\text{Gamma} = \frac{\partial^2 V}{\partial S^2} = \frac{\partial N(d_1)}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{\tau}}$$

3.3 Vega

对于 vega 而言，calls 和 puts 相同

$$\text{Vega} = \frac{\partial V}{\partial \sigma} = SN'(d_1)\sqrt{\tau}$$

$$\begin{aligned}
\text{Vega} &= \frac{\partial C}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left[SN(d_1) - Ke^{-r\tau}N(d_2)\right] \\
&= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial \sigma} \\
&= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r\tau}N'(d_2)\left[\frac{\partial d_1}{\partial \sigma} - \sqrt{\tau}\right] \qquad \text{（对 } d_2 = d_1 - \sigma\sqrt{\tau} \text{ 求导）} \\
&= \left[SN'(d_1) - Ke^{-r\tau}N'(d_2)\right]\frac{\partial d_1}{\partial \sigma} + Ke^{-r\tau}N'(d_2)\sqrt{\tau} \\
&= SN'(d_1)\sqrt{\tau} \qquad \text{（已知 } SN'(d_1) = Ke^{-r\tau}N'(d_2)\text{）}
\end{aligned}$$