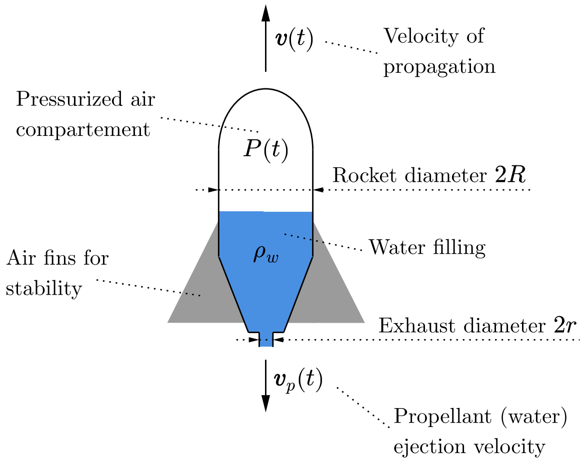


# Water Rocket Physics

An analysis of the physical principles of a water rocket


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# Contents

# 1 Introduction

The principle upon which rockets rely is simply Newton's third law, also known as the *actio est reactio* principle. Newton's third law states, that whenever there are two entities $A$ and $B$ and $A$ exerting a force $\boldsymbol{F}_{A\to B}$ upon $B$, there is also a force $\boldsymbol{F}_{B\to A}$ which is exerted on $A$ by $B$ related to $\boldsymbol{F}_{A\to B}$ as

$$\boldsymbol{F}_{A\to B} = -\boldsymbol{F}_{B\to A}. \tag{1}$$

A rocket in general is therefore a prime example of Newton's third law, but also of the conservation of momentum, meaning that all momenta $\boldsymbol{p}_i(t)$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$ in the system ≪rocket≫ add up to zero at all times $t$, that is $\sum_i \boldsymbol{p}_i(t) = 0$. Using the second law of Newton, the momenta $\boldsymbol{p}_i(t)$ in the system can be connected to forces $\boldsymbol{F}_i(t)$ as

$$\boldsymbol{F}_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}_i(t), \tag{2}$$

which allows for a derivation of the equation of motion for a rocket.

One of the first people to find and publish a form of the so-called rocket equation, namely Russian rocket scientist Konstantin Tsiolkovsky, proposed the following thought experiment. Imagine sitting on a rowing boat loaded with several individual stones; by an unhappy accident, you loose your oars, giving rise to the problem that your boat is no longer maneuverable. Remembering the physics education you received in school, you immediately have an idea to solve this problem looking at the stones loaded in your boat. By throwing the stones into the water in opposite direction of where you want to go one after another, you can propel the boat safely to shore. This is because by throwing a stone into the water, you exert a force $\boldsymbol{F}_{\text{boat}\to\text{water}}$ on the boat, but due to Newton's third law, an opposite force $\boldsymbol{F}_{\text{water}\to\text{boat}} = -\boldsymbol{F}_{\text{boat}\to\text{water}}$ is exerted by the water on the boat, propelling it to shore. In this case, the stones served as a propellant, but with a water rocket, also water can be used as a propellant - in fact, any massive substance can be used as a propellant for a rocket or other vehicles; the only condition for a propellant is, that it is ejected from the vehicle with high velocity, generating thrust to move the vehicle forward.

Using water as a propellant - rather than hot gases ejected from a jet engine - these Newtonian principles can be turned into action by means of building a water rocket.

# 2 Setup

A water rocket is essentially a container partially filled by water, where the remaining space contains pressurized air. The water rocket consists of a single opening at the bottom, through which the water contained in the rocket is ejected due to pressure chamber in the rocket, which is pressurized by several atmospheres, creating quite a significant pressure gradient with respect to the environment.

The water rocket is filled to approximately one third by water and by two thirds with pressurized air; the pressurization can be undertaken by means of a bicycle pump.

Using a suitable release mechanism, the rocket can be allowed to depressurize, resulting in the ejection of the water from the rocket and therefore producing thrust for the rocket; thrust is generated - as briefly explained in the introduction - by application of Newton's second and third laws of mechanics. A sketch of the setup may be found in fig. 1.

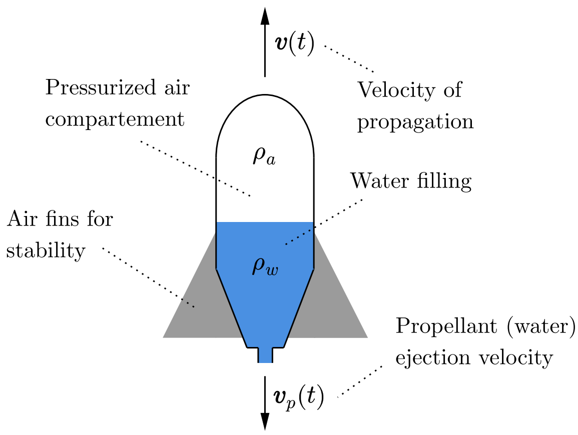



**Figure 1:** Sketch of a water rocket. Such a water rocket can be built using simple tools and resources; a water bottle, some styrofoam for the fins, hot glue and some hosing tools and equipment.

# 3 Analysis

## 3.1 Physics and mathematics preliminaries

### 3.1.1 Bernoulli equation

The Bernoulli equation relates the pressure $\boldsymbol{P}(\boldsymbol{x}, t)$ of a fluid element along a streamline with the respective velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ and height $h(\boldsymbol{x}, t)$ above ground of the fluid element. The Bernoulli equation states, that

$$|\boldsymbol{P}(\boldsymbol{x}, t)| + \rho g h(\boldsymbol{x}, t) + \frac{1}{2}\rho \boldsymbol{v}(\boldsymbol{x}, t)^2 = \text{const.} \tag{3}$$

holds. Consider all quantities at a fixed time $t$ and at two places $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ along a streamline; let $P(\boldsymbol{x}_1, t) \doteq \boldsymbol{P}_1$, $P(\boldsymbol{x}_2, t) \doteq \boldsymbol{P}_2$, $h(\boldsymbol{x}_1, t) \doteq h_1$, $h(\boldsymbol{x}_2, t) \doteq h_2$, $\boldsymbol{v}(\boldsymbol{x}_1, t) \doteq \boldsymbol{v}_1$ and $\boldsymbol{v}(\boldsymbol{x}_2, t) \doteq \boldsymbol{v}_2$; then, the Bernoulli equation can also be stated in the form

$$|\boldsymbol{P}_1| + \rho h_1 g + \frac{1}{2}\rho \boldsymbol{v}_1^2 = |\boldsymbol{P}_2| + \rho h_2 g + \frac{1}{2}\rho \boldsymbol{v}_2^2, \tag{4}$$

where $g \approx 9.81\,\mathrm{ms}^{-2}$ is the gravitational acceleration. The Bernoulli equation in these formulations only holds for incompressible fluids, which is to say that the density $\rho$ across the fluid essentially stays the same everywhere in the system under consideration at all times.

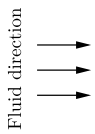
### 3.1.2 Drag equation

The so-called drag equation relates the velocity $\boldsymbol{v}(t)$ of an object to the force $\boldsymbol{F}_d(t)$ exerted on it by friction between the object and the surrounding medium. The drag equation is given by

$$\boldsymbol{F}_d(\boldsymbol{x}, t) = -\frac{1}{2}\rho(\boldsymbol{x}) C_d A \frac{\boldsymbol{v}(t)^3}{|\boldsymbol{v}(t)|}, \tag{5}$$

where $\rho(\boldsymbol{x})$ is the density of the surrounding medium, $C_d$ is the so-called drag coefficient of the object travelling with speed $\boldsymbol{v}(t)$ and $A$ is the cross-sectional area of the object with respect to its propagation direction. By definition, the drag force is directed exactly opposed to the propagation direction of the object at all times $t$.

The drag coefficient $C_d$ is dependent upon the geometry of the object. Different coefficients can be taken from fig. 2. A water rocket made from a plastic bottle combines




| Object | Shape | Drag coefficient $C_d$ |
|---|---|---|
| Sphere | | 0.47 |
| Half-sphere | | 0.42 |
| Long cylinder | | 0.82 |
| Short cylinder | | 1.15 |
| Streamlined body | | 0.04 |

**Figure 2:** Drag coefficients for different object geometries.

the properties of a sphere, of a half-sphere aswell as of a long cylinder; therefore it seems reasonable to calculate an average drag coefficient $\bar{C_d}$ for the water rocket averaging over these three shapes, given by

$$\bar{C_d} = \frac{C_{d,s} + C_{d,hs} + C_{d,lc}}{3} = 0.57. \tag{6}$$

### 3.1.3 Rocket equation

Consider a rocket ascending in positive vertical direction, where the speed of the rocket is denoted by $\boldsymbol{v}(t)$ and the propellant is ejected with velocity $\boldsymbol{v}_p(t)$ in negative vertical direction. The rocket equation can be stated as

$$m(t)\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}m(t)}{\mathrm{d}t}\boldsymbol{v}_p(t)\theta(\tau - t) - \frac{1}{2}\rho(\boldsymbol{x})\boldsymbol{v}(t)^2\bar{C}_d A\boldsymbol{e}_z - m(t)g\boldsymbol{e}_z, \tag{7}$$

where $\tau$ denotes the total time of propellant ejection. Given the stated assumptions and furthermore assuming a constant surrounding density $\rho$ for all places $\boldsymbol{x}$ and times $t$, the rocket velocity is $\boldsymbol{v}(t) = |\boldsymbol{v}(t)|\boldsymbol{e}_z = v(t)\boldsymbol{e}_z$ and the propellant velocity can be written as

$\boldsymbol{v}_p(t) = -|\boldsymbol{v}_p(t)|\boldsymbol{e}_z = -v_p(t)\boldsymbol{e}_z$, simplifying the rocket equation to

$$m(t)\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{\mathrm{d}m(t)}{\mathrm{d}t}v_p(t)\theta(\tau - t) - \frac{1}{2}\rho\bar{C}_d A v(t)^2 - m(t)g. \tag{8}$$

Here, the so-called Heaviside function

$$\theta : \mathbb{R} \to \{0, 1\}, \quad t \mapsto \theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases} \tag{9}$$

was used.

### 3.1.4 Integrals using the Heaviside function

Consider the above defined Heaviside function eq. (9) and three functions defined as

$$g(t) \doteq f(t)\theta(\tau - t), \quad h(t) \doteq f(t)\theta(t - \tau), \quad k(t) \doteq f(\tau)\theta(t - \tau), \tag{10}$$

where $f(t)$ is an arbitrary smooth function. The integral of $g(t)$ with respect to time $t$ can be calculated as

$$\begin{aligned} I_{g(t)} &= \int_0^t g(t')\,\mathrm{d}t' = \int_0^t f(t')\theta(\tau - t')\,\mathrm{d}t' \\ &= \theta(t - \tau)\int_0^\tau f(t')\,\mathrm{d}t' + \theta(\tau - t)\int_0^t f(t')\,\mathrm{d}t'. \end{aligned} \tag{11}$$

The integral of $h(t)$ with respect to time $t$ can be calculated as

$$\begin{aligned} I_{h(t)} &= \int_0^t h(t')\,\mathrm{d}t' = \int_0^t f(t')\theta(t' - \tau)\,\mathrm{d}t' \\ &= \theta(t - \tau)\int_\tau^t f(t')\,\mathrm{d}t' + \theta(\tau - t)\cdot 0 \\ &= \theta(t - \tau)\int_\tau^t f(t')\,\mathrm{d}t'. \end{aligned} \tag{12}$$

The integral of $k(t)$ with respect to time $t$ can be calculated as

$$\begin{aligned} I_{k(t)} &= \int_0^t k(t')\,\mathrm{d}t' = \int_0^t f(\tau)\theta(t' - \tau)\,\mathrm{d}t' \\ &= \theta(t - \tau)f(\tau)\int_\tau^t \mathrm{d}t' + \theta(\tau - t)f(\tau)\cdot 0 \\ &= \theta(t - \tau)f(\tau)\int_\tau^t \mathrm{d}t'. \end{aligned} \tag{13}$$

## 3.2 Solving the rocket equation for a water rocket not experiencing drag force

Solving the rocket equation without taking drag forces into consideration can be done analytically. In this section, the analytical solution for the rocket equation without drag force is derived, as well as examined by example calculations.

### 3.2.1 Formulation of the rocket equation for a water rocket not experiencing drag force

In order to solve the rocket equation without taking into consideration drag forces exerted by the air surrounding the rocket, we start from eq. (8). Not accounting for drag forces means, that the drag coefficient $\bar{C}_d$ is set to zero, that is $\bar{C}_d = 0$. Therefore, the second term in eq. (8) vanishes and we can account for the resulting rocket equation by

$$m(t)\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{\mathrm{d}m(t)}{\mathrm{d}t}v_p(t)\theta(\tau - t) - m(t)g, \tag{14}$$

where $v_p(t) \geq 0$. Furthermore, we assume, that the propellant velocity is constant, i.e. $v_p(t) = v_p$, whereas for the mass of the rocket, a linear relationship to time is presupposed. Hence, the mass $m(t)$ can be accounted for by

$$m(t) = m_0 - bt \quad \Rightarrow \quad \frac{\mathrm{d}m(t)}{\mathrm{d}t} = -b, \tag{15}$$

with $b$ being the mass ejection parameter with units $[b] = \mathrm{kgs}^{-1}$. Using these simplifications, the rocket equation can be further simplified to

$$(m_0 - bt)\frac{\mathrm{d}v(t)}{\mathrm{d}t} = bv_p\theta(\tau - t) - (m_0 - bt)g. \tag{16}$$

Dividing both sides of this equation by $m_0 - bt$ yields

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{bv_p}{m_0 - bt}\theta(\tau - t) - g, \tag{17}$$

which can subsequently be integrated with respect to time $t$.

### 3.2.2 Analytical solution of the rocket equation for a rocket not experiencing drag force

In order to obtain the velocity $v(t)$ of the rocket, eq. (17) has to be integrated with respect to time $t$, that is to say by calculating

$$v(t) = \int_0^t \frac{\mathrm{d}v(t')}{\mathrm{d}t'}\,\mathrm{d}t' = \int_0^t \left[\frac{bv_p}{m_0 - bt'}\theta(\tau - t') - g\right]\mathrm{d}t'. \tag{18}$$

The integral on the right-hand side of the above equation can be split into three parts by application of eq. (11), such that

$$v(t) \overset{\text{eq. (11)}}{=} \underbrace{\theta(t-\tau)\int_0^\tau \frac{bv_p}{m_0 - bt'}\,\mathrm{d}t'}_{I_{v,1}} + \underbrace{\theta(\tau - t)\int_0^t \frac{bv_p}{m_0 - bt'}\,\mathrm{d}t'}_{I_{v,2}} - \underbrace{\int_0^t g\,\mathrm{d}t'}_{I_{v,3}}. \tag{19}$$

Solving the integrals $I_{v,1}$ and $I_{v,2}$ can be solved by means of using the integral

$$\int_0^t \frac{bv_p}{m_0 - bt'}\,\mathrm{d}t' = -v_p\int_{m_0}^{m_0 - bt}\frac{1}{x}\,\mathrm{d}x = v_p \ln\left(\frac{m_0}{m_0 - bt}\right), \tag{20}$$

where the substitution $x(t') \doteq m_0 - bt'$ was performed; hence for $I_{v,1}$, $I_{v,2}$ and $I_{v,3}$ one obtains

$$I_{v,1} = \theta(t-\tau)v_p \ln\left(\frac{m_0}{m_0 - b\tau}\right), \quad I_{v,2} = \theta(\tau - t)v_p \ln\left(\frac{m_0}{m_0 - bt}\right), \quad I_{v,3} = gt. \tag{21}$$

Putting everything together yields

$$v(t) = \theta(t-\tau)v_p \ln\left(\frac{m_0}{m_0 - b\tau}\right) + \theta(\tau - t)v_p \ln\left(\frac{m_0}{m_0 - bt}\right) - gt. \tag{22}$$

In order to obtain the analytical expression for the height $h(t)$ reached by the rocket, eq. (22) has to be integrated with respect to time $t$, that is to say by calculating

$$h(t) = \int_0^t \frac{\mathrm{d}h(t')}{\mathrm{d}t'}\,\mathrm{d}t' = \int_0^t v(t')\,\mathrm{d}t. \tag{23}$$

Using the expression eq. (22), one obtains

$$h(t) = \underbrace{\int_0^t \theta(t'-\tau)v_p \ln\left(\frac{m_0}{m_0 - b\tau}\right)\mathrm{d}t'}_{I_{h,1}} + \underbrace{\int_0^t \theta(\tau - t')v_p \ln\left(\frac{m_0}{m_0 - bt'}\right)\mathrm{d}t'}_{I_{h,2}} - \underbrace{\int_0^t gt'\,\mathrm{d}t'}_{I_{h,3}}. \tag{24}$$

The integral $I_{h,1}$ can be solved by using eq. (13), such that

$$\begin{aligned} I_{h,1} &= \int_0^t \theta(t'-\tau)v_p \ln\left(\frac{m_0}{m_0 - b\tau}\right)\mathrm{d}t' \overset{\text{eq. (13)}}{=} \theta(t-\tau)v_p \ln\left(\frac{m_0}{m_0 - b\tau}\right)\int_\tau^t \mathrm{d}t' \\ &= \theta(t-\tau)v_p \ln\left(\frac{m_0}{m_0 - b\tau}\right)(t-\tau). \end{aligned} \tag{25}$$

The integral $I_{h,2}$ is more complicated to solve; first, it can be decomposed into two separate parts using eq. (11), yielding

$$I_{h,2} \overset{\text{eq. (11)}}{=} \underbrace{\theta(t-\tau)v_p \int_0^\tau \ln\left(\frac{m_0}{m_0 - bt'}\right)\mathrm{d}t'}_{I_{h,2a}} + \underbrace{\theta(\tau - t)v_p \int_0^t \ln\left(\frac{m_0}{m_0 - bt'}\right)\mathrm{d}t'}_{I_{h,2b}}. \tag{26}$$

Solving these integrals requires the solution to an integral of the form

$$\begin{aligned} \int_0^t \ln\left(\frac{m_0}{m_0 - bt'}\right)\mathrm{d}t' &= \frac{m_0}{b}\int_1^{\frac{m_0}{m_0-bt}} \frac{\ln(y)}{y^2}\,\mathrm{d}y = -\frac{m_0}{b}\int_1^{\frac{m_0}{m_0-bt}} \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\ln(y)+1}{y}\right)\mathrm{d}y \\ &= -\frac{m_0}{b}\left[\frac{\ln(y)+1}{y}\right]_1^{\frac{m_0}{m_0-bt}} \\ &= -\frac{m_0}{b}\left[\frac{m_0 - bt}{m_0}\ln\left(\frac{m_0}{m_0 - bt}\right) + \frac{m_0 - bt}{m_0} - 1\right] \\ &= t - \frac{m_0 - bt}{b}\ln\left(\frac{m_0}{m_0 - bt}\right), \end{aligned} \tag{27}$$

where the substitution $y(t') \doteq \frac{m_0}{m_0 - bt'}$ was used. Using this integral, one obtains

$$I_{h,2a} = \theta(t-\tau)v_p \left[\tau - \frac{m_0 - b\tau}{b} \ln\left(\frac{m_0}{m_0 - b\tau}\right)\right] \tag{28}$$

and

$$I_{h,2b} = \theta(\tau - t)v_p \left[t - \frac{m_0 - bt}{b} \ln\left(\frac{m_0}{m_0 - bt}\right)\right], \tag{29}$$

herewith obtaining $I_{h,2}$ as

$$I_{h,2} = \theta(t-\tau)v_p \left[\tau - \frac{m_0 - b\tau}{b} \ln\left(\frac{m_0}{m_0 - b\tau}\right)\right] + \theta(\tau - t)v_p \left[t - \frac{m_0 - bt}{b} \ln\left(\frac{m_0}{m_0 - bt}\right)\right] \tag{30}$$

Hence, the total solution to $h(t)$ is yielded by the expression $h(t) = I_{h,1} + I_{h,2} - I_{h,3}$, given by

$$\begin{aligned} h(t) = \theta(t-\tau)v_p &\left[\ln\left(\frac{m_0}{m_0 - b\tau}\right)(t-\tau) + \tau - \frac{m_0 - b\tau}{b} \ln\left(\frac{m_0}{m_0 - b\tau}\right)\right] \\ &+ \theta(\tau - t)v_p \left[t - \frac{m_0 - bt}{b} \ln\left(\frac{m_0}{m_0 - bt}\right)\right] - \frac{1}{2}gt^2. \end{aligned} \tag{31}$$

#### 3.2.3 Propellant velocity, propellant ejection time and mass ejection parameter

In order to calculate the propellant velocity $v_p$, the propellant ejection time $\tau$ and the mass ejection parameter $b$ needed to perform actual calculations using the solutions of the rocket equation, the Bernoulli equation can be of help; consider therefore fig. 3.

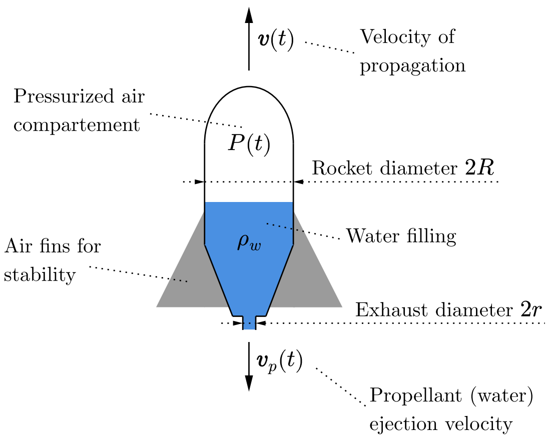



**Figure 3:** Sketch of a water rocket with necessary definitions of quantities to calculate.

In order to use the Bernoulli equation to calculate an approximate expression for the propellant velocity $v_p(t) = v_p$ - which is assumed to be constant over the whole period of propellant ejection -, one first needs to know the function $P(t)$. The function $P(t)$ describes, how the pressure inside the pressurized air chamber of the rocket evolves in time, as the water is ejected through the exhaust. In order to keep things as simple as possible, but still powerful regarding predictions, a linear relationship $P(t) = \alpha_P t + P_0$ for the pressure is assumed, where we require, that $P(\tau) = 0$ and $P(0) = P_0$. This means, that the slope parameter $\alpha_P$ must be given as $\alpha_P = -P_0/\tau$. The pressure $P(t)$ is understood to be the additional pressure to one standard atmosphere, which is to say that it is the pressure indicated by any pump or pressure measuring device. If the propellant ejection velocity $v_p$ is assumed to be constant, it is reasonable to use averaged values with respect to time in calculations leading to an expression for $v_p$. The, we calculate the average pressure $\bar{P}$ over the total time of propellant ejection as

$$\bar{P} = \frac{1}{\tau}\int_0^\tau P(t)\,\mathrm{d}t = \frac{1}{\tau}\int_0^\tau [\alpha_P t + P_0]\,\mathrm{d}t = \frac{1}{2}\alpha_P\tau + P_0 = -\frac{P_0}{2} + P_0 = \frac{P_0}{2}. \tag{32}$$

Since it is reasonable to use average values for any time dependent quantity, we use $P(t) \approx \bar{P}$.

Next, we assume, that the flow velocity of the fluid inside the rocket is zero; furthermore, any gravitational effect within the fluid is neglected. It is to be remembered, that $P(t) \approx \bar{P}$ describes the pressure that exceeds that of one standard atmosphere. Using the Bernoulli equation, one therefore obtains the equation

$$\bar{P} = \frac{1}{2}\rho_w v_p^2 \quad \Leftrightarrow \quad v_p = \sqrt{\frac{2\bar{P}}{\rho_w}} = \sqrt{\frac{P_0}{\rho_w}}. \tag{33}$$

The propellant ejection time $\tau$ is calculated using the assumption, that the total mass of water $m_w$ initially present in the rocket tank is equal to a cylinder of water with a height $v_p\tau$ and a radius of $r$; $r$ being the exhaust radius. Denoting the mass of the empty rocket with $m_r$, it hence follows, that

$$m_w = m_0 - m_r = \rho_w r^2 \pi v_p \tau = r^2\pi\tau\sqrt{P_0\rho_w} \quad \Leftrightarrow \quad \tau = \frac{m_w}{r^2\pi\sqrt{P_0\rho_w}} \tag{34}$$

holds.

Now, it is straightforward to calculate the mass ejection parameter $b$ involved in the mass equation $m(t) = m_0 - bt = m_w + m_r - bt$; for this purpose, we require $m(\tau) = m_r$. Hence, the parameter $b$ is the obtained as

$$m(\tau) = m_w + m_r - b\tau = m_r \quad \Leftrightarrow \quad b = \frac{m_w}{\tau} = r^2\pi\sqrt{P_0\rho_w}. \tag{35}$$

### 3.2.4 Actual calculations and results

Text.

### 3.2.5 Optimizing adjustable parameters for maximal height

Given that the typical timespan

$$\tau = \frac{m_w}{r^2\pi\sqrt{P_0\rho_w}} \tag{36}$$

of water ejection is very short compare to the total flight time of a water rocket, we can restrict the time interval where we look for maximal height to the condition $t > \tau$, since the rocket is very likely to be in upward vertical motion while propellant is still ejected. Due to inertia, the rocket will even be in upward motion for a time period after the last bit of propellant was ejected, hence the restriction $t > \tau$ in time for a search of maximal height optimization. With this restriction, the general solution eq. (31) now just reads as

$$h(t) = v_p\left[\ln\left(\frac{m_0}{m_0 - b\tau}\right)(t-\tau) + \tau - \frac{m_0 - b\tau}{b}\ln\left(\frac{m_0}{m_0 - b\tau}\right)\right] - \frac{1}{2}gt^2, \quad t > \tau. \tag{37}$$

Basically, there are two controllable parameters with a water rocket, the mass of the water $m_w$ with which the rocket was filled; and the pressure $P_0$ the rocket is started with. For further optimization, we first need to find an expression for maximal height $h_{max}(m_w, P_0)$ involving these parameters, so we start by the classical procedure for finding an extremal point of a function:

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = v_p \ln\left(\frac{m_0}{m_0 - b\tau}\right) - gt \stackrel{!}{=} 0, \tag{38}$$

which leads to the time of maximal height $t_{max}$ as

$$t_{max} = \frac{v_p}{g}\ln\left(\frac{m_0}{m_0 - b\tau}\right) \tag{39}$$

by means of rearrangement. Now, we can calculate $h_{max}$ by just inserting $t_{max}$ into the function $h(t)$ for $t$:

$$\begin{aligned} h_{max} \doteq h(t_{max}) &= v_p \ln\left(\frac{m_0}{m_0 - b\tau}\right)\frac{v_p}{g}\ln\left(\frac{m_0}{m_0 - b\tau}\right) - v_p\ln\left(\frac{m_0}{m_0 - b\tau}\right)\tau \\ &\quad + v_p\tau - \frac{m_0 - b\tau}{b}v_p\ln\left(\frac{m_0}{m_0 - b\tau}\right) \\ &= v_p\ln\left(\frac{m_0}{m_0 - b\tau}\right)\left[\frac{v_p}{g}\ln\left(\frac{m_0}{m_0 - b\tau}\right) - \tau - \frac{m_0}{b} + \tau\right] + v_p\tau \\ &= v_p\ln\left(\frac{m_0}{m_0 - b\tau}\right)\left[\frac{v_p}{g}\ln\left(\frac{m_0}{m_0 - b\tau}\right) - \frac{m_0}{b}\right] + v_p\tau. \end{aligned} \tag{40}$$

Now, we can insert the full expressions for the parameters $b$ and $\tau$ as given in eq. (35), finally yielding a function

$$h_{max}(m_w, P_0) = \sqrt{\frac{P_0}{\rho_w}}\ln\left(\frac{m_f + m_w}{m_f}\right)\left[\sqrt{\frac{P_0}{g^2\rho_w}}\ln\left(\frac{m_f + m_w}{m_f}\right) - \frac{m_f + m_w}{r^2\pi\sqrt{P_0\rho_w}}\right] + \frac{m_w}{r^2\pi\rho_w}, \tag{41}$$

which can be optimized with respect to $m_w$ and $P_0$ using the gradient condition and the Hesse matrix. **Go on here.**

### 3.3 Solving the rocket equation for a water rocket with drag force

It is not possible to solve the rocket equation analytically anymore, if drag forces are taken into consideration. In this section, the rocket equation with drag forces is solved, using the fourth-order Runge-Kutta method. Furthermore, example calculations are carried out and examined.

#### 3.3.1 Formulation of the rocket equation for a water rocket experiencing drag force

Text.

#### 3.3.2 Numerical solution of the rocket equation for a rocket experiencing drag force

#### 3.3.3 Actual calculations and results

Text.

## 4 Derivations

### 4.1 Derivation of the Bernoulli equation

Consider a fluid flowing in a pipe as shown in fig. 4; furthermore let $\Delta V$ with mass $\Delta m = \rho \Delta V$ denote a small fluid portion moving along a streamline from the right to the left, where $\rho$ is the constant fluid density. The total work done on/by the fluid portion $W_{tot}$ is given by

$$W_{tot} = \Delta E_{kin} + \Delta E_{pot}. \tag{42}$$

Due to the conservation of energy, the change in potential energy $\Delta E_{pot}$ can be calculated by

$$\Delta E_{pot} = \Delta m g h_2 - \Delta m g h_1 = \rho \Delta V g (h_2 - h_1). \tag{43}$$

The change in kinetic energy $\Delta E_{kin}$ however can be written as

$$\Delta E_{kin} = \frac{1}{2} \Delta m \boldsymbol{v}_2^2 - \frac{1}{2} \Delta m \boldsymbol{v}_1^2 = \frac{1}{2} \rho \Delta V (\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2). \tag{44}$$

The force $\boldsymbol{F}_1$ pushing the fluid element from 1 to $1'$ can be written by $\boldsymbol{F}_1 = A_1 \boldsymbol{P}_1$; furthermore the force $\boldsymbol{F}_2$ opposing the fluid movement from 2 to $2'$ is given by $\boldsymbol{F}_2 = -A_2 \boldsymbol{P}_2$. The work $W_1$ done for the fluid movement $1 \to 1'$ is calculated as $W_1 = |\boldsymbol{F}_1| \Delta x_1 = |\boldsymbol{P}_1| A_1 \Delta x_1 = |\boldsymbol{P}_1| \Delta V$, whereas the work $W_2$ done for the fluid movement $2 \to 2'$ is given by $W_2 = -|\boldsymbol{F}_2| \Delta x_2 = -|\boldsymbol{P}_2| A_2 \Delta x_2 = -|\boldsymbol{P}_2| \Delta V$. The total work $W_{tot}$ can then be written by means of the equation

$$W_{tot} = W_1 + W_2 = \Delta E_{kin} + \Delta E_{pot} = (|\boldsymbol{P}_1| - |\boldsymbol{P}_2|) \Delta V. \tag{45}$$

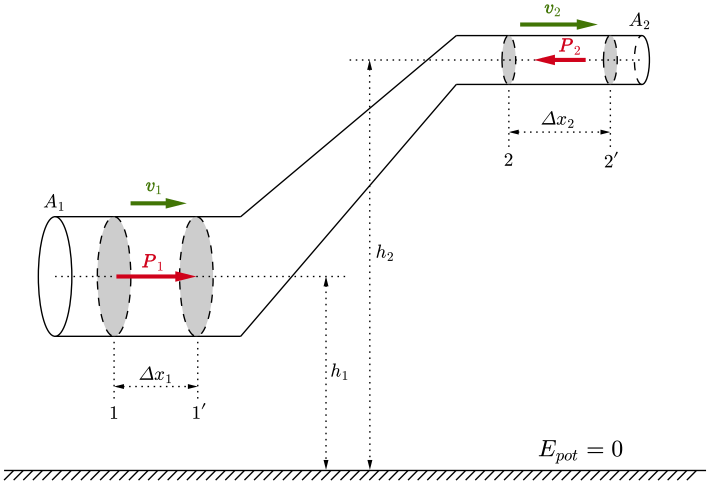



**Figure 4:** Sketch for the derivation of the Bernoulli equation.

Plugging in the expressions for $\Delta E_{pot}$ and $\Delta E_{kin}$, the equation

$$\rho \Delta V g(h_2 - h_1) + \frac{1}{2}\rho\Delta V(\boldsymbol{v}_2^2 - \boldsymbol{v}_1^2) = (|\boldsymbol{P}_1| - |\boldsymbol{P}_2|)\Delta V \tag{46}$$

results. After rearranging terms and dividing by $\Delta V$, the Bernoulli equation

$$|\boldsymbol{P}_1| + \rho g h_1 + \frac{1}{2}\rho\boldsymbol{v}_1^2 = |\boldsymbol{P}_2| + \rho g h_2 + \frac{1}{2}\rho\boldsymbol{v}_2^2 \tag{47}$$

obtains. Assuming, that this equation reflects a stationary state at a fixed time $t$ and further assuming that $\boldsymbol{P}_1 = \boldsymbol{P}(\boldsymbol{x}_1, t)$, $\boldsymbol{P}_2 = \boldsymbol{P}(\boldsymbol{x}_2, t)$, $h_1 = h(\boldsymbol{x}_1, t)$, $h_2 = h(\boldsymbol{x}_2, t)$, $\boldsymbol{v}_1 = \boldsymbol{v}(\boldsymbol{x}_1, t)$ and $\boldsymbol{v}_2 = \boldsymbol{v}(\boldsymbol{x}_2, t)$, the Bernoulli equation can also be written in the form

$$|\boldsymbol{P}(\boldsymbol{x}, t)| + \rho g h(\boldsymbol{x}, t) + \frac{1}{2}\rho\boldsymbol{v}(\boldsymbol{x}, t)^2 = \text{const.} \tag{48}$$

indicating, that pressure, velocity and height of a fluid element sum to a constant value in the specified combination at all times and places along a streamline.

## 4.2 Derivation of the drag equation

Consider a fluid of constant density $\rho$ flowing horizontally around a massive object, as seen in fig. 5. Assuming, that the fluid velocity $|\boldsymbol{v}_2|$ behind the object is zero and that the fluid does not undergo any height change with respect to some reference height, the Bernoulli equation yields

$$|\boldsymbol{P}_1| + \frac{1}{2}\rho\boldsymbol{v}_1^2 = |\boldsymbol{P}_2| \quad \Leftrightarrow \quad \Delta P \doteq |\boldsymbol{P}_2| - |\boldsymbol{P}_1| = \frac{1}{2}\rho\boldsymbol{v}_1^2. \tag{49}$$

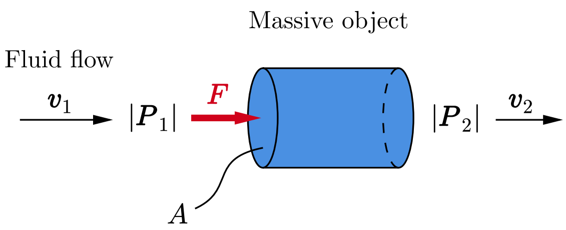



**Figure 5:** Sketch for the derivation of the drag equation.

A change of pressure $\Delta P$ in the fluid going from the front of the object to the back of it implies, that there is a force $F \doteq |\boldsymbol{F}|$ exerted by the fluid on the object. Writing $\boldsymbol{v} \doteq \boldsymbol{v}_1$, the magnitude of $F$ is given by the equation

$$F = A\Delta P = \frac{1}{2}\rho A \boldsymbol{v}^2. \tag{50}$$

In general, the fluid velocity is dependent upon time, therefore $\boldsymbol{v} = \boldsymbol{v}(t)$; furthermore, the density of the fluid can be spatially variable, hence $\rho = \rho(\boldsymbol{x})$. Now, if an object of arbitrary shape moving with the velocity $\boldsymbol{v}(t)$ within a stationary fluid is considered, the drag force $\boldsymbol{F}_d(\boldsymbol{x}, t)$ it experiences is directly opposing the direction of movement and can be written by

$$\boldsymbol{F}_d(\boldsymbol{x}, t) = -\frac{1}{2}\rho(\boldsymbol{x}) C_d A \frac{\boldsymbol{v}(t)^3}{|\boldsymbol{v}(t)|}, \tag{51}$$

where $C_d$ is a scaling factor to be experimentally determined related to the geometry of the object under consideration. Drag coefficients for different object geometries can be taken from fig. 2.

### 4.3 Derivation of the rocket equation

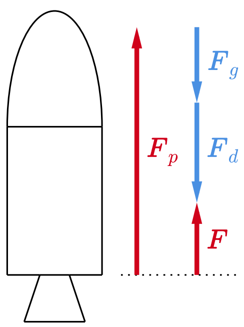



**Figure 6:** Sketch for the derivation of the general rocket equation considering drag force $\boldsymbol{F}_d$, gravitational force $\boldsymbol{F}_g$, propellant force $\boldsymbol{F}_p$ and a net force $\boldsymbol{F}$.

Consider a situation, where one has rocket of mass $m(t)$ ascending in positive vertical direction $\boldsymbol{e}_z$, where the speed of the rocket is denoted by $\boldsymbol{v}(t)$ and the propellant is ejected with velocity $\boldsymbol{v}_p(t)$ in negative vertical direction, that is $-\boldsymbol{e}_z$. This situation is shown in fig. 6, where $\boldsymbol{F}_p$ denotes the force generated by the propellant ejection, $\boldsymbol{F}_g$ is the force due to gravity, $\boldsymbol{F}_d$ is the drag force the rocket experiences and $\boldsymbol{F}$ is the net force acting upon the rocket. In general, all of these forces are time-dependent. First of all, the force due to the propellant ejection can be accounted for by

$$\boldsymbol{F}_p(t) = \boldsymbol{v}_p(t)\frac{\mathrm{d}m(t)}{\mathrm{d}t}\theta(\tau - t), \tag{52}$$

where $\theta : \mathbb{R} \to \{0, 1\}$ is the Heaviside function, $\tau$ is the total time of propellant ejection and $\boldsymbol{v}_p(t)$ is the in general time-dependent ejection velocity of the propellant, which is oriented towards the negative vertical axis; that is to say, against the movement direction of the rocket. Furthermore, the force due to gravity is given as

$$\boldsymbol{F}_g(t) = -m(t)g\boldsymbol{e}_z, \tag{53}$$

where $m(t)$ is the time-dependent mass of the rocket and $g = 9.81\,\mathrm{ms}^{-2}$ is the gravitational acceleration assumed as constant for all reached heights of the rocket. Lastly, the force due to air resistance (drag) is accounted for by the drag equation as

$$\boldsymbol{F}_d(t) = -\frac{1}{2}\rho(\boldsymbol{x})\boldsymbol{v}(t)^2\bar{C}_d A\boldsymbol{e}_z, \tag{54}$$

where $\rho(\boldsymbol{x})$ is the air density at $\boldsymbol{x}$, $\bar{C}_d$ is the drag coefficient of the rocket and $A$ is the cross-sectional area of the rocket with respect to its propagation direction.

From fig. 6 it can be derived, that the net force $\boldsymbol{F}(t)$ is given by the sum

$$\boldsymbol{F}(t) = \boldsymbol{F}_p(t) + \boldsymbol{F}_d(t) + \boldsymbol{F}_g(t). \tag{55}$$

According to Newton's second law, this net force can be written as

$$\boldsymbol{F}(t) = m(t)\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t}, \tag{56}$$

thus leading to the full rocket equation accounting for air resistance obtained by inserting the above expressions for each force into eq. (55), namely

$$m(t)\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}m(t)}{\mathrm{d}t}\boldsymbol{v}_p(t)\theta(\tau - t) - \frac{1}{2}\rho(\boldsymbol{x})\boldsymbol{v}(t)^2\bar{C}_d A\boldsymbol{e}_z - m(t)g\boldsymbol{e}_z. \tag{57}$$

### 4.4 Derivation of the Runge-Kutta scheme

The derivation of the fourth-order Runge-Kutta integration scheme is a lengthy task, which is in principle not complicated, but tedious; therefore only the key steps are outlined here.

The fourth-order Runge-Kutta scheme solves a differential equation of the form

$$\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = \boldsymbol{f}(t, \boldsymbol{y}) \tag{58}$$

numerically. The fourth-order refers to the number of vectors $\boldsymbol{k}_i$, $i \in \mathbb{N}$ used in order to propagate the solution $\boldsymbol{y}(t)$ from a state $t$ to a state $t + \Delta t$ with $\Delta t \in \mathbb{R}$; in the fourth-order case four vectors $\boldsymbol{k}_i$, $i \in \{1, \ldots, 4\}$ are thus needed. The fourth-order Runge-Kutta scheme is given by

$$\begin{aligned}
\boldsymbol{k}_1 &= \boldsymbol{f}(t_k, \boldsymbol{y}_k) \\
\boldsymbol{k}_2 &= \boldsymbol{f}(t_k + {}^1\!/\!_2\Delta t, \boldsymbol{y}_k + {}^1\!/\!_2\boldsymbol{k}_1) \\
\boldsymbol{k}_3 &= \boldsymbol{f}(t_k + {}^1\!/\!_2\Delta t, \boldsymbol{y}_k + {}^1\!/\!_2\boldsymbol{k}_2) \\
\boldsymbol{k}_4 &= \boldsymbol{f}(t_k + \Delta t, \boldsymbol{y}_k + \boldsymbol{k}_3) \\
\boldsymbol{y}_{k+1} = \boldsymbol{y}(t_{k+1}) &= \boldsymbol{y}_k + \frac{\Delta t}{6}\left(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4\right),
\end{aligned} \tag{59}$$

where $\Delta t \in \mathbb{R}$ is the step-size parameter. In fig. 7, a sketch of the functionality for the Runge-Kutta scheme can be seen.

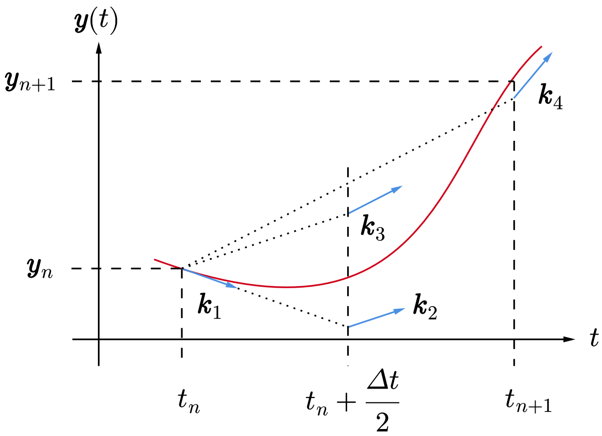



**Figure 7:** Sketch of the functionality for the fourth-order Runge-Kutta scheme. The shown coordinate system is to be understood as simplified representation of the $n$-dimensional vector $\boldsymbol{y}(t)$ in a two-dimensional coordinate system.

In order to derive this result, the ansatz

$$\begin{aligned}
\boldsymbol{k}_1 &= \boldsymbol{f}(t_k, \boldsymbol{y}_k) \\
\boldsymbol{k}_2 &= \boldsymbol{f}(t_k + \alpha_{2,1}\Delta t, \boldsymbol{y}_k + \alpha_{2,1}\boldsymbol{k}_1) \\
\boldsymbol{k}_3 &= \boldsymbol{f}(t_k + \alpha_{3,1}\Delta t + \alpha_{3,2}\Delta t, \boldsymbol{y}_k + \alpha_{3,1}\boldsymbol{k}_1 + \alpha_{3,2}\boldsymbol{k}_2) \\
\boldsymbol{k}_4 &= \boldsymbol{f}(t_k + \alpha_{4,1}\Delta t + \alpha_{4,2}\Delta t + \alpha_{4,3}\Delta t, \boldsymbol{y}_k + \alpha_{4,1}\boldsymbol{k}_1 + \alpha_{4,2}\boldsymbol{k}_2 + \alpha_{4,3}\boldsymbol{k}_3) \\
\boldsymbol{y}_{k+1} = \boldsymbol{y}(t_{k+1}) &= \boldsymbol{y}_k + \Delta t\left(\beta_1\boldsymbol{k}_1 + \beta_2\boldsymbol{k}_2 + \beta_3\boldsymbol{k}_3 + \beta_4\boldsymbol{k}_4\right),
\end{aligned} \tag{60}$$

is made inspired by visual considerations as seen in fig. 7. Next, optimal values for the parameters in the set

$$C \doteq \{\alpha_{2,i}, \alpha_{3,j}, \alpha_{4,k}, \beta_l\},\ i \in \{1\},\ j \in \{1,2\},\ k \in \{1,2,3\},\ l \in \{1,2,3,4\} \tag{61}$$

are desired to be found; to this end, the following steps are carried out as outlined:

(1) Expand $\boldsymbol{y}_{k+1}$ with $t_{k+1} = t_k + \Delta t$ into a third-order Taylor expansion around $t_k$, neglecting terms of order $\mathcal{O}(\Delta t^5)$.

(2) Expand expressions for $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ around $\Delta t = 0$ up to fourth order, neglecting terms of order $\mathcal{O}(\Delta t^5)$.

(3) Calculate $\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + \Delta t(\beta_1 \boldsymbol{k}_1 + \beta_2 \boldsymbol{k}_2 + \beta_3 \boldsymbol{k}_3 + \beta_4 \boldsymbol{k}_4)$ with the expanded k-vectors.

(4) Force this expression to resemble the Taylor expansion from step 1 by equaling expressions of same order in $\Delta t$.

(5) Get conditional equations for coefficient set $C$.

Using this procedure, one ends up with the fourth-order Runge-Kutta scheme as stated above.