Tutorial 1 Solutions

COMP2048

March 29, 2020

Problem 1

Recall that a rational number is a number we can write in the form $\frac{a}{b}$, where $a, b \in \mathbf{Z}$. We wish to proceed by contradiction, so we assume that

$$\sqrt{2} = \frac{a}{b}$$

Where $a, b \in \mathbf{Z}$ and $\gcd(a, b) = 1$ (ie fraction is in lowest terms). Now we can rearrange to get:

$$\begin{aligned} \sqrt{2} &= \frac{a}{b} \\ \implies 2 &= \frac{a^2}{b^2} \\ \implies 2b^2 &= a^2 \end{aligned}$$

So a^2 is even, which implies a is even, so we can write $a = 2k$ for some integer k. So:

$$\begin{aligned} 2b^2 &= (2k)^2 \\ 2b^2 &= 4k^2 \\ \implies b^2 &= 2k^2 \end{aligned}$$

So b^2 is even, which implies b is even. As a and b are both even, they have a common factor of 2 which contradicts our initial assumption. Therefore, $\sqrt{2}$ cannot be a rational number.

Problem 2

As with problem 1, we proceed by contradiction. From your discrete math course, you may recall that we define two sets as having the same size (or *cardinality*) if there exists some way to create a bijective mapping between them. In this case, we're looking at the natural numbers $\mathbf{N}$ and the powerset of the natural numbers $\mathcal{P}(\mathbf{N})$ (i.e all the subsets of the natural numbers). We assume that some bijective mapping between these two sets exists, so:

$$\begin{gathered} 0 \rightarrow S_0 \\ 1 \rightarrow S_1 \\ 2 \rightarrow S_2 \\ \ldots \end{gathered}$$

This mapping maps every natural number to a subset of the natural numbers, and every subset has a number which maps to it. Now consider the following representation of these subsets, where a 1 bit denotes that the natural number is included in the subset, and a 0 bit denotes that it is excluded (Note that we are assuming the rows of the table continue infinitely, and that every subset of the naturals exists in the table thanks to our bijective mapping).

	0	1	2	3	4	5	6	7	8 ⋯
S_0	$\boxed{1}$	1	1	1	1	1	1	1	1 ⋯
S_1	1	$\boxed{0}$	1	0	1	0	1	0	1 ⋯
S_2	0	1	$\boxed{0}$	1	0	1	0	1	0 ⋯
S_3	0	0	1	$\boxed{1}$	0	1	0	1	0 ⋯
⋯									

Now select the diagonal bits and invert them to produce a new subset:

	0	1	2	3 ⋯
S_A	0	1	1	0 ⋯

This subset cannot exist in our original table, as it is different to every row in at least one spot (as the subset is constructed from inverting diagonal bits). This implies that our mapping does not have a number which maps to this subset, a contradiction, so no such mapping exists. Therefore, the powerset of the naturals is larger than the naturals themselves.

Problem 3

a) Easily solved by the 'borrow binge' strategy, where we simply have nodes with negative values repeatedly perform the borrow action until they are no longer negative.

b) Some easy heuristics:

1. If the total number of dollars is negative, the game is unsolvable.
2. If we have some disconnected component with a negative total number of dollars, the game is unsolvable.

And a much less easy theorem (see this article):
Let $g = \#edges - \#vertices + 1$ and N be the total number of dollars in the graph, then if $N \geq g$ the game is always winnable.

Problem 4

Assume we have two integer sequences $A = \{a, a^2, a^3, \cdots\}$, $B = \{b, b^2, b^3, \cdots\}$ such that $a \neq b$ and $a^k = b$ for some k (ie the sets have the same elements, but with a different cyclic order). First note that from Fermat's little theorem we have that every element has an inverse, ie there exists some a^{-1} such that $aa^{-1} = 1$, then:

$$\begin{aligned} a^k &= b \\ \implies (a^k)^{-1} &= b^{-1} \\ a^k a &= bb \text{ as we know the sets have the same elements with some cyclic shift} \\ \implies (a^k)^{-1} a^k a &= b^{-1} bb \\ \implies a &= b \end{aligned}$$

So in fact we have that $a = b$, so the integer sequence is unique.