The University of Queensland
AUSTRALIA
CREATE CHANGE

Pattern Analysis

Fourier Analysis

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V1.0

> “The methods of theoretical physics should be applicable to all those branches of thought in which the essential features are expressible with numbers.”
>
> Paul M. Dirac (link)
> (1902-1984)

Change of Basis

- Data/Signals as a Vector with some basis $\mathbf{e}_i$

$$x(t) = \sum_{i}^{N} a_i \mathbf{e}_i$$

- We can change the basis to a more convenient set $\boldsymbol{\alpha}_i$ by projecting the data to these new basis set

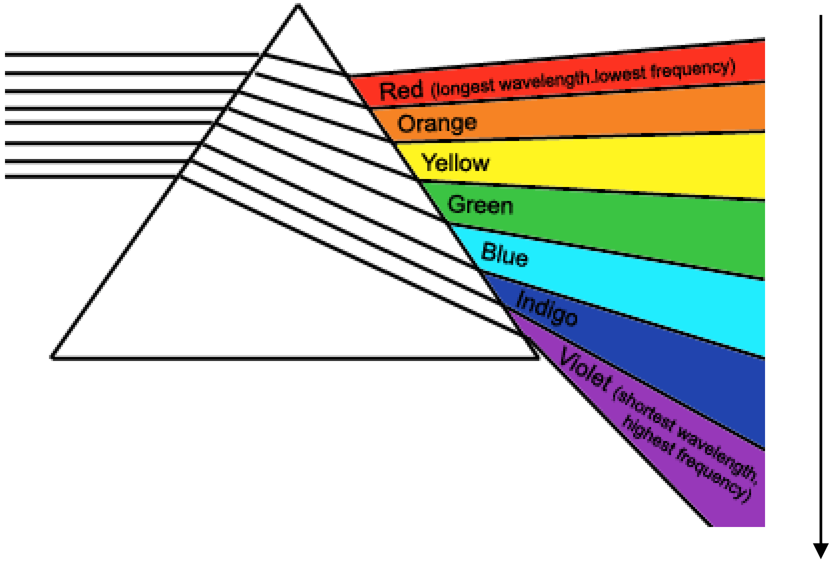
$$\hat{x}(t) = \sum_{i}^{N} a_i (\mathbf{e}_i \cdot \boldsymbol{\alpha}_i)$$

- These bases can be predefined, such as trigonometric functions resulting in the Fourier Transform. Or we can compute new ones based on some property of the data/signal, such as the variance with respect to the mean, resulting in Eigen-decomposition.

Fourier Transform

Refraction of Sunlight

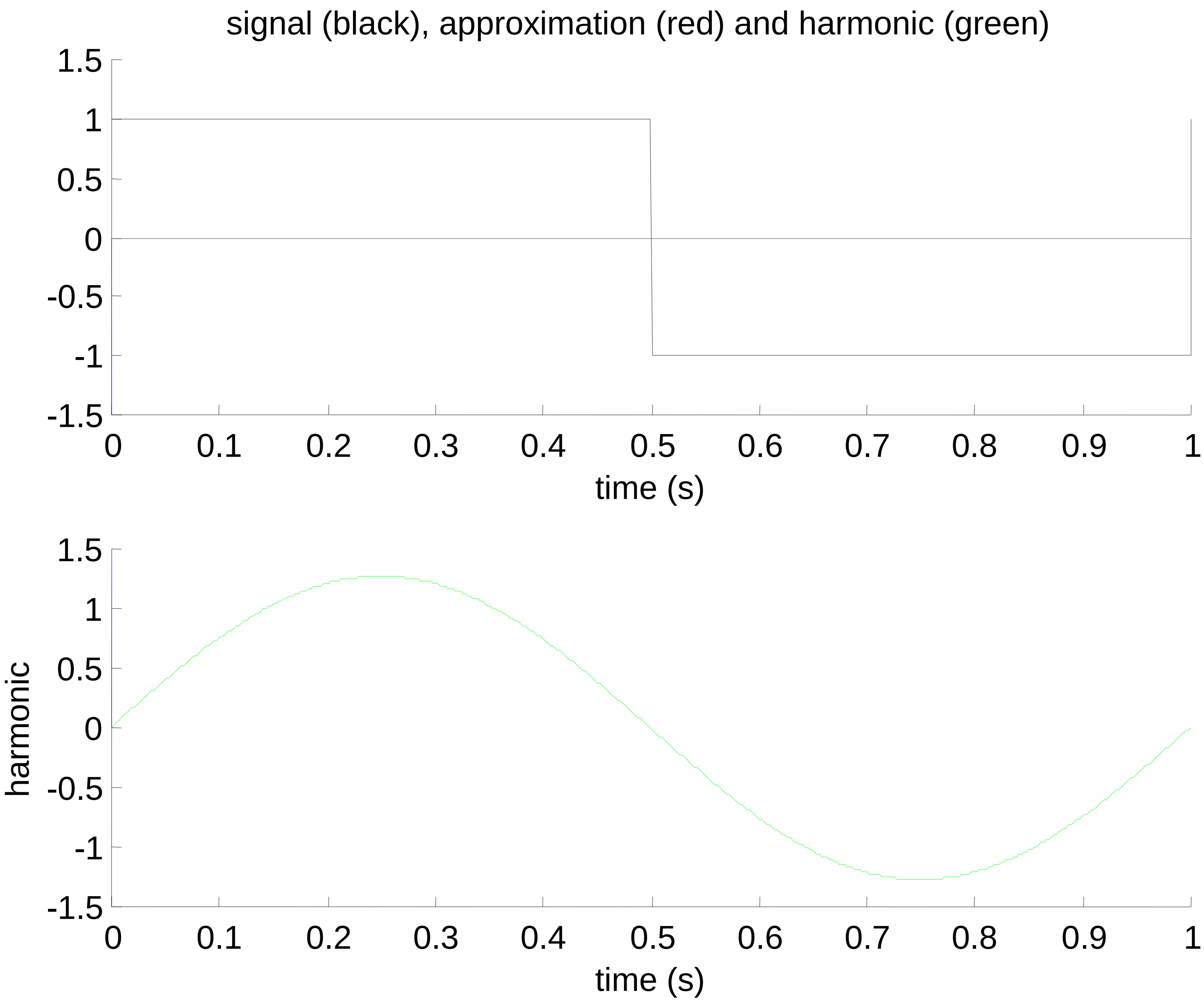
White light
(all wavelengths)



Colour spectrum

$f = 1/\lambda$

Think of a Fourier Transform like a prism:
"Destructs a source signal into its constituent frequencies"



signal (black), approximation (red) and harmonic (green)
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time (s)
harmonic
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
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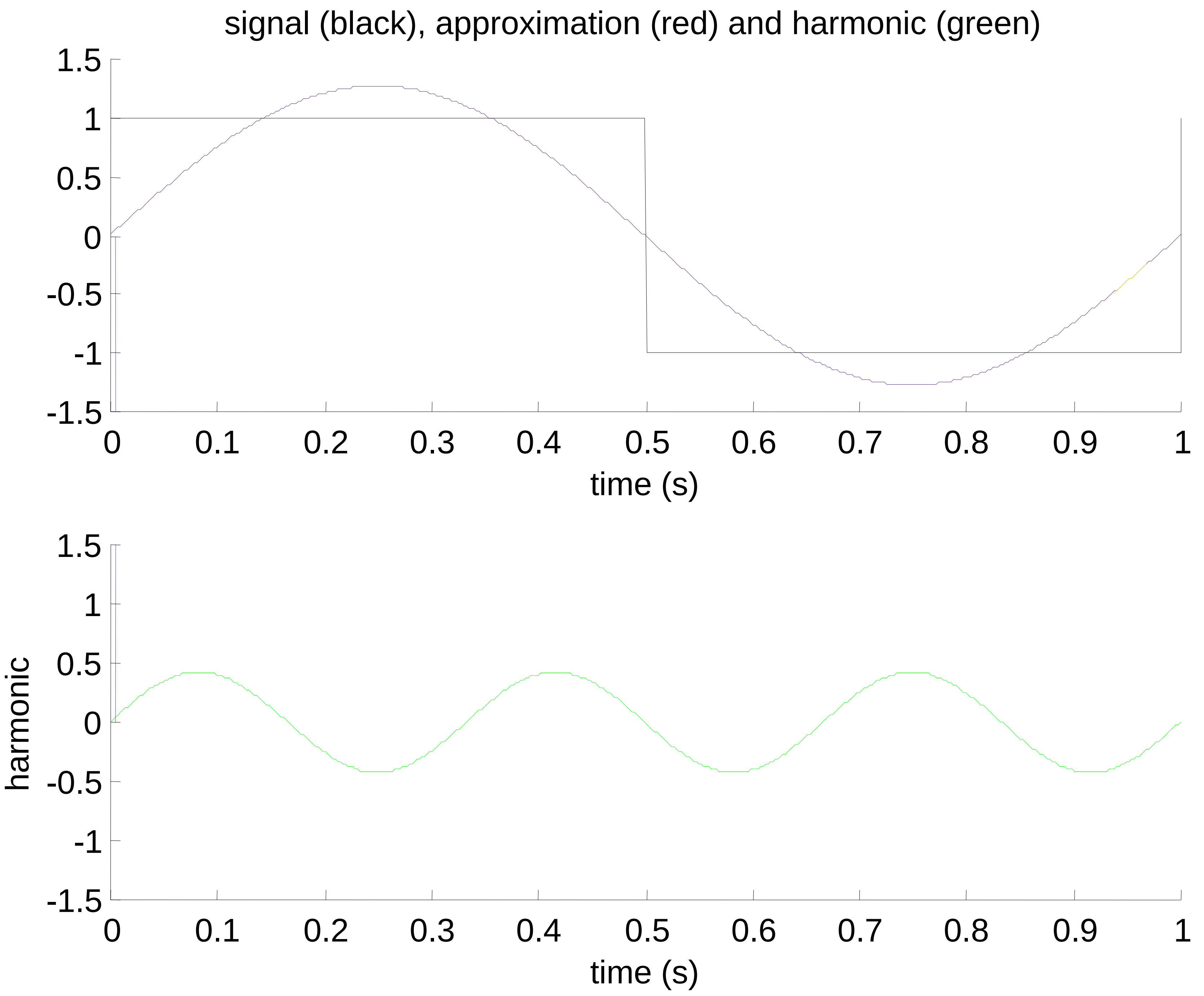
time (s)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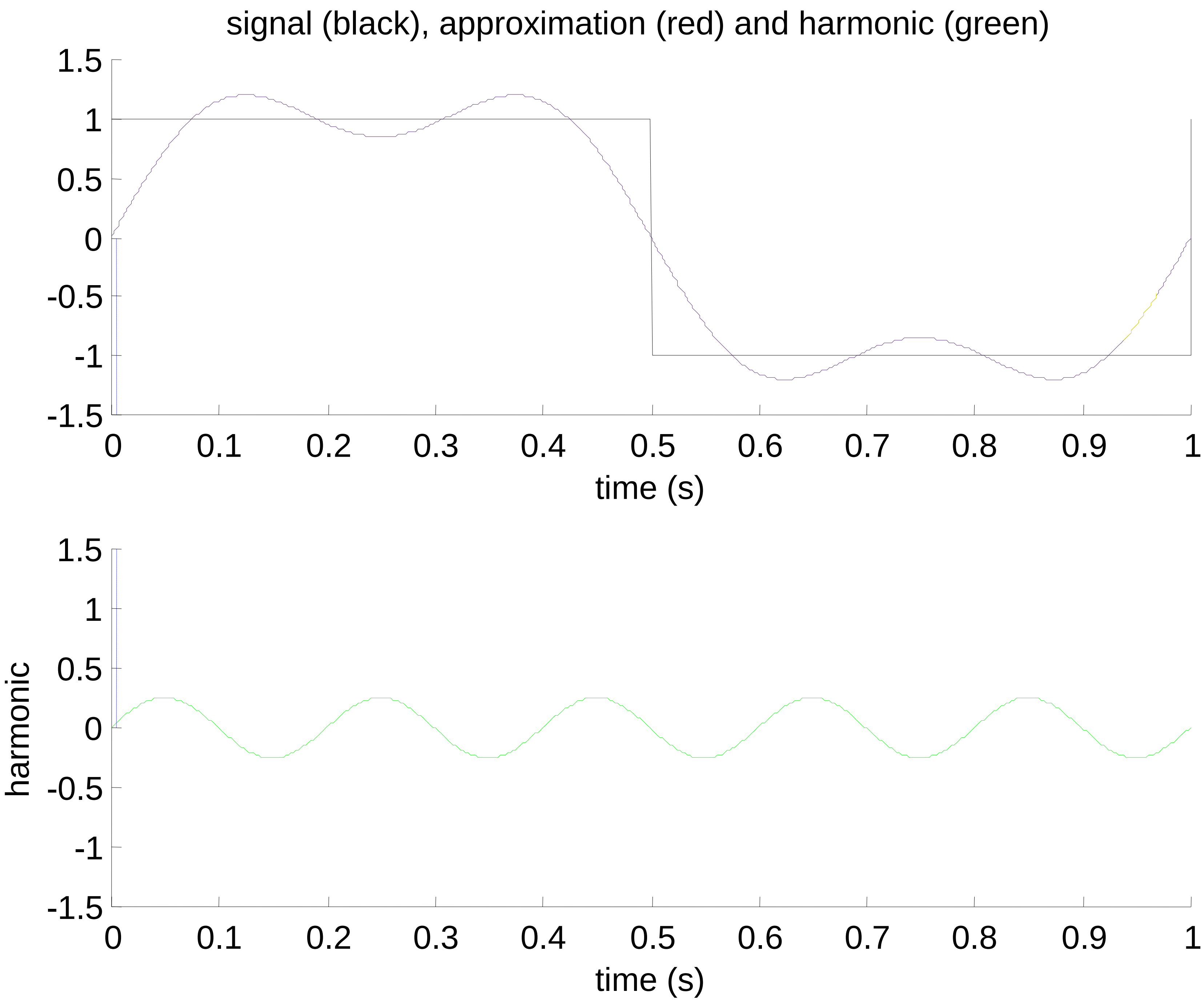
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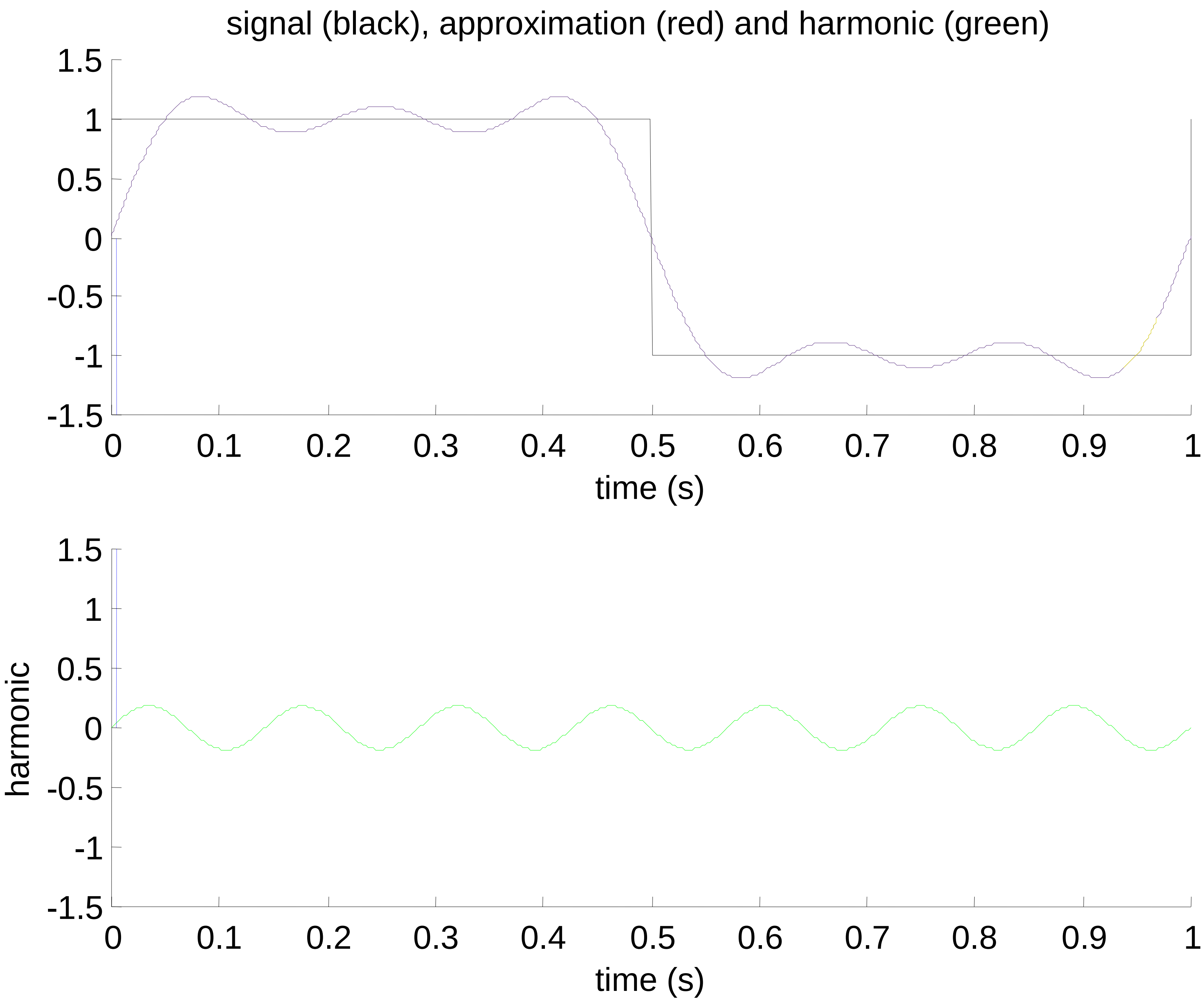
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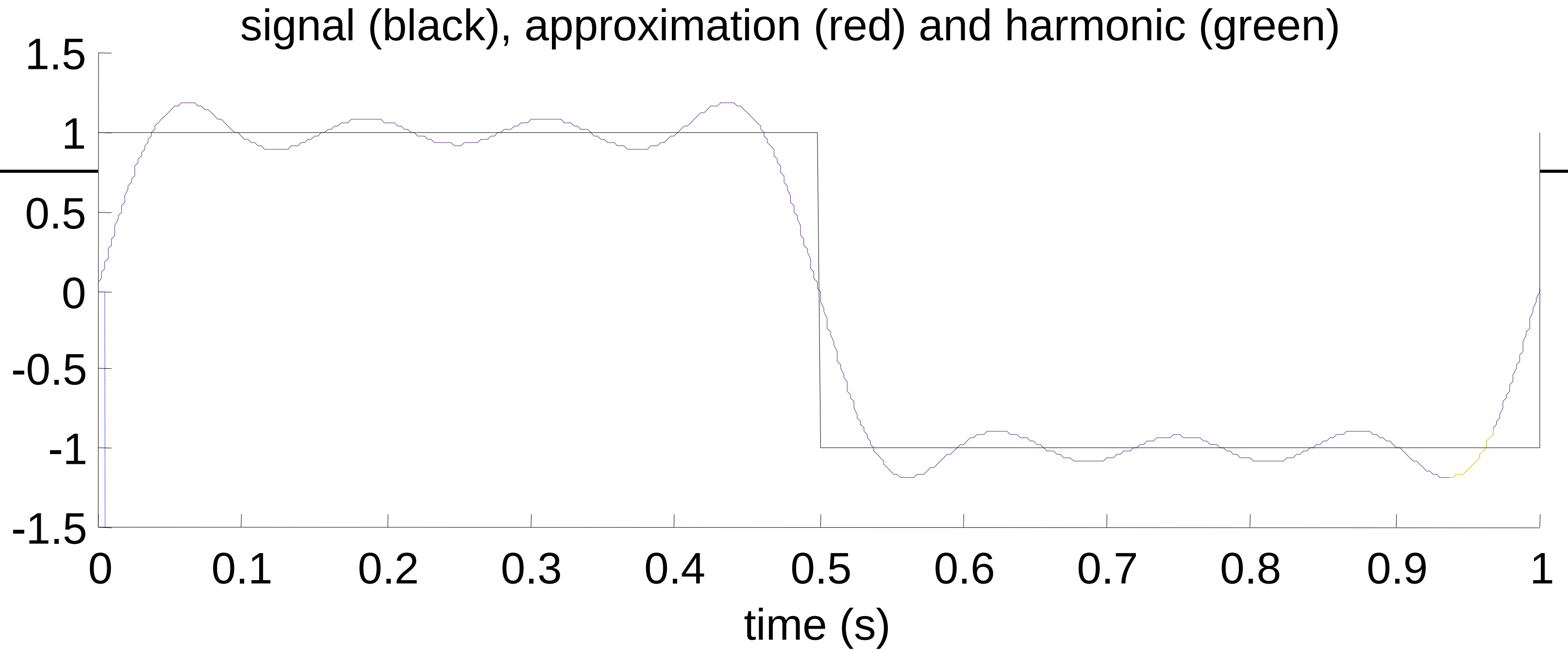
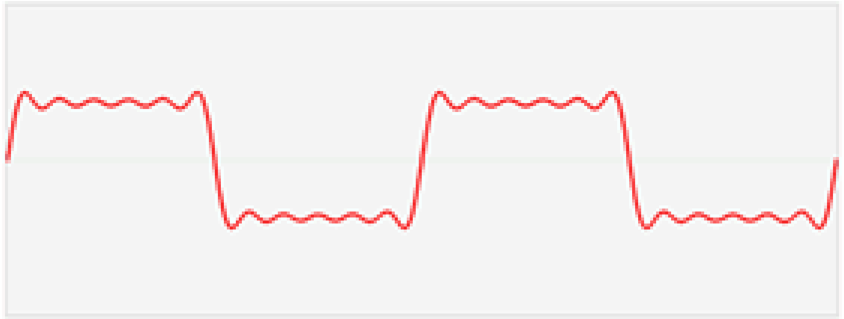


- Approximation with 1st, 3rd, 5th, & 7th Harmonics added, note:
  - ‘Ringing’ on edges due to series truncation
  - Often referred to as **Gibb’s phenomenon**
- Fourier series **converges** to original signal if
  - **Dirichlet conditions** satisfied
  - Closer approximation with more harmonics

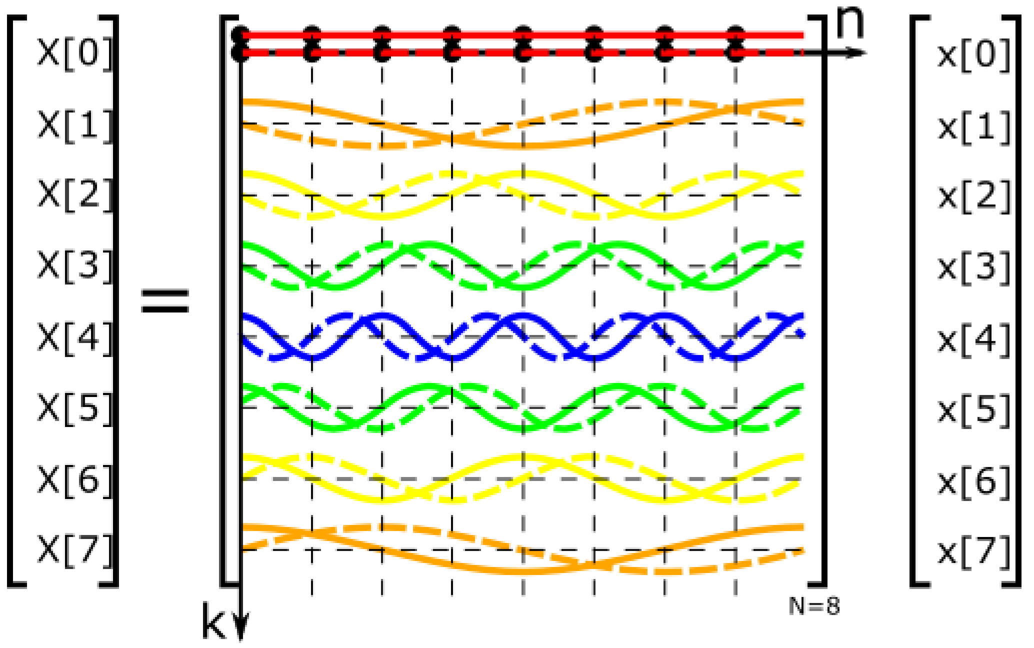
Fractals

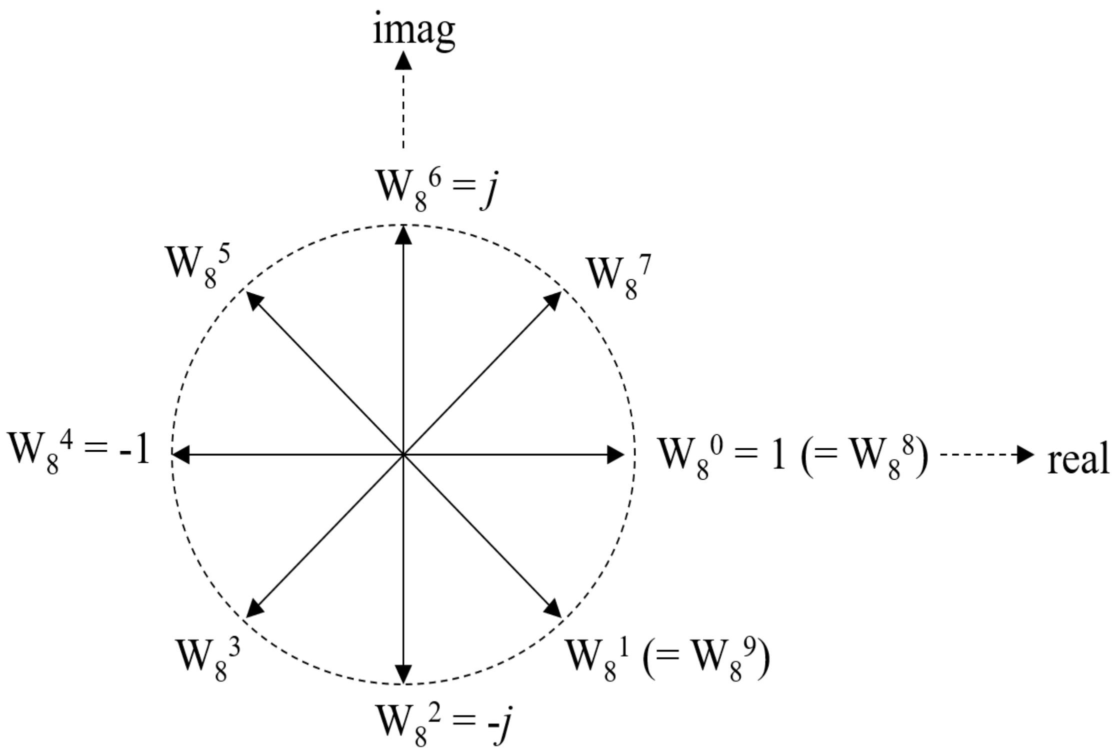


By: Andrés Cabrera and Karl Yerkes.
http://w2.mat.ucsb.edu/201A/nb/Sinusoids%20and%20Phasors.html

Discrete Fourier Transform

- Discrete Fourier Transform (DFT) is computed as the dot product of the signal with the finite sections of the unit circle, known as the Nth roots of unity
- It results is a harmonic representation of the signal as a linear combination of sines and cosines

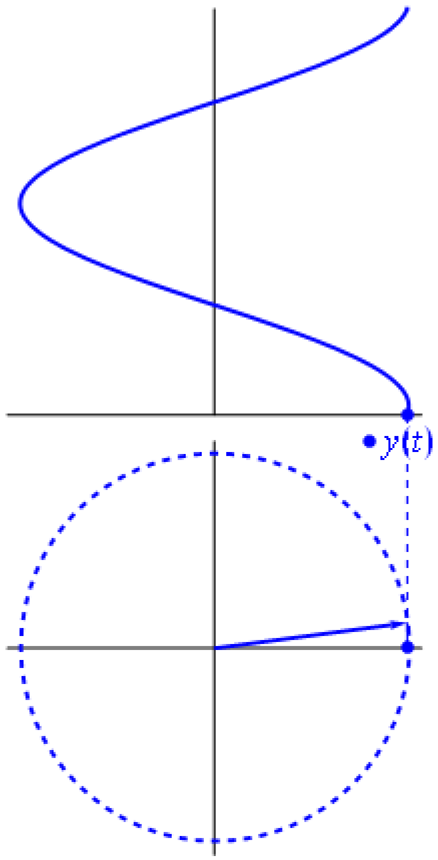




Discrete Fourier Transform

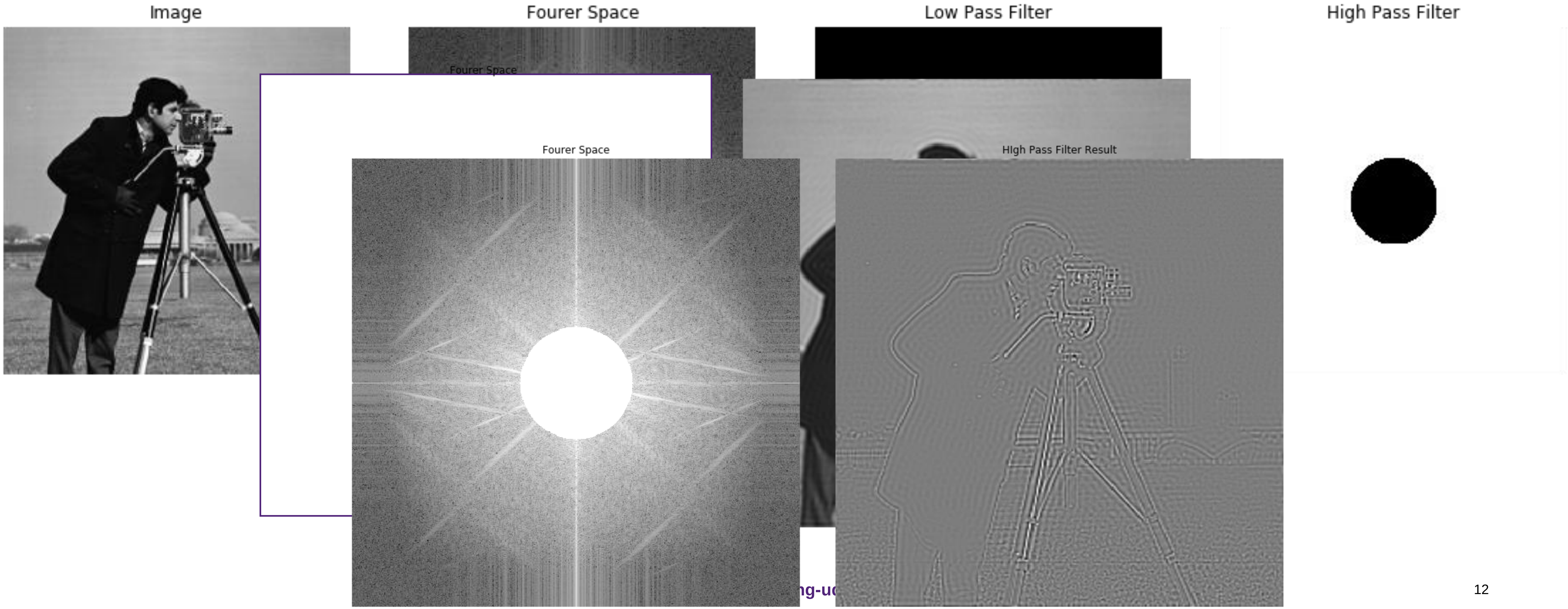
- The harmonics are a valid set for computing this representation because they form an ortho-normal basis
- This is because they are Nth roots of unity over the unit circle is $\alpha^{2\pi ik/N}$, where N is the length of the signal
- Complex values are required because unique values of the sine/cosine curves are determined only when both amplitude and phase are known
- The Discrete Fourier Transform is then

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-2\pi ink}/_{N}$$



Filtering

- Fourier space has many advantages, including filtering and convolution




Fourier
Visualized

Conclusion

- Fourier transform projects data onto the unit circle
- It maps harmonics to the data
- The resulting Fourier space is a one stop shop for filtering and preprocessing operations, such as smoothing, downsampling and edge detection
- Natural images are sparsely represented in the Fourier domain
- Fourier features are ideal for signals and data from physical systems

What's Next?

How can we model the variation in the data or patterns? Using Eigen-Decomposition…

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Thank you

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