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Pattern Analysis

Convolution

“All mathematics is is a language that is well tuned, finely honed, to describe patterns; be it patterns in a star, which has five points that are regularly arranged, be it patterns in numbers like 2, 4, 6, 8, 10 that follow very regular progression.”

Brian Greene (link)

(1963-)

Dr Shekhar “Shakes” Chandra

V1.0

Convolution

Given two continuous sequences in one dimension $f(t)$ and $g(t)$, the convolution $f * g$ is computed as

$$f(t) * g(t) \triangleq \underbrace{\int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau}_{(f*g)(t)}$$

Given two discrete data or signal sequences in one dimension f and g, the discrete convolution $f * g$ is computed as

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[n-m] g[m]$$

If we assume that the discrete sequences are periodic, the circular convolution can be computed as

$$(f * g_N)[n] = \sum_{m=0}^{N-1} f[m] g_N[n-m]$$

Convolution

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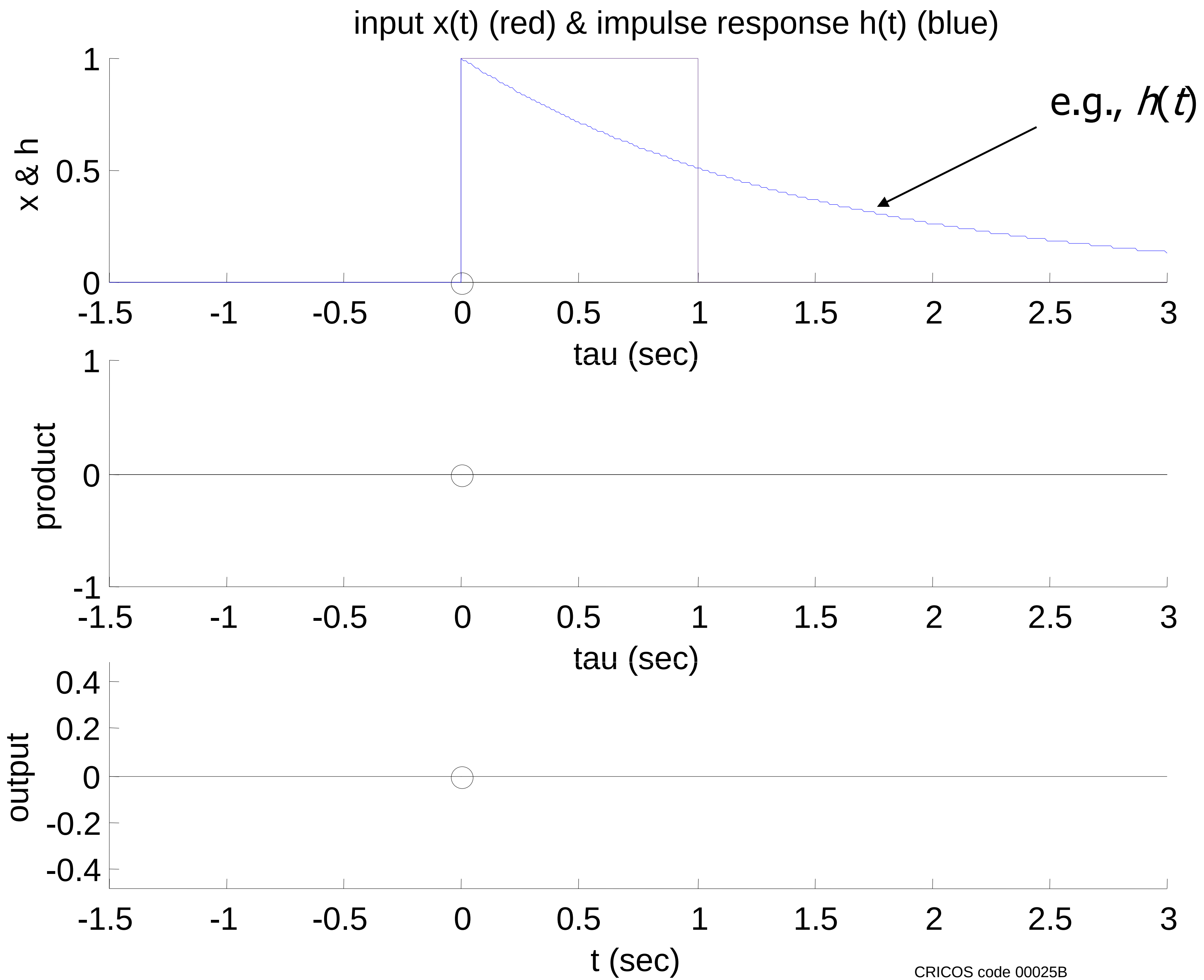
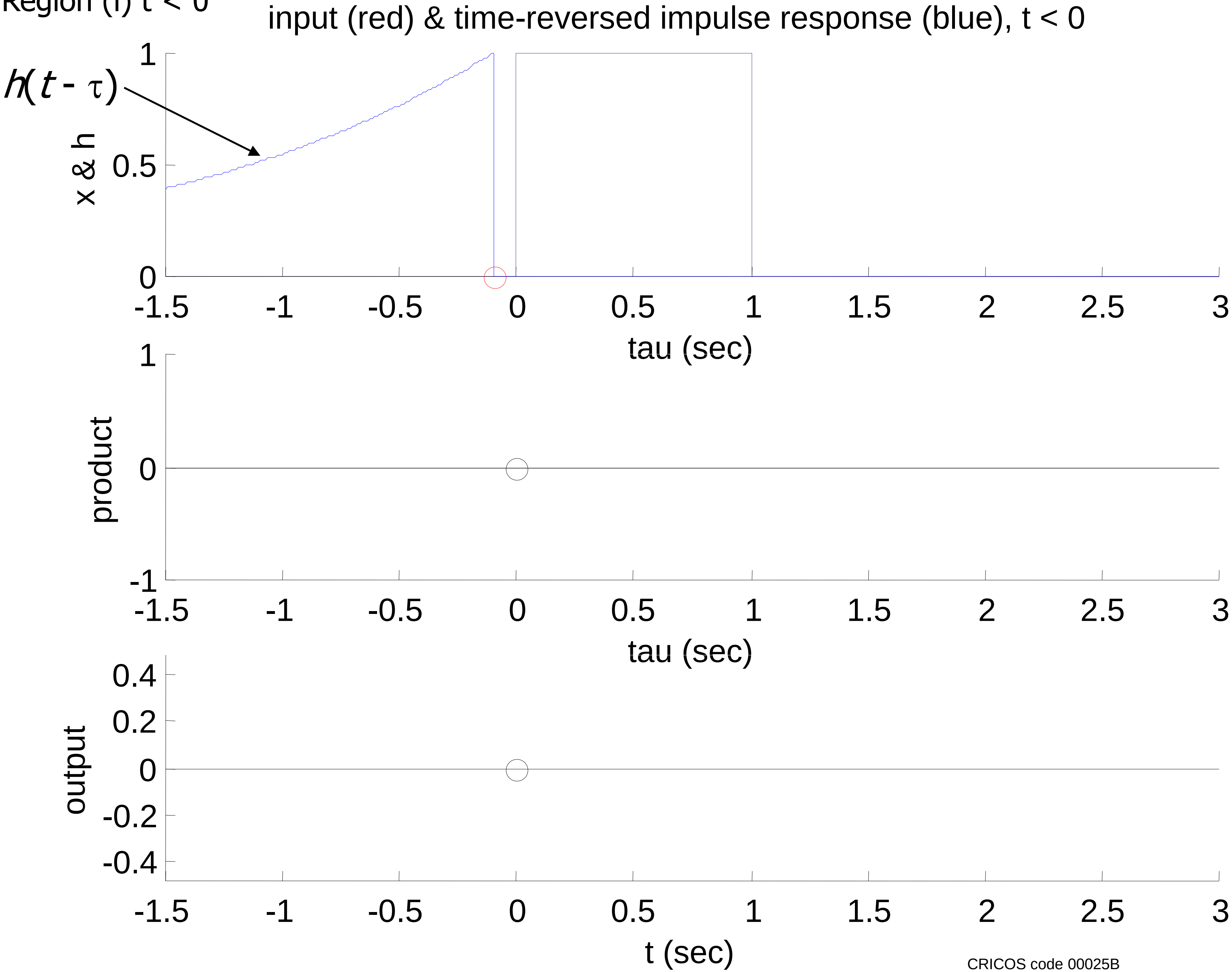
It's Just A Sliding Window Dot Product!

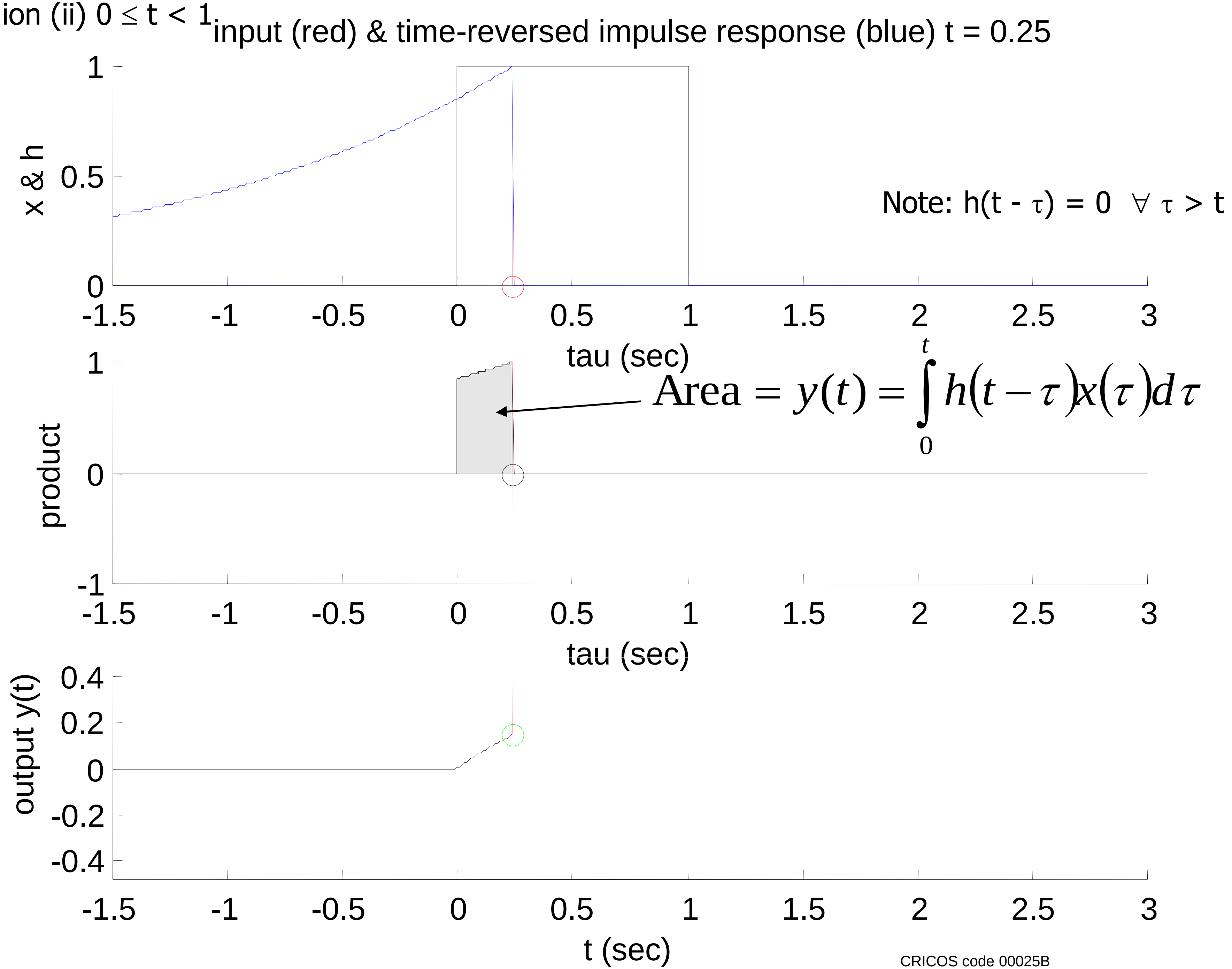

input x(t) (red) & impulse response h(t) (blue)
e.g., h(t)
x & h
1
0.5
0
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
tau (sec)
product
1
0
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
tau (sec)
output
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
t (sec)

Region (i) t < 0

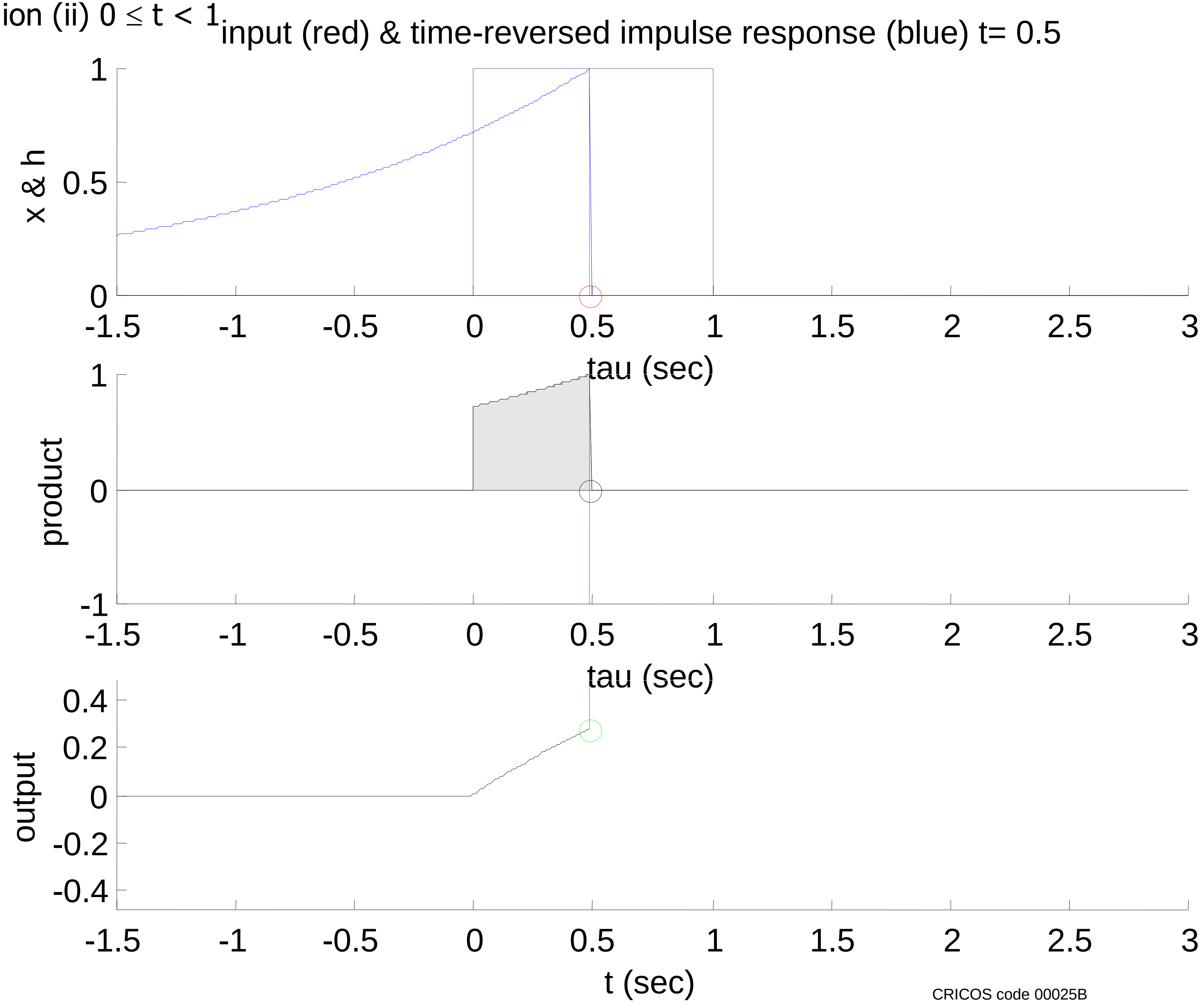


input (red) & time-reversed impulse response (blue), t < 0
h(t - τ)
x & h
1
0.5
0
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
tau (sec)
product
1
0
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
tau (sec)
output
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
t (sec)

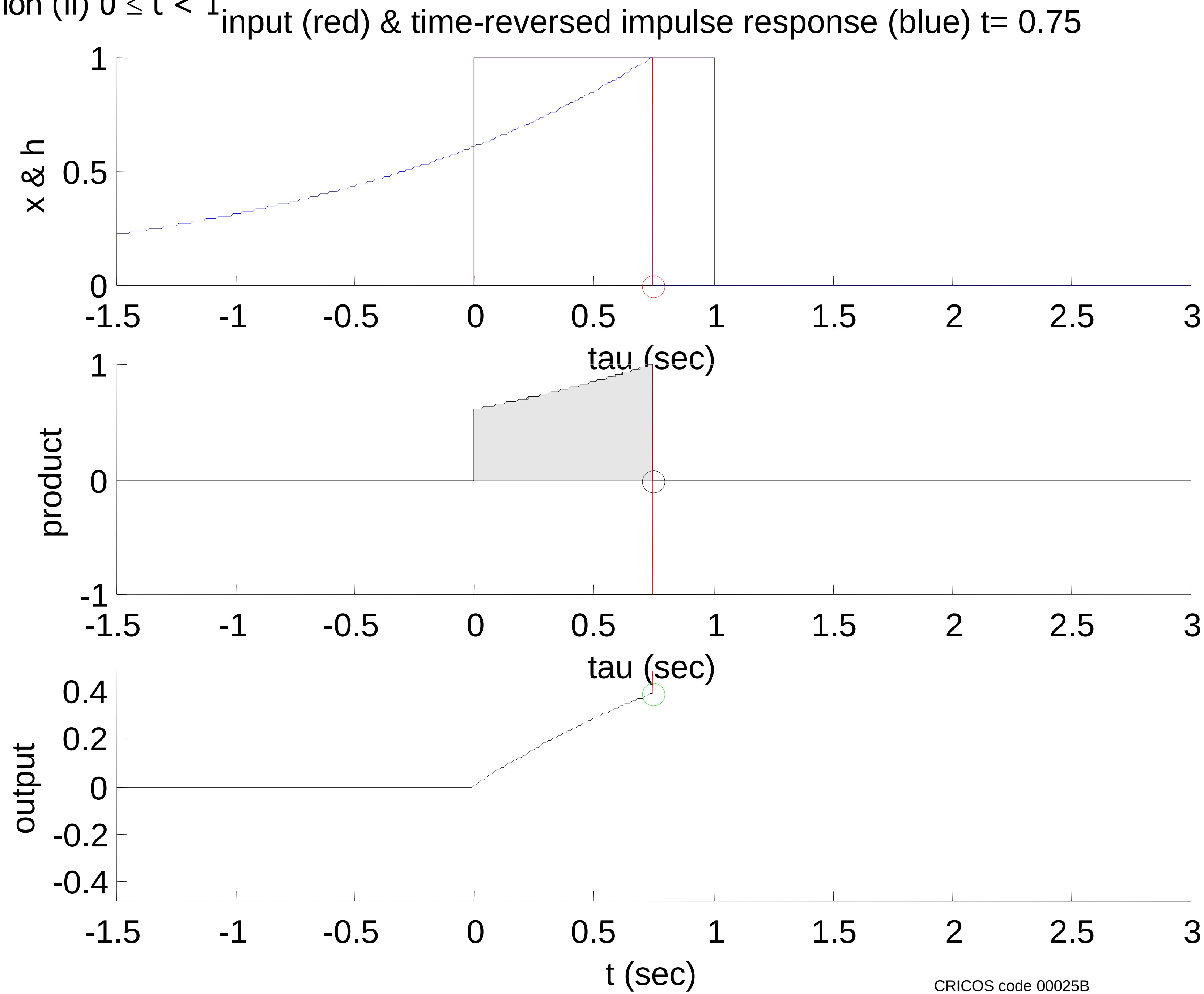
Region (ii) $0 \le t < 1$



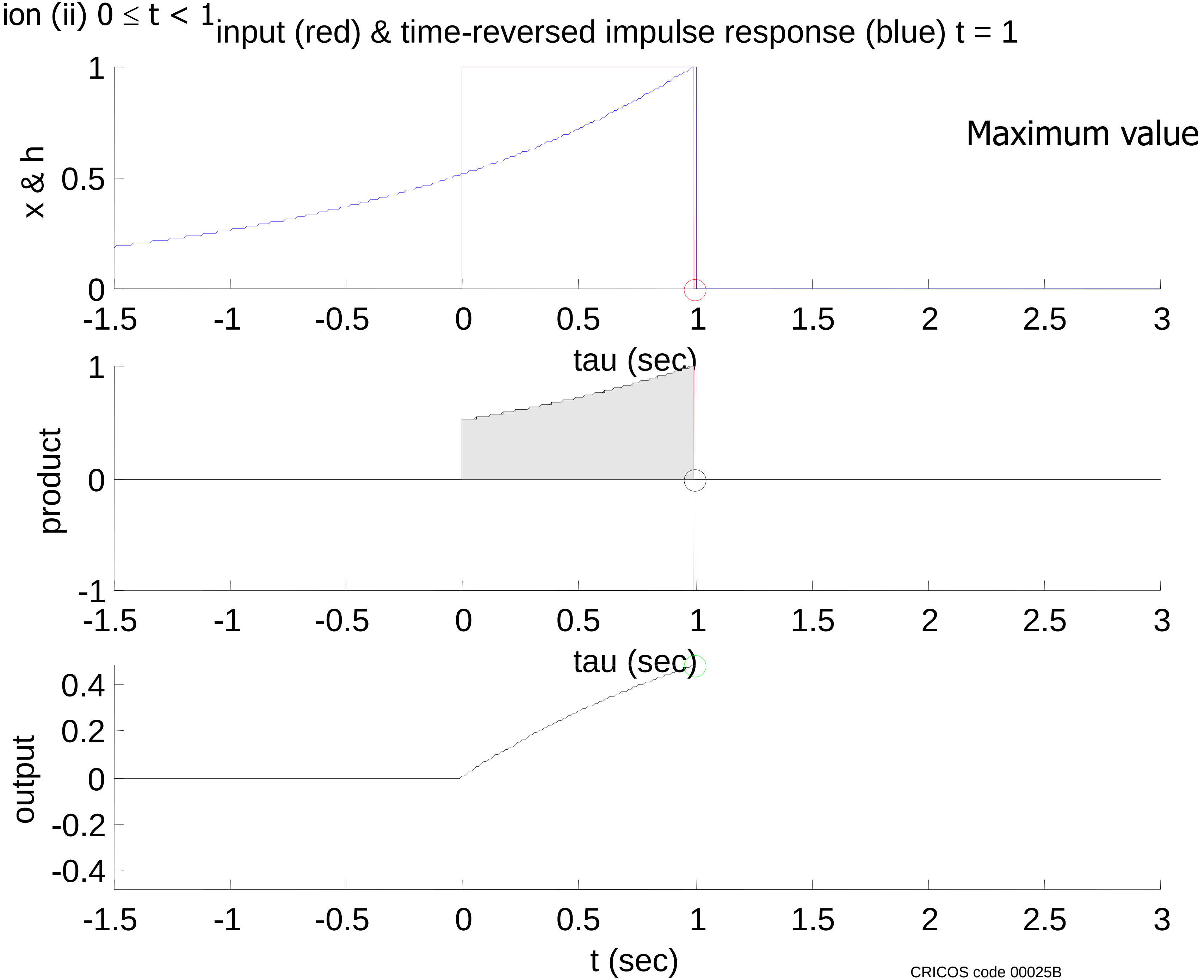
Region (ii) $0 \le t < 1$



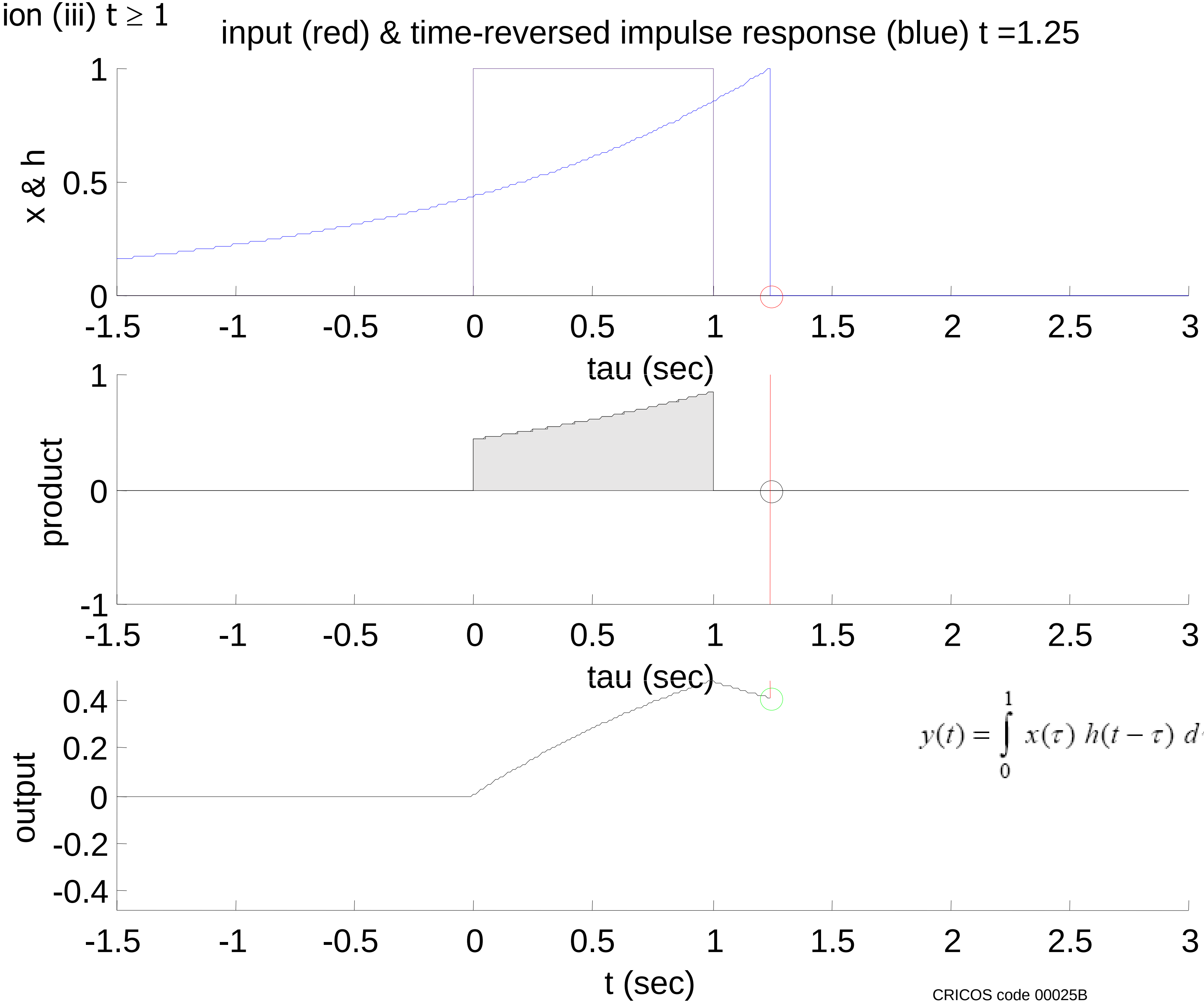
Region (ii) $0 \leq t < 1$



Region (ii) $0 \le t < 1$

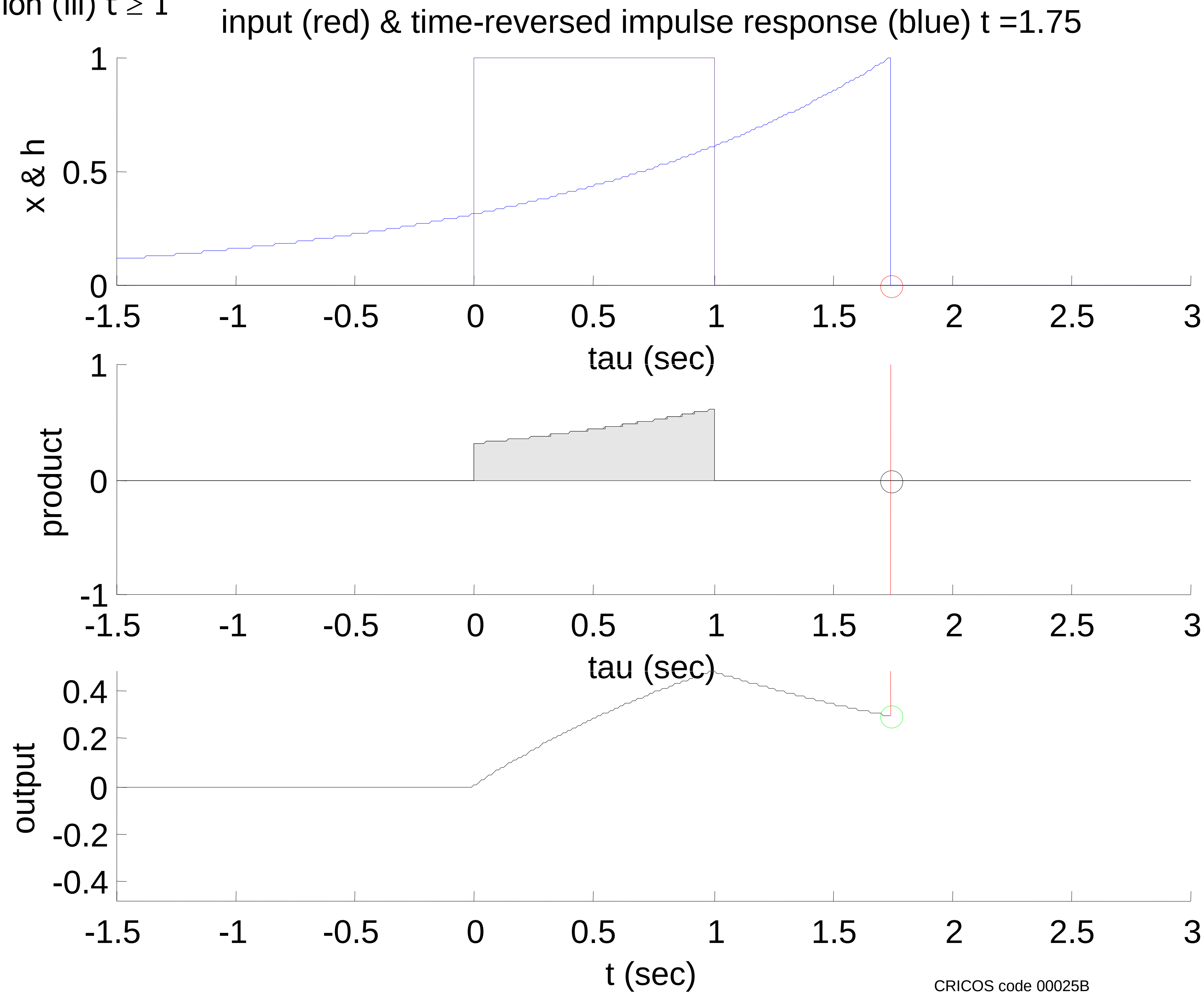


Region (iii) $t \geq 1$

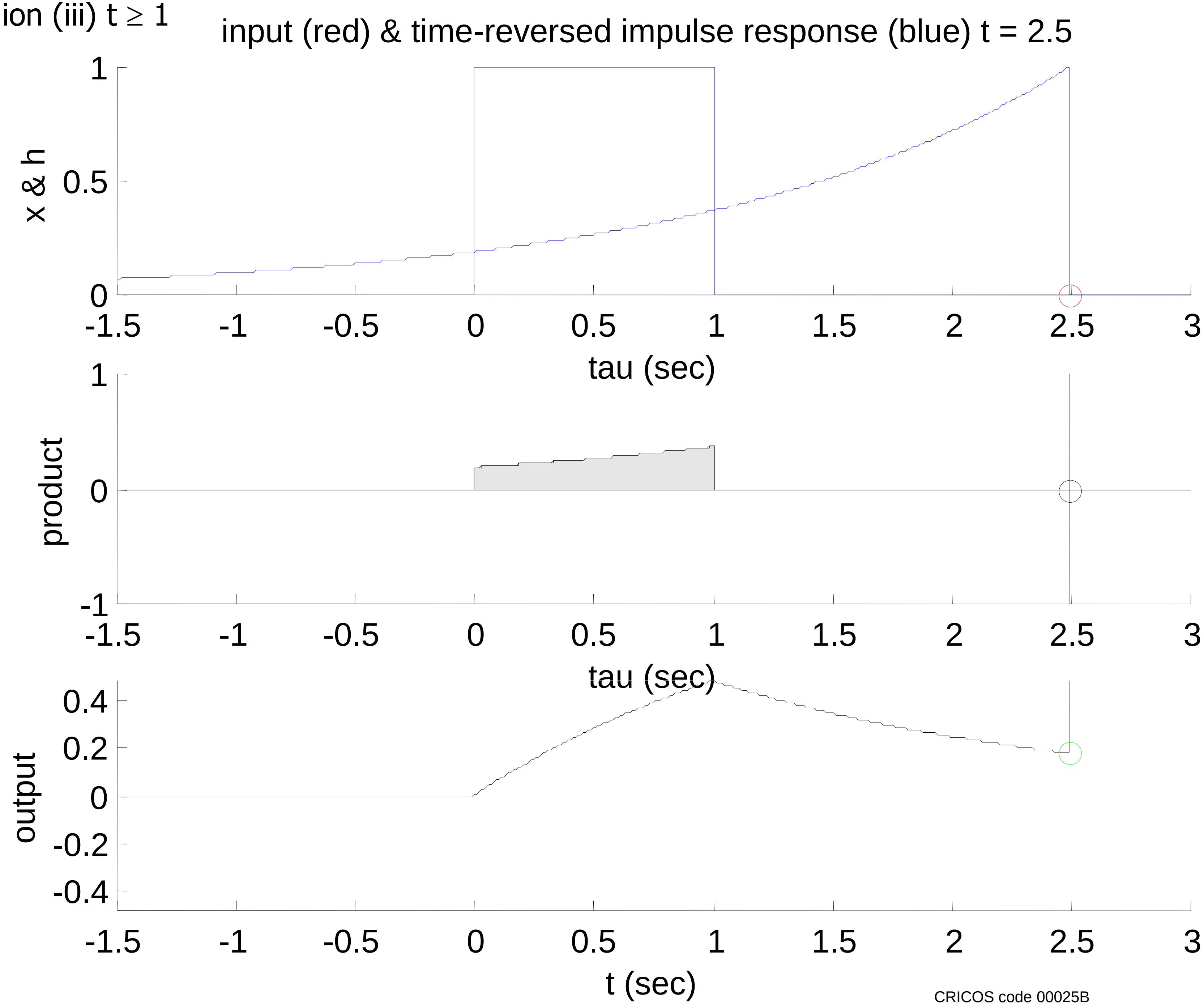


$$y(t) = \int_0^1 x(\tau)\ h(t-\tau)\ d\tau$$

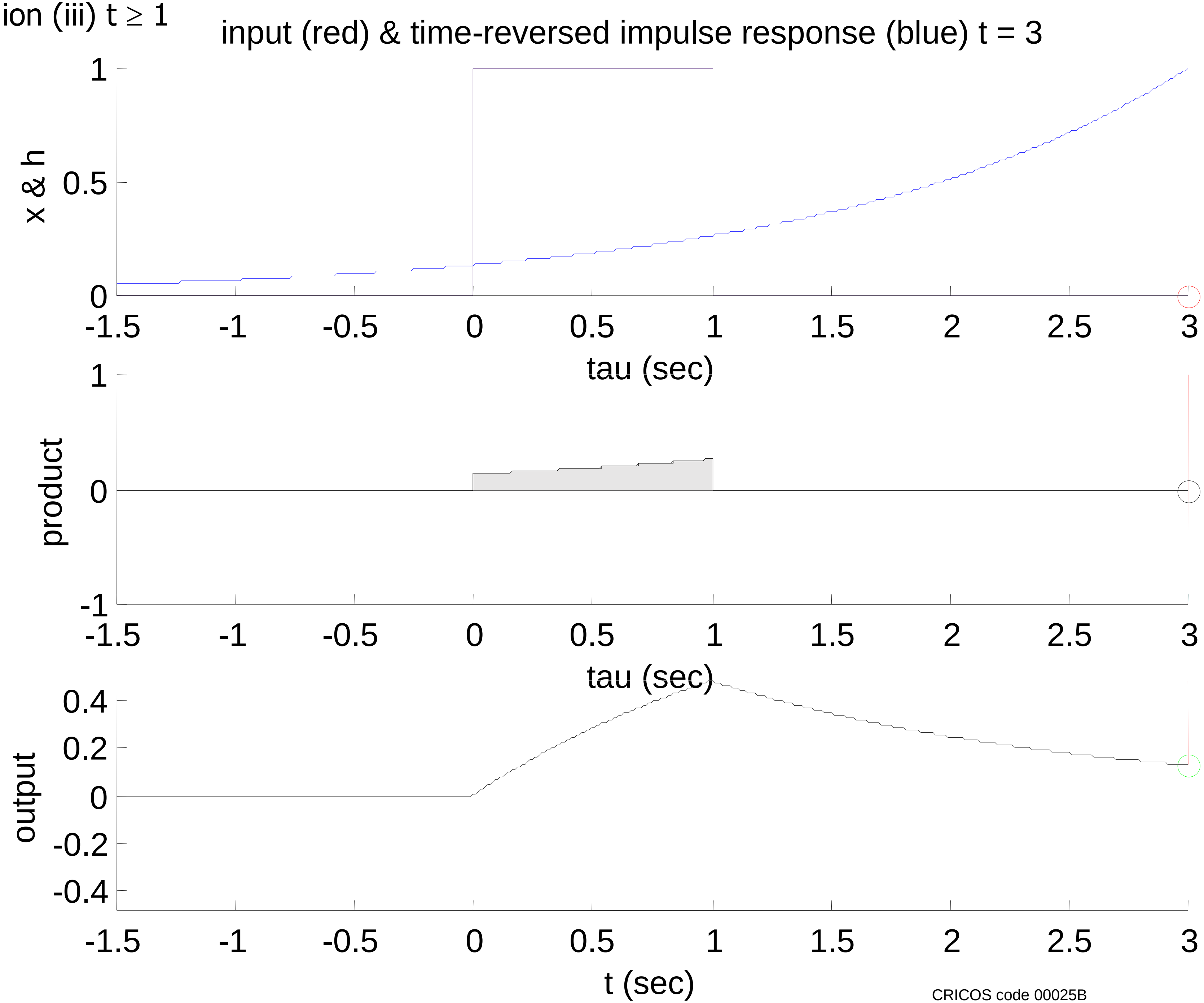
Region (iii) $t \geq 1$

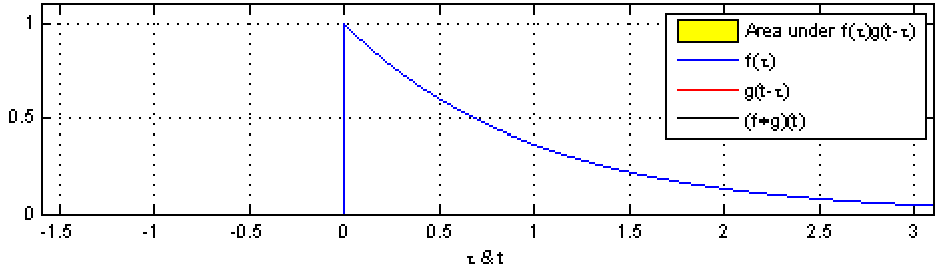


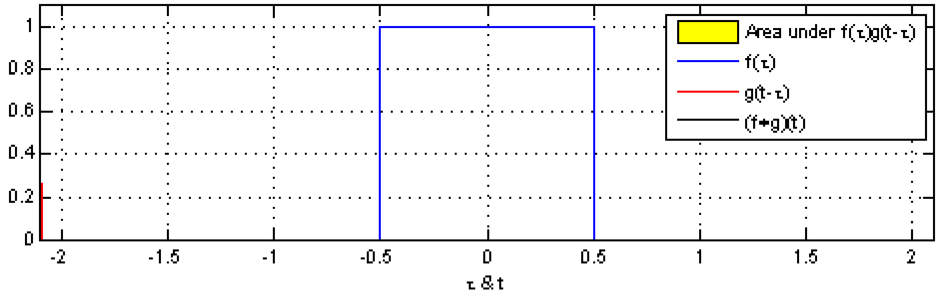
Region (iii) $t \geq 1$

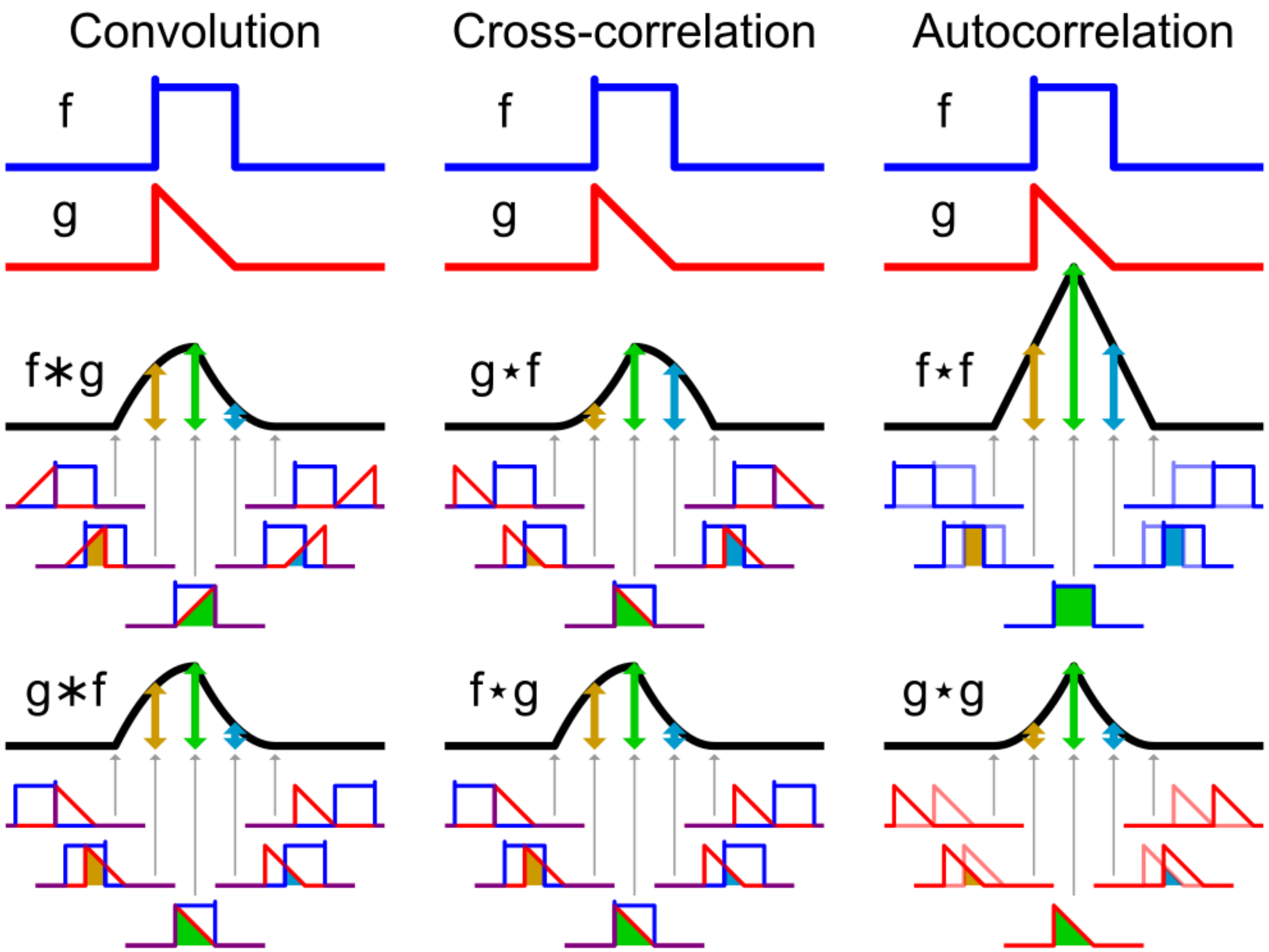


Region (iii) $t \geq 1$




Area under f(τ)g(t-τ)
f(τ)
g(t-τ)
(f∗g)(t)
τ & t


Area under f(τ)g(t-τ)
f(τ)
g(t-τ)
(f∗g)(t)
τ & t



Convolution
Cross-correlation
Autocorrelation
f
g
f∗g
g∗f
g⋆f
f⋆g
f⋆f
g⋆g

Discrete-time Convolution

$$y[n] = \sum_{m=0}^{\infty} h[n-m]x[m] = h[n] * x[n]$$

Example, $x[n]$ = {1, 2, 3} and $h[n]$ = {1, 0.5, 0.25}

1. Time reverse $h[n]$: $h[-m]$ = {0.25, 0.5, 1}

2. Apply $h[\mathrm{n}-m]$ to $x[m]$:

								y[n] = [
			1	2	3			
$h[-m]$ $n = 0$:	0.25	0.5	1					1
$h[1-m]$ $n = 1$:		0.25	0.5	1				2.5
$h[2-m]$ $n = 2$:			0.25	0.5	1			4.25
$h[3-m]$ $n = 3$:				0.25	0.5	1		2
$h[4-m]$ $n = 4$:					0.25	0.5	1	0.75
								]

Discrete-time Convolution

What is the length of y[n]?

- If x[n] is length N
  - And h[n] is length M
- Then y[n] will be length N+M-1

So, for the previous example

- N = M = 3 and so y[n] is length 3+3-1 = <u>5</u>

In MATLAB:

```
x = [1 2 3];
h = [1 0.5 0.25];
y = conv(x,h)   % Note: same as conv(h,x)
y =    1.0000    2.5000    4.2500    2.0000    0.7500
```

Matrix Formulation of Convolution

$$y = \mathbf{H}x$$

Where **H** is a Toeplitz Matrix

$$\begin{bmatrix} 0.75 \\ 2 \\ 4.25 \\ 2 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.5 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 & 0.25 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Concise maths, but not how you implement it!

Convolution Theorem

Convolution in the time-domain is equivalent to multiplication in the Fourier and Laplace domains

$$
\begin{aligned}
x_1(t) * x_2(t) &= L^{-1}\{X_1(s)X_2(s)\} \\
&= F^{-1}\{X_1(w)X_2(w)\}
\end{aligned}
$$

$$
\begin{aligned}
L\{x_1(t)x_2(t)\} &= \frac{1}{2\pi j} X_1(s) * X_2(s) \\
F\{x_1(t)x_2(t)\} &= \frac{1}{2\pi} X_1(w) * X_2(w)
\end{aligned}
$$

e.g., frequency modulation

Circular and Linear Convolution

Let's revisit our simple MATLAB example

Now let's do the convolution via multiplication in Frequency domain

Different answer!

- Wrong length ☹

```
x = [1 2 3];
h = [1 0.5 0.25];
y = conv(x,h)
y = 1  2.5  4.25   2   0.75
```

```
X = fft(x); % go to Freq. domain!
H = fft(h);
% Do element-wise (dot) multiply
Y = H.*X;
y = ifft(Y);
y = 3   3.25   4.25
```

This is 'circular' convolution

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Circular and Linear Convolution

Problem:

- x[n] is assumed periodic by the DFT
  - Period incorrect
    - Overlaps with h[n]
  - More on this later…

Solution:

- Zero pad to correct length (M+N-1)
- Period of x[n] correct
  - No overlap ☺
    - Linear convolution

1 2.5 4.25 2 0.75 ← linear

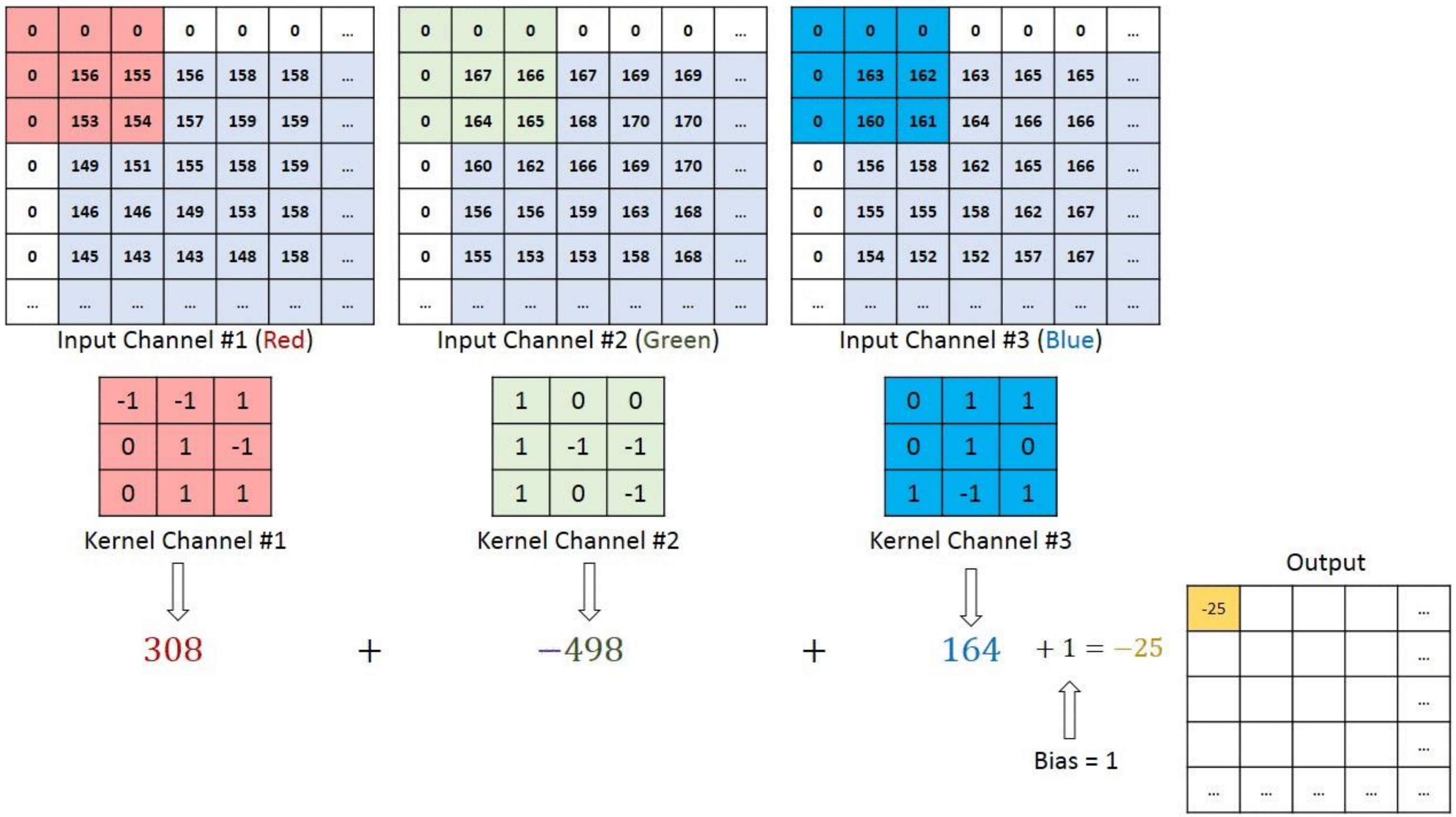
\+ \+

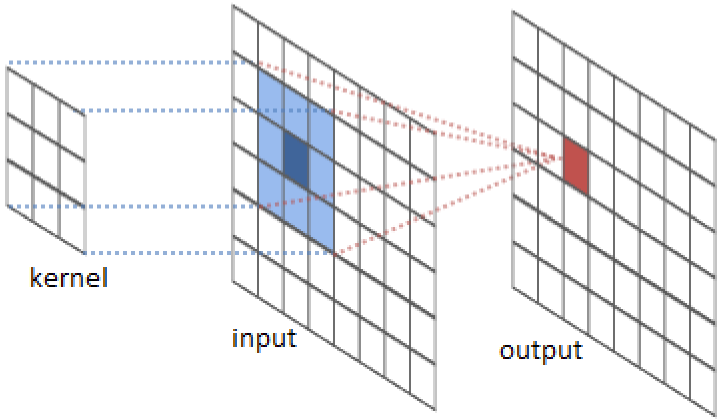
⇓

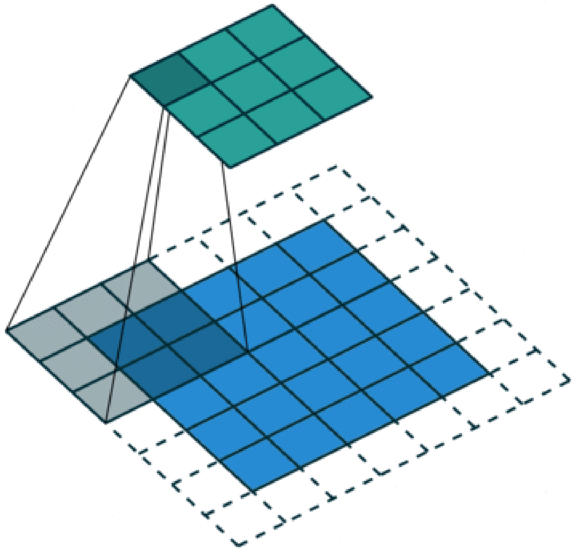
3 3.25 4.25 ← circ

```
x = [1 2 3 0 0];
h = [1 0.5 0.25 0 0];
X = fft(x);
H = fft(h);
Y = H.*X;
y = ifft(Y)
y = 1    2.5   4.25   2    0.75
```

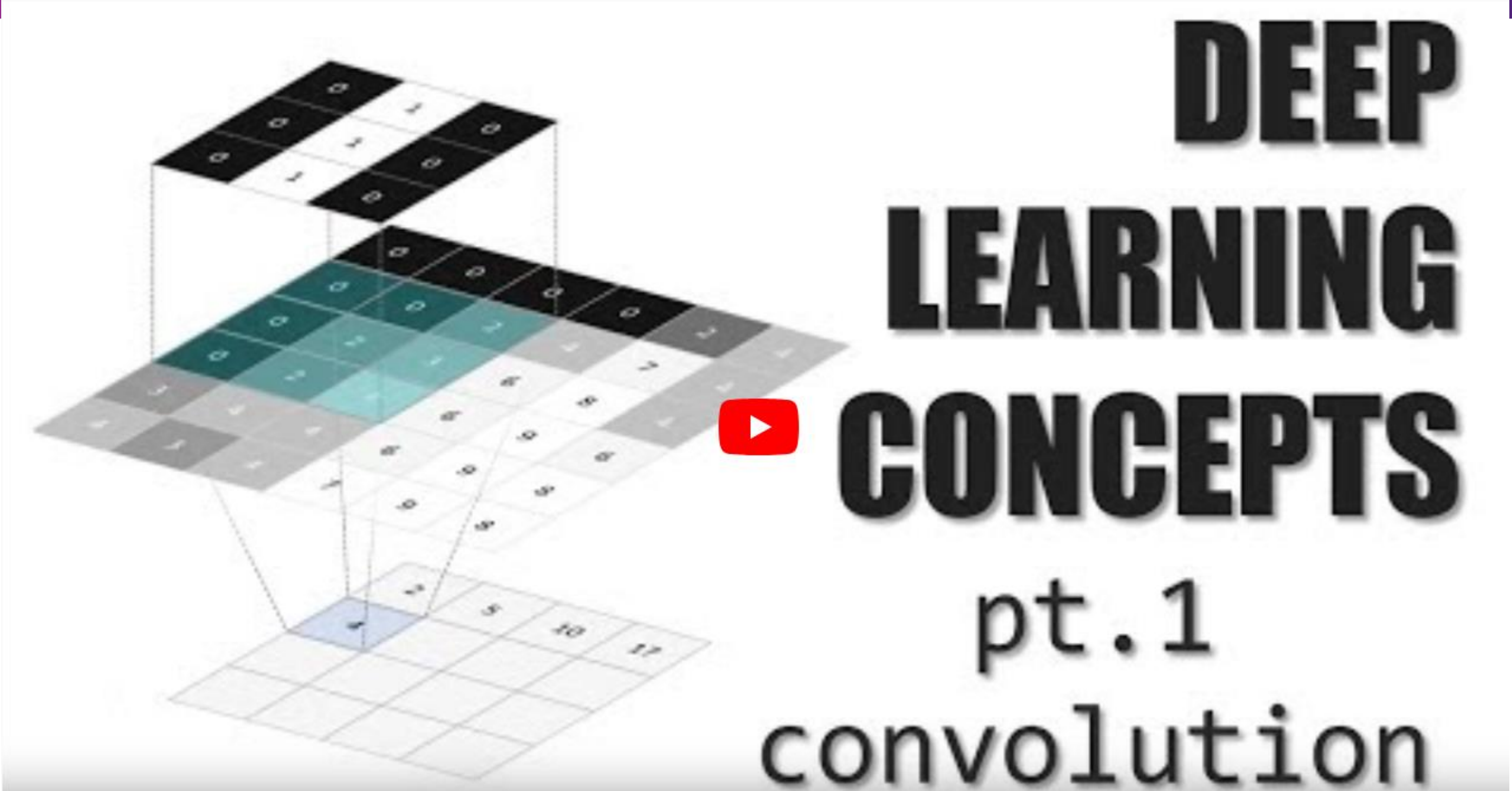
Image Convolution







Links
http://colah.github.io/posts/2014-07-Understanding-Convolutions/
https://towardsdatascience.com/a-comprehensive-guide-to-convolutional-neural-networks-the-eli5-way-3bd2b1164a53


DEEP
LEARNING
CONCEPTS
pt.1
convolution

Conclusion

- Convolution is an important operation in many fields
- We can process a per pixel filtering operation across data very efficiently using convolutions
- Careful consideration have to be made between convolutions, correlations, circular convolutions and auto correlations
- Convolution theorem can be used to efficiently implement discrete convolutions

What’s Next?

Convolutions form the key part in the success of deep learning in computer vision and image analysis. We will develop the concept of convolutional neural networks in the next lecture …

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Thank you

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