Pattern Analysis

A Deep Learning Approach

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"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

Godfrey H. Hardy (1877-1947)

CHAPTER 2

Symmetry

"So our problem is to explain where symmetry comes from. Why is nature so nearly symmetrical? No one has any idea why."

Richard Feynman (1918-1988)

Have you ever asked why flowers are beautiful? What gives them their beauty. No doubt that colour plays a role, but there is one aspect that has the strongest effect. It is because they are symmetrical. In fact, the reason why flowers are symmetrical is because bees like symmetry, while they don't even see any colour! Bees are one of the most wonderful of species in that they employ symmetry is nearly everything they do. In their honeycombs, the way they cover an entire area to find pollen and how they build their hives.

The key principle governing patterns in nature is symmetry. In this chapter, we will see how we can quantify symmetry scientifically and exhaustively. We will see that there is an optimal way to represent symmetry and mathematician Évariste Galois [1830] (1811-1832) invented it, just before he died in duel at the early age of twenty! We will begin by observing symmetry and the most intuitive way to do this is to visualise it in what we experience in a daily lives - Geometry.

2.1 Geometry

We can consider everything around us as geometry, objects moving in three dimensional (3D) space. For our purposes, we can consider the simpler two dimensional (2D) case and begin with the simplest object, a line (or more technically a line segment) as shown in figure 2.1 on the following page(a). The line exists even without a coordinate system, but we are accustomed to viewing it within such as system as shown in figure 2.1(b). A number of areas of science depend on the properties of this simple line, such as the theory of gravity, which we will explore later in this book.

For now consider the simple act of trying to measure the length $\boldsymbol{c}$ of this line. We could use Pythagoras' theorem $\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$, but that depends on knowing the lengths $\boldsymbol{a}$ and $\boldsymbol{b}$, which in turn depends on the coordinate system we choose to define. However, there is no

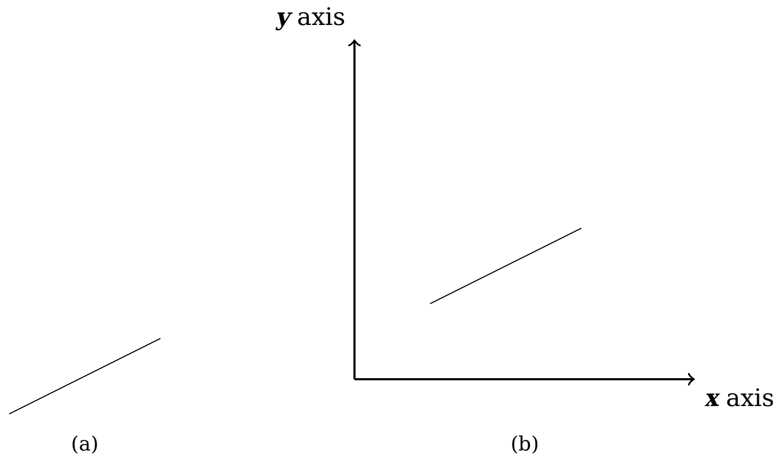

Figure 2.1: The simplest non-trivial geometric object, a line.

unique way to do this, as we can define the unit distance of the coordinate with many differing lengths. Consider that even the unit of measure we use called a metre is not absolute, it was in fact defined by a platinum-iridium bar of fixed length at a fixed temperature in the mid 20th century, and currently defined as the distance travelled by light in some fraction of a second. But even the speed of light varies depending on the medium it is travelling in, which is the same phenomenon that causes light to refract.

We can mitigate the differences between coordinate systems by determining the scaling between the various systems. For example, let us assume that there are three different coordinate systems and the line has integer coordinates, so that the length of the line is also an integer. This is called a Pythagorean triple, such as $\boldsymbol{a} = 3$, $\boldsymbol{b} = 4$ and $\boldsymbol{c} = 5$. Let one of the other coordinate systems be scaled 2 times larger than your coordinate system and the other 3 times more. Relative to your point of view, where $\boldsymbol{a}_1^2 + \boldsymbol{b}_1^2 = \boldsymbol{c}_1^2$, we can view the other coordinate systems as scaled versions

$$
\begin{aligned}
\frac{\boldsymbol{a}_2^2}{2} + \frac{\boldsymbol{b}_2^2}{2} &= \frac{\boldsymbol{c}_2^2}{2} \\
\frac{\boldsymbol{a}_3^2}{3} + \frac{\boldsymbol{b}_3^2}{3} &= \frac{\boldsymbol{c}_3^2}{3}
\end{aligned}
$$

This normalisation effectively allows you to convert between the different coordinate systems provided you know the scaling factors. Note that this scaling of integer coordinates also provides a way to introduce rational numbers through geometry rather than the number line, since the coefficients of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are integer ratios. These types of descriptions for lines and points, as well as precise definitions of parallel lines, were first constructed by Euclid [0BCE].

However, this method becomes very cumbersome if the scalings are different per dimension and when other considerations need to be taken into account, such as motion. The elegant solution to these considerations is to use coordinate free systems for representing geometry and we will cover this later via tensors. But for the moment, we have a way of ensuring that all observers can agree on the length of the line regardless of their coordinate systems via scaling.

2.2 Invariance

Having agreed on the length of the line, the multiple observers can experiment with the line, apply operations and measure the outcomes. If we assume that the coordinates of the end points of the line are defined as (1,1) and (3,2), we can observe the following about the line:

1. moving the line anywhere (e.g. adding (4,4) to both end points) does not alter the length of the line

2. rotating the line by 180 degrees does not alter the line at all

We can say that the line we have defined is ***invariant*** under those operations. Those operations leave the line unchanged and expose to us the symmetry of the line, namely the line is symmetrical along the direction of the line.

To provide a more concrete example, consider a triangle constructed from three of our lines, whose vertices are coloured green, yellow and blue. Given the fixed positions $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, we can explore rotations around the centre of the triangle as shown as figure 2.2. The rotation of the

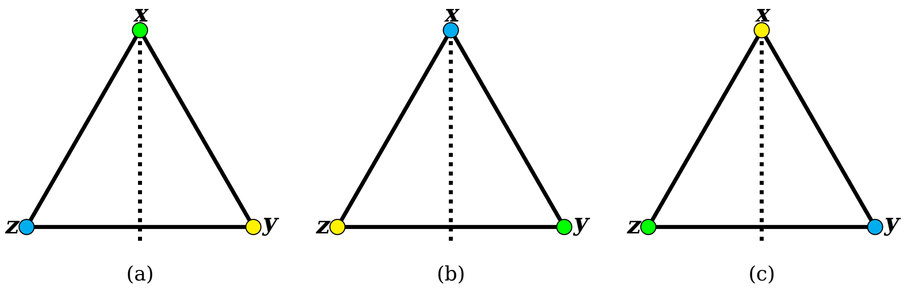

Figure 2.2: A triangle with coloured vertices showing rotational symmetry. (a)-(c) shows rotations of the triangle around its centre with respect to the fixed points $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ and the (dotted) lines of reflective symmetry.

triangle as shown in figure 2.2 leaves the triangle unchanged but the vertices permuted. This invariance to a rotation of 60 degrees reveals a symmetry of the object. We have uncovered a clue that applying some operations may leave our geometry objects unchanged and therefore reveal the symmetries involved. What operations do we mean?

2.3 Transformations

The experiments above were limited to translation (movement) and rotation. These operations are examples of what we call in mathematics as transformations. They are a mapping of coordinates to new coordinates just like applying a function to a variable in algebra. Felix Klein [1893] constructed a generalised form of geometry that superseded that constructed by Euclid [0BCE]. He defined symmetry in geometry as those properties of geometric objects that remain invariant to transformations.

For example, let us go back to the triangle shown in figure 2.2. We saw that the triangle has rotational symmetry around the centre. We can also observe that the triangle has reflective

	I	U	V	X	Y	Z
I	I	U	V	X	Y	Z
U	U	V	I	Z	X	Y
V	V	I	U	Y	Z	X
X	X	Y	Z	I	U	V
Y	Y	Z	X	V	I	U
Z	Z	X	Y	U	V	I

Table 2.1: The mapping of transformation pairs for the triangle, where ***X***, ***Y*** and ***Z*** represent the rotations to those fixed points starting from ***X***. The symbols ***I***, ***U*** and ***V*** represent the identity (nothing happens), reflection and rotation respectively.

symmetry around a line drawn perpendicular to each side through each vertex (see dotted lines in figure 2.2 on the preceding page). Each of these operations can be seen as a transformation of the triangle. For example, a clockwise rotation of the triangle will move the green vertex to the yellow vertex starting from ***x*** and ending up at ***y*** (from figure 2.2 on the previous page(a) to figure 2.2 on the preceding page(b)). Likewise, the reflections of the triangle along the perpendicular (dotted) lines with respect to the vertices are shown in figure 2.3. This gives

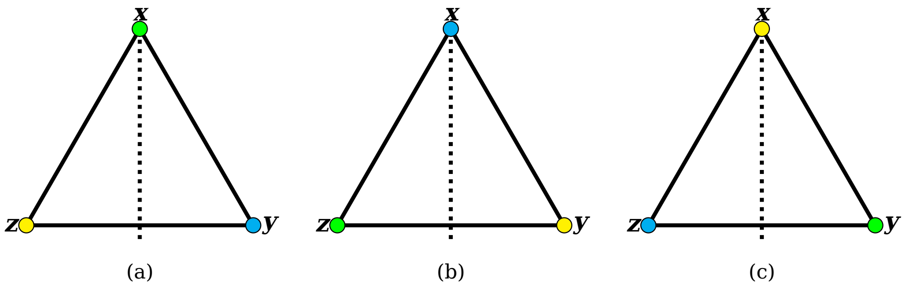

Figure 2.3: A triangle with coloured vertices showing reflective symmetry. (a)-(c) shows reflections of the triangle along the perpendicular (dotted) line with respect to the fixed points ***x***, ***y*** and ***z***.

us a total of six unique symmetries that leave the triangle unchanged after transformations, where we have included the identity transformation that does nothing. But does this cover ***all*** the possible transformations? What happens when we compose transformations together?

If we label the operations of reflective symmetry about ***x***, ***y*** and ***z*** as ***X***, ***Y***, ***Z*** respectively and ***U***, ***V*** as the rotations clockwise and anti-clockwise respectively, we can draw a table of the possible pairs of transformations and note the resulting configuration of the triangle obtained from them and label this accordingly. We can then observe any patterns that show up. This composition or 'multiplication' table, also known as a Cayley table is shown in table 2.1.

We can observe a startlingly fact that all possible composed pairs of transformations simply lead to the result of another known transformation. For example, two counter-clockwise rotations ***V***, read from the table as the row ***V*** and the column ***V*** is equivalent to a single clockwise rotation ***U***. Or that a reflection ***X*** and a reflection ***Y*** is equivalent to a rotation ***V***.

In fact, the table we have constructed is exhaustive and let's us map any possible transformation of the object that leave the object unchanged. In other words, the composition $\boldsymbol{A} \cdot \boldsymbol{B}$ of two operations $\boldsymbol{A}$ and $\boldsymbol{B}$ is always in table 2.1 on the preceding page. This is effectively a quantification of that objects ***entire*** symmetry as a single construct called a Group. This is the central idea of the theory of Galois Groups or simply Groups or in our case a symmetry Group.

2.4 Groups

1 **Definition (Groups [Galois, 1830])**
A Group G is composed of a set of elements and a composition or multiplication operation $\cdot$, which abide by the following properties:

1. ***The multiplication result $C = A \cdot B$ is also part of the same set.***
2. ***There is an identity element I in the set, so that $A \cdot I = A$.***
3. ***There exists an inverse operation, so that $A \cdot A^{-1} = I$.***
4. ***The multiplication is associative, so that $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.***

2.5 Applications

2.5.1 Symmetry Breaking

2.5.2 Standard Model of Particle Physics

Abbreviations

2D two dimensional ... 5
3D three dimensional ... 5
RF random forest ... 19
CNN convolutional neural network ... 27

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