SF2822 Applied Nonlinear Optimization, 2020/2021
Project assignment 1D
Due Tuesday April 27 2021 23.59

Instructions for the project assignments are given in the course PM. Additional clarifications are given below.

- Discussion between the groups is encouraged, but each group must individually solve the assignments. It is *not* allowed to use solutions made by others in any form.
- Instructions for the report:
  - The report should have a leading title page where the project name and the group members' names, personal number and e-mail addresses are clearly stated.
  - The report should be written using a suitable word processor.
  - The contents should be such that another student in the course, who is not familiar with the project, should be able to read the report and easily understand:
    1. What is the problem? What it the problem background? This does *not* mean a copy of the project description, but rather a suitable summary of necessary information needed in order to understand the problem statement.
    2. How has the group chosen to formulate the problem mathematically? What assumptions have been made? If these assumptions affect the solution, this should be noted.
    3. What is the meaning of constraints, variables and objective function in the mathematical formulation?
    4. What is the solution of the formulated optimization problem? If suitable, refer the mathematical solution to the terminology of the (non-mathematical) problem formulation. (There could be more than one optimization problem.)
  - Most project descriptions contain a number of questions to be answered in the report. The report *must* contain the answers to these questions. They should, however, in a natural way be part of the content of the report and not be given in a "list of answers". The purpose of the questions is to suggest suitable issues to consider in the part of the report where the results are interpreted and analyzed. Make use of your knowledge from the course when formulating the problem and analyzing the results. Additional interpretations are encouraged as well as generalizations and other ways of modeling the problem.
  - A suggested outline of the report is as follows:
    1. Possibly a short abstract.
    2. Problem description and background information.
    3. Mathematical formulation.
    4. Results and analysis (interpretation of results).
    5. A concluding section with summary and conclusions.

    Deviations from the outline can of course be done.
  - GAMS code should not be part of the report, and should not be referred to in the report.

- Each group should upload the following documents via the Canvas page of the course no later than by the deadline of the assignment:
  - The report as a pdf file.
  - GAMS files.

  Please upload your documents as individual pdf and gms files, and not as zip files.
- Each student should fill out a paper copy of the self assessment form and hand in at the beginning of the presentation lecture.

In certain applications one is facing an optimization problem where the constraints and/or the objective function is on a form that makes standard methods within mathematical programming non-applicable. For example, the constraints may consist of ordinary or partial differential equations. The problem is then infinite-dimensional. One way of attacking such a problem is to discretize functions and approximate the derivatives. The resulting finite-dimensional problem is then solved. There are also methods that are not based on discretization. One example is the Pontryagin maximum principle, which is handled in the course *SF2852 Optimal control*.

The project concerns transferring a satelite from one circular orbit around the sun to another one. The aim is to make the new orbit have as large radius as possible and the manoeuvre has to be completed in a given time span. The exercise is to a large extent inspired by an example in Moyer and Pinkham [2, pp. 91–105] and Ho [1, pp. 66–69].

Fundametal mechanics in polar coordinates gives the equations of motion for the satellite, see Figure 1, according to

$$\begin{aligned}
\dot{r}(\tau) &= v_r(\tau),\\
\dot{v}_r(\tau) &= \frac{v_t(\tau)^2}{r(\tau)} - \frac{\mu}{r(\tau)^2} + \frac{T(\tau)\sin u(\tau)}{m(\tau)},\\
\dot{v}_t(\tau) &= -\frac{v_r(\tau)v_t(\tau)}{r(\tau)} + \frac{T(\tau)\cos u(\tau)}{m(\tau)},
\end{aligned}$$

where $\dot{r}(\tau)$ denotes the time derivative of $r(\tau)$ and

$$\begin{aligned}
\tau &= \text{time variable},\\
r(\tau) &= \text{radial distance from the sun},\\
v_r(\tau) &= \text{radial component of speed vector},\\
v_t(\tau) &= \text{tangential component of speed vector},\\
u(\tau) &= \text{driving force direction (angle)},\\
T(\tau) &= \text{driving force},\\
m(\tau) &= \text{mass of satellite (including fuel)},\\
\mu &= \text{constant of gravity multiplied by the mass of the sun.}
\end{aligned}$$

Further, let t_f denote the time span available for the manoeuvre. To ensure that the satellite is in a new circular orbit at the end time t_f, the radial component of the speed vector has to be zero, and there has to be balance between the centrifugal force and the force of gravity. These two conditions may be expressed as $v_r(t_f) = 0$ and in addition

$$\frac{m(t_f)v_t(t_f)^2}{r(t_f)} = \frac{\mu m(t_f)}{r(t_f)^2}.$$

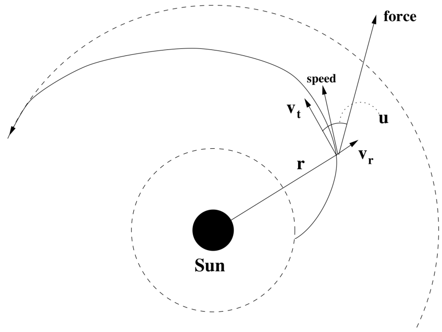

Figure 1: Schematic picture of the problem.

To avoid numerical difficulties, we scale the problem by introducing the variable t and the functions $x_1(t)$, $x_2(t)$ and $x_3(t)$ according to

$$t = \frac{\tau}{t_f}, \qquad x_1(t) = \frac{r(\tau)}{r_0}, \qquad x_2(t) = \frac{r_0^2}{\mu t_f} v_r(\tau), \qquad x_3(t) = \sqrt{\frac{r_0}{\mu}} v_t(\tau),$$

where $r_0 = r(0)$. We thus scale the time so that it ranges from zero to one. In addition, the radius is mesured relative to the radius of the original orbit, and the tangential speed is measured relative to the intial tangential speed. The equations of motion may in these coordinates with the scaled time be expressed as

$$\begin{aligned}
\dot{x_1}(t) &= \frac{\mu t_f^2}{r_0^3} x_2(t),\\
\dot{x_2}(t) &= \frac{x_3(t)^2}{x_1(t)} - \frac{1}{x_1(t)^2} + \frac{r_0^2}{\mu} \frac{T(t)}{m(t)} \sin u(t),\\
\dot{x_3}(t) &= -\frac{\mu t_f^2}{r_0^3} \frac{x_2(t) x_3(t)}{x_1(t)} + \frac{t_f \sqrt{r_0}}{\sqrt{\mu}} \frac{T(t)}{m(t)} \cos u(t).
\end{aligned}$$

The above equations referer to the case when the engine power is constant, and when the engines are running continuously. This makes $T(t) \equiv T$ and $m(t) = m_0 - |\dot{m}| t_f t$, where m_0 is the mass of the satellite at $t = 0$ and $|\dot{m}|$ is the fuel consumption per second. By introducing the dimensionless constants

$$c_1 = \frac{T r_0^2}{\mu m_0}, \qquad c_2 = \frac{|\dot{m}| \sqrt{\mu}}{T \sqrt{r_0}}, \qquad c_3 = \frac{t_f \sqrt{\mu}}{\sqrt{r_0^3}},$$

we may write the equations as

$$\begin{aligned}
\dot{x_1}(t) &= c_3^2 x_2(t),\\
\dot{x_2}(t) &= \frac{x_3(t)^2}{x_1(t)} - \frac{1}{x_1(t)^2} + \frac{\sin u(t)}{1/c_1 - c_2 c_3 t},\\
\dot{x_3}(t) &= -c_3^2 \frac{x_2(t) x_3(t)}{x_1(t)} + \frac{c_3 \cos u(t)}{1/c_1 - c_2 c_3 t}.
\end{aligned}$$

Numerical values of the required constants are given in Table 1.

Constant	Value	Unit
μ	$1.327 \cdot 10^{20}$	m^3/s^2
r_0	$1.496 \cdot 10^{11}$	m
m_0	$4.53 \cdot 10^{3}$	kg
$\lvert\dot{m}\rvert$	$6.76 \cdot 10^{-5}$	kg/s
t_f	$1.668 \cdot 10^{7}$	s
T	3.77	N

Table 1: Parameter values.

Basic exercises

1. Formulate the above outlined problem of transferring the satellite from the initial orbit to an orbit with as large radius as possible as an optimization problem. Remember to include all boundary conditions.

2. Transform the problem to a nonlinear programming problem by discretizing the problem in a suitable manner. Solve the resulting nonlinear program. Are you able to say anything about local or global optimality of the computed solution?

Advanced exercises

3. In practice, only a certain fraction of the initial mass of the satellite m_0 is fuel. Assume that one is able to adjust the driving force of the engine during the manoeuvre to any level between 0 and T, and in addition that the fuel consumption is proportional to the driving force of the engine. The proportionality constant is given by the requirement that the driving force T should give fuel consumption $|\dot{m}|$. Formulate this problem, discretize and solve the resulting discrete problem. Compare the solution to the previous solution for different values of the fraction of the initial mass that consists of fuel.

Remark: Possibly certain difficulties may arise when solving the GAMS model in Exercise 3, when the driving force is close to zero. If so, try to avoid these difficulties. In addition, try to explain why the difficulties arise.

4. Let 20% of the initial mass of the satellite consist of fuel and assume that the ratio between the supplied amount of fuel per time step and the driving force is the same as before. Study how the solution depends on the maximum driving force.

5. Investigate the role of choice of discretization method by trying some other method or methods. Does the solution appear to be sensitive to the choice of method or is this choice of marginal importance?

Good luck!

References

[1] A. E. Bryson, Jr. and Y.-C. Ho. *Applied Optimal Control*. Blaisdell Publishing Company, 1969.

[2] H. G. Moyer and G. Pinkham. Several trajectory optimization techniques. part ii: Application. In A. V. Balakrishnan and L. W. Neustadt, editors, *Computing methods in optimization problems*, pages 91–105. Academic Press, 1964.