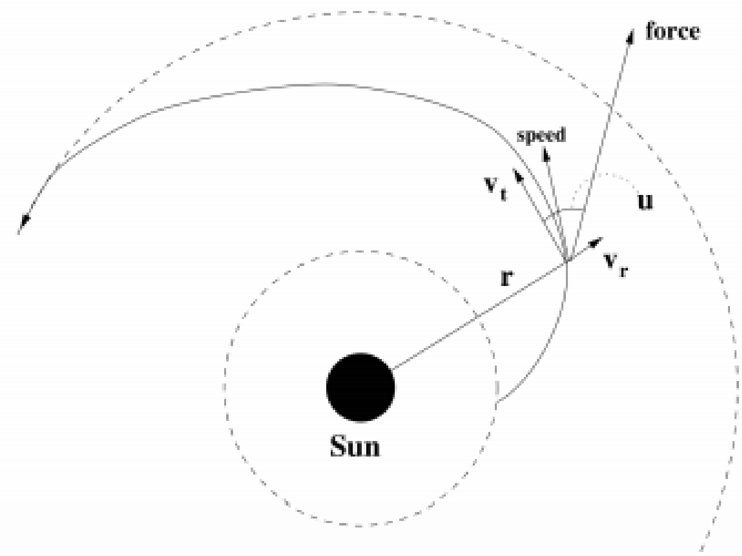

Applied Nonlinear Optimization

SF2822

Project Assignment 1: 1D

Group 1D2

Sigurlaug Rún Jónsdóttir
Daniel Liu
Matilda Strandh

Stockholm

27. april 2021

1 Abstract

This report presents a model on how to maximize the radius of the orbit of a satellite transitioning from one inner orbit to an outer orbit. The model used is based on discretization of time, which allows the constraints to be expressed as a nonlinear programming problem. The results show that when the driving force is constant to a certain level, the optimal scaled radius is 1.512. The report also examines what happens when either the fuel capacity or the maximum driving force of is changed. It is suggested that if one seeks to improve the satellite, one should improve the fuel capacity and the maximum driving force simultaneously. Lastly, the numerical properties of the problem are discussed, where the importance of precise calculations are argued for.

2 Problem Description

The purpose of this paper is to optimize a transition of a satellite around the sun from one orbit to another larger orbit. The objective is to maximize the radius of the new orbit at the end of a given time period by controlling the thrust and driving angle of the satellite. Three scenarios are considered. In the first scenario, it is assumed that the thrust is constant and always maximized, and the only controllable variable is the driving direction. In the second scenario, the fuel capacity of the satellite is limited and the thrust needs to be adjusted accordingly. The report examines how different limits on fuel capacity affects the optimal radius. In the third scenario, the fuel capacity is fixed, and it is examined how allowing for more or less maximum engine force affects the optimal radius. The models are formulated and approximated as nonlinear programming problems, and solved using GAMS.

2.1 Definition of Physical Quantities

We first define the physical quantities:

$$
\begin{aligned}
\tau &= \text{time variable,}\\
r(\tau) &= \text{the radial distance between the satellite and the sun at time } \tau,\\
v_r(\tau) &= \text{the radial velocity of the satellite at time } \tau,\\
v_t(\tau) &= \text{the tangential velocity of the satellite at time } \tau,\\
T(\tau) &= \text{the thrust at time } \tau,\\
u(\tau) &= \text{the thrust angle at time } \tau,\\
m(\tau) &= \text{the mass of the satellite, including fuel, at time } \tau,\\
\mu &= \text{the constant of gravity multiplied by the mass of the sun.}
\end{aligned}
$$

A schematic for the problem is illustrated in Figure 1. Fundamental mechanics in polar coordinates gives the equations of motion for the satellite, where $\dot{r}(\tau)$, $\dot{v}_r(\tau)$, $\dot{v}_t(\tau)$ denote the time derivatives $r(\tau)$, $v_r(\tau)$, $v_t(\tau)$ respectively, according to

$$\begin{aligned}
\dot{r}(\tau) &= v_r(\tau), \\
\dot{v}_r(\tau) &= \frac{v_t(\tau)^2}{r(\tau)} - \frac{\mu}{r(\tau)^2} + \frac{T(\tau)\sin u(\tau)}{m(\tau)}, \\
\dot{v}_t(\tau) &= -\frac{v_r(\tau)v_t(\tau)}{r(\tau)} + \frac{T(\tau)\cos u(\tau)}{m(\tau)}.
\end{aligned}$$

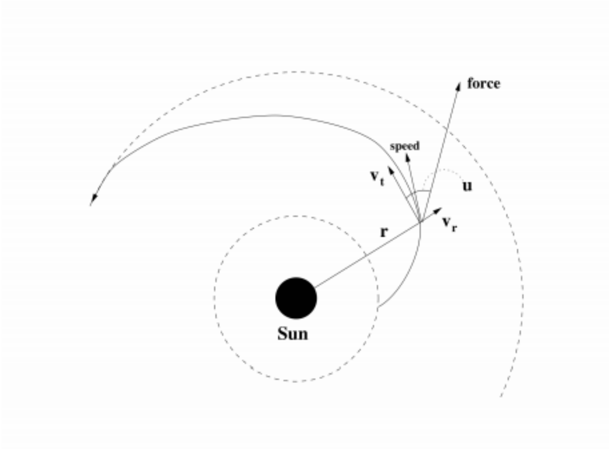

Figure 1: *Schematic picture of the problem*

Let t_f denote the fixed time span within which the transition to the large orbit should be completed. Two conditions for the satellite to be considered as in a new orbit are that the radial velocity is zero and the centrifugal force is equal to the force of gravity. These conditions may be expressed as

$$v_r(t_f) = 0 \text{ and } \frac{m(t_f)v_t(t_f)^2}{r(t_f)} = \frac{\mu m(t_f)}{r(t_f)^2}.$$

2.2 Scaling and Parameters

To avoid numerical difficulties, the time variable τ can be scaled so that it ranges from 0 to 1, denoted by the variable t. We introduce the variables $x_1(t)$, $x_2(t)$, $x_3(t)$ according to

$$t = \frac{\tau}{t_f},\ x_1(t) = \frac{r(\tau)}{r_0},\ x_2(t) = \frac{r_0^2}{\mu t_f} v_r(\tau),\ x_3(t) = \sqrt{\frac{r_0}{\mu}} v_t(\tau).$$

Here, r_0 denotes the initial radius. Using these new variables, the conditions that indicate the satellite entering a new orbit can instead be expressed as $x_2(1) = 0$ and $x_3(1) = x_1(1)^{-\frac{1}{2}}$. We also define some other constants, which are listed in Table 1.

The time derivatives of $x_1(t)$, $x_2(t)$, $x_3(t)$ can now be expressed as

$$\begin{aligned}
\dot{x}_1(t) &= \frac{\mu t_f^2}{r_0^3} x_2(t), \\
\dot{x}_2(t) &= \frac{x_3(t)^2}{x_1(t)} - \frac{1}{x_1(t)^2} + \frac{r_0^2}{\mu} \frac{T(t)}{m(t)} \sin u(t), \\
\dot{x}_3(t) &= -\frac{\mu t_f^2}{r_0^3} \frac{x_2(t) x_3(t)}{x_1(t)} + \frac{t_f \sqrt{r_0}}{\sqrt{\mu}} \frac{T(t)}{m(t)} \cos u(t),
\end{aligned}$$

Table 1: *Parameters to the problem*

Constant	Value	Unit	Description
μ	$1.327 \cdot 10^{20}$	m^3/s^2	constant of gravity multiplied by mass of sun
r_0	$1.496 \cdot 10^{11}$	m	initial radius
m_0	$4.53 \cdot 10^{3}$	kg	initial mass of satellite
T	$3.77 \cdot 10^{11}$	N	maximum thrust
$\lvert\dot{m}\rvert$	$6.76 \cdot 10^{-5}$	kg/s	fuel consumption per second when driving at force T
t_f	$1.668 \cdot 10^{7}$	s	fixed time span

3 Mathematical Formulation

3.1 Theoretical Formulation of Problem

We first consider the case where the thrust is constant and $T(t) \equiv T$. When the thrust is constant, the fuel consumption per second is also constant and we can express the mass at any time as $m(t) = m_0 - |\dot{m}| t_f t$. We notice that while the problem likely needs to contain many variables, all variables are directly or indirectly dependent on two variables: The time t and the thrust direction $u(t)$. The time t is obviously not controllable, which makes the thrust direction $u(t)$ the only controllable variable. This means that the decision of the thrust direction $u(t)$ at every time point is the main interest in any practical application. Thus, the approach to the problem should be to express all variables as functions of $u(t)$ and t, directly or indirectly.

Assuming that the distance and velocities have been scaled to be expressed as $x_1(t)$, $x_2(t)$, $x_3(t)$, the objective function should to maximize $x_1(1)$. The value of $x_1(1)$ can be expressed as the initial value plus all of the changes in $x_1(t)$ until time 1. In fact, any function at time t can be expressed as the initial value plus the integral of the derivative from time 0 to time t.

$$x(t) = x(0) + \int_0^t \dot{x}(\sigma)\, d\sigma$$

The constraint of the optimization problem can be categorized into three categories.

- Definitions. These constraints are the definitions for the variables $x_1(t)$, $x_2(t)$, $x_3(t)$, their derivatives $\dot{x}_1(t)$, $\dot{x}_2(t)$, $\dot{x}_3(t)$, and the mass of the satellite $m(t)$.
- End conditions. These constrains identify when the satellite has entered a new orbit. Namely, it is necessary that the radial component of the speed vector be zero, and there has to be balance between the centrifugal force and the force of gravity.
- Initial values. These are the values of the variables $x_1(t)$, $x_2(t)$, $x_3(t)$ at time 0.

As for the initial values, they all can be expressed as constants. Since the radius of the prior orbit is r_0, it is implied that $x_1(0) = 1$. Since the orbit is assumed to be circular, it is implied that the radial component of the speed vector is 0, or $x_2(0) = 0$. To obtain the initial value $x_3(0)$, we assume that the initial tangential velocity is constant and utilize the equation for the velocity of a satellite moving about a central body in circular motion, $v_t(0) = \sqrt{\frac{\mu}{r_0}}$. Inserting this into the expression for $x_3(t)$, we obtain that $x_3(0) = 1$.

Lastly, we assume that we only allow the direction of the thrust to be forward by setting the constraint $-\frac{\pi}{2} \leq u(t) \leq \frac{\pi}{2}$. The theoretical optimization problem is formulated as

$$\begin{aligned}
\text{maximize} \quad & x_1(1) = x_1(0) + \int_0^1 \dot{x}_1(\sigma)\, d\sigma \\
\text{subject to} \quad & x_1(t) = x_1(0) + \int_0^t \dot{x}_1(\sigma)\, d\sigma, && t \in [0,1], \\
& x_2(t) = x_2(0) + \int_0^t \dot{x}_2(\sigma)\, d\sigma, && t \in [0,1], \\
& x_3(t) = x_3(0) + \int_0^t \dot{x}_3(\sigma)\, d\sigma, && t \in [0,1], \\
& \dot{x}_1(t) = \frac{\mu t_f^2}{r_0^3} x_2(t), && t \in [0,1], \\
& \dot{x}_2(t) = \frac{x_3(t)^2}{x_1(t)} - \frac{1}{x_1(t)^2} + \frac{r_0^2}{\mu}\frac{T}{m(t)} \sin u(t), && t \in [0,1], \\
& \dot{x}_3(t) = -\frac{\mu t_f^2}{r_0^3}\frac{x_2(t)x_3(t)}{x_1(t)} + \frac{t_f\sqrt{r_0}}{\sqrt{\mu}}\frac{T}{m(t)} \cos u(t), && t \in [0,1], \\
& m(t) = m_0 - |\dot{m}| t_f t, && t \in [0,1], \\
& x_2(1) = 0, \\
& x_3(1) = \frac{1}{\sqrt{x_1(1)}}, \\
& x_1(0) = 1, \\
& x_2(0) = 0, \\
& x_3(0) = 1, \\
& -\frac{\pi}{2} \leq u(t) \leq \frac{\pi}{2}, && t \in [0,1].
\end{aligned} \tag{1}$$

This problem is difficult to solve, since the variables $x_1(t)$, $x_2(t)$, $x_3(t)$ and their time derivatives $\dot{x}_1(t)$, $\dot{x}_2(t)$, $\dot{x}_3(t)$ are recursively expressed by each other. Furthermore, integrals are difficult to solve directly with numerical solvers since they theoretically require infinitesimal time steps.

3.2 Discretization into a Nonlinear Programming Problem when Thrust is Constant

One way to approximate the solution to Problem 1 is to discretize the time into small finite time steps. Let $k = 0, 1, \ldots, n$ denote the order of a time step. Furthermore, since t is scaled so that it ranges between 0 and 1, we can express

$$\Delta t = \frac{1}{n} \text{ and } t_k = k \cdot \Delta t$$

Instead of programming integrals, we can approximate them by expressing each integral as a Riemann sum. A variable $x(t)$ at time t_k can be approximated as

$$x(t_k) = x(0) + \int_0^{t_k} \dot{x}(\sigma)\, d\sigma \approx x(0) + \Delta t \sum_{j=0}^{k-1} \dot{x}(t_j),\ k = 1, \ldots, n$$

This implies that

$$x(t_{k+1}) = x(0)+\Delta t \sum_{j=0}^{k} \dot{x}(t_j) = x(0)+\Delta t \sum_{j=0}^{k-1} \dot{x}(t_j)+\Delta t\cdot\dot{x}(t_k) = x(t_k)+\Delta t\cdot\dot{x}(t_k),\ k = 0, \ldots, n-1.$$

In other words, we can calculate the value of a variable $x(t)$ at time t_k recursively by using the explicit Euler method, i.e. setting $x(t_{k+1}) = x(t_k) + \Delta t \cdot \dot{x}(t_k)$. To convert Problem 1 from an integral problem into a nonlinear programming problem, we simply replace the "definition" constraints in Problem 1 by the recursive equalities. The rest of the constraints are the same, but are now discretized to be calculated at specific times. The discretized nonlinear programming problem can be expressed as

$$\begin{array}{lll}
\text{maximize} & x_1(1) & \\
\text{subject to} & x_1(t_{k+1}) = x_1(t_k) + \Delta t \cdot \dot{x}_1(t_k), & k = 0, 1 \ldots, n-1, \\
& x_2(t_{k+1}) = x_2(t_k) + \Delta t \cdot \dot{x}_2(t_k), & k = 0, 1 \ldots, n-1, \\
& x_3(t_{k+1}) = x_3(t_k) + \Delta t \cdot \dot{x}_3(t_k), & k = 0, 1 \ldots, n-1, \\
& \dot{x_1}(t_k) = \dfrac{\mu t_f^2}{r_0^3} x_2(t_k), & k = 0, 1 \ldots, n, \\
& \dot{x_2}(t_k) = \dfrac{x_3(t_k)^2}{x_1(t_k)} - \dfrac{1}{x_1(t_k)^2} + \dfrac{r_0^2}{\mu} \dfrac{T}{m(t_k)} \sin u(t_k), & k = 0, 1 \ldots, n, \\
& \dot{x_3}(t_k) = -\dfrac{\mu t_f^2}{r_0^3} \dfrac{x_2(t_k) x_3(t_k)}{x_1(t_k)} + \dfrac{t_f \sqrt{r_0}}{\sqrt{\mu}} \dfrac{T}{m(t_k)} \cos u(t_k), & k = 0, 1 \ldots, n, \\
& m(t_k) = m_0 - |\dot{m}| t_f t_k, & k = 0, 1 \ldots, n, \\
& x_2(1) = 0, & \\
& x_3(1) = \dfrac{1}{\sqrt{x_1(1)}}, & \\
& x_1(0) = 1, & \\
& x_2(0) = 0, & \\
& x_3(0) = 1, & \\
& -\dfrac{\pi}{2} \leq u(t_k) \leq \dfrac{\pi}{2}, & k = 0, 1 \ldots, n.
\end{array} \tag{2}$$

3.3 Mathematical Formulation when thrust is not Constant and Fuel is Limited

In earlier sections we have assumed that the thrust has a constant value T. Now we can instead allow the thrust to be a controllable variable that can vary between 0 and T. Let $0 \leq p(t_k) \leq 1$ denote the proportion of the maximum thrust at time t_k, so that the thrust is defined as $T(t) \equiv p(t)T$. Furthermore, we assume that the fuel consumption per second is proportional to the thrust, and that a thrust of T should yield a fuel consumption of $|\dot{m}|$. Thus, we can define the fuel consumption per second at a certain time t_k as $t_f|\dot{m}|p(t_k)$. The factor t_f is used to scale the time measure τ into the time measure t, which ranges from 0 to 1.

Since the fuel consumption per second is no longer assumed to be constant, we have to express the mass of the satellite in another way. We use the same discretization method as described in Section 3.2. Then the instantaneous fuel consumption at time t_k can be expressed as

$$\lim_{\Delta t \to 0} \Delta t \cdot t_f|\dot{m}|p(t_k).$$

Thus, we can approximate the mass of the satellite at any time t_k as

$$m(t_{k+1}) = m_0 - \Delta t \sum_{j=0}^{k-1} t_f|\dot{m}|p(t_k),\ k = 0, 1, \ldots n-1.$$

Note that in the specific case when $p(t) = 1$ for all $t \in (0,1)$, then the mass of the satelite is given by the explicit expression $m(t) = m_0 - |\dot{m}|t_f t$, which is the same as when the thrust is constant. We can also express the mass as a recursive expression, which is used as a constraint in the nonlinear program

$$m(t_{k+1}) = m(t_k) - \Delta t \cdot t_f|\dot{m}|p(t_k)$$

We also want to look at how the amount of available fuel has an impact on the optimal solution, and also how possible engine enchantments would impact the optimal solution. We do that by allocating certain amount of the mass of the satellite as fuel. First, we investigate how the optimal solution varies as we change the amount of available fuel, while letting the thrust range from 0 to T. Then, we keep the amount of fuel fixed but vary the value of the maximum thrust.

3.3.1 Limiting the Amount of Fuel

In addition to the equations above, we need to define how much of the mass is fuel and add a constraint such that the satellite can not burn up more fuel than it has. We therefore must add:

- A constant m_{fuel} where we can choose how much of the initial mass is fuel. Here $q \in (0,1)$ represents the percentage of the initial mass that is fuel:

$$m_{fuel} = q \cdot m_0$$

- A constraint that ensures that we will not burn up more fuel than we have:

$$m_{fuel} + m(t_k) \geq m_0, \qquad k = 0, 1, \ldots, n$$

The mathematical formulation of this problem is given as follows:

$$\begin{array}{lll}
\text{maximize} & x_1(1) & \\
\text{subject to} & x_1(t_{k+1}) = x_1(t_k) + \Delta t \cdot \dot{x}_1(t_k), & k = 0, 1 \ldots, n-1, \\
& x_2(t_{k+1}) = x_2(t_k) + \Delta t \cdot \dot{x}_2(t_k), & k = 0, 1 \ldots, n-1, \\
& x_3(t_{k+1}) = x_3(t_k) + \Delta t \cdot \dot{x}_3(t_k), & k = 0, 1 \ldots, n-1, \\
& \dot{x}_1(t_k) = \dfrac{\mu t_f^2}{r_0^3} x_2(t_k), & k = 0, 1 \ldots, n, \\
& \dot{x}_2(t_k) = \dfrac{x_3(t_k)^2}{x_1(t_k)} - \dfrac{1}{x_1(t_k)^2} + \dfrac{r_0^2}{\mu} \dfrac{p(t_k) T}{m(t_k)} \sin u(t_k), & k = 0, 1 \ldots, n, \\
& \dot{x}_3(t_k) = -\dfrac{\mu t_f^2}{r_0^3} \dfrac{x_2(t_k) x_3(t_k)}{x_1(t_k)} + \dfrac{t_f \sqrt{r_0}}{\sqrt{\mu}} \dfrac{p(t_k) T}{m(t_k)} \cos u(t_k), & k = 0, 1 \ldots, n, \\
& m(t_{k+1}) = m(t_k) - \Delta t \cdot t_f |\dot{m}| p(t_k), & k = 0, 1 \ldots, n, \\
& x_2(1) = 0, & \\
& x_3(1) = \dfrac{1}{\sqrt{x_1(1)}}, & \\
& x_1(0) = 1, & \\
& x_2(0) = 0, & \\
& x_3(0) = 1, & \\
& m_{fuel} + m(t_k) \geq m_0, & k = 0, 1, \ldots, n, \\
& 0 \leq p(t_k) \leq 1, & k = 0, 1, \ldots, n, \\
& -\dfrac{\pi}{2} \leq u(t_k) \leq \dfrac{\pi}{2}, & k = 0, 1 \ldots, n.
\end{array} \tag{3}$$

3.3.2 Varying the Maximum Thrust

For this next problem we fix the fuel to be 20% of the initial mass. Since the fuel is now fixed we can study how the solution depends on the maximum thrust. There are few things we need to add or change:

- The constant m_{fuel} is fixed by setting q to be 0.2.

$$m_{fuel} = 0.2 \cdot m_0$$

- Since m_{fuel} is now constant we can rewrite the constraint that makes sure that the satellite doesn't use up more fuel than it has as

$$m(t_k) \geq 0.8 m_0, \qquad k = 0, 1, \ldots, n$$

- There is a new positive constant p_{max}, which is a new upper bound for the variable $p(t_k)$. By allowing for p_{max} to exceed 1, we allow the maximum thrust to be greater than T, while keeping the same relationship between fuel consumption and thrust.

$$0 \leq p(t_k) \leq p_{max}, \qquad k = 0, 1, \ldots, n.$$

With these changes the mathematical formulation is given below:

$$
\begin{aligned}
\text{maximize}\quad & x_1(1) \\
\text{subject to}\quad & x_1(t_{k+1}) = x_1(t_k) + \Delta t \cdot \dot{x}_1(t_k), && k = 0, 1 \ldots, n-1, \\
& x_2(t_{k+1}) = x_2(t_k) + \Delta t \cdot \dot{x}_2(t_k), && k = 0, 1 \ldots, n-1, \\
& x_3(t_{k+1}) = x_3(t_k) + \Delta t \cdot \dot{x}_3(t_k), && k = 0, 1 \ldots, n-1, \\
& \dot{x}_1(t_k) = \frac{\mu t_f^2}{r_0^3} x_2(t_k), && k = 0, 1 \ldots, n, \\
& \dot{x}_2(t_k) = \frac{x_3(t_k)^2}{x_1(t_k)} - \frac{1}{x_1(t_k)^2} + \frac{r_0^2}{\mu} \frac{p(t_k)T}{m(t_k)} \sin u(t_k), && k = 0, 1 \ldots, n, \\
& \dot{x}_3(t_k) = -\frac{\mu t_f^2}{r_0^3} \frac{x_2(t_k) x_3(t_k)}{x_1(t_k)} + \frac{t_f \sqrt{r_0}}{\sqrt{\mu}} \frac{p(t_k)T}{m(t_k)} \cos u(t_k), && k = 0, 1 \ldots, n, \\
& m(t_{k+1}) = m(t_k) - \Delta t \cdot t_f |\dot{m}| p(t_k), && k = 0, 1 \ldots, n, \\
& x_2(1) = 0, \\
& x_3(1) = \frac{1}{\sqrt{x_1(1)}}, \\
& x_1(0) = 1, \\
& x_2(0) = 0, \\
& x_3(0) = 1, \\
& m(t_k) \geq 0.8 m_0, && k = 0, 1, \ldots, n, \\
& 0 \leq p(t_k) \leq p_{max}, && k = 0, 1, \ldots, n, \\
& -\frac{\pi}{2} \leq u(t_k) \leq \frac{\pi}{2}, && k = 0, 1 \ldots, n.
\end{aligned}
\tag{4}
$$

3.4 Other Discretization Methods

In the formulation above, we have used explicit Euler method, so that a variable $x(t_k)$ is calculated by the recursive formula

$$x(t_{k+1}) = x(t_k) + \Delta t \cdot \dot{x}(t_k)$$

This is not the only way of approximating the change of variables between time points. We could also use implicit Euler method, where each variable $x(t)$ is calculated by the formula

$$x(t_{k+1}) = x(t_k) + \Delta t \cdot \dot{x}(t_{k+1}),$$

or the trapezoidal method, where $x(t)$ is calculated by the formula

$$x(t_{k+1}) = x(t_k) + \Delta t \cdot \frac{\dot{x}(t_k) + \dot{x}(t_{k+1})}{2}$$

By changing the discretization method in our mathematical model, we can investigate whether or not the solution is sensitive to which discretization method we use.

4 Results

4.1 Results for when the Thrust is Constant

As mentioned in Section 3, the variable of primary interest when the thrust is constant is $u(t)$. We assume that the angle can be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because we want the direction of thrust to always be forward. A practical reason for this could be for example that the satellite only has jet engines on one side. Figure 2 shows the optimal $u(t)$ from time $t = 0$ to $t = 1$. The figure shows that the optimal strategy is to continuously increase the driving angle until the largest possible positive angle is reached, keep that angle until it needs to switch to the largest possible negative angle.

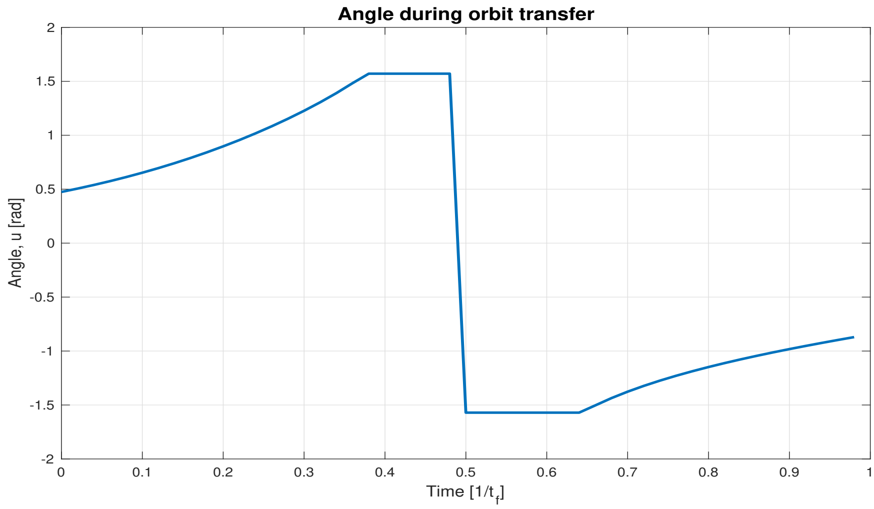

Figure 2: *The angle between the thrust and the tangential velocity direction for the satellite during the orbit transfer. The figure shows the angle on the vertical axis and the normalized time on the horizontal axis.*

Another result is how the radius changes over the time period. Figure 3 shows that the radius always increases during the time period, and that the slope of the graph is the steepest in the middle. The interpretation of the figure is that the satellite accelerates (in the radial component of the velocity) in the beginning and slows down in the end of the time period, while maintaining the highest velocity in the middle of the time period. The optimal normalized radius at time $t = 1$, when discretizing the problem into 500 time steps, is 1.512.

We can say several things about the optimality of the solution. The output of GAMS states "Optimal solution. Reduced gradient less than tolerance". This means that the solution is a local maximizer. More specifically, the solution is an interior point of the feasible set where all directions are feasible and descent. However, we cannot be sure if the solution is a global maximizer since the feasible set is not convex. One way to test for global optimality is to experiment with different values for which the GAMS solver starts its search, which we by default have set to 0 for $\dot{x}_1(t_k)$, $\dot{x}_2(t_k)$, $\dot{x}_3(t_k)$, $u(t_k)$. For example, we could set the starting value as a very large positive or negative number and examine if they converge to the same solution regardless of starting point. In our case we have tried by setting the starting value of $\dot{x}_1(t_k)$, $\dot{x}_2(t_k)$, $\dot{x}_3(t_k)$ to the extreme numbers -10000 and 10000, and the starting

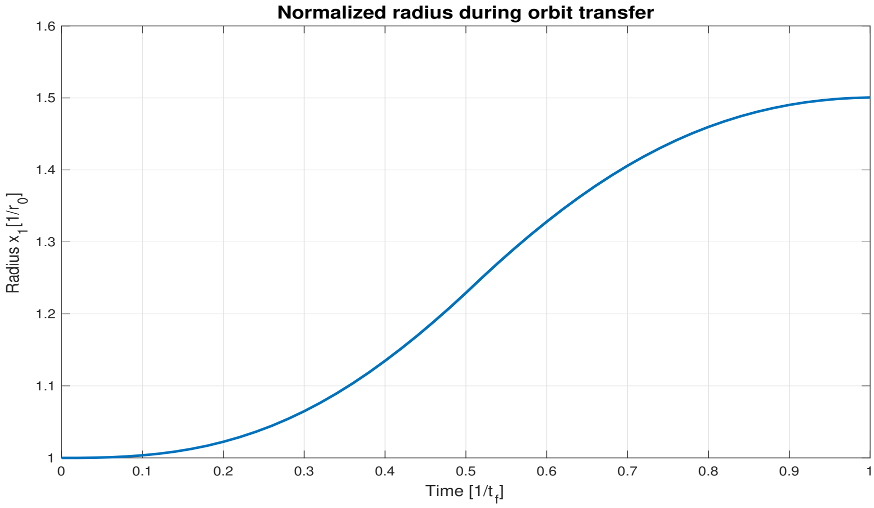

Figure 3: *The scaled radius of the satellite during the orbit transfer. The radius is shown on the vertical axis and the normalized time on the horizontal axis.*

value of $u(t_k)$ to the boundary values $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The tests do not yield the same optimal solution and optimal value, which means that we cannot conclude that the optimal solution in the results are globally optimal. However, none of the other given solutions yield a larger optimal value than the solution presented in the results, which means it could possibly be a globally optimal solution.

4.2 Results for when Fuel is Limited

We now consider the results for when we do not require the thrust to be constant, and examine the optimal value for some initial fuel values. Figure 4 shows that the optimal value increases as the initial fuel level increases. The figure shows that the optimal radius converges towards a value as the initial fuel tank is expanded. In fact, when the initial fuel level is 0.25 m_0 or higher, the optimal radius is 1.512, which is the same as when the thrust is constant.

Instead of only having one controllable variable, we can now control two variables: $u(t)$ and $T(t)$. Figure 5 shows the optimal angle $u(t)$ during the time period, and Figure 6 shows the optimal thrust, expressed by the proportionality constant $p(t)$, at each time.

When the initial fuel mass is $0.25m_0$ or more, then not only is the optimal radius the same as the constant case, but the optimal solutions is also the same, which is to always use the maximum thrust T. Figure 6 further shows that for all initial fuel masses, it is optimal to either use the maximum thrust or to not use any thrust at all. Since fuel consumption is affected by the thrust used, the results suggest that it is optimal to use the fuel at the beginning and at the end of the time period, while saving fuel in the middle of the period.

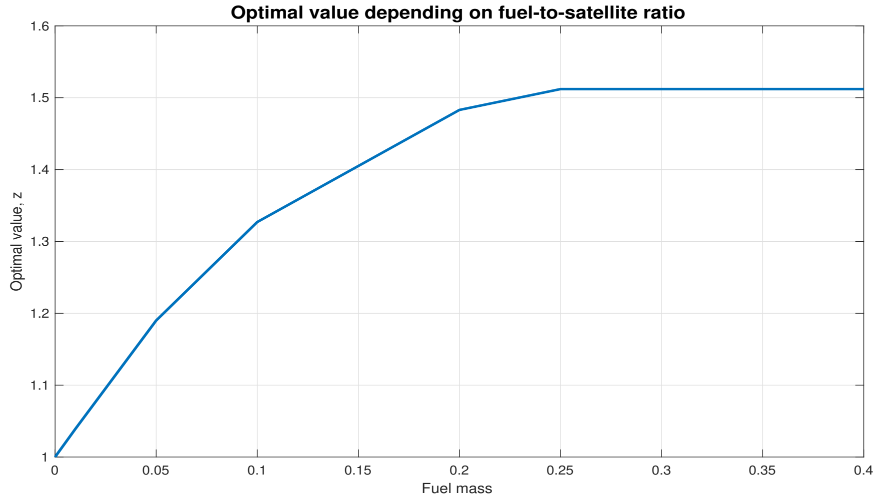

Figure 4: *The optimal value for different percentages of fuel mass. The vertical axis shows the optimal value and the horizontal axis shows the ratio of mass fuel to the initial mass of the satellite.*

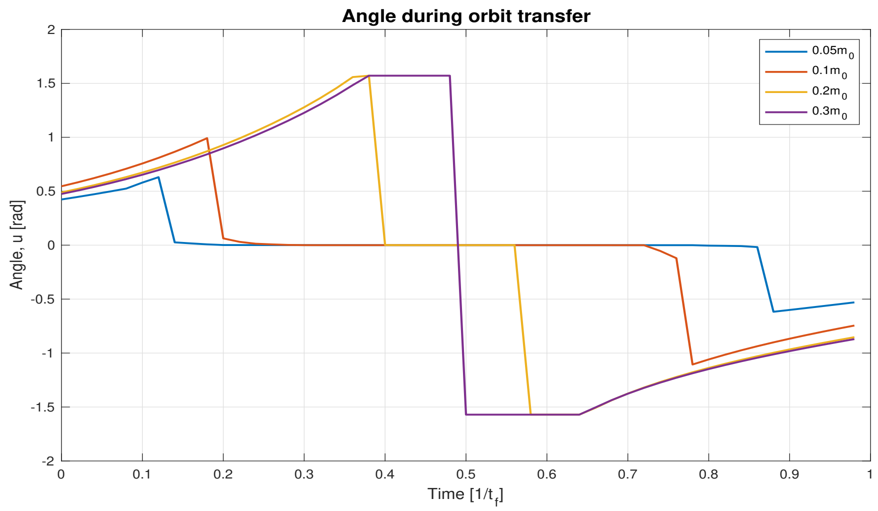

Figure 5: *The angle between the thrust and the tangential velocity direction for the satellite during the orbit transfer for different percentages of the fuel mass. The figure shows the angle on the vertical axis and the normalized time on the horizontal axis.*

4.3 Results for when Maximum Driving Force is Changed

For the results in this subsection, the initial fuel is fixed to $0.2m_0$, and p_{max} is unlimited. Figure 7 shows that the optimal radius seems to converge as we increase p_{max}, which is similar to when the fuel is limited. The pattern of the optimal solution is also very similar to the case when the fuel is limited. The program attempts to maximize the thrust at the beginning and at the end of the time period, possibly leaving a period in the middle where no thrust is applied. This is true for any p_{max}.

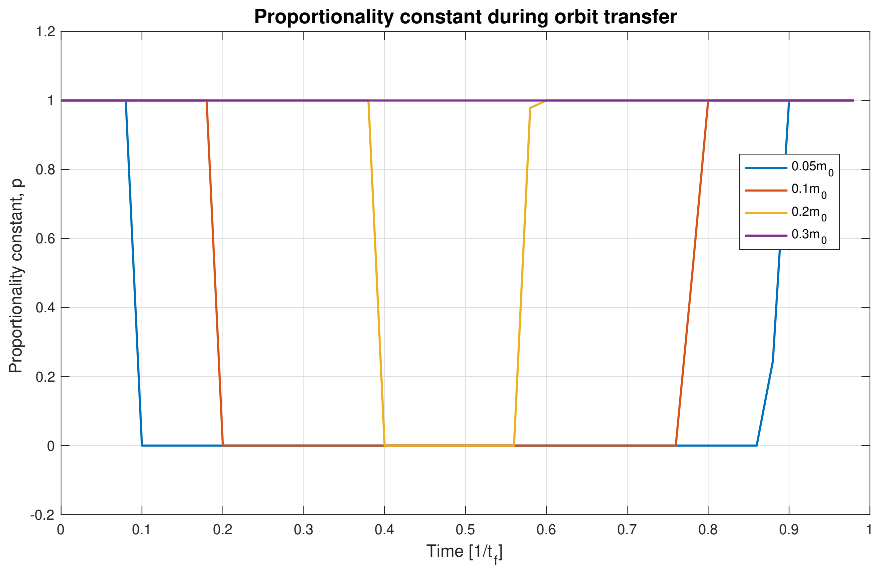

Figure 6: *The proportionality constant $p(t)$ for different percentages of fuel mass. The constant goes between 0 and 1 which regulates the thrust and fuel consumption accordingly.*

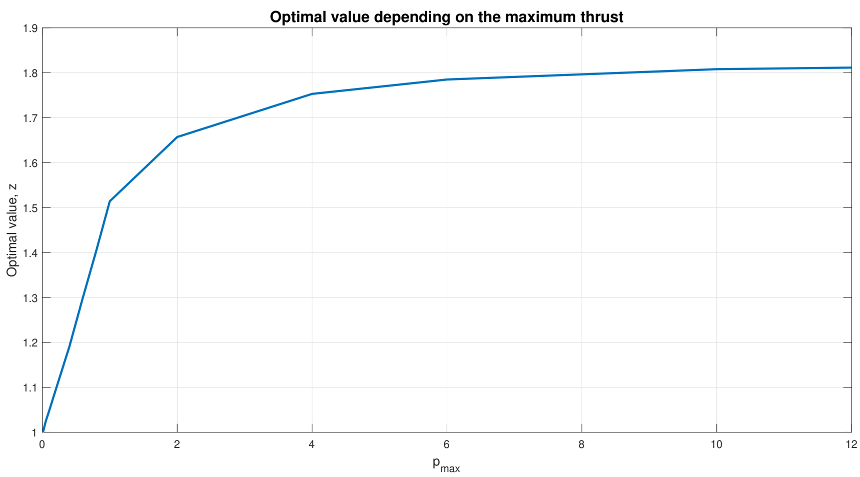

Figure 7: *The optimal value of the radius ratio for different maximal values of p_{max} which gives the thrust, $p(t)T$, and fuel consumption, $p(t)|\dot{m}|$.*

5 Discussion

5.1 Discussion of Results

In order to optimize the transition of the satellite from one orbit to another and maximize the radius of the new orbit, there are few things that are important to control and all work together, such as the angle of the satellite during the orbit transfer, the amount of fuel it has, and the value of the maximum driving force.

The results show that if the satellite has enough fuel, it will use maximum thrust throughout the transfer, in order to get the maximum radius of the new orbit. If the satellite does not have enough fuel to have the maximum thrust all the time, it is optimal to use maximum thrust in the beginning, no thrust in the middle, and maximum thrust in the end. Thus, it is never optimal to use less than maximum thrust, unless there is some leftover fuel that is not enough for maximal thrust. If the satellite can use the maximal thrust in the beginning, then it is able to maintain the velocity while it is not using any thrust.

When the fuel mass is finite we notice that the transfer is similar to the Hohmann transfer, where the initial burn is done in parallel to the tangential of the initial orbit. This changes the circular orbit into an elliptical orbit. When the satellite has reached the point furthest away from the sun, the second burn is done to change into the larger circular orbit. This explains why it is critical to apply thrust in the beginning and the end of the time period rather than in the middle of the time period.

If fuel capacity of the satellite is fixed but the maximum thrust is changeable, we can see that the optimal value increases quite fast with the lower values of $p(t)$ but converges for the higher values. When increasing either the maximum driving force or the initial fuel mass while keeping the other fixed, the optimal radius will increase but eventually converge or remain constant. This happens when either there is not enough fuel to use more thrust, or when the maximum thrust is used during the whole period. It seems that the maximum radius is constrained by either the initial fuel level or the maximum thrust. For engineers, this means that it is of little use to increase one property infinitely without improving the other part. Rather, the fuel capacity and engine thrust should be improved simultaneously to improve the radius.

5.2 Possible Parameter Substitutions

The derivatives $\dot{x}_1(t)$, $\dot{x}_2(t)$, $\dot{x}_3(t)$ can be written both in the form where the time is scaled to range from 0 to 1, and in a dimensionless form. In our solution we chose to disregard the dimensionless form and only use the form of scaled time. The dimensionless constants are given as

$$c_1 = \frac{Tr_0^2}{\mu m_0}, \quad c_2 = \frac{|\dot{m}|\sqrt{\mu}}{T\sqrt{r_0}} \quad \text{and} \quad c_3 = \frac{t_f\sqrt{\mu}}{\sqrt{r_0^3}},$$

and the corresponding equations are

$$\begin{aligned}
\dot{x}_1(t) &= c_3^2 x_2(t),\\
\dot{x}_2(t) &= \frac{x_3(t)^2}{x_1(t)} - \frac{1}{x_1(t)^2} + \frac{\sin u(t)}{1/c_1 - c_2 c_3 t},\\
\dot{x}_3(t) &= -c_3^2\frac{x_2(t)x_3(t)}{x_1(t)} + \frac{c_3\cos u(t)}{1/c_1 - c_2 c_3 t},
\end{aligned}$$

The reason why we decided to disregard this dimensionless formulation is because it only works when the thrust is constant, as in formulations (1) and (2). Once the thrust is not constant anymore, like in formulation (3) and (4), and we multiply it with the proportionality constant $p(t_k)$, we cannot guarantee that the thrust $T(k) = p(t_k)T$ does not become zero. If it does become zero, then $c_1 = 0$ and c_2 will divide by zero, which causes errors.

5.3 Discretization Sensitivity

Even though we know the time derivatives of the variables $x_1(t)$, $x_2(t)$, $x_3(t)$ precisely for all time points, we can only approximate the actual change between time points. This is because we can only discretize the time into a finite number of time periods, and the actual change between these time periods can only be approximated. It can be useful to investigate how different types of ways to approximate the actual change between the discretized periods affect the optimal solution. This is done in Figure 8, where the optimal radius using explicit Euler method, implicit Euler method, and trapezoidal method are shown, for some different number of discretization steps. In the graph, the corresponding problem solved is when $m_{fuel} = 0.2m_0$ and the maximum thrust is T.

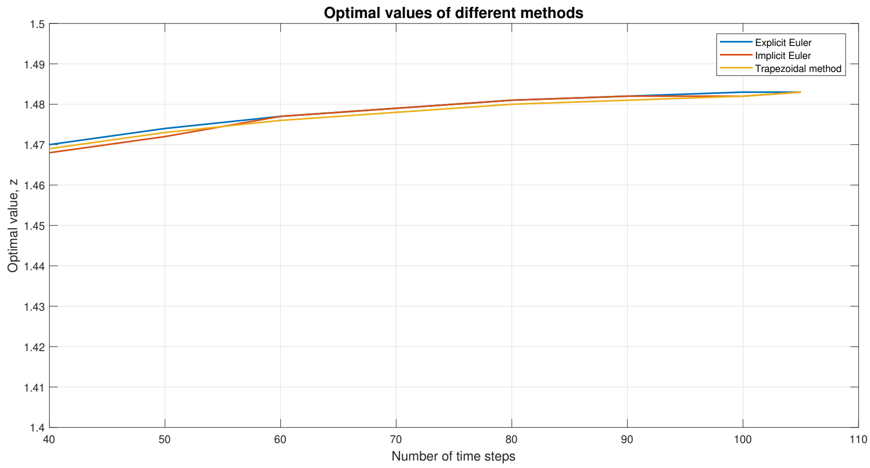

Figure 8: *Optimal value for different choices of discretization methods.*

The difference in the optimal value for the three different discretization methods do not seem to be significant. This means that the problem is not very sensitive to the choice of discretization model. However, Figure 8 shows that the optimal value increases with the number of steps and does not seem to yet converge. This suggests that the number of steps included in the graph is not sufficient to get a precise value. Table 2 shows the optimal value for the same problem using explicit Euler method for some different numbers of time steps. The table suggests that an appropriate number of time steps to use is between 200 and 400, which yields a results accurate to three decimals.

Table 2: *The optimal value for some different numbers of time steps*

Number of steps	Optimal value
50	1.474
100	1.483
200	1.487
400	1.487

6 Conclusion

This report optimizes the radius of a satellite transferring from one inner orbit to another outer orbit. The problem is discretized and formulated as a nonlinear programming problem. The problem has two primary optimizable variables that are of interest: The thrust and the driving angle. This report proposed a control strategy that optimizes the radius at the end of the time period.

When the driving force has a constant value T, then the scaled optimal radius is 1.512. The report also presents what happens if the fuel capacity and maximum thrust of the satellite are changed. The results show that, if the fuel is insufficient for using the maximum thrust for the whole period, then it is optimal to use the maximum thrust in the beginning and the end of the time period, while using no thrust in the middle. Otherwise, it is optimal to always use the maximum thrust. The result also shows that to improve the satellite in the best way, engineers should seek to improve the fuel capacity and the maximum thrust simultaneously.

Lastly, a discussion is made on the use of different numerical methods to approximate the derivatives of the satellites motion. It is concluded that the choice of numerical method is not significant, but that it is important to discretize the time into a sufficient number of time steps. Sending an object far away in space is complex and requires intensive calculations. Thus, getting the numbers correct can save both money and lives in the future.