Advanced Dynamical Systems (MATH60146/70146)

Final Project

Project issued on: February 4, 2025
Due date: March 21, 2025

Term: Spring 2024-25 Max mark: 40 Points

The Lu-Chen System

The *Lu-Chen system* is a three-dimensional continuous-time autonomous dynamical system that exhibits chaotic behavior. It was introduced as a modified version of the classic Lorenz and Chen systems, aimed at exploring alternative chaotic attractors and bifurcation phenomena. The governing equations are:

$$\begin{aligned}\dot{x} &= -ax + ay,\\ \dot{y} &= -xz + cy + x + u,\\ \dot{z} &= xy - bz.\end{aligned}$$

where a, b, c, u are system parameters. Depending on their values, the system undergoes a transition from periodic behavior to chaos via period-doubling bifurcations. The primary objective of this project is to analyze these transitions and understand chaotic properties through computational simulations.

Each student is assigned a unique set of parameters and initial conditions (see the last section).

Part 1: Bifurcation and Chaos Analysis (15 points)

1. *Phase Space and Sensitivity to Initial Conditions* (5)
   (a) Simulate the Lu-Chen system for your assigned parameters using two nearby initial conditions:

   $$(x_0, y_0, z_0) \quad \text{and} \quad (x_0 + \Delta, y_0, z_0)$$

   where Δ is a small perturbation provided in your assigned parameters.

   (b) Plot:
   - A *3D phase space plot* comparing both trajectories.
   - *Time series plots* for $x(t)$, $y(t)$, and $z(t)$ showing how small changes in initial conditions grow over time.

   (c) Discuss:
   - How does a small difference in the initial condition evolve over time?
   - What does this indicate about the system's sensitivity to initial conditions?

2. *Lyapunov Spectrum and Chaos* (5)
   (a) Compute the *Lyapunov exponents* for your assigned parameters. Interpret your results.

   (b) Compare the largest Lyapunov exponent with your phase space plots. Does a positive exponent confirm chaotic behavior?

   (c) Sweep parameter a from 23 to 36 and compute the Lyapunov exponent at each step. Explain the graph and distinguish periodic and chaotic regimes.

3. *Bifurcation Analysis via Poincaré Sections* (5)

   (a) Sweep parameter a from 23 to 36 in steps of 0.1. Plot the bifurcation diagram using Poincaré sections.

   (b) Compare and justify your results with the diagram of Lyapunov exponents in Question 2(c). Describe the observed transitions from periodic behavior to chaos.

   (c) Identify any *period-doubling cascades*. How do they relate to classical routes to chaos?

Part 2: Synchronization and Data-Driven Analysis (25 points)

1. *Secure Communication via Chaotic Synchronization* (10)

   Chaotic synchronization can be used for secure communication by embedding a message into a chaotic signal. Alice wants to send a hidden message to Bob. However, Alice can only transmit a single signal. Bob, knowing Alice's system parameters, must extract the hidden message using synchronization.

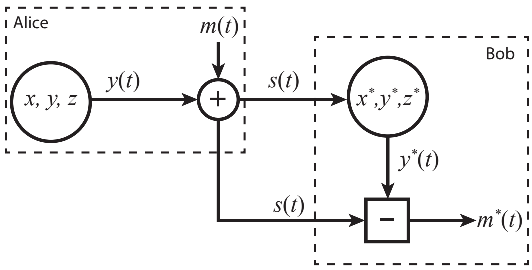

Figure 1: Schematic of the secure communication algorithm using chaotic synchronization.

   (a) Simulate Alice's Lu-Chen system using your assigned parameters to generate the master trajectory. Define a message signal $m(t)$ and embed it into the transmitted signal:

   $$y_{\text{transmit}} = y + m(t).$$

   The message signal should be structured as alternating flat and wave intervals, ensuring that Bob can distinguish the transmitted message.

   Suggested message function:

   $$m(t) = 0.1 \left(\frac{\sin(1.2\pi \sin t)^2}{\pi \sin^2 t} \right) \cos(10\pi \cos(0.9t)) + \text{noise}.$$

   (b) Extract the recovered message from Bob's end. **Hint:** You must introduce a suitable coupling term to synchronize Bob's system. You can first study the system without a message to ensure synchronization.

   (c) Analyze and visualize:
   - Plot the original and recovered message $m(t)$ vs. $m^*(t)$.
   - Show synchronization error $|y_A - y_B|$ over time.
   - Discuss recovery accuracy and limitations. **Hint:** The recovered message may not be perfectly identical to the original due to noise and chaos, but it should resemble the original structure. One can apply a simple thresholding technique to binarize the recovered message for digital communication.

2. *Network Reconstruction from Data* (15)

   You are given time-series data, *network_dynamics_data.txt*, obtained from a network of $N = 5$ interacting dynamical systems, each evolving according to a three-dimensional system of equations. The interactions between the systems occur via *diffusive coupling*, but the specific coupling type and network topology are unknown. Assume that the coupling strength and function are identical for all connections.

Data Description: The provided dataset consists of time-series measurements of the five coupled systems. The data is structured as follows:

- The first column represents time (in seconds).
- The following columns contain the state variables:

$$x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_5, y_5, z_5.$$

Your task is to:

- **Reconstruct the network topology** (i.e., determine the adjacency matrix).
- **Identify the coupling function** (e.g., identity matrix, sinusoidal coupling, x-coupled, etc.).
- **Estimate the coupling strength**.
- **Clearly explain your approach** for estimating the topology, function, and strength of the coupling.
- **Discuss potential real-world examples** of coupled dynamical systems where such an interaction scheme might occur.

Hints and Notes:

- **On Numerical Derivatives:** Since you will use numerical derivatives, the reconstructed system will not produce an exact match to the time series when simulated. However, the detected connections should be correct, as the system is not synchronized.
- **On Methodology:** You can use any reconstruction method, such as **Compressed Sensing**, **Lasso**, or **SINDy**. Depending on the methodology, the recovered system may vary slightly. Small numerical errors are acceptable and should not significantly affect the final topology reconstruction.
- **On Justification:** It is not sufficient to simply present the reconstructed topology properties. You must clearly show how each step in your method led to the final results. Provide explanations, intermediate results, and visualization techniques that support your findings if available.

Submission Guidelines:

- Submit a single PDF document containing:
  - All code, figures, and analysis.
  - Answers to theoretical questions.
  - Your unique parameter assignments.

Best wishes

Student Parameter Assignments

Each student is assigned unique parameters a, b, c, u and two nearby initial conditions:

Username	a	b	c	u	x_0 perturbation
mlh124	25.00	3.01	20.00	-15.10	± 0.001
kbe24	25.10	3.02	20.15	-15.12	± 0.001
hl4124	25.20	3.00	20.30	-15.14	± 0.001
jkt24	25.30	3.03	20.45	-15.16	± 0.001
eg1123	25.40	2.99	20.60	-15.18	± 0.001
el921	25.50	3.04	20.75	-15.20	± 0.001
yz9621	25.60	3.05	20.90	-15.22	± 0.001
ch1121	25.70	3.01	21.05	-15.24	± 0.001
aw721	25.80	3.00	21.20	-15.26	± 0.001
dlz21	25.90	2.98	21.30	-15.28	± 0.001
cl3821	26.00	3.02	21.00	-15.30	± 0.001
rk720	26.10	3.06	20.85	-15.32	± 0.001
ww1021	26.20	3.04	20.70	-15.34	± 0.001
ra921	26.30	3.02	20.55	-15.36	± 0.001
ab3321	26.40	3.01	20.40	-15.10	± 0.001
gv221	26.50	3.00	20.25	-15.12	± 0.001
qy421	26.60	3.03	20.10	-15.14	± 0.001
yf1021	26.70	3.05	20.00	-15.16	± 0.001
al2021	26.80	2.99	20.15	-15.18	± 0.001
bm221	26.90	3.04	20.30	-15.20	± 0.001
djp20	27.00	3.06	20.45	-15.22	± 0.001
ep1119	27.10	3.02	20.60	-15.24	± 0.001
yh2420	27.20	3.01	20.75	-15.26	± 0.001
av520	27.30	3.00	20.90	-15.28	± 0.001