Advanced Dynamical Systems (MATH60146/70146)

Coursework 3

Homework issued on: February 7, 2025
Due date: February 14 ♡, 2025

Duration: 1 week | Spring 2024-25 | Max mark: 12 Points

Objective: This coursework focuses on extracting a discrete return map from the Lorenz system using Poincaré sections.

1. **Extracting a Return Map from the Lorenz System** (12)

The **Lorenz system** is a well-known chaotic system introduced by Edward Lorenz in 1963. The following equations govern it:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(\rho - z) - y, \quad \dot{z} = xy - \beta z.$$

where $(\sigma, \rho, \beta) = (10, 28, \frac{8}{3})$ are system parameters.

(a) Simulate the Lorenz system over 2000 seconds and visualize the trajectory in 3D. Additionally, plot 2D projections in the (x, y), (x, z), and (y, z) planes. Initial condition: $(x_0, y_0, z_0) = (1.0, 1.0, 1.0)$.

(b) Instead of defining a traditional Poincaré section, we adopt an alternative approach: *Locate all local maxima of $z(t)$.* This approach captures the points where the trajectory reaches the top of each oscillation. Store the sequence of z-maxima: z_n, $z_{n+1}, \ldots$.

Hint: Traditional Poincaré sections might be difficult to define when the system exhibits complex structures, such as multiple scrolls in the Lorenz attractor. A well-known technique in such cases is to extract extremal values (local maxima or minima) of a single variable instead.

(c) Construct a return map by plotting z_{n+1} versus z_n.

(d) Fit a function to approximate the iterated map. Instead of a global polynomial fit, use a *piecewise approach*: - Identify the peak of the return map (the largest z_{n+1}). - Fit a quadratic function separately to the left and right branches of the return map.

Hint: Since the Lorenz system has two distinct scrolls, the sequence of maxima z_n is generated from oscillations between both scrolls. As a result, fitting the entire return map with a single function is not straightforward. The structure is better approximated using a piecewise function, where separate quadratic functions model the left and right branches.

(e) Print the reconstructed piecewise function in the following form:

$$z_{n+1} = \begin{cases} a_{\ell,2} z^2 + a_{\ell,1} z + a_{\ell,0}, & \text{if } z_n \leq z_{\max} \\ a_{r,2} z^2 + a_{r,1} z + a_{r,0}, & \text{if } z_n > z_{\max} \end{cases}$$

where $z_{\max}$ is the peak of the return map, and $(a_{\ell,2}, a_{\ell,1}, a_{\ell,0})$ and $(a_{r,2}, a_{r,1}, a_{r,0})$ are the coefficients for the left and right branches, respectively.

(f) Iterate the reconstructed function starting from two nearby initial conditions, $z_0^{(1)}$ and $z_0^{(2)}$, and plot their time evolution over 100 iterations.

Questions to Explore:

- How does the iterated sequence behave over time?
- What does this return map reveal about the chaotic structure of the Lorenz system?
- Compare it to the tent map: Does the system exhibit a similar sensitivity to initial conditions?

2. **Bonus Question*: Effect of Parameter Changes on the Return Map**

   The behavior of the Lorenz system is highly sensitive to its parameters. Consider modifying the Lorenz system by slightly changing the parameters, keeping the system within the chaotic regime. Compare the fitted polynomial function across different parameter values. Address the following:

   - How do the coefficients of the polynomial change?
   - Does the overall shape of the return map change significantly?
   - Can you intuitively infer any relationship between the dynamical regime and the return map structure?

   *Bonus questions are designed as an optional challenge for students who wish to explore deeper insights into chaotic dynamics. While they are primarily for enrichment, they can also contribute to the final grade in specific cases. If a student's total score is very close to the next grade boundary (within 1-2 points), completing the bonus question may help bridge the gap. However, bonus questions cannot contribute more than 2 additional points in total.

Submission Guidelines: Submit a single PDF document containing the codes, solutions, mathematical foundations, justifications and figures.

Best wishes