PUMaC 2007

devo1ution

July 2022

1 Problem

(PUMaC) Given two sequences x_n and y_n defined by $x_0 = y_0 = 7$,

$$x_n = 4x_{n-1} + 3y_{n-1}\text{and}$$
$$y_n = 3y_{n-1} + 2x_{n-1},$$

find $\lim_{n\to\infty} \frac{x_n}{y_n}$.

2 Solution

Writing out the formulas for x_1, x_2, y_1 and y_2 one might observe that x_i can be written as $ax_0 + (a-1)y_0$ and y_i can be written as $ay_0 + (a-1)x_0$. Let's try to construct a sequence a such that $x_n = a_i x_{n-i} + (a_i - 1)y_{n-i}$. Our base case is $a_1 = 4$. We can use induction for $n \in [2, \infty)$:

$$\begin{aligned} x_n &= a_{n-1}x_1 + (a_{n-1} - 1)y_1 \\ &= a_{n-1}(4x_0 + 3y_0) + (a_{n-1} - 1)(3y_0 + 2x_0) \\ &= (6a_{n-1} - 2)x_0 + (6a_{n-1} - 3)y_0 \\ &\Rightarrow a_n = 6a_{n-1} - 2 \end{aligned}$$

Because the two coefficients are a_n and $a_n - 1$, using this sequence is a valid way to compute x_n.

We can construct a similar sequence b for y_n:

$$\begin{aligned}
y_n &= b_{n-1}y_1 + (b_{n-1} - 1)x_1 \\
&= b_{n-1}(3y_0 + 2x_0) + (b_{n-1} - 1)(4x_0 + 3y_0) \\
&= (6b_{n-1} - 3)x_0 + (6b_{n-1} - 4)x_0 \\
&\Rightarrow b_n = 6b_{n-1} - 3
\end{aligned}$$

In order to get a formula for a_n, we can expand out the terms to get $4 * 6^{n-1} - 2 * (6^{n-2} + 6^{n-3}...1)$. The second term is simply $2 * \frac{6^{n-1}-1}{5}$, so the whole thing is $\frac{18*6^{n-1}+2}{5} = \frac{3*6^n+2}{5}$. We can get a similar formula for b_n: $b_n = \frac{2*6^n+3}{5}$. Now $\frac{x_n}{y_n} = \frac{7(2a_n-1)}{7(2b_n-1)} = \frac{2a_n-1}{2b_n-1} = \frac{6*6^n-1}{4*6^n+1}$. As n becomes larger the constant terms become negligible, so this fraction is simply $\frac{3}{2}$.