A model for the solar system using ordinary differential equations

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October 26, 2020

Abstract

In this project we determine numerically the paths of the planetary bodies in the solar system, using the Euler and velocity Verlet methods. Various parameters and known constants are tested, such as the escape velocity of the Earth from the gravitational field of the Sun which is found to correspond accurately with the analytic value. The three-body-problem with Jupiter is simulated with different masses of Jupiter, transitioning into a binary star-like system at a mass $1000M_j$. A generalized force of gravity is simulated for stability, corresponding largely with theory. We simulate and plot the perihelion precession of Mercury with and without relativistic correction, and find a correspondence with the observed values to an error of $\approx 10''$. The code has been created in an object oriented way, to ensure re-usability of functions and to learn better ways to structure code. Further work could involve generalizing the code even more to be applicable to for instance molecular dynamics.

1 Introduction

The solar system is a complicated system with many interacting bodies, and it is impossible to solve analytically for instance the path of every object in the solar system. However, this complexity allows for a numerical approach in the solutions. In particular, the planetary motions in the solar system can be modelled as coupled differential equations, and these equations can be written in general forms, such that the solutions can be found using general numerical methods. Differential equations are ubiquitous in physics, appearing in almost every field in some fashion, and so the methods used in this project have wide-ranging applications.

Another important aspect of this project is object oriented programming. We will work to generalize the code used so that it may be reused in different parts of both this project and possibly another. The system looked at here is particularly suited to object oriented programming as we need to work on multiple sets of systems, with just Earth-Sun, Earth-Sun-Jupiter, the entire Solar system, the Mercury-Sun system with relativistic corrections, and the Earth-Sun system with a generalized force of gravity. Rewriting code for each and every one of these systems would take a large amount of time, and also be prone to errors. Furthermore, if the code is generalized enough, we could reuse large parts of it for other similar systems, such as molecular dynamics where the forces are also conservative and therefore of a similar form.

We will solve first the Earth-Sun system, ignoring all other planets and objects, using the Euler method and the velocity Verlet method, comparing the two solutions and their performance. Then, having made code that can simulate a specific system, testing various quantities such as the escape velocity and conservation of angular momentum to ensure that the algorithms work well, we will extend the framework to a three-body problem, and further to the entire solar system. Finally we check the perihelion precession of Mercury, relating to relativistic calculations of gravity.

First we review some important astronomical and mechanical laws, before explaining the method with which the solar system is simulated. We present images of the simulated solar system, testing escape velocity, different forms of the gravitational force, and the perihelion precession of Mercury with relativistic corrections. We compare the Forward Euler and Velocity Verlet method on stability and energy conservation.

2 Theory

2.1 The differential equations

Newton's gravitational law governs the trajectory of the objects in the solar system, and is given by

$$\vec{F}_{ij} = G\frac{M_i M_j}{r^2} \tag{1}$$

where G is the gravitational constant, M_i and M_j are the masses of the considered objects and r is the distance between them. By employing Newton's second law of motion the acceleration due to gravitational pull is given by

$$\vec{a} = GM_i\frac{x}{r^3}\vec{r}, \tag{2}$$

where $\vec{r}$ gives the position of the body.

We now consider Newton's second law of motion on the form

$$m\frac{d^2x(t)}{dt^2} = -kx(t), \tag{3}$$

where k is the force constant.

We rewrite equation (3) using $x(t) = y^{(1)}(t)$ and $\frac{dy^{(1)}(t)}{dt} = y^{(2)}(t)$ such that

$$my^{(2)}(t) = -ky^{(1)}(t), \tag{4}$$

Suppose we have an initial value for the function $y(t)$ given by $y_0 = y(t = t_0)$. Then, using the step size $h = \frac{b-a}{N}$ for the space $[a, b]$ with N defining the number of steps needed to go from a to b, we can move one step along $y(t)$ to find

$$y_1 = y(t_1 = t_0 + h) \tag{5}$$

and so on. Depending on how well behaved the function is in the defined domain $[a, b]$, there might be a need for adaptive steps, however, in our case, a fixed step size will suffice.

Next we will generalize equation (5), by writing in terms of steps, so that we get

$$y_{i+1} = y(t = t_i + h) = y(t_i) + h\Delta(t_i, y_i(t_i)) + \mathcal{O}(h^{p+1}), \tag{6}$$

where $\mathcal{O}(h^{p+1})$ is the truncation error. To determine Δ, we can Taylor expand (6), which gives us

$$y_{i+1} = y(t = t_i + h) = y(t_i) + h\left(y'(t_i) + ... + y^p(t_i)\frac{h^{p-1}}{p!}\right) + \mathcal{O}(h^{p+1}) \tag{7}$$

2.2 Conservation of angular momentum

Kepler's 2nd law,

> A radius vector joining any planet to the sun sweeps out equal areas in equal lengths of time [1],

can be used to show conservation of angular momentum. We consider a small wedge of the orbit, as can be seen in figure 1.

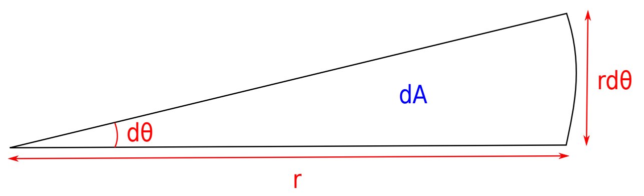


Figure 1: Wedge of Earths orbit.

The area of the wedge is given by

$$dA = \frac{1}{2}rrd\theta.$$

The rate at which area is swept out in the orbit is then

$$\frac{dA}{dt} = \frac{1}{2}rr\frac{d\theta}{dt} = \frac{1}{2}rv_\theta = \text{const}.$$

By inserting the definition of angular momentum, $L = mrv_\theta$, in the equation above, we get

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{m} = \text{const}.$$

Which shows that the angular momentum is conserved.

2.3 Circular Orbits

For the Earth-Sun system, a circular orbit means that the distance between the center of mass of the Sun and the Earth is equal to a constant radius $r_E = 1\text{AU}$. Thus the Newtonian gravitational potential energy

$$E_p(r) = -\frac{GM_\odot M_E}{r} \tag{8}$$

will remain constant for a circular orbit. Since the overall sum of the kinetic energy and potential energy of a system is conserved, conservation of the potential energy also implies conservation of kinetic energy, since in the inertial frame of the Sun, the kinetic and potential energy of the Sun is always constant.

The force between the Earth and the Sun is given by

$$F = M_E a = \frac{GM_\odot M_E}{r^2}. \tag{9}$$

Using the fact that for a circular orbit $a = \frac{v^2}{r}$, we can rewrite (9) to find

$$\begin{aligned} \frac{v^2}{r} &= \frac{GM_\odot}{r^2} \\ v &= \sqrt{\frac{GM_\odot}{r}}. \end{aligned} \tag{10}$$

This can also be used to find a convenient expression for $GM_\odot$. Since $v_E = 2\pi \frac{\text{AU}}{\text{Year}}$, we can use equation (10) to find

$$GM_\odot = 4\pi^2 \frac{\text{AU}^3}{\text{Year}^2} \tag{11}$$

This lends itself to a natural length and time scale of the system, using AU and years, leading to simplified equations.

2.4 Escape velocity

The kinetic energy of a body escaping the gravitational influence of another is given by

$$K = \frac{1}{2} m v_e^2, \tag{12}$$

where m is the mass of the the escaping body and v_e is the escape velocity.

If we then consider the force between two objects $F = G\frac{Mm}{r^2}$, then the work needed to move an infinitesimally small distance dr is given by $dW = F dr$. The work needed to move from a distance r_0 to a distance infinitely far away can so be found by

$$W = \int_{r_0}^{\infty} F\, dr = \int_{r_0}^{\infty} G\frac{Mm}{r^2}\, dr = \left[-\frac{GMm}{r} \right]_{r_0}^{\infty} = \frac{GMm}{r_0}, \tag{13}$$

were we define M as the mass of the stationary object, r as the distance between the two objects and G as the gravitational constant.

To escape the gravitational influence of another body, the kinetic energy of the escaping body needs to be equal to the work required to move against the gravity of the stationary object from r_0 to $r = \infty$. That means that $K = W$, such that

$$\frac{1}{2} m v_e^2 = \frac{GMm}{r_0} \tag{14}$$

$$v_e^2 = \frac{2GM}{r_0} \tag{15}$$

$$v_e = \sqrt{\frac{2GM}{r_0}}, \tag{16}$$

which is the solution for the escape velocity. For the Earth-Sun system, inserting $r_0 =$ 1AU, $M = M_\odot$ and equation (11) yields $v_e = 2\sqrt{2}\pi \frac{\text{AU}}{\text{yr}}$.

2.5 Generalization of Newton's gravitational law

The modified gravitational force is given by

$$\vec{F} = G\frac{M_{\odot}M_{\text{Earth}}}{r^{\beta}}\vec{e}_r \tag{17}$$

where $\vec{r}$ is the position vector of the Earth relative to the sun and with $\beta \in [2, 3]$. The corresponding potential can then be written

$$V = \frac{1}{\beta - 1}\frac{M_{\odot}M_{\text{Earth}}}{r^{\beta-1}} = \frac{1}{\alpha}\frac{GM_{\odot}M_{\text{Earth}}}{r^{\alpha}} \tag{18}$$

$\alpha = \beta - 1$

The position vector of the Earth is given by

$$\vec{r} = r(\cos\theta, \sin\theta).$$

We define the Lagrangian of the system, given by

$$\mathcal{L} = \frac{1}{2}M_{\text{Earth}}(\dot{r}^2 + r^2\dot{\theta}^2) - \left(-\frac{1}{\alpha}\frac{GM_{\odot}M_{\text{Earth}}}{r^{\alpha}}\right).$$

In appendix A.2 we use conservation of total energy and angular momentum to show that the Lagrangian can be reduce to a one dimensional Lagrangian

$$\mathcal{L} = \frac{1}{2}M_{\text{Earth}}\dot{r}^2 - V_{\text{eff}}(r),$$

where the effective potential is given by

$$V_{\text{eff}}(r) = \frac{1}{2}\frac{l^2}{M_{\text{Earth}}}\frac{1}{r^2} - \frac{1}{\beta - 1}\frac{GM_{\odot}M_{\text{Earth}}}{r^{\alpha}}.$$

In appendix A.2, we also show that we must have $\beta < 3$ to have a stable equilibrium for Earths orbit.

2.6 Perihelion precession

Adding a relativistic correction to the Newtonian gravitational force will change the orbit of the object, which in general is not closed without the inverse square form of the Newtonian gravitational force. For small corrections, this means that the object will follow an almost elliptical orbit, which can be thought of as an elliptical orbit that rotates slowly. Mercury, as the nearest planet to the sun, experiences the greatest perihelion drift. The observed perihelion shift is 43 arc-seconds per century [2]. We add a relativistic correction to the general Newtonian gravitation equation equation (1) and find the corrected gravitational force on Mercury, given by

$$F_G = \frac{GM_\odot M_{\text{Mercury}}}{r^2}\left(1 + \frac{3l^2}{r^2c^2}\right), \tag{19}$$

with $l = |\vec{r} \times \vec{v}|$, the angular momentum per unit mass and c the speed of light in a vacuum.

The angle of the perihelion can be calculated by

$$\theta_p = \arctan\left(\frac{y_p}{x_p}\right),$$

where y_p and x_p is the position of Mercury at perihelion.

3 Method

3.1 Algorithms

There are two numerical iterative methods used in this project to compute the path of the celestial bodies, the forward Euler and the velocity Verlet methods. Following is a short explanation of both methods.

3.1.1 Forward Euler

In this project we are working with coupled differential equations of the form

$$\begin{aligned} x'(t) &= v(t, x) \\ v'(t, x) &= a(t, x), \end{aligned} \tag{20}$$

where x is the position of a celestial body, v is the velocity of this body, and a is the acceleration. We know that for a conservative force such as gravity, there is no dependence on the velocity of the body, and thus the acceleration is also independent of the velocity. We discretize the functions $x(t), v(t, x), a(t, x)$ with the time values $t = \{t_0, t_1, \ldots, t_n\}$, such that $\frac{t_n - t_0}{n} = h$ and $t_{i+1} = t_i + h$. Now we canuse equation (7) with $p = 1$, expanding the discretized functions $x(t_i) = x_i$ and $v(t_i) = v_i$ as

$$\begin{aligned} x_{i+1} =& x_i + hx_i' + \mathcal{O}(h^2) \\ v_{i+1} =& v_i + hv_i' + \mathcal{O}(h^2). \end{aligned}$$

Using the differential equations equation (20) and suppressing the truncation error, we can write this as

$$\begin{aligned} y_{i+1} =& y_i + hv_i \\ v_{i+1} =& v_i + ha_i. \end{aligned} \tag{21}$$

With equation (21), simulating the solar system is possible, albeit with an error in the form of conservation of energy. Euler's method does not conserve energy, but with a simple modification, we obtain the Euler-Cromer method [3],

$$
\begin{aligned}
y_{i+1} =& y_i + hv_{i+1} \\
v_{i+1} =& v_i + ha_i,
\end{aligned} \tag{22}
$$

and this does conserve the energy of the system. In both equation (21) and equation (22) we suppress the truncation error $\mathcal{O}(h^2)$. When computing a large number of steps $n \propto \frac{1}{h}$, the total error is of the order $\mathcal{O}(h)$.

3.1.2 Velocity Verlet

The velocity Verlet method is a commonly used integration algorithm which calculates the velocity and position at the same value of the time variable [4]. It is frequently used to calculate trajectories of particles in molecular dynamics simulations, and is also well suited to calculate the trajectories of our celestial bodies. The method is easy to implement and offers good numerical stability, as well as other properties that are important in physical systems such as time-reversibility and area preserving properties.

The velocity Verlet method can be represented by the recursive relation

$$
\begin{aligned}
y_{i+1} =& y_i + hv_i + h^2 a_i + \mathcal{O}(h^3) \\
v_{i+1} =& v_i + \frac{h}{2}(a_{i+1} + a_i) + \mathcal{O}(h^3).
\end{aligned} \tag{23}
$$

note that a_{i+1} depends on the position y_{i+1}. Thus, we have to calculate the position at the updated time t_{i+1} before we compute the next velocity.

The advantage of the velocity Verlet method over the Euler-Cromer method is its accuracy. The error in equation (23) is $\mathcal{O}(h^3)$, such that when accumulating over many steps $n \propto \frac{1}{h}$, the total error is $\mathcal{O}(h^2)$, which is smaller than for the Euler-Cromer method above. The velocity Verlet method also conserves energy as a symplectic method [4].

3.2 Comparison of Algorithms

From equation (21) and equation (22) we see that the Euler and Euler-Cromer method both require 4 floating point operations (FLOPs) for each point for each dimension, in addition to the number of FLOPs from calculating the acceleration. The acceleration requires 6 FLOPs as seen in equation (2), but this is only true for a 2-body problem. Additional planets will add 6 FLOPs each to the calculation of each planet's acceleration. In total then, the Forward Euler and Euler-Cromer methods require 10 FLOPs per dimension.

Meanwhile, the velocity Verlet method requires 8 FLOPs from equation (23), assuming h^2 and $\frac{h}{2}$ are precalculated. Furthermore, v_{i+1} requires both a_i and a_{i+1}, but a_{i+1} can be saved in the memory to be used for the calculation of the next point. In total therefore, the Verlet method should require 14 FLOPs per dimension. In our program, we have left out certain precalculations as well as calculating the force separately from acceleration, which requires an additional two FLOPs to add and remove the mass of each planet. Thus, the expected time differences will be somewhat different, as each point of acceleration requires an additional two FLOPs, the values h^2 and $\frac{h}{2}$ are not precalculated and the acceleration is computed twice for each point. Thus the true number of FLOPs in our program should be 12 for the Euler methods, and 26 for the velocity Verlet method.

3.3 Implemented classes/object orientation

In this work, we make use of the advantages of writing object oriented code by implementing classes. We implement the `planet` class for setting up the planets. Further, we implement the `solver` class for solving n-body problems for the particular differential equation of particles (planets) interacting in only a conservative gravitational field, using either the velocity Verlet, Euler's forward or Euler-Cromer method.

3.4 Data

The initial conditions used to start the differential equation solver is extracted from the HORIZON Web-Interface, provided by the Jet Propulsion Laboratory (NASA) at the California institute of technology [5]. The extracted data has length unit AU and velocity unit AU per day. We convert the velocities to AU per year for our simulation.

4 Results

Here we present figures showing the results of the simulations we have produced of the solar system.

4.1 Earth-Sun

We simulate the Earth-Sun system with gravity acting on both the Sun and the Earth, comparing the velocity Verlet method with the forward Euler method. This is shown in figure 2b. The Euler-Cromer method was also used for comparison, shown in equation (22).

For circular motion we expect from equation (11) and equation (10) that the velocity that yields circular motion for the Earth around the Sun is $2\pi\frac{\text{AU}}{\text{Year}}$. This is also the velocity used in the various simulations with circular motion of the Earth.

4.2 Algorithm Timing

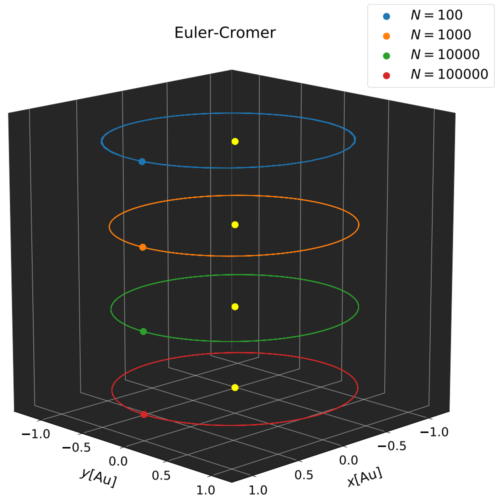
Running 10 simulations of three years of the Earth-Sun system yields average timings of ≈ 22.54 seconds for the velocity Verlet method, and ≈ 20.33 seconds for the forward Euler method. The difference is surprisingly small, despite the Verlet method requiring ostensibly more than twice as many floating point operations as the Euler method. Since the velocity Verlet method is also much more accurate with a total error $\mathcal{O}(h^2)$ compared with the Euler method at $\mathcal{O}(h)$, we use the velocity Verlet method in all subsequent simulations of the solar system.

4.3 Generalized Newtonian Gravity

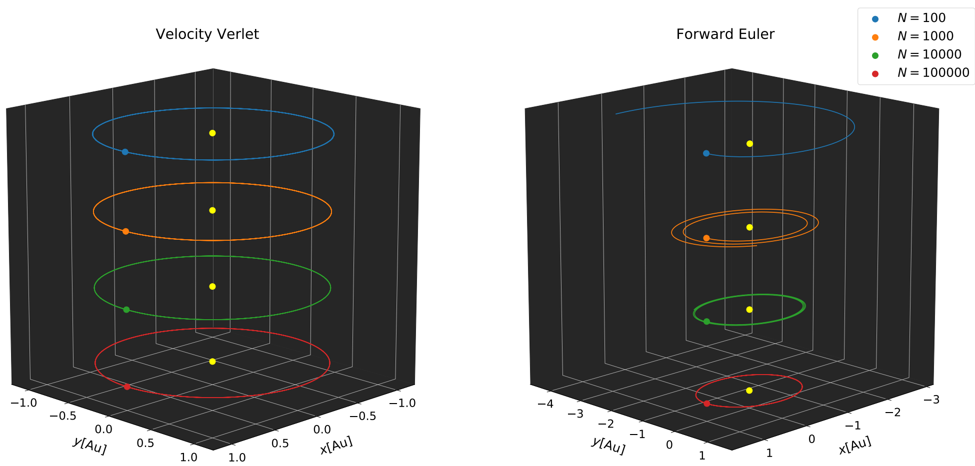
Generalizing the gravitational force as in equation (A1), we simulated the Earth-Sun system over three years for varying values of β as plotted in figure 3.

4.4 Escape Velocity

To find the escape velocity of the Earth in the Earth-Sun system numerically, simple trial and error was used. The results are shown in figure 4, and the final value of the initial velocity v_0 that was sufficient for the Earth to leave the solar system is $v_0 = 8.885764876\frac{\text{AU}}{\text{year}}$. Comparing with the analytic value $2\sqrt{2}\pi \approx 8.885765876\frac{\text{AU}}{\text{year}}$, we see a very small difference of order 10^{-6}.



(a)



(b)

Figure 2: (a) This figure shows the same simulation as figure 2b, but with the Euler-Cromer method. The Euler-Cromer method is stable even for small numbers of time steps as expected for a symplectic method. (b) Here is the Earth-Sun system simulated over 3 years using the Euler and Verlet methods. The Earth does not form a closed orbit for the Forward Euler method, and is slowly escaping the gravitational well of the Sun, especially for small numbers simulation points. The velocity Verlet method remains stable at even very few points.

4.5 Three-body Problem

We have implemented a model consisting of the Sun, Earth and Jupiter, employing the velocity Verlet method. Figure 5 shows the simulated orbits of the celestial bodies in our model for 10 years. The top subfigure shows the simulation with Jupiter's usual mass, M_j. We also model the scenarios where Jupiter's mass is $10M_j$ and $1000M_j$, which is

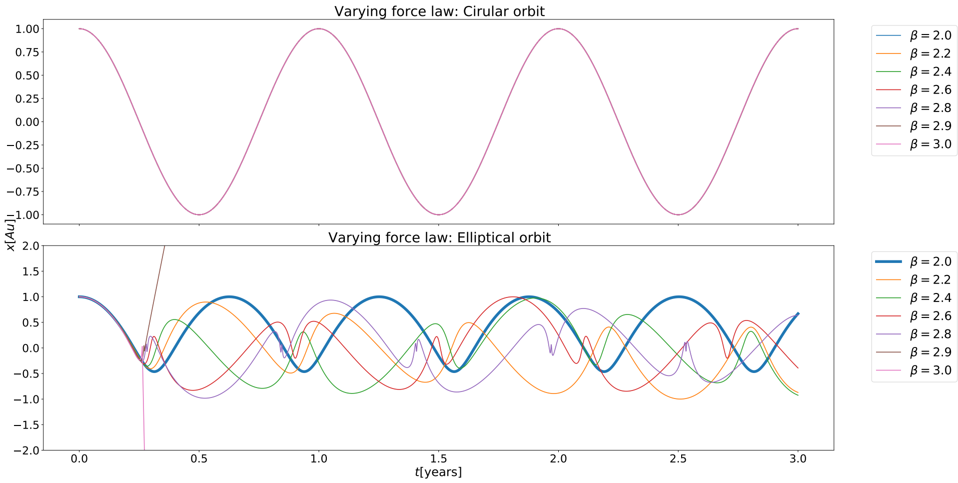


Figure 3: This plot shows the x-axis movement of the Earth in the Sun-Earth system simulated over three years, with varying values of β for generalized Newtonian gravity. For an elliptical orbit, the system varies greatly with β, and loses stability for $\beta \to 3$.

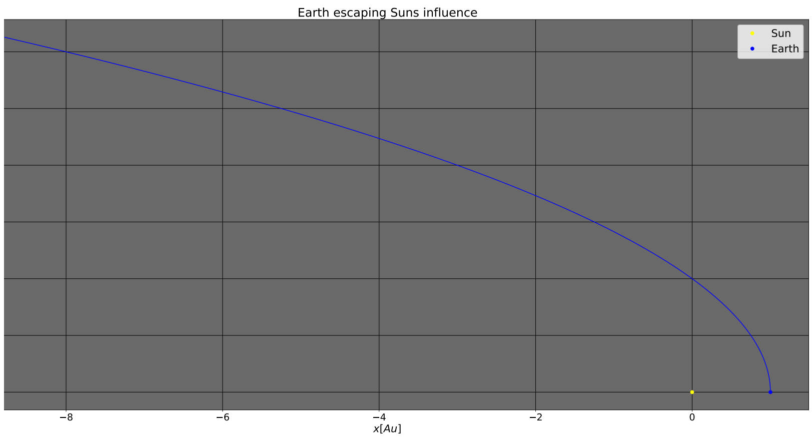


Figure 4: Pictured is a simulation over 3 years, with the initial velocity of the earth set to $v_0 = 8.885764876 \frac{\text{AU}}{\text{year}}$. By trial and error, this value seems to be enough for the Earth to leave the Solar system.

shown in the middle and bottom subfigure, respectively. As one can see from figure 5, a mass of $10M_j$ would drag the Earth closer to the Sun, while a mass of $1000M_j$ causes the Earth to leave the system entirely, while Jupiter and the Sun start orbiting each other, similar to a binary star system.

4.5.1 Extending to the entire solar system

Extending the framework of the three-body problem, the entire solar system can be simulated. 248 years of simulation allows for one full orbit of Pluto, as shown in figure 6. Since the innermost planets have orbits that are difficult to make out, they are shown more accurately in figure 7.

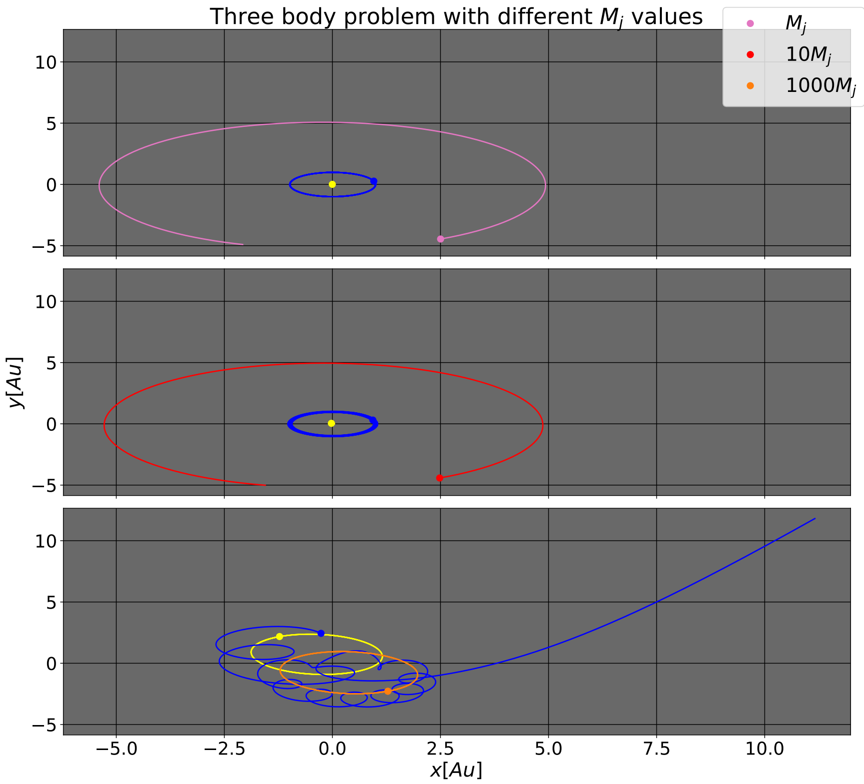


Figure 5: Plot of the simulated orbit of the Sun, Earth and Jupiter system for 10 years with: **top:** Jupiter's usual mass, M_j, **middle:** $10M_j$ **bottom:** $1000M_j$. The initial position of the Sun is not in the center in the bottom subfigure, as Jupiter is so massive in this case that the center of mass is moved to almost equidistantly between it and the Sun.

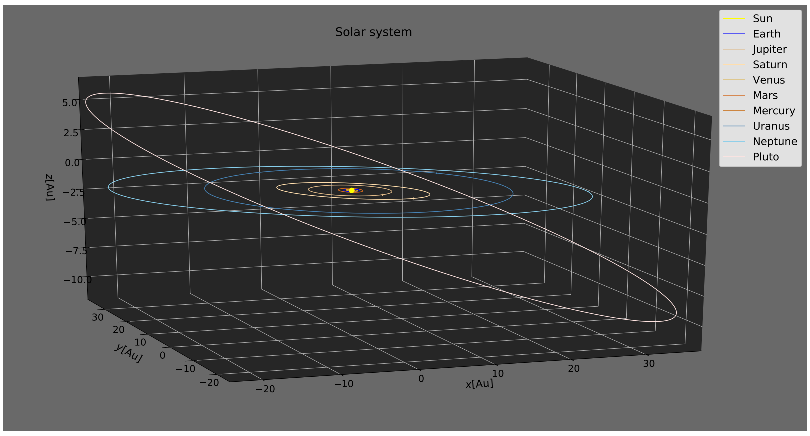


Figure 6: Simulation of all the planets in the solar system. The time is 248 years.

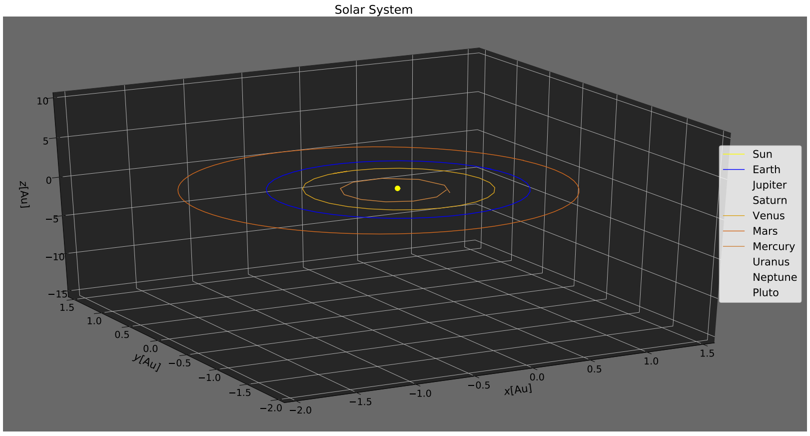


Figure 7: The same plot from figure 6, zoomed in to reveal the innermost planets.

4.6 Perihelion Precession

Applying the relativistic correction in equation (19) to the classical Newtonian equation of gravity, we simulated one century of only Mercury and the Sun. Starting at perihelion, the point closest to the Sun, we simulated with relativistic corrected gravity and without. The angles between the coordinates of the first perihelion and the coordinates of subsequent perihelions are plotted in figure 8, against the observed $43''$ precession per century.

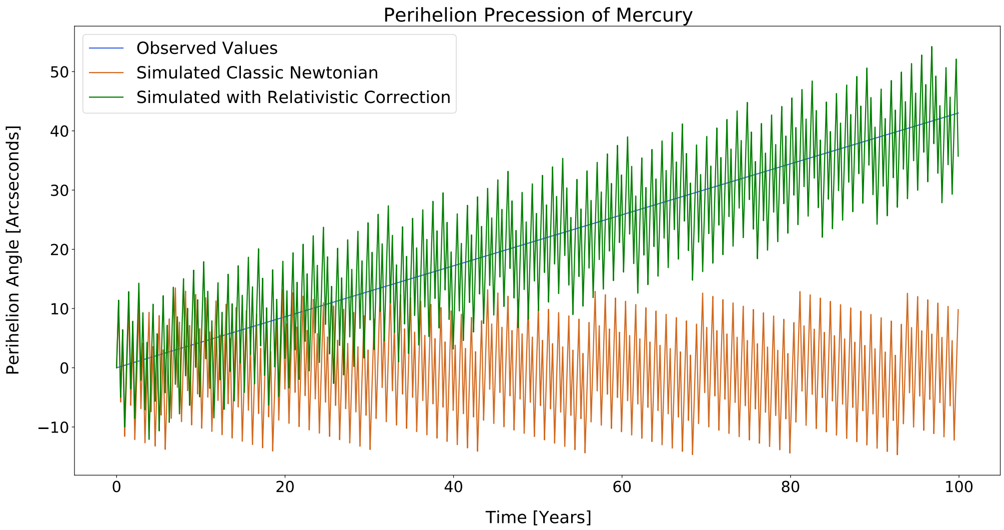


Figure 8: Plot of the perihelion angle of Mercury as a function of time, for observed and simulated values, with and without relativistic correction. While the simulated data is noisy, it is clear that the relativistic correction follows the observed values closely.

5 Discussion

5.1 Earth-Sun System

We see in figure 2b the results of using the Forward Euler method, as the Earth in this simulation gradually escapes the Sun's gravitational field. This is in accordance with section 3.1.1, as the Forward Euler method should not conserve energy and will therefore in general not lead to closed orbits. As the step size decreases and the number of time steps increase, the orbit becomes more stable, as the error in the energy grows smaller. Meanwhile, the velocity Verlet method remains stable for all step sizes shown, as expected from a symplectic method.

Figure 4 shows the Earth with an initial velocity of 8.885764876AU/yr. This velocity was found through trial and error to be approximately enough for the Earth to escape the gravitational pull of the sun.

5.2 Generalized Newtonian Gravity

In figure figure 3, we show the x-coordinate of the Earth over time, with different values of the parameter β. As β approaches 3, the gravitational force seems to be unable to keep the Earth in a bound state, as expected from the discussion in appendix A.2. However, it seems that for a circular orbit, there is almost no dependence on β. In truth, there is a small gradual decline in the radius over time for $\beta = 3$, but it is not resolvable in figure 3. It might be of interest to explore longer term simulations of circular orbits, or with a smaller step size h to see the rate of decline and how long it takes before the system loses stability.

5.3 Three-body Problem

The three-body problem illustrated in figure 5 shows the Jupiter-Earth-Sun system for three different values for the mass of Jupiter. We see that for the ordinary case, the Earth's orbit is largely unaffected, while for $10M_j$ the Earth's orbit is no longer closed. For the case of $1000M_j$, Jupiter has a mass similar to that of the Sun $M_\odot \approx 1047M_j$ [6], which is why the Sun and Jupiter begin orbiting around each other, with the Earth leaving the system behind entirely. This is essentially a system with two stars. The case of $10M_j$ seems to keep stability for the Earth, although figure 5 only shows the simulation for 10 years. It could be that for a longer simulation, the system loses stability.

5.4 Perihelion Precession

We expect precession to occur when adding the relativistic correction in equation (19) to the classical Newtonian gravitational force, as the orbit should be almost closed elliptical. The observed perihelion precession of Mercury is about $43''$ per century greater than what the Newtonian contributions from the other bodies in the solar system should add up to [2]. Thus we expect that with only the Sun-Mercury system, including the relativistic correction should show a large contribution towards this additional precession. figure 8 shows that the relativistic corrected path of Mercury fits well with this additional precession, oscillating with $\approx 10''$ about the observed slope.

To resolve the difference of 43 arcseconds at all, we expect to need at least a resolution of 86 arcseconds. In a given orbit, there are 1296000 arcseconds, and in a century Mercury makes approximately 400 orbits. Thus, to get above the minimum precision of $86''$, we need more than $n \approx 6 \cdot 10^6$ time steps. In figure 8, a total of $n = 3 \cdot 10^7$ time steps were used. Thus we can expect to resolve about $\frac{3 \cdot 10^7}{400 \cdot 1296000} \approx 17$ arcseconds, and it is clear that this is enough to get a clear picture of the relativistic contribution to the precession from the figure, also corresponding with the error being of the order 10 arcseconds as in the figure.

These results seem to indicate that General Relativity can explain most of the additional perihelion precession observed on Mercury, if not all. With even higher number of points, it should be possible to get a more accurate slope for the relativistic contribution to see if there are any other corrections needed, and to lower the resolution even further.

6 Conclusion

We have made various simulations of the Solar system, attaining a deeper understanding of object oriented programming and the value of being able to reuse code in different subsystems. The simulations have largely corresponded with theory, for both two-body, three-body and many-body problems. The case of Jupiter increased to near-Sun mass lead to a system similar to a binary star system with the Earth leaving the Solar system, while for $10M_j$ the system seemed perturbed but stable with the Earth changing orbit slowly. Simulating a longer time period might show whether the orbit of the Earth would remain stable.

With a generalized Newtonian force of gravity we saw instabilities at β values approaching 3, in accordance with theory. For circular orbits the system is noticeably more stable even for higher values of β, and future work could involve testing longer time periods to see the rate of decline, or different number of time steps.

The simulated perihelion precession of Mercury over the course of a century was found to oscillate about the observed slope with an error of about $10''$. Simulating the perihelion precession more accurately involves only increasing the number of points, and thus time spent running the program. This could be of interest in order to more accurately determine if general relativity can explain the whole $43''$ precession.

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A Appendix

A.1 Source code

Github repository with codes and figures can be found at `https://github.com/idadue/ComputationalPhysics/tree/master/project3`.

A.2 Generalization of Newton's gravitational law

The modified gravitational force is given by

$$\vec{F} = G\frac{M_\odot M_{\text{Earth}}}{r^\beta}\vec{e}_r, \tag{A1}$$

where $\vec{r}$ is the position vector of the Earth relative to the sun and with $\beta \in [2, 3]$. The corresponding potential can then be written

$$V = \frac{1}{\beta - 1}\frac{M_\odot M_{\text{Earth}}}{r^{\beta-1}} = \frac{1}{\alpha}\frac{GM_\odot M_{\text{Earth}}}{r^\alpha}, \tag{A2}$$

with $\alpha = \beta - 1$.

First we look at this problem in two dimensions, where the Earth has two degrees of freedom. The system can be described by two generalized coordinates r and θ. The position vector of the Earth is given by

$$\vec{r} = r(\cos\theta, \sin\theta).$$

We want to investigate the stability of Earths orbit for different values of β. The Lagrangian of the system is given by

$$\mathcal{L} = \frac{1}{2}M_{\text{Earth}}(\dot{r}^2 + r^2\dot{\theta}^2) - \left(-\frac{1}{\alpha}\frac{GM_\odot M_{\text{Earth}}}{r^\alpha}\right).$$

It is trivial to see that we have two constants of motion because we have conservation of total energy and angular momentum. We set up Lagrange's equation for θ, which gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0.$$

The angular momentum of the Earth relative to the sun is given by

$$\frac{\partial L}{\partial\dot{\theta}} = M_{\text{Earth}}r^2\dot{\theta} \equiv l.$$

We use the equation above to eliminate the $\dot{\theta}$ dependence in the equation of motion for the r-coordinate, which gives

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0 \\
M_{\text{Earth}}\ddot{r} - \left(\frac{l^2}{M_{\text{Earth}} r^3} - \frac{GM_\odot M_{\text{Earth}}}{r^\alpha}\right) &= 0,
\end{aligned}
$$

where we have used that

$$
\dot{\theta} = \frac{l}{M_{\text{Earth}} r^2}.
$$

We get the same equation of motion from a one dimensional Lagrangian

$$
\mathcal{L} = \frac{1}{2} M_{\text{Earth}} \dot{r}^2 - V_{\text{eff}}(r),
$$

where the effective potential is defined by

$$
V_{\text{eff}}(r) = \frac{1}{2}\frac{l^2}{M_{\text{Earth}}}\frac{1}{r^2} - \frac{1}{\beta - 1}\frac{GM_\odot M_{\text{Earth}}}{r^\alpha} \equiv \frac{1}{2}\frac{k}{r^2} - \frac{1}{\alpha}\frac{c}{r^\alpha}.
$$

In a stable equilibrium, this effective potential must have a minimum. Thus, we set the derivative with respect to r equal to zero:

$$
\begin{aligned}
\frac{dV_{\text{eff}}}{dr} &= 0 \\
-\frac{k}{r^3} + \frac{c}{r^{\alpha+1}} &= 0 \\
r^{\alpha-2} &= \frac{c}{k}
\end{aligned}
$$

To have a minimum we get the condition

$$
\begin{aligned}
\frac{d^2 V_{\text{eff}}}{dr} &> 0 \\
3\frac{k}{r^4} - (\alpha+1)\frac{c}{r^{\alpha+2}} &> 0 \\
&\vdots \\
\beta < 3r^{\alpha-2}\frac{k}{c} &= 3,
\end{aligned}
$$

which gives $\beta < 3$ for a stable equilibrium. The Earth will have a stable orbit for all $\beta < 3$, while the r-coordinate will diverge.