CENTRO DE ESTUDIOS ECONÓMICOS

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Macroeconomics 3

Topic 3: Government Budget Deficits, the Current Account and Ricardian Equivalence

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Disclaimer: I am not the original intellectual author of the material presented in these notes. The content is strongly based on a combination of lecture notes, textbook references, and personal annotations for learning purposes. Any errors or omissions are entirely my own responsibility.

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3.1 Introduction and Aims

"Blessed are the young, for they shall inherit the national debt"
Herbert Hoover

- Up to this point, we've been relying on the representative agent model to understand the current account and the key economic forces that shape it.
- While this model forms the backbone of the theoretical approach in our course, it's not the only one. There are other important frameworks used in macroeconomics—especially in international contexts.
- One key implication of the representative agent framework is that it supports the idea of **Ricardian Equivalence**.
- When Ricardian Equivalence holds, government borrowing (or deficits) doesn't change national saving—because people adjust their private saving in response.
- To explore what happens when this equivalence breaks down, we'll turn to a different modeling approach: the **Overlapping Generations (OLG)** framework.
- In OLG models, individuals live for a limited time—typically two periods—while the government is modeled as living forever.
- **Intuition:** Ricardian Equivalence relies on agents internalizing future taxes. But in OLG models, each generation lives only briefly and may not fully account for future fiscal burdens—breaking the equivalence.
- To explore why Ricardian Equivalence may fail, we'll build a standard two-period Overlapping Generations (OLG) model. In this setup, taxes and government borrowing influence individuals' consumption decisions.
- From this framework, several key insights emerge:
  1. Budget deficits that are financed by tax cuts can significantly influence a country's current account—contrary to Ricardian Equivalence predictions.
  2. A nation's productivity level plays a central role in shaping its saving behavior, which in turn affects the current account.
  3. The distribution of taxes across different generations can create short-run effects on a country's external balance.
  4. In an open-economy setting with two countries (where the interest rate adjusts endogenously), tax policy can shift global savings and investment patterns—altering the world interest rate, capital flows, and the allocation of resources across borders.
- **Intuition:** OLG models let us capture how intergenerational dynamics and policy timing affect macroeconomic outcomes like saving, borrowing, and the current account—especially in open economies.

Reading

- Obstfeld and Rogoff (1996), *Chapter 3*, Sections 3.1, 3.2, and 3.6.
- Seater (1993), "Ricardian Equivalence", *Journal of Economic Literature*, 31, pp. 142–190.

3.2 The Ricardian Equivalence Outcome under the Representative-Agent Framework

- Why are government budget deficits considered neutral in representative agent models?
- Let's assume, as before, that the population size is constant and equal to 1.
- This means that individual-level quantities are equivalent to aggregate economy-wide quantities.
- We begin with the familiar two-period model introduced in Topic 1.
- When investment is present, the intertemporal budget constraint of the representative individual is:

$$C_1 + I_1 + \frac{C_2 + I_2}{1+r} = Y_1 - T_1 + \frac{Y_2 - T_2}{1+r}. \tag{1}$$

- Previously, we assumed a balanced government budget in each period:

$$G_1 = T_1 \quad \text{and} \quad G_2 = T_2,$$

where T is a lump-sum tax.

- Now let's relax that assumption. The government's intertemporal budget constraint becomes:

$$G_1 + \frac{G_2}{1+r} = T_1 + \frac{T_2}{1+r}. \tag{2}$$

- **Intuition:** In representative agent models, what matters is the present value of taxes, not their timing. So even if taxes shift across periods, consumption choices remain unchanged under Ricardian Equivalence.
- Equation (2) tells us that if the government starts with zero debt, then its spending is limited to the present value of its tax revenues.
- Substituting (2) into the individual's intertemporal budget constraint (1) gives:

$$C_1 + I_1 + \frac{C_2 + I_2}{1+r} = Y_1 - G_1 + \frac{Y_2 - G_2}{1+r}.$$

- This is exactly the same equation we'd get if the government balanced its budget in every period.

- As a result, the representative agent behaves as if the government budget were balanced at all times—making the same consumption and investment choices.
- Government budget imbalances don't affect individual decision-making, as long as G_1 and G_2 are unchanged. There's no impact on real allocation.
- This result hinges on lump-sum taxes: postponing taxes only shifts them forward with interest, which the same taxpayer eventually pays.
- Therefore, what matters for consumption is the present value of government spending—not whether the government runs a deficit.
- In other words, under these conditions, government borrowing doesn't change consumption choices.
- While rescheduling taxes changes when the government saves, it doesn't change how much it saves in total—so national or aggregate saving is unaffected.
- **Intuition:** Ricardian Equivalence holds because the representative agent fully internalizes government borrowing: they save more today knowing they'll face future taxes.
- Private saving is defined as:
$$S_P = Y - T - C$$
- Government saving (i.e., the budget surplus) is:
$$S_G = T - G$$
- Total national saving (private plus government) is therefore:
$$S_P + S_G = Y - C - G$$
- Since consumption remains unchanged under Ricardian Equivalence, national saving also stays the same.
- This happens because any change in government saving is exactly offset by an opposite change in private saving.
- For example, if the government cuts taxes in period 1 by dT (and raises them in period 2 by $(1 + r)dT$), the private sector increases its saving by dT in period 1—anticipating the future tax hike.
- As a result, the agent's optimal consumption path remains unchanged, despite the timing of taxes being altered.
- This leads to a central conclusion: the timing of taxes (and government budget balances) doesn't affect real allocation in the economy.
- This idea is known as the **Ricardian Equivalence of debt and taxes**.

- **Intuition:** Ricardian Equivalence holds because forward-looking agents smooth consumption and fully internalize future tax liabilities, neutralizing any effect of debt-financed tax changes on saving or consumption.

- We now turn to the infinite-horizon setup from Topic 2 to explore the Ricardian Equivalence result in a more general setting.

- The representative agent's asset accumulation is described by:

$$B_{t+1}^p - B_t^p = Y_t + rB_t^p - T_t - C_t - I_t,$$

where B_t^p is the stock of financial assets at the end of period $t-1$, and T_t represents lump-sum taxes.

- Assuming a constant interest rate r, we can derive the agent's intertemporal (lifetime) budget constraint as:

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(C_s+I_s)+\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T}B_{t+T+1}^p=(1+r)B_t^p+\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s-T_s)$$

- Rearranging:

$$\Rightarrow \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(C_s+I_s)=(1+r)B_t^p+\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s-T_s)$$

- The condition

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T}B_{t+T+1}^p=0$$

is the standard transversality condition (TVC), ensuring that the agent cannot accumulate infinite wealth.

- **Intuition:** The present value of consumption and investment must equal the present value of income minus taxes, plus initial wealth. Ricardian Equivalence emerges because only the present value of taxes matters—not their timing.

- The government's period-by-period budget constraint is:

$$B_{t+1}^G - B_t^G = T_t + rB_t^G - G_t$$

- This leads to the government's lifetime (intertemporal) budget constraint:

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}G_s+\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T}B_{t+T+1}^G=(1+r)B_t^G+\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}T_s$$

- Rearranged:

$$\Rightarrow \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}G_s=(1+r)B_t^G+\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}T_s$$

- As with households, we assume the transversality condition (TVC) holds:

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T} B^{G}_{t+T+1} = 0$$

- **Interpretation:** Just like in the two-period model, the present value of government consumption must equal the present value of tax revenues plus the initial net asset position. If $B^G_t < 0$, the government starts with debt.

- **Intuition:** This constraint ensures fiscal solvency. The government can shift taxes over time, but it can't spend beyond its lifetime resources.

- The net foreign asset position of the economy is the sum of private and government assets:

$$B = B^p + B^G$$

- We can validate Ricardian Equivalence in this context by combining the household and government lifetime budget constraints and using $B = B^p + B^G$:

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(C_s + I_s) = (1+r)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s)$$

- This expression shows that only the present value of government purchases G influences equilibrium.

- How those purchases are financed—through taxes or debt—has no effect on individual consumption or investment decisions.

- Therefore, simply changing the timing of lump-sum taxes doesn't affect the open economy's equilibrium.

- Since the representative agent fully internalizes the government's budget constraint, it doesn't matter whether net foreign assets are held by the private sector or by the government.

- For instance, if the government receives a transfer of foreign assets from the private sector, it can lower taxes by an equivalent amount—leaving the agent's disposable income unchanged.

- **Intuition:** In representative agent models, what matters is government spending, not how it's financed. Ricardian Equivalence ensures that asset ownership shifts between public and private hands without real economic impact.

3.3 Reasons for Ricardian non-Equivalence

- In this section, we explore the assumptions under which Ricardian Equivalence may break down.

1. Distortionary taxes

- Distortionary taxation is one of the main reasons Ricardian Equivalence can fail.
- For instance, in models where households choose how much to work (i.e., labor supply is endogenous), taxes on labor income distort incentives.
- Such taxes can reduce labor supply and saving, prompting households to adjust their consumption behavior in response.
- If the government cuts taxes today by borrowing (raising future taxes), households may anticipate higher future marginal tax rates.
- In response, they might work more today (when taxes are low) and less in the future, thus altering the timing of labor and consumption.
- This shift is called intertemporal substitution in labor supply. It creates real effects by changing how tax distortions are spread across periods.
- As a result, changes in the timing of taxes (even if lump-sum) now matter—violating Ricardian Equivalence.

- **Intuition:** When taxes influence behavior (e.g., labor supply), the timing of taxation matters. Shifting taxes over time leads households to change their work and consumption choices, breaking the neutrality implied by Ricardian Equivalence.

2. Credit (borrowing) Restrictions

- Our baseline model assumes that households can borrow and lend freely at the same interest rate as the government.
- In practice, however, households may face **credit constraints**, which can lead to a breakdown in Ricardian Equivalence.
- For example, even if someone expects to earn much more in the future, they might not be able to borrow today—especially if they lack sufficient collateral.
- A credit-constrained consumer is thus unable to smooth consumption over time. Instead, they spend all of their current income today, even though they'd prefer to borrow against future earnings.
- If the government cuts taxes today and finances it by borrowing, credit-constrained households will increase their current consumption—despite knowing that future taxes will be higher.
- In this way, the government effectively relaxes private borrowing constraints by using its own access to capital markets to shift consumption from the future to the present on behalf of constrained households.
- **Intuition:** Credit constraints prevent households from fully smoothing consumption. A tax cut financed by debt gives them extra liquidity—so their behavior changes, violating Ricardian Equivalence.

3. Finite Horizons for the Household

- In our baseline model, we assumed that households and the government shared the same time horizon.
- But in reality, governments continue to exist beyond the lives of any single generation of households.
- Many current taxpayers (such as the elderly) may reasonably believe that future generations—not themselves—will bear the burden of repaying public debt through future taxation.
- So, shifting from tax-financed to debt-financed government spending effectively pushes the cost of financing onto future generations.
- **As a result, a debt-financed tax cut raises the perceived wealth of current generations, increasing their consumption.**
- However, it's important to note: just because lifespans are finite doesn't automatically mean Ricardian Equivalence fails.
- **If future generations are linked to the current generation through bequests**, then Ricardian Equivalence may still hold. This connection maintains an implicit infinite horizon across generations.
- For instance, if parents care about the well-being of their children (and their children care about theirs), then tax savings today may be passed forward via bequests—offsetting future tax burdens.
- **Intuition:** When people don't live forever, they may ignore future taxes and consume more today. But if they leave bequests and care about their heirs, Ricardian Equivalence can still work through intergenerational links.

3.4 The Diamond Overlapping Generations (OLG) Model

- Before considering an open economy OLG model and the breakdown of Ricardian Equivalence, let us start by developing the famous closed economy **OLG Diamond model**.
- **Key questions:**
  - Does the economy converge to the steady state for any initial condition K_0?
  - If yes, is this steady state Pareto efficient? (i.e., competitive equilibrium = social planner)

Assumptions

- Time is discrete and runs to infinity.
- Each generation of individuals lives for two periods, denoted y (young) and o (old), and a new generation is born each period.
- Households born on date t are assumed to have a separable utility function:

$$U(c_t^y, c_{t+1}^o) = u(c_t^y) + \beta u(c_{t+1}^o)$$

- where c_t^y denotes consumption when young of someone born on date t and c_{t+1}^o denotes consumption of the same person when old in period $t+1$.
- As usual, $0 < \beta < 1$ denotes the discount factor.
- Individuals within the same generation are treated identically—no intra-generational differences.
- The period utility function $u(C)$ is assumed to be strictly increasing, concave, and twice differentiable.
- **Exponential population growth:** The number of young individuals in period t, denoted L_t, grows at a constant rate n:

$$L_t = (1+n)L_{t-1} \Rightarrow L_t = (1+n)^t L_0,$$

  where L_0 is the initial population size.
- Time is discrete and infinite ($t = 0, 1, 2, \ldots$).
- Output is a homogeneous good that can be consumed or invested in capital.
- Capital is the only asset and is entirely owned by households. It is rented to firms.
- There are three perfectly competitive markets: for output, labor, and capital services.
- Production is carried out by competitive firms with constant returns to scale, using an aggregate production function:

$$Y_t = F(K_t, L_t).$$

- Only the young supply labor, inelastically providing one unit each. They receive the equilibrium wage rate w_t. The trade-off between labor and leisure is considered exogenous.
- Capital fully depreciates after use: $\delta = 1$.
- The model assumes perfect foresight—there is no uncertainty.

Intuition: This framework abstracts from heterogeneity and focuses on generational behavior over time. Each young agent works and saves for old age, shaping capital accumulation and intertemporal dynamics. The lack of uncertainty and full depreciation simplify the analysis.

Firms' Problems

- Firms maximize profit by choosing labor and capital.
- First express the production function in *intensive form*:

$$Y_t = F(K_t, L_t) = L_t F(k_t, 1) \equiv L_t f(k_t),$$

  where $k_t \equiv \frac{K_t}{L_t}$ is the capital-labor ratio.

- Profit maximization requires that the marginal product of capital equals the rental rate of capital:

$$F_K(K_t, L_t) = \frac{\partial L_t f(k_t)}{\partial K_t} = f'(k_t) = R_t, \tag{5}$$

- And firms hire labor up to the point where the marginal product of labor equals the wage:

$$F_L(K_t, L_t) = \frac{\partial L_t f(k_t)}{\partial L_t} = f(k_t) - k_t f'(k_t) = w_t. \tag{6}$$

- Note that the gross return to saving equals the rental rate of return from capital:

$$1 + r_t = R_t = f'(k_t) \tag{7}$$

- i.e., the owner of K_t units of physical capital receives a real net rate of return on capital:

$$\frac{R_t K_t - \delta K_t}{K_t} = R_t - \delta,$$

- where δ is the depreciation rate of capital
- No-arbitrage requires that capital and other assets (e.g., loans) yield the same rate of return

$$R_t - \delta = r_t,$$

- where r_t is the market interest rate
- Since by assumption $\delta = 1$, equation (7) automatically follows

Full procedure
Firm's maximization problem:

$$\max_{K_t, L_t} \{F(K_t, L_t) - R_t K_t - w_t L_t\}$$

Assume intensive form of production:

$$Y_t = F(K_t, L_t) = L_t f(k_t), \quad \text{where} \quad k_t = \frac{K_t}{L_t}$$

Lagrangian: (not necessary here since unconstrained, but we proceed with FOCs)

First-order conditions:

$$\frac{\partial \Pi}{\partial K_t}: \quad \frac{\partial}{\partial K_t}\left[L_t f\left(\frac{K_t}{L_t}\right) - R_t K_t - w_t L_t\right] = f'(k_t) - R_t = 0 \Rightarrow R_t = f'(k_t)$$

$$\frac{\partial \Pi}{\partial L_t}: \quad \frac{\partial}{\partial L_t}\left[L_t f\left(\frac{K_t}{L_t}\right) - R_t K_t - w_t L_t\right] = f(k_t) - k_t f'(k_t) - w_t = 0 \Rightarrow w_t = f(k_t) - k_t f'(k_t)$$

No-arbitrage and depreciation condition:

$$\text{Owner of capital receives: } \frac{R_t K_t - \delta K_t}{K_t} = R_t - \delta$$

$$\text{No-arbitrage: } R_t - \delta = r_t \quad \Rightarrow \quad R_t = r_t + \delta$$

If $\delta = 1$, **then:**

$$R_t = r_t + 1 \quad \Rightarrow \quad f'(k_t) = R_t = 1 + r_t$$

Summary:

$$R_t = f'(k_t) = 1 + r_t$$
$$w_t = f(k_t) - k_t f'(k_t)$$

Intuition: Firms choose capital and labor so that each input's marginal product equals its price—capital earns its marginal return, and workers earn their marginal productivity. This ensures **efficient resource allocation in competitive markets.** The equilibrium interest rate and wage reflect the productivity of capital and labor, respectively.

Households Problem

- Savings S_t by the young in period t are determined from the solution of the following maximization problem:

$$\max_{C_t^y, C_{t+1}^o} \; u(C_t^y) + \beta u(C_{t+1}^o),$$

- subject to:

$$C_t^y + S_t = w_t, \qquad C_{t+1}^o = (1 + r_{t+1}) S_t.$$

- Eliminating S_t yields the intertemporal budget constraint:

$$C_t^y + \frac{C_{t+1}^o}{1 + r_{t+1}} = w_t.$$

- Using this budget constraint to substitute for C_{t+1}^o in the utility function yields the following unconstrained maximization problem:

$$\max_{C_t^y} \; u(C_t^y) + \beta u\left((1 + r_{t+1}) w_t - (1 + r_{t+1}) C_t^y\right).$$

- The FOC for this problem yields the familiar *consumption Euler equation*:
$$u'(C_t^y) = \beta(1 + r_{t+1})u'(C_{t+1}^o). \tag{8}$$
- Together, the Euler equation (8) and the intertemporal budget constraint characterize the optimal consumption path.
- Combining the Euler equation and the intertemporal budget constraint, one obtains an implicit function that determines savings per person:
$$S_t = S(w_t, 1 + r_{t+1}). \tag{9}$$
- Since only the young save, total savings is $TS_t = S_t \cdot L_t$.
- The law of motion for the capital stock is consequently:
$$K_{t+1} = (1 - \delta)K_t + I_t \Rightarrow K_{t+1} = I_t = TS_t = S_t \cdot L_t = L_t \cdot S(w_t, 1 + r_{t+1}). \tag{10}$$

Full procedure
Maximization problem:

$$\max_{C_t^y, C_{t+1}^o} \; u(C_t^y) + \beta u(C_{t+1}^o) \quad \text{s.t.} \quad C_t^y + S_t = w_t, \quad C_{t+1}^o = (1 + r_{t+1})S_t$$

Step 1: Substitute the second constraint into the first:

$$C_t^y + \frac{C_{t+1}^o}{1 + r_{t+1}} = w_t$$

Step 2: Rearranged intertemporal budget constraint:

$$C_{t+1}^o = (1 + r_{t+1})(w_t - C_t^y)$$

Unconstrained maximization:

$$\max_{C_t^y} \; u(C_t^y) + \beta u\left((1 + r_{t+1})(w_t - C_t^y)\right)$$

Lagrangian (optional, but implied):

$$\mathcal{L}(C_t^y) = u(C_t^y) + \beta u\left((1 + r_{t+1})(w_t - C_t^y)\right)$$

First-order condition:

$$\frac{\partial \mathcal{L}}{\partial C_t^y} = u'(C_t^y) - \beta(1 + r_{t+1})u'\left(C_{t+1}^o\right) = 0$$

Euler equation:

$$u'(C_t^y) = \beta(1 + r_{t+1})u'(C_{t+1}^o)$$

Saving:

$$S_t = w_t - C_t^y$$

Capital accumulation:

$$K_{t+1} = S_t \cdot L_t$$

Intuition: The Euler equation shows how individuals optimally trade off consumption today versus tomorrow.

It balances the marginal benefit of consuming now with the discounted, interest-adjusted benefit of saving and consuming later.

Equilibrium

A perfect foresight equilibrium is represented by sequences of aggregate capital stock, household consumption and prices $\{K_t, c_t^y, c_t^o, r_t, w_t\}$ satisfying the firm optimality conditions (6), (7); household optimality conditions (8) and (9); and the law of motion for capital (10).

- Divide (10) by L_{t+1}:

$$\frac{K_{t+1}}{L_{t+1}} = k_{t+1} = \frac{L_t S(w_t, 1 + r_{t+1})}{L_{t+1}}.$$

- Since $\frac{L_{t+1}}{L_t} = 1 + n$:

$$\Rightarrow k_{t+1} = \frac{S(w_t, 1 + r_{t+1})}{1 + n}.$$

- Now using (6) and (7) to substitute out w_t and $1 + r_{t+1}$ yields:

$$k_{t+1} = \frac{S\left(f(k_t) - k_t f'(k_t), f'(k_{t+1})\right)}{1 + n}. \tag{11}$$

- This non-linear first-order difference equation is the fundamental law of motion of the OLG economy.

- A steady state, $k_{t+1} = k_t = k^*$, is given by the solution of (11):

$$k^* = \frac{S\left(f(k) - k f'(k), f'(k)\right)}{1 + n}.$$

- Since the savings function $S(\cdot, \cdot)$ can take any form, the difference equation (11) can lead to complicated dynamics, and multiple steady states.

Full procedure

Objective: Derive the law of motion for capital in per capita terms in the OLG economy.

Step 1: Start with aggregate law of motion for capital (from households savings)

$$K_{t+1} = I_t = TS_t = S_t \cdot L_t = L_t \cdot S(w_t, 1 + r_{t+1})$$

Step 2: Convert to per capita capital by dividing both sides by L_{t+1}:

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{L_t \cdot S(w_t, 1 + r_{t+1})}{L_{t+1}}$$

Step 3: Use population growth $L_{t+1} = (1 + n)L_t \Rightarrow \frac{L_t}{L_{t+1}} = \frac{1}{1+n}$:

$$k_{t+1} = \frac{S(w_t, 1 + r_{t+1})}{1 + n}$$

Step 4: Express w_t and $1 + r_{t+1}$ from firm's optimality conditions (6) and (7):

$$w_t = f(k_t) - k_t f'(k_t), \quad 1 + r_{t+1} = f'(k_{t+1})$$

Step 5: Plug into the law of motion for capital:

$$k_{t+1} = \frac{S\left(f(k_t) - k_t f'(k_t), f'(k_{t+1})\right)}{1+n}$$

Step 6: Define steady state by setting $k_{t+1} = k_t = k^*$:

$$k^* = \frac{S\left(f(k) - k f'(k), f'(k)\right)}{1+n}$$

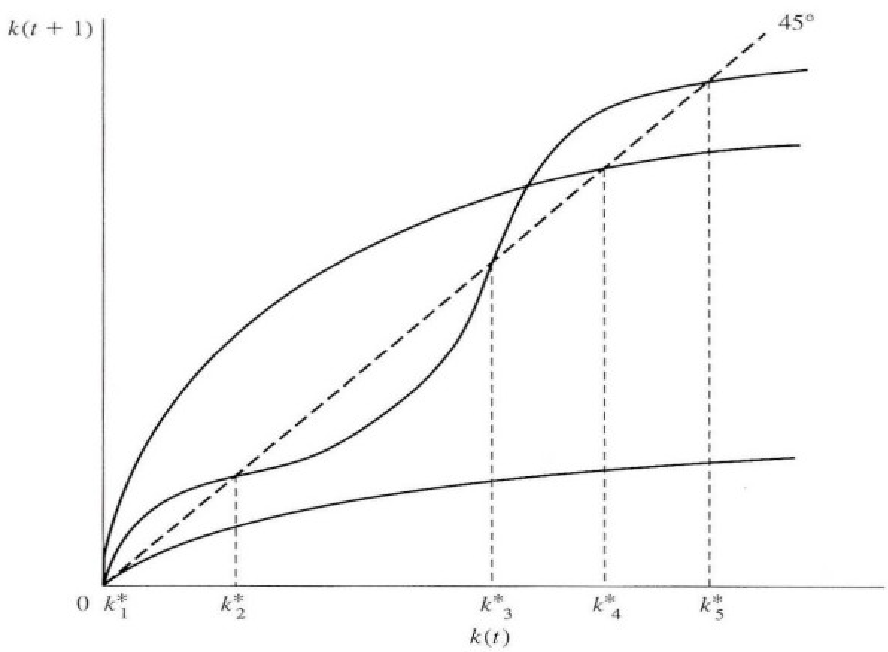

Figura 1: Phase diagram of capital accumulation in the OLG model. The steady state k^* is determined by the intersection where $k_{t+1} = k_t$, derived from the capital accumulation law $k_{t+1} = \frac{S(w_t, 1+r_{t+1})}{1+n}$. The dynamic behavior of capital depends on the shape of the savings function and production technology.

- The graph shows the dynamic path of capital accumulation: how today's capital-labor ratio $k(t)$ determines tomorrow's $k(t+1)$.
- Steady states occur where the capital accumulation curve intersects the 45° line; these are points where $k(t+1) = k(t)$.
- Multiple intersections imply multiple steady states—some stable, some unstable—reflecting that the economy can converge to different long-run outcomes depending on initial conditions.

Restrictions on Utility and Production Functions

- To describe the steady-state equilibrium and how the economy moves over time, we assume individuals have CRRA (Constant Relative Risk Aversion) preferences:

$$U(C_t^y, C_{t+1}^o) = \frac{(C_t^y)^{1-\theta} - 1}{1-\theta} + \beta \left(\frac{(C_{t+1}^o)^{1-\theta} - 1}{1-\theta} \right),$$

  which lets us compare utility from consuming when young vs. old in a consistent way.

- The utility curvature is governed by $\theta > 0$, and output is produced using a Cobb-Douglas production function:

$$f(k) = k^\alpha,$$

  a common form where α captures capital's share in production.

- The capital share satisfies $0 < \alpha < 1$. From intertemporal optimization, we derive the Euler equation:

$$\frac{C_{t+1}^o}{C_t^y} = \left[\beta(1 + r_{t+1})\right]^{\frac{1}{\theta}},$$

  which equates the marginal utility tradeoff between consuming today and tomorrow.

- Given the lifetime budget constraint, we can solve for optimal consumption and savings:

$$C_t^y = \left(\frac{\varphi_{t+1} - 1}{\varphi_{t+1}} \right) w_t, \qquad S_t = w_t - C_t^y = \frac{w_t}{\varphi_{t+1}},$$

$$\text{where } \varphi_{t+1} \equiv 1 + \beta^{\frac{1}{\theta}} (1 + r_{t+1})^{\frac{1-\theta}{\theta}} > 1.$$

  This tells us how a worker splits their wage between present and future consumption.

- We can analyze how savings respond to changes in factor prices through these derivatives:

$$S_W \equiv \frac{\partial s_t}{\partial w_t} = \frac{1}{\varphi_{t+1}} \in (0, 1),$$

$$S_R \equiv \frac{\partial s_t}{\partial R_{t+1}} = \left(\frac{1-\theta}{\theta} \right) \left[\beta R_{t+1}^{\frac{1}{\theta}} \right] \frac{s_t}{\varphi_{t+1}}.$$

  These expressions tell us how sensitive savings are to changes in wages and interest rates.

- The sign of S_R depends on the value of θ: it's positive if $\theta < 1$, negative if $\theta > 1$, and exactly zero if $\theta = 1$. So preferences determine whether higher returns encourage or discourage savings.

- Since $R = 1 + r$, it determines the relative price of future consumption. A higher R means consumption when old is cheaper compared to now.

- A rise in R affects savings behavior through two channels: income and substitution effects.

- **Income effect**: If returns go up, your future consumption from a given amount of saving is worth more, so you might save less.
- **Substitution effect**: A higher return makes it more rewarding to postpone consumption, encouraging more saving when young.
- When $\theta > 1$, the income effect dominates. People want to consume more in both periods and thus save less.
- When $\theta < 1$, the substitution effect dominates. People prefer to shift consumption to the future and save more today.
- When $\theta = 1$, corresponding to log utility, the two effects cancel out perfectly—so changes in interest rates do not influence saving.
- **Intuition**: Preferences (through θ) shape how people trade off consumption over time. The return on savings (via r) affects this decision depending on whether substitution or income motives are stronger.
- The special case where $\theta = 1$ corresponds to logarithmic preferences. This is very commonly used, so from now on we adopt log utility for simplicity and tractability.
- Under log utility, the savings function simplifies to:

$$S_t = \frac{\beta}{1+\beta} w_t.$$

- Plugging this into the capital accumulation equation (11), we obtain:

$$k_{t+1} = \frac{S_t}{1+n} = \frac{\beta w_t}{(1+\beta)(1+n)} = \frac{\beta(1-\alpha)k_t^{\alpha}}{(1+\beta)(1+n)}. \tag{12}$$

  This expresses how capital per worker evolves over time in terms of its current value.

- Two steady-state equilibria can result from this dynamic equation:
  1. A trivial steady state: $k^* = 0$
  2. A non-trivial steady state:

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}.$$

- Given that output per worker is $y = \frac{Y}{L} = k^{\alpha}$, the steady-state level of output per worker becomes:

$$y^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{\alpha}{1-\alpha}}.$$

- Equations like (12) generally can't be solved explicitly over time, but we can still learn about the dynamics using a qualitative tool known as a *Phase Diagram*.

- Our main goal in this type of analysis is to determine whether the economy will converge toward the non-trivial steady-state $k^* > 0$ over time.

- To study the local stability of each steady state, we evaluate the slope (derivative) of the capital accumulation function at the steady state:

$$f'(k_t) = \frac{\alpha\beta(1-\alpha)k_t^{\alpha-1}}{(1+\beta)(1+n)} > 0. \tag{13}$$

  This tells us how sensitive future capital is to today's capital.

- Since the derivative is positive, we have *monotonic dynamics* — capital either increases or decreases smoothly, without oscillations.

- Evaluating the slope at the non-trivial steady state $k^* > 0$, we find:

$$0 < f'(k^* > 0) = \alpha < 1,$$

  which implies that the steady state is locally stable — the system gradually converges toward it.

- The concavity of the transition function helps us visualize its dynamics. Because:

$$f''(k_t) = \frac{\alpha\beta(1-\alpha)(\alpha-1)k_t^{\alpha-2}}{(1+\beta)(1+g)(1+n)} < 0,$$

  we know the function is strictly concave for all $k_t > 0$.

- This strict concavity means the adjustment path is smooth and diminishing over time.

- Therefore, as long as we start with a positive capital stock $k_0 > 0$, the economy will converge globally to the non-trivial steady state.

Full procedure

Utility maximization problem:

$$\max_{C_t^y, C_{t+1}^o} \quad U(C_t^y, C_{t+1}^o) = \frac{(C_t^y)^{1-\theta}-1}{1-\theta} + \beta\frac{(C_{t+1}^o)^{1-\theta}-1}{1-\theta}$$

Subject to:

$$C_t^y + \frac{C_{t+1}^o}{1+r_{t+1}} = w_t$$

Lagrangian:

$$\mathcal{L} = \frac{(C_t^y)^{1-\theta}-1}{1-\theta} + \beta\frac{(C_{t+1}^o)^{1-\theta}-1}{1-\theta} + \lambda\left(w_t - C_t^y - \frac{C_{t+1}^o}{1+r_{t+1}}\right)$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial C_t^y}: \quad (C_t^y)^{-\theta} - \lambda = 0 \quad \Rightarrow \quad \lambda = (C_t^y)^{-\theta}$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1}^o}: \quad \beta (C_{t+1}^o)^{-\theta} - \lambda \cdot \frac{1}{1+r_{t+1}} = 0$$

$$\Rightarrow \quad \beta (C_{t+1}^o)^{-\theta} = \lambda \cdot \frac{1}{1+r_{t+1}}$$

$$\Rightarrow \quad \beta (C_{t+1}^o)^{-\theta} = (C_t^y)^{-\theta} \cdot \frac{1}{1+r_{t+1}}$$

Euler equation:

$$\left(\frac{C_{t+1}^o}{C_t^y}\right)^{-\theta} = \frac{1}{\beta(1+r_{t+1})} \quad \Rightarrow \quad \frac{C_{t+1}^o}{C_t^y} = [\beta(1+r_{t+1})]^{\frac{1}{\theta}}$$

Lifetime budget constraint:

$$C_t^y + \frac{C_{t+1}^o}{1+r_{t+1}} = w_t$$

Substitute $C_{t+1}^o = [\beta(1+r_{t+1})]^{\frac{1}{\theta}} C_t^y$:

$$C_t^y + \frac{[\beta(1+r_{t+1})]^{\frac{1}{\theta}} C_t^y}{1+r_{t+1}} = w_t$$

Factor out C_t^y:

$$C_t^y \left(1 + \frac{[\beta(1+r_{t+1})]^{\frac{1}{\theta}}}{1+r_{t+1}}\right) = w_t$$

Define $\varphi_{t+1} \equiv 1 + \beta^{\frac{1}{\theta}} (1+r_{t+1})^{\frac{1-\theta}{\theta}}$:

$$C_t^y = \frac{w_t}{\varphi_{t+1}}, \quad S_t = w_t - C_t^y = w_t \left(\frac{\varphi_{t+1}-1}{\varphi_{t+1}}\right)$$

$$\varphi_{t+1} = \frac{w_t}{C_t^y}$$

Summary:

$$\frac{C_{t+1}^o}{C_t^y} = [\beta(1+r_{t+1})]^{\frac{1}{\theta}}$$

$$C_t^y = \left(\frac{\varphi_{t+1}-1}{\varphi_{t+1}}\right) w_t$$

$$S_t = \frac{w_t}{\varphi_{t+1}}$$

$$\varphi_{t+1} = 1 + \beta^{\frac{1}{\theta}} (1+r_{t+1})^{\frac{1-\theta}{\theta}} > 1$$

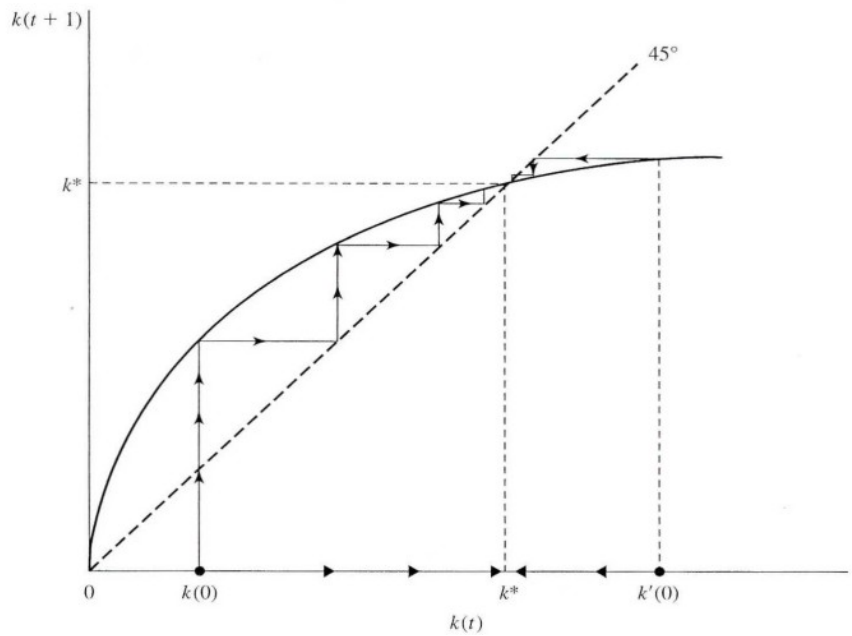

Figura 2: Phase diagram showing the dynamics of capital accumulation. The arrows indicate convergence toward the non-trivial steady state k^*.

- The curved line represents the capital accumulation equation $k_{t+1} = f(k_t)$, while the 45° line shows points where $k_{t+1} = k_t$; their intersection at k^* is the steady state.
- Arrows illustrate that regardless of whether the initial capital stock is below or above k^*, the economy gradually converges toward the steady state — confirming its stability.

Capital Over-Accumulation and Dynamic Inefficiency

- Before continuing, it's important to ask whether the competitive equilibrium in our OLG model is Pareto optimal — that is, whether resources are allocated efficiently across generations.
- In the standard overlapping generations (OLG) model, competitive equilibrium doesn't always lead to Pareto efficiency. In fact, whenever the steady-state interest rate r is less than the population growth rate n, the economy exhibits *dynamic inefficiency.*
- In such a case, the capital stock is too high. Reducing it from the competitive steady state would raise the consumption levels of all generations, improving welfare.
- To explore this, let's work with the OLG model under log utility and Cobb-Douglas production.
- Start from the economy-wide resource constraint:

$$C_t = Y_t - I_t = Y_t - TS_t = F(K_t, L_t) - K_{t+1}$$

- Divide through by labor L_t, using lower-case letters to denote per-worker terms:

$$\Rightarrow \quad c_t \equiv \frac{C_t}{L_t} = f(k_t) - (1+n)k_{t+1}$$

$$\Rightarrow \quad c^* = f(k^*) - (1+n)k^*$$

- Therefore, the derivative of steady-state consumption with respect to capital is:

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n)$$

- The **golden rule** level of capital is the one that maximizes consumption. At that point:

$$\frac{\partial c^*}{\partial k^*} = 0 \quad \Rightarrow \quad f'(k_{\text{gold}}) = (1+n)$$

- For a Cobb-Douglas production function, this gives:

$$k_{\text{gold}} = \left[\frac{\alpha}{1+n}\right]^{\frac{1}{1-\alpha}}$$

- Meanwhile, the competitive economy settles at:

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

- Firms maximize profits by equating marginal product of capital to the gross interest rate:

$$1 + r^* = R^* = f'(k^*) = \alpha k^{\alpha-1} = \frac{\alpha}{1-\alpha}\left(\frac{1+\beta}{\beta}\right)(1+n)$$

- Therefore, we can alternatively express the steady-state capital-labor ratio as:

$$k^* = \left[\frac{\alpha}{1+r}\right]^{\frac{1}{1-\alpha}}$$

- **Intuition:** If the interest rate is too low relative to population growth, the economy is "saving too much" — accumulating more capital than is socially optimal. The golden rule defines the capital level that maximizes consumption. If the competitive outcome exceeds that, reducing capital raises everyone's consumption.

- If $r < n$, then $k^* > k_{\text{gold}}$ and $\frac{\partial c^*}{\partial k^*} < 0$, which implies that the competitive economy is dynamically inefficient!

- In this case, the economy accumulates too much capital — meaning that by reducing savings, we could raise total steady-state consumption.

- This outcome may seem surprising, especially since the model assumes perfect competition and no externalities — yet inefficiency still arises from intergenerational trade-offs.
- **Intuition:** When population grows faster than the return on capital, saving becomes excessive from a social standpoint. Cutting capital improves everyone's consumption, even in a frictionless competitive setup.

Full procedure

Step 1: Resource constraint in per capita terms:

$$C_t = Y_t - I_t = Y_t - K_{t+1} \quad \Rightarrow \quad c_t \equiv \frac{C_t}{L_t} = f(k_t) - (1+n)k_{t+1}$$

At steady state:

$$k_{t+1} = k^* \quad \Rightarrow \quad c^* = f(k^*) - (1+n)k^*$$

Step 2: Marginal condition for dynamic efficiency:

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n)$$

Golden rule:

$$\frac{\partial c^*}{\partial k^*} = 0 \quad \Rightarrow \quad f'(k_{\text{gold}}) = 1+n$$

Step 3: Cobb-Douglas production function:

$$f(k) = k^\alpha \quad \Rightarrow \quad f'(k) = \alpha k^{\alpha-1}$$

Apply golden rule:

$$\alpha k_{\text{gold}}^{\alpha-1} = 1+n \quad \Rightarrow \quad k_{\text{gold}} = \left(\frac{\alpha}{1+n}\right)^{\frac{1}{1-\alpha}}$$

Step 4: Competitive steady state capital level:

$$k^* = \left(\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}\right)^{\frac{1}{1-\alpha}}$$

Step 5: Compare marginal returns and condition:

$$f'(k^*) = 1+r \quad \text{(by firm's FOC)}$$

If $r < n$, then:

$$1+r < 1+n \Rightarrow f'(k^*) < f'(k_{\text{gold}}) \Rightarrow k^* > k_{\text{gold}} \Rightarrow \frac{\partial c^*}{\partial k^*} < 0$$

Conclusion:

$$\text{If } r < n, \text{ the competitive economy is dynamically inefficient.}$$

Intuition for Dynamic Inefficiency

- Individuals living at time t face market prices that are determined by the existing capital stock, which results from past generations' saving behavior.
- In this way, the capital stock today reflects decisions made by earlier generations — shaping the economic environment faced by the current one.
- This creates a **pecuniary externality**: the decisions of previous generations indirectly affect the welfare of current generations by influencing market prices.
- Pecuniary externalities capture how price changes (from others' decisions) affect a household's utility — not through direct consumption effects, but indirectly via the market mechanism.
- These externalities are the root cause of dynamic inefficiency. While pecuniary effects always exist in general equilibrium, they are usually second-order and do not cause inefficiencies.
- However, in the OLG framework, pecuniary externalities can become first-order when there is a continuous flow of new agents entering the economy each period.
- Dynamic inefficiency stems from the economy accumulating too much capital over time.
- This happens because each young generation needs to save for their own retirement, which leads to more and more capital being accumulated.
- Ironically, the more they save, the lower the return on capital becomes — which can unintentionally encourage the next generation to save even more.
- The reduced return on capital is itself a pecuniary externality — a consequence of today's saving decisions that distorts tomorrow's incentives.
- If the economy had better ways to provide for retirement (like pensions), it could avoid this over-accumulation problem.
- A social planner is not constrained by market returns in determining how much old agents can consume. This flexibility allows for better intertemporal allocation.
- Therefore, when the capital stock exceeds the golden rule (i.e., when $r < n$), a planner could increase overall welfare by reallocating resources rather than relying solely on saving.
- This inefficiency doesn't stem from typical frictions or market failures. Instead, it arises from the economy's intergenerational structure — making it a unique case called **dynamic inefficiency**.
- **Intuition:** In OLG models, current saving affects future interest rates, which in turn shape future saving. If this loop leads to excessive capital and declining returns, the economy becomes dynamically inefficient — even in a perfectly competitive setup.

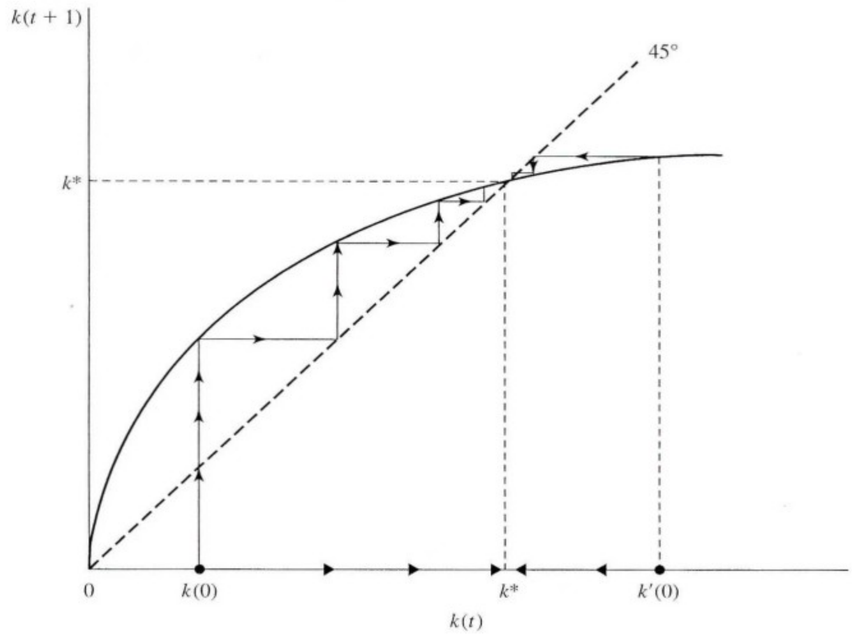

Figura 3: Phase diagram showing the dynamics of capital accumulation. The arrows indicate convergence toward the non-trivial steady state k^*.

- Log utility and Cobb-Douglas production technology
- Provided $r < n$, the steady-state capital level satisfies $k^* > k_{\text{gold}}$
- This implies over-accumulation of capital, making the steady state inefficient
- The result is dynamic inefficiency driven by a pecuniary externality that leads to welfare loss

3.5 Government Budget Deficits in an OLG Model

- This section shows how a simple overlapping generations (OLG) model breaks the connection between the planning horizon of private individuals and that of the government.
- As a result, the classical Ricardian Equivalence fails in this model. Changes in lump-sum taxation can shift the economy's equilibrium because individuals do not internalize the government's intertemporal budget constraint.
- The OLG model also gives very different predictions about private saving behavior compared to standard representative agent models studied so far.

Assumptions: A Small Open Economy Endowment Economy

- Each generation lives for two periods: once as a young worker (denoted Y) and then as an old retiree (denoted O). A new generation is born in every period.

- For simplicity, we normalize the population so that each generation consists of one individual.
- The utility function of a person born at date t is given by:

$$U(c_t^Y, c_{t+1}^O) = \log(c_t^Y) + \beta \log(c_{t+1}^O),$$

which captures preferences over consumption when young and old, with a discount factor β.

- Here, c_t^Y is consumption during youth (period t), and c_{t+1}^O is consumption during old age (period $t+1$) for the same individual.
- We assume log utility and perfect foresight — individuals fully understand the future economic environment.
- Let τ_t^Y represent net lump-sum taxes paid when young at time t, and τ_{t+1}^O when old at time $t+1$. Negative values indicate net transfers (i.e., receiving rather than paying taxes).
- This framework allows for the possibility that taxation may differ by age even within the same period: $\tau_t^Y \neq \tau_t^O$.
- We assume that there is no intergenerational transfer of wealth — in other words, no inheritances are allowed.
- **Intuition:** This setup isolates the role of government taxation and saving decisions over the life cycle in a clean environment. By using log utility, we simplify the algebra while still capturing intertemporal trade-offs. Allowing differential taxes across age groups lets us analyze how fiscal policy affects individual choices in the absence of bequests.
- The individual's lifetime budget constraint is:

$$c_t^Y + \frac{c_{t+1}^O}{1+r} = y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}. \tag{15}$$

- Substituting this into the utility function gives the following maximization problem:

$$\max_{c_t^Y} \log(c_t^Y) + \beta \log\left(y_{t+1}^O - \tau_{t+1}^O + (1+r)y_t^Y - (1+r)\tau_t^Y - (1+r)c_t^Y\right)$$

- Solving this yields the first-order condition (Euler equation):

$$c_{t+1}^O = \beta(1+r)c_t^Y \tag{16}$$

- The path of consumption over time depends on the relative magnitude of β and $\frac{1}{1+r}$:
  - If $\beta > \frac{1}{1+r}$, consumption rises with age (upward-tilting path)
  - If $\beta < \frac{1}{1+r}$, consumption falls with age (downward-tilting path)

- Using both the Euler equation (**??**) and budget constraint (**??**), we derive the explicit consumption functions:

$$c_t^Y = \left(\frac{1}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right) \tag{17}$$

$$c_{t+1}^O = (1+r)\left(\frac{\beta}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right) \tag{18}$$

- **Intuition:** The household spreads consumption across periods by smoothing marginal utility. The Euler equation shows how much old-age consumption is worth in terms of current consumption, while the closed-form solutions clarify how taxes and income from both periods shape optimal behavior.

- Although the individual's optimization problem looks similar to that in a representative agent model, the OLG framework differs because there are always two generations alive at each point in time.

- Therefore, to analyze aggregate outcomes, we must sum over both the young and old individuals living in the economy during period t.

- Aggregate consumption is given by:

$$C_t = c_t^Y + c_t^O.$$

- To derive the government's budget constraint, we must account for the fact that the young and old can face different taxes.

- The government budget constraint is:

$$B_{t+1}^G - B_t^G = \tau_t^Y + \tau_t^O + rB_t^G - G_t, \tag{19}$$

  where:

  - G_t: government consumption/spending
  - B_t^G: government assets (if negative, this is public debt)
  - τ_t^Y, τ_t^O: taxes paid by the young and old generations

- The transversality condition (TVC) on government assets requires:

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T B_{t+T+1}^G = 0,$$

  ensuring that debt doesn't explode in the long run.

- **Intuition:** In contrast to representative agent models, the OLG setting features overlapping generations that jointly determine aggregate variables. Fiscal policy must balance not just intertemporal budgets, but also how taxes and transfers affect both cohorts simultaneously.

- Using equation (??) and imposing the transversality condition, the government's intertemporal budget constraint becomes:

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} G_s = (1+r)B_t^G + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(\tau_s^Y + \tau_s^O)$$

- This has the same form as in the representative agent model, since aggregate tax revenue is just the sum $T = \tau^Y + \tau^O$.

- To simplify the analysis, we assume that all relevant variables are constant over time:

$$y_t^Y, y_t^O, \tau_t^Y, \tau_t^O, G_t \text{ are constant.}$$

- In this case, each young agent is identical across generations. So, both young and old individuals have constant consumption:

$$c^Y = \left(\frac{1}{1+\beta}\right)\left(y^Y - \tau^Y + \frac{y^O - \tau^O}{1+r}\right)$$

$$c^O = (1+r)\left(\frac{\beta}{1+\beta}\right)\left(y^Y - \tau^Y + \frac{y^O - \tau^O}{1+r}\right)$$

- Thus, aggregate consumption is also constant over time:

$$C = c^Y + c^O = \left(\frac{1+\beta(1+r)}{1+\beta}\right)\left(y^Y - \tau^Y + \frac{y^O - \tau^O}{1+r}\right)$$

- **Intuition:** With constant incomes, taxes, and preferences, each generation behaves identically, and consumption remains stable over time. Government debt dynamics do not disrupt this equilibrium as long as the intertemporal budget constraint and transversality condition are satisfied.

- The government's lifetime budget constraint can be written as:

$$G = rB^G + \tau^Y + \tau^O.$$

- Substituting this into the aggregate consumption equation to eliminate τ^O, we obtain:

$$C = \left(\frac{1+\beta(1+r)}{1+\beta}\right)\left(y^Y + \frac{y^O - G - r\tau^Y + rB^G}{1+r}\right)$$

- Importantly, we don't need the assumption $\beta = \frac{1}{1+r}$ to achieve constant aggregate consumption.

- While individuals may experience changing consumption over their lifetime, each person is replaced by an identical new agent after two periods. This means the overall cross-sectional pattern of consumption remains steady.

- Therefore, the *cross-sectional profile* of consumption across cohorts is constant, even if individuals' consumption paths tilt over time.
- Aggregate consumption depends not only on public spending G, but also on how taxes are distributed and the size of government assets (or debt).
- **Conclusion:** Because private agents do not internalize the government's intertemporal budget — especially in an overlapping generations setting — **Ricardian Equivalence fails**.
- **Intuition:** In the OLG model, the government's debt and tax choices affect aggregate demand — not because people are irrational, but because each generation is economically distinct. Without dynastic links, agents do not offset government borrowing with saving.

Full procedure

Step 1: Lifetime budget constraint

$$c_t^Y + \frac{c_{t+1}^O}{1+r} = y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}$$

Step 2: Utility maximization (log utility)

$$\max_{c_t^Y} \log(c_t^Y) + \beta \log(c_{t+1}^O)$$

Substitute c_{t+1}^O from budget constraint:

$$c_{t+1}^O = (1+r)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r} - c_t^Y\right)$$

Objective becomes:

$$\max_{c_t^Y} \log(c_t^Y) + \beta \log\left((1+r)\left[y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r} - c_t^Y\right]\right)$$

First-order condition:

$$\frac{d}{dc_t^Y}\left[\log(c_t^Y) + \beta \log(c_{t+1}^O)\right] = 0$$
$$\Rightarrow \frac{1}{c_t^Y} - \beta \cdot \frac{(1+r)}{c_{t+1}^O} = 0$$
$$\Rightarrow c_{t+1}^O = \beta(1+r)c_t^Y$$

Euler equation:

$$c_{t+1}^O = \beta(1+r)c_t^Y \qquad \text{(Euler)}$$

Step 3: Solve consumption from budget and Euler equation

Plug Euler into budget constraint:

$$c_t^Y + \frac{\beta(1+r)c_t^Y}{1+r} = y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}$$

Simplify:

$$c_t^Y(1+\beta) = y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}$$

Solve for c_t^Y:

$$c_t^Y = \left(\frac{1}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right)$$

Then:

$$c_{t+1}^O = \beta(1+r)c_t^Y = (1+r)\left(\frac{\beta}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right)$$

Final consumption functions:

$$c_t^Y = \left(\frac{1}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right) \tag{17}$$

$$c_{t+1}^O = (1+r)\left(\frac{\beta}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right) \tag{18}$$

Step 4: Aggregate consumption (2 generations alive)

$$C_t = c_t^Y + c_t^O$$

If all variables constant over time:

$$y_t^Y = y^Y, \quad y_{t+1}^O = y^O, \quad \tau_t^Y = \tau^Y, \quad \tau_{t+1}^O = \tau^O$$

$$C = \left(\frac{1+\beta(1+r)}{1+\beta}\right)\left(y^Y - \tau^Y + \frac{y^O - \tau^O}{1+r}\right)$$

Step 5: Government budget constraint

$$G = rB^G + \tau^Y + \tau^O$$

Substitute into consumption equation to eliminate τ^O:

$$C = \left(\frac{1+\beta(1+r)}{1+\beta}\right)\left(y^Y + \frac{y^O - G - r\tau^Y + rB^G}{1+r}\right)$$

Key result: Ricardian equivalence fails.

Government Saving, Private Saving and the Current Account

- How do changes in the government's budget deficit affect the nation's current account balance?
- The current account reflects the change in the economy's net foreign asset position.

- In a setting without physical investment, all net assets are simply financial claims on foreigners — meaning no domestic capital accumulation complicates the picture.
- Let total national assets be:
$$B_t = B_t^p + B_t^G,$$
where B^p is private sector foreign assets and B^G is government foreign assets.
- Then, the current account is defined as:
$$CA_t = B_{t+1} - B_t = B_{t+1}^p + B_{t+1}^G - (B_t^p + B_t^G) = (B_{t+1}^p - B_t^p) + (B_{t+1}^G - B_t^G). \tag{20}$$
- This shows that the current account equals total net saving in the economy — i.e., the sum of:
$$S_t^p = B_{t+1}^p - B_t^p \quad \text{(private saving)}$$
$$S_t^G = B_{t+1}^G - B_t^G \quad \text{(government saving)}$$
- Alternatively, one can still define the current account from the national income identity:
$$CA_t = rB_t + Y_t - C_t - G_t,$$
but equation (**??**) provides a cleaner decomposition for our purpose.
- **Intuition:** The current account measures how much the country as a whole is saving or dissaving relative to the rest of the world. When the government runs a deficit (i.e., saves less or dissaves), this tends to reduce the current account — unless offset by higher private saving.
- The question we now explore is: For a given government policy, how is aggregate private saving determined?
- At the end of period t, the private financial assets in the economy equal the savings of the young in that period, since the old consume all remaining wealth and die with zero assets.
- The young of period t start with no assets, so their saving is:
$$S_t^Y = B_{t+1}^p \tag{21}$$
- The old of period t de-accumulate all assets they previously saved in youth (or repay debt), so their saving is:
$$S_t^O = -S_{t-1}^Y = -B_t^p \tag{22}$$
- Therefore, total net private saving is:
$$S_t^p = S_t^Y + S_t^O = B_{t+1}^p - B_t^p$$
- This is simply the sum of saving by the young and dissaving by the old — which together determine how private net foreign assets evolve over time.

- **Intuition:** In the OLG model, private saving isn't just what current agents set aside — it's the net result of what the young are saving today minus what the old are withdrawing. This two-generation dynamic gives rise to aggregate saving behavior quite different from representative agent models.

- A key implication of equation (**??**) is that the economy's net foreign assets at the end of period t are the sum of savings by the young and the government:

$$B_{t+1} = B_{t+1}^p + B_{t+1}^G = S_t^Y + B_{t+1}^G$$

- In the special case where $\beta(1+r) = 1$, individual consumption paths are flat:

$$c_t^Y = c_{t+1}^O$$

- From equations (17) and (18), this implies:

$$c_t^Y = \left(\frac{1}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right)$$

$$c_{t+1}^O = \left(\frac{1}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right)$$

- With this, saving by the young becomes:

$$S_t^Y = y_t^Y - \tau_t^Y - c_t^Y = \frac{\beta}{1+\beta}\left[\left(y_t^Y - \tau_t^Y\right) - \left(\frac{y_{t+1}^O - \tau_{t+1}^O}{1+r}\right)\right] = B_{t+1}^p \tag{23}$$

- **Intuition:** When preferences and returns are aligned (i.e., $\beta(1+r) = 1$), agents choose a perfectly smooth consumption path. In this case, saving by the young exactly bridges the gap between their own needs and anticipated old-age consumption — allowing us to directly express saving as a simple function of lifetime resources.

- Consolidating equations (**??**) and (**??**), which relate to the saving behavior of the young and old respectively, we derive the expression for total private saving:

$$S_t^p = B_{t+1}^p - B_t^p = S_t^Y - S_{t-1}^Y = \frac{\beta}{1+\beta}\left[\Delta(y_t^Y - \tau_t^Y) - \Delta(y_{t+1}^O - \tau_{t+1}^O)\right] \tag{24}$$

- Here, the disposable income changes are defined as:

$$\Delta(y_t^Y - \tau_t^Y) = (y_t^Y - \tau_t^Y) - (y_{t-1}^Y - \tau_{t-1}^Y)$$

$$\Delta(y_{t+1}^O - \tau_{t+1}^O) = (y_{t+1}^O - \tau_{t+1}^O) - (y_t^O - \tau_t^O)$$

- **Interpretation:** This expression tells us that total private saving in the economy is driven by how the disposable income of different age cohorts changes over time.

- **Implication:**

- When the disposable income of the young increases more than that of the old, i.e.,

$$\Delta(y_t^Y - \tau_t^Y) > \Delta(y_{t+1}^O - \tau_{t+1}^O),$$

total private saving rises.

- This aligns with the idea that rising productivity or income for the young (relative to the old) leads to more saving by the young and less dissaving by the old.

- **Intuition:**

  - Aggregate private saving captures how the life-cycle income profile of households evolves.
  - The model highlights that private saving is not just a function of current income, but also of intertemporal shifts in income across generations.
  - Hence, economies with rapidly increasing productivity among the young tend to exhibit higher private saving rates.

- A rise in expected aggregate output growth raises the aggregate savings rate when the young are net savers. Higher growth influences the overall saving rate through a scale effect — it increases the wealth accumulated by young savers relative to the wealth de-accumulated by the old.

- This outcome does *not* occur in a representative agent framework, because in that setup higher expected future income reduces the incentive to save today.

- **Intuition:** The OLG model reveals a key departure from representative agent models — here, forward-looking young agents save more under growth because they do not internalize the reduced saving needs of future generations.

Full procedure
Government Saving, Private Saving, and the Current Account:
Step 1: Define the current account as the change in total net foreign assets

$$CA_t = B_{t+1} - B_t$$

Step 2: Decompose total assets into private and government components

$$B = B^p + B^G \Rightarrow CA_t = B_{t+1}^p - B_t^p + B_{t+1}^G - B_t^G$$

Step 3: Define savings for private and government sectors

$$S_t^p = B_{t+1}^p - B_t^p, \quad S_t^G = B_{t+1}^G - B_t^G \Rightarrow CA_t = S_t^p + S_t^G$$

Step 4: Define private saving from young and old

$$S_t^Y = B_{t+1}^p \quad \text{(young save)}$$
$$S_t^O = -B_t^p \quad \text{(old dissave)}$$
$$S_t^p = S_t^Y + S_t^O = B_{t+1}^p - B_t^p$$

Step 5: Using Euler equation and budget constraint, we solve for saving of young

$$S_t^Y = y_t^Y - \tau_t^Y - c_t^Y = \frac{\beta}{1+\beta}\left[(y_t^Y - \tau_t^Y) - (y_{t+1}^O - \tau_{t+1}^O)\right] \Rightarrow B_{t+1}^p = S_t^Y$$

Step 6: Total private saving becomes

$$S_t^p = \frac{\beta}{1+\beta}\left[\Delta(y_t^Y - \tau_t^Y) - \Delta(y_{t+1}^O - \tau_{t+1}^O)\right]$$

Step 7: Total net foreign assets now satisfy

$$B_{t+1} = S_t^Y + B_{t+1}^G \Rightarrow CA_t = S_t^p + S_t^G$$

Summary: Private saving depends on changes in disposable income across generations. High growth in young's income leads to higher saving, increasing net foreign assets.

The Timing of Taxes: An Example

- In OLG models, the timing of taxes can have significant effects on both aggregate consumption and the current account.
- Consider an example: suppose the government finances a *one-time transfer payment* (a "gift") using debt.
- This transfer is split equally between the current young and the current old generation.
- In a representative agent model, such a debt-financed transfer would have no real effect — neither on consumption, nor the current account, nor welfare. Why?
- Because a forward-looking representative agent anticipates that future taxes will need to repay the debt. As a result, the agent saves the transfer entirely to offset this expected liability.
- **Key Insight:** In our OLG framework, the two-period-lived individuals *do not fully internalize* future taxes, especially since the young and old are different people. Hence, this intertemporal neutrality breaks down.
- **Conclusion:** The timing of taxes *does matter* in the OLG model. A bond-financed gift can change both current consumption and the current account.

Description of the Fiscal Policy

- Suppose that in period $t = 0$, the government lowers the per capita taxes paid by both the young and old by $d/2$, financing its higher budget deficit in period 0 by selling bonds worth d to each of the current young.
- That is, the current tax bill of the young falls to $\tau_0^Y - \frac{d}{2}$ and the current tax bill of the old falls to $\tau_0^O - \frac{d}{2}$.

- The government's net end-of-period assets B_1^G consequently decline to $B_1^G - d$.
- We will assume that the tax burden due to future interest payments on the added debt rd is split evenly between young and old generations.
- That is, for all $t \geq 1$, per capita taxes on the young rise to $\tau_t^Y + \frac{rd}{2}$ and per capita taxes on the old rise to $\tau_t^O + \frac{rd}{2}$.
- **What are the consequences of such a policy?** Unlike the representative agent model, aggregate consumption rises in the short run and falls in the long run.
- Let variables with asterisks * denote the economy's path after the fiscal policy is implemented.
- The period 0 old clearly consume their entire "gift," so:

$$c_0^{O*} = c_0^O + \frac{d}{2}. \tag{25}$$

- The young in period 0 receive the same gift of $d/2$ as the old, but they do *not* consume it all at once. Why?
    1. They prefer to smooth their consumption over both periods of life.
    2. They will face higher taxes of $rd/2$ in old age, so the net benefit of the gift is smaller for them.
- **Without the gift**, the young's consumption demand is:

$$c_0^Y = \left(\frac{1}{1+\beta}\right)\left(y_0^Y - \tau_0^Y + \frac{y_1^O - \tau_1^O}{1+r}\right)$$

- **With the gift**, the young adjust their consumption:

$$c_0^{Y*} = c_0^Y + \frac{1}{1+\beta}\left[\frac{d}{2} - \left(\frac{1}{1+r}\right)\frac{rd}{2}\right] = c_0^Y + \frac{1}{1+\beta}\left(\frac{1}{1+r}\right)\frac{d}{2} \tag{26}$$

- **Intuition:** the young consume *less* of the gift upfront because they expect to be taxed later. They internalize that some of today's benefit will need to be repaid.
- Adding the old's and young's consumption responses:

$$c_0^{O*} + c_0^{Y*} - (c_0^O + c_0^Y) = \left[1 + \frac{1}{(1+\beta)(1+r)}\right]\frac{d}{2}$$

- **Conclusion:** Aggregate consumption in period 0 increases due to the fiscal transfer, but the total rise is *less than* the total size of the transfer d, since part of it is saved in anticipation of future taxes.

- While period 0 aggregate consumption has risen, what happens next periods?
- The period 1 old generation still enjoys higher consumption (from (18)):

$$c_{t+1}^0 = (1+r)\left(\frac{\beta}{1+\beta}\right)\left(y_t^Y - \tau_t^Y + \frac{y_{t+1}^0 - \tau_{t+1}^0}{1+r}\right),$$

- Simplifying gives:

$$c_1^{0*} = c_1^0 + (1+r)\frac{\beta}{1+\beta}\left[\frac{d}{2} - \left(\frac{1}{1+r}\right)\frac{rd}{2}\right] = c_1^0 + \left(\frac{\beta}{1+\beta}\right)\frac{d}{2}. \tag{27}$$

- However, the period 1 young and all later generations lose, since higher taxes reduce their lifetime income:

$$-\left[\frac{rd}{2} + \left(\frac{1}{1+r}\right)\frac{rd}{2}\right] = -\frac{(2r+r^2)}{1+r}\frac{d}{2}.$$

- Their consumption when young falls to:

$$c_t^{Y*} = c_t^Y - \frac{1}{1+\beta}\frac{(2r+r^2)}{1+r}\frac{d}{2}. \tag{28}$$

- And their consumption when old falls to:

$$c_t^{0*} = c_t^0 - (1+r)\frac{\beta}{1+\beta}\frac{(2r+r^2)}{1+r}\frac{d}{2} = c_t^0 - \frac{\beta}{1+\beta}(2r+r^2)\frac{d}{2}. \tag{29}$$

- *Intuition:* The initial old gain because they enjoy higher consumption, but future generations face lower lifetime resources due to distortionary taxes. This leads to reduced consumption both when young and old.
- Combining (27) and (28) (the latter for $t = 1$), we see that aggregate period 1 consumption changes by:

$$c_1^{0*} + c_1^{Y*} - (c_1^0 + c_1^Y) = \left[\frac{\beta}{1+\beta} - \frac{1}{1+\beta}\left(\frac{2r+r^2}{1+r}\right)\right]\frac{d}{2},$$

- This expression has an ambiguous sign.
- Combining (28) and (29), it is straightforward to show that aggregate consumption is unambiguously lower from period 2 onwards:

$$c_2^{0*} + c_2^{Y*} - (c_2^0 + c_2^Y) = -\frac{1}{1+\beta}[2r+r^2]\left[\beta + \frac{1}{1+r}\right]\frac{d}{2} < 0.$$

- The new government budget constraint has real effects: transfer and tax policies shift income across generations.
- Generations alive at time 0 gain, since they receive a net positive transfer (financed by future generations), which raises their consumption.

- Future generations, however, are "hurt" by the budget deficit. Their consumption is lower, but this does not affect period 0 aggregate consumption because these cohorts had not yet been born.
- *Intuition:* The deficit redistributes across generations. The initial old benefit, but starting from period 2, aggregate consumption is strictly lower, as the tax burden reduces the resources of all later generations.
- Ricardian Equivalence fails because government borrowing shifts current taxes from today's generation onto unrelated future generations who will be born later.
- We can solve for the path of the current account:

$$CA_t = rB_t + Y_t - C_t - G_t.$$

- Since output and government spending are constant, and net foreign assets in period 0 are given, the only element of the current account equation that can change is aggregate consumption.
- Hence, the current account change in period 0 is negative but smaller than 1:

$$CA_0^* - CA_0 = -\Big[c_0^{0*} + c_0^{Y*} - (c_0^0 + c_0^Y)\Big] = -\left[1 + \frac{1}{(1+\beta)(1+r)}\right]\frac{d}{2}.$$

- To find the period 1 current account change, notice that the increase in net foreign assets at the start of period 1 equals the change in the period 0 current account.
- Therefore:

$$CA_1^* - CA_1 = r\Big[CA_0^* - CA_0\Big] - \Big[c_1^{0*} + c_1^{Y*} - (c_1^0 + c_1^Y)\Big] = -\frac{\beta}{1+\beta}(1+r)\frac{d}{2}.$$

- This remains negative in period 1.
- *Intuition:* Borrowing today raises consumption for the current old, but shifts the repayment burden to the future. As a result, the current account worsens immediately and stays negative, reflecting intergenerational redistribution.
- It is easy to verify that there is no further change in the current account for $t > 1$, since from that point all generations share the debt burden equally.
- Therefore, the current account only worsens temporarily but returns to its original path after two periods.
- However, the period 0 fiscal deficit leaves permanent effects:
  - For those born in or after period 1, consumption is lower in both periods of life.
  - The higher current account deficits in periods 0 and 1 mean that government indebtedness reduces the economy's net foreign asset position.

- The higher consumption enjoyed by those alive in period 0 is financed by accumulating foreign debt, which future generations must repay.

▪ *Intuition:* The deficit creates only a short-lived deterioration in the current account, but its true legacy is permanent: future generations suffer lower lifetime consumption because they must service the debt incurred to finance period 0's higher consumption.

Full procedure
Policy at $t = 0$:

$$\tau_0^{Y*} = \tau_0^Y - \frac{d}{2}, \qquad \tau_0^{O*} = \tau_0^O - \frac{d}{2}, \qquad B_1^{G*} = B_1^G - d,$$

$$\tau_t^{Y*} = \tau_t^Y + \frac{rd}{2}, \qquad \tau_t^{O*} = \tau_t^O + \frac{rd}{2} \qquad (t \geq 1).$$

Period 0 old: consume gift

$$c_0^{0*} = c_0^0 + \frac{d}{2}. \tag{25}$$

Period 0 young: baseline demand (from (17))

$$c_0^Y = \frac{1}{1+\beta}\left(y_0^Y - \tau_0^Y + \frac{y_1^O - \tau_1^O}{1+r}\right).$$

PV income change for $t = 0$ young

$$\Delta \text{PV} = \frac{d}{2} - \frac{1}{1+r}\frac{rd}{2}.$$

Young consumption response

$$\Delta c_0^Y = \frac{1}{1+\beta}\left[\frac{d}{2} - \frac{1}{1+r}\frac{rd}{2}\right] = \frac{1}{(1+\beta)(1+r)}\frac{d}{2},$$

$$c_0^{Y*} = c_0^Y + \Delta c_0^Y = c_0^Y + \frac{1}{1+\beta}\left[\frac{d}{2} - \frac{rd}{2(1+r)}\right]. \tag{26}$$

Aggregate period 0 consumption change

$$(c_0^{0*} + c_0^{Y*}) - (c_0^0 + c_0^Y) = \frac{d}{2} + \frac{1}{1+\beta}\left[\frac{d}{2} - \frac{rd}{2(1+r)}\right] = \boxed{\left[1 + \frac{1}{(1+\beta)(1+r)}\right]\frac{d}{2}}.$$

3.6 Global Effects of Government Deficits

- Up to now we studied a small open economy with an exogenous world interest rate.
- We now ask: how do tax policies of large countries affect the global economy?
- To address this, we set up a two-country equilibrium model with overlapping generations.
- Assumptions:
  - Capital is perfectly mobile across countries, but labor is not (no migration or immigration).
  - This is a two-country extension of the closed-economy Diamond model from Section 3.4.
- There are two countries: Home (H) and Foreign (F). Both share identical preferences and technologies. Foreign variables are denoted with an asterisk *.
- As before, each agent lives for two periods: supplies labor when young and nothing when old.
- *Intuition:* Moving from a small to a large open economy changes the role of fiscal policy: now government deficits can influence the world interest rate, transmitting effects across both countries through capital flows.
- In Home (Foreign), the young generation born at date t has N_t (N_t^*) members.
- The levels of the two populations may differ, but both have the same net growth rate n:

$$N_t = (1+n)N_{t-1}, \qquad N_t^* = (1+n)N_{t-1}^*.$$

- Assumption: the young pay taxes, the old do not.
- The savings problem of a young person in Home is:

$$U(c_t^Y, c_{t+1}^O) = \log c_t^Y + \beta \log c_{t+1}^O,$$

  subject to

$$c_t^Y + s_t^Y = w_t - \tau_t^Y, \qquad c_{t+1}^O = (1 + r_{t+1})s_t^Y.$$

- The FOC gives the standard Euler equation. Combined with the budget constraint, savings in Home are:

$$s_t^Y = \frac{\beta}{1+\beta}(w_t - \tau_t^Y). \tag{30}$$

- Similarly, for Foreign:

$$s_t^{Y*} = \frac{\beta}{1+\beta}(w_t^* - \tau_t^{Y*}).$$

- *Intuition:* Each young agent allocates income between present and future consumption, saving a fixed fraction $\frac{\beta}{1+\beta}$ of net labor income. Both Home and Foreign economies follow the same logic, so any fiscal policy in one country directly affects savings and capital flows globally.

- Assume both countries have a Cobb–Douglas production function:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}, \qquad 0 < \alpha < 1.$$

- Under competitive markets, factors earn their marginal products.

- Marginal product of capital (rental rate of return):

$$R_t = F_K(K_t, L_t) = \alpha K_t^{\alpha-1} L_t^{1-\alpha} = \alpha k_t^{\alpha-1}.$$

- Marginal product of labor (wage rate):

$$w_t = F_L(K_t, L_t) = (1-\alpha) K_t^{\alpha} L_t^{-\alpha} = (1-\alpha) k_t^{\alpha}. \tag{31}$$

- Define $k_t \equiv K_t / L_t$ as the capital–labor ratio.

- With integrated world capital markets, the capital–labor ratio must be the same across both countries, since technologies are identical. Denote this common value as k^W.

- Then:

$$R_t = \alpha k_t^{\alpha-1} = \alpha (k_t^*)^{\alpha-1} = \alpha (k^W)^{\alpha-1}. \tag{32}$$

- *Intuition:* In equilibrium, capital mobility ensures a common world interest rate. Wages differ only through population sizes, while returns to capital depend on the shared world capital–labor ratio.

- Assume initially no government debt, so taxes are zero: $\tau_t^Y = \tau_t^{Y*} = 0$.

- Without government debt, global equilibrium requires that aggregate savings of the young equal the world supply of capital available next period:

$$K_{t+1} + K_{t+1}^* = N_t s_t^Y + N_t^* s_t^{Y*}. \tag{33}$$

- With immobile labor, market clearing requires:

$$L_t = N_t, \qquad L_t^* = N_t^*.$$

- From (30) and (31), equilibrium saving of a young Home resident is:

$$s_t^Y = \frac{\beta}{1+\beta}(w_t - \tau_t^Y) = \frac{\beta}{1+\beta} w_t = \frac{\beta(1-\alpha)}{1+\beta} (k_t^W)^{\alpha}.$$

- Similarly, for a young Foreign resident:

$$s_t^{Y*} = \frac{\beta(1-\alpha)}{1+\beta}(k_t^W)^\alpha.$$

- *Intuition:* With no government debt, both Home and Foreign young save the same fixed share of net labor income. Since labor is immobile but capital is mobile, savings pool globally and determine the world capital–labor ratio.

- Using the equilibrium condition (33):

$$K_{t+1} + K_{t+1}^* = N_t s_t^Y + N_t^* s_t^{Y*},$$

and substituting savings expressions:

$$K_{t+1} + K_{t+1}^* = \frac{\beta(1-\alpha)}{1+\beta}(k_t^W)^\alpha(N_t + N_t^*).$$

- Divide both sides by the world labor force $N_t + N_t^*$, and note:

$$\frac{K_{t+1} + K_{t+1}^*}{N_t + N_t^*} = (1+n)\frac{K_{t+1} + K_{t+1}^*}{N_{t+1} + N_{t+1}^*} = (1+n)\frac{K_{t+1} + K_{t+1}^*}{L_{t+1} + L_{t+1}^*} = (1+n)k_{t+1}^W.$$

- Therefore, the law of motion for the world capital–labor ratio is:

$$k_{t+1}^W = \frac{\beta(1-\alpha)}{(1+n)(1+\beta)}(k_t^W)^\alpha. \tag{34}$$

- *Intuition:* World savings by the young finance future capital. Since population grows at rate n, effective capital per worker evolves through a non-linear difference equation, determining the global dynamics of k^W.

Full procedure

Step 1: Demography

$$N_t = (1+n)N_{t-1}, \qquad N_t^* = (1+n)N_{t-1}^*, \qquad L_t = N_t,\ L_t^* = N_t^*.$$

Step 2: Household problem (young pay taxes, old pay none)

$$\max_{s_t^Y}\ \log c_t^Y + \beta \log c_{t+1}^O \quad \text{s.t.} \quad c_t^Y + s_t^Y = w_t - \tau_t^Y,\ \ c_{t+1}^O = (1+r_{t+1})s_t^Y.$$

FOC (Euler) ⇒ savings rule

$$\frac{1}{c_t^Y} = \beta\frac{1+r_{t+1}}{c_{t+1}^O} \ \Rightarrow\ s_t^Y = \frac{\beta}{1+\beta}(w_t - \tau_t^Y), \qquad s_t^{Y*} = \frac{\beta}{1+\beta}(w_t^* - \tau_t^{Y*}). \tag{30}$$

Step 3: Technology and factor prices (Cobb–Douglas)

$$Y_t = K_t^\alpha L_t^{1-\alpha}, \quad 0<\alpha<1, \quad k_t \equiv K_t/L_t.$$

$$R_t = F_K = \alpha k_t^{\alpha-1}, \qquad w_t = F_L = (1-\alpha)k_t^{\alpha}. \tag{31}$$

Step 4: Integrated world capital (identical technologies)

$$k_t = k_t^* = k_t^W \;\Rightarrow\; R_t = \alpha(k_t^W)^{\alpha-1} = R_t^*. \tag{32}$$

Step 5: Baseline with no government debt

$$\tau_t^Y = \tau_t^{Y*} = 0.$$

Step 6: World capital accumulation / market clearing

$$K_{t+1} + K_{t+1}^* = N_t s_t^Y + N_t^* s_t^{Y*}. \tag{33}$$

Substitute (30) and (31) with $\tau = 0$, $k_t = k_t^W$

$$s_t^Y = \frac{\beta}{1+\beta}(1-\alpha)(k_t^W)^{\alpha}, \qquad s_t^{Y*} = \frac{\beta}{1+\beta}(1-\alpha)(k_t^W)^{\alpha}.$$

$$\Rightarrow\; K_{t+1} + K_{t+1}^* = \frac{\beta(1-\alpha)}{1+\beta}(k_t^W)^{\alpha}\,(N_t + N_t^*).$$

Step 7: Law of motion for world capital–labor ratio

$$k_{t+1}^W = \frac{K_{t+1} + K_{t+1}^*}{L_{t+1} + L_{t+1}^*} = \frac{K_{t+1} + K_{t+1}^*}{N_{t+1} + N_{t+1}^*} = \frac{\beta(1-\alpha)}{1+\beta}(k_t^W)^{\alpha}\,\frac{N_t + N_t^*}{N_{t+1} + N_{t+1}^*}.$$

$$N_{t+1} + N_{t+1}^* = (1+n)(N_t + N_t^*) \;\Rightarrow\; \boxed{k_{t+1}^W = \frac{\beta(1-\alpha)}{(1+n)(1+\beta)}(k_t^W)^{\alpha}}. \tag{34}$$

- **Step 1: Steady-state values.** The steady state satisfies $k_{t+1}^W = k_t^W = \bar{k}^W$.
- From (34), there are two steady states:

$$\bar{k}^W(1) = 0, \qquad \bar{k}^W(2) = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}.$$

- **Step 2: Local stability of the non-trivial steady state.** The derivative of (34) is:

$$f'(k_t^W) = \frac{\alpha\beta(1-\alpha)}{(1+n)(1+\beta)}(k_t^W)^{\alpha-1}. \tag{35}$$

- Evaluating at the non-trivial steady state $\bar{k}^W(2)$:

$$f'(\bar{k}^W(2)) = \alpha. \tag{36}$$

- Since $0 < \alpha < 1$, the non-trivial steady state is locally stable under Cobb–Douglas technology. Moreover, monotonic convergence occurs because $f'(k^W) > 0$.

- *Intuition:* The world capital–labor ratio converges to a positive steady state. Stability is guaranteed because the slope at the steady state is $\alpha < 1$, ensuring dampened adjustments over time.

- **Step 3: Draw the Phase Diagram**

- From (34), the first derivative is:

$$\frac{dk_{t+1}^W}{dk_t^W} = \frac{\alpha\beta(1-\alpha)}{(1+n)(1+\beta)}(k_t^W)^{\alpha-1} > 0,$$

  so the transition curve is upward-sloping for all $k_t^W > 0$.

- Next, compute the second derivative:

$$\frac{d^2k_{t+1}^W}{d(k_t^W)^2} = \frac{\alpha\beta(1-\alpha)(\alpha-1)}{(1+n)(1+\beta)}(k_t^W)^{\alpha-2} < 0,$$

  which shows the transition curve is strictly concave for all $k_t^W > 0$.

- *Intuition:* The dynamics of the world capital–labor ratio follow a concave, upward-sloping transition path. This guarantees convergence towards the steady state from below and rules out oscillatory dynamics.

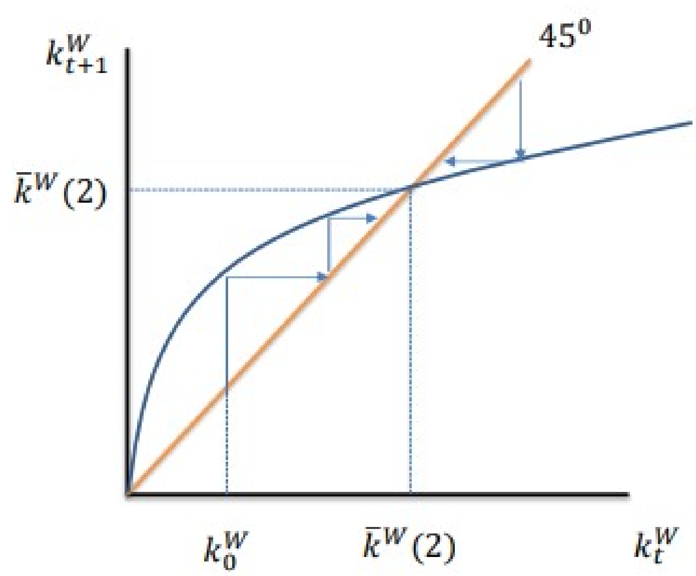

Figura 4: Phase diagram

- The phase diagram shows that the world capital–labor ratio k_t^W converges monotonically towards the stable steady state $\bar{k}^W(2)$.

- If the economy starts with $k_0^W < \bar{k}^W(2)$, capital accumulates gradually along the transition curve until the steady state is reached.

Full procedure

Step 1: Steady state(s)

From law of motion:

$$k_{t+1}^W = \frac{\beta(1-\alpha)}{(1+n)(1+\beta)}(k_t^W)^\alpha.$$

Steady state requires $k_{t+1}^W = k_t^W = \bar{k}^W$:

$$\bar{k}^W = \frac{\beta(1-\alpha)}{(1+n)(1+\beta)}(\bar{k}^W)^\alpha.$$

Trivial solution:

$$\bar{k}^W(1) = 0.$$

Non-trivial solution:

$$\bar{k}^W(2) = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}.$$

—

Step 2: Local stability of non-trivial steady state

Derivative of transition function:

$$f'(k_t^W) = \frac{d}{dk_t^W}\left(\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}(k_t^W)^\alpha\right).$$

$$f'(k_t^W) = \frac{\alpha\beta(1-\alpha)}{(1+n)(1+\beta)}(k_t^W)^{\alpha-1}. \tag{35}$$

At $\bar{k}^W(2)$:

$$f'(\bar{k}^W(2)) = \alpha. \tag{36}$$

Since $0 < \alpha < 1$, steady state is locally stable.

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Step 3: Phase diagram properties

First derivative positive:

$$\frac{dk_{t+1}^W}{dk_t^W} = \frac{\alpha\beta(1-\alpha)}{(1+n)(1+\beta)}(k_t^W)^{\alpha-1} > 0,$$

$\Rightarrow$ transition curve is upward sloping.

Second derivative:

$$\frac{d^2k_{t+1}^W}{d(k_t^W)^2} = \frac{\alpha\beta(1-\alpha)(\alpha-1)}{(1+n)(1+\beta)}(k_t^W)^{\alpha-2} < 0,$$

$\Rightarrow$ transition curve is strictly concave.

Result: Dynamics are monotonic and concave, converging to $\bar{k}^W(2)$.

- We now analyze the global effect of government deficits and debts.
- Suppose the Home government, starting with zero net assets, issues a positive quantity of debt to the current old as a transfer (a "gift").
- The Home government then taxes all future young generations so that the government debt-to-labor ratio remains constant:

$$-\frac{B_t^G}{N_t} = \bar{d}.$$

- With no government consumption, the government budget constraint is:

$$B_{t+1}^G = (1 + r_t)B_t^G + N_t \tau_t^Y,$$

where τ_t^Y is the tax per unit of labor and r_t is the endogenous interest rate.

- Divide through by N_{t+1}:

$$\bar{d} = -\frac{B_{t+1}^G}{N_{t+1}} = (1 + r_t)\frac{N_t}{N_{t+1}}\bar{d} - \frac{N_t}{N_{t+1}}\tau_t^Y.$$

- Since $\frac{N_t}{N_{t+1}} = \frac{1}{1+n}$, we obtain:

$$\bar{d} = \frac{(1 + r_t)\bar{d} - \tau_t^Y}{1 + n},$$

which rearranges to:

$$\tau_t^Y = (r_t - n)\bar{d}. \tag{37}$$

- *Intuition:* Issuing debt today requires future generations to pay higher taxes. The tax rate depends positively on the interest rate and debt burden, and negatively on population growth, which helps dilute the debt.
- Combining (37) with (30) gives the new saving function for the Home young:

$$s_t^Y = \frac{\beta}{1 + \beta}(w_t - \tau_t^Y) = \frac{\beta}{1 + \beta}(w_t - (r_t - n)\bar{d}).$$

- Using (31) and (32):

$$s_t^Y = \frac{\beta}{1 + \beta}\Big[(1 - \alpha)(k_t^W)^\alpha - (\alpha(k_t^W)^{\alpha-1} - n)\bar{d}\Big]. \tag{38}$$

- The savings of Foreign young, who are untaxed, remain:

$$s_t^{Y*} = \frac{\beta}{1 + \beta}(1 - \alpha)(k_t^W)^\alpha.$$

- With government debt, world market-clearing requires young savers to hold both capital and Home government debt. Hence (33) becomes:

$$K_{t+1} + K^*_{t+1} - B^G_{t+1} = N_t s^Y_t + N^*_t s^{Y*}_t. \tag{39}$$

- Substituting (38) and the Foreign saving function into (39), and dividing by the total labor force $N_t + N^*_t$, yields:

$$k^W_{t+1} = \frac{\beta}{(1+n)(1+\beta)}\Big\{(1-\alpha)(k^W_t)^\alpha - \Big[\alpha(k^W_t)^{\alpha-1} - n\Big]\bar{d}\Big\} - \bar{d}.$$

- *Intuition:* Government debt crowds out capital accumulation by diverting savings into public bonds. Taxes on Home residents reduce their savings, and equilibrium requires that global savings now finance both capital and government debt.

- Define the Home country's share of the world labor force as:

$$x \equiv \frac{N_t}{N_t + N^*_t}.$$

- Since

$$\frac{\partial k^W_{t+1}}{\partial k^W_t} > 0 \quad \text{and} \quad \frac{\partial^2 k^W_{t+1}}{\partial (k^W_t)^2} < 0,$$

  the transition curve in the phase diagram is strictly concave.

- As before, the system has one stable steady state and one unstable steady state.

- The following phase diagram illustrates the dynamics of world capital after the Home government introduces public debt.

- *Intuition:* With debt issuance, the shape of the transition curve remains concave, preserving the existence of a stable steady state. However, the steady-state level of world capital is now lower, reflecting crowding out by government debt.

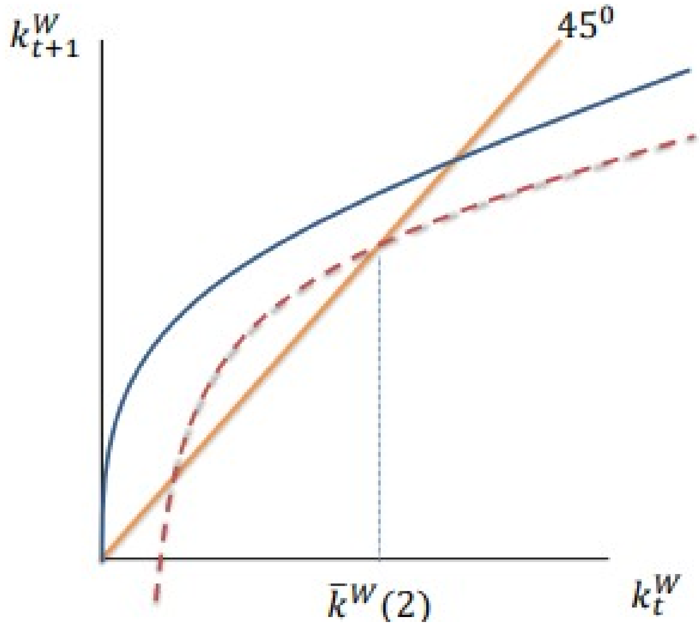

Figura 5: Phase diagram

- The dashed curve shows that the introduction of Home government debt shifts the transition path of world capital downward, reducing the long-run steady-state level of k^W.
- As a result, the economy still converges to a stable steady state, but at a lower capital–labor ratio due to the crowding-out effect of public debt.
- The stable steady-state capital ratio $\bar{k}^W(2)$ is lower in the presence of public debt.
- The other steady state $\bar{k}^W(1)$ is no longer at zero, but remains unstable.
- Domestic public debt reduces saving by the young and diverts global savings into government paper assets. As a result, steady-state capital intensity falls in both Home and Foreign.
- Consequently, the world interest rate rises.
- With higher world interest rates, global investment declines.
- **Key insight:** When a large country runs a fiscal deficit (e.g. the USA) and capital markets are internationally integrated, the debt crowds out capital accumulation both abroad and at home.
- *Intuition:* Government debt in a large economy not only reduces its own capital stock but also depresses global investment, transmitting the crowding-out effect worldwide.
- The world interest rate depends on the global capital–labor ratio:

$$R_t = \alpha \cdot (k_t^W)^{\alpha-1}.$$

- An increase in R_t reduces world investment I_t^W and total world saving S_t^W:

$$\uparrow R_t \ \Rightarrow \downarrow I_t^W = \downarrow S_t^W.$$

- Under Ricardian Equivalence:

$$\downarrow S^G, \ \uparrow S^p, \ \bar{S}^W \ \text{(no effect on world saving)}.$$

- In the OLG economy:

$$\downarrow S^G, \ \downarrow S^p, \ \downarrow S^W = \downarrow I^W.$$

- Government debt therefore crowds out private saving in both Home and Foreign.
- **Question:** Is this crowding-out effect beneficial?
- *Intuition:* Unlike in Ricardian equivalence, debt in an OLG world lowers global savings and investment, raising interest rates and reducing capital accumulation everywhere.
- The welfare effect of government debt depends on the initial steady state.
- If the economy was dynamically inefficient ($r < n$), introducing government debt raises welfare.
- By reducing capital accumulation, debt moves the world economy closer to the golden rule level of capital.
- Conversely, if $r \geq n$, government debt lowers welfare.
- *Intuition:* Debt can be welfare-improving when it corrects overaccumulation of capital, but harmful otherwise since it reduces capital below the efficient level.

Full procedure

Policy + targets:

$$-\frac{B_t^G}{N_t} = \bar{d}, \qquad G_t = 0.$$

Government budget:

$$B_{t+1}^G = (1 + r_t) B_t^G + N_t \tau_t^Y.$$

Divide by $N_{t+1} = (1 + n) N_t$ and use $-\frac{B_t^G}{N_t} = \bar{d}$:

$$\bar{d} = -\frac{B_{t+1}^G}{N_{t+1}} = \frac{(1 + r_t)\bar{d} - \tau_t^Y}{1 + n} \quad \Rightarrow \quad \boxed{\tau_t^Y = (r_t - n)\bar{d}.} \tag{37}$$

Factor prices (Cobb–Douglas):

$$w_t = (1 - \alpha)(k_t^W)^\alpha, \qquad r_t = \alpha (k_t^W)^{\alpha - 1}.$$

Home saving rule (use (30)):

$$s_t^Y = \frac{\beta}{1+\beta}(w_t - \tau_t^Y) = \frac{\beta}{1+\beta}\Big[(1-\alpha)(k_t^W)^\alpha - (\alpha(k_t^W)^{\alpha-1} - n)\bar{d}\Big]. \tag{38}$$

Foreign saving (untaxed):

$$s_t^{Y*} = \frac{\beta}{1+\beta}(1-\alpha)(k_t^W)^\alpha.$$

World asset market with public debt:

$$K_{t+1} + K_{t+1}^* - B_{t+1}^G = N_t s_t^Y + N_t^* s_t^{Y*}. \tag{39}$$

Per world worker. Let

$$x \equiv \frac{N_t}{N_t + N_t^*}, \qquad \frac{N_{t+1}}{N_{t+1} + N_{t+1}^*} = \frac{N_t}{N_t + N_t^*} = x.$$

$$k_{t+1}^W - \frac{B_{t+1}^G}{N_{t+1} + N_{t+1}^*} = \frac{N_t s_t^Y + N_t^* s_t^{Y*}}{(1+n)(N_t + N_t^*)}.$$

Use $\frac{B_{t+1}^G}{N_{t+1}} = -\bar{d} \;\Rightarrow\; \frac{B_{t+1}^G}{N_{t+1}+N_{t+1}^*} = -x\bar{d}$:

$$k_{t+1}^W + x\bar{d} = \frac{x s_t^Y + (1-x)s_t^{Y*}}{1+n}.$$

Substitute s_t^Y, s_t^{Y*}:

$$\boxed{k_{t+1}^W = \frac{\beta}{(1+n)(1+\beta)}\Big[(1-\alpha)(k_t^W)^\alpha - x\Big(\alpha(k_t^W)^{\alpha-1} - n\Big)\bar{d}\Big] - x\bar{d}.}$$

Slope and curvature:

$$\frac{\partial k_{t+1}^W}{\partial k_t^W} = \frac{\beta\alpha(1-\alpha)}{(1+n)(1+\beta)}\Big[(k_t^W)^{\alpha-1} + x\bar{d}\,(k_t^W)^{\alpha-2}\Big] > 0,$$

$$\frac{\partial^2 k_{t+1}^W}{\partial (k_t^W)^2} = \frac{\beta\alpha(1-\alpha)}{(1+n)(1+\beta)}\Big[(\alpha-1)(k_t^W)^{\alpha-2} + x\bar{d}\,(\alpha-2)(k_t^W)^{\alpha-3}\Big] < 0.$$

Steady states:

$$k_{t+1}^W = k_t^W = \bar{k}^W \quad \Rightarrow \quad \text{one unstable near 0, one stable } \bar{k}^W_{\text{debt}} < \bar{k}^W_{\text{no debt}}.$$

World interest and crowding out:

$$R_t = \alpha(k_t^W)^{\alpha-1}, \qquad \bar{k}^W \downarrow \Rightarrow R\uparrow,\; S^W = I^W \downarrow.$$

Welfare (dynamic efficiency):

$$\text{Golden rule: } R = n.$$

$$\boxed{r < n \;\Rightarrow\; \text{debt } (\bar{d} > 0) \text{ raises welfare (reduces overaccumulation);} \qquad r \geq n \;\Rightarrow\; \text{debt lowers welfare.}}$$