CENTRO DE ESTUDIOS ECONÓMICOS

Maestría en Economía 2024–2026

Macroeconomics 3

Homework 1: The Intertemporal Current Account Model

PRESENTED BY: José Daniel Fuentes García

PROFESSOR: Stephen McKnight

Índice

Problem 1

Consider a **two-period small open endowment economy**. Suppose that the representative agent has **lifetime utility** given by:

$$U = \log(C_1) + \beta \log(C_2),$$

where C_t is the **consumption level** in period $t = \{1, 2\}$, and $0 < \beta < 1$ is the **subjective discount factor**. In each period, the agent receives with certainty an **endowment** Y_t. Initially assume that there is **no government consumption** in the economy ($G_t = 0$).

a)

If r denotes the **exogenously given world real interest rate** for borrowing and lending from the world financial market, derive the **period 1 and period 2 budget constraints** and the **intertemporal budget constraint**. Assume that the economy's **initial net foreign assets are zero**.

ANSWER:

Assumptions:

- C_t: consumption in period t, $t = 1, 2$
- Y_t: endowment (income) in period t
- B_t: NFA in t
- r: world interest rate

Initial condition: $B_0 = 0$
Terminal condition: $B_2 = 0$

Period 1 Budget Constraint: [Consumption = Income]

$$\boxed{C_1 + B_1 = Y_1 + (1 + r)B_0}$$

Since $B_0 = 0$, this simplifies to:

$$C_1 + B_1 = Y_1 \tag{1}$$

Period 2 Budget Constraint:

$$C_2 + B_2 = Y_2 + (1+r)B_1$$

Since $B_2 = 0$, this becomes:

$$C_2 = Y_2 + (1+r)B_1 \tag{2}$$

From equation (1), solve for B_1:

$$B_1 = Y_1 - C_1 \tag{3}$$

Substitute (3) into (2):

$$C_2 = Y_2 + (1+r)(Y_1 - C_1) \tag{4}$$

Expanding:

$$C_2 = Y_2 + (1+r)Y_1 - (1+r)C_1 \tag{5}$$

Rearranging terms:

$$\boxed{(1+r)C_1 + C_2 = (1+r)Y_1 + Y_2} \tag{6}$$

Divide both sides by $(1+r)$ to get the **intertemporal budget constraint**:

$$\boxed{C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}} \quad \text{[Intertemporal BC]}$$

b)

Given the budget constraints derived above, solve the agent's maximization problem.

ANSWER:

Intertemporal Budget Constraint (from above):

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \quad \text{[Intertemporal BC]} \tag{1}$$

Maximization problem:

$$\max_{C_1, C_2} \log C_1 + \beta \log C_2 \quad \text{s.t. constraint (1)} \tag{2}$$

Lagrangian:

$$\mathcal{L} = \log C_1 + \beta \log C_2 - \lambda \left(C_1 + \frac{C_2}{1+r} - Y_1 - \frac{Y_2}{1+r} \right) \tag{3}$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_1} : \quad \frac{1}{C_1} - \lambda = 0 \Rightarrow \lambda = \frac{1}{C_1} \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial C_2} : \quad \frac{\beta}{C_2} - \lambda \cdot \frac{1}{1+r} = 0 \Rightarrow \lambda = \frac{\beta(1+r)}{C_2} \tag{5}$$

Euler Equation (from (4) = (5)):

$$\frac{1}{C_1} = \frac{\beta(1+r)}{C_2} \Rightarrow C_2 = \beta(1+r)C_1 \tag{6}$$

Substitute (6) into (1):

$$C_1 + \frac{\beta(1+r)C_1}{1+r} = Y_1 + \frac{Y_2}{1+r} \Rightarrow (1+\beta)C_1 = Y_1 + \frac{Y_2}{1+r} \tag{7}$$

Optimal C_1^*:

$$\boxed{C_1^* = \frac{(1+r)Y_1 + Y_2}{(1+\beta)(1+r)}} \tag{8}$$

Optimal C_2^* (using equation (6)):

$$\boxed{C_2^* = \beta(1+r)C_1^* = \frac{\beta\left((1+r)Y_1 + Y_2\right)}{1+\beta}} \tag{9}$$

Optimal asset holdings $B_1^* = Y_1 - C_1^*$:

$$\boxed{B_1^* = Y_1 - C_1^* = \frac{\beta Y_1 - \frac{Y_2}{1+r}}{1+\beta} = \frac{\beta(1+r)Y_1 - Y_2}{(1+\beta)(1+r)}} \tag{10}$$

c)

If $\beta = 0{,}8$, $r = 0{,}25$, $Y_1 = 5$ and $Y_2 = 10$, derive the optimal consumption level in each period.

ANSWER:

Optimal C_1^*:

$$C_1^* = \frac{Y_1 + \dfrac{Y_2}{1+r}}{1+\beta}$$

$$C_1^* = \frac{5 + \dfrac{10}{1{,}25}}{1+0{,}8}$$

$$C_1^* = \frac{5+8}{1{,}8}$$

$$C_1^* = \frac{13}{1{,}8}$$

$$\boxed{C_1^* = \frac{65}{9} \approx 7{,}22}$$

Optimal C_2^*:

$$C_2^* = \beta(1+r)C_1^*$$
$$C_2^* = 0{,}8 \cdot 1{,}25 \cdot 7{,}22$$
$$C_2^* = 1 \cdot 7{,}22$$
$$\boxed{C_2^* = \frac{65}{9} \approx 7{,}22}$$

Both consumption levels are the same because preferences and the interest rate align so that the agent perfectly **smooths consumption** across periods.

d)

Define the current account for this economy. Using the results obtained from **(c)** above, calculate the current account in each period.

ANSWER:

From the period budget constraint:

$$C_t + B_t = Y_t + (1+r)B_{t-1}$$

Rearranged:

$$B_t - B_{t-1} = Y_t - C_t + rB_{t-1}$$

Define the current account:

$$CA_t \equiv B_t - B_{t-1}$$

Substitute:

$$CA_t = Y_t - C_t + rB_{t-1}$$

Using: $\beta = 0{,}8, \quad r = 0{,}25, \quad Y_1 = 5, \quad Y_2 = 10, \quad C_1 = C_2 = \frac{65}{9}$

Solve for B_1:

$$B_1 = Y_1 - C_1 = 5 - \frac{65}{9} = \boxed{-\frac{20}{9}}$$

Period 1 Current Account:

$$\boxed{CA_1 = -\frac{20}{9} \approx -2{,}22}$$

Period 2 Current Account:

$$\boxed{CA_2 = \frac{20}{9} \approx 2{,}22}$$

Checks:
Current Account (CA):

$$CA_1 = Y_1 - C_1 \quad (\text{since } B_0 = 0)$$

$$CA_2 = Y_2 - C_2 + rB_1$$

Trade Balance (TB):

$$TB_1 = Y_1 - C_1, \qquad TB_2 = Y_2 - C_2$$

Intertemporal consistency:

$$CA_2 = -CA_1$$

(Because $CA_1 + CA_2 = B_2 - B_0 = 0$ *with* $B_0 = B_2 = 0$ *)*

e)

Define the autarky interest rate and use the parameter values given in **c)** above to calculate its value.

ANSWER:

Definition (Autarky Interest Rate):
The autarky interest rate r_A is such that there is no trade, i.e.:

$$B_1 = 0 \quad \Rightarrow \quad C_1 = Y_1, \quad C_2 = Y_2$$

Euler Equation:

$$C_2 = \beta(1+r)C_1$$

Autarky condition:

$$Y_2 = \beta(1+r_A)Y_1$$

Solve for r_A:

$$1 + r_A = \frac{Y_2}{\beta Y_1}$$
$$r_A = \frac{Y_2}{\beta Y_1} - 1$$

Substitute: $\beta = 0{,}8, \quad Y_1 = 5, \quad Y_2 = 10$

$$r_A = \frac{10}{0{,}8 \cdot 5} - 1$$
$$r_A = \frac{10}{4} - 1$$
$$r_A = 2{,}5 - 1$$
$$\boxed{r_A = 1{,}5}$$

Check: Use optimal consumption formula with r_A
Check C_1^*:

$$C_1^* = \frac{Y_1 + \frac{Y_2}{1+r_A}}{1+\beta}$$
$$C_1^* = \frac{5 + \frac{10}{2{,}5}}{1{,}8}$$
$$C_1^* = \frac{5+4}{1{,}8}$$
$$C_1^* = \frac{9}{1{,}8}$$
$$C_1^* = 5 = Y_1$$

Check C_2^*:

$$C_2^* = \beta(1 + r_A)C_1^*$$
$$C_2^* = 0{,}8 \cdot 2{,}5 \cdot 5$$
$$C_2^* = 10 = Y_2$$

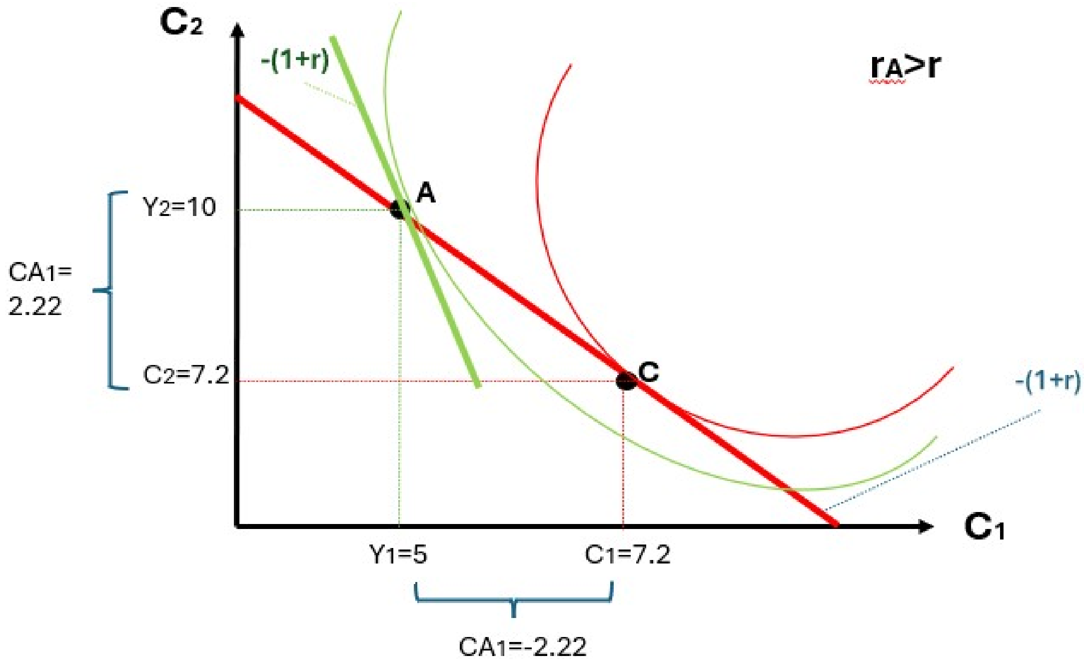
Check $B_1 = Y_1 - C_1^* = 0$

f)

Depict your results showing equilibrium consumption levels under autarky, equilibrium consumption levels for the small open economy and the current account for both periods.

ANSWER:

Figura 1: r Autrky vs r Open Economy



Fuente: Elaboración propia

g)

Assume now that there is a temporary increase in government spending. Specifically, suppose that the government consumes the amount $G_1 = 2$ in period 1 and nothing in period 2 ($G_2 = 0$).

ANSWER:

New budget constraint (with government spending):

$$C_1 + \frac{C_2}{1+r} = (Y_1 - G_1) + \frac{Y_2 - G_2}{1+r}$$

Given:

$$\beta = 0{,}8, \quad r = 0{,}25, \quad Y_1 = 5, \quad Y_2 = 10, \quad G_1 = 2, \quad G_2 = 0$$

$$Y_1 - G_1 = 3, \quad Y_2 - G_2 = 10$$

Utility: $U = \log C_1 + \beta \log C_2$

Optimal consumption (as in part b, replacing Y_t with $Y_t - G_t$):

$$C_1^* = \frac{Y_1 - G_1 + \dfrac{Y_2 - G_2}{1+r}}{1+\beta}$$
$$C_1^* = \frac{3 + \dfrac{10}{1{,}25}}{1{,}8}$$
$$C_1^* = \frac{3+8}{1{,}8}$$
$$C_1^* = \frac{11}{1{,}8} = \frac{55}{9} \approx 6{,}11$$
$$\boxed{C_1^* = \frac{55}{9} \approx 6{,}11}$$

$$C_2^* = \beta(1+r)C_1^*$$
$$C_2^* = 0{,}8 \cdot 1{,}25 \cdot \frac{55}{9}$$
$$C_2^* = \frac{55}{9} \approx 6{,}11$$
$$\boxed{C_2^* = \frac{55}{9} \approx 6{,}11}$$

Comparison with no-G case:

$$C_1^* = C_2^* = \frac{65}{9} \approx 7{,}22 \quad \Rightarrow \quad \text{Fall in consumption: } \frac{10}{9} \approx 1{,}11$$

Current Account: (using $CA_t = Y_t - C_t - G_t + rB_{t-1}$, with $B_0 = 0$)
Period 1:

$$CA_1 = (Y_1 - G_1) - C_1^*$$
$$CA_1 = 3 - \frac{55}{9}$$
$$CA_1 = -\frac{28}{9} \approx -3{,}11$$
$$\boxed{CA_1 = -\frac{28}{9} \approx -3{,}11}$$

Period 2:

$$CA_2 = (Y_2 - G_2) - C_2^* + rB_1$$
$$CA_2 = 10 - \frac{55}{9} + 0{,}25 \cdot \left(-\frac{28}{9}\right)$$
$$CA_2 = \frac{35}{9} - \frac{7}{9} = \frac{28}{9} \approx 3{,}11$$
$$\boxed{CA_2 = \frac{28}{9} \approx 3{,}11}$$

Note: $CA_2 = -CA_1$, as required by the condition $B_2 = B_0 = 0$.

A temporary increase in government spending in period 1 reduces the **resources available for private consumption** today, shifting the **intertemporal budget constraint inward**. Since **households prefer to smooth consumption over time**, they "spread the pain" by lowering both **current and future consumption** rather than absorbing the entire adjustment in period 1.

As a result, the economy runs a **current account deficit** in the first period, **borrowing from abroad** to finance part of the government's higher spending, and later repays through a **current account surplus** in the second period. This ensures that the **intertemporal budget constraint holds** while keeping **consumption as smooth as possible**.

Problem 2

Consider a two-period small open endowment economy. Suppose that the representative agent has lifetime utility given by:

$$U = u(C_1) + \beta u(C_2),$$

where $\mathbf{0} < \beta < \mathbf{1}$ is the subjective discount factor. Suppose the agent receives an endowment $\mathbf{Y_1} = \mathbf{1}$ and $\mathbf{Y_2} = \mathbf{x}$. Assume that government consumes the amount $\mathbf{G_1} \geq \mathbf{0}$ and $\mathbf{G_2} \geq \mathbf{0}$ but balances its budget every period.

a)

Derive the intertemporal budget constraint and the current account for this economy (assume that the economy's initial net foreign assets are zero).

ANSWER:

Assumptions:

- C_t: consumption in period t, $t = 1, 2$
- $Y_1 = 1,\ Y_2 = x$: endowment income
- G_t: government spending in each period
- $T_t = G_t$: Balanced Budget
- B_t: net foreign asset position at end of period t
- r: world interest rate
- $\beta \in (0, 1)$: discount factor

Initial and Terminal Conditions

$$B_0 = 0 \quad \text{(initial NFA)}$$

$$B_2 = 0 \quad \text{(terminal NFA)}$$

Period 1 Budget Constraint

$$C_1 + B_1 = Y_1 - T_1$$

$$C_1 + B_1 = 1 - G_1 \quad (1)$$

Period 2 Budget Constraint

$$C_2 + B_2 = Y_2 - T_2 + (1+r)B_1$$

$$C_2 = x - G_2 + (1+r)B_1 \quad (2)$$

Substitute B_1 from (1):

$$B_1 = 1 - G_1 - C_1 \quad (3)$$

Into (2):

$$C_2 = x - G_2 + (1+r)(1 - G_1 - C_1) \quad (4)$$

Expand (4):

$$C_2 = x - G_2 + (1+r)(1 - G_1) - (1+r)C_1 \quad (5)$$

Rearrange (5):

$$(1+r)C_1 + C_2 = (1+r)(1 - G_1) + x - G_2 \quad (6)$$

Divide both sides by $(1+r)$:

$$C_1 + \frac{C_2}{1+r} = (1 - G_1) + \frac{x - G_2}{1+r} \quad (7)$$

Intertemporal Budget Constraint

$$\boxed{C_1 + \frac{C_2}{1+r} = (1 - G_1) + \frac{x - G_2}{1+r}} \quad \text{[Intertemporal BC]}$$

Current Account — Period 1

$$CA_1 = Y_1 - G_1 - C_1 \quad (8)$$

$$CA_1 = 1 - G_1 - C_1 \quad (9)$$

$$\boxed{CA_1 = 1 - G_1 - C_1}$$

Current Account — Period 2

$$CA_2 = Y_2 - G_2 - C_2 \quad (10)$$

$$CA_2 = x - G_2 - C_2 \quad (11)$$

$$\boxed{CA_2 = x - G_2 - C_2}$$

b)

If the period utility function is given by

$$u(C) = \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \text{where } \sigma > 0,\ \sigma \neq 1,$$

derive the Euler equation for this economy. What is the autarky interest rate and how does it determine whether the economy runs a current account surplus or deficit?

ANSWER:

Given: $u(C) = \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},\ \sigma > 0,\ \sigma \neq 1$

Maximization Problem

Maximize: $U = u(C_1) + \beta u(C_2)$

Subject to (IBC): $C_1 + \frac{C_2}{1+r} = (1 - G_1) + \frac{x - G_2}{1+r}$

Lagrangian

$$L = u(C_1) + \beta u(C_2) + \lambda\left[(1 - G_1) + \frac{x - G_2}{1+r} - C_1 - \frac{C_2}{1+r}\right]$$

First-Order Conditions

$$\frac{\partial L}{\partial C_1} = u'(C_1) - \lambda = 0 \quad \text{(FOC1)}, \qquad \frac{\partial L}{\partial C_2} = \beta u'(C_2) - \lambda\frac{1}{1+r} = 0 \quad \text{(FOC2)}$$

From FOC1–FOC2: $u'(C_1) = \beta(1+r)u'(C_2)$.

Marginal Utility

$$u'(C) = C^{-\frac{1}{\sigma}}$$

Euler Equation

$$\boxed{C_1^{-\frac{1}{\sigma}} = \beta(1+r)\, C_2^{-\frac{1}{\sigma}}}$$

Equivalently,

$$\left(\frac{C_2}{C_1}\right)^{\frac{1}{\sigma}} = \beta(1+r) \quad \Rightarrow \quad \frac{C_2}{C_1} = [\beta(1+r)]^{\sigma}.$$

Autarky Interest Rate

Autarky: $B_1 = 0 \Rightarrow C_1 = 1 - G_1,\ C_2 = x - G_2$.

Plug into Euler:

$$(1-G_1)^{-\frac{1}{\sigma}} = \beta(1+r_a)(x-G_2)^{-\frac{1}{\sigma}} \quad \Rightarrow \quad \beta(1+r_a) = \left(\frac{x-G_2}{1-G_1}\right)^{\frac{1}{\sigma}}.$$

$$\boxed{r_a = \frac{1}{\beta}\left(\frac{x-G_2}{1-G_1}\right)^{\frac{1}{\sigma}} - 1}$$

CA Sign Condition

If $r < r_a$:

$$[\beta(1+r)]^{\sigma} < \frac{x-G_2}{1-G_1}$$

$$\Longrightarrow \frac{C_2}{C_1} < \frac{x-G_2}{1-G_1}$$

$$\Longrightarrow C_1 > 1-G_1$$

$$\Longrightarrow CA_1 = 1-G_1-C_1 < 0 \quad \text{(Deficit)}$$

If $r > r_a$:

$$[\beta(1+r)]^{\sigma} > \frac{x-G_2}{1-G_1}$$

$$\Longrightarrow \frac{C_2}{C_1} > \frac{x-G_2}{1-G_1}$$

$$\Longrightarrow C_1 < 1-G_1$$

$$\Longrightarrow CA_1 = 1-G_1-C_1 > 0 \quad \text{(Surplus)}$$

- When $r < r_a$: the agent **borrows** to consume more today, causing a **CA deficit in period 1** and a **CA surplus in period 2**.
- When $r > r_a$: the agent **saves**, consuming less today, leading to a **CA surplus in period 1** and a **CA deficit in period 2**.
- In both cases, the intertemporal budget constraint ensures the **net foreign asset position returns to zero** by the end.
- If $r = r_a$: there is no incentive to trade intertemporally, so **consumption equals income in each period** and **CA = 0** throughout.

c)

Assume that $\beta(1+r) = 1$. Using the autarky interest rate derived in (b), show whether the country runs a current account surplus or deficit in period 1 when:

1. $G_1 = G_2 = 0$, and $x < 1$.
2. $G_1 > 0,\ G_2 = 0$, and $x = 1$.

ANSWER (1):

Recall: Autarky Interest Rate r_a
From earlier:

$$1 + r_a = \frac{1}{\beta}\left(\frac{x - G_2}{1 - G_1}\right)^{\frac{1}{\sigma}}$$

Plug in $G_1 = G_2 = 0$:

$$1 + r_a = \frac{1}{\beta}\left(\frac{x}{1}\right)^{\frac{1}{\sigma}} = \frac{1}{\beta}x^{\frac{1}{\sigma}}$$

Compare with World Interest Rate r
Given:

$$\beta(1+r) = 1 \quad \Rightarrow \quad 1 + r = \frac{1}{\beta}$$

Compare:

$$1 + r_a = \frac{1}{\beta}x^{\frac{1}{\sigma}} \quad \text{vs.} \quad 1 + r = \frac{1}{\beta}$$

Since $x < 1$ and $\sigma > 0$, we get:

$$x^{\frac{1}{\sigma}} < 1$$

$$\Longrightarrow \quad 1 + r_a < 1 + r$$

$$\Longrightarrow \quad r_a < r$$

Interpretation

$$r > r_a \quad \Rightarrow \quad \text{agent saves in period 1}$$

$$\Rightarrow \quad C_1 < Y_1$$

$$\Rightarrow \quad CA_1 > 0$$

Conclusion: The country runs a **current account surplus in period 1**.

- Given $\beta(1+r) = 1$, the consumer is **indifferent between consuming today or tomorrow** only if income is equal across periods.
- Since $x < 1$, **future income is lower**, so the autarky interest rate satisfies $r_a < r$, making **saving attractive**.
- This leads the agent to **consume less than income today**, generating a **current account surplus in period 1**.

ANSWER (2):

Autarky Interest Rate

Recall:

$$1 + r_a = \frac{1}{\beta}\left(\frac{x - G_2}{1 - G_1}\right)^{\frac{1}{\sigma}}$$

Substitute $x = 1,\ G_2 = 0$:

$$1 + r_a = \frac{1}{\beta}\left(\frac{1}{1 - G_1}\right)^{\frac{1}{\sigma}}$$

Since $G_1 > 0$, then

$$\frac{1}{1 - G_1} > 1$$

$$\implies \left(\frac{1}{1 - G_1}\right)^{\frac{1}{\sigma}} > 1$$

$$\implies 1 + r_a > \frac{1}{\beta} = 1 + r$$

$$\implies r_a > r$$

Interpretation

$$r_a > r$$

$$\implies \text{agent prefers to } \textbf{borrow} \text{ (consume more today)}$$

$$\implies C_1 > Y_1 - G_1$$

$$\implies CA_1 < 0$$

Conclusion: The country runs a **current account deficit in period 1**.

- With $x = 1$, income is symmetric across periods.
- Since $G_1 > 0$, **net resources are lower today**.
- This makes **future resources relatively more attractive**, so $r_a > r$.
- The agent **borrows to smooth consumption** "spreading the pain".
- Thus, **consumption exceeds income in period 1**, leading to a **current account deficit**.

Problem 3

Consider a two-period small open endowment economy. Assume that the period utility function is given by

$$u(C) = \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \sigma > 0,\ \sigma \neq 1.$$

Further assume that the world interest rate r is set at such a level such that the small open economy is a net borrower b_1 in period 1.

Now suppose that the government introduces capital controls thereby restricting the amount the small open economy can borrow from the rest of the world. Let $\bar{b}_1$ denote the level of borrowing in the presence of capital controls.

a)

Show for which values of debt will these capital controls affect equilibrium consumption in the small open economy.

ANSWER:

Assumptions:

- Two-period small open economy
- Utility: $u(C) = \dfrac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \sigma > 0,\ \sigma \neq 1$
- World interest rate r
- Initial borrowing without restrictions: b_1^*

- Capital controls impose: $b_1 \leq \bar{b}_1$

Maximizing Problem
Maximize:

$$U = u(C_1) + \beta u(C_2)$$

Subject to (Recall IBC):

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \quad (1)$$

Euler Equation

$$C_1^{-\frac{1}{\sigma}} = \beta(1+r)\, C_2^{-\frac{1}{\sigma}} \quad (2)$$

$$\frac{C_2}{C_1} = [\beta(1+r)]^{\sigma} \quad (3)$$

Substitute into (1)

$$C_1 + \frac{[\beta(1+r)]^{\sigma}\, C_1}{1+r} = Y_1 + \frac{Y_2}{1+r} \quad (4)$$

$$C_1 \left[1 + \frac{[\beta(1+r)]^{\sigma}}{1+r}\right] = Y_1 + \frac{Y_2}{1+r} \quad (5)$$

Optimal Consumption

$$C_1^* = \frac{Y_1 + \frac{Y_2}{1+r}}{1 + \frac{[\beta(1+r)]^{\sigma}}{1+r}} \quad (6)$$

Optimal Borrowing

$$b_1^* = Y_1 - C_1^* \quad (7)$$

Capital Control Constraint
If

$$b_1^* > \bar{b}_1 \quad (8)$$

then the constraint binds.
New consumption under controls:

$$C_1 = Y_1 - \bar{b}_1 \quad (9)$$

$$C_2 = Y_2 + (1+r)\bar{b}_1 \quad (10)$$

Conclusion:

$$\boxed{\text{Capital controls affect equilibrium consumption iff } b_1^* > \bar{b}_1}$$

- Without controls, the agent chooses optimal borrowing b_1^* to **smooth consumption over time**.
- If the government imposes a tighter cap $\bar{b}_1 < b_1^*$, the agent *cannot borrow as much* and must **reduce consumption in period 1**.
- This constraint **breaks optimal smoothing**, so equilibrium consumption changes only when controls **bind** — that is, when $b_1^* > \bar{b}_1$.

b)

What will the interest rate be in the presence of capital controls?

ANSWER:

Assume: Capital Controls Bind

$$b_1 = \bar{b}_1 \quad (1)$$

Period 1 Consumption

$$C_1 = Y_1 - \bar{b}_1 \quad (2)$$

Period 2 Consumption

$$C_2 = Y_2 + (1+r)\bar{b}_1 \quad (3)$$

Euler Equation (Always Holds):

$$C_1^{-\frac{1}{\sigma}} = \beta(1+r)\, C_2^{-\frac{1}{\sigma}} \quad (4)$$

Substitute (2) and (3) into (4):

$$(Y_1 - \bar{b}_1)^{-\frac{1}{\sigma}} = \beta(1+r)\left(Y_2 + (1+r)\bar{b}_1\right)^{-\frac{1}{\sigma}} \quad (5)$$

Rearrange:

$$\frac{(Y_2 + (1+r)\bar{b}_1)^{-\frac{1}{\sigma}}}{(Y_1 - \bar{b}_1)^{-\frac{1}{\sigma}}} = \beta(1+r) \quad (6)$$

Raise both sides to power $-\sigma$:

$$\frac{Y_2 + (1+r)\bar{b}_1}{Y_1 - \bar{b}_1} = \left[\beta(1+r)\right]^{\sigma} \quad (7)$$

Final Expression (Implicit Equation for r):

$$\boxed{\frac{Y_2 + (1+r)\bar{b}_1}{Y_1 - \bar{b}_1} = \left[\beta(1+r)\right]^{\sigma}}$$

This is the **implicit equation for** r in the presence of binding capital controls.

- r **adjusts** so that the agent, forced to borrow only $\bar{b}_1$, is still **optimizing consumption**.
- It **balances** how much .^{ex}tra future consumption*"the agent gets from borrowing today, given the constraint.
- Thus, it is the **interest rate that makes the Euler equation hold** under the imposed borrowing limit.

c)

Analyze the effect of a temporary increase in the endowment of the economy. Will consumption smoothing take place in the presence of capital controls?

ANSWER:

Define the Shock
Only in Period 1
Original: Y_1, Y_2
Shock:

$$Y_1' = Y_1 + \Delta, \quad \Delta > 0, \qquad Y_2' = Y_2$$

Unconstrained Optimum
With no capital controls, the agent solves:

$$C_1 + \frac{C_2}{1+r} = Y_1' + \frac{Y_2'}{1+r}$$

Smoothing condition (Euler equation):

$$C_1^{-\frac{1}{\sigma}} = \beta(1+r)\, C_2^{-\frac{1}{\sigma}}$$

$\Rightarrow$ Agent **spreads the shock Δ across both periods**.
With Capital Controls
Let $\bar{b}_1$ be binding before the shock.
New income: $Y_1' = Y_1 + \Delta$
Borrowing limit unchanged: $b_1 = \bar{b}_1$
Then:

$$C_1 = Y_1' - \bar{b}_1 = (Y_1 + \Delta) - \bar{b}_1$$

$$C_2 = Y_2 + (1+r)\bar{b}_1$$

$\Rightarrow$ **Consumption rises only in period 1**.
$\Rightarrow$ **No adjustment in period 2**.
Conclusion

$$\boxed{\text{No consumption smoothing under capital controls if } \bar{b}_1 \text{ still binds after the shock.}}$$

- With capital controls binding, the agent **cannot borrow or lend freely** to reallocate income over time.
- When income **rises temporarily in period 1**, they **cannot save the extra** to shift consumption to period 2.
- As a result, they are **forced to consume the entire increase immediately**, which **breaks consumption smoothing**.

Problem 4

Consider a small open production economy with an infinite horizon. Suppose the production technology is given by

$$Y_t = A_t F(K_t)$$

where A_t denotes a positive productivity parameter and $F(K_t)$ is a strictly increasing and strictly concave function.

Capital accumulates according to

$$K_{t+1} = K_t + I_t$$

and net foreign assets accumulate via:

$$B_{t+1} = Y_t + (1+r)B_t - C_t - I_t.$$

a)

Derive the first-order conditions for this problem

ANSWER:

Maximizing Problem

$$\max_{\{C_t, K_{t+1}, B_{t+1}\}} \sum_{s=t}^{\infty} \beta^{s-t} u(C_t)$$

Subject to

$$B_{t+1} = A_t F(K_t) + (1+r)B_t - C_t - I_t \quad (1)$$

$$K_{t+1} = K_t + I_t$$

$$\implies I_t = K_{t+1} - K_t \quad (2)$$

Substitute (2) into (1)

$$B_{t+1} = A_t F(K_t) + (1+r)B_t - C_t - (K_{t+1} - K_t) \quad (3)$$

$$B_{t+1} = A_t F(K_t) + (1+r)B_t - C_t - K_{t+1} + K_t \quad (4)$$

Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \Big[u(C_t) + \lambda_t \Big(A_t F(K_t) + (1+r)B_t - C_t - K_{t+1} + K_t - B_{t+1} \Big) \Big] \quad (5)$$

First-Order Condition C_t

$$\frac{\partial L}{\partial C_t} = \beta^t \Big[u'(C_t) - \lambda_t \Big] = 0$$

$$\implies \lambda_t = u'(C_t) \quad (6)$$

First-Order Condition B_{t+1}

$$\frac{\partial L}{\partial B_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1}(1+r) = 0$$

$$\implies \lambda_t = \beta(1+r)\lambda_{t+1} \quad (7)$$

First-Order Condition K_{t+1}

$$\frac{\partial L}{\partial K_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \Big(A_{t+1} F'(K_{t+1}) + 1 \Big) = 0$$

$$\implies \lambda_t = \beta \lambda_{t+1} \Big(A_{t+1} F'(K_{t+1}) + 1 \Big) \quad (8)$$

Final First-Order Conditions

$$\boxed{\begin{aligned} &(1) \quad \lambda_t = u'(C_t) \\ &(2) \quad \lambda_t = \beta(1+r)\lambda_{t+1} \\ &(3) \quad \lambda_t = \beta \lambda_{t+1} \Big(A_{t+1} F'(K_{t+1}) + 1 \Big) \end{aligned}}$$

b)

Derive the intertemporal budget constraint and TVC condition.

ANSWER:

Budget Constraint

$$B_{t+1} = A_t F(K_t) + (1+r)B_t - C_t - I_t \quad (1)$$

From capital accumulation

$$K_{t+1} = K_t + I_t$$

$$\implies \quad I_t = K_{t+1} - K_t \quad (2)$$

Substitute (2) into (1)

$$B_{t+1} = A_tF(K_t) + (1+r)B_t - C_t - (K_{t+1} - K_t) \quad (3)$$

$$B_{t+1} = A_tF(K_t) + (1+r)B_t - C_t - K_{t+1} + K_t \quad (4)$$

Solve forward for B_{t+1}
Start with (1):

$$B_{t+1} = A_tF(K_t) + (1+r)B_t - C_t - I_t$$

Divide both sides by $(1+r)^{t+1}$:

$$\frac{B_{t+1}}{(1+r)^{t+1}} = \frac{A_tF(K_t)}{(1+r)^{t+1}} + \frac{B_t}{(1+r)^t} - \frac{C_t}{(1+r)^{t+1}} - \frac{I_t}{(1+r)^{t+1}} \quad (5)$$

Repeat forward up to T, then sum both sides:

$$\frac{B_{T+1}}{(1+r)^{T+1}} = \frac{B_t}{(1+r)^t} + \sum_{s=t}^{T} \frac{A_sF(K_s) - C_s - I_s}{(1+r)^{s+1}} \quad (6)$$

Intertemporal Budget Constraint (IBC)
Let $T \to \infty$. Then

$$\lim_{T\to\infty} \frac{B_{T+1}}{(1+r)^{T+1}} = 0 \quad \text{(TVC)} \quad (7)$$

Thus,

$$\frac{B_t}{(1+r)^t} = \sum_{s=t}^{\infty} \frac{C_s + I_s - A_sF(K_s)}{(1+r)^{s+1}} \quad \text{(IBC)}$$

Or equivalently:

$$\sum_{s=t}^{\infty} \frac{C_s + I_s}{(1+r)^{s+1}} = \sum_{s=t}^{\infty} \frac{A_sF(K_s)}{(1+r)^{s+1}} + \frac{B_t}{(1+r)^t}$$

Transversality Condition (TVC)

$$\lim_{t\to\infty} \frac{B_t}{(1+r)^t} = 0$$

This rules out Ponzi schemes (indefinitely rolling over debt).

c)

Assuming $(1+r)\beta = 1$ where $0 < \beta < 1$ is the subjective discount factor, characterize the perfect foresight path for a constant value of A_t (i.e., $A_t = A$ for all $t = 0, 1, 2, \ldots$).

ANSWER:

Given

$$(1+r)\beta = 1$$

$$A_t = A \quad \forall t \quad \text{(constant productivity)}$$

Consumption Euler Equation

$$u'(C_t) = \beta(1+r)u'(C_{t+1}) \quad (1)$$

Since $\beta(1+r) = 1$:

$$u'(C_t) = u'(C_{t+1}) \quad (2)$$

Implication: Constant Consumption

$$C_t = C_{t+1} = C_{t+2} = \cdots \quad \Rightarrow \quad C_t = \bar{C} \quad (3)$$

Investment Euler Condition

$$F'(K_{t+1}) = rA \quad (4)$$

With $A_t = A$:

$$F'(K_{t+1}) = r \quad \forall t \quad (5)$$

Capital Stock Constant

$$K_{t+1} = \bar{K} \quad (6)$$

$$I_t = K_{t+1} - K_t = 0 \quad (7)$$

Output Constant

$$Y_t = AF(K_t) = AF(\bar{K}) = \bar{Y} \quad (8)$$

Budget Constraint in Steady State

$$B_{t+1} = AF(\bar{K}) + (1+r)B_t - \bar{C} \quad (9)$$

$$B_{t+1} = (1+r)B_t + \bar{Y} - \bar{C} \quad (10)$$

Steady State Condition (No Debt Explosion)

$$\lim_{t\to\infty} \frac{B_t}{(1+r)^t} = 0 \quad \Longrightarrow \quad \bar{C} = \bar{Y} \quad (11)$$

Then:

$$B_{t+1} = (1+r)B_t + \bar{Y} - \bar{Y} = (1+r)B_t \quad \Longrightarrow \quad B_t = 0 \quad (12)$$

(assuming initial $B_0 = 0$)

Perfect Foresight Path

$$\boxed{\begin{aligned} &C_t = \bar{C} = AF(\bar{K}) \\ &K_t = \bar{K} \quad \text{such that } F'(\bar{K}) = r \\ &I_t = 0 \\ &B_t = 0 \end{aligned}}$$

Perfect foresight path: **no growth, constant consumption, efficient capital allocation**.

d)

Suppose that in period $t = -1$ the economy is in the equilibrium derived in part (c) above. Suppose that in period 0 a temporary positive productivity shock occurs such that A rises from A to A^*, and then returns to its initial level in period 1. Show that this results in an increase in consumption and period 0 saving, an initial improvement in the trade balance, and a current account surplus.

ANSWER:

Assumptions:

- At $t = -1$, the economy is in steady state:

$$A_{-1} = A, \quad C_t = \bar{C}, \quad K_t = \bar{K}, \quad B_t = 0$$

- At $t = 0$, a temporary productivity shock occurs:

$$A_0 = A^* > A, \quad A_1 = A$$

Effect on Output

$$Y_0 = A^* F(K_0) > AF(K_0) = \bar{Y} \quad (1)$$

$\Rightarrow$ Output is temporarily higher in period 0.

Household Behavior

Euler equation:

$$u'(C_0) = \beta(1 + r)u'(C_1) \quad (2)$$

Since $C_1 = \bar{C}$, the household smooths consumption:

$$C_0 \uparrow, \quad C_0 < Y_0 \quad \Longrightarrow \quad \text{saves the difference} \quad (3)$$

Period 0 Saving

Budget constraint:

$$B_1 = Y_0 + (1 + r)B_0 - C_0 - I_0 \quad (4)$$

In steady state: $B_0 = 0,\ I_0 = 0,\ K_1 = K_0 = \bar{K}$.
So:

$$B_1 = Y_0 - C_0 > 0 \quad (5)$$

$\Rightarrow$ The agent accumulates foreign assets (**saves**).

Trade Balance and Current Account

Trade balance:

$$TB_0 = Y_0 - C_0 - I_0$$

Since $I_0 = 0$ and $Y_0 > C_0$:

$$TB_0 > 0 \quad \text{(trade balance improves)}$$

Current account:

$$CA_0 = TB_0 + rB_0 = TB_0 + 0 = TB_0 \quad \Longrightarrow \quad CA_0 > 0$$

Results of a Temporary Productivity Shock in t

$$\boxed{\begin{array}{ll} C_0 \uparrow & \text{(consumption increases)} \\ B_1 > 0 & \text{(period 0 saving)} \\ TB_0 > 0 & \text{(trade balance improves)} \\ CA_0 > 0 & \text{(current account surplus)} \end{array}}$$