CENTRO DE ESTUDIOS ECONÓMICOS

Maestría en Economía 2024–2026

Macroeconomics 3

Problem Set 2: Dynamic Programming

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Problem 1

[The Cass-Koopman Model]

Consider the following optimal growth problem. A consumer wants to maximize consumption:

$$\sum_{t=0}^{\infty} \beta^t u(C_t),$$

subject to a given value of the capital stock $K_0 > 0$, and a transition law:

$$C_t + K_{t+1} = f(K_t),$$

where $0 < \beta < 1$, $f'(K) > 0$, $f''(K) < 0$, $u'(C) > 0$, $u''(C) < 0$, $u'(0) = +\infty$.

a)

Let the *state variable* be defined as K_t and the *control variable* as K_{t+1}. Set up the Bellman equation for this problem and derive the Euler equation.

ANSWER:

Assumptions
Objective:

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t u(C_t)$$

Constraint:

$$C_t + K_{t+1} = f(K_t), \qquad K_0 > 0 \text{ given}$$
$$0 < \beta < 1, \quad f'(K) > 0, \quad f''(K) < 0$$
$$u'(C) > 0, \quad u''(C) < 0, \quad u'(0) = +\infty$$

Bellman Equation

$$V(k_t) = \max_{c_t} \{u(c_t) + \beta V(k_{t+1})\}$$

Subject to:

$$c_t + k_{t+1} = f(k_t)$$

Solve constraint for c_t:

$$c_t = f(k_t) - k_{t+1}$$

Substitute into Bellman:

$$V(k_t) = \max_{k_{t+1}} \{u(f(k_t) - k_{t+1}) + \beta V(k_{t+1})\}$$

First-Order Condition (FOC)
Define:

$$\mathcal{L}(k_{t+1}) = u(f(k_t) - k_{t+1}) + \beta V(k_{t+1})$$

Differentiate:

$$\frac{d}{dk_{t+1}}\left[u(f(k_t) - k_{t+1}) + \beta V(k_{t+1})\right]$$

$$= \frac{du}{dc_t} \cdot \frac{dc_t}{dk_{t+1}} + \beta \cdot \frac{dV}{dk_{t+1}}$$

$$= u'(f(k_t) - k_{t+1}) \cdot (-1) + \beta V'(k_{t+1})$$

$$= -u'(c_t) + \beta V'(k_{t+1})$$

Set equal to zero:

$$-u'(c_t) + \beta V'(k_{t+1}) = 0$$

$$u'(c_t) = \beta V'(k_{t+1})$$

Envelope Condition
From:

$$V(k_t) = u(f(k_t) - k_{t+1}) + \beta V(k_{t+1})$$

Differentiate w.r.t. k_t:
Use chain rule:

$$\frac{dV}{dk_t} = u'(f(k_t) - k_{t+1}) \cdot f'(k_t)$$

$$V'(k_t) = u'(c_t) \cdot f'(k_t)$$

Euler Equation
From FOC:

$$u'(c_t) = \beta V'(k_{t+1})$$

From envelope:

$$V'(k_{t+1}) = u'(c_{t+1}) \cdot f'(k_{t+1})$$

Substitute:

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1})$$

$$\boxed{u'(C_t) = \beta\, u'(C_{t+1})\, f'(K_{t+1})}$$

b)

Now suppose that $u(C_t) = \log(C_t)$ and $f(K_t) = AK_t^\alpha$, where $A > 0$ and $0 < \alpha < 1$. Using these particular functional forms again solve for the Euler equation.

ANSWER:

Assumptions:

Utility:

$$u(C_t) = \log(C_t)$$

Production:

$$f(K_t) = AK_t^\alpha, \quad A > 0, \quad 0 < \alpha < 1$$

Resource constraint:

$$C_t + K_{t+1} = AK_t^\alpha$$

Derivation

Bellman equation:

$$V(K_t) = \max_{K_{t+1}} \{\log(C_t) + \beta V(K_{t+1})\}$$

Substitute constraint:

$$C_t = AK_t^\alpha - K_{t+1}$$

$$V(K_t) = \max_{K_{t+1}} \{\log(AK_t^\alpha - K_{t+1}) + \beta V(K_{t+1})\}$$

First-order condition:

Define objective:

$$L(K_{t+1}) = \log(AK_t^\alpha - K_{t+1}) + \beta V(K_{t+1})$$

Differentiate w.r.t. K_{t+1}:

$$\frac{d}{dK_{t+1}} \log(AK_t^\alpha - K_{t+1}) = -\frac{1}{AK_t^\alpha - K_{t+1}} = -\frac{1}{C_t}$$

$$\frac{d}{dK_{t+1}} \beta V(K_{t+1}) = \beta V'(K_{t+1})$$

Set FOC:

$$-\frac{1}{C_t} + \beta V'(K_{t+1}) = 0 \quad \Rightarrow \quad \frac{1}{C_t} = \beta V'(K_{t+1}) \tag{1}$$

Envelope condition:

Recall:

$$C_t = AK_t^\alpha - K_{t+1}$$

Differentiate Bellman w.r.t. K_t:

$$\frac{dV}{dK_t} = \frac{d}{dK_t} \log(AK_t^\alpha - K_{t+1}) = \frac{1}{C_t} \cdot \frac{d}{dK_t}(AK_t^\alpha) = \frac{1}{C_t} \cdot A\alpha K_t^{\alpha-1}$$

$$V'(K_t) = \frac{A\alpha K_t^{\alpha-1}}{C_t} \tag{2}$$

Advance one period in (2):

$$V'(K_{t+1}) = \frac{A\alpha K_{t+1}^{\alpha-1}}{C_{t+1}} \tag{3}$$

Substitute (3) into (1):

$$\frac{1}{C_t} = \beta \cdot \frac{A\alpha K_{t+1}^{\alpha-1}}{C_{t+1}}$$

Multiply both sides by C_{t+1}:

$$\frac{C_{t+1}}{C_t} = \beta A\alpha K_{t+1}^{\alpha-1}$$

Final Euler Equation:

$$\boxed{\frac{C_{t+1}}{C_t} = \beta A\alpha K_{t+1}^{\alpha-1}}$$

c)

Using the method of value function iteration show that the optimal policy is to have capital move according to the difference equation

$$K_{t+1} = A\beta\alpha K_t^{\alpha}.$$

[Hint: as a starting point set $V_0 = 0$.]

ANSWER:

Assumptions

$$u(C_t) = \log(C_t) \quad (A1)$$

$$f(K_t) = AK_t^{\alpha}, \quad A > 0,\ 0 < \alpha < 1 \quad (A2)$$

$$C_t + K_{t+1} = f(K_t) = AK_t^{\alpha} \ \Rightarrow\ C_t = AK_t^{\alpha} - K_{t+1} \quad (A3)$$

$$V_0(k) = 0 \quad (A4)$$

$$\boxed{K' \ \equiv\ K_{t+1}}$$

First Iteration: $V_1(k)$

$$V_1(k) = \max_{K'}\{\log(Ak^{\alpha} - K') + \beta V_0(K')\} \quad (1)$$

Substitute $V_0(K') = 0$:

$$V_1(k) = \max_{K'} \log(Ak^\alpha - K') \quad (2)$$

Differentiate:

$$\frac{d}{dK'} \log(Ak^\alpha - K') = -\frac{1}{Ak^\alpha - K'} \quad (3)$$

Always negative $\Rightarrow$ maximum at $K' = 0$ (4)
Substitute:

$$V_1(k) = \log(Ak^\alpha) \quad (5)$$

Use log rules:

$$\log(Ak^\alpha) = \log A + \alpha \log k \quad (6)$$

$$V_1(k) = \log A + \alpha \log k \quad (7)$$

Second Iteration: $V_2(k)$

$$V_2(k) = \max_{K'} \{\log(Ak^\alpha - K') + \beta V_1(K')\} \quad (8)$$

Substitute $V_1(K') = \log A + \alpha \log K'$:

$$V_2(k) = \max_{K'} \{\log(Ak^\alpha - K') + \beta\alpha \log K' + \beta \log A\} \quad (9)$$

Differentiate:

$$\frac{d}{dK'}[\log(Ak^\alpha - K') + \beta\alpha \log K'] = -\frac{1}{Ak^\alpha - K'} + \frac{\beta\alpha}{K'} \quad (10)$$

Set (10)=0:

$$-\frac{1}{Ak^\alpha - K'} + \frac{\beta\alpha}{K'} = 0 \quad (11)$$

Rearrange:

$$\beta\alpha K' = \frac{1}{Ak^\alpha - K'} \quad (12)$$

Cross-multiply:

$$\beta\alpha(Ak^\alpha - K') = K' \quad (13)$$

Distribute:

$$\beta\alpha Ak^\alpha - \beta\alpha K' = K' \quad (14)$$

Group:

$$\beta\alpha Ak^\alpha = K'(1 + \beta\alpha) \quad (15)$$

Solve:

$$K' = \frac{\beta\alpha}{1 + \beta\alpha} Ak^\alpha \quad (16)$$

Define:

$$\theta_2 := \frac{\beta\alpha}{1 + \beta\alpha} \quad \Rightarrow \quad K' = \theta_2 Ak^\alpha \quad (17)$$

Substitute back:

$$V_2(k) = \log(Ak^\alpha - \theta_2 Ak^\alpha) + \beta\alpha \log(\theta_2 Ak^\alpha) + \beta \log A \quad (18)$$

Simplify first log:

$$Ak^\alpha(1-\theta_2) \quad \Rightarrow \quad \log(A(1-\theta_2)k^\alpha) \quad (19)$$

$$\log(A(1-\theta_2)k^\alpha) = \log A + \log(1-\theta_2) + \alpha \log k \quad (20)$$

$$\log(\theta_2 Ak^\alpha) = \log\theta_2 + \log A + \alpha \log k \quad (21)$$

So:

$$V_2(k) = \log A + \log(1-\theta_2) + \alpha \log k \quad (22)$$

$$+\beta\alpha(\log\theta_2 + \log A + \alpha \log k) + \beta \log A \quad (23)$$

Group terms in $\log k$:

$$\alpha \log k + \beta\alpha^2 \log k = \alpha(1+\beta\alpha)\log k \quad (24)$$

Define constants:

$$\delta_2 = \log A + \log(1-\theta_2) + \beta\alpha\log\theta_2 + \beta\alpha\log A + \beta\log A \quad (25)$$

Final expression:

$$V_2(k) = \alpha(1+\beta\alpha)\log k + \delta_2 \quad (26)$$

General Form

From (7) and (26):

$$V_n(k) = \alpha\left(\sum_{i=0}^{n-1}(\beta\alpha)^i\right)\log k + \delta_n \quad (27)$$

As $n \to \infty$:

$$\sum_{i=0}^{\infty}(\beta\alpha)^i = \frac{1}{1-\beta\alpha} \quad (28)$$

Therefore:

$$V(k) = \frac{\alpha}{1-\beta\alpha}\log k + \delta \quad (29)$$

Optimal Policy Function

Bellman:

$$V(k) = \max_{K'}\{\log(Ak^\alpha - K') + \beta V(K')\} \quad (30)$$

Substitute (29):

$$V(K') = \frac{\alpha}{1-\beta\alpha}\log K' + \delta \quad \Rightarrow \quad \text{ignore constants} \quad (31)$$

$$V(k) = \max_{K'}\{\log(Ak^{\alpha} - K') + \beta\tfrac{\alpha}{1-\beta\alpha}\log K'\} \quad (32)$$

Differentiate:

$$\frac{d}{dK'} = -\frac{1}{Ak^{\alpha} - K'} + \frac{\beta\alpha}{(1-\beta\alpha)K'} \quad (33)$$

Set (33)=0:

$$-\frac{1}{Ak^{\alpha} - K'} + \frac{\beta\alpha}{(1-\beta\alpha)K'} = 0 \quad (34)$$

Rearrange:

$$\frac{\beta\alpha}{1-\beta\alpha}K' = \frac{1}{Ak^{\alpha} - K'} \quad (35)$$

Cross-multiply:

$$\beta\alpha(Ak^{\alpha} - K') = (1-\beta\alpha)K' \quad (36)$$

Expand:

$$\beta\alpha Ak^{\alpha} - \beta\alpha K' = (1-\beta\alpha)K' \quad (37)$$

Group:

$$\beta\alpha Ak^{\alpha} = K'(1-\beta\alpha+\beta\alpha) = K' \quad (38)$$

Solve:

$$K' = \beta\alpha Ak^{\alpha} \quad (39)$$

Final Result:

$$\boxed{K_{t+1} = \beta\alpha AK_t^{\alpha}} \quad \text{[Optimal Policy Function]} \quad (40)$$

d)

Now solve the problem using the guess-and-verify method. Guess that

$$V(K) = E + F\log(K)$$

and verify this guess using the Bellman equation.

ANSWER:

Assumptions

$$u(C_t) = \log(C_t) \quad (A1)$$

$$f(K_t) = AK_t^{\alpha}, \quad A > 0,\ 0 < \alpha < 1 \quad (A2)$$

$$C_t + K_{t+1} = AK_t^{\alpha} \;\Rightarrow\; C_t = AK_t^{\alpha} - K_{t+1} \quad (A3)$$

$$\text{Guess: } V(K) = E + F\log(K) \quad (A4)$$

Bellman Equation

$$V(k) = \max_{K'} \{\log(Ak^\alpha - K') + \beta V(K')\} \quad (1)$$

First-order condition (FOC)
Differentiate objective:

$$\frac{d}{dK'}[\log(Ak^\alpha - K') + \beta V(K')] = 0 \quad (2)$$

Substitute $V(K') = E + F\log(K')$:

$$\frac{d}{dK'}[\log(Ak^\alpha - K') + \beta(E + F\log K')] = 0 \quad (3)$$

Differentiate:

$$-\frac{1}{Ak^\alpha - K'} + \frac{\beta F}{K'} = 0 \quad (4)$$

Rearrange:

$$\beta F K' = \frac{1}{Ak^\alpha - K'} \quad (5)$$

Cross-multiply:

$$\beta F(Ak^\alpha - K') = K' \quad (6)$$

Distribute:

$$\beta F Ak^\alpha - \beta F K' = K' \quad (7)$$

Group:

$$\beta F Ak^\alpha = K'(1 + \beta F) \quad (8)$$

Solve for K':

$$K' = \frac{\beta F}{1 + \beta F} Ak^\alpha \quad (9)$$

Substitute K' into Bellman Equation

$$V(k) = \log\left(Ak^\alpha - \tfrac{\beta F}{1+\beta F} Ak^\alpha\right) + \beta V\left(\tfrac{\beta F}{1+\beta F} Ak^\alpha\right) \quad (10)$$

Factor first log:

$$Ak^\alpha\left(\tfrac{1}{1+\beta F}\right) \quad (11)$$

So:

$$\log(Ak^\alpha - K') = \log A + \log(k^\alpha) - \log(1 + \beta F) \quad (12)$$

$$\log(k^\alpha) = \alpha \log k \quad (13)$$

Now compute:

$$\beta V(K') = \beta[E + F\log(\tfrac{\beta F}{1+\beta F} Ak^\alpha)] \quad (14)$$

Expand:

$$\beta E + \beta F \log(\beta F) + \beta F \log A + \beta F \log(k^\alpha) - \beta F \log(1 + \beta F) \quad (15)$$

$$\beta F \log(k^\alpha) = \beta F \alpha \log k \quad (16)$$

Group Terms in Bellman
From (12) and (15):

$$V(k) = [\log A - \log(1 + \beta F)] + \alpha \log k \quad (17)$$

$$+\beta E + \beta F \log(\beta F) + \beta F \log A - \beta F \log(1 + \beta F) + \beta F \alpha \log k \quad (18)$$

Group log terms:

$$\alpha \log k + \beta F \alpha \log k = \alpha(1 + \beta F) \log k \quad (19)$$

Group constants:

$$\delta = \log A - \log(1 + \beta F) + \beta E + \beta F \log(\beta F) + \beta F \log A - \beta F \log(1 + \beta F) \quad (20)$$

Verify the Guess
From guess (A4):

$$V(k) = E + F \log k \quad (21)$$

Compare with (19)–(20):

$$E + F \log k = \alpha(1 + \beta F) \log k + \delta \quad (22)$$

Match coefficients:

$$F = \alpha(1 + \beta F) \quad (23)$$

Solve:

$$F = \alpha + \alpha\beta F \;\Rightarrow\; F - \alpha\beta F = \alpha \;\Rightarrow\; F(1 - \alpha\beta) = \alpha$$

$$F = \frac{\alpha}{1 - \alpha\beta} \quad (24)$$

Recover Policy Function
Substitute (24) into (9):

$$K' = \frac{\beta F}{1 + \beta F} A k^\alpha = \frac{\beta \cdot \frac{\alpha}{1-\alpha\beta}}{1 + \beta \cdot \frac{\alpha}{1-\alpha\beta}} A k^\alpha \quad (25)$$

Simplify numerator:

$$\frac{\beta \cdot \alpha}{1 - \alpha\beta} A k^\alpha = \frac{\beta\alpha A k^\alpha}{1 - \alpha\beta} \quad (26)$$

Simplify denominator:

$$1+\frac{\beta\cdot\alpha}{1-\alpha\beta}=\frac{1\cdot(1-\alpha\beta)}{1-\alpha\beta}+\frac{\beta\alpha}{1-\alpha\beta} \quad (27)$$

$$=\frac{(1-\alpha\beta)+\beta\alpha}{1-\alpha\beta} \quad (28)$$

$$=\frac{1-\alpha\beta+\alpha\beta}{1-\alpha\beta}=\frac{1}{1-\alpha\beta} \quad (29)$$

Now simplify entire expression:
From (26) and (29), we have:

$$K'=\frac{\frac{\beta\alpha Ak^{\alpha}}{1-\alpha\beta}}{\frac{1}{1-\alpha\beta}} \quad (30)$$

Use rule: $\frac{a/b}{c/b}=\frac{a}{c}$

$$K'=\frac{\beta\alpha Ak^{\alpha}}{1} \quad (31)$$

Final Result:

$$\boxed{K'=\beta\alpha Ak^{\alpha}} \quad (32)$$

Problem 2

[The Cake-Eating Problem]

Consider the following cake-eating problem. A consumer is initially endowed with a cake of size $W_0>0$ and wants to decide how much of the cake should be eaten each period:

$$\sum_{t=0}^{\infty}\beta^t u(C_t),$$

where the transition law is given by:

$$C_t+W_{t+1}=W_t,$$

and

$$0<\beta<1,\quad u'(C)>0,\quad u''(C)<0,\quad u'(0)=+\infty.$$

a)

Set up the Bellman equation for this problem.

ANSWER:

Assumptions

$$u(C_t) \text{ is strictly increasing: } u'(C) > 0 \quad (A1)$$
$$u(C_t) \text{ is strictly concave: } u''(C) < 0 \quad (A2)$$
$$u'(0) = +\infty \quad (A3)$$
$$0 < \beta < 1 \quad (A4)$$
$$W_0 > 0 \quad \text{(initial cake stock)} \quad (A5)$$
$$\text{Law of motion: } C_t + W_{t+1} = W_t \quad \Leftrightarrow \quad C_t = W_t - W_{t+1} \quad (A6)$$

Objective:
Maximize lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(C_t) \quad (A7)$$

Bellman Equation
Let the value function $V(W_t)$ denote the maximum lifetime utility from starting with cake stock W_t.

$$C_t = W_t - W_{t+1} \tag{1}$$

The Bellman equation is:

$$V(W_t) = \max_{W_{t+1} \in [0, W_t]} \Big\{ u(W_t - W_{t+1}) + \beta V(W_{t+1}) \Big\} \tag{2}$$

This equation captures the trade-off between: Consuming $C_t = W_t - W_{t+1}$ today (instant utility); Saving W_{t+1} to generate future utility (discounted by β).

b)

Find the Euler equation for this problem.

ANSWER:

Restate Bellman Equation (from part a)

$$V(W_t) = \max_{W_{t+1} \in [0, W_t]} \Big\{ u(W_t - W_{t+1}) + \beta V(W_{t+1}) \Big\} \tag{1}$$

Let

$$C_t = W_t - W_{t+1} \tag{2}$$

First-Order Condition (FOC)
Differentiate (1) with respect to W_{t+1}:

$$\frac{d}{dW_{t+1}} \Big[u(W_t - W_{t+1}) + \beta V(W_{t+1}) \Big] = 0 \tag{3}$$

Chain rule:

$$-u'(W_t - W_{t+1}) + \beta V'(W_{t+1}) = 0 \tag{4}$$

Substitute from (2):

$$-u'(C_t) + \beta V'(W_{t+1}) = 0 \quad \Rightarrow \quad u'(C_t) = \beta V'(W_{t+1}) \tag{5}$$

Envelope Condition
Differentiate (1) with respect to W_t:

$$\frac{dV}{dW_t} = u'(W_t - W_{t+1}) \cdot \frac{d(W_t - W_{t+1})}{dW_t} + \beta V'(W_{t+1}) \cdot \frac{dW_{t+1}}{dW_t} \tag{6}$$

Since $\frac{d(W_t - W_{t+1})}{dW_t} = 1$ and $\frac{dW_{t+1}}{dW_t} = 0$:

$$V'(W_t) = u'(C_t) \tag{7}$$

Advance one period:

$$V'(W_{t+1}) = u'(C_{t+1}) \tag{8}$$

Substitute into FOC (5)

$$u'(C_t) = \beta V'(W_{t+1}) \tag{5 repeated}$$
$$V'(W_{t+1}) = u'(C_{t+1}) \tag{8 repeated}$$

Thus:

$$u'(C_t) = \beta u'(C_{t+1}) \tag{9}$$

$$\boxed{u'(C_t) = \beta u'(C_{t+1})} \quad \text{(Euler Equation)} \tag{10}$$

c)

Suppose that $u(C) = \log(C)$. Using a solution method of your choice find the optimal policy functions.

ANSWER:

Euler Equation
From utility:

$$u(C_t) = \log(C_t) \quad \Rightarrow \quad u'(C_t) = \frac{1}{C_t}$$

Euler equation:

$$u'(C_t) = \beta u'(C_{t+1}) \quad \Rightarrow \quad \frac{1}{C_t} = \beta \cdot \frac{1}{C_{t+1}} \tag{1}$$

Multiplying both sides by $C_t \cdot C_{t+1}$:

$$C_{t+1} = \beta C_t \tag{2}$$

Law of Motion

Budget constraint:

$$C_t + W_{t+1} = W_t \tag{3}$$

Solving for W_{t+1}:

$$W_{t+1} = W_t - C_t \tag{4}$$

Substitute (2) into (4):

$$W_{t+1} = W_t - C_t = W_t - \frac{1}{\beta}C_{t+1} \tag{5}$$

Recursive Consumption Path

From (2):

$$C_{t+1} = \beta C_t, \quad C_{t+2} = \beta^2 C_t, \quad C_{t+3} = \beta^3 C_t$$

In general:

$$C_{t+j} = \beta^j C_t, \quad \forall j \geq 0 \tag{6}$$

Total Resources Consumed

Entire cake is consumed:

$$W_t = \sum_{j=0}^{\infty} C_{t+j} \tag{7}$$

Substitute (6):

$$W_t = \sum_{j=0}^{\infty} \beta^j C_t = C_t \cdot \left(\sum_{j=0}^{\infty} \beta^j \right) \tag{8}$$

Geometric series:

$$\sum_{j=0}^{\infty} \beta^j = \frac{1}{1-\beta}, \quad 0 < \beta < 1 \tag{9}$$

So:

$$W_t = \frac{C_t}{1-\beta} \tag{10}$$

Solve for C_t:

$$C_t = (1-\beta)W_t \tag{11}$$

Back to Law of Motion

From (4):

$$W_{t+1} = W_t - C_t \tag{12}$$

Substitute (11):

$$W_{t+1} = W_t - (1-\beta)W_t = \beta W_t \tag{13}$$

Final Policy Functions

$$\boxed{C_t = (1-\beta)W_t} \tag{14}$$

$$\boxed{W_{t+1} = \beta W_t} \tag{15}$$

d)

Now suppose that

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma} \quad \text{where } \sigma > 0.$$

Guess that the value function and policy function $W_{t+1} = g(W_t)$ have the forms:

$$V(W) = \frac{\alpha W^{1-\sigma}}{1-\sigma}, \qquad g(W) = \mu W$$

for some unknown coefficients $\alpha > 0$ and $0 < \mu < 1$. Derive the first-order condition and envelope condition. Solve for these unknown coefficients.

ANSWER:

Assumptions:

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma}, \quad \sigma > 0$$

$$V(W) = \frac{\alpha W^{1-\sigma}}{1-\sigma}, \quad \alpha > 0$$

$$W_{t+1} = \mu W_t, \quad 0 < \mu < 1$$

$$C_t + W_{t+1} = W_t \quad \text{(resource constraint)}$$

$$0 < \beta < 1 \quad \text{(discount factor)}$$

Bellman Equation

$$V(W_t) = \max_{W_{t+1}} \left\{ \frac{(W_t - W_{t+1})^{1-\sigma}}{1-\sigma} + \beta \cdot \frac{\alpha W_{t+1}^{1-\sigma}}{1-\sigma} \right\} \tag{1}$$

First Order Condition (FOC)

Differentiate (1) w.r.t. W_{t+1}:

$$-(W_t - W_{t+1})^{-\sigma} + \beta\alpha W_{t+1}^{-\sigma} = 0$$

$$(W_t - W_{t+1})^{-\sigma} = \beta\alpha W_{t+1}^{-\sigma} \tag{2}$$

Raise both sides to power $-1/\sigma$:

$$W_t - W_{t+1} = (\beta\alpha)^{-1/\sigma} W_{t+1}$$

$$\frac{W_{t+1}}{W_t} = \frac{1}{1 + (\beta\alpha)^{-1/\sigma}} = \mu \tag{3}$$

Envelope Condition

From guess:

$$V(W_t) = \frac{\alpha W_t^{1-\sigma}}{1-\sigma} \quad \Rightarrow \quad V'(W_t) = \alpha W_t^{-\sigma}$$

From Bellman RHS:

$$V'(W_t) = (W_t - W_{t+1})^{-\sigma}$$

Equating:

$$\alpha W_t^{-\sigma} = (W_t - W_{t+1})^{-\sigma}$$

$$\alpha = (1-\mu)^{-\sigma} \tag{4}$$

Plug into Policy Ratio (Eq. 3)

$$\mu = \frac{1}{1 + (\beta\alpha)^{-1/\sigma}} \tag{3 revisited}$$

Substitute (4):

$$\mu = \frac{1}{1 + \left[\beta(1-\mu)^{-\sigma}\right]^{-1/\sigma}} = \frac{1}{1 + \beta^{1/\sigma}(1-\mu)^{-1}}$$

$$\mu\left(1 - \mu + \beta^{1/\sigma}\right) = 1 - \mu$$

Solution:

$$\mu = \beta^{1/\sigma} \tag{5}$$

From (4):

$$\alpha = (1 - \beta^{1/\sigma})^{-\sigma} \tag{6}$$

Final Answer:

$$\boxed{V(W_t) = \frac{(1-\beta^{1/\sigma})^{-\sigma}\, W_t^{1-\sigma}}{1-\sigma}}$$

$$\boxed{W_{t+1} = \beta^{1/\sigma} W_t}$$

$$\boxed{C_t = (1 - \beta^{1/\sigma}) W_t}$$

Problem 3

[The Hanson Real Business Cycle Closed-Economy Model]
Suppose that the representative agent wants to maximize lifetime utility subject to a given value of the capital stock $K_0 > 0$:

$$\sum_{t=0}^{\infty} \beta^t u(C_t, L_t),$$

where L_t is time t leisure and $L_t = 1 - H_t$, where H_t is labor. The period utility function is given by:

$$u(C_t, 1 - H_t) = \ln C_t + A \ln(1 - H_t),$$

with $A > 0$. The production technology is assumed to be Cobb-Douglas:

$$Y_t = K_t^{\alpha} H_t^{1-\alpha},$$

and capital accumulation follows the process:

$$K_{t+1} = (1 - \delta)K_t + I_t,$$

where $0 < \delta < 1$ is the depreciation rate of capital. Finally, market clearing requires that

$$Y_t = C_t + I_t.$$

a)

Set up the Bellman equation for this problem with K_t as the state variable and K_{t+1} and H_t as the control variables.

ANSWER:

Assumptions:

- State variable: K_t (capital at time t).
- Control variables: H_t (labor at time t), K_{t+1} (capital chosen for next period).
- Consumption: C_t.
- Investment: I_t.
- Output: Y_t.
- Leisure: $L_t = 1 - H_t$.

Utility function:

$$u(C_t, 1 - H_t) = \ln C_t + A \ln(1 - H_t).$$

Production function:

$$Y_t = K_t^{\alpha} H_t^{1-\alpha}.$$

Capital accumulation:

$$K_{t+1} = (1 - \delta)K_t + I_t \quad \iff \quad I_t = K_{t+1} - (1 - \delta)K_t.$$

Market clearing:

$$Y_t = C_t + I_t \quad \Longleftrightarrow \quad C_t = Y_t - I_t.$$

Substituting production and investment:

$$C_t = K_t^\alpha H_t^{1-\alpha} - \Big(K_{t+1} - (1-\delta)K_t\Big).$$

Rewriting:

$$C_t = K_t^\alpha H_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t. \tag{1}$$

Bellman Equation:

The representative agent maximizes lifetime utility:

$$V(K_t) = \max_{K_{t+1},H_t} \Big\{ \ln C_t + A\ln(1-H_t) + \beta V(K_{t+1}) \Big\}, \tag{2}$$

subject to the transition constraint (1):

$$C_t = K_t^\alpha H_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t.$$

b)

Derive the Intertemporal Budget Constraint (IBC)

ANSWER:

Assumptions:

$$Y_t = K_t^\alpha H_t^{1-\alpha} \tag{A1}$$
$$K_{t+1} = (1-\delta)K_t + I_t \tag{A2}$$
$$Y_t = C_t + I_t \tag{A3}$$
$$u(C_t, 1-H_t) = \ln(C_t) + A\ln(1-H_t) \tag{A4}$$

Intertemporal Budget Constraint: From (A2) and (A3),

$$C_t = K_t^\alpha H_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t. \tag{1}$$

Bellman Equation:

$$V(K_t) = \max_{K_{t+1},H_t} \Big\{ \ln(C_t) + A\ln(1-H_t) + \beta V(K_{t+1}) \Big\}, \tag{2}$$

subject to (1). Substituting (1):

$$V(K_t) = \max_{K_{t+1},H_t} \Big\{ \ln\Big(K_t^\alpha H_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t\Big) + A\ln(1-H_t) + \beta V(K_{t+1}) \Big\}. \tag{3}$$

FOC w.r.t. K_{t+1}:

$$\frac{\partial}{\partial K_{t+1}}\Big[\ln(C_t) + A\ln(1-H_t) + \beta V(K_{t+1})\Big] = 0.$$

Since $\frac{\partial C_t}{\partial K_{t+1}} = -1$:

$$-\frac{1}{C_t} + \beta V'(K_{t+1}) = 0 \quad \Longrightarrow \quad \boxed{\frac{1}{C_t} = \beta V'(K_{t+1})}. \tag{4}$$

FOC w.r.t. H_t:

$$\frac{1}{C_t}\cdot\frac{\partial C_t}{\partial H_t} - \frac{A}{1-H_t} = 0.$$

Since $\frac{\partial C_t}{\partial H_t} = K_t^{\alpha}(1-\alpha)H_t^{-\alpha}$:

$$\boxed{\frac{1}{C_t}K_t^{\alpha}(1-\alpha)H_t^{-\alpha} = \frac{A}{1-H_t}}. \tag{5}$$

Envelope Condition:

$$V'(K_t) = \frac{1}{C_t}\Big(\alpha K_t^{\alpha-1}H_t^{1-\alpha} + (1-\delta)\Big).$$

$$\boxed{V'(K_t) = \frac{1}{C_t}\left(\alpha K_t^{\alpha-1}H_t^{1-\alpha} + (1-\delta)\right)}. \tag{6}$$

Final Results:

$$\boxed{\begin{aligned} \frac{1}{C_t} &= \beta V'(K_{t+1}) && (4)\\ \frac{1}{C_t}K_t^{\alpha}(1-\alpha)H_t^{-\alpha} &= \frac{A}{1-H_t} && (5)\\ V'(K_t) &= \frac{1}{C_t}\left(\alpha K_t^{\alpha-1}H_t^{1-\alpha} + (1-\delta)\right) && (6) \end{aligned}}$$

c)

Obtain the optimality conditions for the representative agent

ANSWER:

Assumptions:

$$\begin{aligned}
u(C_t, 1-H_t) &= \ln(C_t) + A\ln(1-H_t), \\
Y_t &= K_t^{\alpha} H_t^{1-\alpha}, \\
K_{t+1} &= (1-\delta)K_t + I_t, \\
Y_t &= C_t + I_t.
\end{aligned}$$

Bellman Equation:

$$V(K_t) = \max_{K_{t+1}, H_t} \Big\{ \ln(C_t) + A\ln(1-H_t) + \beta V(K_{t+1}) \Big\},$$

subject to

$$C_t = K_t^{\alpha} H_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t. \tag{1}$$

FOC w.r.t. K_{t+1}:

$$-\frac{1}{C_t} + \beta V'(K_{t+1}) = 0 \quad \Longrightarrow \quad \boxed{\frac{1}{C_t} = \beta V'(K_{t+1})}. \tag{2}$$

FOC w.r.t. H_t:

$$\frac{1}{C_t}(1-\alpha)K_t^{\alpha} H_t^{-\alpha} = \frac{A}{1-H_t}.$$

$$\boxed{\frac{(1-\alpha)K_t^{\alpha} H_t^{-\alpha}}{C_t} = \frac{A}{1-H_t}}. \tag{3}$$

Envelope Condition:

$$V'(K_t) = \frac{1}{C_t}\Big(\alpha K_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)\Big).$$

$$\boxed{V'(K_t) = \frac{1}{C_t}\Big(\alpha K_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)\Big)}. \tag{4}$$

Shift one period forward:

$$V'(K_{t+1}) = \frac{1}{C_{t+1}}\Big(\alpha K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta)\Big). \tag{5}$$

Euler Equation:

Substitute (5) into (2):

$$\frac{1}{C_t} = \beta \cdot \frac{1}{C_{t+1}}\Big(\alpha K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta)\Big).$$

Multiply through by C_{t+1}:

$$\boxed{\frac{C_{t+1}}{C_t} = \beta\Big(\alpha K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta)\Big)}. \tag{6}$$

Final Optimality Conditions:

$$\boxed{\begin{aligned} \frac{1}{C_t} &= \beta V'(K_{t+1}) && (2) \\ \frac{(1-\alpha)K_t^{\alpha} H_t^{-\alpha}}{C_t} &= \frac{A}{1-H_t} && (3) \\ \frac{C_{t+1}}{C_t} &= \beta\Big(\alpha K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta)\Big) && (6) \end{aligned}}$$

Problem 4

The Small Open Economy Infinite-Horizon Model

Consider the small-open economy model of *Topic 2* (with no government spending and no investment). The representative agent has to maximize lifetime utility:

$$\sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to the sequence of budget constraints:

$$B_{t+1} = (1+r)B_t + Y_t - C_t,$$

where B is net foreign assets, r is the exogenously given world interest rate, Y is output and C is consumption. Suppose throughout that the period utility function is isoelastic:

$$u(C) = \frac{C^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}.$$

First, solve this dynamic optimization problem using the direct (or sequence) approach.

a)

Set up the optimization problem and derive the Euler equation.

ANSWER:

Assumptions

$$u(C) = \frac{C^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}, \quad \sigma > 0,\ \sigma \neq 1 \quad (A1)$$

$$B_{t+1} = (1+r)B_t + Y_t - C_t \quad (A2)$$

Objective

$$\max_{\{C_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \quad (A3)$$

$$\max_{\{C_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \quad \text{s.t. } B_{t+1} = (1+r)B_t + Y_t - C_t \quad (1)$$

Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big[u(C_t) + \lambda_t \Big((1+r)B_t + Y_t - C_t - B_{t+1} \Big) \Big] \quad (2)$$

FOC w.r.t. C_t

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \Big(u'(C_t) - \lambda_t \Big) = 0 \;\Rightarrow\; u'(C_t) = \lambda_t \quad (3)$$

FOC w.r.t. B_{t+1}

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1}(1+r) = 0 \;\Rightarrow\; \lambda_t = \beta(1+r)\lambda_{t+1} \quad (4)$$

Euler Equation

$$u'(C_t) = \beta(1+r) u'(C_{t+1}) \quad (5)$$

Derivative of Utility

$$u'(C) = C^{-1/\sigma} \quad (6)$$

Substitute (6) into (5):

$$C_t^{-1/\sigma} = \beta(1+r)\, C_{t+1}^{-1/\sigma} \quad (7)$$

Solve for C_{t+1}

$$(C_{t+1}/C_t)^{-1/\sigma} = \frac{1}{\beta(1+r)} \;\Rightarrow\; \left(\frac{C_{t+1}}{C_t} \right)^{1/\sigma} = \beta(1+r) \;\Rightarrow\; C_{t+1} = \Big[\beta(1+r) \Big]^{\sigma} C_t \quad (8)$$

$$\boxed{C_{t+1} = \Big[\beta(1+r) \Big]^{\sigma} C_t}$$

b)

Derive the intertemporal budget constraint.

ANSWER:

Per-period budget

$$B_{t+1} = (1+r)B_t + Y_t - C_t, \qquad t = 0, 1, 2, \ldots \quad (1)$$

Discount and sum

$$\frac{B_{t+1}}{(1+r)^{t+1}} = \frac{B_t}{(1+r)^t} + \frac{Y_t - C_t}{(1+r)^t} \tag{2}$$

Sum $t = 0$ to T:

$$\sum_{t=0}^{T} \frac{B_{t+1}}{(1+r)^{t+1}} = \sum_{t=0}^{T} \frac{B_t}{(1+r)^t} + \sum_{t=0}^{T} \frac{Y_t - C_t}{(1+r)^t} \tag{3}$$

Telescoping

$$\frac{B_{T+1}}{(1+r)^{T+1}} - B_0 = \sum_{t=0}^{T} \frac{Y_t - C_t}{(1+r)^t} \tag{4}$$

Rearrange

$$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} = B_0 + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t} - \frac{B_{T+1}}{(1+r)^{T+1}} \tag{5}$$

No-Ponzi / TVC

$$\lim_{T\to\infty} \frac{B_{T+1}}{(1+r)^{T+1}} = 0 \tag{6}$$

Intertemporal budget constraint (date 0)

$$\boxed{\sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t} = B_0 + \sum_{t=0}^{\infty} \frac{Y_t}{(1+r)^t}} \tag{IBC-0}$$

Equivalent, any date t

$$\boxed{\sum_{j=0}^{\infty} \frac{C_{t+j}}{(1+r)^j} = (1+r)B_t + \sum_{j=0}^{\infty} \frac{Y_{t+j}}{(1+r)^j}} t \tag{IBC-}$$

c)

Using the intertemporal budget constraint and the Euler equation, show that the optimal consumption path is:

$$C_t = \frac{r+v}{1+r} W_t$$

where

$$v \equiv 1 - \beta^{\frac{1}{\sigma}} (1+r)^{\frac{1}{\sigma}}$$

and wealth is defined as:

$$W_t \equiv (1+r)B_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s.$$

ANSWER:

Utility and derivative:

$$u(C) = \frac{C^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}, \qquad u'(C) = C^{-1/\sigma}$$

Euler:

$$u'(C_t) = \beta(1+r)\,u'(C_{t+1}) \Rightarrow C_t^{-1/\sigma} = \beta(1+r)\,C_{t+1}^{-1/\sigma} \Rightarrow \frac{C_{t+1}}{C_t} = \big[\beta(1+r)\big]^{\sigma} \equiv 1-v$$

$$\boxed{v \equiv 1 - \beta^{\sigma}(1+r)^{\sigma}}$$

Wealth (date t):

$$\boxed{W_t \equiv (1+r)B_t + \sum_{s=t}^{\infty}\left(\tfrac{1}{1+r}\right)^{s-t} Y_s}$$

IBC (with TVC):

$$\sum_{j=0}^{\infty}\frac{C_{t+j}}{(1+r)^j} = W_t$$

Path from Euler:

$$C_{t+j} = (1-v)^j\,C_t$$

Plug and sum (geometric):

$$C_t\sum_{j=0}^{\infty}\left(\frac{1-v}{1+r}\right)^j = W_t \quad\Rightarrow\quad C_t\cdot\frac{1}{1-\frac{1-v}{1+r}} = W_t \quad\Rightarrow\quad C_t\cdot\frac{1+r}{r+v} = W_t$$

Optimal consumption rule (matches prompt):

$$\boxed{C_t = \frac{r+v}{1+r}\,W_t, \qquad v = 1-\beta^{\sigma}(1+r)^{\sigma}}$$

Second, solve the model using dynamic programming. Suppose the transition law is given by:

$$W_{t+1} = (1+r)\,(W_t - C_t).$$

d)

Set up the Bellman equation for this problem.

ANSWER:

Objects

State:

$$W_t \in \mathcal{W} \subseteq \mathbb{R}_+$$

Control (version 1):

$$C_t \in [0, W_t]$$

Transition:

$$W_{t+1} = (1+r)(W_t - C_t)$$

Bellman (control C_t)

$$V(W_t) = \max_{0 \le C_t \le W_t} \left\{ u(C_t) + \beta\, V\Big((1+r)(W_t - C_t)\Big) \right\}$$

Equivalent form (control W_{t+1})
Solve transition for C_t:

$$C_t = W_t - \frac{W_{t+1}}{1+r}$$

Then:

$$V(W_t) = \max_{W_{t+1} \ge 0} \left\{ u\Big(W_t - \frac{W_{t+1}}{1+r}\Big) + \beta\, V(W_{t+1}) \right\}$$

e)

Using the Euler equation method, show that the policy function for consumption is exactly the same as what you found under the direct approach in part (c) above.

ANSWER:
Euler method $\Rightarrow$ policy equals part (c)
Bellman (control W_{t+1}):

$$V(W_t) = \max_{W_{t+1} \ge 0} \left\{ u\Big(W_t - \frac{W_{t+1}}{1+r}\Big) + \beta V(W_{t+1}) \right\}, \qquad C_t \equiv W_t - \frac{W_{t+1}}{1+r}$$

FOC:

$$\frac{\partial}{\partial W_{t+1}} : \quad u'(C_t)\Big(-\frac{1}{1+r}\Big) + \beta V'(W_{t+1}) = 0 \quad \Longrightarrow \quad u'(C_t) = \beta(1+r)\, V'(W_{t+1})$$

Envelope:

$$V'(W_t) = u'(C_t) \cdot \frac{\partial C_t}{\partial W_t} = u'(C_t) \quad \Longrightarrow \quad V'(W_{t+1}) = u'(C_{t+1})$$

Euler:

$$u'(C_t) = \beta(1+r)\, u'(C_{t+1})$$

CRRA: $u(C) = \frac{C^{1-\sigma}}{1-\sigma}$

$$\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} = \beta(1+r) \quad \Longrightarrow \quad \frac{C_{t+1}}{C_t} = \Big[\beta(1+r)\Big]^{1/\sigma} \equiv 1 - v$$

$$v \equiv 1 - \Big[\beta(1+r)\Big]^{1/\sigma}, \qquad 0 < v < 1$$

Path:

$$C_{t+s} = (1-v)^s C_t$$

IBC (equality by TVC):

$$\sum_{s=0}^{\infty} \frac{C_{t+s}}{(1+r)^s} = W_t$$

Substitute path:

$$C_t \sum_{s=0}^{\infty} \left(\frac{1-v}{1+r}\right)^s = W_t$$

Geometric series:

$$\sum_{s=0}^{\infty} \left(\frac{1-v}{1+r}\right)^s = \frac{1}{1-\frac{1-v}{1+r}} = \frac{1+r}{r+v}$$

Solve C_t:

$$C_t \cdot \frac{1+r}{r+v} = W_t \quad \Longrightarrow \quad \boxed{C_t = \frac{r+v}{1+r} W_t}$$

f)

Now solve the problem using the guess-and-verify method. Hint: guess that

$$V(W) = \frac{F}{\frac{\sigma-1}{\sigma}} W^{\frac{\sigma-1}{\sigma}},$$

where F is an undetermined coefficient.

ANSWER:

Bellman (control W'):

$$V(W) = \max_{W' \geq 0} \left\{ u\Big(W - \frac{W'}{1+r}\Big) + \beta V(W') \right\}, \qquad C \equiv W - \frac{W'}{1+r}$$

Utility (problem's notation):

$$u(C) = \frac{C^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}, \qquad u'(C) = C^{-1/\sigma}$$

Guess:

$$V(W) = \frac{F}{\frac{\sigma-1}{\sigma}} W^{\frac{\sigma-1}{\sigma}} \qquad \Rightarrow \qquad V'(W) = F\, W^{-1/\sigma}$$

FOC wrt W':

$$-\frac{1}{1+r} C^{-1/\sigma} + \beta F\, (W')^{-1/\sigma} = 0 \quad \Rightarrow \quad \left(\frac{C}{W'}\right)^{1/\sigma} = \frac{1}{\beta F(1+r)} \tag{A}$$

Shares: let $C = \kappa W,\ W' = (1+r)(1-\kappa)W$.

$$\left(\frac{\kappa}{(1+r)(1-\kappa)}\right)^{1/\sigma} = \frac{1}{\beta F(1+r)} \tag{B}$$

Envelope & Euler:

$$u'(C_t) = \beta(1+r)u'(C_{t+1}) \ \Rightarrow\ \left(\frac{C_{t+1}}{C_t}\right)^{1/\sigma} = \beta(1+r) \ \Rightarrow\ \frac{C_{t+1}}{C_t} = g, \quad g \equiv [\beta(1+r)]^{\sigma}$$

$$\frac{W_{t+1}}{W_t} = (1+r)(1-\kappa) = g \ \Rightarrow\ \boxed{\kappa = 1 - \frac{g}{1+r} = 1 - \frac{[\beta(1+r)]^{\sigma}}{1+r}} \tag{C}$$

Value matching:

$$\frac{F}{\frac{\sigma-1}{\sigma}} W^{\frac{\sigma-1}{\sigma}} = \frac{(\kappa W)^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}} + \beta \frac{F}{\frac{\sigma-1}{\sigma}} \Big((1+r)(1-\kappa)W\Big)^{\frac{\sigma-1}{\sigma}}$$

$$F = \kappa^{\frac{\sigma-1}{\sigma}} + \beta F\Big((1+r)(1-\kappa)\Big)^{\frac{\sigma-1}{\sigma}} \ \Rightarrow\ \boxed{F = \frac{\kappa^{\frac{\sigma-1}{\sigma}}}{1 - \beta g^{\frac{\sigma-1}{\sigma}}}} \tag{D}$$

Plug (C): $g = [\beta(1+r)]^{\sigma}$, $\kappa = 1 - \dfrac{g}{1+r} = 1 - \beta^{\sigma}(1+r)^{\sigma-1}$.

$$\boxed{F = \frac{\Big(1 - \beta^{\sigma}(1+r)^{\sigma-1}\Big)^{\frac{\sigma-1}{\sigma}}}{1 - \beta^{\sigma}(1+r)^{\sigma-1}} = \Big(1 - \beta^{\sigma}(1+r)^{\sigma-1}\Big)^{-\frac{1}{\sigma}}}$$

Value function (verified):

$$\boxed{V(W) = \frac{F}{\frac{\sigma-1}{\sigma}} W^{\frac{\sigma-1}{\sigma}}, \qquad F = \Big(1 - \beta^{\sigma}(1+r)^{\sigma-1}\Big)^{-\frac{1}{\sigma}}}$$

Implied policy (matches part (c)):

$$C = \kappa W = \left(1 - \frac{[\beta(1+r)]^{\sigma}}{1+r}\right) W = \frac{r+v}{1+r} W, \quad v \equiv 1 - \beta^{\sigma}(1+r)^{\sigma}.$$