Università degli Studi di Padova

Forecast Reconciliation for Hierarchically Organized Data

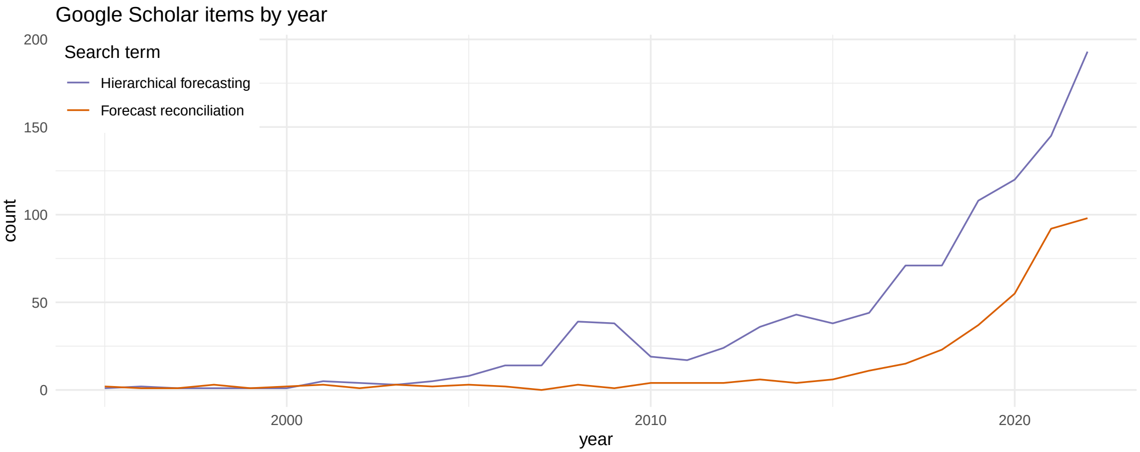
Daniele Girolimetto

- danigiro.github.io
- github.com/daniGiro
- @ daniele.girolimetto@phd.unipd.it

Department of Statistical Sciences, University of Padova (Italy)

March 31st, 2023

Why forecast reconciliation?



Hot topic in the recent years debate on forecasting methodology and practice: several contributions starting from Hyndman et al. (2011)

What is forecast reconciliation

- Post-forecasting process aimed to improve the quality of the base forecasts (however obtained) of a linearly constrained multiple time series by exploiting cross-sectional (e.g., spatial) and/or temporal constraints of the target forecasts
    - cross-sectional constraints
    - temporal constraints
    - cross-temporal constraints (both cross-sectional and temporal)
- Looking for approaches
    - statistically well-grounded (interpretation, properties)
    - feasible, for practical implementation
    - effective, in terms of quality of the results in real-world applications
- Forecasting examples: Sales, Production, Tourism, Energy demand, Healthcare, Real estate, Supply chain ...

Linearly constrained multiple time series

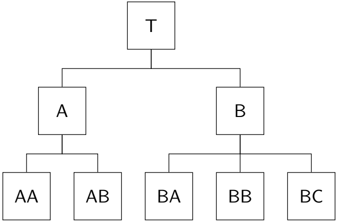
Hierarchical/grouped time series

- A hierarchical time series is a collection of several time series that are linked together in a hierarchical structure (e.g., geographical energy consumption).
- A grouped time series is a collection of time series that are aggregated according to multiple hierarchical structures (e.g., tourism flows grouped by region and purpose of travel)
- Many forecasting applications involve linearly constrained multiple (not only hierarchical/grouped) time series

A cross-sectional (contemporaneous) hierarchical/grouped time series is a collection of n variables for which - at each time - **aggregation relationships** hold. It is a special case of **multiple time series** with exact **linear constraints**

Hierarchical, grouped and linearly constrained time series

Hierarchical time series



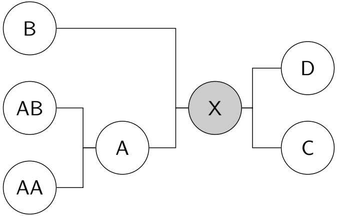
Constraints

$T = A + B$

$A = AA + AB$

$B = BA + BB + BC$

Linearly constrained time series



Constraints

$X = A + B$

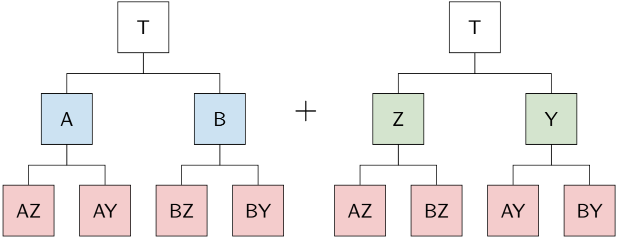
$X = C + D$

$A = AA + AB$

Grouped time series

T	A	B
Z	AZ	BZ
Y	AY	BY

$=$



Constraints

$T = A + B$

$A = AZ + AY$

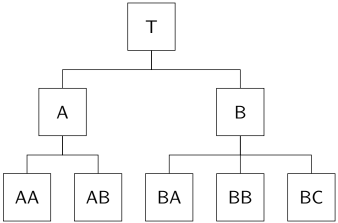
$B = BZ + BY$

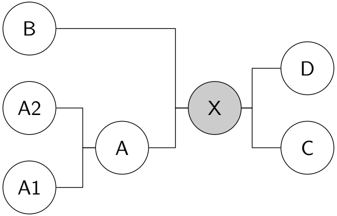
$T = Z + Y$

$Z = AZ + BZ$

$Y = AY + BY$

Forecast reconciliation: a first look





$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$, structural representation $\quad$ $\boldsymbol{U}'\boldsymbol{y}_t = 0$, zero constrained representation

1. Forecast **all series at all levels** of aggregation $\rightarrow$ base forecasts
2. Make the base forecasts **coherent** using least squares $\rightarrow$ reconciled forecasts

Target	Base forecasts		Reconciled forecasts
$\boldsymbol{U}'\boldsymbol{y}_h = 0$	$\boldsymbol{U}'\widehat{\boldsymbol{y}}_h \neq 0$	$\rightarrow$	$\boldsymbol{U}'\widetilde{\boldsymbol{y}}_h = 0$

Optimal forecast reconciliation

Wickramasuriya *et al.* (2019), Panagiotelis *et al.* (2021)

Two equivalent point forecast reconciliation formulae

Structural reconciliation approach

Structural representation

$$\widehat{\boldsymbol{y}}_h = \boldsymbol{S}\boldsymbol{\beta}_h + \boldsymbol{\varepsilon}_h$$

$$\Downarrow$$

$$\widetilde{\boldsymbol{y}}_h = \boldsymbol{S}\left(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\right)^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\widehat{\boldsymbol{y}}_h = \boldsymbol{S}\boldsymbol{G}\widehat{\boldsymbol{y}}_h$$

Projection reconciliation approach

Zero-constrained representation

$$\widehat{\boldsymbol{y}}_h = \boldsymbol{y}_h + \boldsymbol{\varepsilon}_h, \quad \text{s.t.} \quad \boldsymbol{U}'\boldsymbol{y}_h = 0$$

$$\Downarrow$$

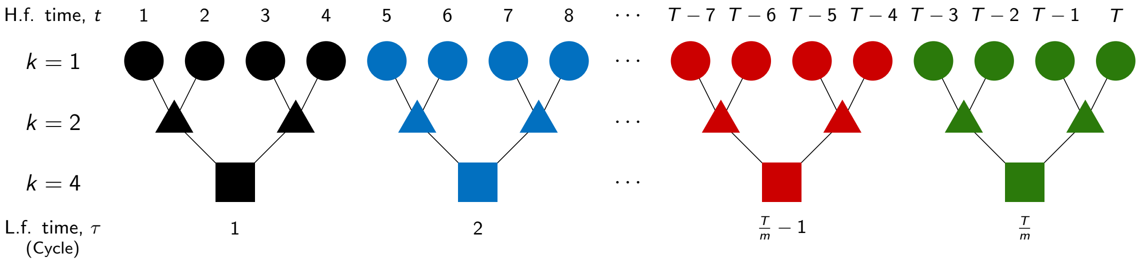
$$\widetilde{\boldsymbol{y}}_h = \left[\boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{U}\left(\boldsymbol{U}'\boldsymbol{W}_h\boldsymbol{U}\right)^{-1}\boldsymbol{U}'\right]\widehat{\boldsymbol{y}}_h = \boldsymbol{M}\widehat{\boldsymbol{y}}_h$$

- The formulation of $\boldsymbol{W}_h = \mathrm{E}(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h')$ is conceptually complex; in practice, approximate forms are used, possibly using in-sample residuals

Temporal hierarchies

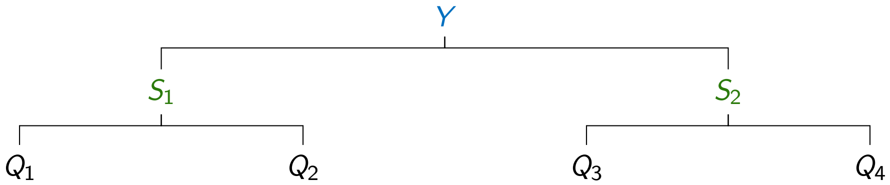
Athanasopoulos et al. (2017), Nystrup et al. (2020)

A temporal hierarchy is built through **non-overlapping aggregation** of the observations of a time series at regular intervals



Quarterly time series ($k = 1$) aggregated to semi-annual ($k = 2$) and annual ($k = 4$): k denotes the aggregation order (e.g., $k \in \mathcal{K} = \{4, 2, 1\}$) and m the frequency of the most disaggregated temporal level (e.g., $m = 4$).

Temporal reconciliation



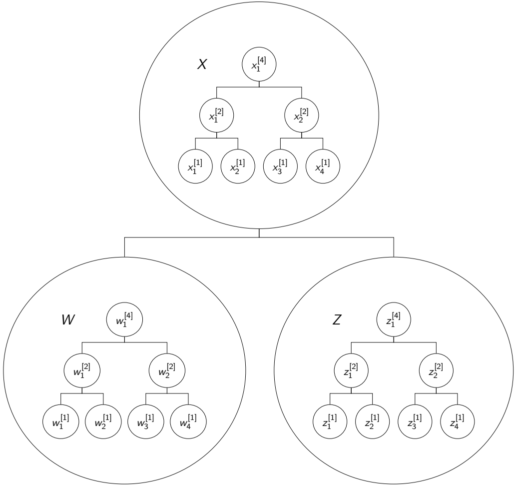
Quarterly hierarchy: quarterly, semi-annual and annual series

- Unlike cross-sectional hierarchies, where n **variables at the same time index** are considered, in temporal hierarchies one deals with **one variable observed at different frequencies**
- Structural representation ($\boldsymbol{x}_\tau = \boldsymbol{R}_1 \boldsymbol{x}_\tau^{[1]}$) and zero constrained representation ($\boldsymbol{Z}_1' \boldsymbol{x}_\tau = 0_{(k^* \times 1)}$) still hold, and may be alternatively used for reconciliation

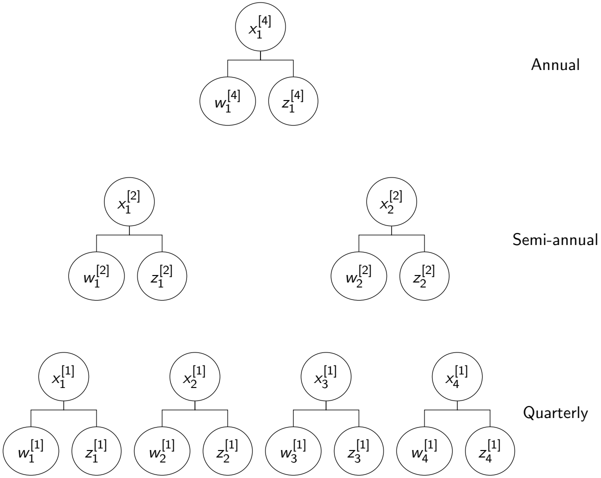
Cross-sectional + Temporal = Cross-temporal

A cross-temporal hierarchy of three quarterly time series ($X = W + Z$)

cross-sectional $\longrightarrow$ temporal



temporal $\longrightarrow$ cross-sectional



Cross-temporal optimal forecast reconciliation

Di Fonzo and Girolimetto (2023)

- The reconciliation formula depends on the full row-rank zero constraints matrix, $\boldsymbol{H}'$
- $\boldsymbol{H}'$ is large and sparse: function of the cross-sectional aggregation matrix ($\boldsymbol{C}$) and of the frequency of the most disaggregated temporal level (m)
- Matrix formulation:

$$\widehat{\boldsymbol{Y}}_h = \boldsymbol{Y}_h + \boldsymbol{E} \longrightarrow \widehat{\boldsymbol{y}}_h = \boldsymbol{y}_h + \boldsymbol{\eta}$$

  with $\boldsymbol{y}_h = \text{vec}\,(\boldsymbol{Y}_h')$, $\boldsymbol{\eta} = \text{vec}\,(\boldsymbol{E}')$ and $\boldsymbol{\Omega} = E\,[\boldsymbol{\eta}\boldsymbol{\eta}']$
- Projection approach:

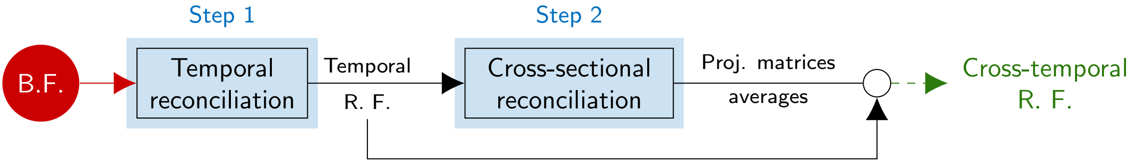
$$\arg\min_{\boldsymbol{y}_h} \left(\widehat{\boldsymbol{y}}_h - \boldsymbol{y}_h\right)' \boldsymbol{\Omega}^{-1} \left(\widehat{\boldsymbol{y}}_h - \boldsymbol{y}_h\right) \qquad \text{s.t.} \qquad \boldsymbol{H}'\boldsymbol{y}_h = 0$$

$$\Rightarrow \quad \widetilde{\boldsymbol{y}} = \left[\boldsymbol{I} - \boldsymbol{\Omega}\boldsymbol{H}\left(\boldsymbol{H}'\boldsymbol{\Omega}\boldsymbol{H}\right)^{-1}\boldsymbol{H}'\right]\widehat{\boldsymbol{y}} = \boldsymbol{M}\widehat{\boldsymbol{y}}$$

- The cross-temporal summing matrix for the structural representation is $\boldsymbol{S}_{ct} = \boldsymbol{S} \otimes \boldsymbol{R}_1$

Two-step approach

Kourentzes and Athanasopoulos, (2019)



tcs (KA): first-temporal-then-cross-sectional reconciliation

Step 1: reconciliation through temporal hierarchies for each single variable
$\rightarrow$ temporally coherent forecasts

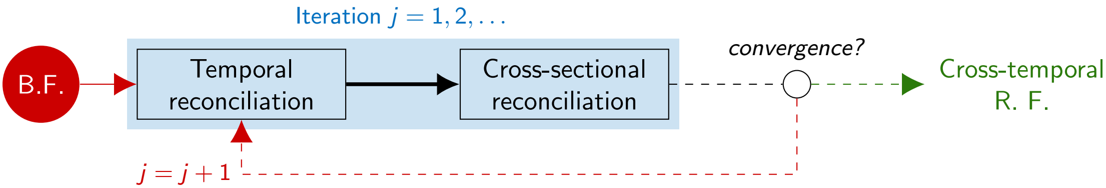
Step 2: time-by-time cross-sectional reconciliation of the previously computed forecasts
$\rightarrow$ cross-sectionally coherent forecasts

$\Rightarrow$ The final reconciled forecasts are calculated starting from the step 1 forecasts through the average of the step 2 projection matrices
$\rightarrow$ cross-temporally coherent forecasts

NB: Sometimes the average of the projection matrices is not needed (Di Fonzo and Girolimetto, 2022)

Iterative cross-temporal point forecast reconciliation

Alternating point forecast reconciliation along one single dimension (Di Fonzo and Girolimetto, 2022)



Iterative first-temporal-then-cross-sectional reconciliation

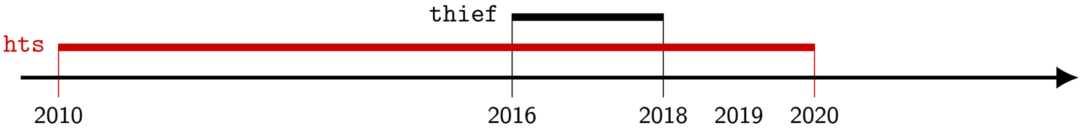
- Each iteration consists in the first two steps of the heuristic KA procedure, until a convergence criterion is met.
- Quick convergence, regardless the first fulfilled dimension
- Possible non-negativity constraints are easily dealt with

Past of reconciliation tools

in **R** (R Core Team, 2021)

`hts` → Cross-sectional structural reconciliation (Hyndman et al. 2021).
→ Available on **CRAN**
→ **First release**: 22/03/2010
→ **Last release**: 30/05/2021 (retired on 2020)

`thief` → Temporal structural reconciliation (Hyndman and Kourentzes, 2018).
→ Available on **CRAN**
→ **First release**: 07/09/2016
→ **Last release**: 24/01/2018

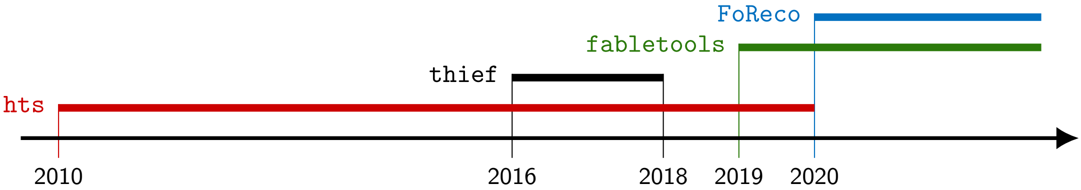


Present and future of reconciliation tools

in **R** (R Core Team, 2021)

`fabletools` → new reference for working with time series (O'Hara-Wild et al. 2021).
→ Only cross-sectional reconciliation is available with `reconcile()` on **CRAN**.
→ Development version for temporal and cross-temporal reconciliation available on **GitHub**.

Waiting for a stable and complete version of `reconcile()` in `fabletools`, `FoReco` is compared with `hts` and `thief`.



What is FoReco?

R package, Di Fonzo and Girolimetto (2022)

- FoReco offers classical (bottom-up and top-down), and modern (optimal and heuristic combination) **forecast reconciliation procedures** for cross-sectional, temporal, and **cross-temporal** **linearly constrained multiple time series**.
- Matrix-based package, exploiting the very sparse nature of the involved matrices
- **Links**:
  - cran.r-project.org/package=FoReco
  - github.com/daniGiro/FoReco
  - danigiro.github.io/FoReco

FoReco

Available on R

First release:	01/10/2020
Last release:	04/07/2022
Next release:	06/2023

+12k

Università degli Studi di Padova

What's next? Lab session!

Daniele Girolimetto

danigiro.github.io
github.com/daniGiro
@ daniele.girolimetto@phd.unipd.it

Department of Statistical Sciences, University of Padova (Italy)

March 31st, 2023

References

Athanasopoulos, G., Hyndman, R. J., Kourentzes, N., & Petropoulos, F. (2017). Forecasting with temporal hierarchies. European Journal of Operational Research, 262, 60–74. https://dx.doi.org/10.1016/j.ejor.2017.02.046.

Di Fonzo, T. & Girolimetto, D. (2023), Cross-temporal forecast reconciliation: Optimal combination method and heuristic alternatives, International Journal of Forecasting 39(1), 39– 57. https://doi.org/10.1016/j.ijforecast.2021.08.004

Di Fonzo, T. & Girolimetto, D. (2022), Enhancements in cross-temporal forecast reconciliation, with an application to solar irradiance forecasts, https://doi.org/10.48550/arXiv.2209.07146.

Girolimetto, D. & Di Fonzo, T. (2022), FoReco: Point Forecast Reconciliation. https://danigiro.github.io/FoReco/

Hyndman, R. J., Ahmed, R. A., Athanasopoulos, G., & Shang, H. L. (2011). Optimal combination forecasts for hierarchical time series. Computational Statistics and Data Analysis, 55(9), 2579–2589. https://dx.doi.org/10.1016/j.csda.2011.03.006.

Hyndman, R. J. e Kourentzes, N. (2018). thief: Temporal HIErarchical Forecasting. url: https://pkg.robjhyndman.com/thief.

Hyndman, R. J., Lee, A., Wang, E. e Wickramasuriya, S. L. (2021). hts: Hierarchical and Grouped Time Series. url: https://cran.r-project.org/package=hts.

References

Kourentzes, N., & Athanasopoulos, G. (2019). Cross-temporal coherent forecasts for Australian tourism. Annals of Tourism Research, 75, 393–409. https://dx.doi.org/10.1016/j.annals.2019.02.001.

Nystrup, P., Lindstro, E., Pinson, P., & Madsen, H. (2020). Temporal hierarchies with autocorrelation for load forecasting. European Journal of Operational Research, 280, 876–888. https://dx.doi.org/10.1016/j.ejor.2019.07.061.

O'Hara-Wild, M., Hyndman, R.J., and Wang, E. (2021). fabletools: Core Tools for Packages in the 'fable' Framework. url: https://CRAN.R-project.org/package=fabletools

Panagiotelis, A., Athanasopoulos, G., Gamakumara, P., & Hyndman, R. J. (2021). Forecast reconciliation: A geometric view with new insights on bias correction. International Journal of Forecasting, 37, 343–359. https://dx.doi.org/10.1016/j.ijforecast.2020.06.004.

R Core Team (2021). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing. Vienna, Austria. url: https://www.r-project.org/.

Wickramasuriya, S. L., Athanasopoulos, G., & Hyndman, R. J. (2019). Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization. Journal of the American Statis- tical Association, 114, 804–819. https://dx.doi.org/10.1080/01621459.2018.1448825.

The cover image has been designed using assets from Freepik.com

Forecast reconciliation: matrix representations

$$\begin{bmatrix} y_{T,t} \\ y_{A,t} \\ y_{B,t} \\ y_{AA,t} \\ y_{AB,t} \\ y_{BA,t} \\ y_{BB,t} \\ y_{BC,t} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{AA,t} \\ y_{AB,t} \\ y_{BA,t} \\ y_{BB,t} \\ y_{BC,t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{T,t} \\ y_{A,t} \\ y_{A1,t} \\ y_{A2,t} \\ y_{B,t} \\ y_{C,t} \\ y_{D,t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$, structural representation

$\boldsymbol{U}'\boldsymbol{y}_t = 0$, zero constrained representation

1. Forecast **all series at all levels** of aggregation $\rightarrow$ base forecasts
2. Make the base forecasts **coherent** using least squares $\rightarrow$ reconciled forecasts

Target	Base forecasts		Reconciled forecasts
$\boldsymbol{U}'\boldsymbol{y}_h = 0$	$\boldsymbol{U}'\widehat{\boldsymbol{y}}_h \neq 0$	$\rightarrow$	$\boldsymbol{U}'\widetilde{\boldsymbol{y}}_h = 0$

$\boldsymbol{H}'$: cross-temporal full row-rank zero constraints matrix

$$\boldsymbol{Y}_\tau = \begin{bmatrix} \boldsymbol{x}_\tau' \\ \boldsymbol{w}_\tau' \\ \boldsymbol{z}_\tau' \end{bmatrix} \quad \boldsymbol{Z}_1' = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{bmatrix} \quad \boldsymbol{I}^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{I}^* & -\boldsymbol{I}^* & -\boldsymbol{I}^* \\ \boldsymbol{Z}_1' & 0 & 0 \\ 0 & \boldsymbol{Z}_1' & 0 \\ 0 & 0 & \boldsymbol{Z}_1' \end{bmatrix} \operatorname{vec}\left(\boldsymbol{Y}_\tau'\right) = 0 \longrightarrow \boldsymbol{H}'\boldsymbol{y}_\tau = 0$$

Iterative convergence criterion

Temporal and the cross-sectional incoherence

- L_1-norm

$$d_{cs} = \|\boldsymbol{U}'\widehat{\boldsymbol{Y}}\|_1 \qquad \text{and} \qquad d_{te} = \|\boldsymbol{Z}_1'\widehat{\boldsymbol{Y}}'\|_1$$

with

$$\|\boldsymbol{X}\|_1 = \sum_{i,j} |x_{i,j}|$$

- L_∞-norm

$$d_{cs} = \|\boldsymbol{U}'\widehat{\boldsymbol{Y}}\|_\infty \qquad \text{and} \qquad d_{te} = \|\boldsymbol{Z}_1'\widehat{\boldsymbol{Y}}'\|_\infty$$

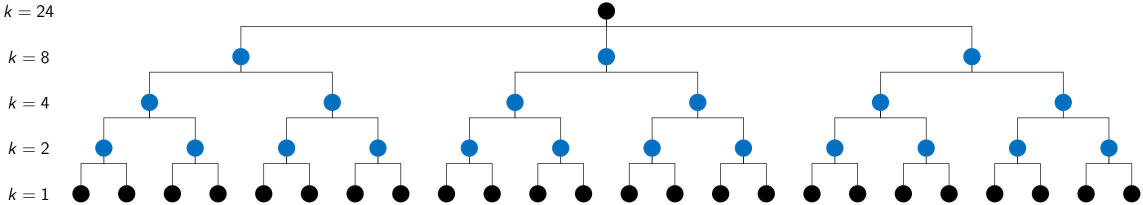
with

$$\|\boldsymbol{X}\|_\infty = \max |x_{i,j}|$$

Hourly grouped time series: two temporal hierarchies

$\mathcal{K} = \{24, 12, 8, 6, 3, 2, 1\}$

Temporal hierarchy $\{24, 8, 4, 2, 1\}$



Temporal hierarchy $\{24, 12, 6, 3, 1\}$