Transforms
Lecture 7 - Laplace Transforms

CS 130 Lecture Slides
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Outline

- Introduction
- Laplace Transforms
- Inverse Laplace Transforms
- Application: Solving IVPs

Introduction

Transforms? What? Why?

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- Some problems are difficult to solve in their original representation.

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Introduction

Transforms? What? Why?

- Some problems are difficult to solve in their original representation.
- Transforms map a function from its original domain into another domain.
- Solving the problem in the target domain can be much easier.
- We will look at two kinds of *integral transforms*: Laplace and Fourier.

Introduction

Laplace and Fourier Transforms:

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- Usually seen in the field of *signal processing*.

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- Usually seen in the field of *signal processing*.
- Signals are functions that convey information about a system (e.g. sounds, images, prices).
- Both map a function from the *time* domain to the *frequency* domain.

Introduction

Laplace and Fourier Transforms:

- Usually seen in the field of *signal processing*.
- Signals are functions that convey information about a system (e.g. sounds, images, prices).
- Both map a function from the *time* domain to the *frequency* domain.
- Fourier is a "subset" of Laplace (a more generalized transform).

Piecewise Continuous Function

A **piecewise continuous function** is a function that has a finite number of breaks in it and does not blow up to infinity anywhere.

Laplace Transforms

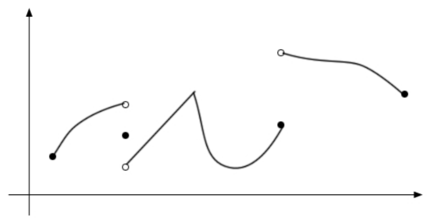

Figure: Example of a piecewise continuous function

Laplace Transform

The **Laplace transform** $\mathcal{L}$ of a piecewise continuous function $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \mathrm{e}^{-st} f(t) dt$$

Given that $c \neq 0$, evaluate the following $\int_0^\infty \mathrm{e}^{ct} dt$.

Solution.

Given that $c \neq 0$, evaluate the following $\int_0^\infty \mathrm{e}^{ct} dt$.

Solution.

1. Convert the improper integral to a limit, then evaluate.

$$\int_0^\infty \mathrm{e}^{ct} dt = \lim_{n \to \infty} \int_0^n \mathrm{e}^{ct} dt$$

Given that $c \neq 0$, evaluate the following $\int_0^\infty e^{ct}dt$.

Solution.

1. Convert the improper integral to a limit, then evaluate.

$$\begin{aligned}\int_0^\infty e^{ct}dt &= \lim_{n\to\infty}\int_0^n e^{ct}dt \\ &= \lim_{n\to\infty}\left(\frac{1}{c}e^{ct}\right)\Bigg|_0^n\end{aligned}$$

Given that $c \neq 0$, evaluate the following $\int_0^\infty \mathrm{e}^{ct}\,dt$.

Solution.

1. Convert the improper integral to a limit, then evaluate.

$$
\begin{aligned}
\int_0^\infty \mathrm{e}^{ct}\,dt &= \lim_{n\to\infty} \int_0^n \mathrm{e}^{ct}\,dt \\
&= \lim_{n\to\infty} \left(\frac{1}{c}\mathrm{e}^{ct} \right) \Big|_0^n \\
&= \lim_{n\to\infty} \left(\frac{1}{c}\mathrm{e}^{cn} - \frac{1}{c} \right)
\end{aligned}
$$

Given that $c \neq 0$, evaluate the following $\int_0^\infty \mathrm{e}^{ct} dt$.

Solution.

1. Convert the improper integral to a limit, then evaluate.

$$\begin{aligned}\int_0^\infty \mathrm{e}^{ct} dt &= \lim_{n\to\infty} \int_0^n \mathrm{e}^{ct} dt \\ &= \lim_{n\to\infty} \left(\frac{1}{c}\mathrm{e}^{ct} \right) \Big|_0^n \\ &= \lim_{n\to\infty} \left(\frac{1}{c}\mathrm{e}^{cn} - \frac{1}{c} \right)\end{aligned}$$

2. If $c < 0$,

$$\int_0^\infty \mathrm{e}^{ct} dt = -\frac{1}{c}$$

Compute $\mathcal{L}\{1\}$.

Solution.

Compute $\mathcal{L}\{1\}$.

Solution.

1. Plug $f(t) = 1$ into the definition.

$$\mathcal{L}\{1\} = \int_0^\infty \mathrm{e}^{-st} dt$$

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Solution.

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$$\mathcal{L}\{1\} = \int_0^\infty \mathrm{e}^{-st} dt$$
$$= -\frac{1}{-s}$$

Compute $\mathcal{L}\{1\}$.

Solution.

1. Plug $f(t) = 1$ into the definition.

$$\mathcal{L}\{1\} = \int_0^\infty \mathrm{e}^{-st} dt$$
$$= -\frac{1}{-s}$$
$$= \frac{1}{s}$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

1. Plug $f(t) = \sin(at)$ into the definition.

$$F(s) = \int_0^\infty \mathrm{e}^{-st} \sin(at) dt$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

1. Plug $f(t) = \sin(at)$ into the definition.

$$F(s) = \int_0^\infty \mathrm{e}^{-st} \sin(at) dt$$

2. Convert the improper integral to a limit.

$$F(s) = \lim_{n \to \infty} \int_0^n \mathrm{e}^{-st} \sin(at) dt$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

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3. Integrating by parts $\int udv = uv - \int vdu$.

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

3. Integrating by parts $\int u dv = uv - \int v du$.

$$u = \mathrm{e}^{-st} \quad , \quad dv = \sin(at)dt$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

3. Integrating by parts $\int udv = uv - \int vdu$.

$$u = \mathrm{e}^{-st} \quad , \quad dv = \sin(at)dt$$

$$du = -s\mathrm{e}^{-st}dt \quad , \quad v = -\frac{1}{a}\cos(at)$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

3. Integrating by parts $\int u dv = uv - \int v du$.

$$u = \mathrm{e}^{-st} \quad , \quad dv = \sin(at)dt$$

$$du = -s\mathrm{e}^{-st}dt \quad , \quad v = -\frac{1}{a}\cos(at)$$

$$F(s) = \lim_{n\to\infty}\Big[-\Big(\frac{1}{a}\mathrm{e}^{-st}\cos(at)\Big)\Big|_0^n - \frac{s}{a}\int_0^n \mathrm{e}^{-st}\cos(at)dt\Big]$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

3. Integrating by parts $\int u dv = uv - \int v du$.

$$u = \mathrm{e}^{-st} \quad , \quad dv = \sin(at)dt$$

$$du = -s\mathrm{e}^{-st}dt \quad , \quad v = -\frac{1}{a}\cos(at)$$

$$F(s) = \lim_{n\to\infty}\left[-\left(\frac{1}{a}\mathrm{e}^{-st}\cos(at)\right)\Big|_0^n - \frac{s}{a}\int_0^n \mathrm{e}^{-st}\cos(at)dt\right]$$

4. Evaluate first term, integrate by parts again.

$$F(s) = \lim_{n\to\infty}\Bigg[\left(\frac{1}{a}(1 - \mathrm{e}^{-sn}\cos(an))\right) - \frac{s}{a}\Bigg(\left(\frac{1}{a}\mathrm{e}^{-st}\sin(at)\right)\Big|_0^n + \frac{s}{a}\int_0^n \mathrm{e}^{-st}\sin(at)dt\Bigg)\Bigg]$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

5. Evaluate second term.

$$F(s) = \lim_{n\to\infty} \left[\left(\frac{1}{a}(1 - \mathrm{e}^{-sn}\cos(an)) \right) - \frac{s}{a}\left(\left(\frac{1}{a}\mathrm{e}^{-sn}\sin(an) \right) + \frac{s}{a}\int_0^n \mathrm{e}^{-st}\sin(at)dt \right) \right]$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

5. Evaluate second term.

$$F(s) = \lim_{n\to\infty}\left[\left(\frac{1}{a}(1 - \mathrm{e}^{-sn}\cos(an))\right) - \frac{s}{a}\left(\left(\frac{1}{a}\mathrm{e}^{-sn}\sin(an)\right)\right.\right.$$
$$\left.\left. + \frac{s}{a}\int_0^n \mathrm{e}^{-st}\sin(at)dt\right)\right]$$

6. Take the limit and simplify.

$$F(s) = \frac{1}{a} - \frac{s}{a}\left(\frac{s}{a}\int_0^n \mathrm{e}^{-st}\sin(at)dt\right)$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

5. Evaluate second term.

$$F(s) = \lim_{n\to\infty}\left[\left(\frac{1}{a}(1-\mathrm{e}^{-sn}\cos(an))\right) - \frac{s}{a}\left(\left(\frac{1}{a}\mathrm{e}^{-sn}\sin(an)\right) + \frac{s}{a}\int_0^n \mathrm{e}^{-st}\sin(at)dt\right)\right]$$

6. Take the limit and simplify.

$$F(s) = \frac{1}{a} - \frac{s}{a}\left(\frac{s}{a}\int_0^n \mathrm{e}^{-st}\sin(at)dt\right)$$

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}F(s)$$

Compute $\mathcal{L}\{\sin(at)\}$.

Solution.

5. Evaluate second term.

$$F(s) = \lim_{n\to\infty}\left[\left(\frac{1}{a}(1-\mathrm{e}^{-sn}\cos(an))\right) - \frac{s}{a}\left(\left(\frac{1}{a}\mathrm{e}^{-sn}\sin(an)\right)\right.\right.$$
$$\left.\left.+\frac{s}{a}\int_0^n \mathrm{e}^{-st}\sin(at)dt\right)\right]$$

6. Take the limit and simplify.

$$F(s) = \frac{1}{a} - \frac{s}{a}\left(\frac{s}{a}\int_0^n \mathrm{e}^{-st}\sin(at)dt\right)$$
$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}F(s)$$
$$= \frac{a}{s^2+a^2}$$

Laplace Transform (Alternate)

The **Laplace transform** $\mathcal{L}$ of a piecewise continuous function $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} \mathrm{e}^{-st} f(t) dt$$

assuming that

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ f(t) & \text{if } t \geq 0 \end{cases}$$

Theorem

Given $f(t)$ and $g(t)$ then,

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

for any constants a and b.

Find the Laplace transform of $f(t) = 6\mathrm{e}^{-5t} + \mathrm{e}^{3t} + 5t^3 - 9$.

Solution.

Find the Laplace transform of $f(t) = 6\mathrm{e}^{-5t} + \mathrm{e}^{3t} + 5t^3 - 9$.

Solution.

1. Use the table!

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

Find the Laplace transform of $f(t) = 6\mathrm{e}^{-5t} + \mathrm{e}^{3t} + 5t^3 - 9$.

Solution.

1. Use the table!

$$\begin{aligned} F(s) &= 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s} \\ &= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s} \end{aligned}$$

Find the Laplace transform of $h(t) = t^2 \sin(2t)$.

Solution.

Find the Laplace transform of $h(t) = t^2 \sin(2t)$.

Solution.

1. Break the function into parts.

   Let $H(s) = \mathcal{L}\{tf(t)\} = -F'(s)$ where $f(t) = t\sin(2t)$

Find the Laplace transform of $h(t) = t^2 \sin(2t)$.

Solution.

1. Break the function into parts.

   Let $H(s) = \mathcal{L}\{tf(t)\} = -F'(s)$ where $f(t) = t\sin(2t)$

2. Use the table!

$$F(s) = \frac{4s}{(s^2+4)^2}$$

Find the Laplace transform of $h(t) = t^2 \sin(2t)$.

Solution.

1. Break the function into parts.

   Let $H(s) = \mathcal{L}\{tf(t)\} = -F'(s)$ where $f(t) = t\sin(2t)$

2. Use the table!

$$F(s) = \frac{4s}{(s^2+4)^2}$$

$$F'(s) = -\frac{12s^2 - 16}{(s^2+4)^3}$$

Find the Laplace transform of $h(t) = t^2 \sin(2t)$.

Solution.

1. Break the function into parts.

   Let $H(s) = \mathcal{L}\{tf(t)\} = -F'(s)$ where $f(t) = t\sin(2t)$

2. Use the table!

$$F(s) = \frac{4s}{(s^2+4)^2}$$
$$F'(s) = -\frac{12s^2-16}{(s^2+4)^3}$$

3. Therefore

$$H(s) = \frac{12s^2-16}{(s^2+4)^3}$$

Inverse Laplace Transforms

Theorem

Given two Laplace transforms $F(s)$ and $G(s)$ then,

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$

for any constants a and b.

Inverse Laplace Transforms
Example 7.6

Find the inverse Laplace transform of $H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$.

Solution.

Inverse Laplace Transforms

Example 7.6

Find the inverse Laplace transform of $H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$.

Solution.

1. Look at each term. Reverse engineer the inverse using the table.

Find the inverse Laplace transform of $H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$.

Solution.

1. Look at each term. Reverse engineer the inverse using the table.
2. Rewrite the transform.

$$H(s) = 19\frac{1}{s-(-2)} - \frac{1}{3}\frac{1}{s-\frac{5}{3}} + \frac{7}{4!}\frac{4!}{s^{4+1}}$$

Find the inverse Laplace transform of $H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$.

Solution.

1. Look at each term. Reverse engineer the inverse using the table.
2. Rewrite the transform.

$$H(s) = 19\frac{1}{s-(-2)} - \frac{1}{3}\frac{1}{s-\frac{5}{3}} + \frac{7}{4!}\frac{4!}{s^{4+1}}$$

3. Take the inverse.

$$h(t) = 19\mathrm{e}^{-2t} - \frac{1}{3}\mathrm{e}^{-\frac{5}{3}t} + \frac{7}{24}t^4$$

Find the inverse Laplace transform of $F(s) = \frac{6s-5}{s^2+7}$.

Solution.

Find the inverse Laplace transform of $F(s) = \frac{6s-5}{s^2+7}$.

Solution.

1. Split the expression.

$$F(s) = \frac{6s}{s^2+7} - \frac{5}{s^2+7}$$

Find the inverse Laplace transform of $F(s) = \frac{6s-5}{s^2+7}$.

Solution.

1. Split the expression.

$$F(s) = \frac{6s}{s^2+7} - \frac{5}{s^2+7}$$

2. Rewrite the transform.

$$F(s) = 6\frac{s}{s^2+(\sqrt{7})^2} - \frac{5}{\sqrt{7}}\frac{\sqrt{7}}{s^2+(\sqrt{7})^2}$$

Inverse Laplace Transforms
Example 7.7

Find the inverse Laplace transform of $F(s) = \frac{6s-5}{s^2+7}$.

Solution.

1. Split the expression.

$$F(s) = \frac{6s}{s^2+7} - \frac{5}{s^2+7}$$

2. Rewrite the transform.

$$F(s) = 6\frac{s}{s^2+(\sqrt{7})^2} - \frac{5}{\sqrt{7}}\frac{\sqrt{7}}{s^2+(\sqrt{7})^2}$$

3. Take the inverse.

$$f(t) = 6\cos(\sqrt{7}t) - \frac{5}{\sqrt{7}}\sin(\sqrt{7}t)$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

1. Factor the denominator.

$$H(s) = \frac{s+7}{(s+2)(s-5)}$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

1. Factor the denominator.

$$H(s) = \frac{s+7}{(s+2)(s-5)}$$

2. Split the term using *partial fractions*.

$$H(s) = \frac{A}{s+2} + \frac{B}{s-5}$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

1. Factor the denominator.

$$H(s) = \frac{s+7}{(s+2)(s-5)}$$

2. Split the term using *partial fractions*.

$$H(s) = \frac{A}{s+2} + \frac{B}{s-5}$$

3. If we add both fractions, the following must be true:

$$\frac{s+7}{(s+2)(s-5)} = \frac{A(s-5)+B(s+2)}{(s+2)(s-5)}$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

4. Simplify.

$$s + 7 = A(s-5) + B(s+2)$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

4. Simplify.

$$\begin{aligned} s + 7 &= A(s-5) + B(s+2) \\ &= As - 5A + Bs + 2B \end{aligned}$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

4. Simplify.

$$
\begin{aligned}
s+7 &= A(s-5)+B(s+2) \\
&= As-5A+Bs+2B \\
&= (A+B)s+(2B-5A)
\end{aligned}
$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

4. Simplify.

$$\begin{aligned} s+7 &= A(s-5) + B(s+2) \\ &= As - 5A + Bs + 2B \\ &= (A+B)s + (2B-5A) \end{aligned}$$

5. Solve the system.

$$\begin{aligned} A + B &= 1 \\ 2B - 5A &= 7 \end{aligned}$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

6. Therefore

$$A = -\frac{5}{7} \quad , \quad B = \frac{12}{7}$$

$$H(s) = \frac{-\frac{5}{7}}{s+2} + \frac{\frac{12}{7}}{s-5}$$

Find the inverse Laplace transform of $H(s) = \frac{s+7}{s^2-3s-10}$.

Solution.

6. Therefore

$$A = -\frac{5}{7} \quad , \quad B = \frac{12}{7}$$

$$H(s) = \frac{-\frac{5}{7}}{s+2} + \frac{\frac{12}{7}}{s-5}$$

7. Take the inverse.

$$h(t) = -\frac{5}{7}\mathrm{e}^{-2t} + \frac{12}{7}\mathrm{e}^{5t}$$

Inverse Laplace Transforms

Factor in Denominator	Term in Partial Fraction Decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

1. Note that:

$$s^3 = (s-0)^3$$

Inverse Laplace Transforms

Example 7.9

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

1. Note that:

$$s^3 = (s-0)^3$$

2. Split the term using *partial fractions*.

$$G(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4s+5}$$

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

1. Note that:

$$s^3 = (s-0)^3$$

2. Split the term using *partial fractions*.

$$G(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4s+5}$$

3. If we add both fractions, the following must be true:

$$25 = As^2(s^2+4s+5) + Bs(s^2+4s+5) + C(s^2+4s+5) + (Ds+E)s^3$$

Inverse Laplace Transforms
Example 7.9

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

Inverse Laplace Transforms
Example 7.9

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

4. Distributing and collecting like terms:

$$\begin{aligned} 25 = {} & (A+D)s^4 + (4A+B+E)s^3 \\ & + (5A+4B+C)s^2 + (5B+4C)s + 5C \end{aligned}$$

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

4. Distributing and collecting like terms:

$$\begin{aligned} 25 = {} & (A+D)s^4 + (4A+B+E)s^3 \\ & + (5A+4B+C)s^2 + (5B+4C)s + 5C \end{aligned}$$

5. Solve the system.

$$\begin{aligned} A + D &= 0 \\ 4A + B + E &= 0 \\ 5A + 4B + C &= 0 \\ 5B + 4C &= 0 \\ 5C &= 25 \end{aligned}$$

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

6. Therefore

$$A = \frac{11}{5} \quad , \quad B = -4 \quad , \quad C = 5 \quad , \quad D = -\frac{11}{5} \quad , \quad E = -\frac{24}{5}$$

$$G(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4s+5}$$

$$= \frac{1}{5}\left(\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11s+24}{s^2+4s+5}\right)$$

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

7. Make numerators look like the formulas.

$$G(s) = \frac{1}{5}\left(\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11s+24}{s^2+4s+5}\right)$$

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

7. Make numerators look like the formulas.

$$\begin{aligned} G(s) &= \frac{1}{5}\left(\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11s+24}{s^2+4s+5}\right) \\ &= \frac{1}{5}\left(\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11(s+2-2)+24}{(s+2)^2+1}\right) \end{aligned}$$

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

7. Make numerators look like the formulas.

$$
\begin{aligned}
G(s) &= \frac{1}{5}\Big(\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11s+24}{s^2+4s+5}\Big) \\
&= \frac{1}{5}\Big(\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11(s+2-2)+24}{(s+2)^2+1}\Big) \\
&= \frac{1}{5}\Big(\frac{11}{s} - \frac{20}{s^2} + \frac{25\frac{2!}{2!}}{s^3} - \frac{11(s+2)}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}\Big)
\end{aligned}
$$

Find the inverse Laplace transform of $G(s) = \frac{25}{s^3(s^2+4s+5)}$.

Solution.

7. Make numerators look like the formulas.

$$\begin{aligned} G(s) &= \frac{1}{5}\left(\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11s+24}{s^2+4s+5}\right) \\ &= \frac{1}{5}\left(\frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11(s+2-2)+24}{(s+2)^2+1}\right) \\ &= \frac{1}{5}\left(\frac{11}{s} - \frac{20}{s^2} + \frac{25\frac{2!}{2!}}{s^3} - \frac{11(s+2)}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}\right) \end{aligned}$$

8. Take the inverse.

$$g(t) = \frac{1}{5}\left(11 - 20t + \frac{25}{2}t^2 - 11\mathrm{e}^{-2t}\cos(t) - 2\mathrm{e}^{-2t}\sin(t)\right)$$

Application: Solving IVPs

Theorem

Suppose that $f, f', f'', \cdots, f^{(n-1)}$ are all continuous functions and $f^{(n)}$ is a piecewise continuous function. Then,

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Application: Solving IVPs
Example 7.10

The Laplace transform of the first two derivatives:

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

Application: Solving IVPs
Example 7.11

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

1. Take the Laplace transform of each term.

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

1. Take the Laplace transform of each term.

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$
$$[s^2Y(s) - sy(0) - y'(0)] - 10[sY(s) - y(0)] + 9[Y(s)] = \frac{5}{s^2}$$

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

1. Take the Laplace transform of each term.

$$\begin{aligned} \mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} &= \mathcal{L}\{5t\} \\ [s^2Y(s) - sy(0) - y'(0)] - 10[sY(s) - y(0)] + 9[Y(s)] &= \frac{5}{s^2} \end{aligned}$$

2. Plug in the initial conditions.

$$s^2Y(s) - s(-1) - 2 - 10sY(s) + 10(-1) + 9Y(s) = \frac{5}{s^2}$$

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

1. Take the Laplace transform of each term.

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$
$$[s^2Y(s) - sy(0) - y'(0)] - 10[sY(s) - y(0)] + 9[Y(s)] = \frac{5}{s^2}$$

2. Plug in the initial conditions.

$$s^2Y(s) - s(-1) - 2 - 10sY(s) + 10(-1) + 9Y(s) = \frac{5}{s^2}$$
$$(s^2 - 10s + 9)Y(s) + s - 12 = \frac{5}{s^2}$$

Application: Solving IVPs

Example 7.11

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

3. Solve for $Y(s)$.

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

3. Solve for $Y(s)$.

$$
\begin{aligned}
Y(s) &= \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)} \\
&= \frac{5+12s^2-s^3}{s^2(s-9)(s-1)}
\end{aligned}
$$

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

4. Compute the partial fractions.

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

4. Compute the partial fractions.

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$\begin{aligned} 5 + 12s^2 - s^3 = {} & As(s-9)(s-1) + B(s-9)(s-1) \\ & + Cs^2(s-1) + Ds^2(s-9) \end{aligned}$$

Application: Solving IVPs
Example 7.11

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

4. Compute the partial fractions.

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$\begin{aligned} 5 + 12s^2 - s^3 &= As(s-9)(s-1) + B(s-9)(s-1) \\ &\quad + Cs^2(s-1) + Ds^2(s-9) \\ &= (A + C + D)s^3 + (B - 10A - C - 9D)s^2 \\ &\quad + (9A - 10B)s + 9B \end{aligned}$$

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

5. Solve the system.

$$\begin{aligned} A + C + D &= -1 \\ B - 10A - C - 9D &= 12 \\ 9A - 10B &= 0 \\ 9B &= 5 \end{aligned}$$

Application: Solving IVPs

Example 7.11

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

6. Therefore

$$A = \frac{50}{81} \quad , \quad B = \frac{5}{9} \quad , \quad C = \frac{31}{81} \quad , \quad D = -2$$

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

Solve the following IVP.

$y'' - 10y' + 9y = 5t$ with $y(0) = -1$ and $y'(0) = 2$.

Solution.

6. Therefore

$$A = \frac{50}{81} \quad , \quad B = \frac{5}{9} \quad , \quad C = \frac{31}{81} \quad , \quad D = -2$$

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

7. Get the inverse transform.

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}\mathrm{e}^{9t} - 2\mathrm{e}^{t}$$

References

Dawkins, P.
Paul's Online Math Notes.
http://tutorial.math.lamar.edu

End of Lecture 7