Transforms

Lecture 8 - Fourier Series and Transforms

CS 130 Lecture Slides
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Outline

- Fourier Series
- Fourier Transforms

Recall:

Taylor Series

The **Taylor series** for $f(x)$ about $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Fourier Series

Fourier Series

Given a function $f(t)$ (with certain properties), its **Fourier series** representation on the interval $-L \leq x \leq L$ is

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Fourier Series (cont.)

where (for $m = 1, 2, \ldots$)

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$A_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx$$

and

$$B_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

Find the Fourier series for $f(x) = L - x$ on $-L \leq x \leq L$.

Solution.

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Solution.

1. Solve for the coefficients.

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$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx$$
$$= \frac{1}{2L} \int_{-L}^{L} (L - x)dx$$

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1. Solve for the coefficients.

$$\begin{aligned}
A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2L}\int_{-L}^{L} (L - x)dx \\
&= \frac{1}{2L}\left[(Lx)\Big|_{-L}^{L} - \left(\frac{x^2}{2}\right)\Big|_{-L}^{L}\right]
\end{aligned}$$

Find the Fourier series for $f(x) = L - x$ on $-L \leq x \leq L$.

Solution.

1. Solve for the coefficients.

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\ &= \frac{1}{2L} \int_{-L}^{L} (L - x) dx \\ &= \frac{1}{2L} \left[(Lx)\Big|_{-L}^{L} - \left(\frac{x^2}{2}\right)\Big|_{-L}^{L} \right] \\ &= \frac{1}{2L} \left[2L^2 - 0 \right] \end{aligned}$$

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Solution.

1. Solve for the coefficients.

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2L}\int_{-L}^{L}(L-x)dx \\ &= \frac{1}{2L}\left[(Lx)\Big|_{-L}^{L} - \left(\frac{x^2}{2}\right)\Big|_{-L}^{L}\right] \\ &= \frac{1}{2L}\left[2L^2 - 0\right] \\ &= L \end{aligned}$$

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Integrate by parts $\int udv = uv - \int vdu$.

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$$u = (L - x) \quad , \quad dv = \cos\left(\frac{n\pi x}{L}\right)dx$$

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1. Solve for the coefficients.

$$\text{IBP} = \frac{L(L-x)}{n\pi} \sin\left(\frac{n\pi x}{L}\right) + \int \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) dx$$

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Therefore

$$A_n = \frac{1}{L}\left[\frac{L(L-x)}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L}$$

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$$= \frac{1}{L}\left[-\frac{L^2}{n^2\pi^2}\cos(n\pi) - \left[-\frac{2L^2}{n\pi}\sin(n\pi) - \frac{L^2}{n^2\pi^2}\cos(n\pi)\right]\right]$$

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$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

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2. Plug coefficients in Fourier series.

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

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$$= L + \sum_{n=1}^{\infty} \frac{2L(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

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$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

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$$A_n = \frac{1}{L}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L}$$

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1. Solve for the coefficients.

$$\begin{aligned}\text{IBP} &= \frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \int \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\end{aligned}$$

Therefore

$$\begin{aligned}A_n &= \frac{1}{L}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L} \\ &= \frac{1}{L}\left[\frac{L^2}{n^2\pi^2}\cos(n\pi) - \frac{L^2}{n^2\pi^2}\cos(n\pi)\right] \\ &= 0\end{aligned}$$

Find the Fourier series for $f(x) = x$ on $-L \leq x \leq L$.

Solution.

Find the Fourier series for $f(x) = x$ on $-L \leq x \leq L$.

Solution.

1. Solve for the coefficients.

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

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$$= -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)$$

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$$= -\frac{2L(-1)^n}{n\pi} = \frac{2L(-1)^{n+1}}{n\pi}$$

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2. Plug coefficients in Fourier series.

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

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Solution.

Find the Fourier series for $f(x) = x^2$ on $-L \leq x \leq L$.

Solution.

1. Solve for the coefficients.

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

Find the Fourier series for $f(x) = x^2$ on $-L \leq x \leq L$.

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1. Solve for the coefficients.

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\ &= \frac{1}{2L} \int_{-L}^{L} x^2 dx \\ &= \frac{1}{2L} \left(\frac{x^3}{3} \right) \Bigg|_{-L}^{L} \end{aligned}$$

Find the Fourier series for $f(x) = x^2$ on $-L \le x \le L$.

Solution.

1. Solve for the coefficients.

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\begin{aligned}
A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2L}\int_{-L}^{L} x^2 dx \\
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&= \frac{1}{2L}\left(\frac{L^3}{3} - \frac{(-L)^3}{3}\right)
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&= \frac{Ln^2\pi^2x^2 + 2L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^2x}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)
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Therefore

$$A_n = \frac{1}{L}\left[\frac{Ln^2\pi^2x^2 + 2L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^2x}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L}$$

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&= \frac{Ln^2\pi^2x^2 + 2L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^2x}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)
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&= \frac{1}{L}\left[\frac{2L^3}{n^2\pi^2}\cos(n\pi) + \frac{2L^3}{n^2\pi^2}\cos(n\pi)\right] = \frac{4L^2(-1)^n}{n^2\pi^2}
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&= -\frac{Ln\pi x^2 - 2L^2 x}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right) - \frac{2L^3}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)
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Therefore

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Find the Fourier series for $f(x) = x^2$ on $-L \leq x \leq L$.

Solution.

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2. Plug coefficients in Fourier series.

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

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&= \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right)
\end{aligned}
$$

Fourier Transforms

Fourier Transform

The Fourier transform $\mathcal{F}$ of a function $f(t)$ is defined as

$$\mathcal{F}\{f(t)\} = F(k) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-2\pi ikt}dt$$

References

Dawkins, P.
Paul's Online Math Notes.
http://tutorial.math.lamar.edu

End of Lecture 8