Mathematical Implementation of Black Scholes Equation

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Introduction

Options, Puts and Calls

An **option** is a financial instrument which provides its owner the **right (NOT THE OBLIGATION) to buy or sell** an underlying asset at an agreed price. To own an option contract, you must pay a premium. There are two types of options:

- **Call option:** the buyer of a call option has the right to buy an asset at an agreed price.
- **Put option:** the buyer of a put option has the right to sell an asset at an agreed price.

From this definition we can differentiate two possible positions (or strategies):

- **Long option** (you are the buyer of an option): If you are long you purchase the right to buy or sell (**but no obligation**) an underlying asset and you pay a premium to the seller of an option.
- **Short option** (you are the seller of an option): If you are long you purchase the right to buy or sell (**but no obligation**) an underlying asset and you pay a premium to the seller of an option.

All options have an **expiration date** and the **time until maturity** is the remaining time an option has left until expiration.

To put all of these definitions into context let's take a look at the next example:

John is an expert in computers and thinks the price of an **ABC** model keyboard, which current price is **$10,** will sky-rocket within one year. He already has a keyboard but would like to benefit from this rise, so he offers his friend **Mike** to buy ABC in 1 year for **$20.** Mike would be very happy to sell for that amount now but the future is uncertain, so he is reluctant to agree. John really believes in his prediction, so offers Mike $5 for the right to buy the mouse and Mike agrees.

This example is an example of a **call option** with:

- Strike price: $20
- Expiration Date: 1 year
- Premium: $5

But a question arises. How much money would be the right premium for this contract?

There are different factors which affect the premium in stock markets:

- **Volatility** σ: a measure of uncertainty of the stock. The higher σ the higher chances of a change in the price of the stock and therefore of profits/losses. So the premium goes up.
- **Change in Price of the underlying asset:** If the price of the asset goes up, so does the premium for a call. The opposite for a put.

- **Time to maturity**: the shorter the lifespan of an option the lower the price as there is less chances of considerable variation in the price of the underlying asset.
- **Strike price:** The higher the strike price the less the price of a call option as there is less chances of it becoming profitable. The opposite for a put option.
- **Dividends and interest-rate:** the first one we will not consider in this paper, the later has a positive correlation with option pricing.

Black-Sholes Model

Putting together all this variable gave rise to the Black-Scholes model, also known as the Black-Scholes-Merton (BSM) model. It is one of the most important concepts in modern financial theory. This mathematical equation estimates the theoretical value of derivatives based on other investment instruments, considering the impact of time and other risk factors.

If you are short, you sell the right to buy or sell an underlying asset. Therefore, if the owner of the option decides to exercise their right, you have the obligation to either buy or sell (depending on if it's a call or a put) the underlying asset.

Some interesting facts about the BSM:

- The Black-Scholes equation is a partial differential equation
- It is widely used to price options contracts.
- It requires five input variables: the strike price of an option, the current stock price, the time to expiration, the risk-free rate, and the volatility.
- Though usually accurate, the BSM makes certain assumptions. Therefore, the prices might not always match the real-world results.
- The standard BSM model is only used to price European options.

The Black-Scholes model makes certain assumptions:

- No dividends are paid out during the life of the option.
- Markets are random (i.e., market movements cannot be predicted).
- There are no transaction costs in buying the option.
- The risk-free rate and volatility of the underlying asset are known and constant.
- The returns on the underlying asset are log-normally distributed.
- The option is European and can only be exercised at expiration.

Black-Scholes Equation:

The Black-Scholes equation is the partial differential equation that governs the price evolution of European stock options in financial markets operating according to the dynamics of the BSM.
The equation is:

$$\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} + rS\frac{\delta V}{\delta S} - rV = 0$$

For this project we will only focus on call options, but results for put options can easily be obtained by slightly manipulating the boundary conditions.

Where:

- V: Premium or price of the option
- t: Time
- σ: Volatility
- S: Underlying asset price
- r: Interest-rate risk free (ex: euribor)
- τ: Time to expiration
- T: Date of expiration
- $\tau = T - t$

To solve this formula, we came across different methods to solve partial differential equations but, in this project, we will focus on the explicit finite-difference method. To create an explicit method, we must make a change of variable on t. A we see above $\tau = \mathrm{T} - t$ and hence $\frac{\partial V}{\partial \tau} = -\frac{\partial V}{\partial t}$. Thus,

$$\mathbf{(0)} \qquad \boldsymbol{\frac{\delta V}{\delta t} = \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} + rS\frac{\delta V}{\delta S} - rV}$$

With the initial conditions (for a Call option)

$$(0.1) \qquad C(0,t) = 0$$

$$(0.2) \qquad C(S,\mathrm{T}) = \max\{S - K, 0\}$$

$$(0.3) \qquad C(S,t) = S - Ke^{-r(\mathrm{T}-t)} = S - Ke^{-r\tau}$$

Recall the calculus definition of a derivative:

$$\frac{dy(t)}{dt} = lim_{h\to\infty}\frac{y(t+h)-y(t)}{h} \quad \text{and} \quad \frac{dy(t)}{dt} = lim_{h\to\infty}\frac{y(t+h)-y(t-h)}{2h}$$

Which we can approximate as

$$(1)\,\frac{dy(t)}{dt} \approx \frac{y(t+h)-y(t)}{\Delta t} \quad \text{and} \quad (2)\,\frac{dy(t)}{dt} \approx \frac{y(t+h)-y(t-h)}{2\Delta t}$$

Respectively.

Also, for the second derivative we get

$$(3)\ \frac{d^2y(t)}{dt^2} \approx \frac{y(t+h) - 2y(t) + y(t-h)}{\Delta t^2}$$

By substituting into (0), (1) into the first partial, (3) to the second partial and (2) into the third partial with the corresponding variables we get:

$$(4)\quad \frac{V(S,t+\Delta t)-V(t)}{\Delta t}=\frac{1}{2}\sigma^2 S^2[\frac{V(S+\Delta S,t)-2V(S,t)+V(S-\Delta S,t)}{\Delta S^2}]+rS\frac{V(S+\Delta S,t)-V(S-\Delta S,t)}{2\Delta S}-rV$$

To put it in a more computational style

$$(5)\quad \frac{V_{n,j+1}-V_{n,j}}{\Delta t}=\frac{1}{2}\sigma^2 S_n^2[\frac{V_{n+1,j}-2V_{n,j}+V_{n-1,j}}{\Delta S^2}]+rS_n\frac{V_{n+1,j}-V_{n-1,j}}{2\Delta S}-rV_{n,j}$$

From the initial condition (0.1) we know we will start at $S_0 = 0$ hence:

$$(6)\quad S_{n,j}=n\Delta S \quad \text{as} \quad S_0=0$$

Which simplifies (5) into

$$(7)\quad \frac{V_{n,j+1}-V_{n,j}}{\Delta t}=\frac{1}{2}\sigma^2 n^2[V_{n+1,j}-2V_{n,j}+V_{n-1,j}]+rn[V_{n+1,j}-V_{n-1,j}]-rV_{n,j}$$

And,

$$(8)\quad V_{n,j+1}=\frac{1}{2}\sigma^2 n^2[V_{n+1,j}-2V_{n,j}+V_{n-1,j}]\Delta t+rn[V_{n+1,j}-V_{n-1,j}]\Delta \mathrm{t}-rV_{n,j}\Delta t+V_{n,j}$$

Finally, by combining coefficients we get

$$(9)\quad \boldsymbol{V_{n,j+1}}=\frac{\Delta t}{2}(\sigma^2 n^2-rn)\,\boldsymbol{V_{n-1,j}}+[1-(\sigma^2 n^2+r)\Delta t]\boldsymbol{V_{n,j}}+\frac{\Delta t}{2}(\sigma^2 n^2+rn)\boldsymbol{V_{n+1,j}}$$

MATLAB Algorithm

```
% Algorithm to calculate the price of a European option

% Parameters
r       = 0.2;        % Interest rate
sigma   = 0.25;       % Volatility of the underlying
Nt      = 1600;       % Number of time steps
Ns      = 160;        % Number of asset price steps
Smax    = 20;         % Maximum asset price considered
Smin    = 0;          % Minimum asset price considered
T       = 1;          % Maturation (expiry) of the contract
E       = 10;         % Exercise price of the underlying

% Stepper variables
dt      = (T/Nt);                       % Time step
ds      = (Smax-Smin)/Ns;               % Price step

% Initializing the matrix of the option value
V(1:Ns+1, 1:Nt+1) = 0.0;

% Create an array with the input values of the price and the time to
expiration
S               = Smin+(0:Ns)*ds;
tau             = (0:Nt)*dt;

% Initial conditions prescribed by the European Call payoff at expiry:
V(S,tau=0) =max (S-E,0)
V(1:Ns+1,1)     = max (S-E, 0) ;

% Boundary conditions prescribed by the European Call:
V(1,1: Nt+1)    = 0;    % V(0, t) =0
V(Ns+1,1:Nt+1)  = Smax-E*exp(-r*tau);
% V(S, t) = S-Exp[-r (T-t)] as S -> infininty

% Implementing the explicit algorithm
for j = 1:Nt    % Time loop
    for n = 2:Ns    % Asset loop
        V(n,j+1) = 0.5*dt*(sigma*sigma*n*n-r*n)*V(n-1,j)+(1-
dt*(sigma*sigma*n*n+r))*V(n,j)+0.5*dt*(sigma*sigma*n*n+r*n)*V(n+1,j);
    end
end

% Figure of the values of the option, V(S,tau), as a function of S at
3 different times: tau=0(t=T), tau=T/2(t=T/2) and tau=T(t=0).
figure (1)
```

```
plot   (S, V(:,1), 'r-', S, V(:,round(Nt/2)), 'g-', S, V(:,Nt+1), 'b-
');
xlabel ('S');
ylabel ('V(S, tau)');
title  ('European Call Option within the Explicit Method');

% 3D plots of the Value of the option, V(S, tau)
figure (2)
mesh   (tau, S, V);
xlabel ('tau');
ylabel ('S');
title  ('European Call Option value, V(S,tau), within the Explicit
Method');
```

References

https://www.investopedia.com/terms/b/blackscholes.asp
https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_equation