Stochastic Processes: Itô Integration

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Contents

1 Introduction

For this research paper, we focused on having a solid foundation of knowledge on Stochastic Processes which helped us later on in understanding more complex concepts of Ito integration and its applications. Some financial background, especially on stock pricing and options, was required as well in order to be able to work with the concepts related to this field. In this paper, we will introduce the basis and define stochastic processes. We will classify these processes into discrete-time and continuous-time and also provide examples for each one. Here the concepts of the Random Walk, Markov Chain, Brownian motion and Ito Integration will be introduced and will be used to explain the modeling of stock prices and to derive The Black and Sholes Equation. Toward the end of the paper, we will provide computer-programmed simulations and discuss how our work and research could be even furthered into more complex models.
Toward the end of the paper, we will discuss future research and work, and computer-programmed simulations.

It is very important to understand the following concept:

Definition 1.1 (Mathematical Modelling)**.** Mathematical modeling refers to the process of creating a mathematical representation of a real-world scenario to make a prediction or provide insight. There is a distinction between applying a formula and the actual creation of a mathematical relationship.

2 Stochastic Processes

Mathematically,a stochastic process is defined as a collection of random variables, indexed by some set, representing time. Random variables by themselves are functions on a sample space $X(\omega)$, $\omega \in \Omega$, without regard to how these might depend on parameters. However, when we use these random variables in situations in which probability evolves with time, then we are talking about a stochastic process.

More formally, we can say that a stochastic process is defined as a collection of random variables defined on a common probability space (Ω, A, P), where Ω is a sample space, A is a σ-algebra, and P is a probability measure. All the random variables are indexed by some set T and take values in the same mathematical space S. Basically, a Stochastic process is a collection of S-valued random variables which we can write as following:

$$\{X(t) : t \in T\}$$

To conclude, a random variable is a function defined on a sample space, which assigns a number to an event $X(\omega) \in R$. A stochastic process on the other hand, is a collection of these random variables depending on a real parameter, which is in most cases the time. Another way to write a stochastic process is: $\{X(t, \omega) : t \in T\}$, showing that we have a function of the two variables $t \in T$ and $\omega \in \Omega$.

There are two main types of SP:

- Discrete-time (Ex. $X_0, X_1, X_2, ...$)
- Continuous-time (Ex. $\{X\}_t, t \geq 0, t \in \mathbb{R}$)

Definition 2.1 (Index Set)**.** The stochastic process's index set or parameter set is denoted by the letter T. An element in the collection is exclusively connected with each probability and random process. The set used to index the random variables is known as the index set. This set will typically be described in terms of time. Accordingly, a stochastic process is said to be in discrete time if its index set has a finite or countable number of elements, or as being in continuous time if the index set is an interval of the real line.

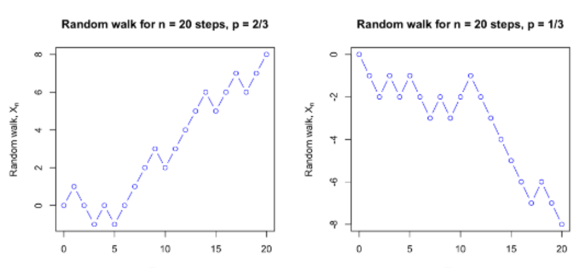
2.1 2.1 Discrete-Time Stochastic Process

There are many examples of discrete-time stochastic processes but for the purpose of this paper, we will focus only on Simple Random Walk, Markov Chain,

Poisson Process, and Martingale process.
We can describe a Stochastic Process by being discrete-time or continuous time. Discrete-time Stochastic Processes model a stochastic event in which the time index set is discrete. In other words, a discrete-time stochastic process models a series of random variables in time, and this time steps are discrete or countable. An example of such a process would be the Simple Random walk, which is a random process and one of the simplest stochastic processes, that describes a path that consists of a succession of random steps on some mathematical space. The simple random walk is a simple but very useful model for lots of processes, like stock prices, sizes of populations, or positions of gas particles.

Definition 2.2 (Simple Random Walk)**.** : In mathematics, a simple random walk is a mathematical model used to describe a path or trajectory that consists of a sequence of random steps in a discrete space.

Example: We are going to consider the following example of a simple random walk, using the integers $\mathbb{Z}$ as our time step. We begin our path at 0 and on each time step we move up or down with probability p and $q = 1 - p$ respectively. In the case of $p = q = \frac{1}{2}$, we call this the simple symmetric random walk. In the following two examples we look at random walks with $n = 20$ time steps and probabilities $p = 23$ and $p = 13$ respectively.



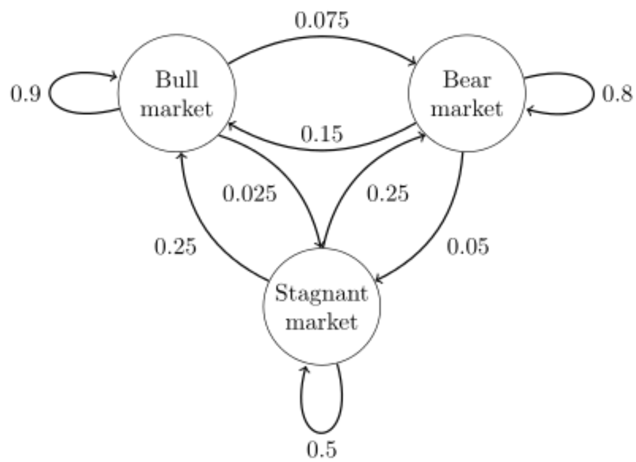
We can write these as a stochastic process (X_n) with discrete time $n = \{0, 1, 2, ...\}$ and discrete space $S = \mathbb{Z}$, where $x_0 = 0$, and for n_0: $X_{n+1} = X_{n+1}$

with probability p or $X_{n+1} = X_{n-1}$ with probability q .

We can see from the definition that X_{n+1}, which represents the future, depends on X_n (the present) but given that ${}_X n$ does not depend on $X_n - 1, X_{n-2}, ..., X_1, X_0$ (the past). Therefore, we can say that the Markov property holds and the simple random walk is a **Discrete-Time Markov Process** or **Markov Chain**.

Definition 2.3 (Markov Process)**.** Is a sequence of possibly dependent random variables $(x_1, x_2, x_3, ...)$ identified by increasing values of a parameter, commonly time—with the property that any prediction of the next value of the sequence (x_n), knowing the preceding states $(x_1, x_2, x_3, ..., x_n)$, may be based on the last state (x_{n-1}) alone. Meaning that, a Markov Process is a stochastic process whose effects of the past on the future are summarized only by the current state.

More formally, we could state that a discrete-time stochastic process $(x_0, x_1, x_2, ...)$ is a Markov Chain if: $P(x_{t+1} = S|x_0, x_1, ..., x_t) = P(x_{t+1} = S|x_t)$, $\forall t \geq 0$ and $\forall S$
A Markov Chain is a stochastic process, with the only difference being that a Markov Chain is "memory-less". That is, (the probability of) future actions are not dependent upon the steps that led up to the present state. This is called the Markov property.



Example: To demonstrate the Markov Chain and the Markov property we are going to look at an example from the stock market. The picture on the left displays the state transitions between the bull, bear and stagnant markets that the stock market might exhibit during a given week. Based on the figure a bull week is followed by another bull market 90% of the time, a bear week 7.5% of the time and a stagnant week the leftover 2.5% of the time. We can label the state space in the following transition matrix for this case:

$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

This distribution can be written as$x^{(n+1)} = x^{(n)} * P$. This means that if at time n the system is in state $x^{(n)}$, then three time periods later, at time $n+3$, the distribution is:

$$x^{(n+3)} = x^{(n+2)} * P = (x^{(n+1)} * P) * P = (x^{(n)} * P) * P^2 = x^{(n)} * P^3$$

In particular if at time n the system is in the bear state, then at time $n + 3$

the distribution

$$x^{(n+3)} = [0\ 1\ 0] \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^3$$

$$= [0\ 1\ 0] \begin{bmatrix} 0.7745 & 0.17875 & 0.04675 \\ 0.3575 & 0.56825 & 0.07425 \\ 0.4675 & 0.37125 & 0.16125 \end{bmatrix} = \begin{bmatrix} 0.0.3575 & 0.56825 & 0.07425 \end{bmatrix}$$

is:

Using the transition matrix we can calculate, for example, the long-term fraction of weeks during which the market is stagnant, or the average number of weeks it will take to go from a stagnant to a bull market. Using the transition probabilities, the steady-state probabilities indicate that 62.5% of weeks will be in a bull market, 31.25% of weeks will be in a bear market and 6.25% of weeks will be stagnant, since

$$\lim_{N\to\infty} P^N = \begin{bmatrix} 0.625 & 0.625 & 0.625 \\ 0.625 & 0.625 & 0.625 \\ 0.625 & 0.625 & 0.625 \end{bmatrix}$$

this finite-state example can be used to represent a Markov Chain, where the probability of reaching a new state only depends on the current state.

Another very important stochastic process due to their mathematical properties and theoretical foundations, is the **Martingale**. It is vastly used in probability theory, financial mathematics, statistical inference and many more. It is also known as a stochastic process that is fair game.

Definition 2.4 (Martingale)**.** A martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time, the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values. Even though we can have both discrete-time and continuous-time martingales, first we will look into the discrete-time definition. A basic definition of a discrete-time martingale is a discrete-time stochastic process (i.e., a sequence of random variables), $x_1, x_2, x_3...$ that satisfies for any time n,

$\mathbb{E}(|X_n|) < \infty$

$\mathbb{E}(X_{n+1}|X_1, ..., X_n) = X_n.$

That is, the conditional expected value of the next observation, given all the past observations, is equal to the most recent observation. We can also look at martingales as sequences. More generally, a sequence $Y_1, Y_2, Y3, ...$ is said to be a martingale with respect to another sequence $x_1, x_2, x_3...$ if for all n,

Similarly, a continuous-time martingale with respect to the stochastic process X_t is a stochastic process Y_t such that for all t,

$\mathbb{E}(|Y_t|) < \infty$

$\mathbb{E}(Y_t|\{X_\tau, \tau \leq s\}) = Y_s \; \forall s \leq t.$

2.2 Continuous-Time Stochastic Process

In probability theory, a continuous stochastic process is a type of stochastic process that is called this way to describe its continuous time or index parameter. Continuous-time processes are way more complex than discrete-time but are in most cases a better and more accurate representation of the real world. More generally, generally, it is a mathematical model that describes the evolution of a system or phenomenon over a continuous interval of time.

It is a collection of random variables indexed by real numbers or a subset of the real line. The continuous-time processes that we will discuss in this paper are the Poisson process and Brownian motion or Wiener process.

Definition 2.5 (Poisson Process)**.** A Poisson Process is a continuous-time stochastic process which is widely used in stochastic processes for modeling the times at which arrivals enter a system. Despite having very simple path-wise properties, this process is very crucial in the study of stochastic calculus and along with Brownian motion, they form the basis for more complicated processes in various fields. We say that a random variable has the Poisson distribution with parameter λ and is denoted by

$$N \approx Po(\lambda)$$

if it takes values in the set of nonnegative integers and

$$\mathbb{P}(N = n)\frac{\lambda^n}{n!}e^{-\lambda}$$

A Poisson process can be used to simulate the arrival of transactions or orders in the setting of the stock market. It is predicated on the idea that trades will consistently arrive at a constant pace across time, independently and at random. A crucial characteristic of a Poisson process is the exponential distribution of the time intervals between trades.
Analysts and academics can learn more about the timing and frequency of trades by using the Poisson process on the stock market. The development of trading techniques, examining liquidity, gauging the effectiveness of the market, and other parts of market analysis can all benefit from this information.
It's crucial to keep in mind that the Poisson process is a streamlined model that relies on independence and a constant arrival rate, which might not accurately reflect the stock market's characteristics. The Poisson process is a simple approximation rather than a complete portrayal of market behavior since real-world financial markets are influenced by complicated factors and interactions among market participants. For this reason, other models and

stochastic processes can give a better representation of the stock market, as it is the simple random walk.
The other continuous-time stochastic process that we are looking at, which can easily be argued to be the most important stochastic model, is the **Brownian Motion** or **Wiener process**. It is used in various fields like science, engineering and more, but we will focus on its applications in finance. Many different parts of the financial market, such as stock markets, foreign exchange markets, the Black-Scholes Model and many more, are all assumed to follow Brownian motion. This means that in all of these markets we have assets that are evolving constantly over continuous intervals of time, while the state of these assets is being altered by random amounts.

Definition 2.6 (Brownian Motion)**.** A standard Brownian motion (Weiner process) is a stochastic process $\{B_t\}$, $t \geq 0^+$ with the following properties:

1. (1) $B_0 = 0$,
2. the function $t \to B_t$ is continuous in t,
3. the process $\{B_t\}$, $t \geq 0$ has stationary, independent increments
4. the increment $B_{t+s} - B_s \sim N(0, t)$

Definition 2.7 (Stationary increments)**.** for any $0 < s$, $t < \infty$, the distribution of the increment $B_{t+s} - B_s$ has the same distribution as $B_t - B_0 = B_t$. The length of the time gap, not the precise starting point, alone determines the distribution of the increment between any two points in time.

Definition 2.8 (Independent increments)**.** For every choice of non-negative real numbers $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq ... \leq s_n < t_n < \infty$, the increment random variables $B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2}, ..., B_{t_n} - B_{s_n}$ are jointly independent. The increments in non-overlapping time intervals do not depend on one another statistically. This means that the process' behavior in one interval does not have an impact on its behavior in a subsequent interval. In other words, a process's future conduct is independent of its past behavior.

3 Modelling Stock Prices

The valuation of publicly traded companies is represented by stock prices, which also act as a significant indicator of market sentiment and economic health. Stock prices are inherently unpredictable due to their dynamic nature, which is influenced by a variety of factors such as investor behavior, macroeconomic trends, and company-specific news. As a result, accurately modeling stock price movements and comprehending them have long been of great theoretical and practical interest.

The complexity and non-linearity of stock prices make modeling them difficult. The market noise that characterizes random fluctuations in stock prices can obscure underlying trends and patterns. Additionally, the abrupt changes and extreme events that can affect stock prices make them naturally volatile. Researchers have created mathematical models to capture the dynamics of stock prices and offer insights for risk management, derivative pricing, and investment decision-making due to the inherent randomness and complexity of stock prices.

For this paper we will show how this can be used to explain stock price fluctuations. It is predicated on the idea that a stock's future price changes will be random and unpredictable. The Simple Random walk model suggests:

1. Price Variations: A stock's price may fluctuate erratically in either direction, with each variation being treated independently of the others. In other words, price changes in the past have no bearing on how prices will move in the future.

2. No Predictable Trends: According to the random walk model, stock price changes do not exhibit any obvious patterns or trends. It means that it is doubtful that efforts to anticipate future prices purely using historical price data will be consistently successful.

3. Efficiency of Markets: The random walk model makes the assumption that markets are efficient, which means that stock prices reflect all avail-

able information promptly and accurately. As a result, it means that continuously outperforming the market through price prediction is challenging.

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We can take a streamlined model where the price of a stock fluctuates at discrete time intervals so that it applies the idea of a simple random walk to stock pricing. The price has an equal chance of going up or down by a specific amount in each interval.
Consider we have a stock whose price at time t is denoted as $P(t)$. In a simple random walk model, the price change in each time interval is represented by a random variable, denoted as ΔP. This random variable can take on two values: $+\Delta P$ (representing an increase in price) or $-\Delta P$ (representing a decrease in price), each with equal probability.

The price of the stock at time $t+1$, $P(t+1)$, is given by:

$$P(t+1) = P(t) + \Delta P.$$

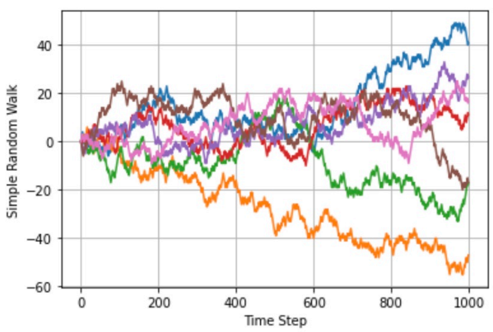
We can go on with this procedure for each time step in order to imitate the movement of the stock price throughout a range of time periods. We generate a sequence of stock values that simulate a straightforward random walk by repeating this technique. We implemented this in python:

Simple Random Walk

```python
# 7 Random Walks
output = np.zeros([1000,8])
x = np.arange(1000)
output[:,0] = x
output

# Loop over every time step (1000) and adds the incrememt provided by random.choices() at every step
for i in range(1,1000):
    output[i,1:] = output[i-1,1:] + random.choices([1,-1], weights = [1, 1], k = 7)


# Plots the 7 sample paths
plt.plot(output[:,0], output[:,1])
plt.plot(output[:,0], output[:,2])
plt.plot(output[:,0], output[:,3])
plt.plot(output[:,0], output[:,4])
plt.plot(output[:,0], output[:,5])
plt.plot(output[:,0], output[:,6])
plt.plot(output[:,0], output[:,7])
plt.xlabel("Time Step")
plt.ylabel("Cummulative Sum")
plt.ylabel("Simple Random Walk")
plt.grid(True)
plt.show()
```



By introducing continuous time and taking the limit as the time interval gets closer to zero, this concept can be expanded to Brownian motion. We take stock price changes over incredibly brief time periods into account rather than discrete time increments.
The price change of the stock in an incredibly small time frame is indicated as dP in the continuous-time mode. We assume that dP follows a random process with a mean of zero and bounded variance, similar to the basic random walk. The stock price at time $t + dt$, denoted as $P(t + dt)$, is given by:

$$P(t + dt) = P(t) + dP$$

. The following presumptions are made in order to model stock prices through

Brownian motion:

1. Infinitesimal time intervals with stationary increments have the same distribution and are not dependent on one another. This presumption is consistent with the notion that stock price fluctuations are arbitrary and unaffected by earlier price changes.

2. The central limit theorem leads to the assumption that the distribution of dP is Gaussian (normal), with a mean of zero and a finite variance. This presumption is in line with the finding that stock price fluctuations frequently follow a bell-shaped distribution.

The stock price process converges to a continuous-time random walk, also referred to as Brownian motion or a Wiener process, by taking the limit as dt approaches zero. Brownian motion accurately depicts the random and ongoing swings in stock prices seen throughout time.

Mathematically, in the context of stock pricing, Brownian motion is often described using stochastic calculus and differential equations, allowing for more sophisticated modeling and analysis of financial processes. Advanced models, such as geometric Brownian motion, which incorporates a drift term, are commonly used to capture additional features of stock price dynamics.

Incorporating this drift term and capturing the exponential growth and volatility seen in many financial phenomena creates, under the premise of continuous compounding, the geometric Brownian motion which is frequently used in the field of finance to model the dynamics of asset prices, such as stock prices. The renowned Black-Scholes option pricing model relies on it as a key element.

The stochastic differential equation given below defines the geometric Brownian motion process:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where:

1. $S(t)$ represents the price of the asset at time t.

2. μ is the drift rate, which captures the average rate of return or growth in the asset price over time.

3. $\sigma$is the volatility, which measures the degree of fluctuation or randomness in the asset price.

4. $dW(t)$ represents the increment of the Wiener process at time t.

5. $\mu S(t)dt$ is the deterministic growth component (average rate of return in the absence of randomness)

6. $\sigma S(t)dt$ is the random component, which shows to what extent has randomness affected the stock the price.

The asset price change over an infinitesimal period of time is described by the geometric Brownian motion equation. We can model the asset's price trajectory over time by integrating this equation.

The following are the characteristics of geometric Brownian motion:

- Log-Normal Distribution: The asset price's logarithm follows a normal distribution, or log-normal distribution. This characteristic results from the fact that a geometric Brownian motion's logarithm is a typical Wiener process.

- Positive Prices: Because an exponential process determines the asset price, it is always positive.

- Path Dependency: The future values of the process depend on its previous values. This property reflects the cumulative effect of random shocks and drift on the price trajectory.

- Non-Stationary: Because the drift term is present, the process' statistical characteristics, such as the mean and variance, alter with time.

We implemented this equation into a python program to simulate a sample of fluctuations of a stock price with an initial value of \$100, drift of 0.05 and a volatility 0.2.

Geometric Brownian Motion

$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$

In [63]:

```
def simulate_stock_price(S0, drift, volatility, T, n_steps):
    dt = T / n_steps
    t = np.linspace(0, T, n_steps)
    dW = np.random.normal(0, np.sqrt(dt), n_steps)

    # Calculate the increments
    increments = (drift - 0.5 * volatility**2) * dt + volatility * dW

    # Calculate the stock price path
    log_returns = np.log(S0) + np.cumsum(increments)
    stock_prices = np.exp(log_returns)

    return t, stock_prices

# Parameters
S0 = 100  # Initial stock price
drift = 0.05  # Average return per time step
volatility = 0.2  # Standard deviation of returns
T = 10  # Time horizon (in years)
n_steps = 500  # Number of time steps (assuming 252 trading days in a year)

# Simulate stock prices
stock_prices = np.zeros((n_steps,5))
for i in range(5):
    t, crnt = simulate_stock_price(S0, drift, volatility, T, n_steps)
    stock_prices[:,i]=crnt
# Plot the results

plt.plot(t, stock_prices[:,0])
plt.plot(t, stock_prices[:,1])
plt.plot(t, stock_prices[:,2])
plt.plot(t, stock_prices[:,3])
plt.plot(t, stock_prices[:,4])
plt.xlabel('Time')
plt.ylabel('Stock Price')
plt.title('Stock Price Simulation')
plt.grid(True)
plt.show()
```

We got the following results:

which really shows stock-like patterns.

A very important property of Brownian motion is that despite being continuous everywhere, it is not differentiable anywhere.

Proof. We consider a small increment $W_{t+\Delta t} - W_t$ that is normally distributed with mean 0 and variance t. Then $E(|W_{t+\Delta t} - W_t|)^2 = \Delta t$, and we can see that the usual size of an increment $|W_{t+\delta t} - W_t|$ is δt. Now as $\Delta t \to 0, \sqrt{\delta t} \to 0$, which supports and explains the continuity of the paths. However, when we take the derivative:

$$\frac{\delta W_t}{\delta t} = \lim_{\Delta t \to 0} \frac{W_{t+\Delta t} - W_t}{\Delta t}$$

we can see that when t is small, the absolute value of the numerator looks like $\sqrt{\Delta t}$ which is much larger than Δt. Therefore the limit does not exist. We can conclude from this that the path of a Brownian motion W_t is nowhere differentiable. $\square$

This forces us to find a different way to be able to integrate stochastic functions, and will introduce us to Ito Calculus.

4 Itô's Calculus

Before introducing Itô's lemma and its derivation it is important to understand some properties of the normal distribution:

- If $Y \sim N(0, 1)$, then $a + bY \sim N(a, b^2)$;
- $Y \sim N(n, V)$, then $a + bY \sim (a + bm, b^2 V)$;
- therefore
$$dS_t = \mu S_t dt + \sigma S_t dW_t \sim N(\mu S_t dt, \sigma^2 S_t^2 dt);$$
- the return
$$\frac{dS}{s} \mu dt + \sigma dW_t \sim N(\mu dt, \sigma^2 dt).$$
- Over longer horizons, the price change is log normally distributed
- The log return
$$d \ln S_t = (\left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \sim N\left(\mu dt - \frac{1}{2}\sigma^2 dt, \sigma^2 dt\right)$$

All of this will be of use for the derivation of the Black-Sholes Model

Definition 4.1 (Itô process)**.** An n-dimensional Itô process is a process which

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

where W_t is a Brownian Motion with drift a and standard deviation b.

The following describes Ito's lemma which will help us in the future derivation of the Black-Sholes equation.

Theorem 4.1 (Alternative Ito's Lemma)**.** *Let $f(t, X_t)$ be an Ito's process which satisfies the stochastic differential equation*

$$dX_t = Z_t dt + y_t dB_t.$$

then $f(t, X_t)$ is also an Ito process with differential

$$df(t, X_t) = [\frac{\partial f}{\partial t} + \frac{\partial f}{dX_t} Z_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2} y_t^2]dt + \frac{\partial f}{\partial X_t} y_t B_t.$$

Lemma 4.2. *Quadratic Variation*

$$\lim_{n\to\infty} \sum_{t=1}^{n} (B(t/n \cdot T) - B(\frac{t-1}{n} T))^2 = T$$

In other words

$$(dB_t^2) = dt$$

Proof. Let $f(t, X_t)$ be a stochastic process. Since X_t is a standard Brownian motion, $X_0 = 0$. Using Taylor expansion on two variables we get,

$$f(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t} dX + \frac{1}{2}\frac{\partial^2}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_t^2}(dX_t)^2 + \frac{\partial^2 f}{\partial f \partial x_t} dt dX_t + ...$$

Recall the quadratic variation of W_t is t, then $(dW_t)^2 = dt$. But all the other terms are smaller than dt and are approximated as zero as illustrates by the figure bellow.

x	dW	dt
dW	dt	0
dt	0	0

Figure: Itô's Table

Using this, we get

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2.$$

Substituting $dX_t = Z_t dt + y_t dB_t$ we get

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t}[Z_t dt + y_t dB_t] + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(Z_t dt + y_t dB_t)^2. \tag{1}$$

And since,

$$(Z_t dt + y_t dB_t)^2 = Z_t^2 (dt)^2 + 2Z_y y_t dt dB_t + y_t^2 (dB_t)^2 = y_t^2 dt$$

Making a substitution to (2)

$$\begin{aligned} df(t, X_t) &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X_t} [Z_t dt + y_t dB_t] + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} y_t^2 \\ &= \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t} Z_t + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} y_t^2 \right] dt + \frac{\partial f}{\partial X_t} y_t dB_t \end{aligned}$$

□

Theorem 4.3 (Ito's Lemma). *Let f be a C^2 function and B_t is a standard Browinan motion, then $\forall t$,*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds$$

and can be written as a differential

$$df(B_t) = f'(B_t) dB_t + \frac{1}{2} f''(B_t) dt . f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) d$$

5 Options, Stocks Black and Sholes

5.1 Stocks and Options

Definition 5.1 (Stock). The capital raised by a company or corporation through the issue and subscription of shares.

Definition 5.2 (Options). is a financial instrument that provides its owner the right *(NOT THE OBLIGATION)* to buy or sell an underlying asset at an agreed price. To own an option contract, you must pay a premium. There are two types of options:

- Call option: the buyer of a call option has the right to buy an asset at an agreed price.

- Put option: the buyer of a put option has the right to sell an asset at an agreed price.

From this definition we can differentiate two possible positions (or strategies):

- Long option (you are the buyer of an option): If you are long you purchase the right to buy or sell (but no obligation) an underlying asset and you pay a premium to the seller of an option.

- Short option (you are the seller of an option): If you are long you purchase the right to buy or sell (but no obligation) an underlying asset and you pay a premium to the seller of an option.

All options have an expiration date and the time until maturity is the remaining time an option has left until expiration.
To put all of these definitions into context let's take a look at the next example: John is an expert in computers and thinks the price of an ABC model keyboard, which currently price is \$10, will sky-rocket within one year. He already has a keyboard but would like to benefit from this rise, so he offers his friend Mike to buy ABC in 1 year for \$20. Mike would be very happy to sell for that amount now but the future is uncertain, so he is reluctant to agree. John really believes in his prediction, so offers Mike \$5 for the right to buy the mouse and Mike agrees.
This example is an example of a call option with:

- Strike price: \$20

- Expiration Date: 1 year

- Premium: \$5

But a question arises. *How much money would be the right premium for this contract?*
There are different factors that affect the premium in stock markets:

- **Volatility σ:** a measure of uncertainty of the stock. The higher σ the higher chances of a change in the price of the stock and therefore of profits/losses. So the premium goes up.

- **Change in Price of the underlying asset:** If the price of the asset goes up, so does the premium for a call. The opposite for a put.
- **Time to maturity**: the shorter the lifespan of an option the lower the price as there are fewer chances of considerable variation in the price of the underlying asset.
- **Strike price:** The higher the strike price the less the price of a call option as there are fewer chances of it becoming profitable. The opposite for a put option.
- **Dividends and interest-rate:** the first one we will not consider in this paper, the latter has a positive correlation with option pricing.

5.2 Black-Sholes Model

Putting together all these variables gave rise to the Black-Scholes model, also known as the Black-Scholes-Merton (BSM) model. It is one of the most important concepts in modern financial theory. This mathematical equation estimates the theoretical value of derivatives based on other investment instruments, considering the impact of time and other risk factors.
If you are short, you sell the right to buy or sell an underlying asset. Therefore, if the owner of the option decides to exercise their right, you have the obligation to either buy or sell (depending on if it's a call or a put) the underlying asset.
Some interesting facts about the BSM:

- The Black-Scholes equation is a partial differential equation
- It is widely used to price options contracts.
- It requires five input variables: the strike price of an option, the current stock price, the time to expiration, the risk-free rate, and the volatility.
- Though usually accurate, the BSM makes certain assumptions. Therefore, the prices might not always match the real-world results.

- The standard BSM model is only used to price European options.

The Black-Scholes model makes certain assumptions:

- No dividends are paid out during the life of the option.
- Markets are random (i.e., market movements cannot be predicted).
- There are no transaction costs in buying the option.
- The risk-free rate and volatility of the underlying asset are known and constant.
- The returns on the underlying asset are log-normally distributed.
- The option is European and can only be exercised at expiration.

5.2.1 Black-Sholes Formula

Definition 5.3 (Black-Sholes-Merton Formula)**.** . This formula estimates the prices of European call and put options. The main difference between American and European options is that the former can be exercised at any moment whilst the later can only be exercised on the expiration date. This formula can also be used price non-dividend paying assets.

The formula is as follows.

$$C(S_0, t) = S_0 N(d_1) - K e^{-r(T-t)} N(d_2), \tag{2}$$

$$d_1 = \frac{\ln \frac{S_0}{K} + (\left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$= \frac{\ln \frac{S_0}{K} + (\left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

- S_0 is the stock price;

- $C(S_0, t)$ is the price of the call option as a formulation of the stock and time;
- K is the exercise price;
- $(T - t)$ is the time to maturity, or time until expiration;
- $N(d_1)$ and $N(d_2)$ are cumulative distribution functions for a standard normal distribution;
- σ represents the underlying volatility;
- r is the risk-free interest rate.

5.2.2 Black-Sholes Equation:

The Black-Scholes equation is the partial differential equation that governs the price evolution of European stock options in financial markets operating according to the dynamics of the BSM.
The equation is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3}$$

Proof. Let ω denote the units of stock share and γ the units of cash in a portfolio. Then ω_t, γ_t denote the amount of stocks and cash at time t. Then, the value of the portfolio at time t (V_t) is given by

$$V_t = \omega_t S_t + \gamma_t rPdt$$

where S_t is the price of stock, $rPdt$ is the amount of interest which can be possibly earned by the owned cash for a time dt. Observe that

$$\frac{\partial V_t}{\partial t} = \gamma_t rPdt,\ \frac{\partial V_t}{\partial s} = \omega_t,\ \frac{\partial^2 V_t}{\partial s^2} = 0.$$

By Ito's lemma with slightly different notation

$$df = \left[\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2} y_t^2\right] dt + \sigma S_t \frac{\partial f}{\partial S_t} dZ_t \tag{4}$$

Now we substitute V_t for f and the derivative computed before to get

$$dV_t = (\gamma_t r P dt + \mu S_t \omega_t + \frac{1}{2}\sigma^2 \cdot 0) dt + \sigma S_t \omega_t dZ_t \tag{5}$$

$$= (\gamma_t r P + \mu S_t \omega_t) dt + \sigma S_t \omega_t dZ_t$$

We need to get a formula for γ and ω and the way to do it is equate coefficients from (5) and (6). We first compare the coefficients of dZ_t

$$\sigma S_t f_{S_t} = \sigma S_t \omega, \ \frac{\partial f}{\partial S_t} = f_{S_t} = \omega.$$

Thus

$$V_t = f = \frac{\partial f}{\partial S_t} S_t + \gamma_t P, \ \gamma_t = \frac{1}{P}[f - \frac{\partial f}{\partial S_t} S_t].$$

By plugging ω and γ into (6) and compare coefficients of (5) and (6) for dt

$$\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2} = \mu \frac{\partial f}{\partial S_t} S_t + \frac{1}{P}[f - \frac{\partial f}{\partial S_t} S_t] r P.$$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2} = r f - \frac{\partial f}{\partial S_t} S_t r$$

thus,

$$\frac{\partial f}{\partial t} + r \frac{\partial f}{\partial S_t} S_t + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S_t^2} = r f.$$

$\square$

For this project, we will only focus on call options, but results for put options can easily be obtained by slightly manipulating the boundary conditions. Where:

- V: Premium or price of the option
- t: Time

- σ: Volatility
- S: Underlying asset price
- r: Interest-rate risk-free (ex: Euribor)
- τ: Time to expiration
- T: Date of expiration
- $\tau = T - t$

5.2.3 Computing the Black and Sholes Equation

To solve this, we came across different methods to solve partial differential equations. In this project we will focus on the explicit finite-difference method. To create an explicit method, we must make a change of variable on t. As we see above$\tau = T - t$ and hence
$\frac{\partial V}{\partial \tau} = -\frac{\partial V}{\partial t}$. Thus substituting into (1),

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{6}$$

With boundary conditions:

$$C(0,t) = 0 \tag{7}$$

$$C(S,T) = max\{S - K, 0\} \tag{8}$$

$$C(S,t) = S - Ke^{-r(T-t)} = S - Ke^{-r\tau} \tag{9}$$

Recall the calculus definition of a derivative:

$$\frac{dy}{dt} = \lim_{h \to \infty} \frac{y(t+h) - y(t)}{h} \tag{10}$$

and

$$\frac{dy(t)}{dt} = \lim_{h \to \infty} \frac{y(t+h) - y(t-h)}{2h} \tag{11}$$

Which we can approximate as

$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{\Delta t} \tag{12}$$

and

$$\frac{dy(t)}{dt} \approx \frac{y(t+h) - y(t-h)}{2\Delta t} \tag{13}$$

respectively.
Also, for the second derivative, we get

$$\frac{d^2y(t)}{dt^2} \approx \frac{y(t+h) - 2y(t) + y(t-h)}{\Delta t^2} \tag{14}$$

By substituting (2), (9) into the first partial, (11) into the second part, and (10) into the third part with the corresponding variables we get:

$$\frac{V(S, t+\Delta t) - V(t)}{\Delta t}$$
$$= \frac{1}{2}\sigma^2 S^2 \left[\frac{V(S+\Delta S, t) - 2V(S,t) + V(S-\Delta S, t)}{\Delta S^2}\right] + rS\left[\frac{V(S+\Delta S, t) - V(S-\Delta S, t)}{2\Delta S}\right] - rV$$

In a more computational style

$$\frac{V_{n,j+1} - V_{n,j}}{\Delta t} = \frac{1}{2}\sigma^2 S_n^2 \left[\frac{V_{n+1,j} - 2V_{n,j} + V_{n-1,j}}{\Delta S^2}\right] + rS_n\left[\frac{V_{n+1,j} - V_{n-1,j}}{2\Delta S}\right] - rV_{n,j} \tag{15}$$

From the initial condition (3) we know we will start at $S_0 = 0$ hence:

$$S_{n,j} = n\Delta S \text{ as } S_0 = 0 \tag{16}$$

Which simplifies (11) into

$$\frac{V_{n,j+1} - V_{n,j}}{\Delta t} = \frac{1}{2}\sigma^2 n^2 \left[V_{n+1,j} - 2V_{n,j} + V_{n-1,j}\right] + rn\left[V_{n+1,j} - V_{n-1,j}\right] - rV_{n,j} \tag{17}$$

and,

$$V_{n,j+1} = \frac{1}{2}\sigma^2 n^2 \left[V_{n+1,j} - 2V_{n,j} + V_{n-1,j}\right] \Delta t + rn \left[V_{n+1,j} - V_{n-1,j}\right] \Delta t - rV_{n,j}\Delta t + V_{n,j} \tag{18}$$

Finally, by combining coefficients we get

$$\boldsymbol{V_{n,j+1}} = \frac{\Delta t}{2}(\sigma^2 n^2 - rn)\boldsymbol{V_{n-1,j}} + [1 - (\sigma^2 n^2 + r)\Delta t]\boldsymbol{V_{n,j}} + \frac{\Delta t}{2}(\sigma^2 n^2 + rn)\boldsymbol{V_{n+1,j}}. \tag{19}$$

We used MatLab and develop a script to find the solution to this partial differential equation:

MATLAB Algorithm

```
% Algorithm to calculate the price of a European option

% Parameters
r       = 0.2;        % Interest rate
sigma   = 0.25;       % Volatility of the underlying
Nt      = 1600;       % Number of time steps
Ns      = 160;        % Number of asset price steps
Smax    = 20;         % Maximum asset price considered
Smin    = 0;          % Minimum asset price considered
T       = 1;          % Maturation (expiry) of the contract
E       = 10;         % Exercise price of the underlying

% Stepper variables
dt      = (T/Nt);                        % Time step
ds      = (Smax-Smin)/Ns;                % Price step

% Initializing the matrix of the option value
V(1:Ns+1, 1:Nt+1) = 0.0;

% Create an array with the input values of the price and the time to
expiration
S               = Smin+(0:Ns)*ds;
tau             = (0:Nt)*dt;

% Initial conditions prescribed by the European Call payoff at expiry:
V(S,tau=0) =max (S-E,0)
V(1:Ns+1,1)     = max (S-E, 0) ;

% Boundary conditions prescribed by the European Call:
V(1,1: Nt+1)    = 0;    % V(0, t) =0
V(Ns+1,1:Nt+1)  = Smax-E*exp(-r*tau);
% V(S, t) = S-Exp[-r (T-t)] as S -> infininty

% Implementing the explicit algorithm
for j = 1:Nt    % Time loop
    for n = 2:Ns    % Asset loop
        V(n,j+1) = 0.5*dt*(sigma*sigma*n*n-r*n)*V(n-1,j)+(1-
dt*(sigma*sigma*n*n+r))*V(n,j)+0.5*dt*(sigma*sigma*n*n+r*n)*V(n+1,j);
    end
end

% Figure of the values of the option, V(S,tau), as a function of S at
3 different times: tau=0(t=T), tau=T/2(t=T/2) and tau=T(t=0).
figure (1)
```

```
plot   (S, V(:,1), 'r-', S, V(:,round(Nt/2)), 'g-', S, V(:,Nt+1), 'b-
');
xlabel ('S');
ylabel ('V(S, tau)');
title  ('European Call Option within the Explicit Method');

% 3D plots of the Value of the option, V(S, tau)
figure (2)
mesh   (tau, S, V);
xlabel ('tau');
ylabel ('S');
title  ('European Call Option value, V(S,tau), within the Explicit
Method');
```

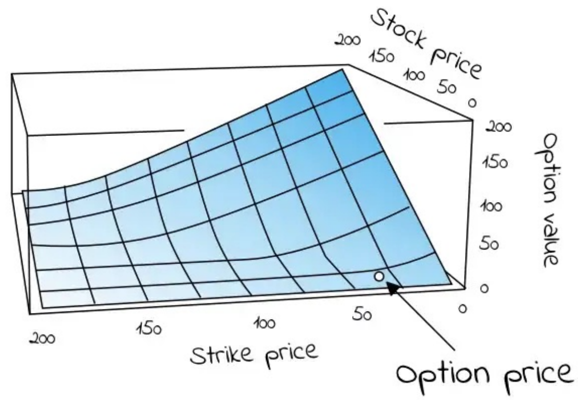


Figure 1: Visual representation of a Call Option as represented by BS model

6 Future Research

Our goals for future research are to further understand and advance the applications of Ito integration in finance by using computer-programmed simulations of real-life data and information. This would include addressing Ito's applications in financial risk management where we can explore how stochastic processes and diffusion models can be used to model and estimate various types of risks, including market risk, credit risk, and operational risk. Through the use of code simulations we could also consider Model Calibration as another possible research candidate. Implementing Ito integration in code allows you to calibrate models to real-world data. You can use optimization techniques or Bayesian estimation methods to estimate model parameters by minimizing the difference between simulated and observed data. Another way to further our research even more is to extend past the financial applications of Ito and apply it to other fields as well. For example, in the field of physics and engineering, stochastic calculus and ito integration play a very important role in involving random and probabilistic components that are required to study mechanics, quantum field theory, signal processing and control systems. When extending research on Ito integration to these areas, it is important to adapt the mathematical framework to suit the specific requirements of the field and the characteristics of the systems being studied. Collaboration with experts in these respective domains can also enhance the interdisciplinary nature of the research and lead to novel insights and applications.

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