Stochastic Processes: Ito Integration

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Senior Seminar II
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May 18, 2023

Introduction to Stochastic Processes

A a stochastic process is defined as a collection of random variables, indexed by some set, representing time.

$$\{X(t) : t \in T\}$$

[Random Variable] A random variable is a function defined on a sample space, which assigns a number to an event $X(\omega)$ in $\mathbb{R}$.

Two types of stochastic processes:

- Discrete-time: (Ex. $X_0, X_1, X_2 ...$)
- Continuous-time(Ex. $\{X\}_t, t \geq 0, t \in \mathbb{R}$)

We will introduce the following stochastic processes:

- Simple Random Walk
- Markov Chain
- Martingale
- Brownian Motion

Simple Random Walk

In mathematics, a simple random walk is a mathematical model used to describe a path or trajectory that consists of a sequence of random steps in a discrete space.

Properties of Simple Random Walk:

- Discrete-time process
- Each step is independent and identically distributed
- State space is usually the set of integers

Markov Chain

A Markov chain is a stochastic process where the future state depends only on the current state and is independent of the past history.
More formally, we could state that a discrete-time stochastic process $(x_0, x_1, x_2, ...)$ is a Markov Chain if:

$$P(x_{t+1} = S | x_0, x_1, ..., x_t) = P(x_{t+1} = S | x_t), \forall t \geq 0 \text{ and } \forall S$$

Properties of Markov Chains:

- Discrete-time or continuous-time process
- Memoryless property (Markov property)
- State space can be discrete or continuous

Martingale

A martingale is a stochastic process that models a fair game, where the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values.
Formally, it is a stochastic process where:

$$E[|X_n|] < \infty \text{ and } E[X_{n+1}|X_1, ..., X_n] = X_n$$

Properties of Martingales:

- Discrete-time or continuous-time process
- Fair game property
- Commonly used in finance and probability theory

Brownian Motion

A Standard Brownian Motion is a s a stochastic process $\{B_t\}, t \geq 0^+$ with the following properties:

1. 1. $B_0 = 0$,
2. the function $t \to B_t$ is continuous in t,
3. the process $B_t, t \geq 0$ has stationary and independent increments
4. the increment $B_{t+s} - B_s \sim N(0, t)$

Properties of Brownian Motion:

- Continuous-time process
- Increments are normally distributed
- Used to model various phenomena, including stock prices

Stationary and Independent Increments

- Stationary Increments: for any $0 < s, t < \infty$, the distribution of the increment $B_{t+s} - B_s$ has the same distribution as $B_t - B_0 = B_t$
- Independent Increments: For every choice of non-negative real numbers $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq ... \leq s_n < t_n < \infty$, the increment random variables $B_{t_1} - B_{s_1}$, $B_{t_2} - B_{s_2}$, ..., $B_{t_n} - B_{s\,n}$ are jointly independent. .

Application to Stock Pricing

Why Simple Random Walk can model stock prices:

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1. Price Variations: A stock's price may fluctuate erratically in either direction, with each variation being treated independently of the others. In other words, price changes in the past have no bearing on how prices will move in the future.
2. No Predictable Trends: According to the random walk model, stock price changes do not exhibit any obvious patterns or trends. It means that it is doubtful that efforts to anticipate future prices purely using historical price data will be consistently successful.
3. Efficiency of Markets: The random walk model makes the assumption that markets are efficient, which means that stock prices reflect all available information promptly and accurately. As a result, it means that continuously outperforming the market through price prediction is challenging.

Application to Stock Pricing II

We refined our model by applying Geometric Brownian Motion:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where:

1. $S(t)$ represents the price of the asset at time t.
2. μ is the drift rate, which captures the average rate of return or growth in the asset price over time.
3. σ is the volatility, which measures the degree of fluctuation or randomness in the asset price.
4. $dW(t)$ represents the increment of the Wiener process at time t.
5. $\mu S(t)dt$ is the deterministic growth component (average rate of return in the absence of randomness)
6. $\sigma S(t)dt$ is the random component, which shows to what extent has randomness affected the stock the price.

Ito's Lemma

Ito's Lemma is a formula used in stochastic calculus to find the differential of a function of a stochastic process. It plays a key role in deriving mathematical models for financial derivatives.

The formula for Ito's Lemma is given by:

$$df(t,x) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2$$

where f is a function of time t and the stochastic process $x(t)$.

Derivation of Black-Scholes Equation

The Black-Scholes equation is a partial differential equation used to price European-style options.

The derivation involves a Ito process (Geometric Brownian Motion in this case) and applying Ito's Lemma, with assumptions about the behavior of the underlying asset price we get the resulting partial differential equation. The resulting equation provides a theoretical framework for option pricing and hedging strategies.

- V: Premium or price of the option
- t: Time
- σ: Volatility
- S: Underlying asset price
- r: Interest-rate risk-free (ex: Euribor)
- τ: Time to expiration
- T: Date of expiration
- $\tau = T - t$

Boundary Conditions

$$C(0,t) = 0$$

$$C(S,T) = max\{S - K, 0\}$$

$$C(S,t) = S - Ke^{-r(T-t)} = S - Ke^{-r\tau}$$

Partial Differential Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The End

Thank You!