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ELASTO-PLASTIC SOLUTIONS OF ENGINEERING PROBLEMS 'INITIAL STRESS', FINITE ELEMENT APPROACH

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SUMMARY

The paper presents first a general formulation of the elasto-plastic matrix for evaluating stress increments from those of stresses for any yield surface with an associated flow rule. A new 'initial stress' computational process is proposed which is shown (1) to yield more rapid convergence than alternative approaches (2) to permit large load increments without violating the yield criteria and thus simply to establish lower bound solutions. Several solutions showing stress distribution, strain development and growth of plastic enclaves are given both for the von Mises and for Coulomb (Drucker) type yield surfaces. Load reversal and thermoplastic behaviour are dealt with.

INTRODUCTION

It is probably true to say that the recent development of numerical methods in general, and of the finite element method in particular, permits solutions to be obtained for any rationally conceived constitutive laws of the material behaviour. Of practical interest in this context are the numerous problems in which plasticity plays a dominating part. Such problems range in application from machine technology, through structural applications, to geophysics. The materials for which solutions are required may exhibit a variety of yield surfaces. One of the objectives of this present paper is to provide a unified treatment of general viability so that with a minimum of programming effort, diverse situations can be accommodated.

In this context it is essential that the method should be able to deal with problems of ideal or work hardening plasticity.

As the interest in a particular solution may be in the prediction of displacements and strains at various stages of the loading, in the development of plastic zones or in the prediction of residual strain distribution on load removal it is important that all these should be adequately represented. In particular it is important that the method should be able to follow both loading and unloading cycles without difficulty, reproducing fully the elasto–plastic behaviour.

Quite frequently the only information required by the designer is that of determining the collapse situation. Here quite simple computational processes suffice and there is no apparent need to go through all the stages of a complete elasto–plastic solution. However, for various important situations only bounding answers can be achieved and indeed kinematic (upper) bound is usually the only one readily achieved. Here it is hoped that the methodology presented will be of interest as it will invariably provide an assessment of the equilibrium (lower) bounds.

Various computational procedures have been used with success for a limited range of elasto-plastic problems utilizing the finite element approach. Two main formulations appear. In the first, during an increment of loading, the increase of plastic strain is computed and treated as an *initial strain* for which the elastic stress distribution is adjusted.[1,2] This approach manifestly

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fails if ideal plasticity is postulated or if the degree of hardening is small. The second approach is that in which the stress-strain relationship in every load increment is adjusted to take into account plastic deformations. The work of Pope,[3] Swedlow,[4] Marcal and King,[5] Reyes and Deere[6] and Popov and others[7] falls into this category. With a properly specified *elasto-plastic matrix* this incremental elasticity approach can successfully treat ideal as well as hardening plasticity.

From the computational point of view the '*incremental elasticity*' process has one serious disadvantage. At each step of computation the stiffness of the structure is changed and iterative processes of solution are necessary to avoid excessive computer times. In this paper an alternative approach which we shall refer to as the '*initial stress*' process is developed. By using the fact that even in ideal plasticity increments of strain prescribe uniquely the stress system (while the reverse is not true for ideal plasticity) an adjustment process is derived in which '*initial stresses*' are distributed elastically through the structure.

This approach permits the advantage of initial processes (in which the basic elasticity matrix remains unchanged) to be retained. The process appears to be the most rapidly convergent. It will be found that no special treatment of unloading cycles is now required. The method will be fully described later but is in principle similar to the treatment of cracking materials described elsewhere.[8]

A comprehensive bibliography on the finite element method in general and on non-linearity in particular will be found in a text.[9] No description of the finite element method or of the nomenclature is thus called for here.

SOME BASIC CONCEPTS OF PLASTICITY

Yield surface

It is quite generally postulated, as an experimental fact, that yielding can occur only if the stresses $\{\sigma\}$ satisfy the general yield criterion

$$F(\{\sigma\}, \kappa) = 0 \tag{1}$$

In this a vectorial notation is used for stress components and κ is a hardening parameter.

This yield condition can be visualised as a surface in n-dimensional hyper space of stress with the position of the surface dependent on the instantaneous value of the parameter κ (Figure 1).

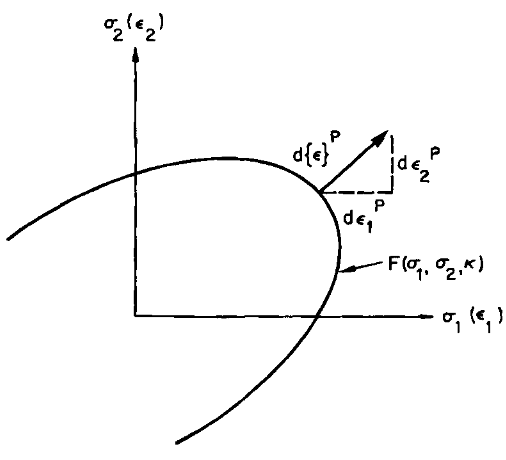

Figure 1. Yield surface and normality criterion in two dimensional stress space

Flow Rule

von Mises[10] first suggested the basic constitutive relation defining the plastic strain increments in relation to the yield surface. Heuristic arguments for the validity of the relationship proposed have been given by various workers in the field[11,12] and at the present time the following hypothesis appears to be generally accepted:

If $\delta\{\varepsilon\}_p$ denotes the increment of plastic strain then

$$\delta\{\varepsilon\}_p = \lambda \frac{\partial F}{\partial\{\sigma\}} \tag{2}$$

or for any component n

$$\delta\varepsilon_{n_p} = \lambda \frac{\partial F}{\partial \sigma_n}$$

In this λ is a proportionality constant, as yet undetermined. The rule is known as the *normality* principle because relation (2) can be interpreted as requiring the normality of the plastic strain increment vector to the yield surface in the hyper space of n stress dimensions.

Total Stress–strain relations

During an infinitesimal increment of stress, changes of strain are assumed to be divisible into elastic and plastic parts. Thus

$$\delta\{\varepsilon\} = \delta\{\varepsilon\}_e + \delta\{\varepsilon\}_p \tag{3}$$

The elastic strain increments are related to stress increments by a symmetric matrix of constants [D] known as the 'elasticity matrix'. Thus

$$\delta\{\varepsilon\}_e = [\mathbf{D}]^{-1}\delta\{\sigma\} \tag{4}$$

We can thus write (3) as

$$\delta\{\varepsilon\} = [\mathbf{D}]^{-1}\delta\{\sigma\} + \frac{\partial F}{\partial\{\sigma\}} . \lambda \tag{5}$$

When plastic yield is occurring the stresses are on the yield surface given by (1). Differentiating this we can write

$$0 = \frac{\partial F}{\partial \sigma_1}\delta\sigma_1 + \frac{\partial F}{\partial \sigma_2}\delta\sigma_2 + \ldots + \frac{\partial F}{\partial \kappa}\,\mathrm{d}\kappa$$

or

$$0 = \left\{\frac{\partial F}{\partial\{\sigma\}}\right\}^T \delta\{\sigma\} + A\lambda \tag{6}$$

in which

$$A = \frac{\partial F}{\partial \kappa}\,\mathrm{d}\kappa . \frac{1}{\lambda} \tag{7}$$

Equations (5) and (6) can be written in a single symmetric matrix form as

$$\left\{\begin{matrix} \delta\{\varepsilon_1\} \\ \\ \delta\{\varepsilon_2\} \\ \vdots \\ \overline{0} \end{matrix}\right\} = \left[\begin{array}{c|c} & \dfrac{\partial F}{\partial \sigma_1} \\ [\mathbf{D}]^{-1} & \dfrac{\partial F}{\partial \sigma_2} \\ & \vdots \\ \hline \dfrac{\partial F}{\partial \sigma_1} \quad \dfrac{\partial F}{\partial \sigma_2} \cdots & A \end{array}\right] \left\{\begin{matrix} \delta\sigma_1 \\ \\ \delta\sigma_2 \\ \vdots \\ \overline{\lambda} \end{matrix}\right\} \tag{8}$$

This form is convenient for use directly provided that 'A' is not zero as shown in a particular form by Marcal and King.[5] Alternatively λ can be eliminated (taking care not to multiply or divide by A which may be zero in general). This results in an explicit expansion which determines the *stress changes* in terms of imposed *strain changes*.

with
$$\delta\{\sigma\} = [\mathbf{D}]^*_{ep}\,\delta\{\varepsilon\} \tag{9}$$

$$[\mathbf{D}]^*_{ep} = [\mathbf{D}] - [\mathbf{D}]\left\{\frac{\partial F}{\partial\{\sigma\}}\right\}\left\{\frac{\partial F}{\partial\{\sigma\}}\right\}^T [\mathbf{D}]\left[A + \left\{\frac{\partial F}{\partial\{\sigma\}}\right\}^T \quad [\mathbf{D}]\left\{\frac{\partial F}{\partial\{\sigma\}}\right\}\right]^{-1} \tag{10}$$

The elasto-plastic matrix $[\mathbf{D}]^*_{ep}$ takes the place of the elasticity matrix $[\mathbf{D}]$ in incremental analysis. It is symmetric, positive definite, and the expression (9) is valid whether or not 'A' takes on a zero value.

Significance of parameter 'A'

Clearly for ideal plasticity with no hardening 'A' is simply zero.

If hardening is considered, attention must be given to the nature of the parameter (or parameters) κ on which the shifts of the yield surface depend.

With a 'work hardening' material κ is taken to be represented by the amount of plastic work done during plastic deformation. Thus

$$d\kappa = \sigma_1\, d\varepsilon_1^p + \sigma_2\, d\varepsilon_2^p + \ldots = \{\sigma\}^T d\{\varepsilon\}_p \tag{11}$$

substituting the flow rule (2) we have simply

$$d\kappa = \lambda\{\sigma\}^T \frac{\partial F}{\partial\{\sigma\}} \tag{12}$$

By equation (7) we now see that λ disappears and we can write

$$A = \frac{\partial F}{\partial \kappa}\{\sigma\}^T \frac{\partial F}{\partial\{\sigma\}} \tag{13}$$

a strictly determinate form if explicit relationship between F and κ is known.

An illustrative example

To illustrate some of the concepts consider the well known von Mises yield surface. This is given by

$$F = [\tfrac{1}{2}(\sigma_1 - \sigma_2)^2 + \tfrac{1}{2}(\sigma_2 - \sigma_3)^2 + \tfrac{1}{2}(\sigma_3 - \sigma_1)^2 + 3\sigma_4^2 + 3\sigma_5^2 + 3\sigma_6^2]^{\frac{1}{2}} - \bar{\sigma} \tag{14}$$

in which suffixes 1, 2, 3 refer to the normal stress components and 4, 5, 6 to shear stress components.

On differentiation it will be found that

$$\begin{aligned} \frac{\partial F}{\partial \sigma_1} = \frac{3\sigma_1'}{2\bar{\sigma}}, \quad \frac{\partial F}{\partial \sigma_2} = \frac{3\sigma_2'}{2\bar{\sigma}}, \quad \frac{\partial F}{\partial \sigma_3} = \frac{3\sigma_3'}{2\bar{\sigma}} \\ \frac{\partial F}{\partial \sigma_4} = \frac{3\sigma_4}{\bar{\sigma}}, \quad \frac{\partial F}{\partial \sigma_5} = \frac{3\sigma_5}{\bar{\sigma}}, \quad \frac{\partial F}{\partial \sigma_6} = \frac{3\sigma_6}{\bar{\sigma}} \end{aligned} \tag{15}$$

in which the dashes stand for deviatoric stresses i.e.

$$\sigma_1' = \sigma_1 - \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3} \text{ etc.}$$

The quantity $\bar{\sigma} = \bar{\sigma}\ (\kappa)$ is the uniaxial stress at yield. If a plot of the uniaxial test giving $\bar{\sigma}$ versus the *plastic* uniaxial strain ε_{up} is available then

$$\mathrm{d}\kappa = \bar{\sigma}\ \mathrm{d}\varepsilon_{up}$$

and

$$\frac{\mathrm{d}F}{\mathrm{d}\kappa} = \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\kappa} = \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\varepsilon_{up}} \cdot \frac{1}{\bar{\sigma}} = \frac{H'}{\bar{\sigma}} \tag{16}$$

in which H′ is the slope of the plot at the particular value of $\bar{\sigma}$.

On substituting into (13) we obtain after some transformation simply

$$A = H' \tag{17}$$

This re-establishes the well known Prandtl-Reuss stress strain relations.

'Corners' of a yield surface

It happens, not infrequently, that the yield surface is defined not by a single continuous (and convex) function but by a series of functions:

$$F_1, F_2 \ldots F_n$$

the state of strain below the yield limit being defined by negative values of all the functions F.

For most of the bounding surface only a single condition such as $F_m = 0$ will define the yield surface and the previously written flow rules (and elasto–plastic matrices) apply.

At a 'corner' of the yield surface we may have however the condition that

$$F_h = \ldots = F_m = 0$$

Here the Koiter generalisation[13] replaces equation (2) giving

$$d\{\varepsilon_p\} = \lambda_h \left\{\frac{\partial F_h}{\partial\{\sigma_h\}}\right\} + \ldots + \lambda_m \left\{\frac{\partial F_m}{\partial\{\sigma_m\}}\right\} \tag{18}$$

where λ_i are positive constants (Figure 2).

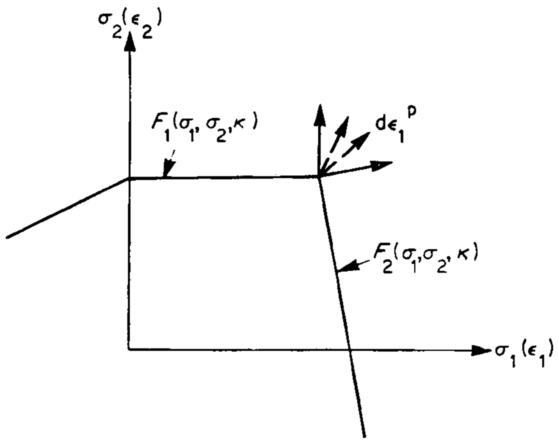

Figure 2. Corners in a yield surface. Graphical interpretation of Koiter's criterion

Matrices of type equation (8) can once again be written now with several undetermined parameters λ. Procedures similar to those above will yield new forms for the elasto–plastic matrix applicable at such corners.

The computation of singular points on yield surfaces is best avoided by a suitable choice of continuous surfaces which usually can with a good degree of accuracy represent the true conditions.

Application

The 'elasto–plastic' matrix has been given in a general form in equation (10). Particular forms will obviously depend on the problem at hand. A generalized stress vector was used throughout as the process is equally applicable to full three dimensional stress fields, to special two dimensional situations or to moment-curvature relations in plate bending. An explicit form of a plane strain matrix for a Prandtl-Reuss material has recently been published by Yamada and others.[14] In this paper various forms of this matrix will be used in different examples.

THE 'INITIAL STRESS' COMPUTATIONAL PROCESS

The expressions derived in the previous section describe fully the stress–strain relation in the elasto–plastic state. The essential non-linearity is evident from the equation (10) with the 'elasto–plastic' matrix being dependent on the state of *total stress*. Some 'piecewise' linearization process is then required. In the 'incremental elasticity' method small load increments are prescribed and in each the material treated as quasi-elastic, with a constant 'elasto–plastic' matrix. The numerical values prescribing this matrix may either correspond to the initial stress values of the increment or by adjustment may be made to coincide to average stresses in the increment (5). Whichever approximation is used different elastic problems are posed at each load increment necessitating changes of the stiffness matrices at each step. This computational inconvenience is overcome by an essentially different approach which we shall call the 'initial stress' method.

The 'initial stress' process which is used in this paper approaches the solution of a non-linear problem as a series of approximations. In the first place during a load increment a purely elastic problem is solved determining an increment of strain $\Delta\{\varepsilon\}'$ and of stress $\Delta\{\sigma\}'$ at every point of the continuum (or structure).

The non-linearity implies however that for the increment of *strain* found, the stress increment will in general not be correct. If the true increment of stress possible for the given strain is $\Delta\{\sigma\}$ then the situation can only be maintained by a set of *body forces* equilibrating the 'initial' stress system $\Delta\{\sigma\}'-\Delta\{\sigma\}$.

At the second stage of the computation this body force system can be removed by allowing the structure (with unchanged elastic properties) to deform further. An additional set of strain and, corresponding, stress increments is caused. Once again these are likely to exceed those permissible by the non-linear relationship and the redistribution of equilibrating body forces has to be repeated.

If the process converges then finally within an increment the full non-linear compatibility and equilibrium conditions will be satisfied just as they are in an 'incremental elasticity' solution.

It appears 'a priori' in an elastic–plastic situation that the process is a natural one making use of the fact that for any prescribed strain the increment of stress is a determinate one and one that changes slowly. Indeed application shows that convergence is rapid, three or four cycles of re-distribution (iteration) being necessary in any increment.

As for each cycle the *same* elastic problem is being solved then, clearly, if use is made of a partial invertion of the elastic equation very rapid computer times will result.

Obviously the process is of a general applicability not limited to the elasto–plastic situation. Indeed if the situation is such that a limit on stress is imposed without a corresponding strain relationship a single increment of load can be used to achieve a final solution, as for instance has been done in 'tension cut-off' situations.[8]

In full elasto–plastic situations it is generally necessary to proceed in a series of load increments to follow the appropriate flow rules. If however a single load increment is used it will be found that an *approximate lower* bound is achieved, the final solution satisfying equilibrium and yield criteria but not necessarily following the current strain development. The use of such bounding solutions in practice is important.

For the elasto–plastic situation the steps during a typical load increment can be summarized as follows:

1. Apply load increment and determine elastic increments of stress $\{\Delta\sigma'\}_1$ and strain $\{\Delta\varepsilon'\}_1$ which correspond.
2. Add $\{\Delta\sigma'\}_1$ to stresses existing at start of increment $\{\sigma_0\}$ to obtain $\{\sigma'\}$. Check whether $F\{\sigma'\}<0$ (with κ referring to the initial value at start of increment). If above satisfied only elastic strain changes occur and *process is stopped*, if not proceed to 3.
3. If $F\{\sigma'\}\geqslant 0$ and also $F\{\sigma_0\} = 0$ (i.e. element was in yield at start of increment) find $\{\Delta\sigma\}_1$ by equation (9).

   $$\{\Delta\sigma\}_1 = [\mathbf{D}]^*_{e.p.}\ \{\Delta\varepsilon'\}_1$$

   with $[\mathbf{D}]^*_{e.p.}$ computed from equation (9) with stresses $\{\sigma'\}$.

   Evaluate stress which has to be supported by body forces

   $$\{\Delta\sigma''\}_1 = \{\Delta\sigma'\}_1 - \{\Delta\sigma\}_1$$

Store current stress $\{\sigma\} = \{\sigma'\} - \{\Delta\sigma''\}_1$
and current strain $\{\varepsilon\} = \{\varepsilon'\} + \Delta\{\varepsilon'\}_1$

4. If $F\{\sigma\}>0$ but $F\{\sigma_0\}<0$
find the intermediate stress value at which yield begins and compute increment $\{\Delta\sigma\}_1$ by equation (9) starting from that point. Then proceed as in 3.
5. Compute nodal forces corresponding to the equilibrating body forces. These are given for any element by*
$$\{P\}_1^e = \int [B]^T \{\Delta\sigma''\}_1 \, d(\text{vol})$$
6. Resolve using original elastic properties and the load system $\{P\}$ to find $\{\Delta\sigma'\}_2$ and $\{\Delta\varepsilon'\}_2$.
7. Find current value of κ
8. Repeat steps 2 to 6, etc.

The cycling is terminated when the nodal forces of 5 reach sufficiently small values. If this is not achieved in a predetermined number of cycles (20 in our case) collapse condition is deemed to have been achieved and the process is stopped.

This brief description shows the necessary modifications to any standard finite element elastic program to enable it to deal with the elasto–plastic situation.

The process is illustrated graphically in a two dimensional stress space in Figure 3. Note that after a few cycles the resulting stress is always brought back to the yield surface.

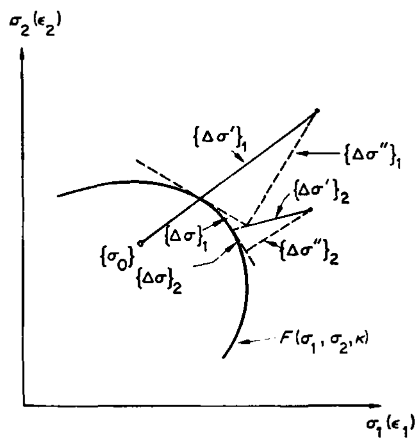

Figure 3. Graphical interpretation of the 'initial stress' process

It is evident (*vide* step 2) that if an unloading occurs the program will automatically follow the elastic unloading process until a new intersection with the yield surface again occurs. Thus complete cycles of load can be dealt with.

By 'load' increments, implicitly any set of external loads or internal initial strains is meant. Thus thermal problems fall readily into the method described.

* The standard notation of the finite element process is used here, reference 9 in which the matrix [B] defines strains in terms of nodal displacements—see Appendix.

Any specified increments can be used from the start of loading. It is convenient however to start the incremental process only when first yield has occurred and in the program this allows the subsequent load increments to be related to the load at which first yield is noted.

SPECIAL FORMS OF THE ELASTO-PLASTIC RELATIONSHIP

Most generally the yield criterion is established in the 'six dimensional' stress space as a function of all the six stress components. When dealing with more restrictive problems such as prescribed by cases of plane stress, plane strain or axial symmetry an appropriate specialization of the yield surface to the more limited freedom has to be made.[22] On occasion the special form may be described directly as, for example, when yield surfaces for plates and shells are considered.[15] Alternatively a purely formal operation will yield the appropriate result.

Consider the general relationship (8) written in terms of the six three dimensional stress components listed as

$$\sigma_1 = \sigma_x \quad \sigma_2 = \sigma_y \quad \sigma_3 = \sigma_z \quad \sigma_4 = \tau_{xy} \quad \sigma_5 = \tau_{yz} \quad \sigma_6 = \tau_{zx}$$

and let us now turn our attention to the special cases.

Plane Stress

If z is chosen as the direction normal to the plane we note immediately that

$$\delta\sigma_z = \partial\tau_{yz} = \delta\tau_{zx} = 0$$

and that appropriate *columns* may be deleted from the relationship. The *rows* corresponding to these stress components cease to be of interest in the two dimensional analysis and we are left with the special form of equation (8)

$$\begin{Bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\gamma_{xy} \\ \hline 0 \end{Bmatrix} = \left[\begin{array}{ccc|c} & & & \dfrac{\partial F}{\partial \sigma_x} \\ & [\mathbf{D}]^{-1} & & \dfrac{\partial F}{\partial \sigma_y} \\ & & & \dfrac{\partial F}{\partial \tau_{xy}} \\ \hline \dfrac{\partial F}{\partial \sigma_x} & \dfrac{\partial F}{\partial \sigma_y} & \dfrac{\partial F}{\partial \tau_{xy}} & A \end{array}\right] \begin{Bmatrix} \delta\sigma_x \\ \delta\sigma_y \\ \delta\tau_{xy} \\ \hline \lambda \end{Bmatrix} \tag{19}$$

where [D] stands for the simple elasticity plane stress matrix and F is the 'cross-section' of the yield surface with $\sigma_z = \tau_{yz} = \tau_{zx} = 0$. The same transformation equation (10) will obviously be still used to eliminate λ.

Although a plastic strain ε_z^p now occurs it is not necesary to record this as with a work hardening situation assumed, its contribution to equation (11) is zero.

Plane strain

Once again two shear stress components (τ_{yz} and τ_{zx}) become zero and rows and columns corresponding to these can be omitted. The normal stress σ_z is however no longer zero and the condition that

$$\varepsilon_z = 0$$

has to be imposed.

The equation can now be written as

$$\begin{Bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\gamma_{xy} \\ 0 \\ \hline 0 \end{Bmatrix} = \left[\begin{array}{c|c} & \dfrac{\partial F}{\partial \sigma_x} \\ & \dfrac{\partial F}{\partial \sigma_y} \\ [\mathbf{D}_0]^{-1} & \dfrac{\partial F}{\partial \tau_{xy}} \\ & \dfrac{\partial F}{\partial \sigma_z} \\ \hline \text{symmetric} & A \end{array}\right] \begin{Bmatrix} \delta\sigma_x \\ \delta\sigma_y \\ \delta\tau_{xy} \\ \delta\sigma_z \\ \hline \lambda \end{Bmatrix} \tag{20}$$

If the matrix of elastic constant $[\mathrm{D}_0]^{-1}$ is written as

$$[\mathbf{D}_0]^{-1} = [a_{ij}]$$

then it is easy to verify that on elimination, equation (20) can be reduced to

$$\begin{Bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\gamma_{xy} \\ \hline 0 \end{Bmatrix} = \left[\begin{array}{c|c} & \left(\dfrac{\partial F}{\partial \sigma_x} - \dfrac{a_{14}}{a_{44}}\dfrac{\partial F}{\partial \sigma_z}\right) \\ [\mathbf{D}]^{-1} & \left(\dfrac{\partial F}{\partial \sigma_y} - \dfrac{a_{24}}{a_{44}}\dfrac{\partial F}{\partial \sigma_z}\right) \\ & \left(\dfrac{\partial F}{\delta \tau_{xy}} - \dfrac{a_{34}}{a_{44}}\dfrac{\partial F}{\partial \sigma_z}\right) \\ \hline \text{symmetric} & \left(A - \dfrac{1}{a_{44}}\left(\dfrac{\partial F}{\partial \sigma_z}\right)^2\right) \end{array}\right] \begin{Bmatrix} \delta\sigma_x \\ \delta\sigma_y \\ \delta\tau_{xy} \\ \hline \lambda \end{Bmatrix}$$

In this $[\mathrm{D}]^{-1}$ is the usual reduced plane strain elastic matrix. For an isotropic material

$$a_{44} = \frac{1}{E}, \quad a_{14} = a_{24} = -\frac{\nu}{E} \text{ and } a_{34} = 0$$

Elimination of λ can still be carried out in the manner of equation (10) noting however that appropriate substitutions have to be made. In particular A is now replaced by

$$\left\{A - \frac{1}{a_{44}}\left(\frac{\partial F}{\partial \sigma_z}\right)^2\right\}$$

and does not become zero even for ideal plasticity.

As σ_z is now no longer zero it is necessary to keep a record of it in the computation as plastic strains in the normal direction will now occur.

Axial symmetry

Here the situation is once again more simple as four stress and strain components have non zero values and only two shear stress and strain components vanish. The form of the relationship will be identical with that of equation (20) but with the fourth non zero strain component.

PLANE STRESS AND STRAIN WITH von MISES YIELD CRITERON—EXAMPLES

Perforated plate with and without strain hardening

This plane stress test example was studied by both the finite element method[5,14] and experimentally[16] and is therefore of value in assessing the accuracy and efficiency of the different approaches. In Figure 4 (see page 86) the mesh of simple triangular elements is shown together with plastic enclaves for both strain hardening and ideal plasticity cases. The results are in substantial agreement with those of the previous investigations.

In solutions (b) and (c) increments of load equal to 0·2 of the load at first yield were used. It was of interest to investigate a *one step process* in which the final load was reached in a single increment. Figure 4 (d) shows the final zone of plasticity thus obtained which, surprisingly, differs but little from the more correct small increment solution.

In Figure 5 (see page 87) the development of the maximum strain is compared with results of the finite element 'partial stiffness' method as used by Marcal and King[5] and experimental results.[16] The results of the initial stress method are slightly closer to experiment for the same load increments. The single step solution gives again an exceedingly good estimate of this strain. Indeed the surprising insensitivity of the results to the magnitude of the increment is illustrated in Figure 6 (see page 87).

If large increments are used directly with the partial stiffness approach[5] then meaningless results are in general obtained.

Table I shows some comparison of the computational times involved in the two approaches.

Table I. Comparison of Computer Time

Problem	No. of Nodes	No. of Elements	$\frac{H'}{E}$	Load Increment	No. of Increments	No. of Interations within an Increment	Computer Time in mts. I.C.T. 1905 Initial Stress Method	Partial Stiffness Method
PERFORATED PLATE (PLANE STRESS)				0·1L	7	3	18	31
						5	23	
				0·2L	7	4	21	43
						6	29	
			0·032	0·4L	4	6	27	—
				1L		30	31	failed to converge
	94	149				4	22	30
				0·1L	7	6	30	
			0	0·2L	7	5	21	50
						7	31	
NOTCHED SPECIMEN (PLANE STRAIN)				0·1L	14	5	72	—
	94	149	0					
				0·2L	11	6	67	69

Note: L = Load at first yield.

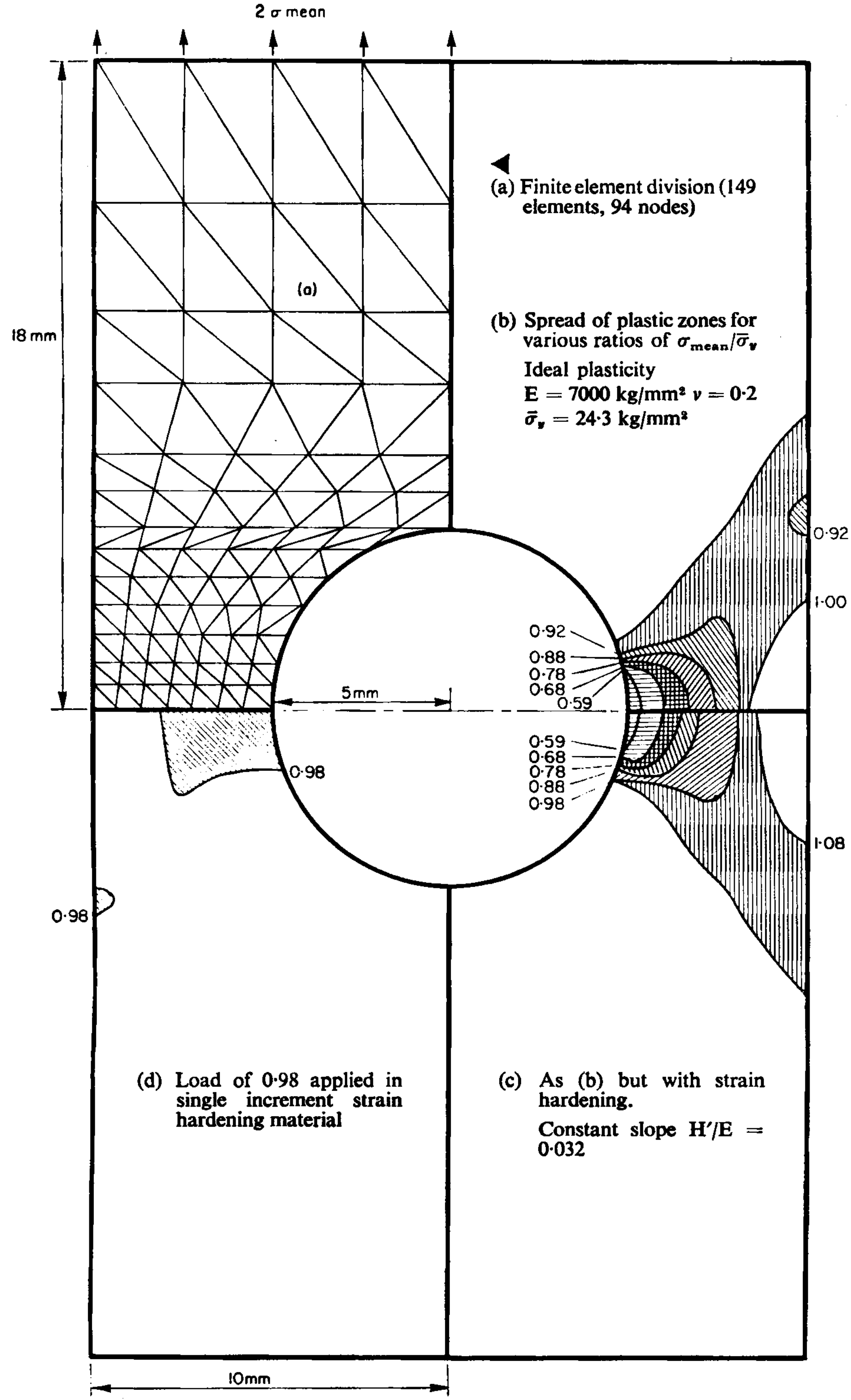

Figure 4. Perforated tension strip (plane strip)

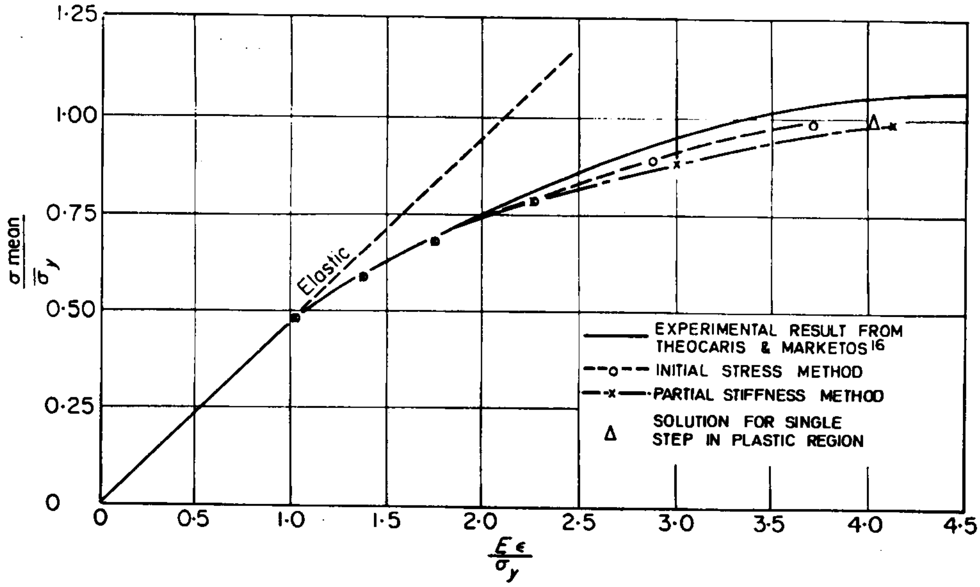

Figure 5. Perforated plate—strain hardening material
Development of maximum strain at point of first yield
$H'/E = 0{\cdot}032$
Load increment $= 0{\cdot}2 \times$ first yield load

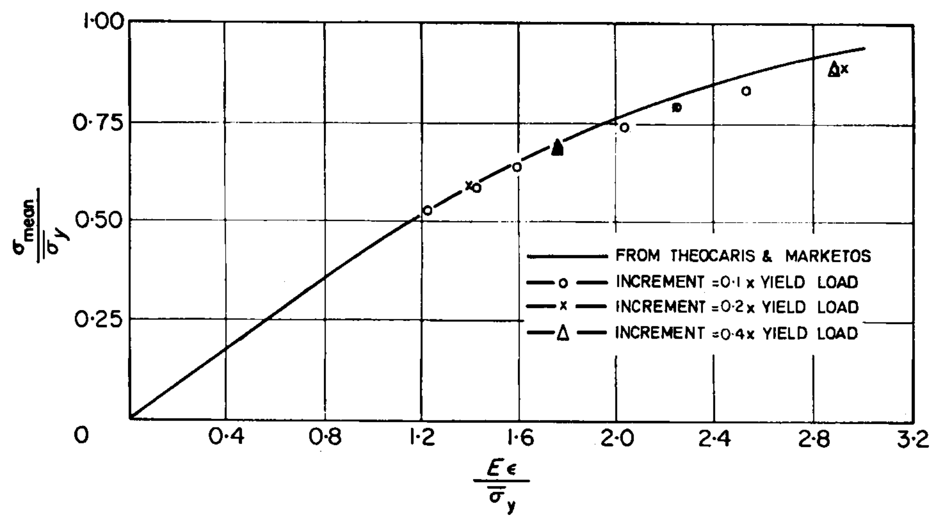

Figure 6. Perforated plate—strain hardening material
As Figure 5 showing the effect of load increment on 'initial stress' solution

Figure 7 shows how closely the stresses at a particular element follow the yield criterion specified. This indeed appears to be a particularly satisfactory feature of the method used.

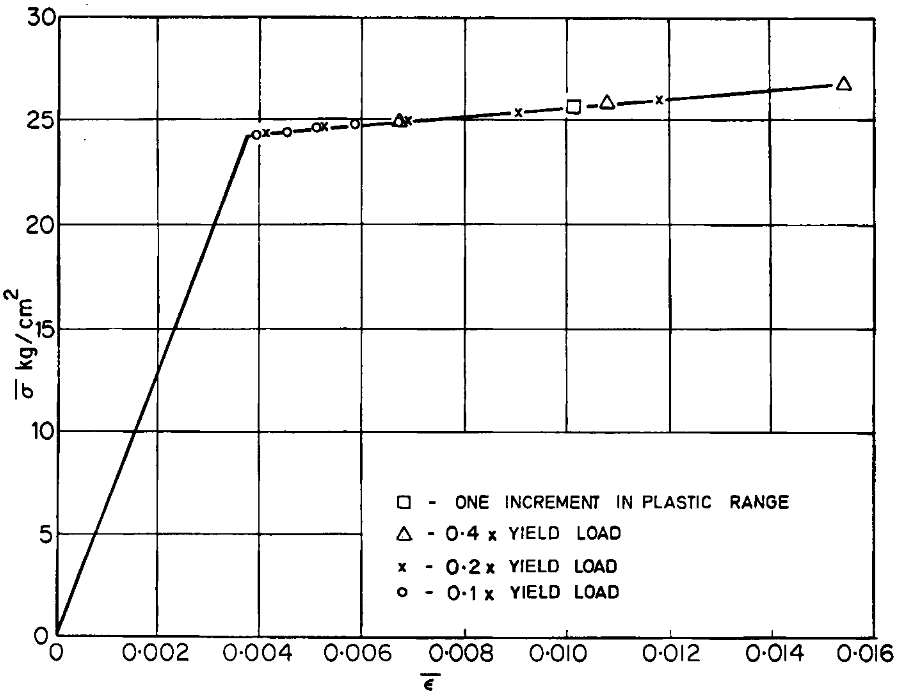

Figure 7. Perforated plate—strain hardening material
Effective stress versus effective strain in element which yielded first

Notched plane strain specimen—perfect plasticity

Figures 8 and 9 show some results for this case and compare the results with those previously obtained.[5] Once again the insensitivity to load increment size should be noticed.

Deep cantilever beam. Plane stress and ideal plasticity

The third example concerns the behaviour of a simple cantilever beam under a uniform load on its top surface. Figure 10 shows the rather coarse mesh used and the spread of the plastic zones.

The 'collapse' load was estimated by conventional beam theory with a 'hinge' developing at the support.

Plastic zones (Figure 10) and deflections (Figure 11) are shown for different ratios of the actual to the 'collapse' load. When the load was incremented up to the full value *no convergence could be obtained* with full 20 cycles and it is presumed that the state of steadily increasing deformation was reached. The last point represents a good estimate of the lower bound of the collapse load.

Deep cantilever beam—load reversal

Up to now only a monotonic load increase was considered. To illustrate that the process is available for load cycling without any modification the load on the previous example was first removed from a particular increment and then entirely reversed.

Figure 12 shows the deflection-load path and Figure 13 the stress patterns. It is of interest to note the residual stress distribution—not dissimilar to that predicted by conventional beam theory.

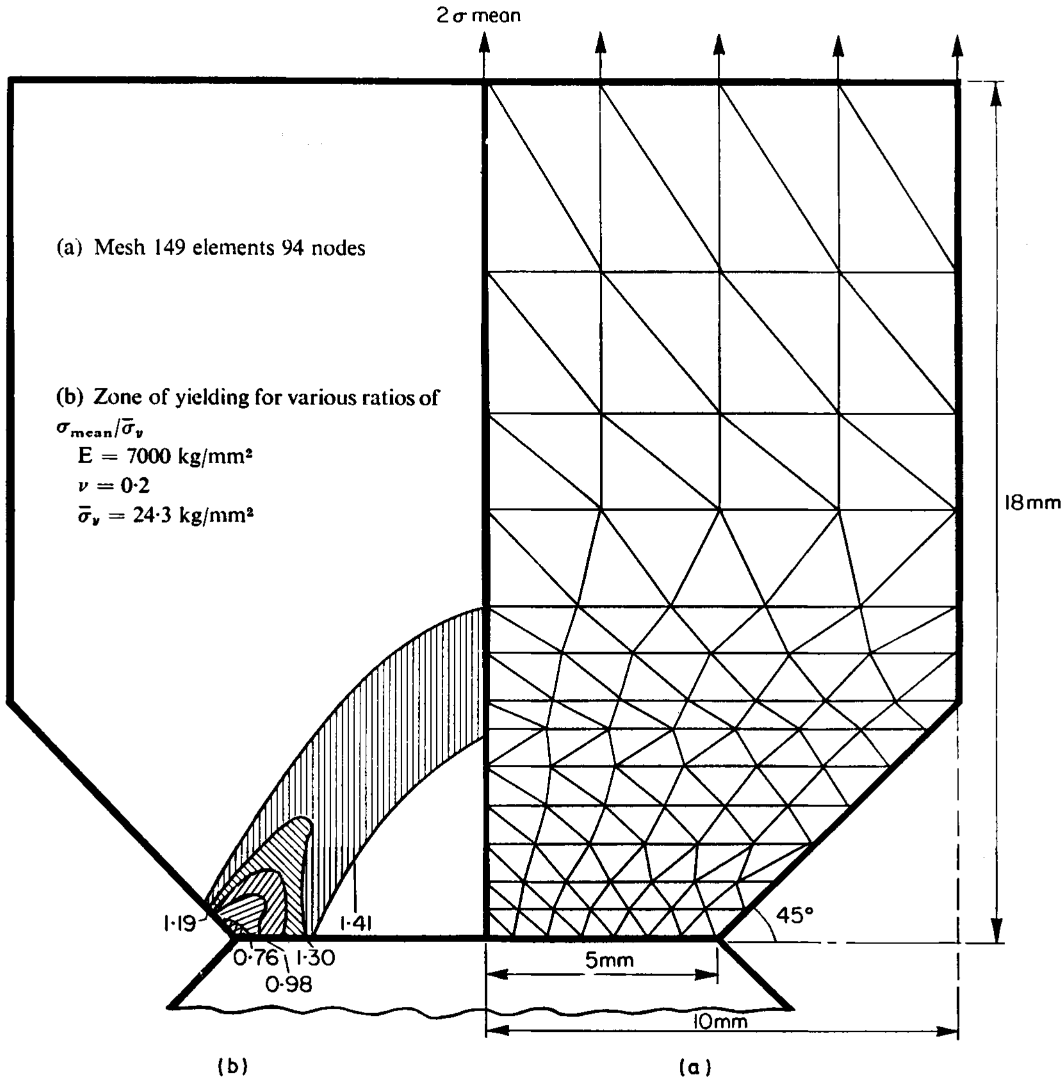

Figure 8. Notched specimen. Plane strain with ideal plasticity

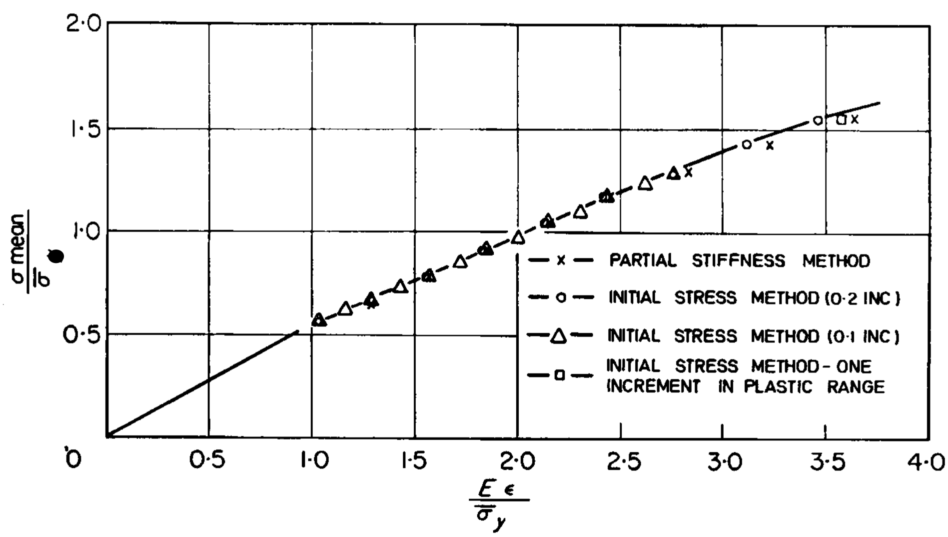

Figure 9. Notched specimen. Development of maximum strain for different load increments

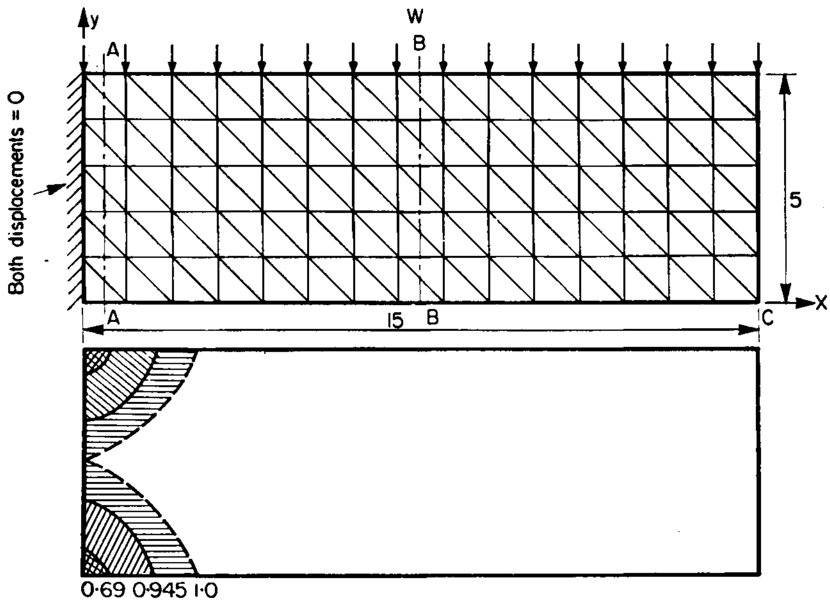

Figure 10. Cantilever Beam—Plane stress, ideal plasticity. The spread of plastic zones for different ratios of w/w_c when w_c is calculated as from plastic beam theory. w_c = collapse load

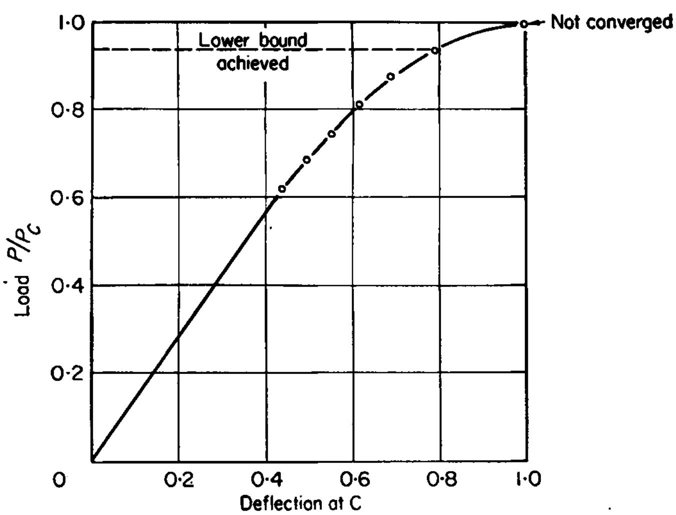

Figure 11. Cantilever beam
Deflection versus w/w_c

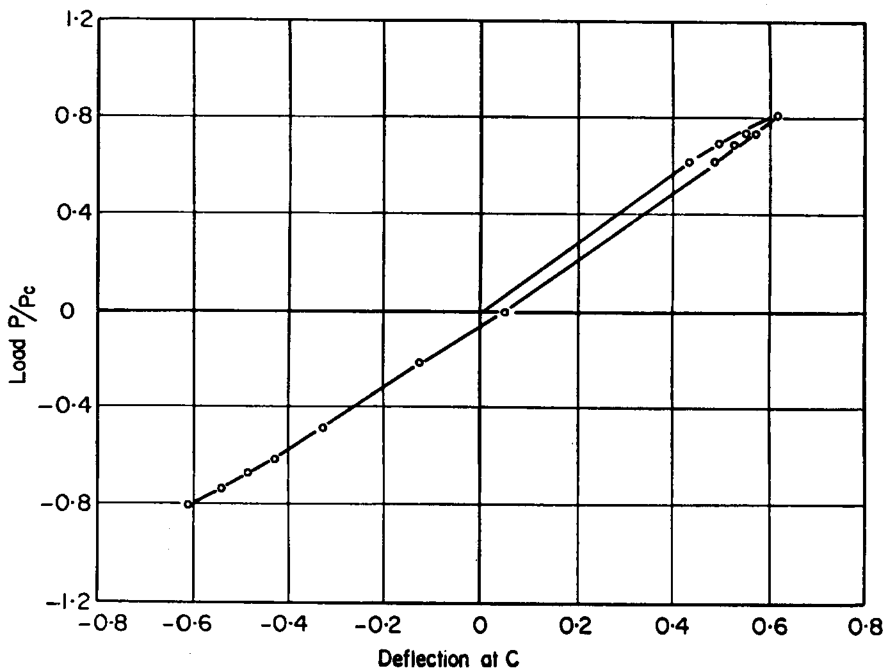

Figure 12. Cantilever beam
Deflections for load reversal

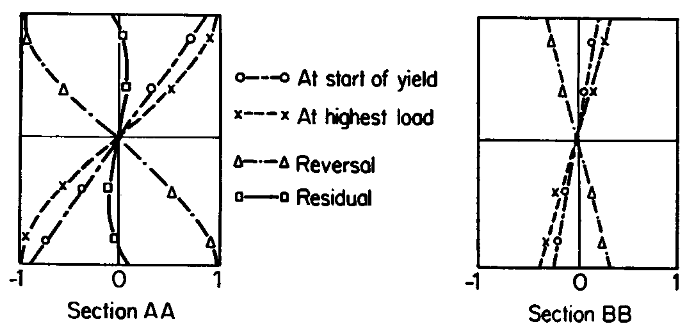

Figure 13. Cantilever beam
σ_x stress distribution at various stages of loading—unloading

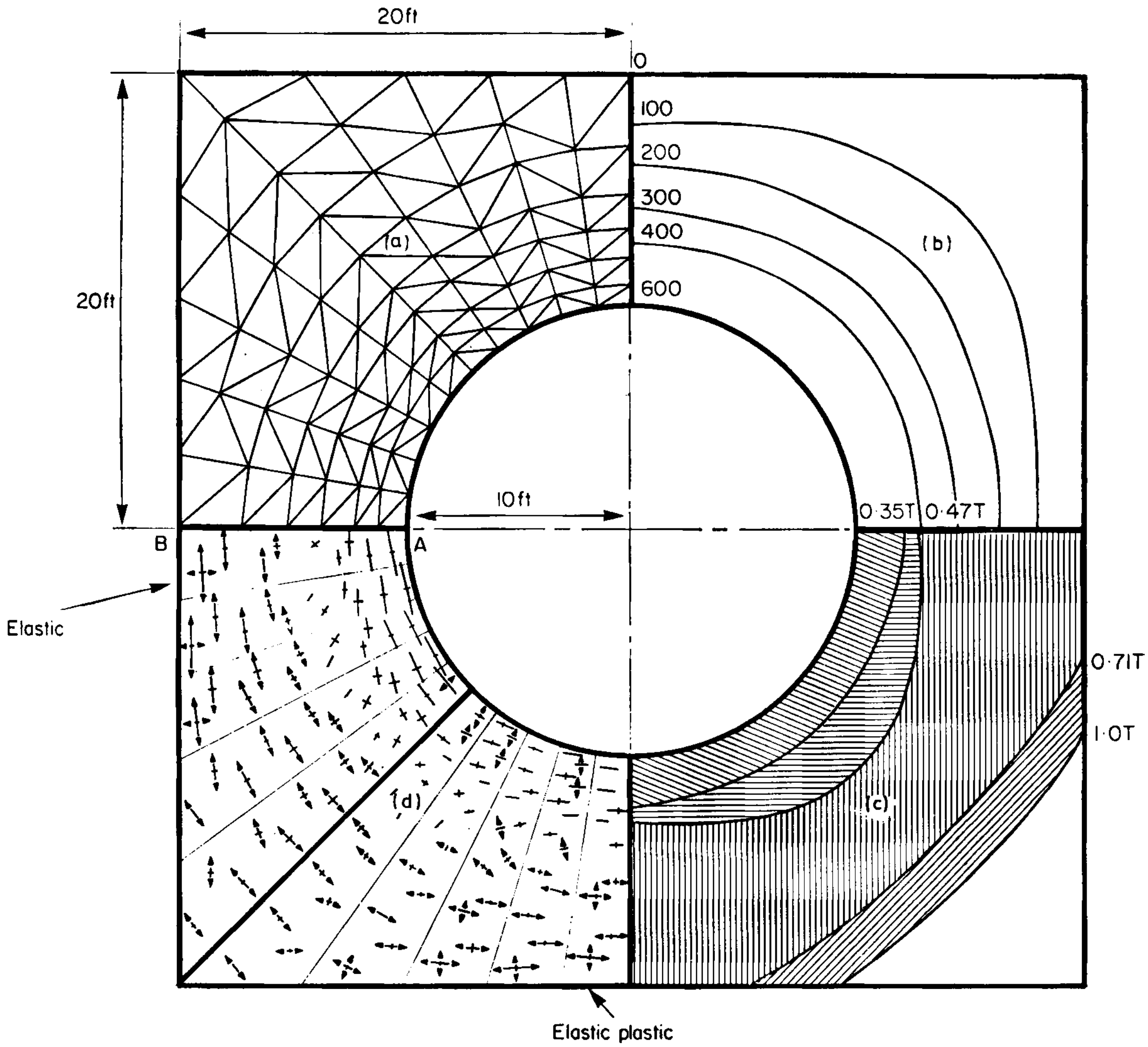

Figure 14. Thermal stress (plane strain)

(a) Mesh used 142 elements 88 nodes
$E = 30 \times 10^6$ lb/in^2
$\nu = 0{\cdot}3$
$\bar{\sigma}_y = 1{,}665{,}000$ lb/in^2
(b) Final temperature state
(c) Spread of plastic zones
(d) Stress distribution
Elastic and Elasto–plastic (both for same temperature state)

Thermo-plastic behaviour

To illustrate that no difficulties are encountered with thermo–plastic behaviour an illustrated problem is shown in Figure 14. A steady temperature shown is for simplicity assumed to be applied in several increments.

The yield stress was taken as temperature independent although inclusion of temperature effects on the yield surface would have caused no major difficulties.

PLANE STRAIN WITH COULOMB TYPE YIELD SURFACE—EXAMPLES

For concrete, rock and soil the yield surface depends not only on the deviatoric but also on the 'hydrostatic' stress components. The best known 'law' approximating to the true observed behaviour is that of Coulomb which specifies yield when the following relation between shear and normal stresses on any plane is reached.

$$\tau_{sn} = C + \sigma_n \tan \phi \tag{22}$$

A more 'manageable' form approximating to this relationship was suggested by Drucker[17] and can be written in the following form·

$$F = \alpha J_1 + J_2^{\frac{1}{2}} - K = 0 \tag{23}$$

where J_1 is the first invariant

$$J_1 = \sigma_x + \sigma_y + \sigma_z$$

and J_2 is the second invariant given as

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] + \sigma_4^2 + \sigma_5^2 + \sigma_6^2$$

in which α and K are constants depending on cohesion and friction of the material. The constants of equation (23) are related to the standard ones of equation (22) as

$$\alpha = \frac{\tan \phi}{\sqrt{(9 + 12 \tan^2 \phi)}} \qquad K = \frac{3C}{\sqrt{(9 + 12 \tan^2 \phi)}}$$

Other possible forms are discussed in some detail by Bishop[18] but for the present purposes the formulation given by Drucker will suffice.

It is a simple matter of algebra to evaluate the appropriate differentials of the yield surface $F = 0$ and use in the general formulation already discussed.

A word of warning is perhaps due at this stage. In the first place the yield surface of ideal non-hardening type has been assumed. This is not true in general for such materials as at first yield work hardening followed by work softening occurs in practice. The latter presents several difficulties[14] and will be dealt with in another publication. For practical purposes the approximation of ideal plasticity is valid providing the total strains developed are limited.

The second difficulty is that of the 'normality' principle built into the general program as described. This for a material with an appreciable value of ϕ implies a continuing volume increase in plastic deformation—a fact at variance with experimental evidence.[20] Alternative 'non-associated' flow rules could be adopted[21] but these are by no means verified.

Three problems of the category described are solved.

A circular underground opening

A circular excavation was studied under the same yield criterion by Reyes and Deere[6] by a rather elaborate 'incremental elasticity' approach. Figure 15 shows the finite element mesh and the spread of plastic zones. The opening is assumed so deep that uniform initial stresses

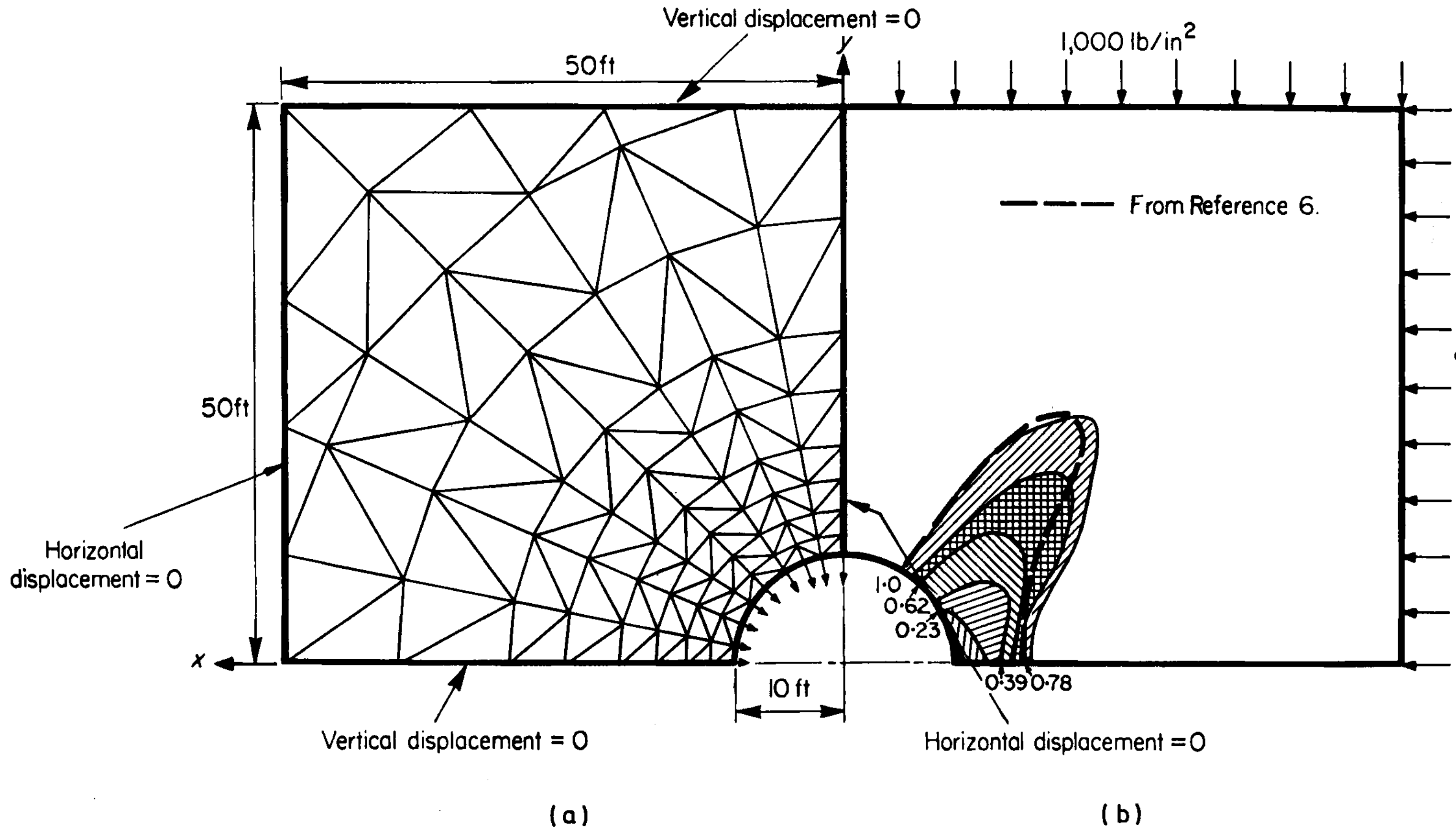

Figure 15. A circular underground opening

(a) Mesh and boundary condition 140 elements 87 nodes
Initial stress σ_{0x} = 250 lb/in^2 σ_{0y} = 1000 lb/in^2
E = 500,000 lb/in^2 ν = 0·2
c = 280 lb/in^2 ϕ = 30°
(b) Spread of plastic zones and comparison[6].

existed before excavation with horizontal component equal to 0·25 of the vertical. The spread of plastic zones is due to progressive removal of boundary loads on excavation.

Figure 16 shows the stress distribution resulting as well as a stress distribution which could have arisen under purely elastic conditions. Note the appreciable reduction of tensile stresses which now have to be limited to a value $C/\tan \phi$ by the yield criterion.

A lined tunnel

This example only differs from the previous in two respects. Firstly a more realistic tunnel shape is adopted and secondly the tunnel is lined with a material assumed to be elastic throughout. The excavation loads are 'externally supported' during the process of lining and any stresses in this are due to subsequent removal of the support.

Figures 17 and 18 show similar results to those of the previous two figures.

A strip foundation

Figure 19 illustrates the solution to this problem of a strip loading on a half space.

The spread of plastic zones can be observed and contrasted with the usual mechanism of failure assumption.

Bounds for the collapse load on this example have been estimated according to Finn[23] and are

Lower bound 18,040 lb.

Upper bound 46,300 lb.

In the computation the problem did not converge at a load of 23,000 lb., hence the highest estimate of the lower bound is given by the previous step, i.e. 20,000 lb.

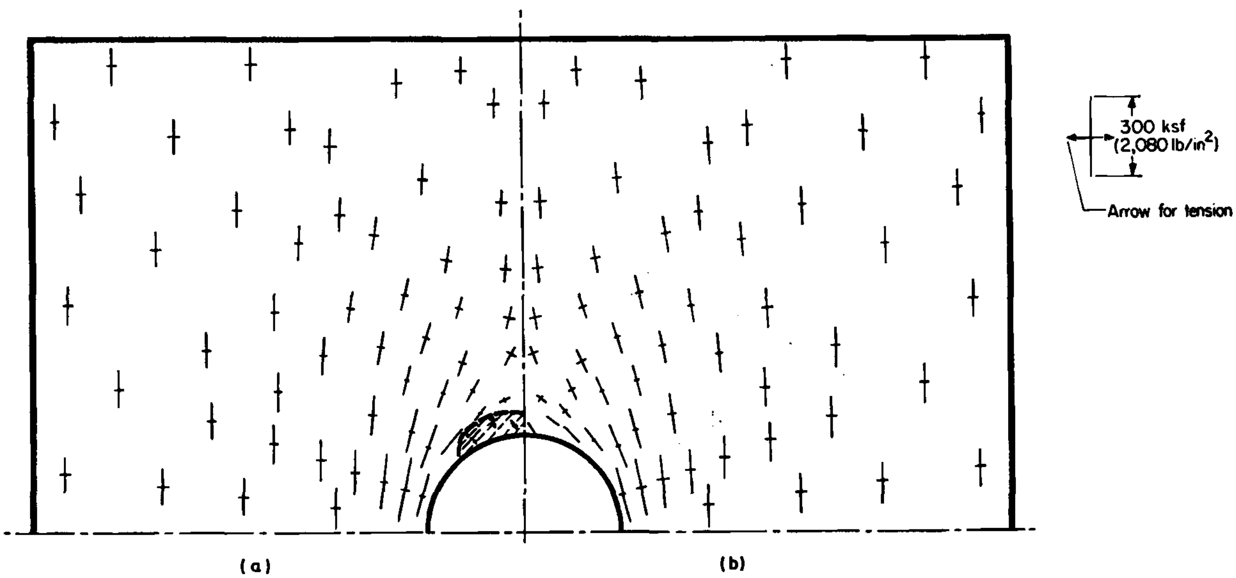

Figure 16. A circular underground opening

(a) Elastic stress distribution on full load
(b) Elasto-plastic stress distribution

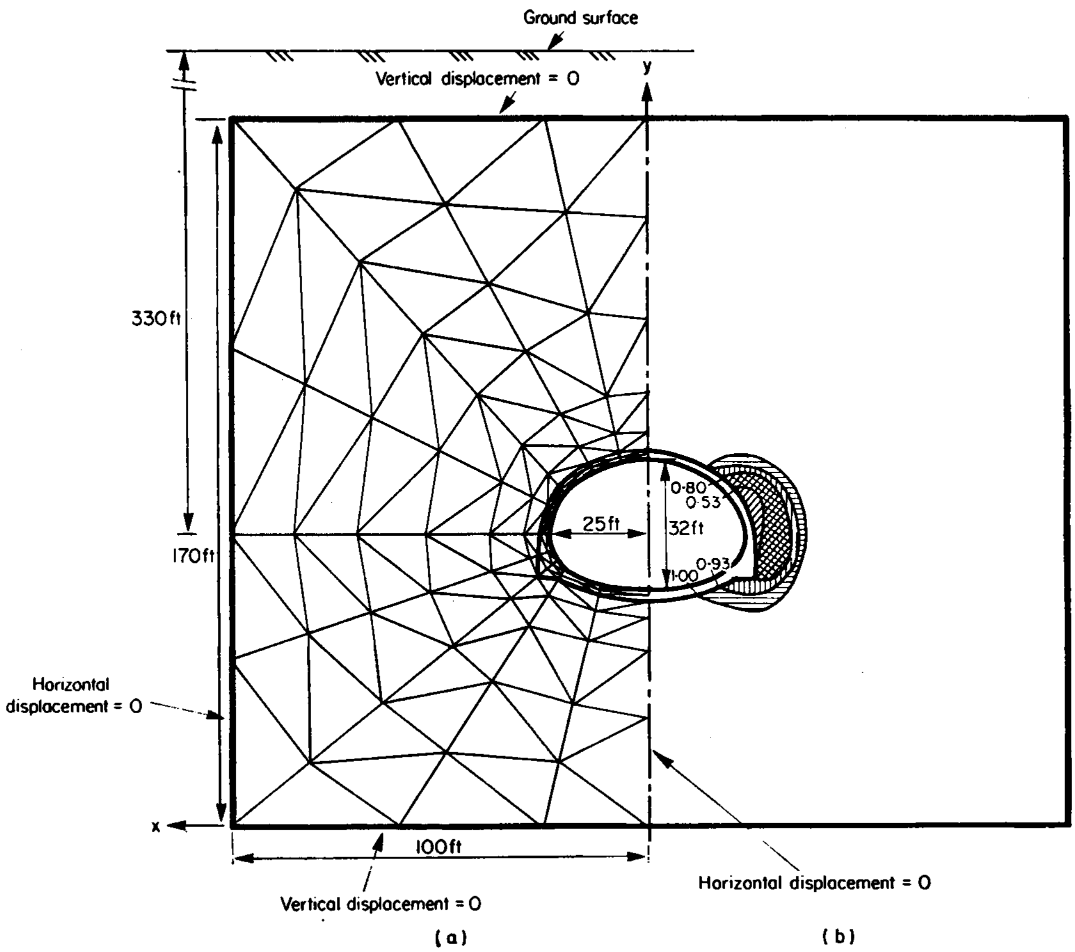

Figure 17. A lined tunnel

(a) Mesh 153 elements 94 nodes

Lining $E = 3 \times 10^6$ lb/in^2 $\nu = 0{\cdot}15$

Rock $E = 5 \times 10$ lb/in^2 $\nu = 0{\cdot}20$ $C = 140$ lb/in^2 $\phi = 30°$

Initial stress $\sigma_{yo} = \gamma h$ with origin of h at 400 ft.
$\sigma_{xo} = 0{\cdot}2\ \gamma h$ $\gamma = 150$ lbs/ft^3

(b) Spread of plastic zones

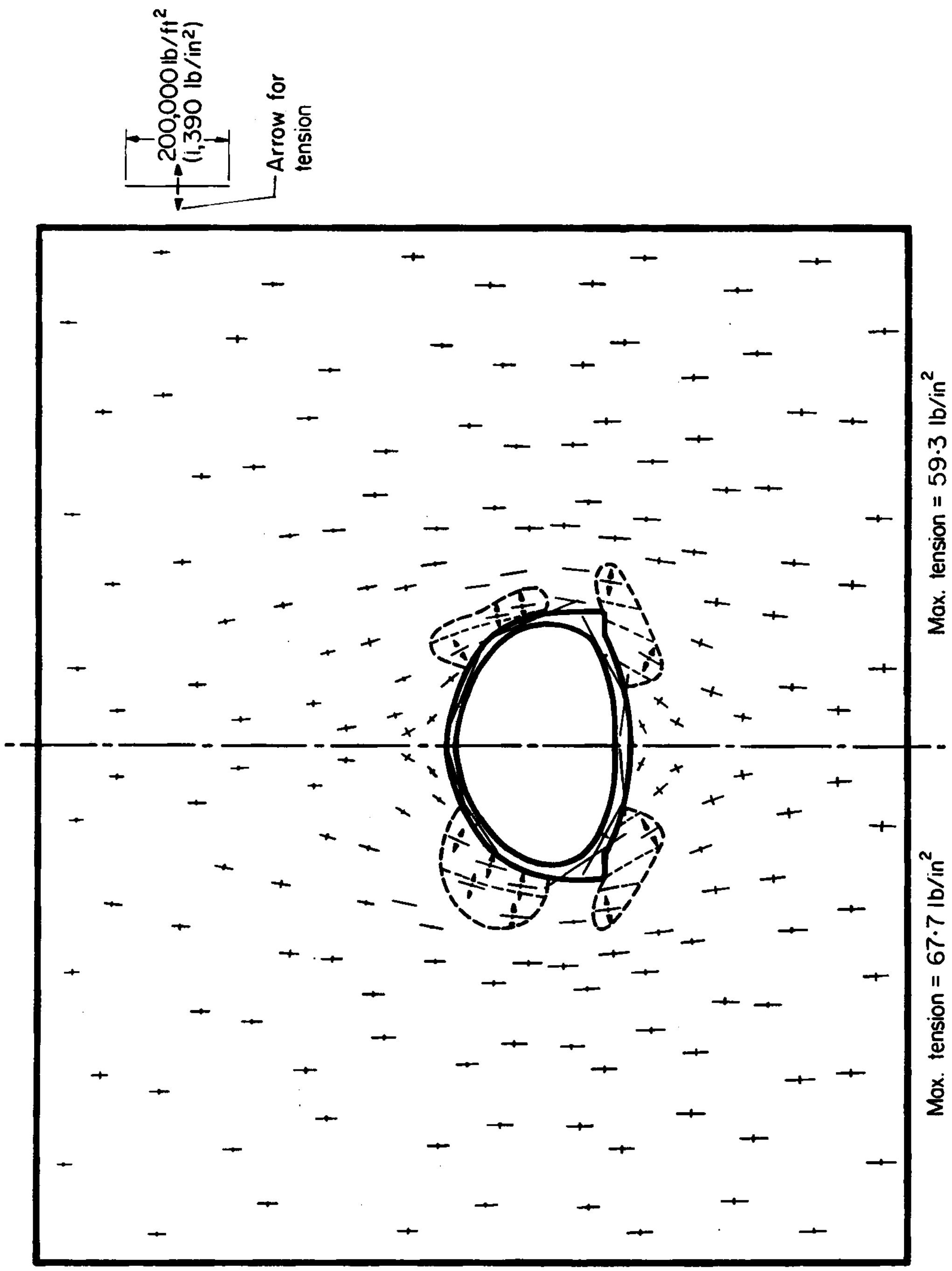

Figure 18. A lined tunnel. (a) Elastic stress distribution; (b) Elasto-plastic stress distribution

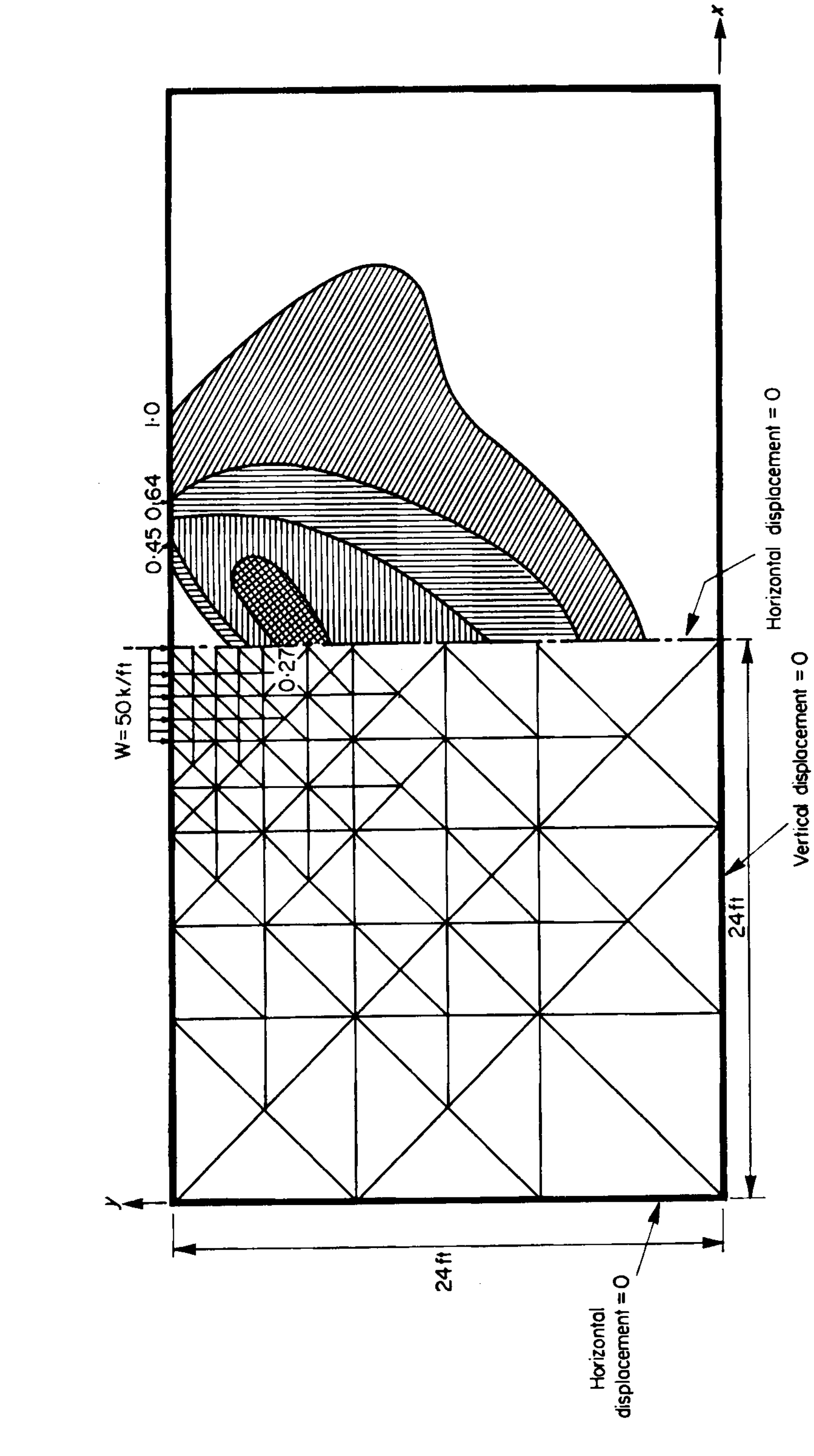

Figure 19. A strip foundation

(a) Mesh 134 elements 80 nodes
$E = 1 \times 10^6$ lb/in^2
$\nu = 0{\cdot}35$
$C = 7$ lb/in^2
$\phi = 35°$
(b) Spread of plastic zones

APPENDIX

Determination of nodal forces corresponding *to 'initial' stresses.*

The displacement vector $\{\mathbf{f}\}$ at any point of an element is defined in terms of its nodal displacements $\{\delta\}^e$ as

$$\{\mathbf{f}\} = [\mathbf{N}]\{\delta\}^e$$

where [N] are the chosen 'shape functions'.

Thus the total strains $\{\varepsilon\}$ will now also be defined by substitution into strain-displacement relations as

$$\{\varepsilon\} = [\mathbf{B}]\{\delta\}^e$$

If now a system of initial stresses $\{\Delta\sigma\}$ exists within an element and is to be balanced by a set of nodal forces $\{P\}_i^e$ we must have for any variation $\delta\{\delta\}^e$ the equality of internal and external work.

Thus

$$\{P_i\}^{e^T}\,\delta\{\delta\}^e = \int\{\Delta\sigma\}^T\,\delta\{\varepsilon\}\,\mathrm{d(vol.)} = \left[\int\left\{\Delta\{\sigma\}^T\,[\mathbf{B}]\,\mathrm{d(vol.)}\right\}\right]\,\delta\{\sigma\}^e$$

or

$$\{P_i\}^e = \int[\mathbf{B}]^T\,\Delta\{\sigma\}\,\mathrm{d(vol.)}$$

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