

Game Theory, Spring 2024

Lecture # 6

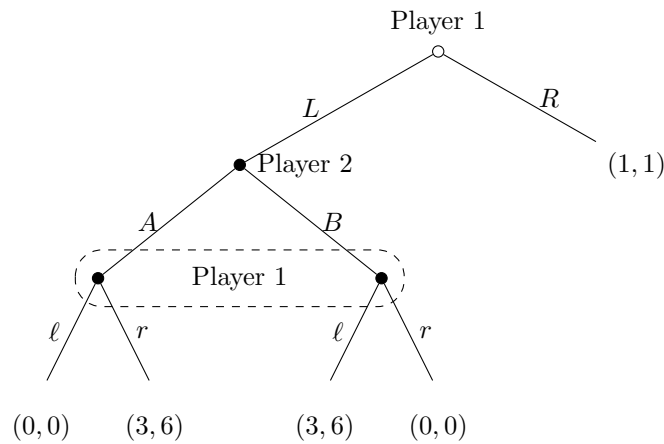
Daniil Larionov

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1 Mixed and behavior strategies

Example 4. Consider the following extensive-form game:



Recall that a pure strategy is a function $\sigma_i : I_i \mapsto \sigma_i(I_i) \in A(I_i)$ that maps an information set to an action available in this information set. In [Example 4](#), the set of pure strategies for player 1 is $S_1 = \{L\ell, Lr, R\ell, Rr\}$, where $L\ell$ stands for $\sigma_1(\{\emptyset\}) = L$ and $\sigma_1(\{LA, LB\}) = \ell$, and Lr stands for $\sigma_1(\{\emptyset\}) = L$ and $\sigma_1(\{LA, LB\}) = r$ etc. The definition of a mixed strategy is standard:

Definition 1 (Mixed strategy). A mixed strategy is a probability distribution over pure strategies.

In [Example 4](#), the following is a mixed strategy: $\frac{1}{4}L\ell + \frac{1}{4}Lr + \frac{1}{4}R\ell + \frac{1}{4}Rr$.

In extensive-form games, it is often more convenient to think about randomization in terms of behavior strategies:

Definition 2 (Behavior strategy). *A behavior strategy is a function that maps each information set into a probability distribution over the actions available at that information set, i.e. $\sigma_i : I_i \mapsto \sigma_i(I_i) \in \Delta(A(I_i))$.*

In [Example 4](#), the following is a behavior strategy:

$$\sigma_1(\{\emptyset\}) = \frac{2}{3}L + \frac{1}{3}R \text{ and } \sigma_1(\{LA, LB\}) = \frac{1}{2}\ell + \frac{1}{2}r.$$

Mixed and behavior strategies are equivalent in games of *perfect recall*.

1.1 Weak perfect Bayesian equilibria in mixed/behavior strategies

Let us find a weak perfect Bayesian equilibrium in mixed strategies in the game of [Example 4](#). Suppose player 1 believes that she is at history LA with probability μ and at history LB with probability $1 - \mu$. We will construct an equilibrium, in which player 1 randomizes between ℓ and r according to $p\ell + (1 - p)r$. The expected payoffs of player 1 are:

$$\begin{aligned} \ell : 0\mu + 3(1 - \mu) &= 3(1 - \mu), \\ r : 3\mu + 0(1 - \mu) &= 3\mu. \end{aligned}$$

By indifference, we have $3(1 - \mu^*) = 3\mu^*$, hence $\mu^* = \frac{1}{2}$. Suppose the information set $\{LA, LB\}$ is reached with positive probability. Bayes' rule then implies that player 2 plays $\frac{1}{2}A + \frac{1}{2}B$. Player 2 therefore has to be indifferent between A and B :

$$\begin{aligned} \ell : 0p + 6(1 - p) &= 6(1 - p), \\ r : 6p + 0(1 - p) &= 6p. \end{aligned}$$

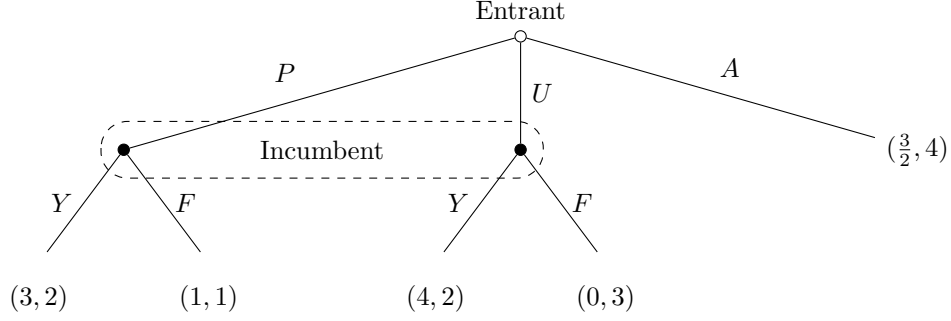
By indifference we have $6(1 - p^*) = 6p^*$, hence $p^* = \frac{1}{2}$.

If player 1 plays L , her expected payoff is $3\mu^* = 1.5$, which is higher than the payoff from R , hence player 1 plays L and the information set $\{LA, LB\}$ is

indeed achieved with positive probability, this the following is a weak perfect Bayesian equilibrium:

$$\left(\sigma_1(\{\emptyset\}) = L, \sigma_1(\{LA, LB\}) = \frac{1}{2}\ell + \frac{1}{2}r, \sigma_2(\{L\}) = \frac{1}{2}A + \frac{1}{2}B; \mu^* = \frac{1}{2} \right).$$

Example 5. Consider the following extensive-form game:



Let us determine the weak perfect Bayesian equilibria of the game in [Example 5](#). Suppose the incumbent believes that she is at history P with probability μ and at history U with probability $1 - \mu$. The expected payoffs of the incumbent are then given by:

$$Y : 2\mu + 2(1 - \mu) = 2,$$

$$F : 1\mu + 3(1 - \mu) = 3 - 2\mu.$$

It is optimal to choose Y whenever $\mu \geq \frac{1}{2}$, and vice versa. Iff $\mu = \frac{1}{2}$, the incumbent is indifferent between Y and F . We consider three cases.

Case 1: the incumbent plays Y , hence $\mu^* \geq \frac{1}{2}$. In this case, the entrant will play U and the information set $\{P, U\}$ will be reached with probability 1. Bayes' rule then implies $\mu^* = 0$, which is a contradiction, hence there is no such weak perfect Bayesian equilibrium.

Case 2: the incumbent plays F , hence $\mu^* \leq \frac{1}{2}$. In this case, the entrant will play A and the information set $\{P, U\}$ will be reached with probability 0, hence $((A, F), \mu^* \in [0, \frac{1}{2}])$ are weak perfect Bayesian equilibria.

Case 3: the incumbent randomizes according to $pY + (1 - p)F$, hence $\mu^* = \frac{1}{2}$.

The expected utilities of the entrant are given by:

$$P : 3p + 1(1 - p) = 2p + 1,$$

$$U : 4p + 0(1 - p) = 4p,$$

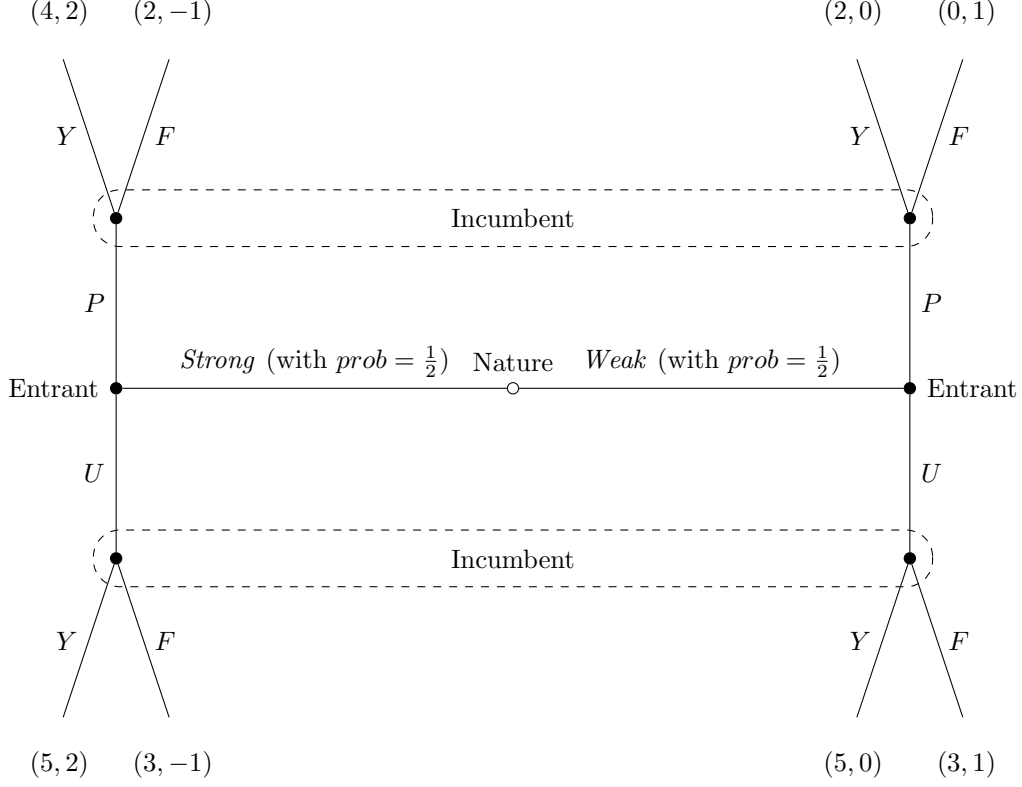
$$A : \frac{3}{2}.$$

We distinguish two subcases:

- **Case 3.1:** the information set $\{P, U\}$ is reached with positive probability. Bayes' rule then implies that the entrant plays $qP + qU + (1 - 2q)A$ for some $q > 0$, hence the entrant has to be indifferent between P and U , which is guaranteed whenever $2p^* + 1 = 4p^*$ or $p^* = \frac{1}{2}$ with the resulting payoff of 2, which exceeds the payoff from A , implying that $q^* = \frac{1}{2}$. $((\frac{1}{2}P + \frac{1}{2}U, \frac{1}{2}Y + \frac{1}{2}F); \mu^* = \frac{1}{2})$ is a weak perfect Bayesian equilibrium.
- **Case 3.2:** the information set $\{P, U\}$ is reached with probability 0. The entrant then plays A . It is optimal for the entrant to play A whenever $\frac{3}{2} \geq 2p^* + 1$ and $\frac{3}{2} \geq 8p^*$, which is equivalent to $p^* \leq \frac{1}{4}$. Hence for every $p^* \in [0, \frac{1}{4}]$ the following is a weak perfect Bayesian equilibrium: $((L, p^*Y + (1 - p^*)F); \mu^* = \frac{1}{2})$.

2 Signaling games

Example 6. Consider the following signaling game:



The formal definition of the game in [Example 6](#) is as follows:

Definition 3. The signaling game in [Example 6](#) consists of the following:

1. *Players:* $\mathcal{N} = \{\text{Entrant}, \text{Incumbent}\}$.
2. *Histories:* $\mathcal{H} = \{\emptyset, S, W, SP, SPY, SPF, SU, SUY, SUF, WP, WPY, WPF, WU, WUY, WUF\}$.
Terminal histories: $\mathcal{Z} = \{SPY, SPF, SUY, SUF, WPY, WPF, WUY, WUF\}$.
3. *Player function:* $\mathcal{P} : \mathcal{H} \setminus \mathcal{Z} \mapsto \mathcal{N} \cup \{\text{Nature}\}$.

$$\mathcal{P}(\emptyset) = \text{Nature},$$

$$\mathcal{P}(S) = \mathcal{P}(W) = \text{Entrant},$$

$$\mathcal{P}(SP) = \mathcal{P}(SU) = \mathcal{P}(WP) = \mathcal{P}(WU) = \text{Incumbent}.$$

4. *Exogenous uncertainty:* for every h such that $\mathcal{P}(h) = \text{Nature}$, we need to specify $f(\cdot|h) \in \Delta(A(h))$. Here we have $f(S|\emptyset) = f(W|\emptyset) = \frac{1}{2}$.

5. Collections of information sets for each player: $\mathcal{I}_{\text{Entrant}} = \{\{S\}, \{W\}\}$ and $\mathcal{I}_{\text{Incumbent}} = \{\{SU, WU\}, \{SP, WP\}\}$.
6. Payoff functions $u_i : \mathcal{Z} \rightarrow \mathbb{R}$, which map terminal histories to payoff for each player $i \in \mathcal{N}$ (see the game tree for the payoffs).

2.1 Separating equilibria

In a separating equilibrium, different types take different actions. Observe that the weak type will never play P , hence we are looking for a separating equilibrium, in which the weak type plays U and the strong type plays P . Since both information sets are reached with positive probability, the beliefs at both information sets are derived via Bayes' rule: $\mu^*(\text{Strong}|P) = \mu^*(\text{Weak}|U) = 1$. If the incumbent observes P , then her optimal response is Y . If the incumbent observes U , then her optimal response is F . The entrant has no profitable deviations: the weak type never plays P ; if the strong type deviates to U , the incumbent will play F in response, and the game will end up at SUF with the payoff of 3 for the strong type as opposed to the payoff of 4 from playing P . Hence the following is a weak perfect Bayesian equilibrium:

$$\left(\sigma_E(W) = U, \sigma_E(S) = P, \sigma_I(\{SU, WU\}) = F, \sigma_I(\{SP, WP\}) = Y; \mu^*(\text{Strong}|P) = \mu^*(\text{Weak}|U) = 1 \right)$$

2.2 Pooling equilibria

In a pooling equilibrium, all types take the same action. Since the weak type never plays P , we are looking for pooling equilibria, in which both types play U . Since both types play U , the information set $\{SU, WU\}$ is reached with positive probability, and the beliefs at this information set are derived via Bayes' rule: $\mu^*(\text{Strong}|U) = \mu^*(\text{Weak}|U) = \frac{1}{2}$. The expected payoffs of the incumbent at $\{SU, WU\}$ are

$$\begin{aligned} Y : 2\frac{1}{2} + 0\frac{1}{2} &= 1, \\ F : -1\frac{1}{2} + \frac{1}{2}1 &= 0. \end{aligned}$$

The incumbent will therefore choose Y . The entrant has no profitable deviations: the weak never plays P , and the strong type gets 5 at SUY , which is the highest possible payoff for the entrant in this game.

It remains to determine the behavior and the beliefs of the incumbent at the information set $\{SP, WP\}$. Let $\mu^* \equiv \mu^*(\text{Strong}|P)$, the expected payoffs of the

incumbent are:

$$\begin{aligned} Y : 2\mu^* + 0(1 - \mu^*) &= 2\mu^*, \\ F : -1\mu^* + \frac{1}{2}(1 - \mu^*) &= 1 - 2\mu^*. \end{aligned}$$

It is optimal for the incumbent to choose Y for $\mu^* \in [\frac{1}{4}, 1]$ and vice versa. Thus we get two kinds of pooling equilibria:

$$\begin{aligned} &\left(\sigma_E(W) = \sigma_E(S) = U, \sigma_I(\{SU, WU\}) = Y, \sigma_I(\{SP, WP\}) = Y; \mu^*(Strong|U) = \frac{1}{2}, \mu^*(Strong|P) \in [\frac{1}{4}, 1] \right), \\ &\left(\sigma_E(W) = \sigma_E(S) = U, \sigma_I(\{SU, WU\}) = Y, \sigma_I(\{SP, WP\}) = F; \mu^*(Strong|U) = \frac{1}{2}, \mu^*(Strong|P) \in [0, \frac{1}{4}] \right). \end{aligned}$$

2.3 Semi-separating equilibria

We will construct a semi-separating equilibrium, in which the weak type plays U (note that the weak type will never play P , and hence cannot mix) and the strong plays $pP + (1 - p)U$ for some $0 < p < 1$. The beliefs of the incumbent are as follows:

$$\begin{aligned} \mu^*(Strong|P) &= 1 \\ \mu^*(Strong|U) &= \frac{\text{prob}(U|Strong)\text{prob}(Strong)}{\text{prob}(U|Strong)\text{prob}(Strong) + \text{prob}(U|Weak)\text{prob}(Weak)} \\ &= \frac{(1 - p)\frac{1}{2}}{(1 - p)\frac{1}{2} + 1\frac{1}{2}} = \frac{1 - p}{2 - p}. \end{aligned}$$

We therefore have $\mu^*(Strong|U) = 1 - \frac{1-p}{2-p} = \frac{1}{2-p}$.

Let's consider the actions of the incumebent. If the incumbent observes P , then the incumbent will believe that the entrant's type is *Strong*, and will choose Y . Suppose that then incumbent plays $qY + (1 - q)F$ after observing U . The entrant mixes between P and U , and therefore has to be indifferent between P and U :

$$\begin{aligned} P : 4, \\ U : 5q + 3(1 - q) &= 2q + 3. \end{aligned}$$

From indifference, we get $q^* = \frac{1}{2}$, hence the incumbent has to be indifferent between

Y and F :

$$\begin{aligned} Y &: \frac{1-p}{2-p}2 + \frac{1}{2-p}0 = \frac{2-2p}{2-p}, \\ F &: \frac{1-p}{2-p}(-1) + \frac{1}{2-p}1 = \frac{p}{2-p}. \end{aligned}$$

From indifference, we have $p^* = \frac{2}{3}$, hence $\mu^*(Strong|U) = \frac{1-p^*}{2-p^*} = \frac{1-2/3}{2-2/3} = \frac{1}{4}$.

We have constructed the following weak perfect Bayesian equilibrium:

$$\begin{aligned} \left(\sigma_E(W) = U, \sigma_E(S) = \frac{2}{3}P + \frac{1}{3}U, \sigma_I(\{SU, WU\}) = \frac{1}{2}Y + \frac{1}{2}F, \sigma_I(\{SP, WP\}) = Y; \right. \\ \left. \mu^*(Strong|P) = 1, \mu^*(Strong|U) = \frac{1}{4} \right). \end{aligned}$$