Game Theory, Spring 2024

Lecture # 1

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Review of Nash equilibria

Definition 1 (**Strategic form game**). *A strategic form game is given by:*

1. *Players* $i \in \mathcal{I} = \{1, \ldots, I\}$,
2. *Actions* $a_i \in A_i$ *for each player* $i \in \mathcal{I}$,
3. *Payoffs* $u_i(a_i, a_{-i})$ *for each player* $i \in \mathcal{I}$.

Example 1. *Consider the following strategic form game:*

	T	B
T	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

In Example 1 we have:

1. Players: $\mathcal{I} = \{1, 2\}$,
2. Actions: $A_1 = A_2 = \{1, 2\}$,

Definition 2 (**Nash equilibrium in pure strategies**). *An action profile* $(a_1^*, \ldots, a_I^*)$ *is a Nash equilibrium in pure strategies if for all players* $i \in \mathcal{I}$ *we have*

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i', a_{-i}^*) \ \forall a_i' \in A_i.$$

In Example 1 (T, T) and (B, B) are both Nash equilibria in pure strategies.

Example 2. *Consider the following strategic form game:*

	T	B
T	$2, 0$	$0, 2$
B	$0, 1$	$1, 0$

In Example 2 there are no Nash equilibria in pure strategies, which motivates the introduction of mixed strategies.

Definition 3 (**Mixed strategy**)**.** *A mixed strategy σ_i of player i is a probability distributions over player i's actions, $\sigma_i \in \Delta(A_i)$.*

If the players play a profile of mixed strategies $(\sigma_i, \dots, \sigma_I)$, then we can write the payoff of player i as follows:

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{a \in A} \big[\sigma_1(a_1) \times \cdots \times \sigma_I(a_I)\big] u_i(a)$$

Definition 4 (**Nash equilibrium in mixed strategies**)**.** *A mixed strategy profile $(\sigma_1^*, \dots, \sigma_I^*)$ is a Nash equilibrium in mixed strategies if for all players $i \in \mathcal{I}$ we have*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(a_i', \sigma_{-i}^*) \ \forall a_i' \in A_i.$$

This definition almost immediately implies the following

Claim 1. *Suppose σ_i^* is an equilibrium strategy of player i. If $\sigma_i^*(a_i) > 0$ and $\sigma_i^*(a_i') > 0$, then $u_i(a_i, \sigma_{-i}^*) = u_i(a_i', \sigma_{-i}^*)$, or, in words, if player i randomizes between a_i and a_i', then player i has to be indifferent between a_i and a_i'.*

We can use this indifference property to look for a mixed Nash equilibrium in Example 2. Suppose player 1 mixes according to $pT + (1-p)B$, with $0 < p < 1$, then player 1 has to be indifferent between T and B:

$$\begin{aligned} T:&\ 2q + 0(1-q) = 2q, \\ B:&\ 0q + 1(1-q) = 1-q. \end{aligned}$$

Player 1 is indifferent whenever $2q = 1 - q$ or $q = \frac{1}{3}$. If player 2 mixes according to $qT + (1-q)B$, then player 2 has to be indifferent between T and B:

$$\begin{aligned} T: &\ 0p + 1(1-q) = 1 - p, \\ B: &\ 2q + 0(1-q) = 2p. \end{aligned}$$

Player 2 is indifferent whenever $1-p = 2p$ or $p = \frac{1}{3}$. We conclude that $\left(\frac{1}{3}T + \frac{2}{3}B, \frac{1}{3}T + \frac{2}{3}B\right)$ is a Nash equilibrium in mixed strategies in Example 2.

Bayesian games

Definition 5 (Bayesian game). *A Bayesian game (game of incomplete information) is given by:*

1. *Players* $i \in \mathcal{I} = \{1, \ldots, I\}$,
2. *Actions* $a_i \in A_i$ *for each player* $i \in \mathcal{I}$,
3. *Types* $\theta_i \in \Theta_i$ *for each player* $i \in \mathcal{I}$,
4. *A probability distribution over type profiles* $p(\theta_i, \theta_{-i})$,
5. *Payoffs* $u_i(a_i, a_{-i})$ *for each player* $i \in \mathcal{I}$.

Example 3. *Consider the following Bayesian game and suppose that the types of player 2 are equally likely.*

	θ_2^1			θ_2^2	
	T	B		T	B
T	$2,1$	$0,0$	T	$2,0$	$0,2$
B	$0,0$	$1,2$	B	$0,1$	$1,0$

In Example 3 we have:

1. Players $\mathcal{I} = \{1, 2\}$,
2. Actions: $A_1 = A_2 = \{T, B\}$,
3. Types $\Theta_1 = \{\theta_1^1\}$, $\Theta_2 = \{\theta_2^1, \theta_2^2\}$,
4. Probability distribution over type profiles: $p(\theta_1^1, \theta_2^1) = p(\theta_1^1, \theta_2^2) = \frac{1}{2}$,

Definition 6 (**Bayesian strategy**)**.** *A (mixed) Bayesian strategy is a function* $\sigma_i : \Theta_i \to \Delta(A_i)$*, which maps player* i*'s type into a probability distribution over player* i*'s actions.*

Definition 7 (**Bayesian Nash equilibrium**)**.** *A Bayesian strategy profile* $(\sigma_1^*, \ldots, \sigma_I^*)$ *is a Bayesian Nash equilibrium (BNE) if for all players* $i \in \mathcal{I}$ *we have*

$$\sum_{\theta \in \Theta} p(\theta_i, \theta_{-i}) u_i\big(\sigma_i^*(\theta_i), \sigma_i^*(\theta_{-i})\big) \geq \sum_{\theta \in \Theta} p(\theta_i, \theta_{-i}) u_i\big(\sigma_i'(\theta_i), \sigma_i^*(\theta_{-i})\big) \ \forall \sigma_i'.$$

Let us go back to Example 3 and identify its Bayesian Nash equilibria.

	θ_2^1			θ_2^2	
	q_1 T	$(1-q_1)$ B		q_2 T	$(1-q_2)$ B
p T	$2, 1$	$0, 0$	p T	$2, 0$	$0, 2$
$(1-p)$ B	$0, 0$	$1, 2$	$(1-p)$ B	$0, 1$	$1, 0$

1. *BNE in pure strategies.* Observe that the best response of player 2 to T is TB, and the best response of player 2 to B is BT, hence only TB and BT could be pure equilibrium strategies for player 2. Suppose player 2 plays TB, player 1 then gets

$$\begin{aligned} \text{from T}: \ & \frac{1}{2}2 + \frac{1}{2}0 = 1, \\ \text{from B}: \ & \frac{1}{2}0 + \frac{1}{2}1 = \frac{1}{2}, \end{aligned}$$

which means that T is the best response to TB, implying that (T, TB) is a Bayesian Nash equilibrium. Now suppose player 2 plays BT, player 1 then gets:

$$\begin{aligned} \text{from T}: \ & \frac{1}{2}0 + \frac{1}{2}2 = 1, \\ \text{from B}: \ & \frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}, \end{aligned}$$

which means that T is also the best response to BT, and thus there are no other BNE in pure strategies.

2. *BNE in mixed strategies.* Observe first that there is no BNE, in which player 1 plays pure. If player 1 plays pure, then the best response of player 2 is to also

play pure, hence we will be looking at equilibria, in which player one randomizes according to $pT + (1-p)B$. Player 1 then is indifferent between T and B:

$$\begin{aligned} T: &\ \frac{1}{2}\big[2q_1 + 0(1-q_1)\big] + \frac{1}{2}\big[2q_2 + 0(1-q_2)\big] = q_1 + q_2, \\ B: &\ \frac{1}{2}\big[0q_1 + 1(1-q_1)\big] + \frac{1}{2}\big[0q_2 + 1(1-q_2)\big] = 1 - \frac{1}{2}(q_1 + q_2). \end{aligned}$$

Player 1 is indifferent whenever $q_1 + q_2 = 1 - \frac{1}{2}(q_1 + q_2)$, i.e. whenever $q_1 + q_2 = \frac{2}{3}$, which implies that at least one of the types of player 2 mixes between T and B. Consider two cases:

Case 1: suppose type θ_2^1 mixes between T and B, then type θ_2^1 must be indifferent between T and B:

$$\begin{aligned} T: &\ 1p + 0(1-p) = p, \\ B: &\ 0p + 2(1-p) = 2 - 2p. \end{aligned}$$

Type θ_2^1 is indifferent whenever $p = 2 - 2p$, i.e. whenever $p = \frac{2}{3}$.

Case 2: suppose type θ_2^2 mixes between T and B, then type θ_2^2 must be indifferent between T and B:

$$\begin{aligned} T: &\ 0p + 1(1-p) = 1 - p, \\ B: &\ 1p + 0(1-p) = 2p. \end{aligned}$$

Type θ_2^2 is indifferent whenever $1 - p = 2p$, i.e. whenever $p = \frac{1}{3}$.

Observe that both types of player 2 cannot mix at the same time (that would require the same value of p for both types, which it is not). Suppose then that we are in **Case 1**, i.e. that type θ_2^1 mixes between T and B, and $p = \frac{2}{3}$, i.e. player 1 plays $\frac{2}{3}T + \frac{1}{3}B$. Since type θ_2^2 is not indifferent between T and B, we either have $q_2 = 0$ or $q_2 = 1$, but we must have $q_2 = 0$ to satisfy $q_1 + q_2 = \frac{2}{3}$. It implies that $q_1 = \frac{2}{3}$, i.e. type θ_2^1 plays $\frac{2}{3}T + \frac{1}{3}B$. $q_2 = 0$ means that type θ_2^2 plays B, so we need to check that B is a best response for type θ_2^2. The payoff of type θ_2^2 from playing B is 4/3, and the payoff of type θ_2^2 from playing T is 1/3, implying that B is indeed a best response to $\frac{2}{3}T + \frac{1}{3}B$. $\left[\frac{2}{3}T + \frac{1}{3}B, \left(\frac{2}{3}T + \frac{1}{3}B, B\right)\right]$ is therefore a Bayesian Nash equilibrium. The analysis of **Case 2** is left to you as an exercise.