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Technicolor and other QCD-like theories at next-to-next-to-leading order

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Abstract

We calculate the vacuum-expectation-value, the meson mass and the meson decay constant to next-to-next-to-leading-order in the chiral expansion for QCD-like theories with general N_F degenerate flavours for the cases with a complex representation, a real and a pseudoreal representation, i.e. with global symmetry and breaking patters $SU(N_F)_L \times SU(N_F)_R \to SU(N_F)_V$, $SU(2N_F) \to SO(2N)$ and $SU(2N_F) \to Sp(2N_F)$. These calculations should be useful for lattice calculations for dynamical electroweak symmetry breaking and related cases.

1 Introduction

Chiral Perturbation Theory (ChPT) [1, 2, 3] as effective field theory (EFT) for QCD is a very well established method within strong interaction phenomenology. The same method can also be used for different symmetry pattern cases. These can be of interest for theories beyond the standard model. Early papers in this context are the technicolor variations discussed in [4, 5, 6]. Recently lattice calculations have started to explore some of these cases, some recent references are [7]. While one is primarily interested in these theories in the massless limit, lattice calculations are performed with finite masses and the results thus need to extrapolated to zero mass. For these extrapolations EFT is an excellent tool and it is heavily used in fitting results for the pseudoscalar meson octet in the QCD case. For high precision fits it is needed there to go to next-to-next-to-leading-order in the ChPT expansion.

When writing the EFT relevant for dynamical electroweak symmetry breaking one needs to consider different patterns of spontaneous breaking of the global symmetry than in QCD. The resulting set of Goldstone Bosons, or pseudo-Goldstone bosons in the presence of mass terms, is thus also different. The low-energy EFT is thus also different.

In this paper we only discuss cases where the underlying strong interaction is vectorlike and all fermions have the same mass. Here three main patterns of global symmetry show up. A thorough discussion of these cases at tree level or lowest order (LO) is [8]. With a gauge group with N_F fermions in a complex representation we have a global symmetry group $SU(N_F)_L \times SU(N_F)_R$ and we expect this to be spontaneously broken to the diagonal subgroup $SU(N_F)_V$. This is the direct extension of the QCD case. For N_F fermions in a real representation the global symmetry group becomes $SU(2N_F)$ and is expected to be spontaneously broken to $SO(2N_F)$. In the case of two colours and N_F fermions in the fundamental (pseudoreal) representation the global symmetry group is again $SU(2N_F)$ but here is expected to be spontaneously broken to an $Sp(2N_F)$ subgroup. Some earlier references are [9, 10, 11]. The complex case was treated to next-to-leading-order (NLO) in [3] in general and for the quantities considered here in [12]. The pseudo-real case has been done to NLO in [13]. We repeat here both calculations and also extend the third, real, case to NLO by calculating the full infinity structure for all three cases at NLO and giving the NLO Lagrangian.

In addition we also go to NNLO for three explicit quantities, the vacuum-expectation-value, the meson mass and the meson decay constant in the equal mass case. These formulas are our main result. We expect that the NNLO Lagrangian for all cases will be a simple generalization of the one for the complex case given in [14] but the calculation of the general divergence structure, though in principle similar to the one in [15], we have not performed.

In the remainder of this paper we refer to the complex representation case as QCD, the real representation case as adjoint and the pseudo-real case as two-colour or $N_c = 2$. We first discuss in Sect. 2 the three different cases at the underlying fermion (quark) level. Here we introduce explicit external fields as done in [2, 3]. Sect. 3 introduces the LO and NLO effective field theory and we do this using the general formalism derived in [16]. This allows to see how similar the calculations for the three cases are. In Sect. 4 we derive the divergent part at NLO and in Sect. 5 we calculate the NNLO result for the meson mass, meson decay constant and vacuum expectation value. In Sect. 6 we summarize our results.

2 Quark level

This section shortly introduces the quark-level Lagrangian giving the gauge groups and showing how the condensates can be written in the more general cases. A more extensive version of this discussion can be found in [8] and the earlier references [9, 11]. We remind the reader that we only consider cases with an underlying simple vector gauge group and we assume confinement and the formation of a condensate.

We use the notation q_R and q_L for the right- and left-handed fermions respectively. Gauge indices we usually suppress and flavour indices will be indicated when needed.

2.1 QCD

This is the usual case where the fermions are in a complex representation of the gauge group. With flavour indices i the fermion part of the Lagrangian enhanced by external fields is given by

$$\begin{aligned}\mathcal{L} &= \overline{q}_{Li} i\gamma^\mu D_\mu q_{Li} + \overline{q}_{Ri} i\gamma^\mu D_\mu q_{Ri} + \overline{q}_{Li}\gamma^\mu l_{\mu ij} q_{Lj} + \overline{q}_{Ri}\gamma^\mu r_{\mu ij} q_{Rj} \\ &\quad -\overline{q}_{Ri}\mathcal{M}_{ij} q_{Lj} - \overline{q}_{Li}\mathcal{M}^\dagger_{ij} q_{Rj}\,. \end{aligned} \tag{1}$$

The covariant derivative is given by $D_\mu q = \partial_\mu q - iG_\mu q$.

When the external fields vanish there is a $SU(N_F)_L \times SU(N_F)_R$ symmetry in the first two terms which is spontaneously broken to the diagonal subgroup $SU(N_F)_V$. This symmetry can be made local by adding the external fields with the transformations $g_L \times g_R \in SU(N_F)_L \times SU(N_F)_R$:

$$\begin{aligned} q_L &\to g_L q_L, \qquad q_R \to g_R q_R, \qquad \mathcal{M} \to g_R \mathcal{M} g_L^\dagger\,, \\ l_\mu &\to g_L l_\mu g_L^\dagger + i g_L \partial_\mu g_L^\dagger, \qquad r_\mu \to g_R r_\mu g_L^\dagger + i g_R \partial_\mu g_R^\dagger\,. \end{aligned} \tag{2}$$

We have here written q_L and q_R as column vectors in flavour and the external fields l_μ, r_μ and $\mathcal{M} = s - ip$ as matrices in flavour.

For later use, we define the big, $2N_F$, columnvector

$$\hat{q} = \begin{pmatrix} q_R \\ q_L \end{pmatrix} \tag{3}$$

and the big, $2N_F \times 2N_F$, matrices

$$\hat{V}_\mu = \begin{pmatrix} r_\mu & 0 \\ 0 & l_\mu \end{pmatrix} \qquad \hat{\mathcal{M}} = \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^\dagger & 0 \end{pmatrix}, \qquad \hat{g} = \begin{pmatrix} g_R & 0 \\ 0 & g_L \end{pmatrix}. \tag{4}$$

In terms of these the symmetry transformation can be written as

$$\hat{q} \to \hat{g}\hat{q}\,, \qquad \hat{V}_\mu \to \hat{g}\hat{V}_\mu\hat{g}^\dagger + i\hat{g}\partial_\mu\hat{g}^\dagger\,, \qquad \hat{\mathcal{M}} \to \hat{g}\hat{\mathcal{M}}\hat{g}^\dagger\,. \tag{5}$$

Note that the symmetry group is not made larger since $\hat{q}$ contains objects that have different Lorentz properties.

The formation of a flavour neutral condensate $\langle \overline{q}q\rangle = \langle \overline{q}_R q_L\rangle$ + h.c. breaks the full symmetry spontaneously to the diagonal subgroup $SU(N_F)_V$

2.2 Adjoint

If the fermions are in the adjoint representations we can write down a similar Lagrangian as above

$$\begin{aligned}\mathcal{L} &= \mathrm{tr}_c\left(\overline{q}_{Li} i\gamma^\mu D_\mu q_{Li}\right) + \mathrm{tr}_c\left(\overline{q}_{Ri} i\gamma^\mu D_\mu q_{Ri}\right) + \mathrm{tr}_c\left(\overline{q}_{Li}\gamma^\mu l_{\mu ij} q_{Lj}\right) + \mathrm{tr}_c\left(\overline{q}_{Ri}\gamma^\mu r_{\mu ij} q_{Rj}\right) \\ &\quad -\mathrm{tr}_c\left(\overline{q}_{Ri}\mathcal{M}_{ij} q_{Lj}\right) - \mathrm{tr}_c\left(\overline{q}_{Li}\mathcal{M}^\dagger_{ij} q_{Rj}\right)\,.\end{aligned} \tag{6}$$

$\mathrm{tr}_c\,(A)$ means a trace over the gauge group indices and the fermions are a matrix rather than a vector in the gauge group indices and $D_\mu q = \partial_\mu q - iG_\mu q + iqG_\mu$. Here we have the same transformation for the conjugated fermions, $D_\mu \overline{q} = \partial_\mu - iG_\mu\overline{q} + i\overline{q}G_\mu$ with $\overline{q} = q^\dagger\gamma^0$ and the Hermitian conjugate also means that the two gauge-indices are transposed. The symmetries discussed here exist in principle for any real representation for the fermions, not only the adjoint one.

We define the matrix $C = i\gamma^2\gamma^0$ and we can define a new fermion field

$$\tilde{q}_{Ri} \equiv C\overline{q}^T_{Li}\,. \tag{7}$$

The transpose in (7) works on the Dirac (and later also flavour) indices but not on the gauge indices. The field $\tilde{q}_{Ri}$ has the same transformation properties under the gauge group as q_R and is also a right-handed fermion.[1] In terms of the big matrices

$$\hat{q} = \begin{pmatrix} q_R \\ \tilde{q}_R \end{pmatrix}, \hat{V}_\mu = \begin{pmatrix} r_\mu & 0 \\ 0 & -l^T_\mu \end{pmatrix} \qquad \hat{\mathcal{M}} = \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^T & 0 \end{pmatrix}\,. \tag{8}$$

The Lagrangian (6) becomes

$$\mathcal{L} = \mathrm{tr}_c\left(\overline{\hat{q}} i\gamma^\mu D_\mu \hat{q}\right) + \mathrm{tr}_c\left(\overline{\hat{q}}\gamma^\mu \hat{V}_\mu \hat{q}_j\right) - \frac{1}{2}\mathrm{tr}_c\left(\overline{\hat{q}} C\hat{\mathcal{M}}\overline{\hat{q}}^T\right) - \frac{1}{2}\mathrm{tr}_c\left(\hat{q}^T C\hat{\mathcal{M}}^\dagger \hat{q}\right)\,. \tag{9}$$

The Lagrangian (9) has clearly a larger symmetry group, $SU(2N_F)$ as compared to QCD case above when we extend the external fields to the full matrices and have as symmetry transformations:

$$\hat{q} \to \hat{g}\hat{q}\,, \quad \hat{V}_\mu \to \hat{g}\hat{V}_\mu\hat{g}^\dagger + i\hat{g}\partial_\mu\hat{g}^\dagger\,, \quad \hat{\mathcal{M}} \to \hat{g}\hat{\mathcal{M}}\hat{g}^T\,. \tag{10}$$

[1] We have chosen right-handed rather than left-handed in order to end up with transformations for fields that look most like those for the QCD case in [2, 3].

The Vafa-Witten argument shows that also in this case the vector symmetries remain unbroken. We expect again a flavour neutral vacuum condensate $\langle \mathrm{tr}_c\,(\bar{q}q)\rangle$ which can be written as $\langle \mathrm{tr}_c\left(\hat{q}^T C J_S \hat{q}\right)\rangle + \text{h.c.}$ with

$$J_S = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} \tag{11}$$

and I the $N_F \times N_F$ unit matrix. This condensate breaks the the symmetry group down to $SO(2N_F)$.

2.3 $N_c = 2$

The fundamental representation of $SU(2)$ is pseudo-real. The Lagrangian enhanced with external fields reads

$$\begin{aligned} \mathcal{L} &= \overline{q}_{Li} i\gamma^\mu D_\mu q_{Li} + \overline{q}_{Ri} i\gamma^\mu D_\mu q_{Ri} + \overline{q}_{Li}\gamma^\mu l_{\mu ij} q_{Lj} + \overline{q}_{Ri}\gamma^\mu r_{\mu ij} q_{Rj} \\ &\quad -\overline{q}_{Ri}\mathcal{M}_{ij} q_{Lj} - \overline{q}_{Li}\mathcal{M}^\dagger_{ij} q_{Rj}\,. \end{aligned} \tag{12}$$

The covariant derivative is given by $D_\mu q = \partial_\mu q - iG_\mu q$.

We can define a field $\tilde{q}_R$ as in the previous section via

$$\tilde{q}_{R\alpha i} = \epsilon_{\alpha\beta} C \overline{q}^T_{L\beta i}\,, \tag{13}$$

with α, β gauge group indices, $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$ and $C = i\gamma^2\gamma^0$ as defined before. The field $\tilde{q}_R$ is a right handed-handed fermion that transforms as the fundamental representation of $SU(2)$.

In terms of the big matrices

$$\hat{q} = \begin{pmatrix} q_R \\ \tilde{q}_R \end{pmatrix}, \hat{V}_\mu = \begin{pmatrix} r_\mu & 0 \\ 0 & -l^T_\mu \end{pmatrix} \qquad \hat{\mathcal{M}} = \begin{pmatrix} 0 & -\mathcal{M} \\ \mathcal{M}^T & 0 \end{pmatrix}. \tag{14}$$

The Lagrangian (12) becomes

$$\mathcal{L} = \overline{\hat{q}} i\gamma^\mu D_\mu \hat{q} + \overline{\hat{q}}\gamma^\mu \hat{V}_\mu \hat{q}_{Lj} - \frac{1}{2}\overline{\hat{q}}_\alpha C\epsilon_{\alpha\beta}\hat{\mathcal{M}}\overline{\hat{q}}^T_\beta - \frac{1}{2}\hat{q}_\alpha \epsilon_{\alpha\beta} C \hat{\mathcal{M}}^\dagger \hat{q}_\beta\,. \tag{15}$$

This has again much larger symmetry group, $SU(2N_F)$ as compared to QCD case above when we extend the external fields to the full matrices and have as symmetry transformations:

$$\hat{q} \to \hat{g}\hat{q}\,, \quad \hat{V}_\mu \to \hat{g}\hat{V}_\mu\hat{g}^\dagger + i\hat{g}\partial_\mu\hat{g}^\dagger\,, \quad \hat{\mathcal{M}} \to \hat{g}\hat{\mathcal{M}}\hat{g}^T\,. \tag{16}$$

The Vafa-Witten argument shows that also in this case the vector symmetries remain unbroken and we expect again a flavour neutral vacuum condensate $\langle\bar{q}q\rangle$ which can be written as $\langle \hat{q}_\alpha \epsilon_{\alpha\beta} C J_A \hat{q}_\beta\rangle + \text{h.c.}$ with

$$J_A = \begin{pmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} \tag{17}$$

and I the $N_F \times N_F$ unit matrix. This condensate breaks the the symmetry group down to $Sp(2N_F)$.

3 Effective field theory

In this section we will show how the three cases can be brought into an extremely similar form. That will allow to take over directly much of the technology developed for the QCD case to the other cases. We assume the reader to be familiar with ChPT and EFT. Introductions can be found in [17]. We will use the terminology LO, NLO and NNLO for the usual powercounting of order p^2, p^4 and p^6.

3.1 QCD

The Goldstone bosons from the spontaneous symmetry breakdown live in the space of possible vacua. For QCD and generalizations this is in the form of a nonzero vacuum condensate

$$\langle \overline{q}_{Lj} q_{Ri} \rangle = \frac{1}{2} \langle \overline{q} q \rangle \delta_{ij} \,. \tag{18}$$

This vacuum is left unchanged by the vector transformations with $g_L \times g_R \in SU(N_F)_L \times SU(N_F)_R$ and $g_L = g_R$. The unbroken symmetry is $SU(N_F)$. The broken symmetry part of the group are the axial transformations wit $g_R = g_L^\dagger \equiv u$, they rotate the vacuum into

$$\langle \overline{q}_{Lj} q_{Ri} \rangle_{\text{rotated}} = \frac{1}{2} \langle \overline{q} q \rangle U_{ij} \tag{19}$$

with $U = g_R g_L^\dagger = u^2$. The special unitary matrix U describes the space of possible vacua and varies under the symmetry as

$$U \to g_R U g_L^\dagger \,. \tag{20}$$

This matrix U can be used to construct the Lagrangians as was done in [3]. The covariant derivative on U is defined as

$$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu \,. \tag{21}$$

The lowest order Lagrangian is

$$\mathcal{L} = \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \rangle \,, \tag{22}$$

with $\chi = 2 B_0 \mathcal{M}$ and $\langle A \rangle = \text{tr}_F\,(A)$ This has the full global symmetry as can be checked using the transformations (2) and (20). In terms of the pion fields π^a the matrix u can be parametrized as

$$u = \exp\left(\frac{i}{\sqrt{2}\,F} \pi^a T^a \right) \,. \tag{23}$$

The T^a are the generators of $SU(N_F)$ and normalized as $\text{tr}_F \left(T^a T^b \right) = \delta^{ab}$.

Let us now do the same analysis using the general formalism (CCWZ) [16]. We only look at the properties in the neighbourhood of the unit matrix here. For the perturbative treatment we do here that is sufficient. The global symmetry group G has generators T^a which are split up in a set of conserved generators Q^a and broken generators X^a. The Q^a generate the unbroken symmetry group H while the generators X^a generate in a sense the manifold of possible vacua, the quotient G/H . We must now find a way to parametrize the manifold G/H and define covariant derivatives in general. The manifold G/H and the group H we parametrize with

$$\hat{u} = \exp\left(i\phi^a X^a\right) \in G/H\,, \qquad \hat{h} = \exp\left(i\epsilon^a T^a\right) \in H \tag{24}$$

The symmetry transformation we define using the property that any group element $\hat{g}'$ can be written in the form

$$\hat{g}' = \hat{u}'\hat{h}\,, \tag{25}$$

where both $\hat{u}'$ and $\hat{h}$ are unique and of the form (24). The symmetry transformation on $\hat{u}$ by a group element $\hat{g} \in G$ is defined as

$$\hat{u} \to \hat{g}\hat{u}\hat{h}^\dagger \tag{26}$$

where $\hat{h}$ is the $\hat{h}$ of (25) needed to bring $\hat{g}' = \hat{g}\hat{u}$ in the standard form (25). Note that $\hat{h}$ is a nonlinear function of both $\hat{u}$ and $\hat{g}$. It is sometimes called the compensator.

The covariant derivatives are defined by using the fact that any variation $\hat{g}\delta\hat{g}^\dagger$ is an element of the Lie algebra and can be written as a linear combination of the generators. The same is true for $\hat{g}\left(\partial_\mu - i\hat{V}_\mu\right)\hat{g}^\dagger$ if we include external fields $\hat{V}_\mu$ transforming as $\hat{V}_\mu \to \hat{g}\hat{V}_\mu\hat{g}^\dagger + i\hat{g}\partial_\mu\hat{g}^\dagger$. We define [16]

$$\hat{u}^\dagger\left(\partial_\mu - i\hat{V}_\mu\right)\hat{u} \equiv \hat{\Gamma}_\mu - \frac{i}{2}\hat{u}_\mu\,, \qquad \hat{\Gamma}_\mu = \Gamma^a_\mu Q^a\,, \qquad \hat{u}_\mu = u^a_\mu X^a\,. \tag{27}$$

I.e. $\hat{\Gamma}_\mu$ is in the conserved part and $\hat{u}_\mu$ in the broken part of the Lie algebra. The transformation under the group G can be derived from (25) and is

$$\hat{\Gamma}_\mu \to \hat{h}\hat{\Gamma}_\mu\hat{h}^\dagger + \hat{h}\partial_\mu\hat{h}^\dagger\,, \qquad \hat{u}_\mu \to \hat{h}\hat{u}_\mu\hat{h}^\dagger\,. \tag{28}$$

$\hat{u}_\mu$ can be used to construct Lagrangians and covariant derivatives on objects ψ transforming as $\psi \to \hat{h}\psi$ are defined as

$$\hat{\nabla}_\mu\psi = \partial_\mu\psi + \hat{\Gamma}_\mu\psi\,. \tag{29}$$

It can be checked that $\hat{\nabla}_\mu\psi \to \hat{h}\hat{\nabla}_\mu\psi$. The external fields appear as (axial) vector fields $\hat{V}_\mu$ and (pseudo) scalar fields $\hat{\mathcal{M}}$. The external fields $\hat{V}_\mu$ show up in $\hat{u}_\mu$, covariant derivatives $\hat{\nabla}_\mu$ and field strengths $\hat{V}_{\mu\nu} \equiv \partial_\mu\hat{V}_\nu - \partial_\nu\hat{V}_\mu - i\left[\hat{V}_\mu, \hat{V}_\nu\right]$. The latter can be made to transform simpler by defining the objects

$$\hat{f}_{\mu\nu} \equiv \hat{u}^\dagger\hat{V}_{\mu\nu}\hat{u} \to \hat{h}\hat{f}_{\mu\nu}\hat{h}^\dagger\,. \tag{30}$$

$\hat{\mathcal{M}} \to \hat{g}\hat{\mathcal{M}}\hat{g}^\dagger$ can similarly be made into

$$\hat{\chi} \equiv \hat{u}^\dagger\hat{\mathcal{M}}\hat{u} \to \hat{h}\hat{\chi}\hat{h}^\dagger\,. \tag{31}$$

If there exists extra discrete symmetries like parity (P) that leave the unbroken part of the group invariant objects O like $\hat{f}_{\mu\nu}$ can be split into pieces that are independent via $O_\pm \equiv O \pm P(O)$.

In the effective field theory for QCD in terms of $N_F \times N_F$ matrices the notation usually used has the objects with the associated symmetry transformations:

$$
\begin{aligned}
u &= \exp\left(\frac{i}{\sqrt{2}\,F}\pi^a T^a\right) \to g_R u h^\dagger = h u g_L^\dagger\,, \\
\Gamma_\mu &= \frac{1}{2}\left(u^\dagger(\partial_\mu - i r_\mu)u + u(\partial_\mu - l_\mu)u^\dagger\right) \to h\Gamma_\mu h^\dagger + ih\partial_\mu h^\dagger\,, \\
u_\mu &= i\left(u^\dagger(\partial_\mu - i r_\mu)u - u(\partial_\mu - l_\mu)u^\dagger\right) \to h u_\mu h^\dagger\,, \\
\nabla_\mu O &= \partial_\mu O + \Gamma_\mu O - O\Gamma_\mu \to h\nabla_\mu O h^\dagger \quad \text{for} \quad O \to hOh^\dagger\,, \\
\chi_\pm &= u^\dagger \chi u^\dagger \pm u\chi^\dagger u \to h\chi_\pm h^\dagger\,, \\
f_{\pm\mu\nu} &= u l_{\mu\nu} u^\dagger \pm u^\dagger r_{\mu\nu} u\,. \to h f_{\pm\mu\nu} h^\dagger
\end{aligned}
\tag{32}
$$

$l_{\mu\nu}$ and $r_{\mu\nu}$ are the field strengths from l_μ and r_μ. T^a are the $SU(N_F)$ generators. These can be related to the general objects defined in the CCWZ way via

$$
\hat{u} = \begin{pmatrix} u & 0 \\ 0 & u^\dagger \end{pmatrix}, \qquad \hat{u}_\mu = \begin{pmatrix} u_\mu & 0 \\ 0 & -u_\mu \end{pmatrix}, \qquad \hat{\Gamma}_\mu = \begin{pmatrix} \Gamma_\mu & 0 \\ 0 & \Gamma_\mu \end{pmatrix} \cdots\,. \tag{33}
$$

$\chi_\pm$ and $\hat{f}_{\pm\mu\nu}$ are constructed from $\hat{\chi}$ and $\hat{f}_{\mu\nu}$ using parity. These objects have been used to construct the NLO Lagrangian and the NNLO Lagrangian [14]. One of the nontrivial relations used there was

$$
\nabla_\mu u_\nu - \nabla_\nu u_\mu = -f_{-\mu\nu}\,. \tag{34}
$$

In this notation the lowest order Lagrangian is

$$
\mathcal{L}_2 = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle\,. \tag{35}
$$

The NLO Lagrangian derived by [2] reads (here in the version for arbitrary N_F)

$$
\begin{aligned}
\mathcal{L}_4 = {} & L_0\langle u^\mu u^\nu u_\mu u_\nu\rangle + L_1\langle u^\mu u_\mu\rangle\langle u^\nu u_\nu\rangle + L_2\langle u^\mu u^\nu\rangle\langle u_\mu u_\nu\rangle + L_3\langle u^\mu u_\mu u^\nu u_\nu\rangle \\
& + L_4\langle u^\mu u_\mu\rangle\langle\chi_+\rangle + L_5\langle u^\mu u_\mu \chi_+\rangle + L_6\langle\chi_+\rangle^2 + L_7\langle\chi_-\rangle^2 + \frac{1}{2}L_8\langle\chi_+^2 + \chi_-^2\rangle \\
& - iL_9\langle f_{+\mu\nu}u^\mu u^\nu\rangle + \frac{1}{4}L_{10}\langle f_+^2 - f_-^2\rangle + H_1\langle l_{\mu\nu}l^{\mu\nu} + r_{\mu\nu}r^{\mu\nu}\rangle + H_2\langle\chi\chi^\dagger\rangle\,.
\end{aligned}
\tag{36}
$$

3.2 Adjoint

The vacuum in this case can be characterized by the condensate

$$
\langle \hat{q}_i^T C \hat{q}_j\rangle = \frac{1}{2}\langle\overline{q}_L q_R\rangle J_{Sij}\,. \tag{37}
$$

Under the symmetry group $g \in SU(2N_F)$ this moves around as

$$J_S \to g J_S g^T \,. \tag{38}$$

The unbroken part of the group is given by the generators Q^a and the broken part by the generators X^a which satisfy

$$J_S Q^a = -Q^{aT} J_S \,, \qquad J_S X^a = X^{aT} J_S \,. \tag{39}$$

Just as in the QCD case we can now construct a rotated vacuum in general by using the broken part of the symmetry group on the vacuum. This leads to a matrix[2]

$$U = u J_S u^T \to g U g^T \qquad \text{with} \qquad u = \exp\left(\frac{i}{\sqrt{2}\,F}\pi^a X^a\right) \,. \tag{40}$$

The matrix u transforms as in the general $CCWZ$ case as

$$u \to g u h^\dagger \,. \tag{41}$$

The earlier work used the matrix U to describe the Lagrangian [8]. Here we will use the CCWZ scheme to obtain a notation that is formally identical to the QCD case. We add full $2N_F \times 2N_F$ matrices of external fields V_μ and $\hat{M}$. We need to obtain the Γ_μ and u_μ parts of $u^\dagger\left(\partial_\mu - iV_\mu\right)u$. Here several observations are useful. Eqs. (39) have as a consequence that matrices like u satisfy

$$u J_S = J_S u^T \,, J_S u = u^T J_S \,. \tag{42}$$

A general matrix F can be split two parts, one behaving as the broken part, the other as the unbroken part of the group generators. I.e.

$$\begin{aligned}
F &= \overline{F} + \tilde{F} \,, \\
\overline{F} J_S &= -J_S \overline{F}^T \,, \qquad \tilde{F} J_S = \tilde{F}^T J_S \,, \\
\overline{F} &= \frac{1}{2}\left(F - J_S F^T J_S\right) \,, \\
\tilde{F} &= \frac{1}{2}\left(F + J_S F^T J_S\right) \,.
\end{aligned} \tag{43}$$

This means that we obtain

$$\begin{aligned}
u_\mu &= i\left(u^\dagger(\partial_\mu - iV_\mu)u - u(\partial_\mu + iJ_S V_\mu^T J_S)u^\dagger\right) \,, \\
\Gamma_\mu &= \frac{1}{2}\left(u^\dagger(\partial_\mu - iV_\mu)u + u(\partial_\mu + iJ_S V_\mu^T J_S)u^\dagger\right) \,.
\end{aligned} \tag{44}$$

Here we used the properties (42). With these quantities we can construct covariant derivatives and Lagrangians. The formal similarity to the QCD case is obviously there if we also use for the vector external fields

$$l_\mu = -J_S V_\mu^T J_S \,, \qquad r_\mu = V_\mu \,. \tag{45}$$

[2]In Sect. 3.1 we added a hat to many quantities to distinguish the $N_F \times N_F$ and $2N_F \times 2N_F$ matrices. This is not needed here and we only keep the hat explicitly on $\mathcal{M}$.

The analogy goes even further since $v_\mu = r_\mu + l_\mu$ corresponds to the currents from conserved generators and $a_\mu = r_\mu - l_\mu$ to the currents from the spontaneously broken generators. The equivalent quantities to the field strengths are

$$f_{\pm\mu\nu} = J_S u V_{\mu\nu} u^\dagger J_S \pm u V_{\mu\nu} u^\dagger \tag{46}$$

with $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i\left(V_\mu V_\nu - V_\nu V_\mu\right)$ and for the mass matrix

$$\begin{aligned}\chi_\pm &= u^\dagger \chi u^{\dagger T} J_S \pm J_S u^T \chi^\dagger u \\ &= u^\dagger \chi J_S u^\dagger \pm u J_S \chi^\dagger u\,,\end{aligned} \tag{47}$$

with $\chi = 2B_0\hat{\mathcal{M}}$. The Lagrangians at LO and NLO have exactly the same form as (35) and (36) but now with u_μ, $\chi_\pm$ and $f_{\pm\mu\nu}$ as defined in (44), (46) and (47).

3.3 Two colours

The vacuum in this case can be characterized by the condensate $\langle \hat{q}^T_{\alpha i} C \epsilon_{\alpha\beta} \hat{q}_{\beta j}\rangle = \frac{1}{2}\langle \overline{q}_L q_R\rangle J_{Aij}$. Under the symmetry group $g \in SU(2N_F)$ this moves around as $J_A \to g J_A g^T$ .The unbroken part of the group is given by the generators Q^a and the broken part by the generators X^a which satisfy $J_A Q^a = -Q^{aT} J_A\,,\ J_A X^a = X^{aT} J_A$. Just as in the QCD and the adjoint case we construct a rotated vacuum by using the broken part of the symmetry group on the vacuum. This leads to a matrix[3] $U = u J_A u^T \to g U g^T$ with $u = \exp\left(\frac{i}{\sqrt{2}F}\pi^a X^a\right)$ .The matrix u transforms as $u \to g u h^\dagger$ .Ref. [8] used the matrix U to describe the Lagrangian. Here we use the CCWZ scheme. We add full $2N_F \times 2N_F$ matrices of external fields V_μ and $\hat{\mathcal{M}}$ and then need to obtain the Γ_μ and u_μ parts of $u^\dagger\left(\partial_\mu - iV_\mu\right)u$. Matrices like u satisfy $uJ_A = J_A u^T$ and $J_A u = u^T J_A$.

A general matrix F can be split two parts, one behaving as the broken part, the other as the unbroken part of the group generators. I.e.

$$\begin{aligned}F &= \overline{F} + \tilde{F}\,, &\quad \overline{F} J_A = -J_A \overline{F}^T\,, &\quad \tilde{F} J_A^T = \tilde{F}^T J_A\,,\\ \overline{F} &= \frac{1}{2}\left(F - J_A F^T J_A^T\right)\,, &\quad \tilde{F} = \frac{1}{2}\left(F + J_A F^T J_A^T\right)\,. &\end{aligned} \tag{48}$$

Using this, we obtain

$$\begin{aligned}u_\mu &= i\left(u^\dagger(\partial_\mu - iV_\mu)u - u(\partial_\mu + iJ_A V_\mu^T J_A^T)u^\dagger\right)\,,\\ \Gamma_\mu &= \frac{1}{2}\left(u^\dagger(\partial_\mu - iV_\mu)u + u(\partial_\mu + iJ_A V_\mu^T J_A^T)u^\dagger\right)\,.\end{aligned} \tag{49}$$

Covariant derivatives and Lagrangians are constructed as above. The formal similarity to the QCD case is once more obviously if we use for the vector external fields

$$l_\mu = -J_A V_\mu^T J_A^T\,, \qquad r_\mu = V_\mu\,. \tag{50}$$

[3] The formulas in this subsection are almost identical with those in the previous subsection but $J_A^2 = -1$ while $J_S^2 = 1$. We have put in those by introducing J_A^T rather than J_A in a few places.

Again $v_\mu = r_\mu + l_\mu$ corresponds to the currents from conserved generators and $a_\mu = r_\mu - l_\mu$ to the currents from the spontaneously broken generators. The equivalent quantities to the field strengths are

$$f_{\pm\mu\nu} = J_A u V_{\mu\nu} u^\dagger J_A^T \pm u V_{\mu\nu} u^\dagger \tag{51}$$

with $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i\left(V_\mu V_\nu - V_\nu V_\mu\right)$ and for the mass matrix

$$\chi_\pm \quad = \quad u^\dagger \chi u^{\dagger T} J_A^T \pm J_A u^T \chi^\dagger u = u^\dagger \chi J_A^T u^\dagger \pm u J_A \chi^\dagger u\,, \tag{52}$$

with $\chi = 2B_0\hat{\mathcal{M}}$. The Lagrangians at LO and NLO have exactly the same form as (35) and (36) but with u_μ, $\chi_\pm$ and $f_{\pm\mu\nu}$ as defined in this subsection.

4 The divergence structure at NLO

When going beyond tree level renormalization becomes necessary. A thorough discussion of renormalization in ChPT at NNLO can be found in [15, 18]. We use here the same conventions and subtraction procedure. This means that the NLO LECs are replaced by

$$L_i = (c\mu)^{d-4}\left[\Gamma_i \Lambda + L_i^r(\mu)\right]\,, \tag{53}$$

with $\Lambda = 1/(16\pi^2(d-4))$ and $\ln c = -[\ln 4\pi + \Gamma'(1) + 1]/2$. The constants Γ_i were calculated for the QCD case in [3]. The same method can be generalized to the case here. The calculation is extremely similar for all three cases. The method is the same as the one in [3]. We split u in a classical and a quantum part

$$u = u_c e^{i\xi} \quad \text{with} \quad \xi = \sum_a \xi^a X^a\,. \tag{54}$$

The second variation w.r.t. ξ of the LO Lagrangian can be rewritten in the form

$$\mathcal{L} = \frac{F^2}{2}\left(d_\mu \xi^a d^\mu \xi^a - \xi^a \tilde{\sigma}^{ab} \xi^b\right)\,, \tag{55}$$

with $d_\mu \xi^a = \partial_\mu \xi^a + \tilde{\Gamma}_\mu^{ab} \xi^b$. The divergence at one-loop level is given by [3]

$$-\frac{1}{16\pi^2(d-4)}\left(\frac{1}{12}\tilde{\Gamma}_{\mu\nu}^{ab}\tilde{\Gamma}^{ba\mu\nu} + \frac{1}{2}\tilde{\sigma}^{ab}\tilde{\sigma}ba\right)\,. \tag{56}$$

Notice that the indices here run over the broken generators and $\tilde{\Gamma}_{\mu\nu}^{ab} = \partial_\mu \tilde{\Gamma}_\nu^{ab} - \partial_\nu \tilde{\Gamma}_\mu^{ab} + \tilde{\Gamma}_\mu^{ac}\tilde{\Gamma}_\nu^{cb} - \tilde{\Gamma}_\nu^{ac}\tilde{\Gamma}_\mu^{cb}$.

The expansion for all three cases is identical and leads to

$$\begin{aligned}
\tilde{\Gamma}_\mu^{ab} &= -\mathrm{tr}_F\left([X^a, X^b]\Gamma_\mu\right)\,, \\
\tilde{\sigma}^{ab} &= -\frac{1}{8}\mathrm{tr}_F\left(\{X^a, X^b\}\left(\chi_+ + u_\mu u^\mu\right)\right) + \frac{1}{2}\mathrm{tr}_F\left(X^a u_\mu X^b u^\mu\right)\,.
\end{aligned} \tag{57}$$

The difficulty in evaluating (56) is now rewriting the sums over broken generators into traces over the original matrices $u_\mu, \ldots$. In the QCD case, the X^a are $SU(N_F)$ generators and one can use the formulas with the $\mathrm{tr}_F\,(A)$ going from $1, \ldots, N_F$.

$$
\begin{aligned}
&\text{QCD}: \\
\mathrm{tr}_F\,(X^a A X^a B) &= \mathrm{tr}_F\,(A)\,\mathrm{tr}_F\,(B) - \frac{1}{N_F}\mathrm{tr}_F\,(AB)\;, \\
\mathrm{tr}_F\,(X^a A)\,\mathrm{tr}_F\,(X^a B) &= \mathrm{tr}_F\,(AB) - \frac{1}{N_F}\mathrm{tr}_F\,(A)\,\mathrm{tr}_F\,(B)\;.
\end{aligned} \tag{58}
$$

There exist similar formulas for the adjoint case now with $\mathrm{tr}_F\,(A)$ going from $1, \ldots, 2N_F$.

$$
\begin{aligned}
&\text{Adjoint}: \\
\mathrm{tr}_F\,(X^a A X^a B) &= \frac{1}{2}\mathrm{tr}_F\,(A)\,\mathrm{tr}_F\,(B) + \frac{1}{2}\mathrm{tr}_F\left(A J_S B^T J_S\right) - \frac{1}{2N_F}\mathrm{tr}_F\,(AB)\;, \\
\mathrm{tr}_F\,(X^a A)\,\mathrm{tr}_F\,(X^a B) &= \frac{1}{2}\mathrm{tr}_F\,(AB) + \frac{1}{2}\mathrm{tr}_F\left(A J_S B^T J_S\right) - \frac{1}{2N_F}\mathrm{tr}_F\,(A)\,\mathrm{tr}_F\,(B)\;.
\end{aligned} \tag{59}
$$

The equivalent formula for the two-colour case is [13], again with $\mathrm{tr}_F\,(A)$ going from $1, \ldots, 2N_F$.

$$
\begin{aligned}
&2-\text{colour}: \\
\mathrm{tr}_F\,(X^a A X^a B) &= \frac{1}{2}\mathrm{tr}_F\,(A)\,\mathrm{tr}_F\,(B) + \frac{1}{2}\mathrm{tr}_F\left(A J_A B^T J_A\right) - \frac{1}{2N_F}\mathrm{tr}_F\,(AB)\;, \\
\mathrm{tr}_F\,(X^a A)\,\mathrm{tr}_F\,(X^a B) &= \frac{1}{2}\mathrm{tr}_F\,(AB) - \frac{1}{2}\mathrm{tr}_F\left(A J_A B^T J_A\right) - \frac{1}{2N_F}\mathrm{tr}_F\,(A)\,\mathrm{tr}_F\,(B)\;.
\end{aligned} \tag{60}
$$

In all three cases these lead to

$$
\tilde{\Gamma}^{ab}_{\mu\nu} = -\mathrm{tr}_F\left([X^a, X^b]\Gamma_{\mu\nu}\right)\;. \tag{61}
$$

Repetitive use of these identities allows to rewrite (56) in the form of (36). These divergences are then absorbed into the redefinition of the NLO LECs (53). The needed constants Γ_i for the three cases are given in Tab. 1. We agree with [3] for the QCD case, have a small discrepancy with [13] for the two-colour case, our coefficients for Γ_0 are Γ_3 are different. The adjoint case is obtained here for the first time.

5 The calculation: mass, decay constant and condensate

In this section we calculate the corrections to the vacuum expectation value, the meson mass and the decay constant. The calculations in the work on three-flavour ChPT were done using FORM [19] and in the loops an explicit sum over all possible particles was always implemented. For this work we have rewritten the flavour routines used in that

i	QCD	Adjoint	2-colour
0	$N_F/48$	$(N_F+4)/48$	$(N_F-4)/48$
1	$1/16$	$1/32$	$1/32$
2	$1/8$	$1/16$	$1/16$
3	$N_F/24$	$(N_F-2)/24$	$(N_F+2)/24$
4	$1/8$	$1/16$	$1/16$
5	$N_F/8$	$N_F/8$	$N_F/8$
6	$(N_F^2+2)/(16N_F^2)$	$(N_F^2+1)/(32N_F^2)$	$(N_F^2+1)/(32N_F^2)$
7	0	0	0
8	$(N_F^2-4)/(16N_F)$	$(N_F^2+N_F-2)/(16N_F)$	$(N_F^2-N_F-2)/(16N_F)$
9	$N_F/12$	$(N_F+1)/2$	$(N_F-1)/2$
10	$-N_F/12$	$-(N_F+1)/2$	$-(N_F-1)/2$
1'	$-N_F/24$	$-(N_F+1)/4$	$-(N_F+1)/4$
2'	$(N_F^2-4)/(8N_F)$	$(N_F^2+N_F-2)/(8N_F)$	$(N_F^2-N_F-2)/(8N_F)$

Table 1: The coefficients Γ_i for the three cases that are needed to absorb the divergences at NLO. The last two lines correspond to the terms with H_1 and H_2.

work to use a general sum over the flavour indices and since we always calculate in the case where $\mathcal{M} = \mathrm{diag}(\hat{m}, \dots, \hat{m})$ we then use the trace formulas of the previous section to perform the sum.

We have checked that our calculations reproduce all the known results and for the QCD case that all infinities cancel when the NNLO divergence of [15] is used. For the adjoint and two-colour case we observe that the nonlocal divergence cancels as it should.

The diagrams for the vacuum expectation value are shown in Fig. 1. The lowest order is the same for all three cases

$$\langle \overline{q}q \rangle_{\mathrm{LO}} \equiv \sum_{i=1,N_F} \langle \overline{q}_{Ri} q_{Li} + \overline{q}_{Li} q_{Ri} \rangle_{\mathrm{LO}} = -N_F B_0 F^2 \,. \tag{62}$$

We use M^2 as notation for the lowest order meson mass

$$M^2 = 2B_0\hat{m} \tag{63}$$

and in addition the function

$$\overline{A}(M^2) = -\frac{M^2}{16\pi^2} \log \frac{M^2}{\mu^2} \,. \tag{64}$$

The integrals needed at the two-loop level are evaluated with the methods of [20] and they can all be expressed in terms of $\overline{A}(M^2)$.

We express the final result as

$$\langle \overline{q}q \rangle = \langle \overline{q}q \rangle_{\mathrm{LO}} + \langle \overline{q}q \rangle_{\mathrm{NLO}} + \langle \overline{q}q \rangle_{\mathrm{NNLO}} \,. \tag{65}$$

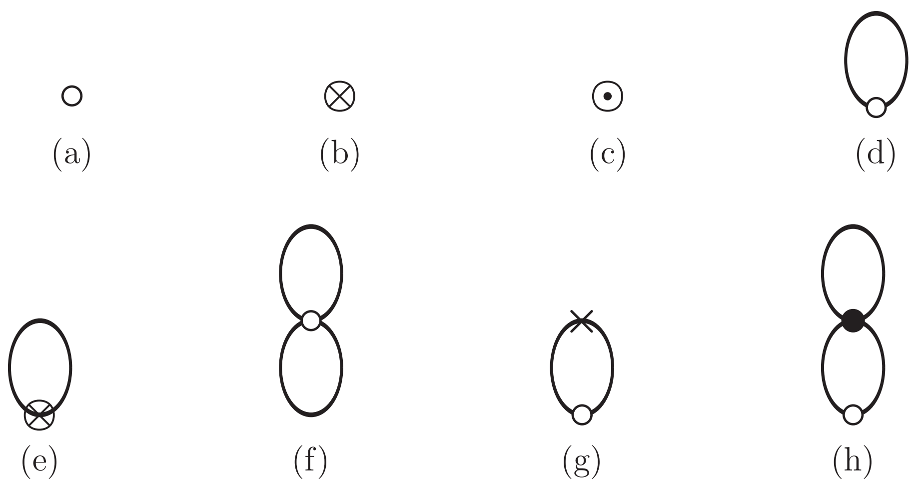

Figure 1: The diagrams up to order p^6 for $\langle \overline{q}q \rangle$. The lines are meson propagators and the vertices are: $\circ$ a p^2 insertion of $\overline{q}q$, $\otimes$ a p^4 insertion of $\overline{q}q$, $\odot$ a p^6 insertion of $\overline{q}q$, $\bullet$ a p^2 vertex and $\times$ a p^4 vertex.

The individual parts can be written in terms of logarithms and analytic contributions as

$$\begin{aligned}
\langle \overline{q}q \rangle_{\text{NLO}} &= \langle \overline{q}q \rangle_{\text{LO}} \left(a_V \frac{\overline{A}(M^2)}{F^2} + b_V \frac{M^2}{F^2} \right) , \\
\langle \overline{q}q \rangle_{\text{NNLO}} &= \langle \overline{q}q \rangle_{\text{LO}} \Bigg(c_V \frac{\overline{A}(M^2)^2}{F^4} + \frac{M^2 \overline{A}(M^2)}{F^4} \left(d_V + \frac{e_V}{16\pi^2} \right) \\
&\qquad + \frac{M^4}{F^4} \left(f_V + \frac{g_V}{16\pi^2} \right) \Bigg) . \qquad (66)
\end{aligned}$$

The coefficients for the three cases are given in Tab. 2. Note that we use the same notation for the LECs in the three cases but they are different LECs and in addition different for different values of N_F. The infinite parts can be absorbed in the NNLO Lagrangian coefficients by writing

$$r_i = (c\mu)^{2(d-4)} \left(r_i^r - \Gamma_i^{(2)} \Lambda^2 - \left(\frac{1}{16\pi^2} \Gamma_i^{(1)} + \Gamma_i^{(L)} \right) \Lambda \right) . \qquad (67)$$

The subtractions needed for the QCD case have been derived in general before in [15]. The adjoint and two-colour case can be made finite by the following:

$$\begin{aligned}
\Gamma_{VA}^{(2)} &= \frac{3}{2} \left(1 - \frac{1}{N_F^2} + 2\frac{1}{N_F} - 2N_F \right) , \\
\Gamma_{VA}^{(L)} &= 24 \left(2N_F^2 + N_F - 1 \right) \left(2L_4^r - 4L_6^r + \frac{1}{N_F} (L_5^r - 2L_8^r) \right) ,
\end{aligned}$$

	QCD
a_V	$N_F - \frac{1}{N_F}$
b_V	$16 N_F L_6^r + 8 L_8^r + 4 H_2^r$
c_V	$\frac{3}{2}\left(-1 + \frac{1}{N_F^2}\right)$
d_V	$-24\,(N_F^2 - 1)\,\left(L_4^r - 2L_6^r + \frac{1}{N_F}(L_5^r - 2L_8^r)\right)$
e_V	$1 - \frac{1}{N_F^2}$
f_V	$48\,(K_{25}^r + N_F K_{26}^r + N_F^2 K_{27}^r)$
g_V	$8\,(N_F^2 - 1)\,\left(L_4^r - 2L_6^r + \frac{1}{N_F}(L_5^r - 2L_8^r)\right)$
	Adjoint
a_V	$N_F + \frac{1}{2} - \frac{1}{2N_F}$
b_V	$32 N_F L_6^r + 8 L_8^r + 4 H_2^r$
c_V	$\frac{3}{8}\left(-1 + \frac{1}{N_F^2} - \frac{2}{N_F} + 2N_F\right)$
d_V	$-12\,(2N_F^2 + N_F - 1)\left(2L_4^r - 4L_6^r + \frac{1}{N_F}(L_5^r - 2L_8^r)\right)$
e_V	$\frac{1}{4}\left(1 - \frac{1}{N_F^2} + \frac{2}{N_F} - 2N_F\right)$
f_V	r_{VA}^r
g_V	$4\,(2N_F^2 + N_F - 1)\left(2L_4^r - 4L_6^r + \frac{1}{N_F}(L_5^r - 2L_8^r)\right)$
	2-colour
a_V	$N_F - \frac{1}{2} - \frac{1}{2N_F}$
b_V	$32 N_F L_6^r + 8 L_8^r + 4 H_2^r$
c_V	$\frac{3}{8}\left(-1 + \frac{1}{N_F^2} + \frac{2}{N_F} - 2N_F\right)$
d_V	$-12\,(2N_F^2 - N_F - 1)\left(2L_4^r - 4L_6^r + \frac{1}{N_F}(L_5^r - 2L_8^r)\right)$
e_V	$\frac{1}{4}\left(1 - \frac{1}{N_F^2} - \frac{2}{N_F} + 2N_F\right)$
f_V	r_{VT}^r
g_V	$4\,(2N_F^2 - N_F - 1)\left(2L_4^r - 4L_6^r + \frac{1}{N_F}(L_5^r - 2L_8^r)\right)$

Table 2: The coefficients $a_V, \ldots, g_V$ appearing in the expansion of the vacuum expectation value.

$$
\begin{aligned}
\Gamma^{(1)}_{VA} &= 0\,,\\
\Gamma^{(2)}_{VT} &= \frac{3}{2}\left(1-\frac{1}{N_F^2}-2\frac{1}{N_F}+2N_F\right)\,,\\
\Gamma^{(L)}_{VT} &= 24\left(2N_F^2-N_F-1\right)\left(2L_4^r-4L_6^r+\frac{1}{N_F}(L_5^r-2L_8^r)\right)\,,\\
\Gamma^{(1)}_{VT} &= 0\,.
\end{aligned}
\tag{68}
$$

This result agrees at NLO with [12] for the QCD case and [13][4] for the 2-colour case. It also agrees for $N_F=3$ at NNLO with [21, 22]. The remaining results are new.

We perform the expansion of the physical meson mass to the same order. The physical mass can be written as

$$
M^2_{\rm phys} = M^2_{\rm LO} + M^2_{\rm NLO} + M^2_{\rm NNLO}\,.
\tag{69}
$$

The lowest order was already given in (63) and is the same for all three cases. The two higher order can be expanded in logarithms and analytical contributions via

$$
\begin{aligned}
M^2_{\rm NLO} &= M^2\left(a_M\frac{\overline{A}(M^2)}{F^2}+b_M\frac{M^2}{F^2}\right)\,,\\
M^2_{\rm NNLO} &= M^2\Bigg(c_M\frac{\overline{A}(M^2)^2}{F^4}+\frac{M^2\overline{A}(M^2)}{F^4}\left(d_M+\frac{e_M}{16\pi^2}\right)\\
&\qquad +\frac{M^4}{F^4}\left(f_M+\frac{g_M}{16\pi^2}+\frac{h_M}{(16\pi^2)^2}\right)\Bigg)\,.
\end{aligned}
\tag{70}
$$

The mass can be calculated by finding the zeros of the inverse propagator, see e.g. the discussion [23]. The relevant one-particle irreducible diagrams are shown in Fig. 2. The coefficients for the three cases are given in Tab. 3. The subtractions needed for the QCD case have been derived in general before in [15]. The adjoint and two-colour case can be made finite by the following:

$$
\begin{aligned}
\Gamma^{(2)}_{MA} &= \frac{1}{2}\left(1-\frac{9}{N_F^2}+\frac{12}{N_F}-7N_F-3N_F^2\right)\,,\\
\Gamma^{(L)}_{MA} &= -8\Bigg[\left(3-\frac{3}{N_F}+2N\right)L_0^r+2\left(-1+2N_F+4N_F^2\right)L_1^r+\left(4+N_F+2N_F^2\right)L_2^r\\
&\quad+\left(3-\frac{3}{N_F}+5N_F\right)L_3^r+\frac{2}{N_F}(2-2N_F-3N_F^2)(2N_FL_4^r+L_5^r)\\
&\quad+4\left(-1+3N_F+4N_F^2\right)L_6^r+\left(10-\frac{10}{N_F}+12N_F\right)L_8^r)\Bigg]\,,\\
\Gamma^{(1)}_{MA} &= -\frac{1}{4}\left(-\frac{5}{3}+\frac{5}{N_F^2}-\frac{5}{N_F}+\frac{67}{12}N_F+\frac{47}{12}N_F^2\right)\,,
\end{aligned}
$$

[4]Those authors used a different normalization for F. Ours corresponds to $F_\pi\approx 93$ MeV for the QCD case and $N_c=3$.

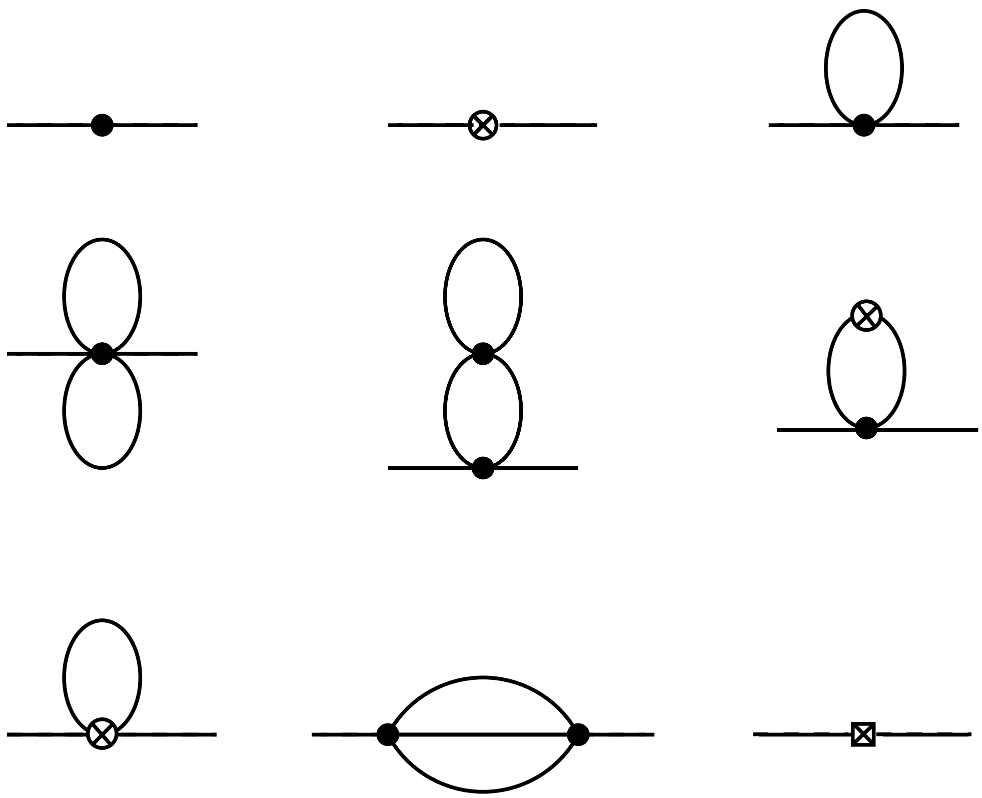

Figure 2: The diagrams up to order p^6 for the meson self energy. The lines are meson propagators and the vertices are: $\bullet$ a p^2 vertex, $\times$ a p^4 vertex and a crossed box a p^6 vertex. The diagrams for the decay constant are the same with one external meson leg replaced by an axial current.

	QCD
a_M	$-\frac{1}{N_F}$
b_M	$8N_F\left(2L_6^r - L_4^r\right) + 8\left(2L_8^r - L_5^r\right)$
c_M	$-\frac{1}{2} + \frac{9}{2N_F^2} + \frac{3}{8}N_F^2$
d_M	$8L_0^r(-\frac{3}{N_F} + N_F) + 8L_1^r(-1 + 2N_F^2) + 4L_2^r(4 + N_F^2) + L_3^r(-\frac{24}{N_F} + 20N_F)$
	$+L_4^r(40 - 16N_F^2) + L_5^r(\frac{40}{N_F} - 16N_F) + L_6^r(-16 + 16N_F^2) + L_8^r(-\frac{80}{N_F} + 32N_F)$
e_M	$-\frac{5}{3} + \frac{4}{N_F^2} + \frac{19}{16}N_F^2$
f_M	$-32K_{17}^r - 16K_{19}^r - 16K_{23}^r + 48K_{25}^r + 32K_{39}^r$
	$+N_F\left(-32K_{18}^r - 16K_{20}^r - 16K_{21}^r + 48K_{26}^r + 32K_{40}^r\right)$
	$+N_F^2\left(-16K_{22}^r + 48K_{27}^r\right) + 64(N_F L_4^r + L_5^r)(N_F L_4^r + L_5^r - 2N_F L_6^r - 2L_8^r)$
g_M	$-\frac{4}{N_F}(L_0^r + L_3^r) + 4L_1^r + 2N_F(2L_0^r + L_3^r) + 2N_F^2 L_2^r$
	$-8[L_4^r - 2L_6^r + \frac{1}{N_F}(L_5^r - 2L_8^r)]$
h_M	$-\frac{1}{4} + \frac{3}{4}\frac{1}{N_F^2} + \frac{169}{384}N_F^2$
	Adjoint
a_M	$\frac{1}{2} - \frac{1}{2N_F}$
b_M	$16N_F(2L_6^r - L_4^r) + 8(2L_8^r - L_5^r)$
c_M	$\frac{3}{8}\left(1 + \frac{3}{N_F^2} - \frac{4}{N_F} + N_F + N_F^2\right)$
d_M	$L_0^r(12 - 12\frac{1}{N_F} + 8N_F) + 8L_1^r(-1 + 2N_F + 4N_F^2)$
	$+4L_2^r(4 + N_F + 2N_F^2) + L_3^r(12 - \frac{12}{N_F} + 20N_F)$
	$+L_4^r(40 - 40N_F - 32N_F^2) + L_5^r(-20 + \frac{20}{N_F} - 16N_F)$
	$+16L_6^r(-1 + 3N_F + 2N_F^2) + L_8^r(40 - \frac{40}{N_F} + 32N_F)$
e_M	$-\frac{2}{3} + \frac{1}{N_F^2} - \frac{3}{4}\frac{1}{N_F} + \frac{77}{48}N_F + \frac{19}{16}N_F^2$
f_M	$r_{MA}^r + 64(2N_F L_4^r + L_5^r)(2N_F L_4^r + L_5^r - 4N_F L_6^r - 2L_8^r)$
g_M	$2L_0^r(1 - \frac{1}{N_F} + 2N_F) + 4L_1^r + 2N_F L_2^r(1 + 2N_F) + 2L_3^r(1 - \frac{1}{N_F} + N_F)$
	$-8(1 - N_F)(L_4^r - 2L_6^r) + 4(1 - \frac{1}{N_F})(L_5^r - 2L_8^r)$
h_M	$-\frac{1}{16} + \frac{3}{16}\frac{1}{N_F^2} - \frac{3}{16}\frac{1}{N_F} + \frac{193}{384}N_F + \frac{169}{384}N_F^2$
	2-colour
a_M	$-\frac{1}{2} - \frac{1}{2N_F}$
b_M	$16N_F(2L_6^r - L_4^r) + 8(2L_8^r - L_5^r)$
c_M	$\frac{3}{8}\left(1 + \frac{3}{N_F^2} + \frac{4}{N_F} - N_F + N_F^2\right)$
d_M	$L_0^r(-12 - 12\frac{1}{N_F} + 8N_F) + 8L_1^r(-1 - 2N_F + 4N_F^2)$
	$+4L_2^r(4 - N_F + 2N_F^2) + L_3^r(-12 - \frac{12}{N_F} + 20N_F)$
	$+L_4^r(40 + 40N_F - 32N_F^2) + L_5^r(20 + \frac{20}{N_F} - 16N_F)$
	$+16L_6^r(-1 - 3N_F + 2N_F^2) + L_8^r(-40 - \frac{40}{N_F} + 32N_F)$
e_M	$-\frac{2}{3} + \frac{1}{N_F^2} + \frac{3}{4}\frac{1}{N_F} - \frac{77}{48}N_F + \frac{19}{16}N_F^2$
f_M	$r_{MT}^r + 64(2N_F L_4^r + L_5^r)(2N_F L_4^r + L_5^r - 4N_F L_6^r - 2L_8^r)$
g_M	$-2L_0^r(1 + \frac{1}{N_F} - 2N_F) + 4L_1^r - 2N_F L_2^r(1 - 2N_F) - 2L_3^r(1 + \frac{1}{N_F} - N_F)$
	$-8(1 + N_F)(L_4^r - 2L_6^r) - 4(1 + \frac{1}{N_F})(L_5^r - 2L_8^r)$
h_M	$-\frac{1}{16} + \frac{3}{16}\frac{1}{N_F^2} + \frac{3}{16}\frac{1}{N_F} - \frac{193}{384}N_F + \frac{169}{384}N_F^2$

Table 3: The coefficients $a_M, \ldots, g_M$ appearing in the expansion of the mass.

$$
\begin{aligned}
\Gamma^{(2)}_{MT} &= \frac{1}{2}\left(1-\frac{9}{N_F^2}-\frac{12}{N_F}+7N_F-3N_F^2\right)\,,\\
\Gamma^{(L)}_{MT} &= -8\Bigg[\left(-3-\frac{3}{N_F}+2N_F\right)L^r_0+2\left(-1-2N_F+4N_F^2\right)L^r_1+\left(4-N_F+2N_F^2\right)L^r_2\\
&\quad+\left(-3-\frac{3}{N_F}+5N_F\right)L^r_3+\frac{2}{N_F}(2+2N_F-3N_F^2)(2N_FL^r_4+L^r_5)\\
&\quad+4\left(-1-3N_F+4N_F^2\right)L^r_6+\left(-10-\frac{10}{N_F}+12N_F\right)L^r_8)\Bigg]\,,\\
\Gamma^{(1)}_{MT} &= -\frac{1}{4}\left(-\frac{5}{3}+\frac{5}{N_F^2}+\frac{5}{N_F}-\frac{67}{12}N_F+\frac{47}{12}N_F^2\right)\,. \qquad (71)
\end{aligned}
$$

This result agrees at NLO with [12] for the QCD case and [13] for the 2-colour case. It also agrees with the masses for two and three flavours in the QCD case as calculated in [24, 18, 23, 25]. The remaining results are new.

We perform the expansion of the physical decay constant to the same order. The decay constant can be written as

$$
F_{\text{phys}} = F_{\text{LO}} + F_{\text{NLO}} + F_{\text{NNLO}}\,. \qquad (72)
$$

The lowest order is $F_{\text{LO}} = F$ and is the same for all three cases. The two higher order can be expanded in logarithms and analytical contributions via

$$
\begin{aligned}
F_{\text{NLO}} &= F\left(a_F\frac{\overline{A}(M^2)}{F^2}+b_F\frac{M^2}{F^2}\right)\,,\\
F_{\text{NNLO}} &= F\Bigg(c_F\frac{\overline{A}(M^2)^2}{F^4}+\frac{M^2\overline{A}(M^2)}{F^4}\left(d_F+\frac{e_F}{16\pi^2}\right)\\
&\quad+\frac{M^4}{F^4}\left(f_F+\frac{g_F}{16\pi^2}+\frac{h_F}{(16\pi^2)^2}\right)\Bigg)\,. \qquad (73)
\end{aligned}
$$

The decay constant can be calculated by computing the one-meson matrix element of the axial current. The diagrams for the wave-function renormalization are the same as those for the mass in Fig. 2 and those for the bare matrix-element are again those of Fig. 2 but with one external meson leg replaced by the axial current. The coefficients for the three cases are given in Tab. 4. The subtractions needed for the QCD case have been derived in general before in [15]. The adjoint and two-colour case can be made finite by the following:

$$
\begin{aligned}
\Gamma^{(2)}_{FA} &= 1-\frac{3}{4}N_F+\frac{1}{4}N_F^2\,,\\
\Gamma^{(L)}_{FA} &= -4\Bigg[\left(-3+\frac{3}{N_F}-2N_F\right)L^r_0+2\left(1-2N_F-4N_F^2\right)L^r_1+\left(-4-N_F-2N_F^2\right)L^r_2\\
&\quad+\left(-3+\frac{3}{N_F}-5N_F\right)L^r_3+\frac{1}{N_F}(N_F-1)(2N_FL^r_4+L^r_5)+8N_F^2L^r_6+4N_FL^r_8\Bigg]\,,
\end{aligned}
$$

	QCD
a_F	$\frac{1}{2}N_F$
b_F	$4N_F L_4^r + 4L_5^r$
c_F	$-\frac{1}{2} - \frac{3}{16}N_F^2$
d_F	$\frac{4}{N_F}(3L_0^r + 3L_3^r - L_5^r) + 4L_1^r - 8L_2^r - 4L_4^r + N_F(-4L_0^r - 10L_3^r - 2L_5^r + 8L_8^r)$
	$+2N_F^2(-4L_1^r - L_2^r - L_4^r + 4L_6^r)$
e_F	$\frac{2}{3} - \frac{1}{2N_F^2} - \frac{59}{96}N_F^2$
f_F	$-8\left(N_F L_4^r + L_5^r\right)^2 + 8(K_{19}^r + K_{23}^r) + 8N_F(K_{20}^r + K_{21}^r) + 8N_F^2 K_{22}^r$
g_F	$\frac{2}{N_F}(L_0^r + L_3^r) - 2L_1^r + N_F(-2L_0^r - L_3^r + 4L_5^r - 8L_8^r) + N_F^2(-L_2^r + 4L_4^r - 8L_6^r)$
h_F	$-\frac{7}{24} + \frac{7}{8N_F^2} + \frac{1}{768}N_F^2$
	Adjoint
a_F	$\frac{1}{2}N_F$
b_F	$8N_F L_4^r + 4L_5^r$
c_F	$-\frac{1}{4} + \frac{3}{16}N_F - \frac{3}{16}N_F^2$
d_F	$L_0^r(-6 + \frac{6}{N_F} - 4N_F) + 4L_1^r(1 - 2N_F - 4N_F^2) - 2L_2^r(4 + N_F + 2N_F^2)$
	$+L_3^r(-6 + \frac{6}{N_F} - 10N_F) - 4L_4^r(1 - N_F + N_F^2) + 2L_5^r(1 - \frac{1}{N_F} - N_F)$
	$+8N_F(2N_F L_6^r + L_8^r)$
e_F	$\frac{7}{24} - \frac{1}{8N_F^2} + \frac{1}{8N_F} - \frac{29}{32}N_F - \frac{59}{96}N_F^2$
f_F	$r_{FA}^r - 8(2N_F L_4^r + L_5^r)^2$
g_F	$L_0^r(-1 + \frac{1}{N_F} - 2N_F) - 2L_1^r + L_2^r(-N_F - 2N_F^2) + L_3^r(-1 + \frac{1}{N_F} - N_F)$
	$8N_F^2(L_4^r - 2L_6^r) + 4N_F(L_5^r - 2L_8^r)$
h_F	$-\frac{7}{96} + \frac{7}{32}\frac{1}{N_F^2} - \frac{7}{32}\frac{1}{N_F} + \frac{19}{256}N_F + \frac{1}{768}N_F^2$
	2-colour
a_F	$\frac{1}{2}N_F$
b_F	$8N_F L_4^r + 4L_5^r$
c_F	$-\frac{1}{4} - \frac{3}{16}N_F - \frac{3}{16}N_F^2$
d_F	$L_0^r(6 + \frac{6}{N_F} - 4N_F) + 4L_1^r(1 + 2N_F - 4N_F^2) - 2L_2^r(4 - N_F + 2N_F^2)$
	$+L_3^r(6 + \frac{6}{N_F} - 10N_F) - 4L_4^r(1 + N_F + N_F^2) - 2L_5^r(1 + \frac{1}{N_F} + N_F)$
	$+8N_F(2N_F L_6^r + L_8^r)$
e_F	$\frac{7}{24} - \frac{1}{8N_F^2} - \frac{1}{8N_F} + \frac{29}{32}N_F - \frac{59}{96}N_F^2$
f_F	$r_{FT}^r - 8(2N_F L_4^r + L_5^r)^2$
g_F	$L_0^r(1 + \frac{1}{N_F} - 2N_F) - 2L_1^r + L_2^r(N_F - 2N_F^2) + L_3^r(1 + \frac{1}{N_F} - N_F)$
	$8N_F^2(L_4^r - 2L_6^r) + 4N_F(L_5^r - 2L_8^r)$
h_F	$-\frac{7}{96} + \frac{7}{32}\frac{1}{N_F^2} + \frac{7}{32}\frac{1}{N_F} - \frac{19}{256}N_F + \frac{1}{768}N_F^2$

Table 4: The coefficients $a_F, \ldots, g_F$ appearing in the expansion of the decay constant.

$$
\begin{aligned}
\Gamma^{(1)}_{FA} &= -\frac{1}{8}\left(\frac{1}{3}-\frac{1}{N_F^2}+\frac{1}{N_F}-\frac{53}{12}N_F-\frac{49}{12}N_F^2\right)\,,\\
\Gamma^{(2)}_{FT} &= 1+\frac{3}{4}N_F+\frac{1}{4}N_F^2\,,\\
\Gamma^{(L)}_{FT} &= -4\Bigg[\left(3+\frac{3}{N_F}-2N_F\right)L_0^r+2\left(1+2N_F-4N_F^2\right)L_1^r+\left(-4+N_F-2N_F^2\right)L_2^r\\
&\quad+\left(3+\frac{3}{N_F}-5N_F\right)L_3^r-\frac{1}{N_F}(1+N_F)(2N_FL_4^r+L_5^r)+8N_F^2L_6^r+4N_FL_8^r\Bigg]\,,\\
\Gamma^{(1)}_{FT} &= -\frac{1}{8}\left(\frac{1}{3}-\frac{1}{N_F^2}-\frac{1}{N_F}+\frac{53}{12}N_F-\frac{49}{12}N_F^2\right)\,. \qquad (74)
\end{aligned}
$$

This result agrees at NLO with [12] for the QCD case and [13] for the 2-colour case. It also agrees with the decay constant for two and three flavours in the QCD case as calculated in [24, 18, 23, 25]. The remaining results are new.

The coefficient of the leading logarithm, $\overline{A}(M^2)^2$ is always determined but note that the coefficient of the subleading logarithm for the vacuum expectation value depends on LECs that can be determined from the masses.

The expansions (66), (70) and (73) have been written in terms of the lowest order mass and decay constant. It is possible to reorder the series in various ways. In particular one can rewrite the series in terms of the physical masses and decay constants instead. The logarithms come from physical particles propagating so the form in terms of physical masses might be preferable. There are some indications that in the case of two-flavour QCD this leads to a better convergence, see e.g. [26]. The physical mass and decay constant expansion is referred to there as the ξ expansion. We thus rewrite (66), (70) and (73) as

$$
O_{\text{phys}} = O_{\text{LO}} + O_{\text{NLO}} + O_{\text{NNLO}}\,, \qquad (75)
$$

with

$$
\begin{aligned}
O_{\text{NLO}} &= O_{\text{LO}}\left(\alpha_O\frac{\overline{A}(M^2_{\text{phys}})}{F^2_{\text{phys}}}+\beta_O\frac{M^2_{\text{phys}}}{F^2_{\text{phys}}}\right)\,,\\
O_{\text{NNLO}} &= O_{\text{LO}}\Bigg(\gamma_O\frac{\overline{A}(M^2_{\text{phys}})^2}{F^4_{\text{phys}}}+\frac{M^2_{\text{phys}}\overline{A}(M^2_{\text{phys}})}{F^4_{\text{phys}}}\left(\delta_O+\frac{\epsilon_O}{16\pi^2}\right)\\
&\quad+\frac{M^4_{\text{phys}}}{F^4_{\text{phys}}}\left(\zeta_O+\frac{\eta_O}{16\pi^2}+\frac{\theta_O}{(16\pi^2)^2}\right)\Bigg)\,. \qquad (76)
\end{aligned}
$$

We do this for $O = V, M, F$ for the vacuum-expectation-value, mass and decay constant. The coefficients in the two expansions are related by

$$
\begin{aligned}
\alpha_O &= a_O\,, \qquad \beta_O = b_O\,, \qquad \gamma_O = c_O + (2a_F - a_M)a_O\,,\\
\delta_O &= d_O + (2b_F - b_M)a_O + (2a_F - a_M)b_O\,, \qquad \epsilon_O = e_O + a_M a_O\,,\\
\zeta_O &= f_O + (2b_F - b_M)b_O\,, \qquad \eta_O = g_O + b_M a_O\,, \qquad \theta_O = h_O\,. \qquad (77)
\end{aligned}
$$

These can be easily evaluated using the results in Tabs. 2 to 4.

6 Conclusions

In this work we have calculated the vacuum expectation value, the meson mass and the meson decay constant in effective field theory to NNLO for the three cases with a simple underlying vector gauge groups and N_F equal mass fermions in the same representation. We discussed the complex case (QCD), real representation (Adjoint) and pseudo-real representation (2-colour).

The three flavour cases have been calculated earlier at NNLO for the QCD case for the mass, decay constant [25, 21] and condensate [21]. For two flavour QCD the NNLO expressions exists for the mass and decay constants [24, 18]. For the N_F flavour case the mass, decay constant and the condensate can be found in [12] to NLO. The NNLO expressions here are new. Note that the equal mass case considered here leads to considerably simpler expressions than those of [25, 23]. We have a slightly different NLO divergence structure for the two-colour case then [8] but agree with their explicit NLO expressions for the mass, decay constant and vacuum expectation value. Again the NNLO expressions here are new. The adjoint case we have extended to NLO in general and to NNLO for the mass, decay constant and vacuum expectation value. Notice that for all three cases the coefficient of the leading logarithm at NNLO is fully determined but that the coefficient of the subleading logarithm at NNLO for the vacuum expectation value depends on LECs that can be determined from the mass at NLO.

The main motivation behind this work is that these expressions should be useful for extrapolations to zero mass in lattice calculations for dynamical electroweak symmetry breaking.

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