Signal and Systems

What is a signal? a function that represents a physical quantity or variable including info about the nature of phenomenon

Sound Signals

3 Mode of Vibration

- Pitch $N f_0$
- Loudness a_N
- Phase Θ_N

Fundemental Frequency of Sound

$$f_0 = \frac{1}{\lambda} \cdot \sqrt{\frac{T}{\mu}} = \frac{1}{2L} \cdot \sqrt{\frac{T}{\rho \cdot \pi \cdot R^2}}$$

ρ = density of mass per volume
T = string tension
λ = wavelength of oscillation.
μ = mass of string per unit of length.

Modes of Oscillation of a String

$1 \rightarrow a_1 \cdot \cos(2\pi \cdot 1 f_0 \cdot t + \Theta_1)$
$N = a_N \cdot \cos(2\pi \cdot N f_0 \cdot t + \Theta_N)$

Fourier Series: every periodical signal can be written as N amount of sinusoids with frequencies / harmonics of a fundemental frequency.

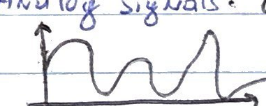
Chapter 1 | Signals

What are the types of Signals?

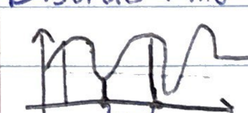
1) casual:
   - real life ; $x(t) \equiv 0 \ \forall t < 0$.
2) non-casual:
   - if $x(t) \neq 0$ for $t < 0$. Basically, graph starts before $t = 0$
3) anti-casual:
   - if $x(t) = 0$ for $t > 0$. Basically, ends at or before $t = 0$).

What are specific types of Signals?

Analog Signals: continuous & real valued.

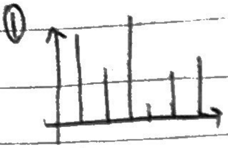


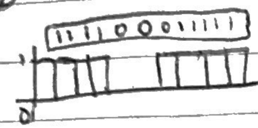
Discrete-time Signals: measures y at discrete times



Tsampling / sampling interval

Digital Signals : amplitudes are approx

① 

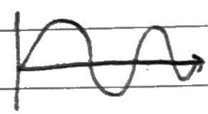
② 

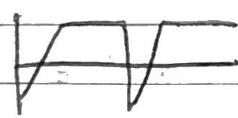
Periodic Signals

T_0 - time int.
f_0 - freq

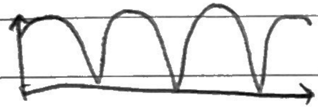
periodicity property: $x(t) = x(t + NT_0)$

$x(t), t \in \mathbb{R}$; for every x interval $y_1 = y_N$.

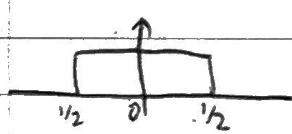
1) 

2) 

Frequency is reciprocal or x^{-1} t_0 period

3) 

Rectangular Pulses / Signal

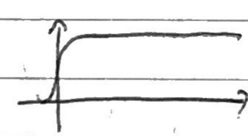


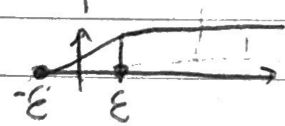
• rect(t), $\Pi(t)$

• $\text{rect}(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & \text{otherwise} \end{cases}$

Step Function / Signal

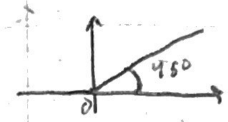
• $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

① 

② 

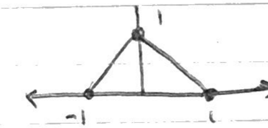
Unit-ramp function / Signal

• $r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$ • $r(t) = t \cdot u(t)$



Unit Triangle Function / Signal

• $\text{tri}(t) \,/\, \Delta(t) = \begin{cases} t+1, & -1 \leq t < 0 \\ -t+1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$



What are ~~[illegible]~~ Basic Operations on Signal / Functions?

Module / Absolute

$|x(t)| \triangleq \begin{cases} x(t), & \text{when } x(t) \geq 0 \\ -x(t), & \text{when } x(t) < 0 \end{cases}$

Time Transformations

Given $x(t)$, you build a signal $y(t) = x(at + b)$

a) Time Shifting

• ($a = 1$, b any value) • $g(t) = x(t_1 - t_d)$

• if $t_d > 0$ shift right

• if $t_d < 0$ shift left

b) Time Scaling

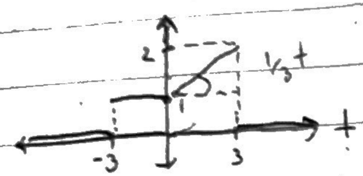
• (a = any value, $b = 0$) • $g(t) = x(at)$

• if $a < 1$, signal gets expansed

• if $a > 1$, signal " compressed

How to write a closed form based from graph?

1)

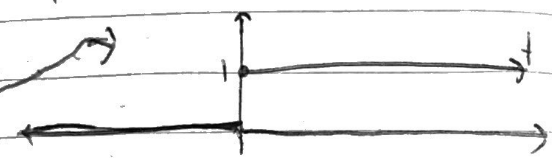


2) Make piece wise function

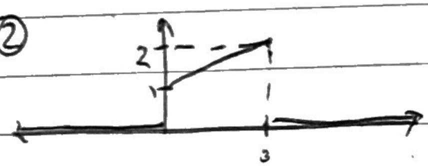
$$x(t) = \begin{cases} 0, & t < -3 \;\forall t \in (-\infty,-3) \\ 1, & -3 \le t < 1 \;\forall t \in (-3,0) \\ 0, & \forall t > 3 \\ \frac{1}{3}t + 1, & \forall t \in (0,3) \end{cases}$$

3) Transform it into step function

① $[u(t+3) - u(t)]$



②



$\left(\frac{1}{3}t + 1\right)\cdot\left(u(t) - u(t-3)\right)$

③ Combine the two gives closed form expression

$$x(t) = \underbrace{[u(t+3) - u(t)]}_{\text{models the activation from -3 to}} + \underbrace{\left(\frac{1}{3}t + 1\right)[u(t) - u(t-3)]}_{\text{models activation from 0 to 3}}$$

Steps for final equation

1) simplify $x(t) = \ldots$ if possible apply ramp function to simplify

$$x(t) = u(t+3) + \frac{1}{3}r(t) - \frac{1}{3}r(t-3) - 2u(t-3)$$

~~stops for here~~

Signal Power and Energy Quick note (More on 77)

P_{ave} and Energy define 3 classes of signals:

- Power Signals: P_{av} is finite and $E \to \infty$.
- Energy Signals: $P_{av} = 0$ and $\mathcal{E} \to$ finite. Pg #22 for proof
- Non-physical signals: $P_{av} = \to \infty$ and $\mathcal{E} \to \infty$.

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t)\, dt \quad \longleftrightarrow \quad |x(t)|^2$$

$$\mathcal{E} = \lim_{T \to \infty} \int_{-T}^{T} |x(t)^2|\, dt$$

if this doesn't approach 0 as $t \to \infty$ then it will converge thus use

if Signal is Periodic only integrate over a single period.

$$P_{av} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2\, dt$$

Signal Transformations

Time shift $\longrightarrow$ $y(t) = x(t - T)$

Time scale $\longrightarrow$ $y(t) = x(at)$

Time reversal $\longrightarrow$ $y(t) = x(-t)$

Generalized $\longrightarrow$ $y(t) = x(at - b)$

Sinusoid Function Extra + Misc

Works for a single cos(bx) and for two, cos(ax)sin(bx)

- To find period / T $\rightarrow$ $T = \frac{2\pi}{|b|}$; b is from $\cos(b \cdot x)$

- if different aka cos(bx) sin(ax) then
  1) find individual freq.
  2) find a value where both complete in period

How to find Inst. and Ave. Power over voltage function

① Instantaneous power / $p(t) = \frac{v(t)^2}{R}$

voltage function ; R ← resistor

Then to find ave. power you must integrate p(t) over one period of it.

$$P_{ave} = \frac{1}{T}\int_0^{T} p(t)\, dt$$

Linear Time-Invariant System (LTI)

A system that follows two main properties

Linearity

i. Scalable property

$c \cdot x(t) \rightarrow \boxed{\text{system}} \rightarrow c \cdot y(t)$

ii Superposition property

$x_N(t) \rightarrow \boxed{\text{system}} \rightarrow y_N(t)$

Time Invariance

$x(t - T) \rightarrow \boxed{\text{system}} \rightarrow y(t - T)$

T being a parameter.

---

impulse response response of a system to an impulse, $\delta(t)$.

impulse: $\delta(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = h(t)$

vs

unit step: $u(t) \rightarrow \boxed{\text{LTI}} \rightarrow y_{step}(t)$ response.

Impulse Response of RC circuit: $h(\tau) = \frac{1}{\tau_c} \cdot e^{-\tau/\tau_c}$.

- $\tau_c = R \cdot C$

Convolution Integral

a weighted, overlapping average of one function as it is shifted over another.

- $x(t) \ast y(t)$ convolution symbol

$x(t) \rightarrow \boxed{LTI} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

- no init. cond.
- impulse response.

Ex: $RC = \tau_c$ $h(t) = \frac{1}{\tau_c} e^{-t/\tau_c} u(t)$, what is $y(t)$?

- assume $\tau_c = 1\,\text{sec}$, $R = 10^3\,\Omega$; $C = 0.001\,\text{mF}$

$h(t) = e^{-t} u(t)$

1) $y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$

2) $= \int_{-\infty}^{\infty} u(\tau) \cdot e^{-(t-\tau)}\, u(t-\tau)\, d\tau$

$= y^{-t} \int_{-\infty}^{\infty} u(\tau) \cdot e^{\tau} \cdot u(t-\tau)\, d\tau$

... solve!

~~Bam~~

Convolution Properties

- Associative: $f \ast (g \ast h) = (f \ast g) \ast h$

When convolving 3 or more function the order doesn't matter.

- Distributive: $f \ast (g + h) = (f \ast g) + (f \ast h)$

You can convolve one function with the sum of two other functions or the other side.

~~Ussociss~~:

Role back to LTI: if impulse, $\delta(t)$ goes $\rightarrow$ LTI then output is $h(t)$, system charact. equation.

Convolution with Impulse

When $f(t)$ is convolved with a shifted impulse $\delta(t-T)$, the result is simply the original function shifted to the location of the impulse:

$f(t) \ast \delta(t-T) = f(t-T)$

Sifting Property

input	system response	output
$x(t)$	$h(t) = \delta(t)$	$x(t) \ast \delta(t) = x(t)$
$x(t)$	$h(t) = \delta(t-T)$	$x(t) \ast \delta(t-T) = x(t-T)$

Key Notes for convolution

$\delta(t) * f(t) = f(t)$ aka sifting property

How to do convolution analytically?

Superposition method

1) $s(t) = h(t) * u(t) = \int_0^t h(\tau)\, d\tau$
2) $y(t) = \sum_{k=1}^{N} A_k\, s(t - t_k)$

t_k: time when change occurs
A_k: magnitude of jump at t_k

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{Circuits} \\ + \\ 0 \text{ init. conds.}\end{array}} \longrightarrow y(t)$$

System

This relationship is a differential equation.

$\boxed{\text{test signal}}$

- $x(t) = \delta(t) = \frac{d}{dt} u(t) \longrightarrow y(t) = h(t)$ / impulse response
- $u(t) \longrightarrow y_{ss}(t)$ = step response

because $\delta(t) = \frac{d}{dt} u(t)$ then $h(t)$ is just $\frac{d}{dt}$ step function.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

Choose whichever one is the easiest to shift using $(t - \tau)$.

Option A

$$\int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

Option B

$$\int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

Causality $h(t) = 0 \quad \forall\, t < 0,\ h(t) \in \mathbb{R}$

Bounded Signal

Signal whose range is finite

BIBO Stability

bounded input, bounded output iff for LTI iff impulse response, $h(t)$, is absolutely integrable.

$$|y(t)| \leq C < \infty$$

Test by $\int_{-\infty}^{\infty} |h(t)|\,dt$
- $\to < \infty$; then Bibo
- $\to \geq \infty$; then not Bibo

Sinusoidal Response of LTI

$$x(t) = A\cos(w_0 t + \theta) \rightarrow \boxed{\text{LTI: } h(t)} \rightarrow y(t) = A\,|H(j w)| \cdot \cos(w_0 t + \theta + \varphi)$$

$w_0 = 2\pi f_0$; θ = phase shift; rad/s.

f_0 in Hz

$\varphi = \angle H(j w_0)$

LTI Systems in Series/Cascade

When systems are connected Output of LTI_N becomes input of LTI_{N+1}.

Time domain: overall impulse response, $h_{eq}(t)$, is the convolution of individual impulse responses:

$$h_{eq} = h_1(t) * h_2(t)$$

Transform domain: convolution in time domain becomes multiplication in freq./transform domain (Laplace)

$$H_{eq}(s) = H_N(s) \cdot H_{N+1}(s)$$

LTI Systems in Parallel

The input signal splits and goes into all systems simultaneously. Final output is sum of individual outputs.

Time domain: sum of individual impulse responses

$$h_{eq} = h_1(t) + h_2(t)$$

Transform domain: sum as well.

$$H_{eq}(s) = H_1(s) + H_2(s)$$

How to solve LTIs in Series & Parallel?

Series

1) Convolution of $h_1(t)$ ~~and~~ $h_2(t)$ = $h_{eq}(t)$
2) Then convolution of $h_{eq}(t)$ and input $x(t)$

Parallel

1) Add $h_1(t)$ and $h_2(t)$ to get $h_{eq}(t)$
2) Then convolute $h_{eq}(t)$ with $x(t)$.

Sinusoidal Signal:

A periodic signal whose wave form is described by sine or cosine function.

General equation: $A\cos(wt + \phi)$

Key characteristics $\longrightarrow$ 1. Amplitude $\rightarrow A\cos(x)$

1) Amplitude: how high signal's height is.

2) **Frequency & Period**

Frequency $\rightarrow$ number of complete cycles wave/signal makes in one second.

Period $\rightarrow$ time it takes for one cycle.

$$f = \frac{1}{T} \;;\; T = \frac{1}{f} \;;$$

How to find T?

$A\cos(wt + \phi)$ $\quad T = \frac{2\pi}{|w|}$; $w = 2\pi \cdot f$

3) **Phase**

starting point of a sinusoidal wave, measured in degrees or radians.

$A\cos(wt + \phi)$

Power & Energy Review

Instanteneous power: power of signal at a specific moment. Watts.

$$p(t) = v(t) \cdot i(t)$$

In a resistor:

- if only have current $\rightarrow p(t) = i^2(t) \cdot R$
- if only have voltage $\rightarrow p(t) = \frac{v^2(t)}{R}$

Energy

Total energy of a signal is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \quad \text{in Joules}$$

Energy signal ≠ Power Signal	
Energy signal has these properties:	Power signal has these properties:
• Total Energy $\rightarrow E < \infty$	• Total Energy $\rightarrow E = \infty$
• $P_{ave} \rightarrow P_{ave} = 0$	• $P_{ave} \rightarrow P_{ave} < \infty$
Signals that are: time limited. ~~the~~	Signals that are: periodic signals, constant signal

Total Energy Dissipated by a Resistor

- if current, $E = \int_{-\infty}^{\infty} i^2(t) \cdot R \, dt = R \int_{-\infty}^{\infty} i^2(t) \, dt$
- if voltage, $E = \int_{-\infty}^{\infty} \frac{v^2(t)}{R} \, dt = \frac{1}{R} \int_{-\infty}^{\infty} v^2(t) \, dt$

General Average Power

1) Find total energy over over time interval, T; T is not T as in period.
2) Then average it by dividing by T.
3) Find it as T approaches ∞.

$$P_{ave} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

What is Periodic signal and how does it change stuff?

Signal is periodic if it repeats itself after some time, T. Smallest T for which that is true is fundemental period.

it doesn't affect inst. power of energy
BUT DOES Pave

if Signal is periodic then its Average Power is given by:

$$P_{ave} = \frac{1}{T}\int_0^T |x(t)|^2 \, dt$$

Real vs Complex Signals

Complex numbers help deal with messy sinusoid functions. For example Euler's formula shows that sinusoids cos wave can be represented by two complex exponentials.

Euler's Formula 1: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

2: $\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$

How to find Stuff using Complex?

1) When squaring complex num it will result in it • its conjugate: $x(t) = e^{j\omega t} \rightarrow E = \int |x(t)|^2$

$\rightarrow x(t) \cdot x^*(t) \rightarrow (A \cdot e^{j\omega t})(A^* \cdot e^{-j\omega t}) = (A \cdot A^*) \cdot (e^{j\omega \ldots}$

$= |A|^2 \cdot 1 = |A|^2$

conjugate symbol

Average Power of Sinusoidal Signal

1) $x(t) = A\cos(\omega t + \phi)$

2) $P_{ave} = \frac{1}{T}\int_0^T |x(t)|^2 dt$

3) $P_{ave} = \frac{A^2}{T}\int_0^T \frac{1}{2}(1 + \cos(2(\omega t + \phi)) dt$

4) $P_{ave} = \frac{A^2}{T} \cdot [T + 0]$

5) $P_{ave} = \frac{A^2}{2}$

same for if voltage or current but!

• if current $P_{ave} = \frac{(A \text{ of } I)^2 \cdot R}{2}$

• if voltage $P_{ave} = \frac{(A \text{ of } V)^2}{2 \cdot R}$

Average Power of Multiple Sinusoidal Signals?

$x(t) = A_1 \cos(w_1 t + \phi_1) + A_2 \sin(w_2 t + \phi_2) + \dots$

Its just their individual sum:

$P_{ave} = \frac{A_1^2}{2} + \frac{A_2^2}{2} + \dots$

Laplace Transform For Signals

Given $x(t)$, its laplace transform is as follows:

$$\int_{0^-}^{+\infty} x(t) \cdot e^{-st} \, dt = X(s)$$

$\cdot\, s \in \mathbb{C}$ or $s \in \sigma + j\omega$ (rect. form)

Dirac delta: if $x(t) = \delta(t)$?

① $\int_{0^-}^{+\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} dt = 1 = X(s)$

Dirac delta: if $x(t) = \delta(t - T)$?

① $\int_{0}^{+\infty} \delta(t - T) e^{-st} dt = e^{-sT} = X(s)$

Unit step: if $x(t) = u(t)$?

① $\int_{0}^{\infty} u(t) e^{-st} dt = \frac{1}{s} = X(s)$

How to use this in convolution?

$x(t) = x_1(t) * x_2(t) \rightarrow \mathcal{L}\{x(t)\} = X_1(s) \cdot X_2(s)$

$\rightarrow Y_s = \mathcal{L}\{y(t)\} = X(s) \cdot H(s)$

Example

$x(t) = u(t)$

$h(t) = e^{-t} u(t)$

① $y(t) = (1 - e^{-t})\, u(t)$

$\downarrow$

② $Y(s) = X(s) \cdot H(s)$

$= \frac{1}{s} \cdot \frac{1}{s+1} = \frac{A}{s} + \frac{B}{s+1}$

③ $y(t) = A\, u(t) + B\, e^{-t} u(t)$

④ Solve for A and B

$\boxed{\text{Use Laplace transform to find } h(t) \text{ in LTI}}$

$x(t) = e^{-2t} u(t)$

$x(t) = u(t)$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

① $y(t) = x(t) * h(t)$

$X(s) / \mathcal{L}(y(t)) = X(s) \cdot H(s)$

② transfer function

$H(s) = \frac{Y(s)}{X(s)}$

③ $h(t) = \mathcal{L}^{-1}\{H(s)\} \rightarrow X(s) = \frac{1}{s+2}$

$Y(s) = \frac{1}{s}$

$H(s) = \frac{1}{s} \cdot \frac{1}{\frac{1}{s+2}} = \frac{s+2}{s} = 1 + \frac{2}{s}$

④ $H(s) = 1 + \frac{2}{s}$

$\downarrow$

$h(t) = \delta(t) + 2u(t)$

Time Scaling

Given $x(t)$, $X(s)$ $t \geq 0$. $y(t) = x(t)\big|_{t \leftarrow at} = x(at)$

$a > 0$ for scaling

Thus,

$$\boxed{\mathcal{L}\{y(t)\} = \mathcal{L}\{x(at)\} = \frac{1}{a} \cdot X\left(\frac{s}{a}\right)}$$

Time Shift

Given, $x(t) \rightarrow X(s)$ ~~and~~ so,

$$\boxed{y(t) = x(t - T) \rightarrow Y(s) = e^{-T \cdot s} \cdot X(s)}$$

$\uparrow$

valuable because if only given $y(t)/Y(s)$ then using this $X(s)/x(t)$ could be found.

Freq. Shift

Given, $x(t) \rightarrow X(s)$

$y(t) = e^{-at} x(t) \Rightarrow X(s + a)$

Ex:

$y(t) = e^{-4t} u(t) \rightarrow X(s+4)$

$\downarrow$

$X(s) = \frac{1}{s+4}$

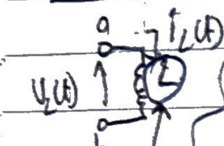
Using laplace with $\boxed{\text{Inductors}}$

$x(t) \rightarrow X(s)$

$y(t) = \frac{d\,x(t)}{dt} \rightarrow Y(s) = s\,X(s) - \boxed{x(0^-)}$

represents current — initial conditions

Visual



① $V_L(s) = \mathcal{L}\{v_L(t)\} = \mathcal{L}\left\{L \cdot \frac{d\,i_L(t)}{dt}\right\} = L \cdot \mathcal{L}\{\dots\}$

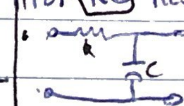
② $V_L(s) = L \cdot \{s \cdot I_L(s) - i_L(0^-)\}$

③ $V_L(s) = \boxed{sL} \cdot I_L(s) - L \cdot i_L(0^-)$

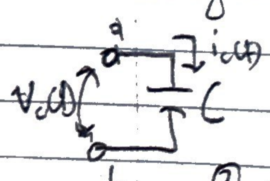
$Z_L = sL$

④ $\boxed{V_L(s) = Z_L(s) \cdot I_L(s) - L \cdot i_L(0^-)}$

For RC $H(s) = \frac{1}{1 + R \cdot C \cdot s}$



Using laplace with $\boxed{\text{Capacitors}}$



① $i_c(t) = C \cdot \frac{d \cdot v_c(t)}{dt}$

② $\mathcal{L}\{i_c(t)\} = I_c(s) = C \cdot \mathcal{L}\left\{\frac{d \cdot v_c(t)}{dt}\right\}$

③ $I_c(s) = C \cdot \{s \cdot V_c(s) - V_c(0)\}$

$Z_c(s) = \frac{1}{sC}$

! $\rightarrow I_c(s) = sC \cdot V_c(s) - C\,v_c(0)$

$$\boxed{I_c(s) = \frac{V_c(s)}{Z_c(s)} - C\,V_c(0^-)}$$

Note on Laplace

Laplace allows to go from solving differential equations to algebraic ones:

$y(t) = x(t) * h(t) \Longleftrightarrow Y(s) = X(s) \cdot H(s)$

What are poles?

A $\boxed{\text{pole}}$ is value of complex variable **s** that makes $H(s)$, transfer function, go to ∞.

$H(s) \leftrightarrow$ transfer func. $\leftrightarrow$ rational function, what does that mean?

→ function that's a fraction of two polynomials

- $N(s)$, $D(s)$. N - numerator, D - denominator.

$H(s) = \frac{N(s)}{D(s)}$

- A pole is a value that makes $D(s) = 0$.

Why are poles important?

Because they inform us about these two:

Stability of a System

- **Stable:** if all poles are in Left Half Plane (the real part of all poles is negative). Natural response will decay to zero.
- **Unstable:** if even one pole is in Right Half Plane (the real part is positive). Natural response will grow exponentially.
- **Marginally stable:** if poles are on imaginary axis ($j\omega$-axis) and are not repeated. Oscillates forever.

Natural Response Poles are natural response of a system. Value of each pole dictates the form of system's response.

Types of Poles?

Distinct Real

Unique non-repeating real numbers.

$X(s)$	$x(t)$
$\frac{A}{s+a}$	$A \cdot e^{-at} u(t)$

Ex. $\frac{A}{s^2+5s+6} \rightarrow s^2+5s+6=0$

$(s+2)(s+3)=0$

poles $= -2, -3$.

Repeated Real

Real numbers that appear more than once.

$X(s)$	$x(t)$
$\frac{A}{(s+a)^N}$	$A \cdot \frac{t^{N-1}}{(N-1)!} e^{-at} u(t)$

Ex: $\frac{A}{s^2-8s+16} \rightarrow (s-4)^2=0$

$(s-4)(s-4)=0$

pole $= 4$ with multiplicity of 2.

Ex. $\frac{A}{s^2+4s+13} \rightarrow s^2+4s+13=0 \rightarrow s = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$s = \frac{-4 \pm \sqrt{-36}}{2} \rightarrow \frac{-4 \pm j6}{2}$

$\sqrt{-1} = j$

Complex Distinct

Unique pairs of complex conjugate roots. Always in pairs of $a \pm jb$

$X(s)$	$x(t)$
$\left[\frac{Ae^{j\theta}}{s+a+jb} + \frac{Ae^{-j\theta}}{s+a-jb}\right]$	$2Ae^{-at}\cos(bt-\theta)u(t)$

Repeated Complex

Complex pairs of conjugate roots that repeat.

$X(s)$	$x(t)$
$\left[\frac{Ae^{j\theta}}{(s+a+jb)^N} + \frac{Ae^{-j\theta}}{(s+a-jb)^N}\right]$	$\frac{2At^{N-1}}{(N-1)!} e^{-at} \cos(bt-\theta) u(t)$

Ex: $\frac{A}{(s^2+9)^2} \rightarrow (s^2+9)^2 = 0$

$(s^2+9)(s^2+9)=0$

$a+jb \rightarrow 0+3j$ and $0-3j$

multiplicity of 2 — multiplicity of 2

Strictly for Periodic Functions

↓

Fourier Series

- Input: periodic signals aka repeating signals.
- Output: a sum of frequencies.

~~Explanation~~

What are F.S. representations for real-valued periodic function $x(t)$?

There are 3 types: ...

Exponential

1) $x(t) = \sum_{N=-\infty}^{\infty} X_N \cdot e^{jN \cdot W_0 \cdot t}$

2) $x_N = |X_N| e^{j\phi_N}$; $x_{-N} = x_N^*$; $\phi_{-N} = -\phi_N$;

3) $|X_N| = C_N/2$; $X_0 = C_0$

4) $\boxed{x_N = \frac{1}{T_0}\int_0^{T_0} x(t) \cdot e^{-jN \cdot W_0 t} \cdot dt}$

$e^{j\infty}$ $\infty = Nwt$ so

$e^{jNwt} = \cos(Nwt) + j\sin(Nwt)$.

Amplitude/Phase

1) $x(t) = C_0 + \sum_{N=1}^{\infty} C_N \cdot \cos(NW_0 t + \phi_N)$

2) $C_N e^{j\phi_N} = a_N - jb_N$

3) $C_N = \sqrt{a_N^2 + b_N^2}$

$$\phi_N = \begin{cases} -\tan^{-1}(b_N/a_N), & a_N > 0 \\ \pi - \tan^{-1}(b_N/a_N), & a_N < 0 \end{cases}$$

Trig

1) $\boxed{x(t) = \underline{a_0} + \sum_{N=1}^{\infty} [a_N \cos(NW_0 t) + b_N \sin(NW_0 t)]}$

$W_0 = \frac{2\pi}{T_0}$

$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt =$

even → $a_N = \frac{2}{T_0}\int_0^{T_0} x(t) \cos(N \cdot W_0 \cdot t)\, dt =$

odd → $b_N = \frac{2}{T_0}\int_0^{T_0, T_0/2} x(t) \sin(N \cdot W_0 \cdot t)\, dt =$

Function is?:	Odd	Even
DC term/A_0	0	NON zero (find it)
Cos / A_N	0	NON zero (find it)
Sine / B_N	NON-zero (find it)	0

Power Average:

$P_{ave} = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = \left(\sum_{N=1}^{\infty} \frac{(c_N)^2}{2}\right) + (C_0)^2$

Variables in Each domain

Time	Laplace	Frequency
t	s	jw

like if given a laplace $s = jw$.

mulas

Parseval's Theorem

$$Energy = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

spectrum

Bandwith of a Signal

- Bandwith at certain percent, P $\Rightarrow \frac{P}{100} \cdot \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$ (spectrum symmetric) $\rightarrow \int_{-B}^{B} |X(\omega)|^2 \, d\omega$

Basically, bandwidth is frequency range required to contain a specified percentage (P) of the signals total energy.

How to find 3dB bandwidth? aka frequency where signal drops by 3dB (power drops by half)

1. find magnitude response $|H(f)| \rightarrow |H(f)| = \sqrt{(Real)^2 + (Imaginary)^2}$ $f = w$
2. find max value of $|H(f)|$ usually at $f = 0$.
3. $|H(f)| = \frac{1}{\sqrt{2}}$ Max (from prev. step)
4. solve for f from prev. step.

if $H(\omega)$ is exponential like

① $|H(\omega)| = A \cdot e^{\dots}$, so magnitude $= |A|$ aka polar form

② rect. form $H(\omega) = a + jb$

Magn. $= \sqrt{a^2 + b^2}$

$$S_h(\omega) = \frac{|H(\omega)|^2}{2\pi}$$

Ex: $|H(\omega)| = \frac{1}{1+j\omega} \Rightarrow H(\omega) = \frac{1}{1+j\omega}$ ① got H in freq. domain

② finds its magnitude $= 1/2\pi$

③ $\frac{1}{1+w} = \frac{1}{2} \cdot \frac{1}{2}\pi \rightarrow \frac{1}{1+w} = \frac{1}{4}\pi$

$\pi + w\pi = 4$

$w\pi = 4 - \pi$

$S_h(\omega) = \frac{|H(\omega)|^2}{2\pi}$

~~$S_h(\omega) = \frac{1}{2\pi} \cdot \frac{1}{1+w^2}$~~

$S_h(\omega) = \frac{1}{2} \cdot A\ max$

$\frac{1}{2\pi} \cdot \frac{1}{1+w^2} = \frac{1}{2} \cdot \frac{1}{2\pi}$ Amplitude found using $S_h(\omega)$ where $w = 0$, since thats the highest point.

$w = \pm 1\ rad/s$

$f_{3db} = \frac{1}{2\pi}\ Hz$

Specifically for aperiodical

Fourier Transform

What is it? Operation that decomposes complex signal into a sum of sin and cos waves.

Why is it useful? Takes signals from time domain to frequency domain making it easier to solve.

Love Fourier Transform

$$\mathcal{F}\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} \, dt \rightarrow$$ apply euler's formula

$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$

frequency domain ($X(j\omega)$) — time domain ($x(t)$)

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt - j \int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt}$$

odd · odd = even

even · even = even

Even Functions

- Given by $x(t) = x(-t)$

$$\boxed{\mathcal{F}\{x(t)\} = X(j\omega) = 2\int_{0}^{\infty} x(t)\cos(\omega t)\, dt + 0}$$

sin part dissapears

even · odd = odd
odd · even = odd

Odd Function

- $x(t) = -x(-t)$

$$X(j\omega) = -2j\int_{0}^{\infty} x(t)\sin(\omega t)\,dt$$

Frequency to Time

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$e^{j\omega t} \rightarrow \cos(\omega t) + j\sin(\omega t)$

$X(j\omega)$: frequency domain function

$$\omega = \frac{2\pi}{T_0}$$

Examples

① $x(t) = e^{-t}u(t)$, find $X(\omega)$:

1. $\mathcal{L}\{e^{-t}u(t)\} \Rightarrow \frac{1}{s+1} = X(s)$

2. $X(s)\Big|_{s=j\omega} = \frac{1}{j\omega+1} = X(\omega)$

② $x(t) = \frac{\sin(10t)}{t}$

in t domain its fucked so go to frequency;

$\mathcal{F}\{x(t)\} = X(\omega) = \pi\, \text{rect}\left(\frac{\omega}{20}\right)$

$$\mathcal{E} = \frac{1}{2\pi}\int_{-10}^{10} \left|\pi\, \text{rect}\left(\frac{\omega}{20}\right)\right|^2 d\omega$$

$\mathcal{E} = \#.$