A synthetic proof of Pappus' theorem in Tarski's geometry

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Abstract. In this paper, we report on the formalization of a synthetic proof of Pappus' theorem. We provide two versions of the theorem: the first one is proved in neutral geometry (without assuming the parallel postulate), the second (usual) version is proved in Euclidean geometry. The proof that we formalize is the one presented by Hilbert in *The Foundations of Geometry* which has been detailed by Schwabhäuser, Szmielew and Tarski in part I of *Metamathematische Methoden in der Geometrie*. We highlight the steps which are still missing in this later version. The proofs are checked formally using the Coq proof assistant. Our proofs are based on Tarski's axiom system for geometry without any continuity axiom. This theorem is an important milestone toward obtaining the arithmetization of geometry which will allow us to provide a connection between analytic and synthetic geometry.

1 Introduction

Several approaches for the foundations of geometry can be used: the synthetic approach and the analytic approach. In the synthetic approach, we start with some geometric axioms such as Hilbert's axioms or Tarski's axioms. In the analytic approach, a field is assumed and geometric objects are defined by their coordinates. The two approaches are interesting: the synthetic approach allows to work in any model of the given axioms and it does not require to assume the existence of a field. The analytic approach has the advantage that definitions of geometric objects and transformations are easier, and the existence of coordinates allows to use algebraic approaches for computations and/or automated deduction. One of the main results which can be expected from a geometry is the arithmetization of this geometry: the construction of the field of coordinates. This is our main objective. Pappus's theorem is very important theorem in geometry as Pappus's theorem holds for some projective plane if and only if it is a projective plane over a commutative field. It is an important milestone in the path toward arithmetization of geometry.

In this paper, we describe the mechanization of a *synthetic proof* of Pappus' theorem in the context of Tarski's geometry.

In our development we formally proved the theorems exposed in the first 15 chapters of Schwabhäuser, Szmielew and Tarski's book: "Metamathematische Methoden in der Geometrie" [SST83], using the Coq proof assistant. To formalize these chapters we had to establish many lemmas that are implicit in Tarski's development. Many of them are of course trivial but essential in a proof assistant, but some of them are not obvious and are missing. For example, to establish the proof of some lemmas, Schwabhäuser, Szmielew and Tarski use implicitly the fact that given a line l, two points not on l, are either on the same side of l or on both sides. We also devoted some chapters to concepts that are not treated in [SST83] such as vectors, quadrilaterals, parallelograms, projections, orientation on a line, and other. We base our formalization on the tactics and lemmas already partially described in [Nar07,BN12,NBB14].

Pappus' theorem is proved in the thirteenth chapter of [SST83]. The proof is based on the one presented by Hilbert [Hil60]. A proof is given in the parallel case and an second one in a non parallel case which is the only one we will treat in this paper.

2 Related work: other formal proofs related to Pappus' theorem

Pappus statement can either be considered as an axiom or a theorem depending on the context. Hessenberg's theorem states the Pappus property implies Desargues property, this has already been formalized in Coq by Bezem and Hendriks using coherent logic [BH08] and by Magaud, Narboux and Schreck using the concept of rank [MNS12] and by Oryszczyszyn and Prazmowski using Mizar [OP90].

We do not present here the first mechanized proof of Pappus' theorem. Pappus' theorem has been proved by the second author using the area method [1] and by F. Pottier and L. Théry using Gröbner's bases [2]. But these proofs can not be used in our context. The proof using the area method is based on an axiom system which contains the axioms of a field and axioms about the ratio of segment length, but we want to prove Pappus' theorem in order to construct the field. The proof using Gröbner's bases is based on the algebraization of the statement which can be justified from a geometric point of view only if we can perform (following Descartes) the arithmetization of geometry and this requires Pappus' theorem. In some sense, all proofs of Pappus' theorem which use the concept of coordinates could be considered somewhat circular. Most of the proofs we found in books are based directly or indirectly on the arithmetization of geometry. For instance the proofs using Thales' theorem or Ceva's theorem or Menelaüs' theorem rely in the fact that the ratio of distances can be defined and manipulated algebraically. The proofs based on homogeneous coordinates require also to have a field. The proof using homothetic transformations often require coordinates to define these transformations.

[1] `http://dpt-info.u-strasbg.fr/~narboux/AreaMethod/AreaMethod.examples_4.html`

[2] `http://www-sop.inria.fr/marelle/CertiGeo/pappus.html`

3 Context

In this section we will first present the axiomatic system we used as a basis for our proofs.

Let us recall that Tarski's axiom system is based on a single primitive type depicting points and two predicates, namely between noted by $[- \ -]$ and congruence noted by $\equiv$. $[A - B - C]$ means that A, B and C are collinear and B is between A and C (and B may be equal to A or C). $AB \equiv CD$ means that the segments AB and CD have the same length. We chose to not use the continuity nor archimedean axiom in our proofs.

Notice that lines can be represented by pairs of distinct points and using the collinearity predicate. Angles can be represented by triple of points and an angle congruence predicate.

A1	Symmetry	$AB \equiv BA$
A2	Pseudo-Transitivity	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
A3	Cong Identity	$AB \equiv CC \Rightarrow A = B$
A4	Segment construction	$\exists E, [A - B - E] \wedge BE \equiv CD$
A5	Five-segments	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $[A - B - C] \wedge [A' - B' - C'] \wedge A \neq B \Rightarrow CD \equiv C'D'$
A6	Between Identity	$[A - B - A] \Rightarrow A = B$
A7	Inner Pasch	$[A - P - C] \wedge [B - Q - C] \Rightarrow$ $\exists X, [P - X - B] \wedge [Q - X - A]$
A8	Lower Dimension	$\exists ABC, \neg[A - B - C] \wedge \neg[B - C - A] \wedge \neg[C - A - B]$
A9	Upper Dimension	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q$ $\Rightarrow [A - B - C] \vee [B - C - A] \vee [C - A - B]$.
A10	Parallel postulate	$\exists XY([A - D - T] \wedge [B - D - C] \wedge A \neq D \Rightarrow$ $[A - B - X] \wedge [A - C - Y] \wedge [X - T - Y])$

Fig. 1. Tarski's axiom system for neutral geometry.

The symmetry axiom (A1 on Table 1) for equi-distance together with the transitivity axiom (A2) for equi-distance imply that the equi-distance relation is an equivalence relation.

The identity axiom for equi-distance (A3) ensures that only degenerated line segments can be congruent to a degenerated line segment. The axiom of segment construction (A4) allows to extend a line segment by a given length. The five-segment axiom (A5) corresponds to the well-known Side-Angle-Side postulate but expressed with betweenness and congruence relations only. The lengths of $\overline{AB}$, $\overline{AD}$ and $\overline{BD}$ fix the angle $\widehat{CBD}$. The identity axiom for betweenness expresses that the only possibility to have B between A and A is to have A and B equal. It also implies that the relation of betweenness is non-strict unlike Hilbert's one. The inner form of the Pasch's axiom is a variant of the axiom Moritz Pasch introduced in [Pas76] to repair the defects of Euclid. It intuitively says that if a line meets one side of a triangle and does not pass through the endpoints of that side, then it must meet one of the other sides of the triangle. The lower 2-dimensional axiom asserts that the existence of three non-collinear points. The upper 2-dimensional axiom means that all the points are coplanar. It is not obvious, but the parallel postulate (A10) is equivalent to the uniqueness of parallels, for the proof see [BNS15].

3.1 Formalization in Coq

Contrary to the formalization of Hilbert's axiom system [DDS00,BN12] which leaves room for interpretation of natural language, the formalization in Coq of Tarski's axiom system is straightforward as the axioms are stated very precisely. We define the axiom system using two type classes. The first one regroup the axioms for neutral geometry in any dimension greater than 1. The second one ensures that the space is of dimension 2. The formalization is given on Figure 2. We work in intuitionist logic but assuming decidability of equality of points. We do not give details about this in this paper see [BNSB14] for further details.

4 Some useful definitions

Before exposing the proof of Pappus' theorem, we need to introduce some definitions involved in this proof. Throughout the first twelve chapters of [SST83] numerous concepts are introduced and many properties are proved about them. We will expose here only the definitions involved in the proof of Pappus' theorem.

The collinearity of three points A B C, noted $[-ABC-]$, is defined using betweenness relation :

Definition 1. `Col`

$$[-ABC-] := [A - B - C] \vee [B - A - C] \vee [A - C - B]$$

The out relation asserts that given three collinear points, two of them are on the same side of the third one.
To assert that A and B are on the same side of O we note : $[O\ -\ AB]$

Definition 2. `out`

$$[O\ -\ AB] := O \neq A \wedge O \neq B \wedge ([O - A - B] \vee [O - B - A])$$

The midpoint relation can be defined using betweenness and segment congruence. We note that M is the midpoint of A and B : $[A \leftarrowtail M \rightarrowtail B]$

Definition 3. `is_midpoint`

$$[A \leftarrowtail M \rightarrowtail B] := [A - M - B] \wedge AM \equiv BM$$

Orthogonality needs three definitions.
The first one, is called *per* and noted $\triangle ABC$:

```
Class Tarski_neutral_dimensionless := {
 Tpoint : Type;
 Bet : Tpoint -> Tpoint -> Tpoint -> Prop;
 Cong : Tpoint -> Tpoint -> Tpoint -> Tpoint -> Prop;
 between_identity : forall A B, Bet A B A -> A=B;
 cong_pseudo_reflexivity : forall A B : Tpoint, Cong A B B A;
 cong_identity : forall A B C : Tpoint, Cong A B C C -> A = B;
 cong_inner_transitivity : forall A B C D E F : Tpoint,
   Cong A B C D -> Cong A B E F -> Cong C D E F;
 inner_pasch : forall A B C P Q : Tpoint,
      Bet A P C -> Bet B Q C ->
      exists x, Bet P x B /\ Bet Q x A;
 five_segments : forall A A' B B' C C' D D' : Tpoint,
    Cong A B A' B' ->
    Cong B C B' C' ->
    Cong A D A' D' ->
    Cong B D B' D' ->
    Bet A B C -> Bet A' B' C' -> A <> B -> Cong C D C' D';
 segment_construction : forall A B C D : Tpoint,
    exists E : Tpoint, Bet A B E /\ Cong B E C D;
 lower_dim : exists A, exists B, exists C,
            ~ (Bet A B C \/ Bet B C A \/ Bet C A B)
}.

Class Tarski_2D `(Tn : Tarski_neutral_dimensionless) := {
  upper_dim : forall A B C P Q : Tpoint,
    P <> Q -> Cong A P A Q -> Cong B P B Q -> Cong C P C Q ->
    (Bet A B C \/ Bet B C A \/ Bet C A B)
}.

(** We replace Tarski's version of the parallel postulate
by the triangle circumscription.
 The proof that these two axioms are
 equivalent can be found in Euclid.v
*)

Class Tarski_2D_euclidean `(T2D : Tarski_2D) := {
  euclid : forall A B C,
    ~ (Bet A B C \/ Bet B C A \/ Bet C A B) ->
    exists CC, Cong A CC B CC /\ Cong A CC C CC
}.

Class EqDecidability U := {
  eq_dec_points : forall A B : U, A=B \/ ~ A=B
}.
```

Fig. 2. Formalization of the axiom system in Coq

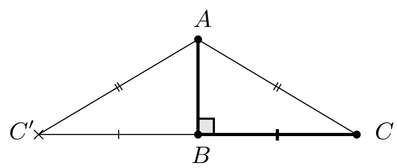

Fig. 3. Definition of the predicate `Per`

Definition 4. `Per`

$$\triangle ABC := \exists C', [C \leftarrowtail B \rightarrowtail C'] \wedge AC \equiv AC'$$

Note that this definition includes degenerated cases since $A = B$ or $C = B$ conforms to the previous definition.

The next definition called *perp_in* asserts that two lines AB and CD are orthogonal and intercepts in a point P. We note : $AB \underset{P}{\perp} CD$

Definition 5. `Perp_in`

$$AB \underset{P}{\perp} CD := A \neq B \ \wedge \ C \neq D \ \wedge \ [-PAB-] \ \wedge \ [-PCD-] \wedge \\ (\forall U\, V, [-UAB-] \Rightarrow [-VCD-] \Rightarrow \triangle UPV)$$

The third definition allows to assert that two lines AB and CD are orthogonal if there exists a point P such as $AB \underset{P}{\perp} CD$.

Definition 6. `Perp`

$$AB \perp CD := \exists P, AB \underset{P}{\perp} CD$$

Tarski introduces the double orthogonality $\perp\!\!\!\perp$ in order to prove Pappus' theorem. This definition asserts that there exists a line passing though P orthogonal to the line AB and CD. We note it $AB \underset{P}{\perp\!\!\!\perp} CD$. In Euclidean geometry, this definition is equivalent to the fact the line AB and CD are parallel but it is not true in neutral geometry.

Definition 7. `Perp2`

$$AB \underset{P}{\perp\!\!\!\perp} CD := \exists X, \exists Y, [-PXY-] \ \wedge \ XY \perp AB \ \wedge \ XY \perp CD$$

The angle congruence relation called *conga* asserts the equality of the measure of two angles, noted : $ABC \mathrel{\widehat{=}} DEF$. It is defined as follows.

Definition 8. `Conga`

$$
\begin{aligned}
ABC \mathrel{\hat{=}} DEF := A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge \\
\exists A', \exists C', \exists D', \exists F', \\
[B - A - A'] \ \wedge \ AA' \equiv ED \\
\wedge\, [B - C - C'] \wedge \ CC' \equiv EF \\
\wedge\, [E - D - D'] \wedge \ DD' \equiv BA \\
\wedge\, [E - F - F'] \wedge \ FF' \equiv BC \\
\wedge\, A'C' \equiv D'F'
\end{aligned}
$$

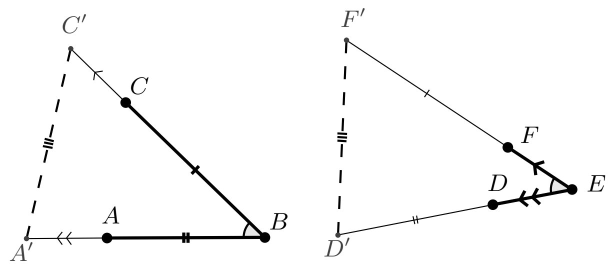

Fig. 4. Definition of angle congruence

Intuitively, two angles are equal, if it is possible to extend them to obtain two congruent triangles.

The `InAngle` relation asserts that a point P is inside an angle ABC. It is noted $P \lessdot ABC$

Definition 9. `InAngle`

$$P \lessdot ABC := A \neq B \wedge C \neq B \wedge P \neq B \wedge \exists X, [A - X - C] \wedge (X = B \vee [B \ - \ XP]$$

Note that the case $X = B$ occurs if ABC is a flat angle when B is between A and C $[A - B - C]$.

Using the $\lessdot$ relation we can define an order relation over angles called *lea* (less_eq_angle) and noted $\hat{\leq}$ and a strict version *lta* (less_than_angle) $\hat{<}$.

Definition 10. `lea`

$$ABC \mathrel{\hat{\leq}} DEF := \exists P, P \lessdot DEF \wedge ABC \mathrel{\hat{=}} DEP$$

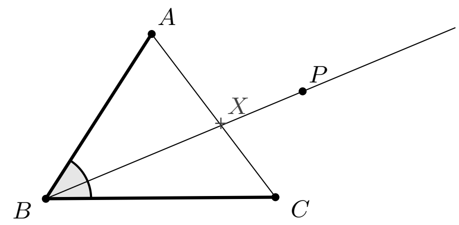

Fig. 5. Definition of the `InAngle` predicate

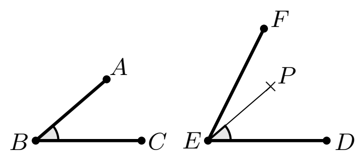

Fig. 6. Definition of the angle comparison predicate

Definition 11. `lta`

$$ABC \mathrel{\widehat{<}} DEF := ABC \mathrel{\widehat{\leq}} DEF \land \neg ABC \mathrel{\widehat{=}} DEF$$

We can now define *acute* angles as angles that are less than a right angle. We note ABC is acute : $\measuredangle ABC$

Definition 12. `acute`

$$\measuredangle ABC := \exists P, \triangle ABP \land ABC \mathrel{\widehat{<}} ABP$$

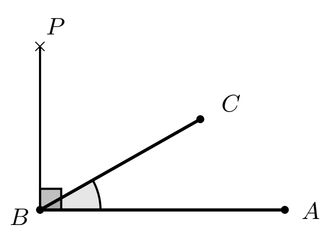

Fig. 7. Definition of acute angle

To end this section we provide the Table 4 which summarizes all our definitions and notations.

Coq	Notation
`Bet A B C`	$[A - B - C]$
`Cong A B C D`	$AB \equiv CD$
`Col A B C`	$[-ABC-]$
`out O A B`	$[O\ -\ AB]$
`is_midpoint M A B`	$[A \leftarrowtail M \rightarrowtail B]$
`Per A B C`	$\triangle ABC$
`Perp_in P A B C D`	$AB \underset{P}{\perp} CD$
`Perp A B C D`	$AB \perp CD$
`Perp2 A B C D P`	$AB \underset{P}{\perp\!\!\perp} CD$
`Conga A B C D E F`	$ABC \mathrel{\widehat{=}} DEF$
`InAngle P A B C`	$P \lessdot ABC$
`lea A B C D E F`	$ABC \mathrel{\widehat{\leq}} DEF$
`lta A B C D E F`	$ABC \mathrel{\widehat{<}} DEF$
`acute A B C`	$\measuredangle ABC$

4.1 Lengths, angles and cosine

Up to now we deal only with congruence relations over segment lengths ($\equiv$) and angle measures ($\widehat{=}$). To prove Pappus' theorem, it is necessary to introduce the notion of length and angle as equivalence class over this congruence relations. This is possible since $\equiv$ and $\widehat{=}$ are equivalence relations.

The length of segments is defined as an equivalence class over $\equiv$ relation.

Definition 13. *length*

$$length(l) := \exists A, \exists B, \forall X\ Y,\ l(X, Y) \iff XY \equiv AB$$

If l is a length ($length(l)$), then l is a predicate such as l(X,Y) is true if and only if $XY \equiv AB$. AB is representative of the length l.

We define a predicate eqL asserting that two lengths are equal :

Definition 14. *eqL*

$$eqL(l_1, l_2) := \forall XY,\ l_1(X, Y) \iff l_2(X, Y)$$

Since we proved that the binary relation eqL is reflexive, symmetric and transitive we can denote $eqL(l_1, l_2)$ by $l_1 = l_2$
In Coq, we use the setoïd rewriting mechanism, we declare the equivalence using:

```
Global Instance eqL_equivalence : Equivalence eqL.
```

We do not use the approach proposed by Cohen in [Coh13] because in our context defining a function to obtain the representative of an equivalence class would require to fix three points as references.

The null length is defined as follows :

Definition 15. *null_length*

$$null\_length(l) := length(l) \wedge \exists A, l(A, A)$$

Similarly we can define angle measure.

Definition 16. *ang*

$$ang(\alpha) := \exists A, \exists B, \exists C, \begin{array}{l} A \neq B \ \wedge \ C \neq B \ \wedge \\ \forall X\ Y\ Z,\ \alpha(X, Y, Z) \Longleftrightarrow ABC \mathrel{\widehat{=}} XYZ \end{array}$$

The predicate `eqA` asserts the equality of two angles :

Definition 17. *eqA*

$$eqA(\alpha_1, \alpha_2) := \forall XYZ,\ \alpha_1(X, Y, Z) \Longleftrightarrow \alpha_2(X, Y, Z)$$

eqA is reflexive, symmetric and transitive, thus we denote $eqA(\alpha_1, \alpha_2)$ by $\alpha_1 = \alpha_2$

The same principle can be applied to define measure of acute angles.

Definition 18. *anga*

$$anga(\alpha) := \exists A, \exists B, \exists C, \measuredangle ABC \ \wedge \ \forall X\ Y\ Z,\ \alpha(X, Y, Z) \Longleftrightarrow ABC \mathrel{\widehat{=}} XYZ$$

The lemma *anga_is_ang* asserts that the measure of an acute angle is the measure of an angle. Therefore as previously we have a predicate for equality between acute angles.

The proof of Pappus' theorem that we formalize is founded on properties of ratio of length and implicitly on the cosine function. The following relation provide a link between two distances and an angle measure without explicitly building the cosine function.

Definition 19. `lcos`

$$\begin{aligned} lcos(lp, l, \alpha) := {} & length(lp) \ \wedge \ length(l) \ \wedge \ anga(\alpha) \ \wedge \\ & (\exists A, \exists B, \exists C, \triangle CBA \ \wedge \ lp(AB) \ \wedge \ l(AC) \ \wedge \ \alpha(BAC)) \end{aligned}$$

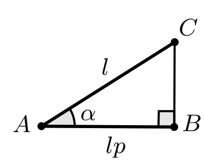

Fig. 8. Definition of `lcos`

Then we can show that length equality and angle equality is compatible with this relation:

Lemma 1. *lcos_morphism*

$$\forall a, b, c, d, e, f, eqL(a, b) \Rightarrow eqL(c, d) \Rightarrow eqA(e, f) \Rightarrow (lcos(a, c, e) \Leftrightarrow lcos(b, d, f))$$

Then we can also define the proper morphism in Coq's syntax :

```
Global Instance lcos_morphism :
 Proper (eqL ==> eqL ==> eqA ==> iff) lcos.
```

Lemma 2. *lcos_existence*

$$\forall \alpha, l, \exists lp, lcos(lp, l, \alpha)$$

Lemma 3. *lcos_unicity*

$$\forall \alpha, l, l_1, l_2, lcos(l_1, l, \alpha) \wedge lcos(l_2, l, \alpha) \Rightarrow eqL(l_1, l_2)$$

Since we have a proof of the existence and the uniqueness of the projected length we can use a functional notation : $\alpha l = lp$ instead of $lcos(lp, l, \alpha)$.
In the mechanization in Coq of this proof we could use Hilbert's ϵ operator to derive Church's ι operator to mimic this notation [Cas07]. But this require adding an axiom such as the `FunctionalRelReification_on` property of the standard library of Coq which states that if we have a functional relation we can obtain the function represented by this relation:

```
Definition FunctionalRelReification_on :=
  forall R:A->B->Prop,
    (forall x : A, exists! y : B, R x y) ->
    (exists f : A->B, forall x : A, R x (f x)).
```

As the proof can be carried without this axiom, we chose the safer option which consists in not using this axiom[3].

Definition 20. *lcos_eq*

$$lcos\_eq(l_1, \alpha_1, l_2, \alpha_2) := \exists\ lp, lcos(lp, l_1, \alpha_1) \wedge lcos(lp, l_2, \alpha_2)$$

Since *lcos_eq* is an equivalence relation we will note $lcos\_eq(l_1, \alpha_1, l_2, \alpha_2)$:

$$\alpha_1 l_1 = \alpha_2 l_2$$

In the proof of Pappus' theorem we will need to deal with two or three applications of the function of arity two implicitly represented by the ternary `lcos` predicate. Given two angles we can apply to a length two consecutive orthogonal projections using the predicate $lcos2$

[3] Note, however that for arithmetization of geometry we will need to use this axiom to obtain the standard axioms of an ordered field expressed using functions instead of relations.

Definition 21. *lcos2*

$$lcos2(lp, l, \alpha_1, \alpha_2) := \exists\, l_1, lcos(l_1, l, \alpha_1) \wedge lcos(lp, l_1, \alpha_2)$$

Using the functional notation : $lcos2(lp, l, \alpha_1, \alpha_2)$ means that $\alpha_2(\alpha_1 l) = lp$.

Given l, α_1, α_2, we proved the existence and the uniqueness of the length lp such that $lcos2(lp, l, \alpha_1, \alpha_2)$.

As previously we can define an equivalence relation $lcos2\_eq$

Definition 22. *lcos2_eq*

$$lcos2\_eq(l_1, \alpha_1, \beta_1, l_2, \alpha_2, \beta_2) := \exists\, lp, lcos2(lp, l_1, \alpha_1, \beta_1) \wedge lcos2(lp, l_2, \alpha_2, \beta_2)$$

We proved that $lcos2\_eq$ is an equivalence relation, thus we can write the relation $lcos2\_eq(l_1, \alpha_1, \beta_1, l_2, \alpha_2, \beta_2)$:

$$\beta_1 \alpha_1 l_1 = \beta_2 \alpha_2 l_2$$

Similarly, given three angles we can apply to a length three consecutive orthogonal projections using the predicate `lcos3` and that is all we will need for the proof of Pappus' theorem. As previously we can define an equivalence relation $lcos3\_eq$ of arity 8 that we denote by:

$$\gamma_1 \beta_1 \alpha_1 l_1 = \gamma_2 \beta_2 \alpha_2 l_2$$

5 Some lemmas involved in the proof of Pappus' theorem

In this section we describe some lemma about the pseudo-cosine function which will be used in the proof of Pappus's theorem. The first lemma shows that two applications of the pseudo-cosine function commute.

Lemma 4. *l13_7*

$$\begin{aligned}
&\forall \alpha, \beta, l, la, lb, lab, lba, \\
&\qquad lcos(la, l, \alpha) \wedge lcos(lb, l, \beta) \wedge lcos(lab, la, \beta) \wedge lcos(lba, lb, \alpha) \\
&\qquad\qquad \Rightarrow eqL(lab,\ lba)
\end{aligned}$$

Using the functional notation we have :

$$\forall \alpha, \beta, l, la, lb, lab, lba,\ \alpha l = la \wedge \beta l = lb \wedge \beta la = lab \wedge \alpha lb = lba \Rightarrow lab = lba$$

From $l13\_7$ we can prove the lemma $lcos2\_comm$ which is a more convenient version :

Lemma 5. *lcos2_comm*

$$\forall \alpha, \beta, lp, l, lcos2(lp, l, \alpha, \beta) \Rightarrow lcos2(lp, l, \beta, \alpha,)$$

In a simplified notation : $\forall \alpha, \beta, lp, l,\ \beta\alpha l = lp \Rightarrow \alpha\beta l = lp$

In the original notation we obtain : $\forall \alpha, \beta, l,\ \beta\alpha l = \alpha\beta l$

From the previous lemma $lcos2\_comm$ we can proof a generalization for the $lcos3$ predicate.

Lemma 6. *lcos3_permut1*

$$\forall \alpha, \beta, \gamma, lp, l, lcos3(lp, l, \alpha, \beta, \gamma) \Rightarrow lcos3(lp, l, \alpha, \gamma, \beta)$$

Lemma 7. *lcos3_permut2*

$$\forall \alpha, \beta, \gamma, lp, l, lcos3(lp, l, \alpha, \beta, \gamma) \Rightarrow lcos3(lp, l, \gamma, \beta, \alpha)$$

Lemma 8. *lcos3_permut3*

$$\forall \alpha, \beta, \gamma, lp, l, lcos3(lp, l, \alpha, \beta, \gamma) \Rightarrow lcos3(lp, l, \beta, \alpha, \gamma)$$

In a more readable notation we have :

$$\forall \alpha, \beta, \gamma, l,\ \gamma\beta\alpha l = \beta\gamma\alpha l$$
$$\forall \alpha, \beta, \gamma, l,\ \alpha\beta\gamma l = \beta\gamma\alpha l$$
$$\forall \alpha, \beta, \gamma, l,\ \gamma\beta\alpha l = \gamma\alpha\beta l$$

It can be proved that the $lcos$ pseudo function is injective in the sense that:

Lemma 9. *13_6*

$$\alpha l_1 = \alpha l_2 \Rightarrow l_1 = l_2$$

From the previous lemma, we can deduce :

Lemma 10. *lcos3_lcos*

$$\forall\ l_1, \alpha_1, l_2, \alpha_2, \beta, \gamma,\ lcos3\_eq(l_1, \alpha_1, \beta, \gamma, l_2, \alpha_2, \beta, \gamma) \Rightarrow lcos\_eq(l_1, \alpha_1, l_2, \alpha_2)$$

In a functional notation we have :
$\forall\ l_1, \alpha_1, l_2, \alpha_2, \beta, \gamma,\ \gamma\beta\alpha_1 l_1 = \gamma\beta\alpha_2 l_2 \Rightarrow \alpha_1 l_1 = \alpha_2 l_2$

6 Proof of Pappus' theorem

We now have all the required ingredients and we can prove the main theorem.

6.1 The statement

The traditional formulation of Pappus theorem is the following:

Theorem 1. *Pappus (euclidean version)*

$$\begin{array}{l}\forall O, A, B, C, A', B', C',\ \neg[-OAA'-] \\ \quad \wedge\ [-OAB-]\ \wedge\ [-OBC-] \\ \quad \wedge\ \ B \neq O\ \ \wedge\ \ C \neq O\ \ \wedge B' \neq O \wedge C' \neq O \\ \quad \wedge [-OA'B'-] \wedge [-OB'C'-] \\ \qquad \wedge AC' \mid\mid CA' \wedge BC' \mid\mid CB' \\ \qquad\qquad \Rightarrow AB' \mid\mid BA' \end{array}$$

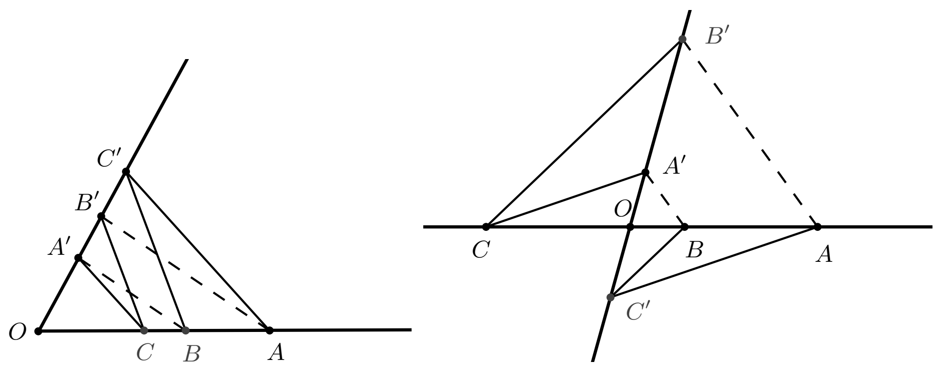

Fig. 9. Two illustrations of Pappus' theorem depending on the configuration of points.

In this paper, we describe the proof of this version which is valid in neutral geometry. To express the statement in neutral geometry, we use the predicate $\perp\!\!\!\perp$ (Definition 7).

Theorem 2. *Pappus (neutral version)*

$$\begin{array}{l}\forall O, A, B, C, A', B', C',\ \neg[-OAA'-] \\ \quad \wedge [-OAB-] \wedge [-OBC-] \wedge B \neq O \wedge C \neq O \\ \quad \wedge [-OA'B'-] \wedge [-OB'C'-] \wedge B' \neq O \wedge C' \neq O \\ \qquad \wedge\ AC' \underset{O}{\perp\!\!\!\perp} CA'\ \wedge\ BC' \underset{O}{\perp\!\!\!\perp} CB' \\ \qquad\qquad \Rightarrow AB' \underset{O}{\perp\!\!\!\perp} BA' \end{array}$$

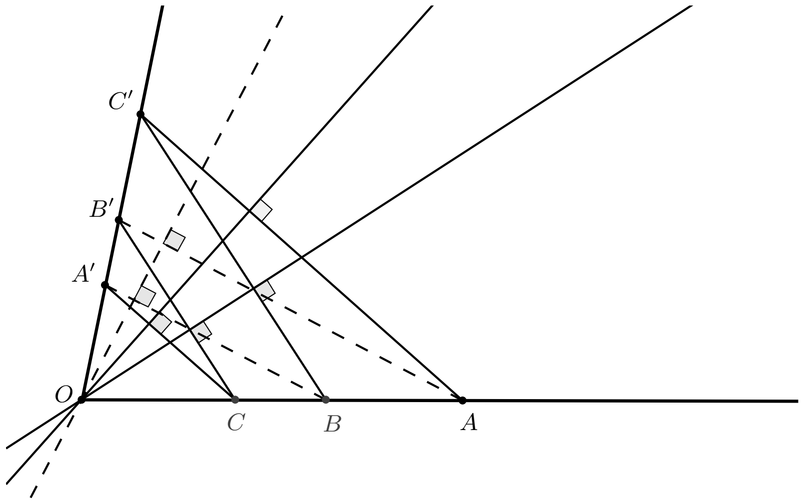

Fig. 10. Main figure for Pappus' theorem in neutral geometry

6.2 Notations

To improve readability of the proofs, we will name the different lengths according to Definition 13 (length).

We will note the length OA : $|OA|$ and name it a. That means $length(a) \wedge a(OA)$.

Similarly :

$$\begin{array}{lll} |OA| = a & |OB| = b & |OC| = c \\ |OA'| = a' & |OB'| = b' & |OC'| = c' \end{array}$$

6.3 Construction

Since $BC' \underset{O}{\perp\!\!\!\perp} CB'$, there exists a line l perpendicular to BC' and CB' passing through O (see Fig.11). l intercepts BC' in L and CB' in L'. The acute angle $C'OL = B'OL'$ is called λ.

The acute angle $COL' = BOL$ is called λ'. Using the previously defined notations we have :

$$\lambda' b = \lambda c' \tag{1}$$

$$\lambda' c = \lambda b' \tag{2}$$

The proof as described in [SST83] and [Hil60] contains a gap here. Indeed it is not trivial to prove that the angles $COL' = BOL$. To prove that the angles are equal we need to prove that the points belongs to the same half lines. In order

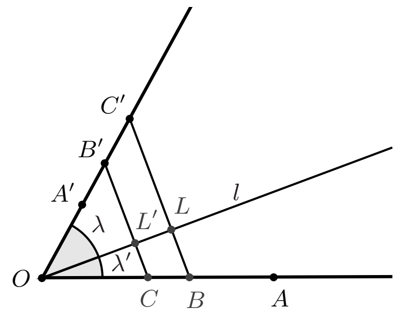

Fig. 11. First notations

to prove this one could think of using the fact that parallel projection preserves betweenness. But remember that we are working in neutral geometry, so parallel projection is not a function. Still we can prove the following lemma about $\perp\!\!\!\perp$ which is valid in neutral geometry:

Lemma 11.

$$\forall OABA'B', [O - A - B] \Rightarrow [-OA'B'-] \Rightarrow \neg[-OAA'-] \Rightarrow AA' \underset{O}{\perp\!\!\!\perp} BB' \Rightarrow [O - A' - B']$$

Since $AC' \underset{O}{\perp\!\!\!\perp} CA'$, there exists m a common perpendicular to line AC' and CA' going through O (see Fig.12). m intercepts AC' in M' and CA' in M. The acute angle $A'OM = C'OM'$ is called μ. The acute angle $COM = AOM'$ is called μ'. To prove these equalities between angles we use lemma 11 again.

As previously we have :

$$\mu' a = \mu c' \tag{3}$$

$$\mu' c = \mu a' \tag{4}$$

We call n the orthogonal line to AB' and passing through O (see Fig.13). n intercepts AB' in N. Similarly acute angle $B'ON$ is called ν and the acute angle AON is called ν'. Translated in terms of lengths, angles and pseudo-cosine it means:

$$\nu b' = \nu' a \tag{5}$$

We will prove that:

$$\nu a' = \nu' b \tag{6}$$

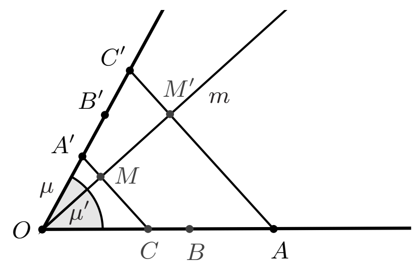

Fig. 12. Second notations

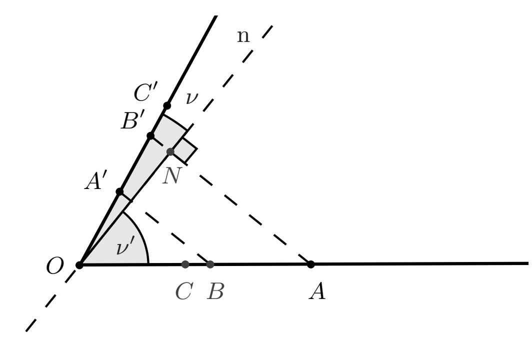

Fig. 13. Third notations

To summarize we have:

$$\begin{aligned}
\lambda' b &= \lambda c' \\
\lambda' c &= \lambda b' \\
\mu' a &= \mu c' \\
\mu' c &= \mu a' \\
\nu' a &= \nu b'
\end{aligned}$$

and we want to prove that $\nu a' = \nu' b$ (6).

$$\begin{aligned}
\lambda'\nu' b = && \nu'\lambda' b && \text{(lcos_2comm)} \\
= && \nu'\lambda c' && (1)
\end{aligned}$$

$$\begin{aligned}
\mu\lambda'\nu' b = && \mu\nu'\lambda c' && \\
= && \nu'\mu\lambda c' && \text{(lcos3_permut)} \\
= && \nu'\lambda\mu c' && \text{(lcos3_permut)} \\
= && \nu'\lambda\mu' a && (3) \\
= && \lambda\mu'\nu' a && \text{(lcos3_permut)} \\
= && \lambda\mu'\nu b' && (5) \\
= && \mu'\nu\lambda b' && \text{(lcos3_permut)} \\
= && \mu'\nu\lambda' c && (2) \\
= && \nu\lambda'\mu' c && \text{(lcos3_permut)} \\
= && \nu\lambda'\mu a' && (4) \\
= && \mu\lambda'\nu a' && \text{(lcos3_permut)}
\end{aligned}$$

Thus we have that $\mu\lambda'\nu' b = \mu\lambda'\nu a'$ and as the pseudo-cosine is injective (see lemma `lcos2_lcos`) we can deduce that $\nu' b = \nu a'$.

At this stage, Schwabhäuser, Szmielew and Tarski define two points N_1 and N_2 the orthogonal projections of A', respectively B on the line ON. Thus we have $\triangle ON_1A'$ and $\triangle ON_2B$. Now it is sufficient to prove that $N_1 = N_2$. In the proof given by Hilbert this is not detailed, the theorem is considered to be proved at this stage.

Since O A B C are collinear Schwabhäuser, Szmielew and Tarski distinguish four different cases depending of the relative positions of O, A, B and C:

1. $[O\ -\ AC]$ and $[O\ -\ BC]$
2. $[O\ -\ AC]$ and $[B - O - C]$
3. $[A - O - C]$ and $[O\ -\ BC]$
4. $[A - O - C]$ and $[B - O - C]$

In our proof, we use a slightly different method. We define the point N' on the line ON such as ON' is of length n'. Two points meet this condition on either side of the point O. We have to distinguish only two cases depending on the relative positions of A, B and O.

1. $[O\ -\ AB]$
2. $[A - O - B]$

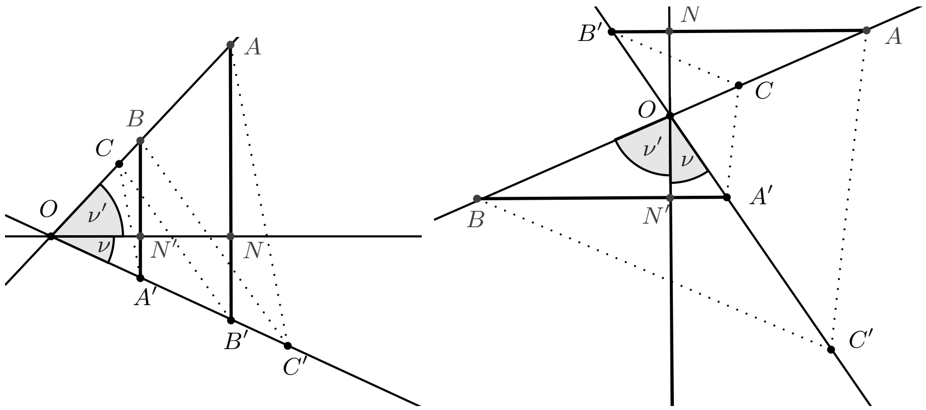

Fig. 14. Case 1 : [O - AB] , Case 2 : [A-O-B]

Then we will have to establish : $\triangle ON'B \wedge \triangle ON'A'$

Case 1 : [O - AB] We build the point N' such as : $|ON'| = n' \wedge [O\ -\ NN']$ by using the lemma *ex_point_lg_out*.

Lemma 12. *ex_point_lg_out*

$$\forall\, l, A, P,\ A \neq P \wedge length(l) \wedge \neg null\_length(l) \Rightarrow \exists B,\ l(A,B) \wedge [A\ -\ BP]$$

Case 2 : [A-O-B] The second case can be proved similarly, but we need to build the point N' such as $[N - O - N']$ and distance ON' is equal to n'. This can be done using the lemma *ex_point_lg_bet*

Lemma 13. *ex_point_lg_bet*

$$\forall\, l, A, M,\ length(l) \Rightarrow \exists B,\ l(M,B) \wedge\ [A - M - B]$$

6.4 Proof of : $\triangle ON'B$

The lemma *lcos_per* helps us to prove $\triangle ON'B$

Lemma 14. *lcos_per*

$$\forall\ A, B, C, lp, l, a, anga(a) \wedge length(l) \wedge length(lp) \\ \wedge lcos(lp, l, a) \wedge l(A,C) \wedge lp(A,B) \wedge a(B,A,C) \Rightarrow \triangle ABC$$

applied in the context :

$$lcos(n', b, \nu') \wedge b(O,B) \wedge n'(O,N') \wedge \nu'(N',O,B) \Rightarrow \triangle ON'B$$

by assumption we already have :

- $\nu' b = n'$
- $|OB| = b$
- $|ON'| = n'$

We have only to prove $\nu'(N',O,B)$. This can be done by proving that $N'OB \widehat{=} NOA$.

Case 1 $[O - AB] \wedge [O - NN']$ In this case to prove $N'OB \widehat{=} NOA$ we apply the lemma *out_conga*.

Lemma 15. *out_conga*

$$\forall A, B, C, A', B', C', A_0, C_0, A_1, C_1,$$
$$ABC \widehat{=} A'B'C' \wedge [B - AA_0] \wedge [B - CC_0] \wedge [B' - A'A_1] \wedge [B' - C'C_1] \Rightarrow A_0BC_0 \widehat{=} A_1B'C_1$$

Applied in the context :

$$NOA \widehat{=} NOA \wedge [O - NN'] \wedge [O - AB] \wedge [O - NN] \wedge [O - AA] \Rightarrow N'OB \widehat{=} NOA$$

Case 2 $[A - O - B] \wedge [N - O - N']$

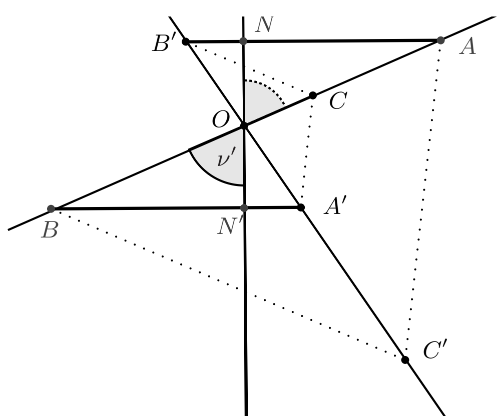

Fig. 15. Case 2 : [A-O-B] and [N-O-N']

In this case to prove $N'OB \widehat{=} NOA$ we have to deal with a pair of vertical angles. This can be done by applying the lemma *l11_13* which say that supplementary angles are congruent if the angles are congruent:

Lemma 16. *l11_13*

$$\forall A, B, C, D, E, F, A', D',$$
$$ABC \widehat{=} DEF \ \wedge\ [A - B - A'] \ \wedge\ A' \neq B \ \wedge\ [D - E - D'] \ \wedge\ D' \neq E \Rightarrow A'BC \widehat{=} D'EF$$

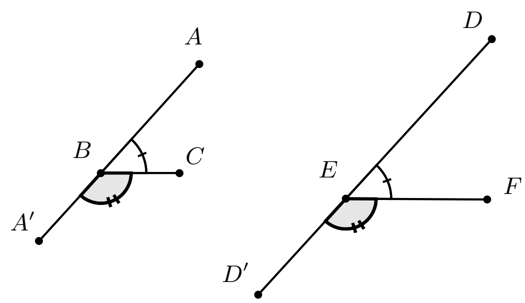

Fig. 16. Congruence of supplementary angles

In the context :

$N'OB' \widehat{=} B'ON' \ \wedge\ [N' - O - N] \ \wedge\ N \neq O \ \wedge\ [A - O - B] \ \wedge\ A' \neq O \Rightarrow NOA \widehat{=} BON'$

6.5 Proof of : $\triangle ON'A'$

We can use a similar approach than previously. But we have before to establish that in the Case 1 we have $[O\ -\ A'B']$ and in the Case 2 we have $[A' - O - B']$.

This result stem from the fact that projections preserves betweenness. Projection properties have been proved in our developments that is not present in Schwabhäuser, Szmielew and Tarski's work.

From this, we deduce two lemmas adapted to the context of the proof.

The lemma $l13\_10\_aux3$ asserts that in the context we can establish :

$$[A - O - B] \Rightarrow [A' - O - B']$$

Lemma 17. *l13_10_aux3*

$$\forall A, B, C, A', B', C', O$$
$$\neg[-OAA'-] \wedge$$
$$B \neq O \ \wedge \ C \neq O \ \wedge \ [-OAB-] \ \wedge \ [-OBC-] \wedge$$
$$B' \neq O \ \wedge \ C' \neq O \ \wedge \ [-OA'B'-] \ \wedge \ [-OB'C'-] \wedge$$
$$BC' \underset{O}{\perp\!\!\!\perp} CB' \ \wedge \ CA' \underset{O}{\perp\!\!\!\perp} AC' \ \wedge \ [A - O - B] \Rightarrow [A' - O - B']$$

The lemma $l13\_10\_aux5$ asserts that in the context we can establish :

$$[O \ - \ AB] \Rightarrow [O \ - \ A'B']$$

Lemma 18. *l13_10_aux5*

$$\forall A, B, C, A', B', C', O,$$
$$\neg[-OAA'-] \wedge$$
$$B \neq O \ \wedge \ C \neq O \ \wedge \ [-OAB-] \ \wedge \ [-OBC-] \wedge$$
$$B' \neq O \ \wedge \ C' \neq O \ \wedge \ [-OA'B'-] \ \wedge \ [-OB'C'-] \wedge$$
$$BC' \underset{O}{\perp\!\!\!\perp} CB' \ \wedge \ CA' \underset{O}{\perp\!\!\!\perp} AC' \ \wedge \ [O \ - \ AB] \Rightarrow [O \ - \ A'B']$$

6.6 Proof of $ON \perp BA'$

Finaly, once we have established $\triangle ON'B$ and $\triangle ON'A'$ we can deduce $ON \perp BA'$ using the lemma *per_per_perp*.

Lemma 19. *per_per_perp*

$$\forall \ O, N', A', B,$$
$$O \neq N' \ \wedge \ A' \neq B \ \wedge \ (A' \neq N' \ \vee \ B \neq N') \ \wedge \ \triangle ON'A' \ \wedge \ \triangle ON'B \Rightarrow$$
$$ON' \perp A'B$$

We have necessarily $A' \neq N' \ \vee \ B \neq N'$ otherwise all the points (O, A, B, C, A', B', C') would be collinear, which is contrary to the hypothesis.
For the same reason we have $A' \neq B$.
On the other hand, $O \neq N'$ since $lcos(n', a', \nu)$ implies that $\nu = A'ON'$ must be an acute angle because of the definition of $lcos$.

Since we have the hypothesis $ON \perp B'A$ and we proved $ON \perp BA'$ we deduce from the definition of $\perp\!\!\!\perp$ that $AB' \underset{O}{\perp\!\!\!\perp} BA'$. QED.

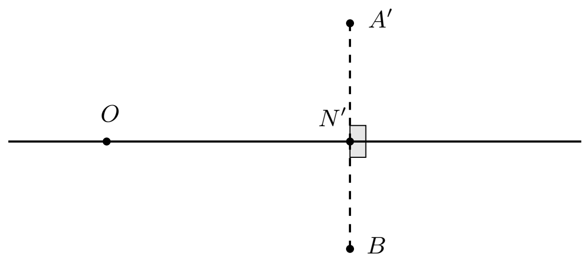

Fig. 17. Case 2 : per_per_perp

7 Some missing lemmas

7.1 About lengths

In the proof, Schwabhäuser, Szmielew and Tarski use a notation by assigning a name to each length like $|OA| = a$. In fact such a notation is valid since, given two points A B, there exists a length l such that $l(AB)$.

In Schwabhäuser, Szmielew and Tarski's work any existence lemma is proved, not even mentioned. Such a lemma is of course trivial, but in the Coq proof assistant, an existence lemma is necessary to assign a name to each length.

Lemma 20. *lg_exists*

$$\forall\ A, B,\ \exists l, length(l) \wedge l(A, B)$$

Conversely, given a length l, we need to prove the existence of two points A and B, such that $l(A, B)$.

Lemma 21. *ex_points_lg*

$$\forall\ l,\ length(l) \Rightarrow \exists A, \exists B, l(A, B)$$

Likewise given a length l and a point A we have a lemma that prove the existence of a point B such that $l(A, B)$

Lemma 22. *ex_point_lg*

$$\forall\ l, A,\ length(l) \Rightarrow \exists B, l(A, B)$$

In the proof, given a length l we have to construct a point B on a half-line AP such that $l(A, B)$.

Lemma 23. *ex_point_lg_out*

$$\forall\ l, A, P\ A \neq P \Rightarrow length(l) \wedge \neg null\_length(l) \Rightarrow \exists B,\ l(A, B) \wedge [A\ -\ BP]$$

Similarly, given a length l we can prolong a segment AP such that $l(P, B)$.

Lemma 24. *ex_point_lg_bet*

$$\forall\ l, A, P,\ length(l) \Rightarrow \exists B,\ l(P, B) \wedge [A - P - B]$$

7.2 About angles

Schwabhäuser, Szmielew and Tarski use a notation by assigning a name to each angle like $COL \widehat{=} BOL$ is called λ. As for lengths, such a notation is valid since, given three points A, B, C there exists angle α such as $\alpha(ABC)$.
In Schwabhäuser, Szmielew and Tarski's proof such trivial lemma doesn't appear, but in the Coq proof assistant an angle existence lemma is necessary to assign a name to each angle.

Lemma 25. *ang_exists*

$$\forall\ A, B, C,\ A \neq B \wedge C \neq B \Rightarrow \exists \alpha, ang(\alpha) \wedge \alpha(A, B, C)$$

Similarly the lemma *anga_exists* works for acute angles :

Lemma 26. *anga_exists*

$$\forall\ A, B, C, A \neq B \wedge C \neq B \wedge \measuredangle ABC \Rightarrow \exists \alpha, anga(\alpha) \wedge \alpha(A, B, C)$$

For completeness we defined some more existence lemmas that doesn't appear in the proof of Pappus' theorem.

- given a point A and an angle α, there exists two points B and C such as $\alpha(A, B, C)$
- given a point B and an angle α, there exists two points A and C such as $\alpha(A, B, C)$
- given two points A, B and an angle α, there exists a point C such as $\alpha(A, B, C)$
- given three points A, B P and an angle α, there exists a point C on the same side of the line AB than P such as $\alpha(A, B, C)$

8 Conclusion

We described a *synthetic proof* of Pappus' theorem for both neutral and euclidean geometry. This is to our knowledge the first formal proof of this theorem using a synthetic approach. This is crucial to obtain a coordinate-free version of the proof of this theorem because this theorem is the main ingredient for building a field and obtaining a coordinate system which will allow the use of the algebraic approaches for automated deduction in geometry. The overall proof consists of approximately 10k lines of proof compared to the proof in [Hil60] which is 2.5 pages long and the version in [SST83] which is 9 pages long.

Availability

The full Coq development is available here: `http://dpt-info.u-strasbg.fr/~narboux/tarski.html`

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