Parallel postulates and decidability of intersection of lines: a mechanized study within Tarski's system of geometry

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Noname manuscript No.
(will be inserted by the editor)

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Abstract In this paper we focus on the formalization of the proof of equivalence between different versions of Euclid's 5^{th} postulate. This postulate is of historical importance because for centuries many mathematicians believed that this statement was rather a theorem which could be derived from the first four of Euclid's postulates and history is rich of incorrect proofs of Euclid's 5^{th} postulate. These *proofs* are incorrect because they assume more or less implicitly a statement which is equivalent to Euclid's 5^{th} postulate and whose validity is taken for granted. Even though these proofs are incorrect the attempt was not pointless because the flawed proof can be turned into a proof that the unjustified statement implies the parallel postulate. In this paper we provide formal proofs verified using the Coq proof assistant that 10 different statements are equivalent to Euclid's 5^{th} postulate. We work in the context of Tarski's neutral geometry without continuity nor Archimedes' axiom. The formalization provide a clarification of the hypotheses used for the proofs. Following Beeson, we study the impact of the choice of a particular version of the parallel postulate on the decidability issues.

Keywords Euclid, parallel postulate, formalization, Tarski's geometry, Coq, foundations of geometry, decidability of intersection

1 Introduction

In this paper we focus on the formalization of results about Euclid's 5^{th} postulate. This postulate is of historical importance because for centuries many mathematicians believed that this statement was rather a theorem which could be derived from the first four of Euclid's postulates. History is rich with incorrect proofs of Euclid's 5^{th} postulate. In 1763, his dissertation written under the guidance of Abraham Gotthelf Kästner, Klügel provides a survey of about 30 attempts to "prove" Euclid's parallel postulate" [Klu63]. Adrien Marie Legendre published a geometry textbook *Eléments de géométrie* in 1774. Each edition of this popular

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book contained an (incorrect) proof of the postulate of Euclid's. Even in 1833, one year after the publication by Bolyai of an appendix about non-euclidean geometry, Legendre was still convinced of the validity of its proofs of Euclid's 5^{th} postulate:

> "Il n'en est pas moins certain que le théorème sur la somme des trois angles du triangle doit être regardé comme l'une de ces vérités fondamentales qu'il est impossible de contester, et qui sont un exemple toujours subsistant de la certiture mathématique qu'on recherche sans cesse et qu'on n'obtient que bien difficilement dans les autres branches des connaissances humaines."[1]

– Adrien Marie Legendre [Leg33]

These proofs are incorrect for different reasons. Some proofs rely on an assumption which is more or less explicit but that the author takes for granted. Some other proofs are incorrect because they rely on an implicit assumption or a circular argument.

Proving the equivalence of different versions of the parallel postulate requires extreme rigor as Richard J. Trudeau has written:

> Pursuing the project faithfully will require that we take the extreme measure of shutting out the entreaties of our intuitions and imaginations - a forced separation of mental powers that will quite understandably be confusing and difficult to maintain [...].

– Richard J. Trudeau [Tru86]

To help us in this task we have a perfect tool which possesses no intuition: a computer. In this paper we provide formal proofs verified using the Coq proof assistant that 10 different statements are equivalent to Euclid's 5^{th} postulate in the theory defined by a subset of the axioms of Tarski's geometry, namely the 2-dimensional neutral geometry without continuity axiom. In this theory, neither of these statements is a theorem nor contradicts the axioms.

More precisely, our formal proofs are based on the first eleven chapters and some results from the twelfth of [SST83] which are valid in neutral geometry. Thoses results have been formalized previously [Nar07,BN12,NBB14] using the Coq proof assistant.

We are of course not the first ones who want to prove the claimed equivalences. In fact, the equivalence between 26 versions of Euclid's 5^{th} postulate can be found in [Mar98]. Greenberg also proves (or leaves as exercises) the equivalence between several versions of the parallel postulate [Gre93]. These proofs are not checked mechanically and sometimes only sketched. We could not reuse directly all these proofs in our context because some proofs in these books use the continuity axiom (see section 5.5).

Following the classical approach to prove that Euclid's 5^{th} postulate is not a theorem of neutral geometry, Timothy Makarios has provided a formal proof of the independence of Tarski's Euclidean axiom [Mak12]. He used the Isabelle proof assistant to construct the Klein-Beltrami model where the postulate is not verified. In the same fashion, a close result is due to Filip Maric and Danijela Petrovic who formalized the complex plane using the Isabelle/HOL proof assistant [MP14].

[1] "It nevertheless certain that the theorem on the sum of the three angles of the triangle should be considered one of those fundamental truths that are impossible to contest and that are an enduring example of mathematical certitude, which is very difficult to obtain in other branches of human knowledge."

Recently, Michael Beeson has also studied the equivalence of different versions of the parallel postulate in the context of a constructive geometry [Bee14].

The paper is structured as follows. In section 2, we describe the axiom system we use and its formalization in Coq. Then in section 3, we describe the different statements we will prove equivalent in section 5 using the lemmas we will present in section 4. We list these simple lemmas because it is difficult to distinguish lemmas which are valid in neutral geometry from statements equivalent to the parallel postulate. Our list will allow the reader to have a clear idea of some statements which are valid in neutral geometry. Most of these lemmas have been proven previously in the formalization of [SST83] by Gabriel Braun and partially described in [Nar07, BN12].

2 The context

In this section we will first present the axiomatic system we used as a basis for our proofs.

It is crucial to be precise about the context and the definitions. This paper is about equivalence properties. But the equivalence are relative to some theory and some logic. We prove the equivalences within the higher order intuitionistic logic of Coq and using Tarski's axiom system for neutral geometry. Using the language of logic, the assertion saying that P and Q are equivalent formally means[2] that : $T \models P \Leftrightarrow Q$ for a given theory T. It could be the case that $T \models P \Leftrightarrow Q$ because both $T \models P$ and $T \models Q$. For every version of the parallel postulates presented in this paper it is also true that $T \not\models P$ but we do not prove the independence results. Defining accurately the theory T is of primary interest because the equivalence results depend on the precise definition of T. For example, Millman and Parker [MP81, p. 226] have shown that the Pythagorean theorem is equivalent to the parallel postulate in the context of an axiom system in the style of Birkhoff axioms based on a ruler and protractor [Bir32]. But there is a (non-archimedean) Pythagorean plane (see example 18.4.3 [Har00, p. 161]) which does not verify the parallel postulate.

2.1 A set of axioms for neutral geometry

Proofs are given within Tarski's system of neutral geometry. We use the axioms given in [SST83] excluding the axiom corresponding to Euclid's 5^{th} postulate, but also an axiom introducing continuity for reasons we will explain in section 5.5. For an explanation of the axioms and their history see [TG99]. Table 1 lists the axioms for neutral geometry. The consistency of this axiom system has been mechanically proven by Makarios [Mak12]. As we already mentioned Makarios has also proven formally the independence of the parallel postulate in this axiom system. These

[2] The meaning of equivalence does not seem to be clear for everyone. For instance, in the french version of the wikipedia page about the parallel postulate (april 2015) one can read that (our own translation) : "These propositions are roughly equivalent to the axiom of parallels. By equivalent, we mean that using some adapted vocabulary, these axioms are true in euclidean geometry but not true in elliptic nor spherical geometry". Thanks to a theorem of Szmielew, it is true that this definition of equivalence is equivalent to the logical one when you have the continuity axiom, but it is far from obvious.

two properties make Tarski's system of geometry suitable for proofs of equivalence of statements of Euclid's parallel postulate.

Let us recall that Tarski's axiom system is based on a single primitive type depicting points and two predicates, namely between noted by —— and congruence noted by $\equiv$. $A—B—C$ means that A, B and C are collinear and B is between A and C (and B may be equal to A or C). $AB \equiv CD$ means that the line-segments $\overline{AB}$ and $\overline{CD}$ have the same length (the orientation does need to be the same). Let us also recall that we chose to not use the continuity axiom in our proofs.

Notice that lines can be represented by pairs of distinct points and using the collinearity predicate. Angles can be represented by a triple of points and an angle congruence predicate.

A1	Symmetry	$AB \equiv BA$
A2	Pseudo-Transitivity	$AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF$
A3	Cong Identity	$AB \equiv CC \Rightarrow A = B$
A4	Segment construction	$\exists E, A—B—E \wedge BE \equiv CD$
A5	Five-segment	$AB \equiv A'B' \wedge BC \equiv B'C' \wedge$ $AD \equiv A'D' \wedge BD \equiv B'D' \wedge$ $A—B—C \wedge A'—B'—C' \wedge A \neq B \Rightarrow CD \equiv C'D'$
A6	Between Identity	$A—B—A \Rightarrow A = B$
A7	Inner Pasch	$A—P—C \wedge B—Q—C \Rightarrow \exists X, P—X—B \wedge Q—X—A$
A8	Lower Dimension	$\exists ABC, \neg A—B—C \wedge \neg B—C—A \wedge \neg C—A—B$
A9	Upper Dimension	$AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q$ $\Rightarrow A—B—C \vee B—C—A \vee C—A—B$.

Table 1: Tarski's axiom system for neutral geometry.

Now, we describe these axioms one by one.

The symmetry axiom (A1 on Table 1) for equidistance together with the transitivity axiom (A2) for equidistance imply that the equidistance relation is an equivalence relation.

The identity axiom for equidistance (A3) ensures that only degenerated line-segments can be congruent to a degenerated line-segment.

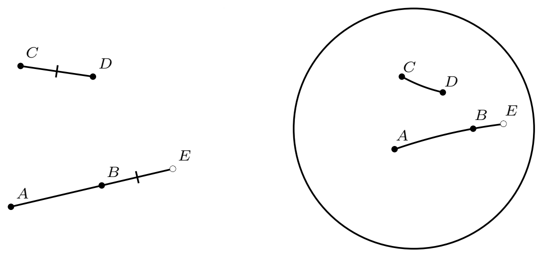


Fig. 1: Axiom of segment construction.

The axiom of segment construction (A4) (Fig. 1[3]) allows to extend a line-segment by a given length.

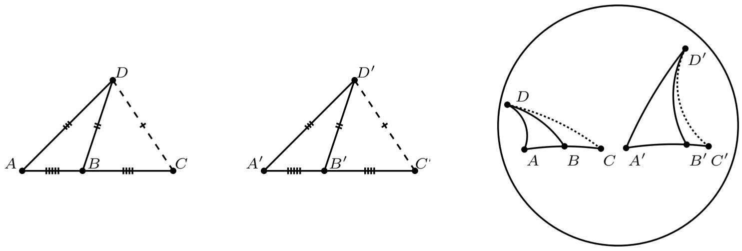


Fig. 2: Five segment axiom.

The five-segment axiom (A5) (Fig. 2) corresponds to the well-known Side-Angle-Side postulate but is expressed with the betweenness and congruence relations only. The lengths of $\overline{AB}$, $\overline{AD}$ and $\overline{BD}$ and the fact that $A—B—C$ fix the angle $\angle CBD$.

The identity axiom for betweenness expresses that the only possibility to have B between A and A is to have A and B equal. It also implies that the relation of betweenness is non-strict unlike Hilbert's one. As Beeson suggests in [Bee15] this choice was probably made to have a reduced number of axioms by allowing degenerated cases of the Pasch's axiom.

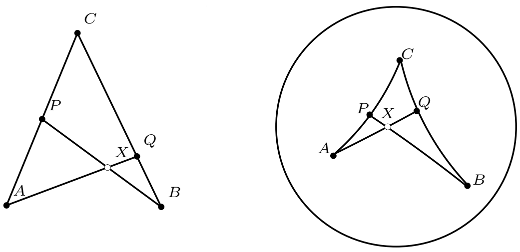


Fig. 3: Pasch's axiom.

The inner form of the Pasch's axiom (Fig. 3) is the axiom Moritz Pasch introduced in [Pas76] to repair the defects of Euclid. It intuitively says that if a line meets one side of a triangle and does not pass through the endpoints of that side,

[3] We will provide figures both in the euclidean model and a non-euclidean model, namely the Poincaré disk model. The figure on the left hand side will illustrate the validity of the axiom or lemma in euclidean geometry. The figure on the right hand side will either depict the validity of the statement in the Poincaré disk model or exhibit a counter-example. We exhibit a counter-example for statements which are equivalent to the parallel postulate.

then it must meet one of the other sides of the triangle. There are three forms of this axiom. Thanks to Gupta's thesis [Gup65] one knows that the inner form and the outer form are equivalent and that both of them allow us to prove the weak form. We will present the outer form and the weak form later on. The inner form enunciates Pasch's axiom without any case distinction. Indeed it indicates that the line BP must meet the triangle ACQ on the side $\overline{AQ}$ as Q is between B and C.

The lower 2-dimensional axiom asserts that the existence of three non-collinear points.

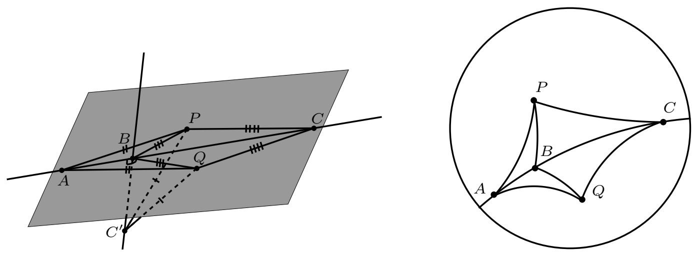


Fig. 4: Upper dimensional axiom.

The upper 2-dimensional axiom (Fig. 4) means that all the points are coplanar. Since A, B and C are equidistant to P and Q which are different they belong to the hyperplane consisting of all the points equidistant to P and Q. Because the upper 2-dimensional axiom specifies that A, B and C are collinear this hyperplane is of dimension 1 and it fixes the dimension of the space to 2. It forbids the existence of the point C' in Fig. 4.

2.1.1 Formalization in Coq

Contrary to the formalization of Hilbert's axiom system [DDS00,BN12] which leaves room for interpretation of natural language, the formalization in Coq of Tarski's axiom system is straightforward as the axioms are stated very precisely. We define the axiom system using two type classes. The first one regroup the axioms for neutral geometry in any dimension greater than 1. The second one ensures that the space is of dimension 2.

We also assume that we can reason by cases on the equality of points (class `EqDecidability`). This axiom does not appear in [SST83] because it is a tautology in classical logic but we work in an intuitionnist setting. We have shown [BNSB14a] previously that decidability of equality implies decidability of the main predicates defined in [SST83] (those that are listed in Table 3) except the existence of intersection of two lines. The formalization is given in Table 2.

```
Class Tarski_neutral_dimensionless := {
 Tpoint : Type;
 Bet : Tpoint -> Tpoint -> Tpoint -> Prop;
 Cong : Tpoint -> Tpoint -> Tpoint -> Tpoint -> Prop;
 between_identity : forall A B, Bet A B A -> A=B;
 cong_pseudo_reflexivity : forall A B : Tpoint, Cong A B B A;
 cong_identity : forall A B C : Tpoint, Cong A B C C -> A = B;
 cong_inner_transitivity : forall A B C D E F : Tpoint,
   Cong A B C D -> Cong A B E F -> Cong C D E F;
 inner_pasch : forall A B C P Q : Tpoint,
      Bet A P C -> Bet B Q C ->
      exists x, Bet P x B /\ Bet Q x A;
 five_segment : forall A A' B B' C C' D D' : Tpoint,
    Cong A B A' B' ->
    Cong B C B' C' ->
    Cong A D A' D' ->
    Cong B D B' D' ->
    Bet A B C -> Bet A' B' C' -> A <> B -> Cong C D C' D';
 segment_construction : forall A B C D : Tpoint,
    exists E : Tpoint, Bet A B E /\ Cong B E C D;
 lower_dim : exists A, exists B, exists C, ~ (Bet A B C \/ Bet B C A \/ Bet C A B)
}.

Class Tarski_2D `(Tn : Tarski_neutral_dimensionless) := {
  upper_dim : forall A B C P Q : Tpoint,
    P <> Q -> Cong A P A Q -> Cong B P B Q -> Cong C P C Q ->
    (Bet A B C \/ Bet B C A \/ Bet C A B)
}.

Class EqDecidability U := {
 eq_dec_points : forall A B : U, A=B \/ ~ A=B
}.
```

Table 2: Formalization of the axiom system in Coq .

3 Equivalent statements of Euclid's parallel postulates

In this section, we give the definitions of some properties which are equivalent to Euclid's parallel postulate. These properties are diverse. Some of these properties are expressed using definitions present in [SST83]. So we also present these definitions in this section. We list in this paper the exact Coq syntax of the axioms, definitions and main theorems without any pretty printing to give the reader the opportunity to check what is the exact statement we proved. For the auxiliary lemmas and all the proofs we will use a classical mathematical notation. The proofs given in this paper serve only as a documentation; the correctness of the results is assured by the mechanical proof checker. Recall that for each postulate, we will provide the figure representing the statement in the euclidean plane and a counterexample in Poincaré's disk model. Having a counterexample in non-euclidean geometry is interesting as Szmielew proved (assuming full continuity) that every statement which is false in hyperbolic geometry and correct in Euclidean geometry is equivalent to the fifth parallel postulate [Szm59].

We start by the official version of the parallel postulate found in [SST83]:

```
Definition tarski_s_parallel_postulate :=
 forall A B C D T : Tpoint,
 Bet A D T -> Bet B D C -> A<>D ->
 exists x, exists y,
 Bet A B x /\ Bet A C y /\ Bet x T y.
```

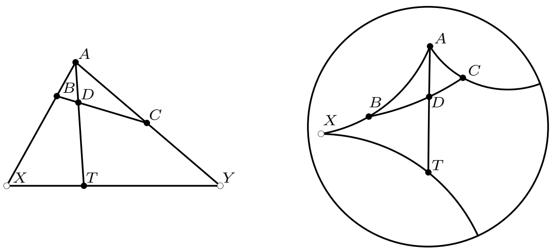


Fig. 5: Tarski's parallel postulate.

The statement (Fig. 5) is an implicit assumption made by Legendre while attempting to prove that Euclid's parallel postulate was a consequence of Euclid's other axioms. Tarski's parallel postulate states that if the point D belongs to the interior of the angle $\angle BAC$ and given a point T further away from A than D on half line AD then one can build a line which intersects the sides of the angle and which goes through T. This version is particularly interesting as it has the advantage that it is easily expressed in term of betweenness and only uses betweenness and that it is valid in dimensions higher than 2.

Among the first definitions which are introduced there is the predicates Col expressing collinearity. It can be defined using only the betweeness predicate (——).

```
Definition Col (A B C : Tpoint) : Prop := Bet A B C \/ Bet B C A \/ Bet C A B.
```

Col $A\,B\,C$ expresses that A, B and C are collinear if and only if one of the three points is between the other two.

Another version (Fig. 6) which is easy to express in the language of Tarski's geometry is the existence of the circumcenter of a triangle. According to Givant [TG99] this is one of the favorite version of Tarski. This version of the axiom is usually attributed to Farkas Bolyai the father of Janos Bolyai who discovered hyperbolic geometry.

```
Definition triangle_circumscription_principle :=
  forall A B C,  ~ Col A B C -> exists CC, Cong A CC B CC /\ Cong A CC C CC.
```

We denote `BetS A B C` by $A \multimap B \multimap C$. It means that B is strictly between A and C. This corresponds to Hilbert's betweenness relation.

```
Definition BetS A B C := Bet A B C /\ A <> B /\ B <> C.
```

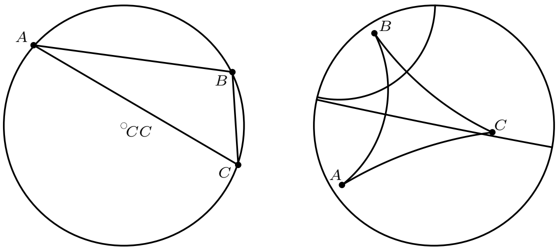


Fig. 6: The triangle circumscription principle.

The original Euclid's 5$^{\text{th}}$ postulate state that (wikipedia's translation):

> If a line-segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

```
Definition euclid_5 :=
  forall P Q R S T U,
  BetS P T Q -> BetS R T S -> BetS Q U R -> ~ Col P Q S ->
  Cong P T Q T -> Cong R T S T ->
  exists I, BetS S Q I /\ BetS P U I.
```

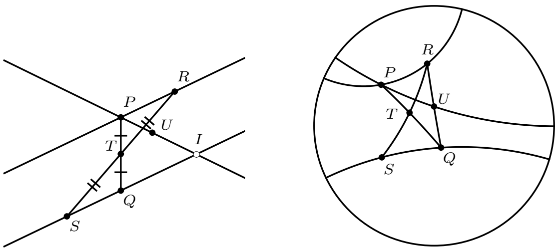


Fig. 7: Euclid 5.

This axiom (Fig. 7) was introduced by Michael Beeson in [Bee14]. It corresponds to the original Euclid's 5$^{\text{th}}$ postulate expressed with only the congruence and strict betweeness relations. He expressed the fact that the two angles sum to less than two right angles by the facts that U belongs strictly to the interior of the angle $\angle RPS$ and that $PRQS$ is a parallelogram (it can be proven in neutral geometry that lines PR and QS are parallel but proving in neutral geometry that angles $\angle RPS$ and $\angle PSQ$ sum to two right angles would be impossible as the fact that the interior angles of a triangle sum to two right angles is itself equivalent to the parallel postulate).

```
Definition strong_parallel_postulate :=
  forall P Q R S T U,
  BetS P T Q -> BetS R T S -> ~ Col P R U ->
  Cong P T Q T -> Cong R T S T ->
  exists I, Col S Q I /\ Col P U I.
```

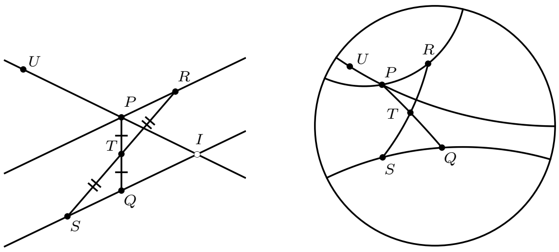


Fig. 8: The strong parallel postulate.

The strong parallel postulate (Fig. 8) is very similar to the previous statement as it was also introduced by Michael Beeson in [Bee14]. This postulate has a weaker conclusion but weaker hypotheses. Indeed we do not know on which side the lines will meet but point U is only required to be outside line PR.

Now we will present probably the most important definition for this work, namely the parallelism. In [SST83] one can find two definitions of parallelism. The common way of defining it is to consider two lines as parallel if they belong to the same plane but do not meet. This implies that we will also define coplanarity. The other definition of parallelism includes the previous one and add the possibility for the lines to be equal. Therefore we will talk about strict parallelism in the first case and about parallelism in the second.

```
Definition coplanar A B C D := exists X,
 (Col A B X /\ Col C D X) \/ (Col A C X /\ Col B D X) \/ (Col A D X /\ Col B C X).
```

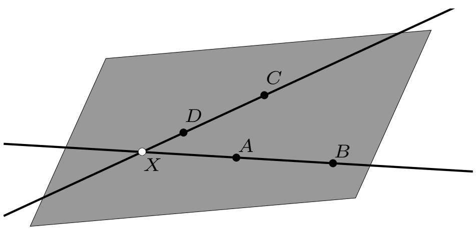


Fig. 9: Definition of `coplanar`.

We denote `coplanar A B C D` by Cp $A\,B\,C\,D$. We did not define coplanarity in the same way as in [SST83]. We chose to express coplanarity as a 4-ary predicate to avoid the definition of a predicate with an arbitrary number of terms. Restricting ourselves to characterize coplanarity of four points we could use lemma 9.33 in [SST83] as a definition of coplanarity. This definition (Fig. 9) states that 4 points are coplanar if two out of these four points form a line which intersect the line formed by the remaining two points. Since we are in a 2-dimensional space 4 points are always coplanar but we still need this predicate as it allows us to do case distinctions on the relative position of 4 given points.

Then we can define the notion of parallelism. Two lines are said to be strictly parallel if the points that define them are coplanar and if these lines do not meet. Since we are in dimension 2 we will be able to prove that the points are coplanar, but we keep this definition because we plan to extend our formalization to dimension n in the future.

```
Definition Par_strict := fun A B C D =>
 A<>B /\ C<>D /\ coplanar A B C D /\ ~ exists X, Col X A B /\ Col X C D.
```

We denote `Par_strict A B C D` by $AB \parallel_s CD$. Note that one could choose other definitions, for instance one could define two lines to be parallel when they are at constant distance, according to Papadopoulos [Pap12], that was the definition used by Posidonius, an early commenter of Euclid's elements. Remember that an implicit change in a definition can have severe consequences in the validity of a proof.

We introduce a new definition which asserts that two lines are parallel if they are strictly parallel or if they are equal since with the previous definition one line is not parallel to itself.

```
Definition Par := fun A B C D =>
 Par_strict A B C D \/ (A<>B /\ C<>D /\ Col A C D /\ Col B C D).
```

We denote `Par A B C D` by $AB \parallel CD$.

Playfair's postulate (Fig. 10) is the one which is well known for the modern reader. It states that there is a unique parallel to a given line going through some point. Note that it does not state the existence of the parallel line but only its uniqueness because the existence can be proved from the axioms of Tarski's neutral geometry as we will see (lemma 28). In [Hil60], Hilbert uses this version of the axiom. Proclus an early commenter of Euclid's *Elements*, already recognized that an incorrect proof of Euclid's postulate by Ptolemy was making this implicit assumption.

```
Definition playfair_s_postulate :=
 forall A1 A2 B1 B2 C1 C2 P,
 Par A1 A2 B1 B2 -> Col P B1 B2 ->
 Par A1 A2 C1 C2 -> Col P C1 C2 ->
 Col C1 B1 B2 /\ Col C2 B1 B2.
```

From Playfair's postulate it is easy to obtain the equivalence with the postulate of transitivity of parallelism.

```
Definition postulate_of_transitivity_of_parallelism :=
 forall A1 A2 B1 B2 C1 C2,
 Par A1 A2 B1 B2 -> Par B1 B2 C1 C2 -> Par A1 A2 C1 C2.
```

This postulate (Fig. 11) would have been inconsistent with the other axiom if we would have taken Euclid's definition of the parallelism(wikipedia's translation):

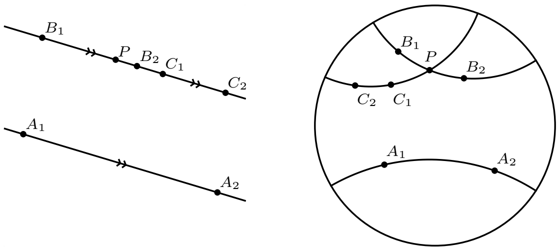


Fig. 10: Playfair's postulate.

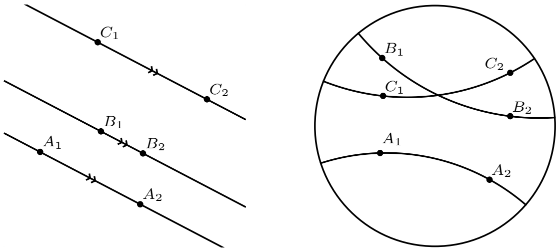


Fig. 11: The postulate of transitivity of parallelism.

> Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in either direction, do not meet one another in either direction.

Indeed it is possible for lines A_1A_2 and C_1C_2 to be equal.

Proclus' postulate states that if a line intersects one of two parallel lines then it intersects the other:

```
Definition proclus_postulate :=
 forall A B C D P Q, Par A B C D -> Col A B P -> ~ Col A B Q ->
 exists Y, Col P Q Y /\ Col C D Y.
```

Although this postulate (Fig. 12) seems similar to the strong parallel postulate only one of the implications required to prove the equivalence is trivial. As a matter of fact for proving that the strong parallel postulate implies Proclus' postulate we prove first that it implies Playfair's postulate.

```
Definition is_midpoint M A B := Bet A M B /\ Cong A M M B.
```

We denote `is_midpoint M A B` by $A\text{-}M\text{-}B$. It states that M is the midpoint of A and B. It is the case when M is between A and B and equidistant from them. It is interesting to notice that the existence of the midpoint appears quite late in [SST83]. This is due to Gupta's proof of the existence of the midpoint of two given points [Gup65] which cannot be done earlier in the development. He managed to give a proof which does not involve the continuity axiom.

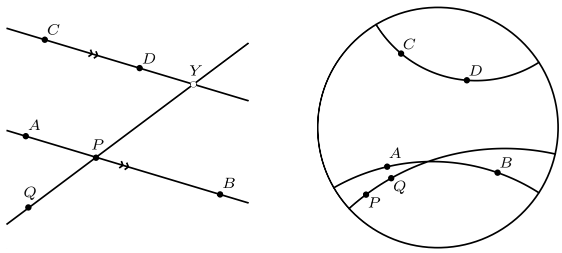


Fig. 12: Proclus' postulate.

```
Definition midpoints_converse_postulate :=
 forall A B C P Q,
 ~ Col A B C -> is_midpoint P B C -> Par A B Q P -> Col A C Q ->
 is_midpoint Q A C.
```

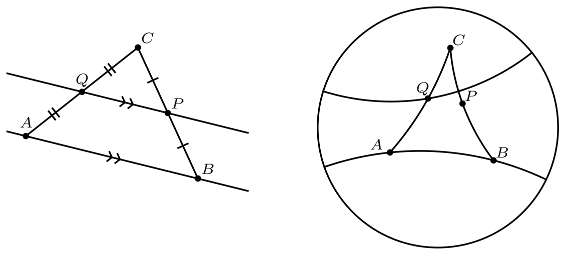


Fig. 13: The midpoint converse postulate.

The midpoint converse postulate (Fig. 13) is the converse of the midpoint theorem, which states that the parallel to a side of a triangle going through the midpoint of another side goes through the third side in its midpoint.

Be careful that the midpoint theorem is valid in **neutral geometry**.

```
Lemma triangle_mid_par : forall A B C P Q,
~Col A B C -> is_midpoint P B C -> is_midpoint Q A C -> Par_strict A B Q P.
```

It is a trivial consequence of lemma 13.1 in [SST83] which is proven in neutral geometry.

Another important notion when working with the parallel postulates is the perpendicularity. In [SST83] the perpendicularity is defined using the existence of a midpoint.

```
Definition Per A B C := exists C', is_midpoint B C C' /\ Cong A C A C'.
```

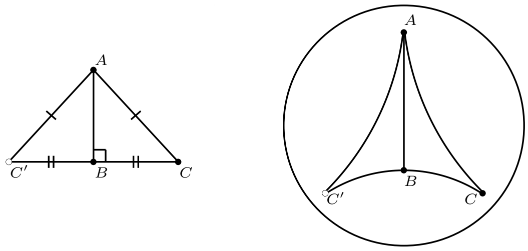


Fig. 14: Definition of `Per`.

We denote `Per A B C` by $\llcorner\ A\,B\,C$. In case B is different from A and C, $\llcorner\ A\,B\,C$ (Fig. 14) means that A, B and C form a right angle. But $\llcorner\ A\,B\,C$ is also true when B is equal to A and/or C. Therefore we need a new definition to avoid this case. Moreover with this definition one needs to know the point in which the perpendicular lines meet.

```
Definition Perp_in X A B C D :=
 A<>B /\ C<>D /\ Col X A B /\ Col X C D /\
 (forall U V, Col U A B -> Col V C D -> Per U X V).
```

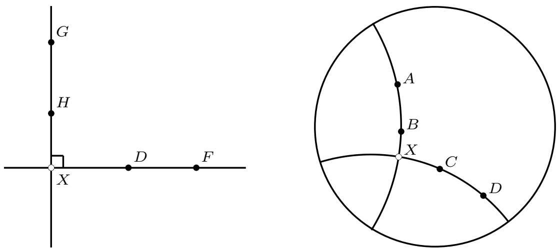


Fig. 15: Definition of `Perp_in`.

We denote `Perp_in X A B C D` (Fig. 15) by $AB \underset{X}{\perp} CD$. It means that lines AB and CD (when we consider lines it is implied that the two points are distinct) meet at a right angle in X. The part of the definition that specifies that any point on the first line together with any point on the second line and the point X form a right angle is essential to the possibility of choosing any couple of different points to represent the lines.

```
Definition Perp A B C D := exists X, Perp_in X A B C D.
```

Most of the time we just want to consider the perpendicularity of two lines without specifying the point in which these two lines meet. In such cases we will use the following predicate that we denote $\perp$. Nevertheless every time we will want to prove that two lines are perpendicular we will have to determine the intersection point.

The following two postulates are properties which would be easy to take for granted and assume implicitly.

```
Definition perpendicular_transversal_postulate :=
 forall A B C D P Q,
 Par A B C D -> Perp A B P Q -> Perp C D P Q.
```

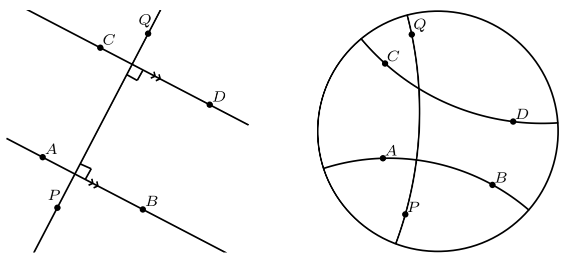


Fig. 16: The perpendicular transversal postulate.

The perpendicular transversal postulate (Fig. 16) states that given two parallel lines, any line perpendicular to the first line is perpendicular to the second line.

Just as for the midpoint theorem the converse of the perpendicular transversal postulate is a lemma in neutral geometry. It corresponds to lemma 12.9 in [SST83].

```
Lemma l12_9 : forall A_1 A_2 B_1 B_2 C_1 C_2,
 Perp A_1 A_2 C_1 C_2 -> Perp B_1 B_2 C_1 C_2 -> Par A_1 A_2 B_1 B_2
```

The last postulate states that two lines perpendicular to two parallel lines are parallel.

```
Definition postulate_of_parallelism_of_perpendicular_tranversals :=
 forall A1 A2 B1 B2 C1 C2 D1 D2,
 Par A1 A2 B1 B2 -> Perp A1 A2 C1 C2 -> Perp B1 B2 D1 D2 -> Par C1 C2 D1 D2.
```

Exactly as for the previous postulate one could think that it can be proven in neutral geometry because of it similarity to lemma 12.9 in [SST83]. Again this statement (Fig. 17) is equivalent to the parallel postulate.

4 A library for neutral geometry

Throughout [SST83] the authors introduce many definitions and prove properties about them. Following are the definitions we will need to state the lemmas we will

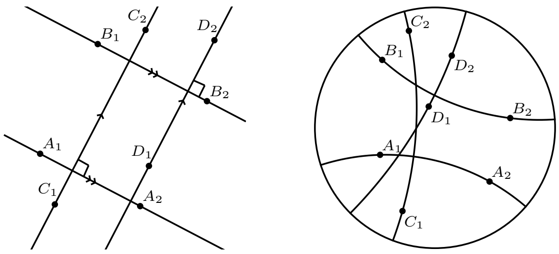


Fig. 17: The postulate of parallelism of perpendicular transversals.

apply to prove the equivalences. We chose not to include trivial lemmas which state permutation properties of the predicates ($AB \parallel CD \Rightarrow CD \parallel AB$ by example). We also decided to omit lemmas allowing to weaken a statement ($AB \|_s CD \Rightarrow AB \| CD$ for instance). One problem one encounters with Tarski's system of geometry is the fact that there is no primitive type *line*. Therefore when considering a line one represents it by two different points. This implies that we need a lemma such as $C \neq D' \Rightarrow AB \parallel CD \Rightarrow \text{Col } C\,D\,D' \Rightarrow AB \parallel CD'$. This kind of lemma are easily proven in neutral geometry. Moreover the proofs of collinearity can be automated by a reflexive tactic that we developed in [BNSB14b]. Therefore we will simply use them implicitly as one would do in a pen and paper proof.

4.1 Lemmas about order relation over points on a line

Lemma 1 (l5.2[4]) $\forall ABCD, A \neq B \Rightarrow A{-}B{-}C \Rightarrow A{-}B{-}D \Rightarrow B{-}C{-}D \vee B{-}D{-}C.$

Lemma 2 (l5.3) $\forall ABCD, A{-}B{-}D \Rightarrow A{-}C{-}D \Rightarrow A{-}B{-}C \vee A{-}C{-}B.$

Lemmas 1 and 2 can be seen as trivial but they were originally axioms in Tarski's system of geometry and they were shown derivable from axioms given on table 1 by Eva Kallin, Scott Taylor, and Tarski in [Tar59].

4.2 Lemmas about lines

Lemma 3 (l6.21) $\forall ABCDPQ, \neg\text{Col } A\,B\,C \Rightarrow C \neq D \Rightarrow \text{Col } A\,B\,P \Rightarrow \text{Col } A\,B\,Q \Rightarrow \text{Col } C\,D\,P \Rightarrow \text{Col } C\,D\,Q \Rightarrow P = Q.$

Lemma 3 (Fig. 18) allows to prove that two points are equal if they are at the intersection of two different lines. It is one of the most useful lemma to prove non-degeneracy conditions when the tactics we developed in [BNSB14b] do not succeed in proving them.

Lemma 4 (l6.25) $\forall AB, A \neq B \Rightarrow \exists C, \neg\text{Col } A\,B\,C.$

Lemma 4 establishes the existence of a point non-collinear to any two given points. This lemma is crucial since it is used together with lemma 9 and lemma 21 to drop or erect perpendicular lines.

[4] The names given in parenthesis are the name as given in [SST83].

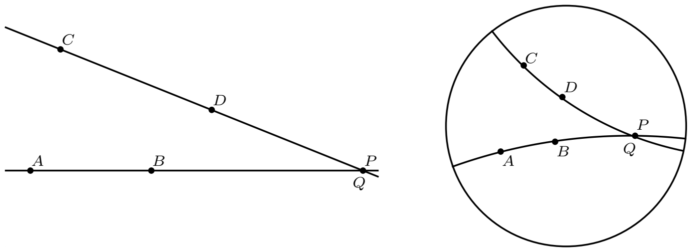


Fig. 18: Figure for lemma 3.

4.3 Lemmas about midpoint

Lemma 5 (l7.8) $\forall AP, \exists P', P{-}A{-}P'$.

Lemma 5 pronounces the existence of the symmetric of a point with respect to another point. In fact this lemma also states the uniqueness of the constructed point. But since we never use the uniqueness property we just present the existence part which corresponds to a special case of the axiom of segment construction.

Lemma 6 (l7.13) $\forall APQP'Q', P'{-}A{-}P \Rightarrow Q'{-}A{-}Q \Rightarrow PQ \equiv P'Q'$.

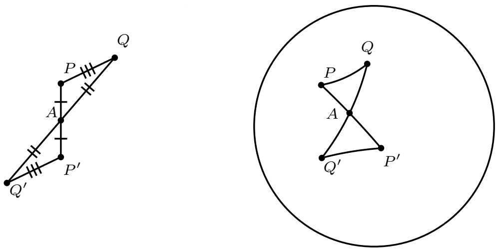


Fig. 19: Figure for lemma 6.

Lemma 6 (Fig. 19) states that the opposite sides of a quadrilateral which has its diagonals intersecting in their midpoint are congruent. There is no non-degeneracy condition for this lemma as it is obviously the case for a degenerated quadrilateral.

Lemma 7 (l7.17) $\forall PP'AB, P{-}A{-}P' \Rightarrow P{-}B{-}P' \Rightarrow A = B$.

Lemma 7 corresponds to the uniqueness of the midpoint of a line-segment and allows to prove that two points are equal if they both are the midpoint of the same line-segment.

Lemma 8 (l7.20) $\forall MAB, \mathrm{Col}\ A\ M\ B \Rightarrow MA \equiv MB \Rightarrow A = B \vee A{-}M{-}B$.

Lemma 8 establishes that if we can prove that a point M is collinear with A and B and equidistant from these same points, then either these points are equal or M is their midpoint.

4.4 Lemmas about orthogonality

Lemma 9 (l8.18) $\forall ABC, \neg\,\mathrm{Col}\ A\,B\,C \Rightarrow \exists X, \mathrm{Col}\ A\,B\,X \wedge AB \perp CX$.

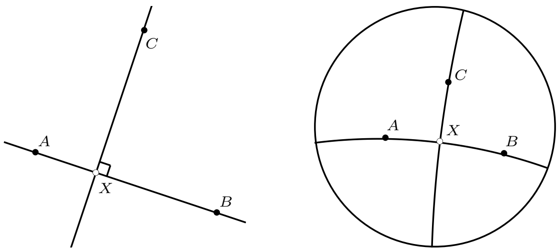


Fig. 20: Figure for lemma 9.

Lemma 9 (Fig. 20) expresses that in neutral geometry it is possible to construct a line perpendicular to a given line passing through a given point. One should be careful to the fact that this only allows us to drop a perpendicular and not erect one.

Lemma 10 (l8.22) $\forall AB, \exists X, A \mathrel{-\!\!+\!\!} X \mathrel{-\!\!+\!\!} B$.

Lemma 10 asserts the existence of the midpoint of any two points. The proof is not trivial as we do not assume any continuity axiom, the usual construction of midpoint using the intersection of two circles and the perpendicular bisector can not be used.

4.5 Lemmas about half-planes

Lemma 11 (l9.6) $\forall ABCPQ, A\text{—}C\text{—}P \Rightarrow B\text{—}Q\text{—}C \Rightarrow \exists X, A\text{—}X\text{—}B \wedge P\text{—}Q\text{—}X$.

Lemma 11 (Fig. 21) is also known as the outer form of the Pasch's axiom. It indicates that the line BP must meet the triangle ACQ on the side $\overline{AQ}$ as C is between B and Q. The outer form of the Pasch's axiom differs from the inner form in that the intersection point lies outside the line-segment $\overline{BP}$ in the outer form unlike in the inner form.

Definition 1 Opposite sides of a line

$$A \underset{P}{\overset{Q}{\text{—/—}}} B := A \neq B \wedge \neg\,\mathrm{Col}\ P\,A\,B \wedge \neg\,\mathrm{Col}\ Q\,A\,B \wedge \exists T, \mathrm{Col}\ T\,A\,B \wedge P\text{—}T\text{—}Q.$$

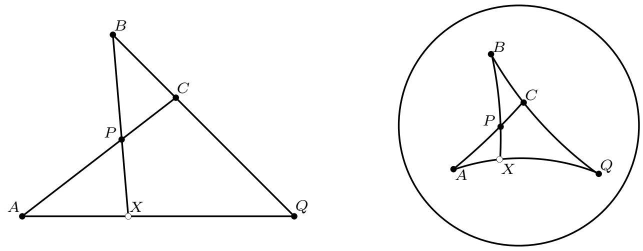


Fig. 21: Figure for lemma 11.

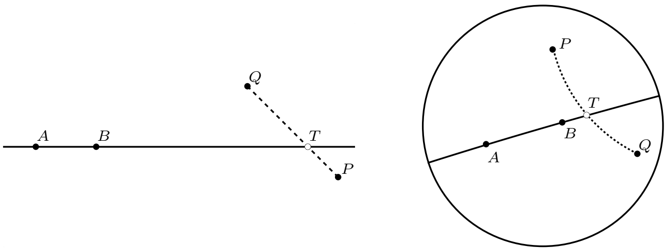


Fig. 22: Figure for definition 1.

A special case of the coplanarity is when the intersection point is between the two points defining one of the lines. In that case one says that these two points are on opposite sides of the other line. $A \overset{Q}{\underset{P}{-\!/\!-}} B$ (Fig. 22) indicates that P and Q are on opposite sides of line AB. This definition being a special case of coplanarity it has the advantage of being valid in more than 2 dimensions.

Definition 2 Same side of a line

$$A \underset{XY}{-\!/\!-} B := \exists Z, A \overset{Z}{\underset{X}{-\!/\!-}} B \wedge A \overset{Z}{\underset{Y}{-\!/\!-}} B.$$

In order to define an analogous to the *plane separation axiom* of Hilbert one needs a predicate expressing that two points are on the same side of line. Two points are said to be on the same side of a line if there exists a third point with which both of the points are on opposite side of this line. $A \underset{PQ}{-\!/\!-} B$ (Fig. 23) indicates that P and Q are on the same side of line AB.

Lemma 12 (l9.8.2) $\forall PQABC, P \overset{C}{\underset{A}{-\!/\!-}} Q \Rightarrow P \underset{AB}{-\!/\!-} Q \Rightarrow P \overset{C}{\underset{B}{-\!/\!-}} Q.$

Lemma 12 (Fig. 24) corresponds to the plane separation axiom of Hilbert. It asserts that when A and C are on opposite sides of a line and B is on the same side of this line as A then B and C are on opposite sides of this line.

Lemma 13 (l9.9) $\forall PQAB, P \overset{B}{\underset{A}{-\!/\!-}} Q \Rightarrow \neg P \underset{AB}{-\!/\!-} Q.$

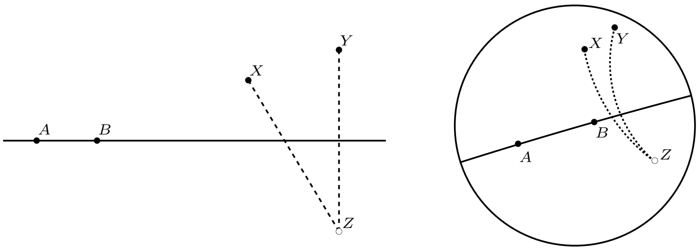


Fig. 23: Figure for definition 2.

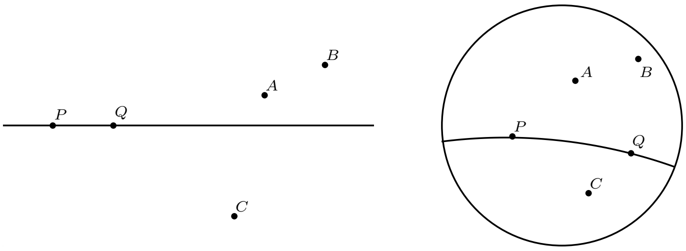


Fig. 24: Figure for lemma 12.

Lemma 14 $\forall PQAB, P \underset{AB}{\not\!\!\relbar\!\!\relbar} Q \Rightarrow \neg P \overset{B}{\underset{A}{\not\!\!\relbar\!\!\relbar}} Q.$

Lemma 15 $\forall PQAB, \neg P \overset{B}{\underset{A}{\not\!\!\relbar\!\!\relbar}} Q \Rightarrow P \underset{AB}{\not\!\!\relbar\!\!\relbar} Q.$

Lemma 13, 14 and 15 are extremely useful to reach a contradiction when one can prove that two points are simultaneously one the same side and on opposite sides of a given line. It is important to note that lemmas 14 and 15 are not in [SST83].

Lemma 16 (l9.13) $\forall PQABC, P \underset{AB}{\not\!\!\relbar\!\!\relbar} Q \Rightarrow P \underset{BC}{\not\!\!\relbar\!\!\relbar} Q \Rightarrow P \underset{AC}{\not\!\!\relbar\!\!\relbar} Q.$

Lemma 16 (Fig. 25) pronounces the pseudo-transitivity of the property of being on the same side of a line. It is often used together with lemma 18 which we present shortly after. We do so when we know the intersection of the line PQ with the lines AB and BC but not with the line AC.

Lemma 17 (l9.17) $\forall ABCPQ, P \underset{AC}{\not\!\!\relbar\!\!\relbar} Q \Rightarrow A{-}B{-}C \Rightarrow P \underset{AB}{\not\!\!\relbar\!\!\relbar} Q.$

Lemma 17 (Fig. 26) states that any point between two points on the same side of a line, is on this same side too. It is often used together with lemma 16. Indeed it can be easier to take a point on the line passing through the two considered points in order to use it with the pseudo-transitivity property of being on the same side of a line.

Predicate $\not\!\!\relbar\!\!\longleftrightarrow$ is among the first definitions and can be defined using only the betweeness predicate and the equality of points.

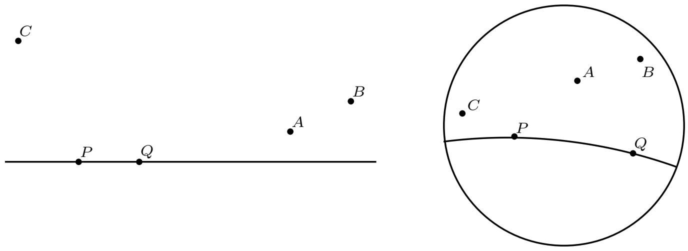


Fig. 25: Figure for lemma 16.

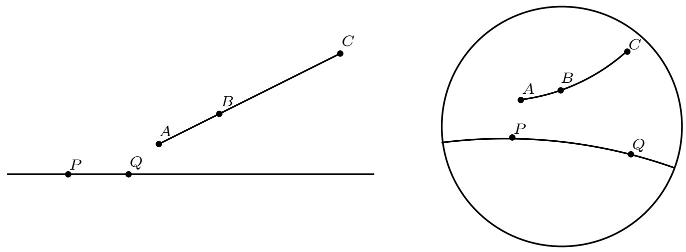


Fig. 26: Figure for lemma 17.

Definition 3 Same side of a point

$$O \mathbin{-\!\!/\!\!/\!\!-} A \leftrightarrow B := O \neq A \wedge O \neq B \wedge (O - A - B \vee O - B - A).$$

$P \mathbin{-\!\!/\!\!/\!\!-} A \leftrightarrow B$ indicates that P belongs to line AB but does not belong to the line-segment $\overline{AB}$.

Lemma 18 (l9.19) $\forall XYABP, X \neq Y \Rightarrow \text{Col } X\,Y\,P \Rightarrow \text{Col } A\,B\,P \Rightarrow (X \underset{AB}{-\!\!/\!\!/\!\!-} Y \Leftrightarrow (P \mathbin{-\!\!/\!\!/\!\!-} A \leftrightarrow B \wedge \neg \text{Col } X\,Y\,A))$.

Lemma 18 (Fig. 27) establishes the equivalence between being on the same side of a line for two points and being on the the same half-line which does not belong to the given line provided that we know the point of intersection of the given line with the half-line.

Lemma 19 (l9.31) $\forall AXYZ, A \underset{YZ}{-\!\!/\!\!/\!\!-} X \Rightarrow A \underset{YX}{-\!\!/\!\!/\!\!-} Z \Rightarrow A \overset{Z}{\underset{X}{-\!\!/\!\!/\!\!-}} Y$.

Lemma 19 (Fig. 28) expresses that Y and Z being on the same side of the line AX in conjunction with X and Y being on the same side of the line AZ implies that X and Z are on opposite sides of the line AY.

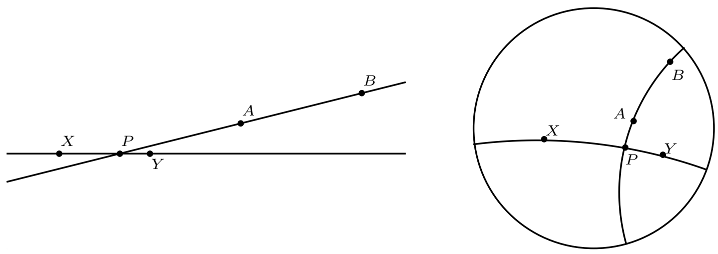


Fig. 27: Figure for lemma 18.

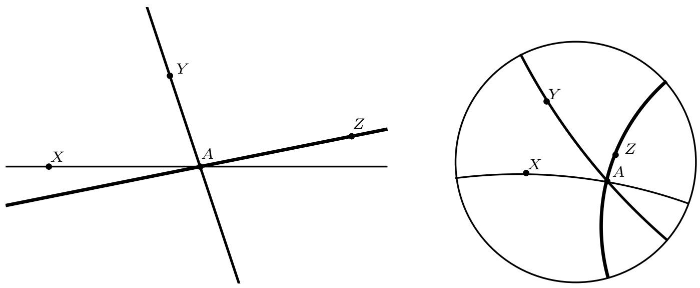


Fig. 28: Figure for lemma 19.

4.6 Lemmas about reflection with respect to a line

Having defined the perpendicularity of lines we can characterize a reflection with respect to a line. $P' \frac{A\bullet}{\bullet B} P$ expresses that P' is the image of P by the reflection with respect to the line AB.

Definition 4 Reflection with respect to a line

$$P' \frac{A\bullet}{\bullet B} P := (\exists X, P \text{-} X \text{-} P' \wedge \mathrm{Col}\ A\,B\,X) \wedge (AB \perp PP' \vee P = P').$$

It is interesting to see that for $P \frac{A\bullet}{\bullet B} P'$ (Fig. 29) to be true when A and B are equal one must have that P and P' are also equal. The authors of [SST83] generalize the definition of the reflection to include this particular case:

Definition 5 Generalized reflection with respect to a line

$$P' \underset{\bullet B}{\overset{A\bullet}{-\!\!/\!\!/\!\!-}} P := A \neq B \wedge P' \frac{A\bullet}{\bullet B} P \vee A = B \wedge P \text{-} A \text{-} P'.$$

When A and B are different $P' \underset{\bullet B}{\overset{A\bullet}{-\!\!/\!\!/\!\!-}} P$ has the same meaning as $P' \frac{A\bullet}{\bullet B} P$. Otherwise it means that P' is the image of P by the point reflection of centre A.

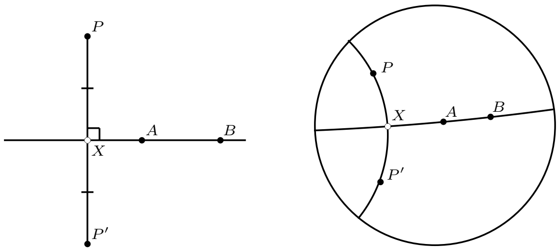


Fig. 29: Figure for definition 4.

Lemma 20 (l10.2) $\forall ABP, \exists P', P' \overset{A\bullet}{\underset{\bullet B}{\rule{2em}{0.4pt}}} P.$

Lemma 20 states the existence of the reflection of a point with respect to a line (this line can be degenerated as we defined $P' \overset{A\bullet}{\underset{\bullet B}{\rule{2em}{0.4pt}}} P$ so that it corresponds to a point reflection of center A when A and B are equal).

Lemma 21 (l10_15) $\forall ABCP, \text{Col } A\,B\,C \Rightarrow \neg\, \text{Col } A\,B\,P \Rightarrow \exists Q, AB \bot QC \wedge A \underset{PQ}{\rule{2em}{0.4pt}} B.$

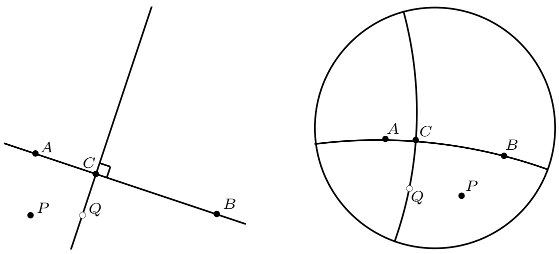


Fig. 30: Figure for lemma 21.

Lemma 21 (Fig. 30) establishes the existence of a point on the perpendicular to a given line passing by a given point which on the same side of the given line as a second given point. It is much more precise than just erecting a perpendicular line as it fixes in which half-plane the constructed point must belong.

Lemma 22 $\forall XYZAB, XY \perp AB \Rightarrow XZ \perp AB \Rightarrow \text{Col } X\,Y\,Z.$

Lemma 22 (Fig. 31) formulates that there is only one line perpendicular to a given line passing through a given point. It is of course only valid in a planar configuration but this is exactly the situation imposed by the axioms related to dimension.

Definition 6 Perpendicular bisector of a line-segment

$$A \overset{P\bullet}{\underset{\bullet Q}{\rule{2em}{0.4pt}}}{}_s B := A \overset{P\bullet}{\underset{\bullet Q}{\rule{2em}{0.4pt}}} B \wedge A \neq B.$$

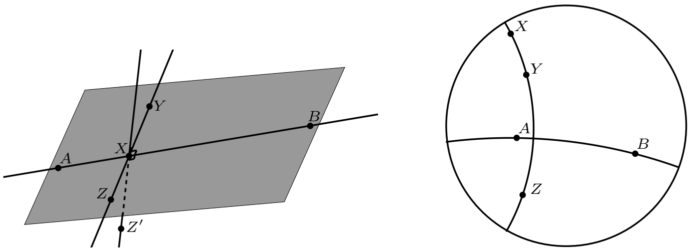


Fig. 31: Figure for lemma 22.

$A \overset{P\bullet}{\underset{\bullet Q}{\text{—}\!\!\!\!\!\!/\!\!\!/\text{—}}}{}_s B$ symbolizes the fact that the line PQ is the perpendicular bisector of the line-segment $\overline{AB}$. This definition was not present in [SST83] but is very similar to $\overset{\bullet}{\underset{\bullet}{\text{—}\!\!\!\!\!/\!\!\!/\text{—}}}$. The only difference is that our definition ensures that A and B are different to obtain a well-defined perpendicular bisector.

Lemma 23 $\forall ABCDE, CD \equiv CE \Rightarrow D \overset{A\bullet}{\underset{\bullet B}{\text{—}\!\!\!\!\!/\!\!\!/\text{—}}}{}_s E \Rightarrow \text{Col } A\,B\,C.$

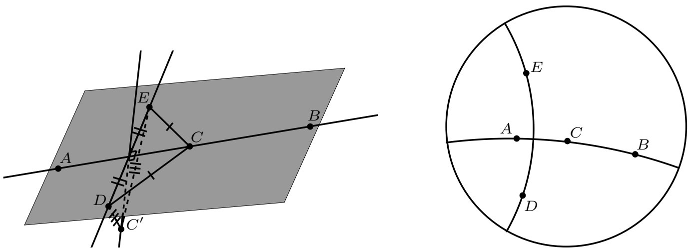


Fig. 32: Figure for lemma 23.

Lemma 23 (Fig. 32) is a consequence of lemma 22. It states that any point which is equidistant to the end points of a line-segment belongs to its perpendicular bisector. As for lemma 22 this statement is false in dimension higher than 2. These last two lemmas do not have a name as they were not present in [SST83]. It is by the way surprising that lemma 22 was absent from it as it was in [Euc98].

4.7 Lemmas about subspaces

Lemma 24 $\forall ABCD, \text{Cp } A\,B\,C\,D.$

Lemma 24 is also absent from [SST83] but it is very similar to lemma 11.69 which establishes that lemma 24 is equivalent to the upper dimensional axiom.

Starting from chapter 10, we assume the upper dimensional axiom, hence we do not need to prove the equivalence.

Lemma 25 $\forall ABCPQ, \neg\,\mathrm{Col}\ C\,Q\,P \Rightarrow \neg\,\mathrm{Col}\ A\,B\,P \Rightarrow A \not{-} Q \not{-} B \Rightarrow \exists X, \mathrm{Col}\ P\,Q\,X \wedge (A \not{-} X \not{-} C \vee B \not{-} X \not{-} C)$.

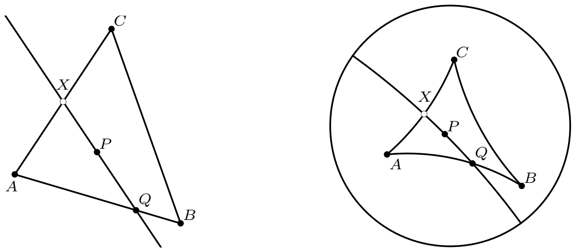


Fig. 33: Figure for lemma 25.

Exactly as the previous lemma, lemma 25 (Fig. 33) was not proven in [SST83]. Its proof can be found in [BN12] as it corresponds to Hilbert's version of Pasch axiom. This proof cannot be done before we have proven lemma 24 as the statement of this lemma is false if P is not coplanar with A, B and C.

4.8 Lemmas about parallelism

Lemma 26 (l12.6) $\forall ABCD, AB \parallel_s CD \Rightarrow A \underset{CD}{-\!\!\not{-}\!\!-} B$.

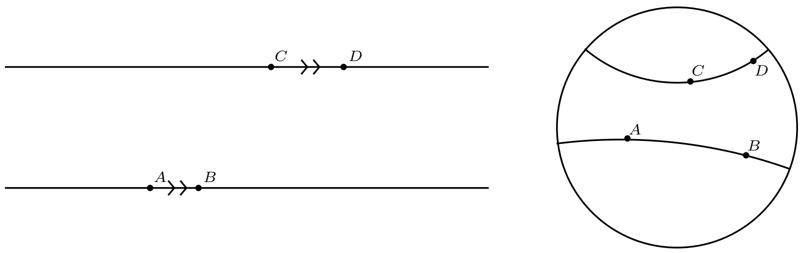


Fig. 34: Figure for lemma 26.

Lemma 26 (Fig. 34) asserts that the points representing a line strictly parallel to another one are on the same side of this latter.

Lemma 27 (l12.9) $\forall A_1A_2B_1B_2C_1C_2, A_1A_2 \perp C_1C_2 \Rightarrow B_1B_2 \perp C_1C_2 \Rightarrow A_1A_2 \parallel B_1B_2$.

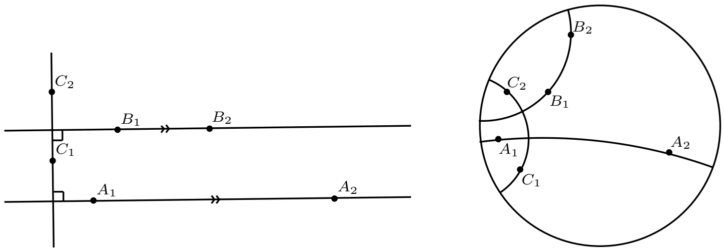


Fig. 35: Figure for lemma 27.

Lemma 27 (Fig. 35) pronounces that two lines which are perpendicular to a third line are parallel. It is interesting to note that while this lemma is valid in neutral geometry, the fact that the parallel to a line which is perpendicular to a third line is also perpendicular to this third line is equivalent to Euclid's parallel postulate.

Lemma 28 (l12.10) $\forall ABP, A \neq B \Rightarrow \exists CD, C \neq D \wedge AB \parallel CD \wedge \text{Col } P\,C\,D.$

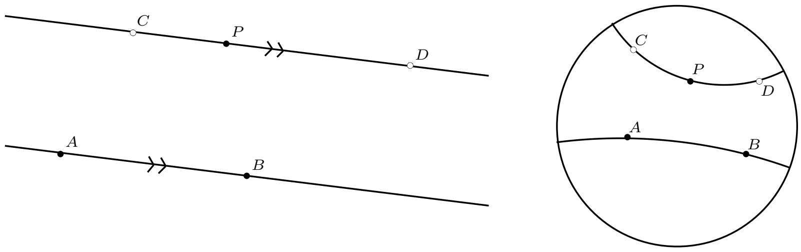


Fig. 36: Figure for lemma 28.

Lemma 28 (Fig. 36) formulates the existence of a line parallel to a given line passing through a given point. This is important because it shows that axioms of neutral geometry are inconsistent with those of elliptic geometry as elliptic geometry assumes that no parallel lines exist.

Lemma 29 (l12.17) $\forall ABCDP, A \neq B \Rightarrow A{-}P{-}C \Rightarrow B{-}P{-}D \Rightarrow AB \parallel CD.$

Lemma 29 (Fig. 37) establishes that if A and B are distinct and if the line-segments $\overline{AC}$ and $\overline{BD}$ have the same midpoint then the lines AB and CD are parallel. It almost corresponds to the fact that the opposite sides of a non-degenerated quadrilateral which has its diagonals intersecting in their midpoint are parallel. To fully correspond to this fact one would need to add the hypothesis that A and D are distinct.

To end this section we provide the table 3 which summarizes all our definitions and notations.

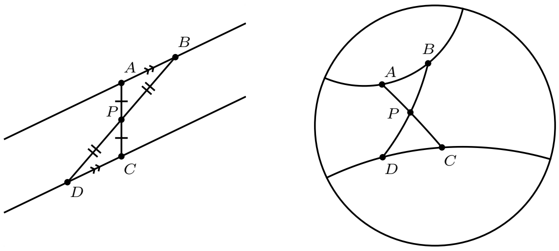


Fig. 37: Figure for lemma 29.

Coq	Notation	Definition
`Bet A B C`	$A\text{—}B\text{—}C$	
`Cong A B C D`	$AB \equiv CD$	
`Col A B C`	Col $A\,B\,C$	$A\text{—}B\text{—}C \vee B\text{—}A\text{—}C \vee A\text{—}C\text{—}B$
`out O A B`	$O \mathrel{-\!\!\!/\!\!\!-} A \leftrightarrow B$	$O \neq A \wedge O \neq B \wedge (O\text{—}A\text{—}B \vee O\text{—}B\text{—}A)$
`is_midpoint M A B`	$A \mathrel{-\!\!\!+\!\!\!-} M \mathrel{-\!\!\!+\!\!\!-} B$	$A\text{—}M\text{—}B \wedge AM \equiv BM$
`Per A B C`	$\llcorner\ A\,B\,C$	$\exists C', C \mathrel{-\!\!\!+\!\!\!-} B \mathrel{-\!\!\!+\!\!\!-} C' \wedge AC \equiv AC'$
`Perp_in P A B C D`	$AB \underset{P}{\perp} CD$	$A \neq B \wedge C \neq D \wedge$ Col $P\,A\,B \wedge$ Col $P\,C\,D \wedge$ $(\forall U\,V, \text{Col } U\,A\,B \Rightarrow \text{Col } V\,C\,D \Rightarrow \llcorner\ U\,P\,V)$
`Perp A B C D`	$AB \perp CD$	$\exists P, AB \underset{P}{\perp} CD$
`is_image_spec P' P A B`	$P' \overset{A\bullet}{\underset{\bullet B}{\text{———}}} P$	$(\exists X, P \mathrel{-\!\!\!+\!\!\!-} X \mathrel{-\!\!\!+\!\!\!-} P' \wedge \text{Col } A\,B\,X) \wedge (AB \perp PP' \vee P = P')$
`is_image P' P A B`	$P' \overset{A\bullet}{\underset{\bullet B}{\mathrel{-\!\!\!/\!\!\!-}}} P$	$A \neq B \wedge P' \overset{A\bullet}{\underset{\bullet B}{\text{———}}} P \vee A = B \wedge P \mathrel{-\!\!\!+\!\!\!-} A \mathrel{-\!\!\!+\!\!\!-} P'$
`two_sides A B P Q`	$A \overset{Q}{\underset{P}{\mathrel{-\!\!\!/\!\!\!-}}} B$	$A \neq B \wedge \neg\text{Col } P\,A\,B \wedge \neg\text{Col } Q\,A\,B \wedge$ $\exists T, \text{Col } T\,A\,B \wedge P\text{—}T\text{—}Q$
`one_side A B X Y`	$A \underset{XY}{\mathrel{-\!\!\!/\!\!\!-}} B$	$\exists Z, A \overset{Z}{\underset{X}{\mathrel{-\!\!\!/\!\!\!-}}} B \wedge A \overset{Z}{\underset{Y}{\mathrel{-\!\!\!/\!\!\!-}}} B$
`coplanar A B C D`	Cp $A\,B\,C\,D$	$\exists X, (\text{Col } A\,B\,X \wedge \text{Col } C\,D\,X) \vee (\text{Col } A\,C\,X \wedge \text{Col } B\,D\,X) \vee (\text{Col } A\,D\,X \wedge \text{Col } B\,C\,X))$
`Par_strict A B X Y`	$AB \parallel_s XY$	$A \neq B \wedge C \neq D \wedge$ Cp $A\,B\,C\,D \wedge$ $\neg\exists X, \text{Col } X\,A\,B \wedge \text{Col } X\,C\,D$
`Par A B X Y`	$AB \parallel XY$	$AB \parallel_s CD \vee (A \neq B \wedge C \neq D \wedge \text{Col } A\,C\,D \wedge$ Col $B\,C\,D)$
`perp_bisect P Q A B`	$A \overset{P\bullet}{\underset{\bullet Q}{\mathrel{-\!\!\!/\!\!\!-}}}_s B$	$A \overset{P\bullet}{\underset{\bullet Q}{\text{———}}} B \wedge A \neq B$
`BetS A B C`	$A \mathrel{-\!\!\!/\!\!\!-} B \mathrel{-\!\!\!/\!\!\!-} C$	$A\text{—}B\text{—}C \wedge A \neq B \wedge B \neq C$

Table 3: Summary of predicates and their notations.

5 Proofs of equivalence

In this section, we prove the equivalences of the following properties.

1. Midpoint converse postulate
2. Postulate of transitivity of parallelism
3. Playfair's postulate
4. Perpendicular transversal postulate
5. Postulate of parallelism of perpendicular tranversals
6. Tarski's parallel postulate

7. Proclus' postulate
8. Euclid's 5^{th} postulate
9. Strong parallel postulate
10. Triangle circumscription principle

We will prove the implications depicted on Fig. 38. One can remark that proving that the postulate of transitivity of parallelism implies Playfair's postulate seems to be redundant. We will see in section 5.4 why this proof is indeed needed. Throughout this work we proved other implications[5] that we will not present. We removed them either because we could find another implication which allowed to simplify the way we prove all of these postulate equivalent or to obtain some results presented in section 5.4.

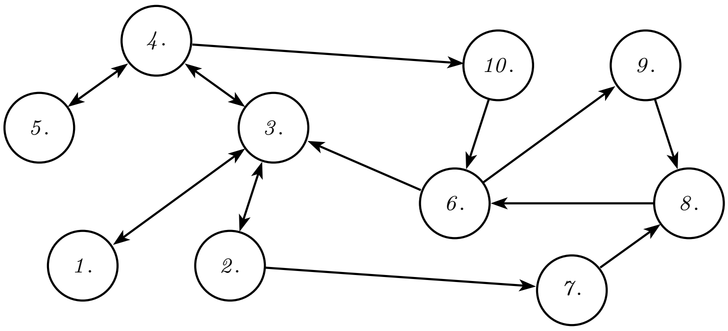


Fig. 38: Overview of the proofs.

5.1 A first equivalence

In this subsection we will prove that the midpoint converse postulate, the postulate of transitivity of parallelism, Playfair's postulate, the perpendicular transversal postulate and the postulate of parallelism of perpendicular tranversals are equivalent.

Proposition 1 *Playfair's postulate is equivalent to the postulate of transitivity of parallelism.*

Proof

[5] We proved that the strong parallel postulate implies Tarski's parallel postulate and Euclid's 5^{th} postulate. We proved that Proclus' postulate implies the postulate of transitivity of parallelism. We proved that Playfair's postulate implies the perpendicular transversal postulate. Finally we also proved that the triangle circumscription principle implies the strong parallel postulate.

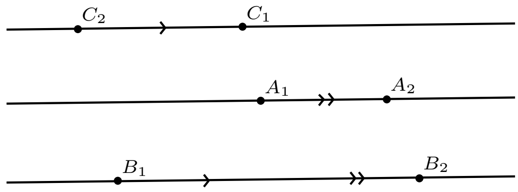


Fig. 39: Playfair's postulate implies the postulate of transitivity of parallelism.

1. Let us first prove that the postulate of transitivity of parallelism follows from Playfair's postulate (Fig. 39). This proof corresponds to a formalization of the proof given in [SST83]. Therefore we may assume that lines A_1A_2 and C_1C_2 are both parallel to line B_1B_2. This implies that A_1A_2 and C_1C_2 are non-degenerated lines by definition of parallelism of lines. We wish to prove that lines A_1A_2 and C_1C_2 are parallel knowing that there is only one line going through a given point and being parallel to a given line. We will distinguish two cases.
   (a) Assuming that A_1, A_2 and C_1 are collinear Playfair's postulate allows to prove that A_1, A_2, C_1 and C_2 are collinear as lines A_1A_2 and C_1C_2 are both parallel to line B_1B_2 and both go through the same point, namely C_1. This implies that lines A_1A_2 and C_1C_2 are parallel since they are non-degenerated lines and equal (by equal we mean that any point collinear with A_1 and A_2 is also collinear with C_1 and C_2 and vice-versa).
   (b) Assuming that A_1, A_2 and C_1 are not collinear we will prove that lines A_1A_2 and C_1C_2 are strictly parallel. To do so it suffices to prove that they cannot intersect since they are non-degenerated lines. We proceed by proof of negation[6] and we may assume that they do in a point P. Playfair's postulate implies that lines A_1A_2 and C_1C_2 are equal as they are both parallel to line B_1B_2 and both pass through point P. This contradicts the fact that A_1, A_2 and C_1 are not collinear and complete the proof that the postulate of transitivity of parallelism follows from Playfair's postulate.

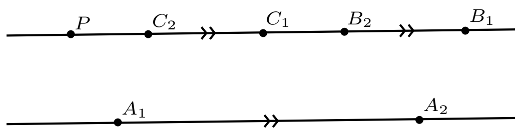


Fig. 40: The postulate of transitivity of parallelism implies Playfair's postulate.

2. We now prove that Playfair's postulate follows from the postulate of transitivity of parallelism (Fig. 40). Our hypotheses are the postulate of transitivity of parallelism, that lines B_1B_2 and C_1C_2 are both parallel to line A_1A_2 and that a point P belongs to both B_1B_2 and C_1C_2. We wish to prove that B_1, B_2, C_1 and C_2 are collinear. From our hypotheses we know that lines B_1B_2 and C_1C_2 are parallel (we make use of the postulate of transitivity of parallelism here). So they are either equal which is exactly what we aim to prove or they do not meet.

[6] We use the expression 'proof of negation' to describe a proof of $\neg A$ assuming A and obtaining a contradiction. For the reader who is not familiar with intuitionistic logic, we recall that this is simply the definition of negation and this proof rule has nothing to do with the proof by contradiction (to prove A it suffice to show that $\neg A$ leads to a contradiction) which is not valid in our intuitionistic setting.

However we now they meet at point P so we may eliminate this case and the proof is complete.

□

Proposition 2 *The perpendicular transversal principle is equivalent to the postulate of parallelism of perpendicular tranversals.*

Proof

1. Let us prove that the postulate of parallelism of perpendicular tranversals follows from the perpendicular transversal principle (Fig. 41). Assuming $A_1A_2 \| B_1B_2$ and $A_1A_2 \perp C_1C_2$ and $B_1B_2 \perp D_1D_2$ we need to show that $C_1C_2 \| D_1D_2$. Using lemma 27 it suffices to prove that $A_1A_2 \perp D_1D_2$ are perpendicular. The perpendicular transversal principle allows us to complete the proof.

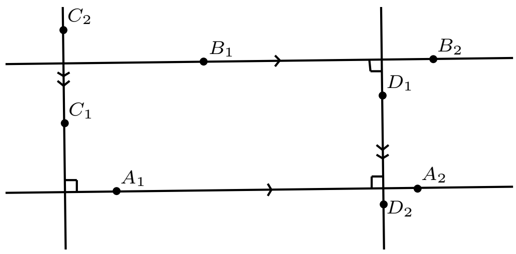


Fig. 41: The perpendicular transversal principle implies the postulate of parallelism of perpendicular tranversals. .

2. Let us now prove that the perpendicular transversal principle follows from the postulate of parallelism of perpendicular tranversals (Fig. 42). So we wish to prove that $CD \perp PQ$ and PQ under the hypotheses that $AB \parallel CD$ and $AB \perp PQ$ knowing that the postulate of parallelism of perpendicular tranversals holds. We first name X the point of intersection of lines AB and PQ. If X is collinear with C and D then the lines AB and CD are equal and we are done. So it remains to prove that $CD \perp PQ$ if C, D and X are not collinear. Using lemma 9 we construct Y such that $CD \perp XY$ and meet in Y. The postulate of parallelism of perpendicular tranversals let us prove that lines $PQ \parallel XY$. Moreover we know that they are equal as Col $P\,Q\,X$ which allows us to complete the proof.

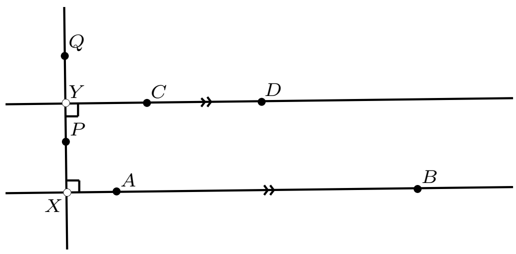


Fig. 42: The postulate of parallelism of perpendicular tranversals implies the perpendicular transversal principle.

□

To prove that the midpoint converse postulate follows from Playfair's postulate we first prove the midpoint theorem.

Lemma 30 *In a non-degenerated triangle ABC where P is the midpoint of the line-segment $\overline{BC}$ and Q the midpoint of the line-segment $\overline{AC}$ the lines AB and PQ are strictly parallel.*

We do not give the proof of the midpoint theorem (Fig. 43) as it is a direct consequence of lemma 13.1 which is proved in [SST83] and was recently formalized by Gabriel Braun. We should point out that this proof is valid in neutral geometry.

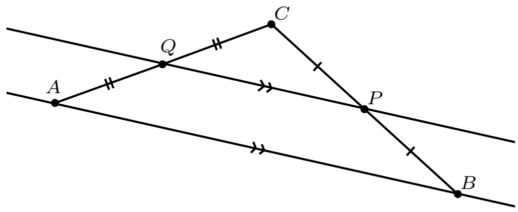


Fig. 43: Figure for lemma 30.

Proposition 3 *The midpoint converse postulate is equivalent to Playfair's postulate.*

Proof

1. Let us first prove that the midpoint converse postulate follows from Playfair's postulate (Fig. 44). So we may assume that ABC is a non-degenerated triangle, that P is the midpoint of the line-segment $\overline{BC}$ and that Q is the intersection point of lines AC and the line which is parallel to line AB and passing through P. Here we may speak of **the** line parallel to a given line and passing given point as Playfair's postulate holds. Lemma 10 let us construct the midpoint X of the line-segment $\overline{AC}$. To conclude we only need to prove that $Q = X$. Uniqueness of intersection (lemma 3) allows us to conclude as they both are at the intersection of lines AC and PQ. The facts that Q and X belong to line AC or that Q belongs to line PQ are trivial. We prove that $AB \parallel PX$ using the midpoint theorem (lemma 30). Finally we can prove that X belongs to line PQ by proving that lines PQ and PX are both parallel to line AB and both pass through point P.

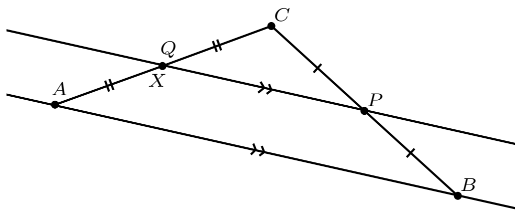


Fig. 44: Playfair's postulate implies the midpoint converse postulate.

2. Let us now prove that Playfair's postulate follows from the midpoint converse postulate (Fig. 45). So assuming that $A_1A_2 \parallel B_1B_2$, $A_1A_2 \parallel C_1C_2$, Col $P\,B_1\,B_2$ and Col $P\,C_1\,C_2$ we wish to prove that B_1, B_2, C_1 and C_2 are collinear. We can first eliminate the cases where line A_1A_2 is equal to B_1B_2 and/or C_1C_2. If all three lines are equal we are trivially done and if two lines are equal and strictly parallel to the third one then we may also conclude as we can also prove that this last case is impossible because the lines meet in P. So we may now assume $A_1A_2 \parallel_s B_1B_2$ and $A_1A_2 \parallel_s C_1C_2$. We can then construct the symmetric point

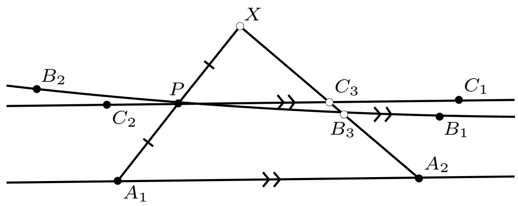


Fig. 45: The midpoint converse postulate implies Playfair's postulate.

X of A_1 with respect to P using lemma 5 as points A_1 and P are different as otherwise it would contradict $A_1A_2 \parallel_s B_1B_2$ since Col $P\,B_1\,B_2$. Now we will prove that there exists a point B_3 on line B_1B_2 which is strictly between A_2 and X. We know that P is either different from B_1 or from B_2 as otherwise it would contradict $A_1A_2 \parallel_s B_1B_2$ since Col $P\,B_1\,B_2$. So to prove the existence of the point B_3 we apply lemma 25 in the triangle A_1A_2X with either line PB_1 or PB_2. We prove the hypotheses of this lemma in the same way in both cases. The hypotheses $\neg$ Col $A_2\,P\,B_1$ and $\neg$ Col $A_1\,X\,B_1$ can be proven by proof of negation. Assuming Col $A_2\,P\,B_1$ would contradict $A_1A_2 \parallel_s B_1B_2$. Assuming Col $A_1\,X\,B_1$, B_1 and P would both be on lines PA_1 and PB_1. From the lemma 3 and provided that $\neg$ Col $P\,A_1\,B_1$ this would imply that $B_1 = P$ which we know to be false. Then to prove $\neg$ Col $P\,A_1\,B_1$ we assume that these three points are collinear which contradicts $A_1A_2 \parallel_s B_1B_2$. Finally we need to prove that B_3 cannot be between A_1 and A_2 which can again be proven by proof of negation. Indeed assuming $A_1 — B_3 — A_2$ would contradict $A_1A_2 \parallel_s B_1B_2$. In the same way we can prove there exists a point C_3 on line C_1C_2 which is strictly between A_2 and X. Now all we need to prove is that B_3 and C_3 are equal and different which will imply that B_1, B_2, C_1 and C_2 are collinear. To do so we prove that both are **the** midpoint[7] of the line-segment $\overline{A_2X}$ and the lemma 7 would allow us to conclude. To prove that they both are the midpoint of this line-segment we use the midpoint converse postulate in the triangle A_1A_2X that we must prove non-degenerated. Again we prove this fact by proof of negation which ends this proof. Assuming these points were collinear would lead to contradict $A_1A_2 \parallel_s B_1B_2$.

□

In the next lemma we have a lengthy proof that Playfair's postulate implies perpendicular transversal postulate. The reason is that the proof we give here does not use the decidability of intersection, this will allow us to obtain some decidability results (see Section 5.4).

Proposition 4 *The perpendicular transversal postulate is equivalent to Playfair's postulate.*

Proof

1. Let us prove that the perpendicular transversal postulate follows from Playfair's postulate first[8] (Fig. 46). So we wish to prove that $CD \perp PQ$ under the hypotheses that $AB \parallel CD$ and $AB \perp PQ$ knowing that Playfair's postulate holds. Let us name X the intersection point of lines AB and PQ. We first eliminate the case when

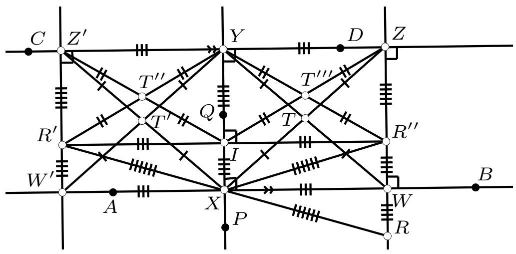


Fig. 46: Playfair's postulate implies the perpendicular transversal postulate.

[7] In neutral geometry the midpoint of a line-segment can be proven to exist and be unique.

[8] This is lemma 12.9 in [SST83] but our proof does not follow the proof given by Tarski.

lines AB and CD are equal since then $CD \perp PQ$ are perpendicular by hypothesis. So we may now assume $AB \parallel_s CD$. Lemma 9 let us construct a point Y on line CD such that $CD \perp XY$ since X is not on CD as otherwise it would contradict $AB \parallel_s CD$. We then eliminate the case when lines PQ and XY are equal since then $CD \perp PQ$ by hypothesis. Then since we know that X is on line PQ we can prove that Y is not on line PQ. Now we will prove that there exists a point R on line PQ but not on lines XY or CD. We know there exists a point on line PQ different from X. This point is not on line XY as otherwise it would contradict $\neg \mathrm{Col}\ P\,Q\,Y$. So if this point is not on line CD we are done. Otherwise we construct its symmetric with respect to X using lemma 5. This new point is obviously of line PQ and cannot be on line XY since the previous one was not. It is also not on line CD because if it was then lines AB and CD would be equal which we know to be false and the proof of the existence of such a point R is complete. So lemma 9 let us construct point Z on line CD such that $CD \perp RZ$. We then construct the midpoint T of the line-segment $\overline{XY}$ using lemma 10 and W the symmetric of point Y with respect to point T. From lemmas 27 and 29 and Playfair's postulate we know that W is on lines AB and RZ. Lemma 5 let us construct R' and W' the symmetric points of R and W with respect to X. W' clearly not being on line CD we can construct Z' on line CD such that $CD \perp W'Z'$. From lemmas 27 and 29 and Playfair's postulate we know that $\mathrm{Col}\ R'\,W'\,Z'$. We then construct the midpoint T' of the line-segment $\overline{W'Y}$ using lemma 10 and X' the symmetric of point Z' with respect to point T'. From lemmas 27 and 29 and Playfair's postulate we know that X' is on lines XW' and XY which allows us to prove that X and X' are equal using lemma 3. Now we construct R'' the symmetric of R' with respect to line XY using lemma 20 and we name I the intersection of lines XY and $R'R''$. We then construct the midpoint T'' of the line-segment $\overline{R'Y}$ using lemma 10 and I' the symmetric of point Z' with respect to point T''. Then either I' and Y are equal and lemma 22 let us prove that I lies on line CD and one can prove using lemma 3 that I and I' are equal as both are on lines CD and IX. Otherwise from lemma 27 and 29 and Playfair's postulate we know that I' is on lines IR' and IY which allows us to prove that $I = I'$ using lemma 3. We then construct the midpoint T''' of the line-segment $\overline{IZ}$ and R''' the symmetric of point Y with respect to point T'''. Then either R''' and Z are equal and lemma 22 let us prove that R'' lies on line CD and one can prove using lemma 8 that either $R'' = R'''$ or that I is the midpoint of the line-segment $\overline{R''R'''}$ which implies that $R'' = R'''$ are equal. To apply lemma 8 we need to prove that $\mathrm{Col}\ I\,R''\,R'''$ which is trivial as all of these points lie on line CD and that $IR'' \equiv IR'''$ which is trivial by the transitivity of the congruence relation and congruences proven using lemma 6. Otherwise lemmas 27, 22 and Playfair's postulate let us prove $\mathrm{Col}\ I\,R''\,R'''$ and we prove that $R'' = R'''$ in the exactly the same way using lemma 8. As Playfair's postulate implies the postulate of transitivity of parallelism we can prove that $R'R'' \parallel WW'$. Now lemma 29 and Playfair's postulate let us prove $\mathrm{Col}\ R\,R''\,W$ which together with the fact that $W \overset{R''}{\underset{R}{\text{---}\!\!\nparallel\!\!\text{---}}} X$ itself proven using lemma 12 with R' (R and R' are on opposite sides of line WX and R' and R'' are on the same side of this line) let us prove R—W—R''. Then a tedious distinction of cases let us prove $RW \equiv R''W$. R—W—R'', $RW \equiv R''W$ and $RX \equiv R''X$ which can be proven using the transitivity of the congruence relation and congruences proven using lemma 6. Finally using this last fact and $AB \perp PQ$ we can prove that $RW \parallel_s RX$ by lemma 27 and we obtain a contradiction.

2. Let us now prove that Playfair's postulate follows from the perpendicular transversal postulate (Fig. 47). So we wish to prove that the points B_1, B_2, C_1 and C_2 are collinear provided that the lines B_1B_2 and C_1C_2 are both parallel to line A_1A_2, and Col $B_1\ B_2\ P$ and Col $C_1\ C_2\ P$. We first eliminate the case where P belongs to line A_1A_2 as then lines B_1B_2 and C_1C_2 are equal to line A_1A_2. Then the points B_1, B_2, C_1 and C_2 are collinear since that $B_1B_2 \parallel C_1C_2$. Now P, A_1 and A_2 are not collinear so using lemma 9 we construct I such that $A_1A_2 \perp PI$ and Col $I\ A_1\ A_2$. The perpendicular transversal postulate tells us that lines B_1B_2 and C_1C_2 are perpendicular to line PI. Finally lemma 22 allows us to conclude.

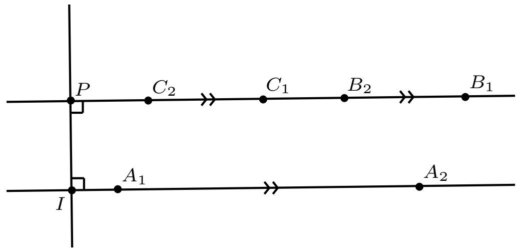


Fig. 47: The perpendicular transversal postulate implies Playfair's postulate.

□

5.2 A second equivalence

In this subsection we will prove that Tarski's parallel postulate, Euclid's 5th postulate and the strong parallel postulate are equivalent.

To prove that Euclid 5 implies the strong parallel postulate we will first prove three following auxiliary lemmas. Note that one may be tempted to think that the following property is trivial, proving it using the properties of parallelograms and Playfair's postulate. But we do not have a proof yet that Euclid 5th postulate implies these usual Euclidean results so we can not use them here.

Lemma 31 *If S is the midpoint of the line-segment $\overline{QQ'}$, T the midpoint of the non-degenerated line-segments $\overline{PQ}$ and $\overline{RS}$, T' the midpoint of the line-segments $\overline{PQ'}$ and $\overline{R'S}$, P does not belong to line QS and Euclid 5 holds then P, R and R' are collinear.*

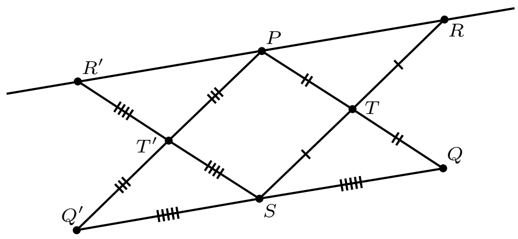
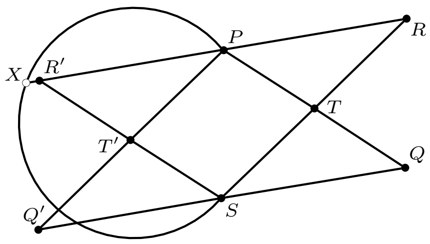


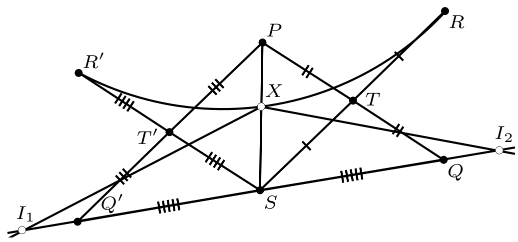
Fig. 48: Figure for lemma 31.

Proof We need to prove that P, R and R' are collinear (Fig. 48). First we can easily prove that P does not belong to line $Q'S$ as QS and $Q'S$ are equal. We can also prove that T' is strictly between P and Q'. Indeed if P and T' were equal the fact that P, Q and S are non collinear would lead to Q, Q' and S being not collinear which is false as S is the midpoint of the line-segment $\overline{QQ'}$. If Q' were equal to T' then the Q' would be equal to P and the hypothesis that P, Q' and S are non collinear would become trivially false. In the same way we can prove that T' is strictly between R' and S. We know that the collinearity is decidable so we can prove that P, R and R' are collinear by deriving a contradiction when assuming that they are not collinear. To do so we prove that P, R, R' and S are coplanar using lemma 24 and we will distinguish the 27 ways for them to be coplanar. From the definition we obtain the intersection point X for either lines PR and $R'S$ or lines PR' and RS or lines PS and RR'. Now for each line the intersection point can have 3 possible place. Either it is between the two points forming the line or the first point is between the intersection point and the second point or the second point is between the intersection point and the first point. From these 27 cases we can derive 3 kinds of contradiction and we will only provide one example for each of them.

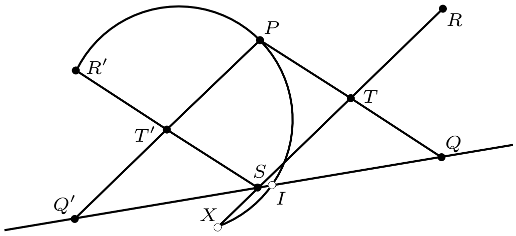
1. The first kind of contradiction is when one can prove that 2 points are at the same time on the same side and on opposite sides of a line. Using lemma 13 or 14 it allows us to derive that we have a contradiction. An example of this case is when the intersection point is on line PS and RR' with R' being between R and X. We already proved that R and R' are on opposite sides of line PS so we need to prove that R and R' are on the same side of line PS. To do so we use lemma 18 together with the facts that lines PS and RR' intersect in point X, that P, R and S are not collinear and that X is collinear with R and R' but outside the line-segment $\overline{RR'}$. The last two statements are easily proven by proof of negation and the fact that R' is between R and X. Indeed if P, R and S were collinear so would P, Q and S which we know to false and if X were equal to R or R' then P, S and R' would be collinear which would imply that P, Q and S are collinear which is false by hypothesis.



2. The second kind of contradiction is when one can prove that 3 points are at the same time collinear and not collinear. An example of this case is when X is between P and S and between R and R'. Then we can apply Euclid 5^{th} postulate to assert the existence of points I_1 and I_2 which belong to line QS and are respectively on lines RX and $R'X$. Provided that we can prove that I_1 and I_2 are different this implies that Q, R and S are collinear which itself implies that P, Q and S are collinear which we know to be false by hypothesis. We prove this statement by proving that S is between I_1 and I_2 using lemma 1 and by proof of negation. Indeed if I_1 and I_2 were equal then S would be equal to I_1 which we know to be false as Euclid 5 assert the existence of point different from the 6 points needed to construct it.



3. The last kind of contradiction is when one can prove that two lines which we know to be strictly parallel intersect. An example of this case is when X is collinear with P and R' and S is between R and X. Then we can prove that P and X are on opposite sides of line $Q'S$. We do so by using lemma 12 together with the facts that R and X are on opposite sides of line $Q'S$ and that P and R are on the same side of line $Q'S$. The first of these statements is a consequence of S being between R and X and P, Q and S not being collinear. The second of these statements is proven using lemma 26 and the fact that lines PR and QS' are strictly parallel. P and X being on opposite sides of line $Q'S$ contradicts the fact that $Q'S$ and PR' are parallel as by definition it means that there is an intersection between lines PR' and $Q'S$.



□

Lemma 32 *If T is the midpoint of the non-degenerated line-segments $\overline{PQ}$ and $\overline{RS}$, U does not belong to line PR, U is on the same side of line PR as S and one the same side of line PS as R and Euclid 5 holds then lines QS and PU intersect.*

Proof We need to prove the existence of the intersection between lines PU and QS (Fig. 49). First we may assume that P, Q and S are not collinear as if they were collinear the intersection point would be P and we would be done. We will distinguish the 27 possible cases for P, Q, R and U to be coplanar (these points can be proven coplanar using lemma 24). We will name X the intersection point. There are 3 kinds of possible cases.

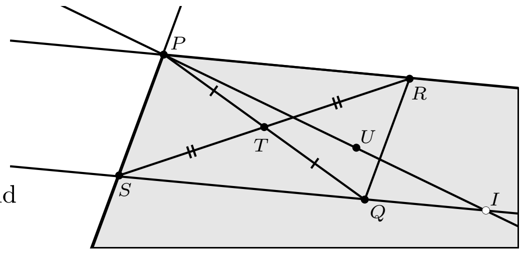
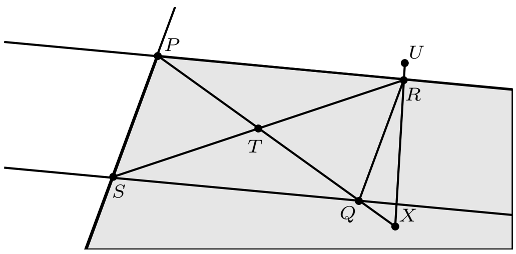


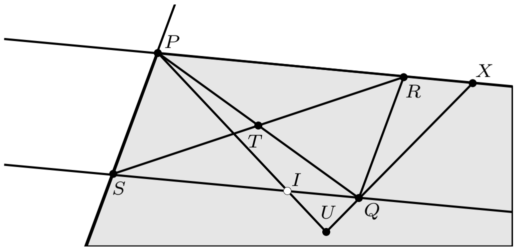
Fig. 49: Figure for lemma 32.

1. The first kind of possible cases is when one can prove that U is not on the same side of either line PR or PS as S or R. An example of this case is when Q is between P and X and R between X and U. We can prove that S and U are on opposite sides of line PR which is sufficient to derive a contradiction using lemma 14. To do so we use lemma 12 together with the fact Q and U are on opposite sides of line PR and that Q and S are on the same side of line PR. The first of these statements can be proven using again lemma 12 and the fact that X and U are on opposite

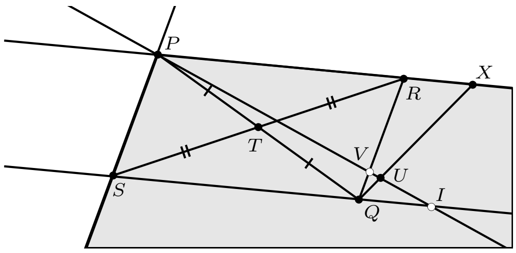


sides of line PR which is a consequence of R being between X and U and P, Q and S being non-collinear and the fact that Q and X being on the same side of line PR. This last fact is proven using lemma 18 together with the fact that lines PR and QX intersecting in P in the desired conditions. The second of theses statements can be proven using lemma 26 and the fact that lines PR and QS are strictly parallel.

2. The second kind of possible cases is when we can assert the existence of the intersection point without having to use Euclid 5. An example of this case is when R is between P and X and Q between X and U. First we may assume that Q, S and U are non-collinear as otherwise the intersection point would be U ad we would be done. We may also assume that Q and X are different as otherwise the intersection would be P and we would be done. We can prove that P and U are on opposite sides of line QS which is enough to prove that lines PU and QS intersect. We do so by using lemma 12 together with the facts that X and U are on opposite sides of line QS and that P and X are on the same side of line QS. The first of these statements is implied by lines QS and XU intersecting in Q with Q being between U and X and Q, S and U being non-collinear. The second of these statements can be proven using lemma 26 and the fact that lines PR and QS are strictly parallel.



3. The last kind of possible cases is when one can assert the existence of a point allowing the use of Euclid 5 to prove the existence of the intersection point. An example of thus case is when R is between P and X and U between Q and X. Then using the inner version of Pasch's axion together with the previous betweenness hypotheses one can assert the existence of the point V being between Q and R and between P and U. We know that P and V are different as otherwise. We can also assume that R and V are different as otherwise the intersection is Q and we are done. Then we know that V is strictly between Q and R and Euclid 5 let us construct the intersection point as V belongs to line PU.



□

Lemma 33 *If O is the midpoint of the line-segments $\overline{I_m I_p}$ and $\overline{PP'}$, O, I_p and J_p are non-collinear, P, I_p and I_m are non-collinear and P, O and J_p are non-collinear then either:*

- *P is on the same side of lines OI_p and OJ_p as J_p and I_p;*
- *P is on the same side of lines OI_m and OJ_p as J_p and I_m;*
- *P' is on the same side of lines OI_m and OJ_p as J_p and I_p;*
- *P' is on the same side of lines OI_m and OJ_p as J_p and I_m.*

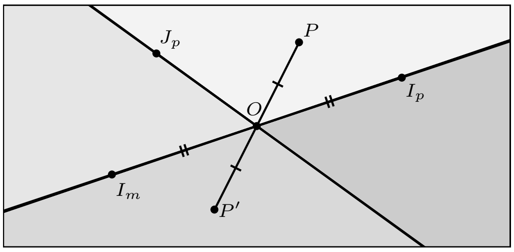


Fig. 50: Figure for lemma 33.

Proof We do not detail the proof of this lemma (Fig. 50) as it corresponds to lemma 9.33 in [SST83]. This lemma states the equivalence between our definition of coplanarity and Tarski's one. In [SST83], 4 points are coplanar if there exists a plane which contains all of these 4 points. He defines a plane by 3 non-collinear points. The first 2 form a line. The plane defined by these 3 points is the union of the sets of points which are on the same side of this line as the third point, or which are on the same side of this line as the symmetric of the third point with respect to this line, or which belong to this line. By lemma 24 we know that O, I_p, J_p and P are coplanar and by our hypotheses we know that P and P' do not belong to lines OI_p and OJ_p. Now either J_p and P or J_p and P' are on the same side of OI_p. Finally either P and I_p (which is equivalent to P' and I_m) or P and I_m (which is equivalent to P' and I_p) are on the same side of OJ_p. □

Proposition 5 *The strong parallel postulate follows from Euclid* 5.

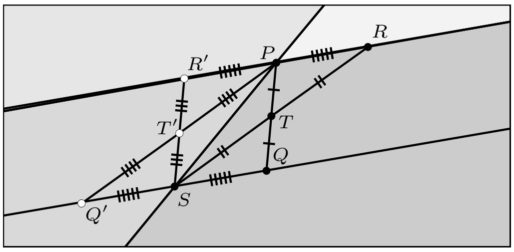


Fig. 51: Figure for proposition 5.

Proof We assume T is the midpoint of the non-degenerated line-segments $\overline{PQ}$ and $\overline{RS}$, that P, R and U are not collinear and that Euclid 5 holds and we wish to prove that lines PU and QS intersect (Fig. 51). We may assume that P, Q and S are not collinear as otherwise P would be the intersection point and we would be done. We may also assume that P, S and U are not collinear as otherwise S would be the intersection point and we would be done. Then we construct the symmetric Q' of Q with respect to S using lemma 5. Lemma 10 let us construct the midpoint T' of the line-segment $\overline{PQ'}$. And we construct R' the symmetric of S with respect to T' using lemma 5. By lemma 31 we know that P, R and R' are collinear. We also know that P, Q' and S are not collinear as otherwise P, Q and S would be collinear which would be a contradiction. We also know that P, R' and U are not collinear as otherwise P, R and U would be collinear which would be a contradiction. From this we easily prove that T' is the midpoint of the non-degenerated line-segments $\overline{PQ'}$ and $\overline{R'S}$. In the same way as in the proof of lemma 31 we can prove that R and R' are on opposite sides of line PS. Then we can prove that P is the midpoint of the non-degenerated line-segment $\overline{RR'}$. The fact that R is different is easily proven as if they were equal the same point would be on both side of line which is impossible. As R and R' are on opposite sides of line PS there exists a point at the intersection of lines RR' and PS which is between R and R'. Lemma 3 allows us to prove that this point is P. So we only need to prove that P

is equidistant from R and R' which we do by using lemma 6 in the parallelograms $PQRS$ and $PQ'R'S$ to obtain that the line-segments $\overline{PR}$ and $\overline{QS}$ are congruent and that the line-segments $\overline{PR'}$ and $\overline{Q'S}$ are congruent and using the fact that S is equidistant from Q and Q'. We construct U' the symmetric of U with respect to P and to apply lemma 33 and obtain 4 simpler cases. In all of these 4 cases we apply lemma 32 which allows us to conclude. This complete the proof that the strong parallel postulate follows from Euclid 5. $\square$

Proposition 6 *Tarski's parallel postulate follows from the strong parallel postulate.*

Proof We assume that A—D—T, B—D—C and $A \neq D$. We wish to prove the existence of points X and Y such that A—B—X, A—C—Y and X—T—Y (Fig. 52). We may also assume that all points are different and that $\neg$ Col $A\,B\,C$ as otherwise the existence of the points X and Y is trivial. Indeed it would suffice to take carefully chosen points amongst A, B, C, D and T. Lemma 5 let us construct the symmetric B' of D with respect to B. Now using lemma 11 together with the facts that A—D—T and B'—B—D we obtain the existence of a point B''' which is between B' and T such that A—B—B'''. Now lemma 10 let us prove the existence of the midpoint M_B of the line-segment $\overline{BT}$ and lemma 5 let us construct the symmetric B'' of B' with respect to M_B. Then we use the strong parallel postulate applied with B, T, B', B'', M_B and B''' asserts the existence of the point X such that Col $A\,B\,X$ and Col $B''\,T\,X$. By distinction of cases on the definition of Col $A\,B\,X$ and Col $B''\,T\,X$ we can show that the points are in the required order (namely A—B—X and B''—T—X) because other cases are impossible by application of Pasch's axiom and because we can show that some points are both on two sides and one side of line AB (we could prove that the strong parallel postulate implies Euclid 5 in the exact same way). Similarly we can assert the existence of the points C', C''', M_C, C'' and Y. So all we need to prove is that X—T—Y. We will distinguish two cases:

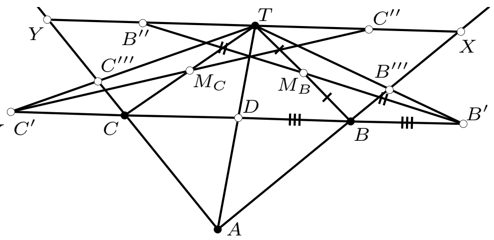


Fig. 52: Figure for proposition 6.

1. Assuming that Col $X\,T\,Y$. Lemma 11 together with the facts A—C—Y and B—D—C implies the existence of point U such that B—U—Y and A—D—U. Lemma 11 together with the facts A—B—X and that B—U—Y implies the existence of point V which is between X and Y such that U is between A and V. And finally from lemma 3 we that T is equal to V as they both are at the intersection of lines XY and AD which are not equal as otherwise A, B and C would be collinear which we know false. To prove that V is on line AD we need to prove that A and U are different. We do so by assuming they are not and seeing that it would imply that A is equal to D which would contradict our hypotheses.
2. Assuming the points X, T and Y are not collinear we can prove that the points T, B'' and Y are not collinear as if they were X, T and Y also would. From the fact that M_C is the midpoint of the non-degenerated line-segments $\overline{CT}$ and $\overline{C'C''}$ we know that the lines BC and TY are strictly parallel using lemma 29.

But from the strong parallel postulate applied with T, B, B'', MB and Y we can prove that these lines intersect which is a contradiction. This completes our proof that Tarski's version of the parallel postulate follows from the strong parallel postulate.

□

The following proof is by Beeson [Bee15].

Proposition 7 (Beeson) *Euclid* 5 *follows from the Tarski's parallel postulate.*

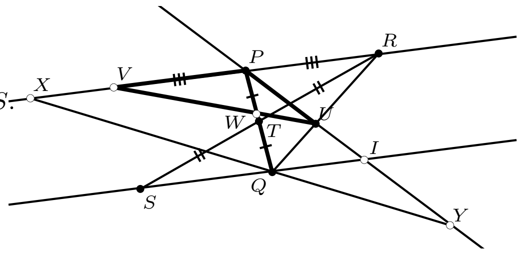


Fig. 53: Figure for proposition 7.

Proof Here we assume that point $P \not\!\!- T \not\!\!- Q$ and $R \not\!\!- T \not\!\!- S$. We also assume that $Q \not\!\!- U \not\!\!- R$ and Col $P\,Q\,S$. Finally we assume that T is equidistant from P and Q and from R and S and that Tarski's version of the parallel postulate holds. And we wish to prove the existence of a point I such that $S \not\!\!- Q \not\!\!- I$ and that $P \not\!\!- U \not\!\!- I$ (Fig. 53). Using lemma 5 we construct point V the symmetric of point R with respect to point P. Using the inner version of Pasch's axiom together with the facts that R—P—V and Q—U—R we obtain the existence of point W such that P—W—Q and U—W—V. All we need to prove in order to apply Tarski's version of the parallel postulate with points P, V, U, W and Q is that $P \neq W$. Assuming the contrary then P would be between U and V which would contradict the fact that $\neg$ Col $P\,Q\,S$ provided that $P \neq V$ (but that cannot be as otherwise P, Q and S would be collinear). So we may use Tarski's version of the parallel postulate to assert the existence of points X and Y such that P—V—X , P—U—Y and X—Q—Y. We may prove that $QS \parallel_s PR$. Since X is on line PR then lemma 26 implies that $Q \underset{PX}{\not\!\!-\!\!-} S$. Now lemma 12 let us prove that $Q \overset{Y}{\underset{P}{\not\!\!-\!\!-}} S$ using the fact that $Q \overset{Y}{\underset{X}{\not\!\!-\!\!-}} S$ which we can prove by using lemma 18 as lines QS and XY intersect in point Q. By definition of being on opposite sides of a line we know that there is a point I on line QS which is between P and Y. Now we will prove that $S \not\!\!- Q \not\!\!- I$ and that $P \not\!\!- U \not\!\!- I$ which will allow us to complete this proof.

1. Let us prove that $P \not\!\!- U \not\!\!- I$ first. We can prove that Col $P\,U\,I$ are collinear since $P \neq Y$ as otherwise P, Q and S would be collinear. So to prove that P—U—I we only need to prove that the I cannot be between P and U and that P cannot be between I and U. If P—I—U then P and U are on opposite sides of line QS which is impossible. Indeed lemma 26 implies that $P \underset{QS}{\not\!\!-\!\!-} R$ as $PR \parallel_s QS$ and lemma 18 implies that R and U are on the same side of line QS as lines QS and RU intersect in Q. Lemma 16 allows us to prove that $Q \underset{PU}{\not\!\!-\!\!-} S$ from the previous two statements. Now if I—P—U then $P \overset{U}{\underset{I}{\not\!\!-\!\!-}} R$ which is also impossible. Indeed lemma 26 implies that I and Q are on the same side of line PR as lines PR and IQ and lemma 18 implies that Q and U are on the same side of line PR as lines PR and QU intersect in R in the desired conditions.

And lemma 16 allows us to prove that I and U are on the same side of line PR from the previous two statements. To prove that $P -\!\!/\!\!- U -\!\!/\!\!- I$ it remains to prove that the points are distinct:

- $P \neq U$ because if P and U were equal then lines QS and PR would not be strictly parallel.
- $U \neq I$ because if U and I were equal then lemma 3 would imply that $Q = U$ which would contradict the fact that $Q -\!\!/\!\!- U -\!\!/\!\!- R$.

2. So it only remains to prove that $S -\!\!/\!\!- Q -\!\!/\!\!- I$. We first remark that $Q \overset{I}{\underset{S}{-\!\!/\!\!-}} R$. Indeed $Q \overset{I}{\underset{P}{-\!\!/\!\!-}} R$ as U which is between P and I is collinear with Q and R and $Q \underset{PS}{-\!\!/\!\!-} R$ which is implied from lemma 26 and the fact that $QR \parallel_s PS$. Lemma 12 allows us to prove that $Q \overset{I}{\underset{S}{-\!\!/\!\!-}} R$. Now to prove that $S - Q - I$ we can prove that S cannot be between Q and I and that I cannot be between Q and S. We do so as both of them would imply that $Q \underset{SI}{-\!\!/\!\!-} R$ which would be a contradiction. Finally $Q \neq S$ and $Q \neq I$ as otherwise it would contradict the non-collinearity part of the definition of being on opposite sides of a line and $S \neq I$ are not equal as otherwise they would be on the same side of line QR.

□

5.3 All the statements are equivalent?

To prove that Playfair's axiom follows from Tarski's parallel postulate Tarski uses implicitely lemma 35. Following is the proof of this lemma which we proved using lemma 34.

Lemma 34 $\forall ABCDI, C \neq D \Rightarrow \text{Col}\ A\,B\,I \Rightarrow \text{Col}\ C\,D\,I \Rightarrow \neg\,\text{Col}\ A\,B\,C \Rightarrow$
$$\exists XY, \text{Col}\ C\,D\,X \wedge \text{Col}\ C\,D\,Y \wedge A \overset{Y}{\underset{X}{-\!\!/\!\!-}} B$$

Proof Knowing that lines AB and CD are not parallel and that C and D are not equal we wish to prove that there exist points X and Y on line CD such that $A \overset{Y}{\underset{X}{-\!\!/\!\!-}} B$. As C and D are not equal at least one of them is different from I. We name this point X and we construct the symmetric point Y of X with respect to I using lemma 5. Obviously $\text{Col}\ C\,D\,X$ and $\text{Col}\ C\,D\,Y$. So all we need to do is to prove $A \overset{Y}{\underset{X}{-\!\!/\!\!-}} B$. The only difficulty in doing so is proving that neither X nor Y belong to line AB. Since we already know that A and B are different and that there exists a point P on line AB such that $X - P - Y$ (namely point I) proving these last two facts would allow us to conclude by definition of being on opposite sides of a line. These facts are proven by proof of negation in a similar way. Assuming $\text{Col}\ X\,A\,B$ we can prove that I and X are equal using lemma 3 which contradicts our hypotheses. This concludes our proof. □

Lemma 35 $\forall ABCDIP, C \neq D \Rightarrow \text{Col}\ A\,B\,I \Rightarrow \text{Col}\ C\,D\,I \Rightarrow$
$$\neg\,\text{Col}\ A\,B\,C \Rightarrow \neg\,\text{Col}\ A\,B\,P \Rightarrow \exists Q, \text{Col}\ C\,D\,Q \wedge A \overset{Q}{\underset{P}{-\!\!/\!\!-}} B$$

Proof Knowing that lines AB and CD are not parallel, that C and D are not equal and that P is not on line AB we wish to prove that there exists point Q on line CD such that $A \overset{Q}{\underset{P}{-\!\!/\!\!-}} B$. We first use lemma 34 to construct points X and Y on line CD such that $A \overset{Y}{\underset{X}{\text{—}}} B$. Then we distinguish two cases. Either $A \overset{X}{\underset{P}{\text{—}}} B$ and we are done as we can take Q to be X or $\neg A \overset{X}{\underset{P}{\text{—}}} B$ and lemmas 15 and 12 allow us to prove that $A \overset{Y}{\underset{P}{\text{—}}} B$ and we are done as we can take Q to be Y. □

We reproduce here the proof given in [SST83].

Proposition 8 (12.11) *Playfair's axiom follows from Tarski's parallel postulate.*

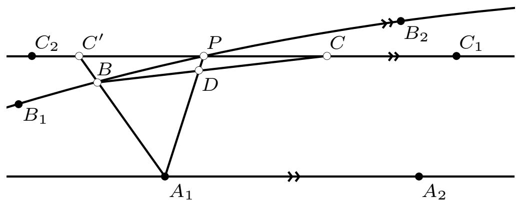


Fig. 54: Figure for proposition 8.

Proof More precisely, we want to prove that any two lines B_1B_2 and C_1C_2 going through a common point P and both parallel to a third line A_1A_2, are equal (Fig. 54). Due to our definition of parallelism, if point P belongs to line A_1A_2, then the three lines are equal. We can then suppose that $\neg \text{Col } P\, A_1\, A_2$ and also that A_1A_2 and B_1B_2, and A_1A_2 and C_1C_2 are strictly parallel. So either lines B_1B_2 and C_1C_2 are equal (if C_1 or C_2 is different from P and belongs to B_1B_2) and we are done as they are parallel or we may assume $\neg \text{Col } B_1\, B_2\, C_1$ and $\neg \text{Col } B_1\, B_2\, C_2$ and we need to find a contradiction. Hence using lemma 35, there exists a point C' on line C_1C_2 which is not on the same side of B_1B_2 as A_1. Name B the point such that A_1—B—C' and $\text{Col } B\, B_1\, B_2$ and lemma 5 let us construct point C symmetric of C' wrt. P (see Fig. 8). By inner Pasch, we know that it exists a point D such that B—D—C and P—D—A_1, and by Tarski's parallel postulate, there are some points X and Y such that: $P\text{—}B\text{—}X \wedge P\text{—}C\text{—}Y \wedge X\text{—}A_1\text{—}Y$. Now, using lemma 26 it follows that P and X are on the same side of A_1A_2 and P and Y are on the same side of A_1A_2. Hence X and Y are on the same side of A_1A_2. By definition of same side we have a point Z such that X and Z and Y and Z are on different sides of A_1A_2. Using the point A_1 we can also show that X and Y are on different sides of A_1A_2. This contradicts the fact that they are one the same side of A_1A_2 (by lemma 9.9). □

Proposition 9 *Tarski's parallel axiom follows from the triangle circumscription principle.*

Proof We assume the existence of the circumcenter of a triangle. We assume that $A—D—T$ and $B—D—C$ and $A \neq D$. We need to show that there are x and y such that $A—B—x$ and $A—C—y$ and $x—T—y$ (Fig. 55). First we may assume that $A \neq B$, $A \neq C$, $A \neq T$, $B \neq C$, $B \neq D$, $B \neq T$, $C \neq D$, $C \neq T$, $D \neq T$. We distinguish two cases.

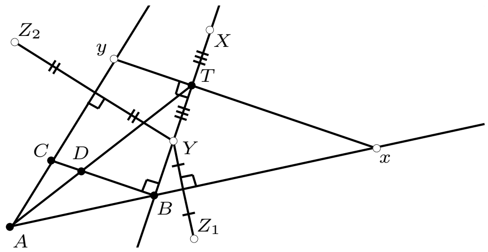


Fig. 55: Figure for proposition 9.

1. Case A, B and C are collinear.
   If $A—B—C$ then we can choose $x = T$ and $y = C$.
   If $B—C—A$ then we can choose $x = B$ and $y = T$.
   Else we have $C—A—B$.
   Using lemma 5.3, we have $B—A—D$ or $B—D—A$. If we have $B—A—D$ then we by lemma 5.2 have $A—C—T$ or $A—T—C$. In the first case we choose B and T. In the second case we choose B and C. Else we have $B—D—A$. Using a similar argument we can conclude in the other cases.
2. Case A, B and C are not collinear. Lemma 9 let us construct the foot Y of the perpendicular to BC going through T. B and C cannot be both equal to Y. Let us consider that $B = Y$ and $C \neq Y$ (we consider that B is the foot of the perpendicular to BC going through T) as the other cases are similar. Lemma 10 asserts the existence of the midpoint Y (Y is not the foot of the perpendicular to BC going through T as we assumed it was B) of the line-segment $\overline{BT}$. Let X be the symmetric of Y wrt. T which can be constructed thanks to lemma 5. Lemma 20 allows the construction of Z_1 being the symmetric of Y wrt. line AB and Z_2 being the symmetric of Y wrt. line AC. We have that $Y \neq Z_1$ and $Y \neq Z_2$ because otherwise A,B and C would be collinear. Using lemma 27, we have that $\neg\,\mathrm{Col}\ X\,Y\,Z_1$ and $\neg\,\mathrm{Col}\ X\,Y\,Z_2$. Using the existence of circumcenter, we can construct x and y, such that $Xx \equiv Yx$ and $Xx \equiv Z_1x$ and $Xy \equiv Yy$ and $Xy \equiv Z_2y$. We have that $\mathrm{Col}\ A\,B\,x$, $\mathrm{Col}\ A\,C\,y$, $\mathrm{Col}\ x\,T\,y$ and $x \neq y$. Moreover we have that $BC \parallel xy$. It remains to show that $A—B—x$, $A—C—y$ and $x—T—y$. By distinction of cases on the definition of $\mathrm{Col}\ x\,T\,y$, $\mathrm{Col}\ A\,B\,x$ and $\mathrm{Col}\ A\,C\,y$ we can show that the points are in the required order because other cases are impossible by application of Pasch's axiom and because we can show that some points are both on two sides and one side of line AB.

□

Lemma 36 *The perpendicular tranversal postulate implies the existence of the perpendicular bisector to any non-degenerate line-segment.*

Proof We want to prove that given a line-segment $\overline{AB}$ there exists a line PQ such that $A \overset{P\bullet}{\underset{\bullet Q}{\text{——}}}_s B$ (Fig. 56). Using lemma 10 one constructs the midpoint M of the line-segment $\overline{AB}$. Then applying both lemma 4 and 9 one constructs the points P' and Q' such that lines AB and $P'Q'$ are perpendicular. Then lemma 28 proves the existence of points P and Q such that lines $P'Q'$ and PQ are parallel and line PQ passes through M. It remains to prove that lines AB and PQ are perpendicular which is true from the perpendicular transversal postulates. □

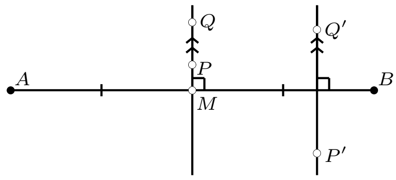


Fig. 56: Figure for lemma 36.

The following proposition is a classic, but we still give the proof because we are in a intuitionistic setting and we want to emphasize the use of the decidability of intersection.

Proposition 10 *The triangle circumscription principle follows from the perpendicular tranversal postulate.*

Proof Given a non-degenerated triangle ABC we wish to prove the existence of point O equidistant to A, B and C (Fig. 57). Lemma 36 let us construct the perpendicular bisector C_1C_2 of the line-segment $\overline{AB}$ and the perpendicular bisector B_1B_2 of the line-segment $\overline{AC}$ since they are non-degenerated line-segment as ABC is a non-degenerated triangle (Fig. 10). We now prove that it is impossible for lines B_1B_2 and C_1C_2 to not intersect to prove the existence of this intersection[9]. Assuming they do not intersect then lines B_1B_2 and C_1C_2 are parallel by definition. Using the perpendicular transversal postulate we can deduce that lines AC and C_1C_2 are perpendicular. Finally lemma 27 establishes that lines AB and AC are parallel as they are both perpendicular to line C_1C_2. This implies that A, B and C are collinear which is false by hypothesis. Since it is impossible for lines B_1B_2 and C_1C_2 to not intersect the decidability of intersection of lines let us assert that O is their intersection which is equidistant from A and B since it belongs to its perpendicular bisector and equidistant from A and C since it belongs to its perpendicular bisector. □

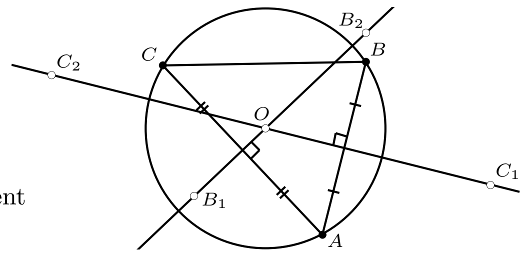


Fig. 57: Figure for proprosition 10.

Proposition 11 *Proclus' postulate follows from the postulate of transitivity of parallelism.*

[9] Note that we use here the decidability of intersection of lines.

Proof So assuming $AB \parallel CD$ and Col $P\,A\,B$ and $\neg$ Col $Q\,A\,B$ we wish to prove that there exists a point belonging to lines CD and PQ (Fig. 58). We know that lines AB and PQ are not parallel from the fact that Col $P\,A\,B$ and $\neg$ Col $Q\,A\,B$. Now we can prove by negation that CD and PQ are not parallel since if they were then by transitivity of parallelism we would be able to prove that lines AB and PQ are parallel which would contradict our hypotheses. Finally lines CD and PQ being not parallel they must intersect[10]. $\square$

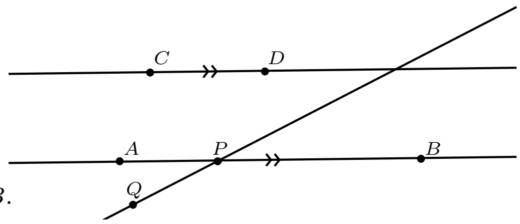


Fig. 58: Figure for proposition 11.

Proposition 12 *The strong parallel postulate follows from Proclus' postulate.*

Proof So assuming $P{-}T{-}Q$, $R{-}T{-}S$, $\neg$ Col $P\,R\,U$, $PT \equiv QT$ and $RT \equiv ST$ we wish to prove that there exists a point I belonging to lines SQ and PU (Fig. 59). We will prove the existence of this point by proving that Col $P\,R\,P$, $\neg$ Col $P\,R\,U$ and $PR \parallel QS$. Then Proclus' postulate will allow us to construct the intersection point. The first of these facts can easily be proven using lemma 29 and the other facts are either trivial or an hypothesis. This completes the proof that the strong parallel postulate follows from Proclus' postulate. $\square$

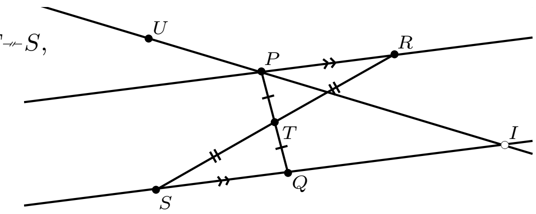


Fig. 59: Figure for proposition 12.

We can now finally prove our main theorem which summarize all the equivalences:

Theorem 1 *In the context of Tarski's neutral geometry, assuming decidability of intersection, the following properties are equivalent:*

1. *Midpoint converse postulate*
2. *Postulate of transitivity of parallelism*
3. *Playfair's postulate*
4. *Perpendicular transversal postulate*
5. *Postulate of parallelism of perpendicular tranversals*
6. *Tarski's parallel postulate*
7. *Proclus' postulate*
8. *Euclid 5^{th} postulate*
9. *Strong parallel postulate*
10. *Triangle circumscription principle*

[10] Note that we use here the decidability of intersection of lines.

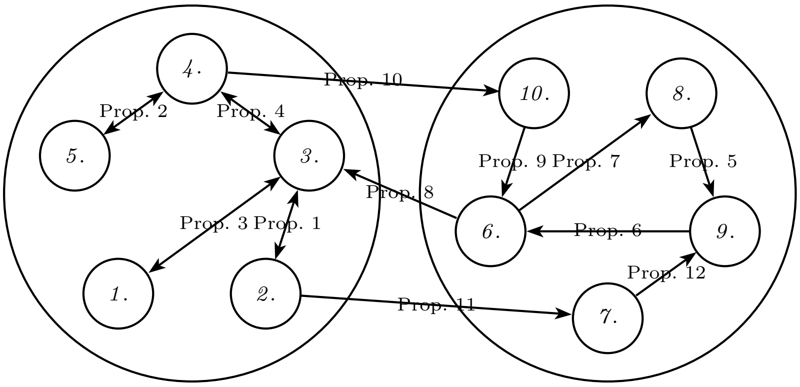


Fig. 60: Overview of the proofs. The postulates 1 to 5 do not imply the decidability of intersection whereas the postulates 6 to 10 imply the decidability of intersection.

The concrete proofs have been done in the previous propositions 1-12. We only provide a graphical summary of the implications (Fig. 60) to convince the reader that we proved enough implications. To convince Coq, we just use the fact the proposition 1-12 imply theorem 1 is a tautology and we use the corresponding Coq tactic.

To formalize this theorem in Coq we need a definition for a n-ary equivalence relation. We use the following definition using lists:

```
Definition all_equiv (l : list Prop) :=
  forall x y, In x l -> In y l -> (x<->y).
```

The final Coq statement is the following:

```
Theorem parallel_postulates:
 decidability_of_intersection ->
 all_equiv  (tarski_s_parallel_postulate::
             triangle_circumscription_principle::
             playfair_s_postulate::
             perpendicular_transversal_postulate::
             par_perp_2_par_property::
             proclus_postulate::
             postulate_of_transitivity_of_parallelism::
             strong_parallel_postulate::
             euclid_5::
             midpoints_converse_postulate::nil).
```

5.4 About decidability of intersection

The attentive reader may have remarked that in the previous section our main theorem was proved under the assumption `decidability_of_intersection`[11]. The definition of the assumption is the following:

```
Definition decidability_of_intersection :=
  forall A B C D,
  (exists I, Col I A B /\ Col I C D) \/
  ~ (exists I, Col I A B /\ Col I C D).
```

This statement is a tautology in classical logic but in our context of Tarski's neutral geometry with decidability of equality, this statement can not be proved. To convice oneself that it can not be proved from the axioms of neutral geometry it suffices to remark that all other existential axioms of Tarski allow only to build points which are not arbitrary far from the given points. To formalize this argument, it requires Herbrand's theorem, this is what has been done in [BBN15][12]. However, some forms of the parallel postulate can be used to prove decidability of intersection but not all. The following two theorems (and Fig. 60) summarize the results. We have two groups of postulates which are equivalent even if we do not assume decidability of intersection. The first group (postulates 1 to 5) does not imply decidability of intersection of lines[13]. The second group (postulates 6 to 10) implies decidability of intersection of lines (from the facts that all of these postulates imply the strong parallel postulate and that the strong parallel postulate implies decidability of intersection of lines by proposition 13[14]). We obtain the same kind of results as Beeson in [Bee14]: all postulates are not equivalent in a intuitionistic setting. But our results are weaker because we assume decidability of equality which is stronger than the Markov's principle that Beeson assumes ($\neg\neg A = B \Rightarrow A = B$).

Proposition 13 *The decidability of intersection of lines follows from the strong parallel postulate.*

[11] It is interesting to remark that the decidability of being on the opposite sides of a line with a given point and the decidability of intersection of lines are not equivalent. Indeed the former is a lemma in our context of Tarski's neutral geometry with decidability of equality but not the latter.

[12] The fact that Tarski's elementary geometry is decidable and enjoy quantifier elimination does not contradict this fact because the proof of Tarski is performed using first-order classical logic.

[13] This fact can be proven in the same fashion as in [BBN15]. We should also point out that to achieve the equivalence between any two of the postulates of the first group we had to either modify ours proofs of implications so that they do not rely on the decidability of intersection of lines or simply find a new proof of the implication. The proposition 4 is an example of the latter.

[14] Since we removed some proofs of implications between postulates throughout this work as explained in the beginning of section 5 we did not always had proofs that any of the postulates 6 to 10 implies the strong parallel postulate. Therefore we also proved that Proclus' postulate and the triangle circumscription principle imply the decidability of intersection of lines.

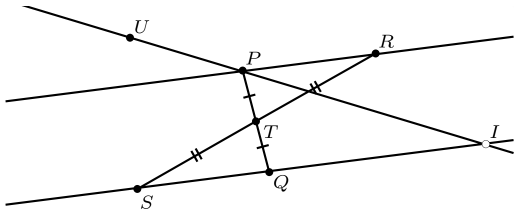


Fig. 61: Figure for proprosition 13.

Proof Given 4 points that we name P, Q, S and U (and not A, B, C and D to work with the same name of those in the definition of the strong parallel postulate) we wish to prove that either there exists a point I such that $\mathrm{Col}\ I\,S\,Q$ and $\mathrm{Col}\ I\,P\,U$ or that there does not exist such a point (Fig. 61). We first elimiate the case of $\mathrm{Col}\ P\,Q\,S$ in which there exists such a point I, namely it is P. So we may assume that $\neg\,\mathrm{Col}\ P\,Q\,S$ and we then eliminate the case of P and U being equal as again there exists such a point I, namely Q (we could have also taken S to be this point). So we may assume P and U to be different. Now we construct the midpoint T of the line-segment $\overline{PQ}$ using lemma 10 and the symmetric point R of S with respect to T using lemma 5. Finally we will distinguish two cases. Either $\neg\,\mathrm{Col}\ P\,R\,U$ and the strong parallel postulate asserts there exists such a point I provided that $P \not- T \not- Q$ and $R \not- T \not- S$ which we easily prove as P and Q are different and $\neg\,\mathrm{Col}\ P\,Q\,S$. The other case is when $\mathrm{Col}\ P\,R\,U$. In this case we can prove that lines QS and PU are strictly parallel using lemma 29 and the fact $\neg\,\mathrm{Col}\ P\,Q\,S$ and by definition of two lines being stricly parallel we know that there does not exist such a point I. □

From the section 5.1 we can prove the following theorem:

Theorem 2 *In the context of Tarski's neutral geometry, without assuming decidability of intersection, the following properties are equivalent:*

1. *Midpoint converse postulate*
2. *Postulate of transitivity of parallelism*
3. *Playfair's postulate*
4. *Perpendicular transversal postulate*
5. *Postulate of parallelism of perpendicular tranversals*

The Coq statement for theorem 2 is the following:

```
Theorem parallel_postulates_without_decidability_of_intersection_of_lines:
 all_equiv  (playfair_s_postulate::
             perpendicular_transversal_postulate::
             postulate_of_parallelism_of_perpendicular_tranversals::
             postulate_of_transitivity_of_parallelism::
             midpoints_converse_postulate::nil).
```

Proposition 13 and theorem 1 allow us to prove the following theorem:

Theorem 3 *In the context of Tarski's neutral geometry, without assuming decidability of intersection, the following properties are equivalent:*

6. *Tarski's parallel postulate*
7. *Proclus' postulate*
8. *Euclid 5^{th} postulate*

9. *Strong parallel postulate*
10. *Triangle circumscription principle*

The Coq statement for theorem 3 is the following:

```
Theorem result_similar_to_beeson_s_one :
   all_equiv (euclid_5::
              strong_parallel_postulate::
              triangle_circumscription_principle::
              proclus_postulate::
              tarski_s_parallel_postulate::nil).
```

5.5 The role of Continuity

In this section, we explain why we did not use the axiom of continuity and the impact on the formalization. First let us recall the continuity axiom given in [SST83]:

Let $\phi(x)$ and $\psi(y)$ be first-order formulas containing no free instances of either a or b.

$$(\exists a \forall x \forall y\ (\phi(x) \wedge \psi(y)) \Rightarrow a\text{—}x\text{—}y) \ \Rightarrow\ (\exists b \forall x \forall y\ (\phi(x) \wedge \psi(y)) \Rightarrow x\text{—}b\text{—}y)$$

The formalization in Coq of such an axiom schema in the higher order logic would require an embedding of first order logic in Coq. A consequence of this axiom is the principle of Archimedes:

> Given four points A,B,C and D, there exist a finite number of points A_1, A_2, ..., A_n on the line AB such that $CD \equiv AA_1 \equiv A_1A_2 \equiv \ldots \equiv A_{n-1}A_n$ and $B\text{—}A\text{—}A_n$.

Using the Archimedean axiom, Legendre has shown that the fact that the sum of angles of triangles is two rights implies Euclid's 5^{th} postulate. Max Dehn, a student of Hilbert, has shown that the principle of Archimedes is necessary for the proof that "the sum of the angles of a triangle is two right" implies Euclid's 5^{th} postulate [Deh00]. Dehn gives a (non Archimedean) model in which Euclid's 5^{th} holds and the sum of angles of a triangle is not always two rights.

We could prove using the axiom of continuity more equivalence between different versions of the parallel postulate. But in this paper we focus on a *synthetic* approach. We do not work in a particular model. We want to provide equivalence results which are valid independently of the continuity because it allows the user of the library to choose which version of the parallel postulate he wants, even if he is interested in working in non-Archimedean models.

6 Conclusion

We have described the formalization within the Coq proof assistant of the proof that 10 versions of the parallel postulate are equivalent. The originality of our proofs relies on the fact that first the equivalence between these different versions are proved in Tarski's neutral geometry without using the continuity axiom nor line-circle continuity and second we work in an intuitionistic logic. Assuming decidability of equality, we clarified the role of the decidability of intersection. We obtained the formal proof that assuming decidability of equality of points some versions of the parallel postulate imply decidability of the intersection of lines. The formal proofs of equivalences consist of about 6k lines of code and rely on 47k lines of code corresponding to a library for neutral geometry. The proofs make heavy use of the tactics developed previously [BNSB14b]. The use of a proof assistant was crucial to check these proofs because it is very easy to produce an incorrect proof in neutral geometry by assuming by error a statement equivalent to the parallel postulate. This work can have two natural extensions. First, we could formalize the proof of equivalence of other versions of the parallel postulate under a continuity assumption or at least Aristotle's axiom as shown in [Gre10]. Second, we could weaken our axiom system to study the fully constructive version of Beeson.

Availability

The full Coq development is available here: `http://geocoq.github.io/GeoCoq/`

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