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A Case Study in Formalizing Projective Geometry in Coq: Desargues Theorem[*]

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Abstract

Formalizing geometry theorems in a proof assistant like Coq is challenging. As emphasized in the literature, the non-degeneracy conditions lead to technical proofs. In addition, when considering higher-dimensions, the amount of incidence relations (e.g. point-line, point-plane, line-plane) induce numerous technical lemmas. In this article, we investigate formalizing projective plane geometry as well as projective space geometry. We mainly focus on one of the fundamental properties of the projective space, namely Desargues property. We formally prove it is independent of projective plane geometry axioms but can be derived from Pappus property in a two-dimensional setting. Regarding at least three dimensional projective geometry, we present an original approach based on the notion of rank which allows to describe incidence and non-incidence relations such as equality, collinearity and coplanarity homogeneously. This approach allows to carry out proofs in a more systematic way and was successfully used to formalize fairly easily Desargues theorem in Coq. This illustrates the power and efficiency of our approach (using only ranks) to prove properties of the projective space.

Key words: formalization, Desargues, rank, projective geometry, Coq, Hessenberg theorem, Hexamys

1. Introduction

This article deals with formalizing projective geometry in a proof assistant such as Coq [1, 7] and studies Desargues property both in the plane and in an at least 3-dimensional setting. In the plane, proofs are performed in a traditional way using points and lines. However, in an at least 3 dimensional space, we use the concept of rank to formally prove Desargues theorem in Coq. In the longer term, the underlying objective of the presented work consists in designing a geometric formal prover able to handle the non-degeneracy conditions in geometry, and especially in geometric constraint solving [13, 15].

Indeed, when designing an object by the mean of geometric constraints, one has to correctly and completely specify the shape of the object. The corresponding constraint system can lead to contradictions and degenerate cases which must be handled properly by the solver as shown in [25]. The structural contradictions are easy to detect, whereas those resulting from geometric theorems are more difficult to deal with.

We choose to focus on projective geometry which is a simple but powerful enough setting to express arbitrarily complex problems as shown in [19]. Moreover, in 3D (or higher), proofs become much more difficult than in 2D: first, Desargues property becomes a theorem and consequently all the projective spaces arise from a division ring; second incidence geometry has to deal not only with points and lines, but also with planes or, more generally, flats; third there is a combinatoric explosion of cases.

For projective plane geometry, we use a traditional approach dealing with points, lines and an incidence relation to formally prove the independence of Desargues property. We then formalize Pappus property as well as hexamys in order to prove Hessenberg theorem, which states Pappus property entails Desargues property in projective plane geometry.

When it comes to at least 3 dimensional projective geometry, we propose to use the concept of rank, which provides a generic way to describe incidence relations and allows to express non-degeneracy conditions nicely. Informally, ranks allow to distinguish between equal/non-equal points, collinear/non-collinear points, coplanar/non-coplanar points, etc. We validate this approach by sucessfully carrying out a mechanized proof (using only ranks) of Desargues theorem which is one of the fundamental theorems of the projective space.

We mechanize the proofs using the Coq proof assistant which implements

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a higher order intuitionistic logic based on type theory. In such a proof assistant, every step of reasoning is proposed by the user (in the form of a *tactic*) but then checked by the system. It dramatically increases the reliability of the proofs compared to paper-and-pencil proofs. In addition, during the development process, being able to change the axiom system easily is very convenient. Proofs can be automatically rechecked by the system and changes only require minor rewriting of the proofs.

However, formal proofs tend to be more technical to write, not leaving out a single piece of details. Therefore we had to design some efficient proof techniques to deal with points and lines in the plane as well as with matroïd and rank properties when the dimension n of the considered space is greater to 2. Developing these techniques, which mainly use and make work together several simple automated tactics is inevitable to make writing formal proofs in Coq tractable.

Related work Proof assistants have already been used in the context of geometry. Numerous papers have emphasized the importance of the problem of degenerate cases in formal geometry [9, 12, 18, 23]. Brandt and Schneider studied how to handle degenerate cases for the orientation predicates in computational geometry using three valued logic [3]. Bezem and Hendricks formalized Hessenberg's theorem in Coq [2]. Guilhot has formalized in Coq a proof of Desargues theorem in affine geometry [12]. Narboux has formalized in Coq the area method of Chou, Gao and Zhang [6, 14, 22] and applied it to obtain a proof of Desargues theorem in affine geometry. Kusak has formalized in Mizar Desargues theorem in the Fanoian projective at least 3-dimensional space [16]. The assumption that the space is Fanoian makes the theorem more specialized than ours. We also carried out some preliminary work on formalizing projective plane geometry in Coq [17]. Finally, the idea of proving projective space theorems with ranks is suggested by Michelucci and Schreck in [20].

Outline The paper is organized as follows. In section 2, we present the axioms for projective geometry and we give an overview of our Coq implementation. In section 3, we explain Desargues property and why it is a fundamental property of projective geometry. Section 4 investigates the role of Desargues property in the case of the generic 2D projective plane and its links with Pappus property through the notion of Hexamys. Section 5 introduces ranks and the associated axiom system for projective space geometry, which is then used to formally prove in Coq that Desargues property holds in at least 3-dimensional projective space.

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2. Axiom Systems for Projective Geometry

Projective geometry is a general setting in the hierarchy of geometries which assumes that two lines in a plane always meet [4, 8]. We first assume that we have two kinds of objects (points and lines). Planes are not basic objects in this axiom system, but are defined within the theory. We then consider a relation ($\in$) between elements of these two sets.

2.1. Axiom System for Projective Plane Geometry

The axiom system for projective plane geometry consists in very few axioms linking abstact points and lines of a plane. Informally, these axioms capture the facts that two different points define one line and two different lines define one point. Moreover, each line contains at least three points and there are at least two lines in the plane. Formally, we have the five following axioms:

Line-Existence $\forall A\ B : Point, \exists l : Line, A \in l \wedge B \in l$

Point-Existence $\forall l\ m : Line, \exists A : Point, A \in l \wedge A \in m$

Uniqueness $\forall A\ B : Point, \forall l\ m : Line,$
$\quad A \in l \wedge B \in l \wedge A \in m \wedge B \in m \Rightarrow A = B \vee l = m$

Three-Points $\forall l : Line, \exists ABC : Point,$
$\quad A \neq B \wedge B \neq C \wedge A \neq C\ \wedge\ A \in l \wedge B \in l \wedge C \in l$

Lower-Dimension-2 $\exists l : Line, \exists m : Line, l \neq m$

The axiom **Lower-Dimension-2** prevents a single line from being a model. It ensures we actually describe a two dimensional projective space.

2.2. Axiom System for Projective Space Geometry

Several axioms remains the same when we consider an at least 3-dimensional space. The required axioms for projective at least 3-dimensional space are listed below:

Line-Existence $\forall A\ B : Point, \exists l : Line, A \in l \wedge B \in l$

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Figure 1: Pasch's axiom

Pasch
$$\forall A\ B\ C\ D : Point, \forall l_{AB}\ l_{CD}\ l_{AC}\ l_{BD} : Line,$$
$$A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D \wedge$$
$$A \in l_{AB} \wedge B \in l_{AB} \wedge C \in l_{CD} \wedge D \in l_{CD} \wedge$$
$$A \in l_{AC} \wedge C \in l_{AC} \wedge B \in l_{BD} \wedge D \in l_{BD} \wedge$$
$$(\exists I : Point, I \in l_{AB} \wedge I \in l_{CD}) \Rightarrow$$
$$(\exists J : Point, J \in l_{AC} \wedge J \in l_{BD})$$

Uniqueness $\forall A\ B : Point, \forall l\ m : Line,$
$$A \in l \wedge B \in l \wedge A \in m \wedge B \in m \Rightarrow A = B \vee l = m$$

Three-Points $\forall l : Line, \exists ABC : Point,$
$$A \neq B \wedge B \neq C \wedge A \neq C \wedge A \in l \wedge B \in l \wedge C \in l$$

Lower-Dimension-3 $\exists l\ m : Line, \forall p : Point, p \notin l \vee p \notin m$

The **Point-Existence** axiom is replaced by **Pasch** axiom. Indeed we need to ensure that two lines always meet only if co-planar. Axiom **Lower-Dimension-3** ensures that there exists two lines which do not meet.

2.3. Formalization in Coq

Implementing both axiom systems in the Coq proof assistant is straightforward. The main difference between the formalization and the axiom system on paper relies in the fact that we need to be carefull about the equality relations and decidability issues. Figure 2 presents the formalization of the projective space in Coq. We note the predicate $\in$ using `Incid`. To enhance modularity we make use of the module system of Coq. In addition, the equality on points (noted `DecPoints.eq`) and lines (noted `line_eq`) are parametrized. As Coq logic is intuitionist, we have to state explicitly which predicates are assumed to be decidable. We assume that we have a set of points with a decidable equality : `DecPoints` is a instance of a `DecidableType`:

```
Module Type DecidableType.
  Parameter t : Set.
  Parameter eq : t -> t -> Prop.
  Axiom eq_refl : forall x : t, eq x x.
```

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```
Module Type ProjectiveSpaceOrHigher (DecPoints: DecidableType).

Definition Point := DecPoints.t.
Parameter Line : Type.

Parameter line_eq : Line -> Line -> Prop.
Axiom line_eq_sym   : forall l m, line_eq l m -> line_eq m l.
Axiom line_eq_trans : forall l m n,
 line_eq l m -> line_eq m n ->  line_eq l n.
Axiom line_eq_refl  : forall l,  line_eq l l.

Parameter Incid : Point -> Line -> Prop.

Axiom incid_dec : forall (A:Point)(l:Line),{Incid A l}+{~Incid A l}.

Axiom line_existence : forall A B : Point,
  {l : Line | Incid A l /\ Incid B l}.

Axiom pasch : forall A B C D:Point, forall lAB lCD lAC lBD :Line,
dist4 A B C D ->
Incid A lAB/\Incid B lAB -> Incid C lCD/\Incid D lCD ->
Incid A lAC/\Incid C lAC -> Incid B lBD/\Incid D lBD ->
(exists I:Point, (Incid I lAB /\ Incid I lCD)) ->
 exists J:Point, (Incid J lAC /\ Incid J lBD).

Axiom uniqueness : forall (A B :Point)(l1 l2:Line),
   Incid A l1 -> Incid B l1  -> Incid A l2 -> Incid B l2 ->
 DecPoints.eq A B \/ line_eq l1 l2.

Axiom three_points :
  forall l:Line,exists A:Point, exists B:Point, exists C:Point,
  dist3 A B C /\ Incid A l /\ Incid B l /\ Incid C l.

Axiom lower_dimension_3 : exists l:Line, exists m:Line,
  forall p:Point, ~Incid p l \/ ~Incid p m.

End ProjectiveSpaceOrHigher.
```

Figure 2: Projective Space Axiom System in Coq

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```
   Axiom eq_sym : forall x y : t, eq x y -> eq y x.
   Axiom eq_trans : forall x y z : t,
                        eq x y -> eq y z -> eq x z.
   Parameter eq_dec : forall x y : t,
                        { eq x y } + { ~ eq x y }.
End DecidableType.
```

The notation `{eq x y}+{~ eq x y}` means that we must know construc-
tively that either $x = y$ or $x \neq y$. As expressed by axiom `incid_dec` we also
assume the decidability of incidence. In Coq syntax `/\`, `\/` and `~` stands
respectively for logic conjunction, disjunction and negation. The notation
`{l : Line | Incid A l /\ Incid B l}` expresses that there exists a line
l going through A and B.

3. Desargues Property

Desargues property is among the most fundamental properties of pro-
jective geometry since in the projective space Desargues property becomes
a theorem and consequently all the projective spaces arise from a division
ring. Let's first recall Desargues statement in projective geometry. Desar-
gues property states that:

*Let E be a projective space and A,B,C,A',B',C' be points in E, if the
three lines joining the corresponding vertices of triangles ABC and $A'B'C'$
all meet in a point O, then the three intersections of pairs of corresponding
sides α, β and γ lie on a line.*
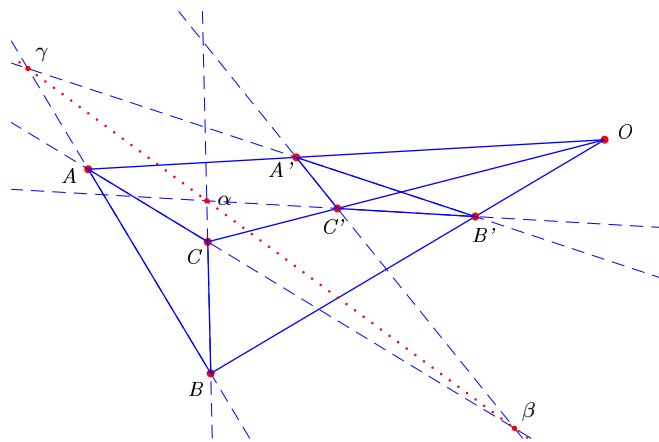
If E is of dimension two, Desargues property is independent from all the projective plane geometry axioms. If E is at least of dimension three, Desargues property always holds.

Even though it can be expressed, it is not provable when E is a plane. Indeed in the 2D case, some projective planes are Desarguesian, for instance Fano's plane, while other planes are not, for instance Moulton plane. The next section investigates the role of Desargues property in projective planes.

4. Projective Plane

In the projective plane, Desargues statement is not necessarily a theorem. We show on the one hand, that Desargues property is compatible with the axioms of projective geometry and, on the other hand that it is independent of them. First we prove that it is a theorem in the smallest projective plane (Fano's plane) and then we show that in a particular model of projective geometry (Moulton plane) it is not a theorem. This shows the independence of Desargues theorem from the axioms of projective plane geometry, which can be regarded as the starting point of non-desarguesian geometry [5]. Finally we formalize the proof of Hessenberg's theorem which demonstrates that in every Pappusian projective plane, Desargues property holds.

4.1. Desargues in Fano's Plane

Fano's plane is the model of projective plane geometry with the least number of points and lines: 7 each. The incidence relation is given by the figure. On the figure, points are simply represented by points, whereas lines are represented by six segments and a circle (DEF). One can verify that the axioms of projective plane geometry (see Section 2.1) hold as shown in [17].
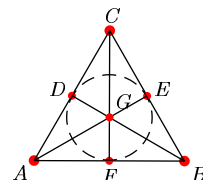
At first sight, proving Desargues property in Fano's plane seems to be straightforward to achieve by case analysis on the 7 points and 7 lines. However, this requires handling numerous cases[2] including many configurations which contradict the hypotheses.

To formalize the property we make use of two kinds of symmetries, a symmetry of the theory and a symmetry of the statement.

---

[2]The most naïve approach would consider 7^7 cases, even with careful analysis it remains untractable to prove all the cases without considering symmetries.

4.1.1. Symmetry of the statement

We first study the special case `desargues_from_ A_specialized` where the point O of Desargues configuration corresponds to A, and the line OA corresponds to ADG, OB to CAE and OC to ADF.

```
Lemma Desargues_from_A_specialized :
 forall P Q R P' Q' R' alpha beta gamma
        lPQ lPR lQR lP'Q' lP'R' lQ'R',
((on_line P Q gamma lPQ) /\ (on_line P' Q' gamma lP'Q')) /\
((on_line P R beta  lPR) /\ (on_line P' R' beta  lP'R')) /\
((on_line Q R alpha lQR) /\ (on_line Q' R' alpha lQ'R')) /\
((on_line A P P' ADG)  /\
 (on_line A Q Q' CAE)  /\
 (on_line A R R' ABF)) /\
~collinear A P Q /\ ~collinear A  P  R  /\ ~collinear A Q R /\
~collinear P Q R /\ ~collinear P' Q' R' /\
 ((P<>P')\/(Q<>Q')\/(R<>R')) ->
collinear alpha beta gamma.
```

Then as Desargues statement is symmetric by permutation of the three lines which intersect in O, we can formalize a proof of slightly more general lemma `desargues_from_A` where the point O of Desargues configuration still corresponds to A but the three lines (`lP`,`lQ` and `lR`) intersecting in O are universally quantified.

```
Lemma Desargues_from_A :
forall P Q R P' Q' R' alpha beta gamma lP lQ lR
       lPQ lPR lQR lP'Q' lP'R' lQ'R',
((on_line P Q gamma lPQ) /\ (on_line P' Q' gamma lP'Q')) /\
((on_line P R beta  lPR) /\ (on_line P' R' beta  lP'R')) /\
((on_line Q R alpha lQR) /\ (on_line Q' R' alpha lQ'R')) /\
((on_line A P P' lP) /\
 (on_line A Q Q' lQ) /\
 (on_line A R R' lR)) /\
~collinear A P Q /\  ~collinear A P R /\ ~collinear A Q R /\
~collinear P Q R /\ ~collinear P' Q' R' /\
 ((P<>P')\/(Q<>Q')\/(R<>R')) ->
 collinear alpha beta gamma.
```

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```
Module swapf3 (M:fano_plane) : fano_plane
with [...]

Definition Point:=M.Point.

Definition A:=M.B. Definition B:=M.E.
Definition C:=M.D. Definition D:=M.F.
Definition E:=M.C. Definition F:=M.G.
Definition G:=M.A.

[...]

Definition ABF:=M.BEG. Definition BCD:=M.DEF.
Definition CAE:=M.BCD. Definition ADG:=M.ABF.
Definition BEG:=M.CAE. Definition CFG:=M.ADG.
Definition DEF:=M.CFG.

[...]

End swapf3.
```

Figure 3: A functor showing an invariance property of Fano's plane.

4.1.2. Symmetry of the theory

The theory of Fano's plane is invariant by permutation of points. It means that, even if it is not obvious from the figure 4.1, all the points plays the same role: if (A, B, C, D, E, F, G) is a Fano's plane then (B, C, D, F, E, G, A) is one as well. We formalize this by building a functor from Fano's theory to itself which permutes the points (Figure 3). Using this functor and `desargues_from_A`, we show that Desargues property holds for any choice for O among the 7 points of the plane.

4.2. Independence of Desargues Property

We now consider a particular projective plane, namely Moulton plane and show that Desargues property does not hold in this plane.

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4.2.1. Moulton plane and its projective counterpart

Moulton plane [21] is an affine plane in which lines with a negative slope are bent (i.e. the slope is doubled) when they cross the y-axis. It can be easily extended into a projective plane.

Moulton plane is an incidence structure which consists in a set of points P, a set of lines L, and an incidence relation between elements of P and elements of L. Points are denoted by couples $(x, y) \in \mathbb{R}^2$. Lines are denoted by couples $(m, b) \in (\mathbb{R} \cup \infty) \times \mathbb{R}$ (where m is the slope - ∞ for vertical lines - and b the y-intercept). The incidence relation is defined as follows:

$$
(x, y) \in (m, b) \iff
\begin{cases}
x = b & \text{if } m = \infty \\
y = mx + b & \text{if } m \geq 0 \\
y = mx + b & \text{if } m \leq 0, x \leq 0 \\
y = 2mx + b & \text{if } m \leq 0, x \geq 0.
\end{cases}
$$

This incidence structure verifies the properties of an affine plane. It can be turned into a projective plane through the following process.

- We extend P with points at the infinite (one *direction point* for each possible slope, including the vertical one); therefore P is $(\mathbb{R} \times \mathbb{R}) \cup \mathbb{R}$.

- We extend the set L of affine lines with a new one which connects all points at the infinite; therefore L is $((\mathbb{R} \cup \infty) \times \mathbb{R}) \cup \infty$.

- We finally extend the incidence relation in order to have all *direction points* and only them incident to the infinite line. We also extend each affine line with a direction point (the one bearing its slope).

This construction leads to a projective plane. The whole process is formally described in Coq and we show that all the axioms of projective plane geometry presented in section 2.1 hold. Most proofs on real numbes rely on using Gröbner basis computation in Coq.

4.2.2. A configuration of Desargues figure where the statement does not hold

We build a special configuration of Desargues for which the property does not hold. This can be achieved in a very algebraic point of view using only coordinates and equations for lines. We first present it that way and then show on a figure why Desargues property does not hold for our configuration.

Let's consider seven points: $O(-4, 12)$, $A(-8, 8)$, $B(-5, 8)$, $C(-4, 6)$, $A'(-14, 2)$, $B'(-7, 0)$ and $C'(-4, 3)$. We then build the points $\alpha(-3, 4)$,
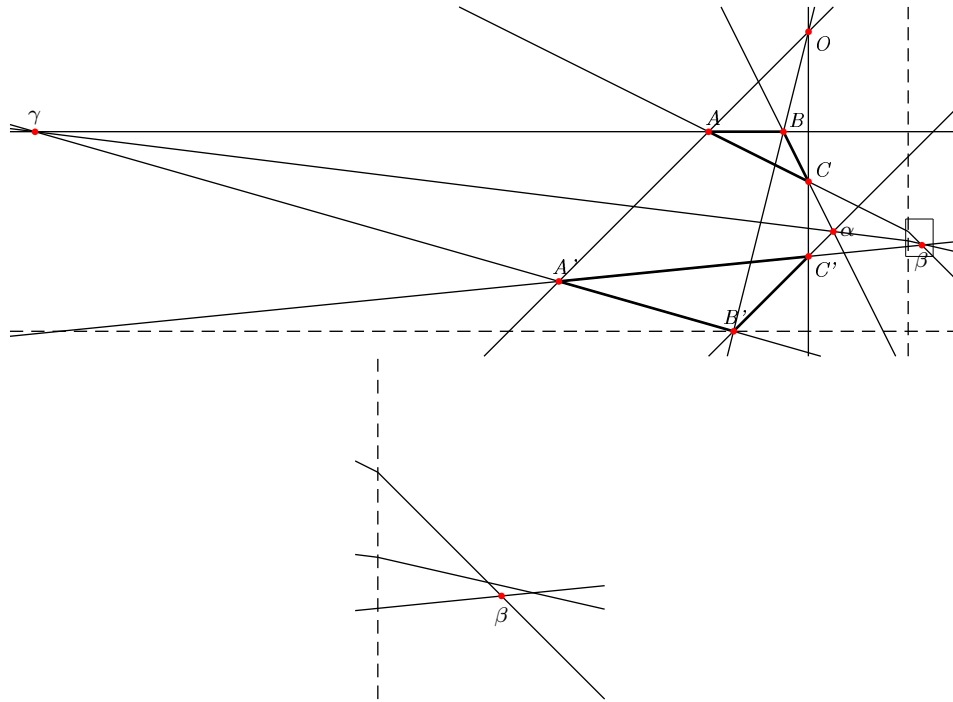
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Figure 4: Counter example to Desargues theorem in Moulton plane

$\beta(6/11, 38/11)$ and $\gamma(-35, 8)$ which are respectively at the intersection of (BC) and $(B'C')$, (AC) and $(A'C')$ and (AB) and $(A'B')$. Then we can check using automated procedures on real numbers computation that there exists no line in Moulton plane which is incident to these 3 points α, β and γ.

Overall Desargues property does not hold in this configuration because only some of the lines are bent. Especially, of the two lines used to build β, one of them is a straight line $(A'C')$ and the other one (AC) is bent. That is what prevents the three points α, β and γ from being on the same line.

Proofs of these lemmas illustrate how combining *automated* and *interactive* theorem proving can be successful.

4.3. Hessenberg's Theorem: Pappus Implies Desargues

In this section, we describe our formalization of Hessenberg's theorem stating that Pappus axiom implies that Desargues property holds in the projective plane. The proof we formalize use the concept of hexamys (*mystic*
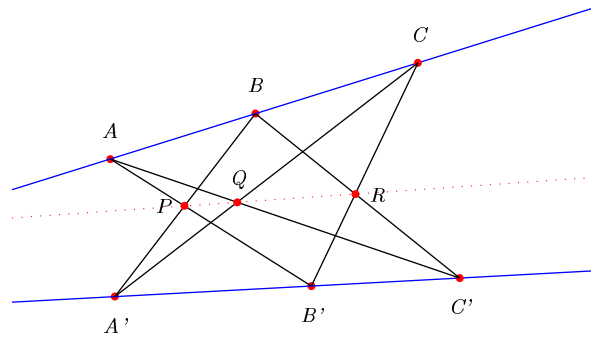
Figure 5: Pappus configuration

hexagrams as named by Pascal). First, we need to define the Pappus property.

4.3.1. Pappus

We say that a plane enjoys Pappus property when:

```
Definition pappus_weak :=
 forall A B C A' B' C' P Q R,
 Col A B C -> Col A' B' C' ->
 all_distinct_6 A B C A' B' C' ->
 is_on_proper_inter P A B' A' B ->
 is_on_proper_inter Q B C' B' C ->
 is_on_proper_inter R A C' A' C ->
 Col P Q R.
```

The property `is_on_proper_inter P A B' A' B` means that the lines AB' and $A'B$ are well defined and not parallel and P is at the intersection. This definition of Pappus configuration assumes that the six points are distinct and that the intersections are all well defined. This also implies that the lines AB and $A'B'$ are distinct. This definition captures the configuration presented in figure 5 without any particular case. But, in field of formal proofs, we must be careful about degenerate cases. Fortunately in the context of projective geometry, the weak version of Pappus property as stated above is also equivalent to this stronger version which assumes that either

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all the intersections are well defined or the six points (A,B,C,A',B',C') are distinct:

```
Definition pappus_strong :=
 forall A B C A' B' C' P Q R,
 (all_distinct_6 A B C A' B' C' \/
 (line A B' <> line A' B /\ A<>B' /\ A'<>B /\
  line B C' <> line B' C /\ B<>C' /\ B'<>C /\
  line A C' <> line A' C /\ A<>C' /\ A'<>C) ) ->
 Col A B C -> Col A' B' C' ->
 is_on_inter P A B' A' B ->
 is_on_inter Q B C' B' C ->
 is_on_inter R A C' A' C ->
 Col P Q R.
```

4.3.2. Mystic hexagram

We say that an hexagon is an *hexamy* if the three intersections of the opposite sides of the hexagon are collinear.

```
Definition is_hexamy A B C D E F :=
  all_distinct_6 A B C D E F /\
  let P:= inter (line B C) (line E F) in
  let Q:= inter (line C D) (line F A) in
  let R:= inter (line A B) (line D E) in
Col P Q R.
```

It is easy to show that every circular permutation of an hexamy is also always an hexamy.

```
Lemma hexamy_rot_left : forall A B C D E F,
 is_hexamy A B C D E F ->  is_hexamy B C D E F A.
```

We say that a plane enjoys the hexamy property if every permutation of an hexamy is also an hexamy. As circular permuations of hexamys are hexamys we just need to assume that if (A, B, C, D, E, F) is an hexamy so is (B, A, C, D, E, F), in Coq syntax:

```
Definition hexamy_prop := forall A B C D E F,
is_hexamy A B C D E F -> is_hexamy B A C D E F.
```
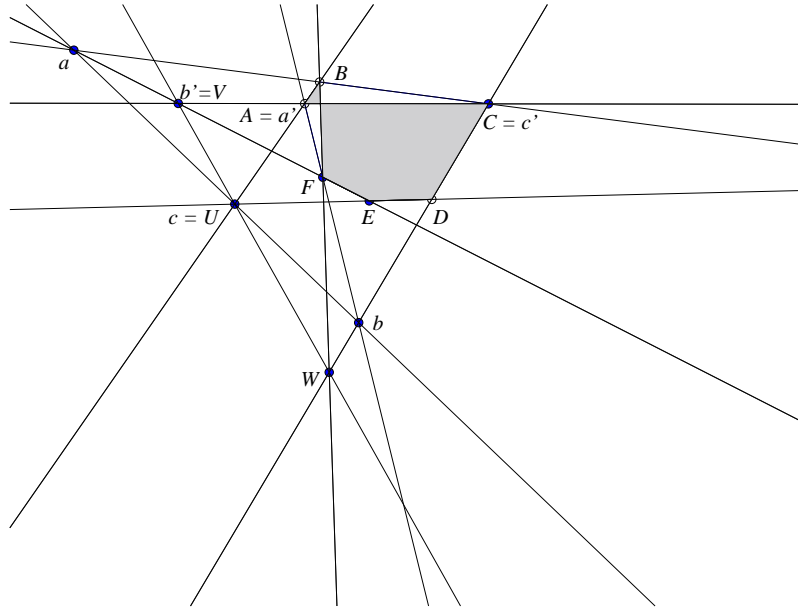
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Figure 6: Pappus property implies hexamy property

We can show that the hexamy property and Pappus property are equivalent.

Lemma 1. *Pappus property implies hexamy property.*

PROOF. Assuming Pappus property we need to show that if (A, B, C, D, E, F) is an hexamy so is (B, A, C, D, E, F). Let U,V and W be the intersections of AB and DE, AC and EF, and CD and BF respectively. We need to show that U, V and W are collinear. Let a, b and c be the intersections of BC and EF, CD and FA, and AB and DE respectively (we have $c = U$). As (A, B, C, D, E, F) is an hexamy, a, b and c are collinear. Let b' be the intersection of EF and AC (we have $b' = V$). We note $a' \equiv A$ and $c' \equiv C$. a', b' and c' are collinear because b' is on line AC. Using Pappus property we know that $ab' \cap a'b$, $ac' \cap a'c$ and $bc' \cap b'c$ are collinear. We have $ab' \cap a'b = F$ since $a'b = AF$ and $ab' = EF$. Similarly, we have $ac' \cap a'c = B$. Hence the third point $bc' \cap b'c$ is on BF. Hence, as $bc' = CD$, we have $bc' \cap b'c = CD \cap BF = W$. The last equality says that $W \in b'c$ but since $b' = V$ and $c = U$, the collinearity of U,V and W holds.

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Lemma 2. *Hexamy property implies Pappus property.*

PROOF. When (A,B,C,A',B',C') is a Pappus configuration, it is easy to show that it is also an hexamy because the intersections of $AB \cap A'B'$ and $BC \cap B'C'$ are equal since the points A,B and C and A', B' and C' are collinear. Using the hexamy property we can show that (A,B',C,A',B,C') is also an hexamy. Hence the Pappus property holds.

4.3.3. Hessenberg theorem

Lemma 3. *Hexamy property implies Desargues property*

PROOF. We distinguish two cases, either $A'B'C'$ is a Cevian triangle of ABC or not. Recall that $A'B'C'$ is a Cevian triangle of ABC if $A' \in BC$, $B' \in AC$, $C' \in AB$ and the lines AA', BB' and CC' are concurrent.
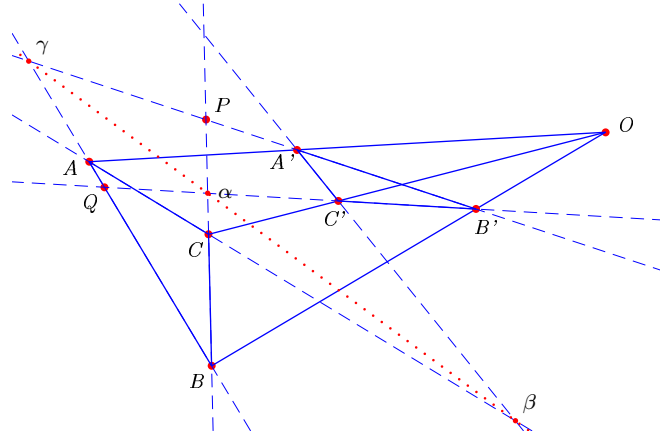
First, we assume that $A'B'C'$ is not a Cevian triangle of ABC. By hypothesis, points O, A and A' are collinear. Similarly, O B and B' are collinear and O C,C' are collinear. We want to prove that α, β and *gamma* are collinear, where $\alpha = BC \cap B'C'$, $\beta = AC \cap A'C'$ and $\gamma = AB \cap A'B'$. The auxiliary points: $P = A'B' \cap BC$, $Q = AB \cap B'C'$ are needed. Now (A, A', P, C, C', Q) is a hexamys since its three opposite sides cut respectively in points O, B and B' which are collinear by hypothesis. Thus (A, Q, C', A', P, C) is an hexamys as well, the opposite sides cut in α β and γ, hence they are collinear.
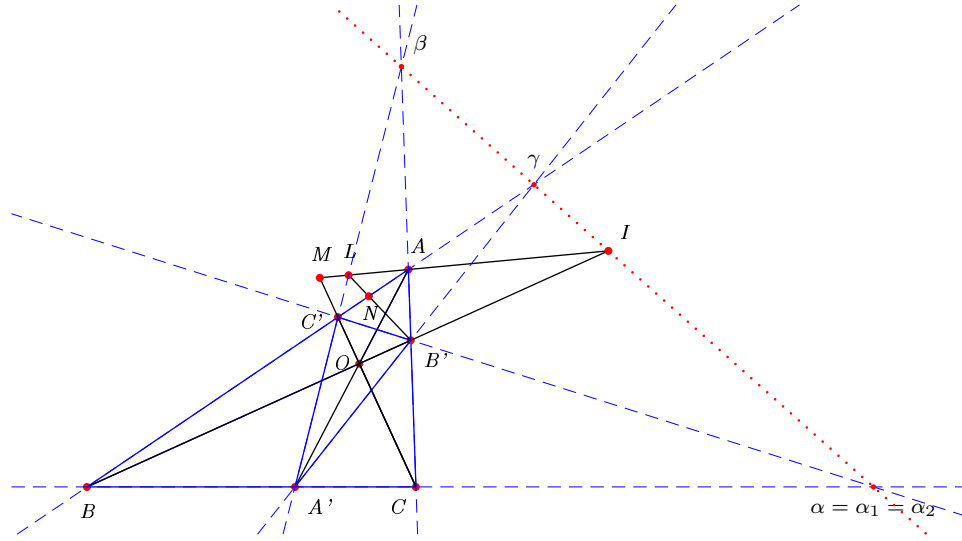
Second, we assume that $A'B'C'$ is a Cevian triangle of ABC. We have to prove that α, β and γ are collinear where $\alpha = BC \cap B'C'$, $\beta = AC \cap A'C'$ and $\gamma = AB \cap A'B'$. We introduce $\alpha_1 = \beta\gamma \cap BC$ and $\alpha_2 = \beta\gamma \cap B'C'$: we have now to prove that $\alpha_1 = \alpha_2$. Auxiliary intersection points are $I = OB \cap \beta\gamma$, $L = A'C' \cap AI$, $N = AB \cap LB'$ and $M = AL \cap CC'$ (it can be shown that these points are well defined). The end of the proof implicitely assumes that $L \neq A'$. If $L = A'$ we perform a proof similar to this one but using a permutation of the statement, if $L = B'$ we permute again the statement, it can not be the case that $L = A' = B' = C'$.

Since L, N and B' are collinear, $(A', L, B', I, A, \gamma)$ is an hexamys, so is $(A', A, \gamma, I, B', L)$, and hence O N and β are collinear. With the second hexamys, (C, B, A, O, I, β) which gives by permutation (C, O, β, I, A, B), we prove that α_1, M and N are collinear. A third hexamys is used to prove that α_2 M and N are collinear: this is (C', B', β, O, I, L) giving (C', O, β, I, L, B'). Thus the lines MN and $\beta\gamma$ have two points in common: α_1 and α_2. Either the lines are equal but then the configuration degenerates and all points belong to the same line; or $\alpha_1 = \alpha_2$ and hence α, β and γ are collinear.

Corollary 1 (Hessenberg). *Pappus property implies Desargues property*

PROOF. Immediate using lemmas 1 and 3.

The formal proofs corresponding to the theorems described in this section still heavily require user-interaction and lack automation. The amount of

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case distinctions required in formal proofs makes them difficult to handle. In the next section, we describe some tactics which we implemented to attempt to deal with non degeneracy conditions.

4.4. Dealing with Non-degeneracy Conditions: Using Tactics

In this section, we describe a tactic whose implementation is simple but which is still powerfull enough to shorten the proofs about degenerate cases. When we know that two points are equal, we can propagate this knowledge. The tactic performs repeated applications of the uniqueness axiom. It proceeds by searching for pattern matching the hypotheses of the two following lemmas which are easy consequences of the uniqueness axiom.

$$
\begin{aligned}
A \in l \wedge B \in l \\
A \in m \wedge B \in m
\end{aligned}
\wedge A \neq B \Rightarrow l = m
$$

$$
\begin{aligned}
A \in l \wedge B \in l \\
A \in m \wedge B \in m
\end{aligned}
\wedge l \neq m \Rightarrow A = B
$$

It is implemented using Coq tactic language as follows:

```
Ltac apply_unicity := match goal with
H1: ?A <> ?B,
H2: ?Incid ?A ?l, H3: ?Incid ?B ?l,
H4: ?Incid ?A ?m, H5: ?Incid ?B ?m |- _ =>
   let id:= fresh in assert (id: l=m);
   try apply (uniq.a1_unique A B l m H1 H2 H3 H4 H5);
   subst l
| H1: ?l <> ?m,
  H2: ?Incid ?A ?l, H3: ?Incid ?A ?m,
  H4: ?Incid ?B ?l, H5: ?Incid ?B ?m |- _ =>
   let id:= fresh in assert (id: A=B);
   try apply (uniq.a2_unique l m A B H1 H2 H3 H4 H5);
   subst A
end.


Ltac collapse := progress (repeat (apply_unicity;
                              CleanDuplicatedHyps)).
```

For example, if we know that A,B,C,A',B', and C' are all distinct and that they form a Pappus configuration and that the line $A'B$ is equal to line

AB' then our `collapse` tactic infers automatically that the lines AB, AB', $A'C$, AB', BC', AC', $B'C$ are all equal. This allows to conclude easily that Pappus theorem holds trivially in this special case (when line $A'B$ is equal to line AB').

5. Projective Space

In this section, we switch from projective plane geometry to projective space geometry. In 2D, a single fact can have numerous representations (e.g. $A \in BC$ vs. $A \in l \land B \in l \land C \in l$). In 3D and more, it is even worse because the langage contains points, lines and planes and all the associated incidence relations. In section 2.2, we presented the standard axiom system for projective space geometry as a reference. But to ease the formalization in Coq, we propose an alternative axiom system based on the notion of rank. Indeed, ranks allow to deal only with points which makes proofs easier in a three dimensionnal setting because we do not handle lines and planes explicitly. This provides an homogeneous description language independent of the dimension and will make proving Desargues property in a at-least 3 dimensional setting much easier.

5.1. Ranks

The concept of rank is a general notion of matroid theory. An integer function rk on E is the rank function of a matroid if and only if the following conditions are satisfied:

R1 $\forall X \subseteq E, 0 \leq rk(X) \leq |X|$ (nonnegative and subcardinal)

R2 $\forall XY \subseteq E, X \subseteq Y \Rightarrow rk(X) \leq rk(Y)$ (nondecreasing)

R3 $\forall XY \subseteq E, rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y)$ (submodular)

In projective geometry, we can define a rank function on sets of points which verify the axioms above: a flat being a set of points closed by the collinearity relation, the rank of a set of points A is the cardinal of a smallest set generating A (see figure 7 for examples).

Using this definition, one can show that every projective space has a matroid structure, but the converse is not true. In the next section, we introduce additional axioms to capture 3D or higher projective geometry but before we need to introduce some lemmas about ranks to ease the proofs.

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$$
\begin{aligned}
&rk\{A, B\} = 1 && A = B \\
&rk\{A, B\} = 2 && A \neq B \\
&rk\{A, B, C\} = 2 && A, B, C \text{ are collinear} \\
& && \text{with at least two of them distinct} \\
&rk\{A, B, C\} \leq 2 && A, B, C \text{ are collinear} \\
&rk\{A, B, C\} = 3 && A, B, C \text{ are not collinear} \\
&rk\{A, B, C, D\} = 3 && A, B, C, D \text{ are co-planar,} \\
& && \text{not all collinear} \\
&rk\{A, B, C, D\} = 4 && A, B, C, D \text{ are not co-planar} \\
&rk\{A, B, C, D, E\} \leq 2 && A, B, C, D, E \text{ are all collinear}
\end{aligned}
$$

Figure 7: Rank statements and their geometric interpretations

Proof techniques using ranks. In this section we describe two proof techniques which are simple but important to simplify formal proofs.

First, all equalities about ranks (say $rk(a) = rk(b)$) are usually proved in two steps: first showing that $rk(a) \leq rk(b)$ and then that $rk(a) \geq rk(b)$. Consequently, when stating a lemma, it is worth being cautious about whether the actual equality is required or if one of the two inequalities is enough to go on with the proofs. This approach allows to avoid numerous technical lemmas when carrying out the formal proofs in the Coq proof assistant.

Second, in the proving process, we make often use of axiom R3. For instance, if we need to prove a statement like:

$$
rk\{A, B, C, D, I\} + rk\{I\} \leq rk\{A, B, I\} + rk\{C, D, I\}
$$

we could be tempted to instantiate axiom R3 with $X := \{A, B, I\}$ and $Y := \{C, D, I\}$. But unfortunately, this statement is not a direct consequence of axiom R3. For instance A may be equal to C and consequently $\{A, B, I\} \cap \{C, D, I\} = \{A, I\}$. Determining the intersection of two finite sets of points requires to distinguish cases about the equality of these points. This leads to intricate proofs in Coq. Therefore, in the rest of this paper, we shall never consider the real set theoretical intersection but a lower approximation of the intersection (noted $\sqcap$).

Definition 1 (Literal intersection). Let L_1 and L_2 be two sets of points. By definition $L_1 \sqcap L_2$ is the intersection of the two sets of points considered syntactically.

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Using literal intersection we can derive a more convenient version of axiom R3 which leads to fewer case distinctions:

Lemma 4 (R3-lit).

$$\forall XY, rk(X \cup Y) + rk(X \sqcap Y) \leq rk(X) + rk(Y)$$

In Coq, it is not possible to define the litteral intersection. To capture the intent of this lemma we rather use:

Lemma 5 (R3-ALT).

$$\forall XYI, I \subseteq X \cap Y \Rightarrow rk(X \cup Y) + rk(I) \leq rk(X) + rk(Y)$$

This lemma will be used heavily in the next sections.

5.1.1. A rank-based axiom system

Contrary to the axiom system shown in section 2.2, we assume that we have only one kind of objects, namely points. To capture the whole projective space, we need to add some new axioms to the matroid's ones:

Rk-Singleton $\forall P : Point, rk\{P\} \geq 1$

Rk-Couple $\forall P\ Q : Point, P \neq Q \Rightarrow rk\{P, Q\} \geq 2$

Rk-Pasch $\forall A\ B\ C\ D, rk\{A, B, C, D\} \leq 3 \Rightarrow$
$$\exists J, rk\{A, B, J\} = rk\{C, D, J\} = 2$$

Rk-Three-Points $\forall A\ B, \exists C, rk\{A, B, C\} = rk\{B, C\} = rk\{A, C\} = 2$

Rk-Lower-Dimension $\exists A\ B\ C\ D, rk\{A, B, C, D\} \geq 4$

The first two ones ensure that the rank function is not degenerate. **Rk-Pasch** is the translation of Pasch's axiom: $rk\{A, B, C, D\} \leq 3$ means these points are coplanar, thus that the two lines AB and CD intersect.

Using this axiom system we formally proved all the axioms of section 2.2. In particular the following lemmas **Rk-Uniqueness** and **Rk-Lower-Dimension** are derivable and can be used to prove the **Uniqueness** and **Lower-Dimension-3** axiom respectively:

Lemma 6 (Rk-Uniqueness).

$$\forall ABCDMP, \begin{pmatrix} rk\{A,B\} = 2 \\ rk\{C,D\} = 2 \\ rk\{A,B,M\} \le 2 \\ rk\{C,D,M\} \le 2 \\ rk\{A,B,P\} \le 2 \\ rk\{C,D,P\} \le 2 \\ rk\{A,B,C,D\} \ge 3 \end{pmatrix} \Rightarrow rk\{M,P\} = 1$$

PROOF. Using R3-ALT we have:

$$rk\{A,B,M,P\} + rk\{A,B\} \le rk\{A,B,M\} + rk\{A,B,P\}$$

Hence $rk\{A,B,M,P\} = 2$, similarly we can show that $rk\{C,D,M,P\} = 2$. Moreover $rk\{A,B,C,D,M,P\} \ge 3$ as $rk\{A,B,C,D\} \ge 3$ and $\{A,B,C,D\} \subseteq \{A,B,C,D,M,P\}$. Finally, using R3-ALT we have:

$$
\begin{array}{ccccccc}
rk\{A,B,C,D,M,P\} & + & rk\{M,P\} & \le & rk\{A,B,M,P\} & + & rk\{C,D,M,P\} \\
3 & + & rk\{M,P\} & \le & 2 & + & 2
\end{array}
$$

Lemma 7 (Rk-Lower-Dimension).

$$\exists ABCD, \forall M, rk\{A,B,M\} \ne 2 \lor rk\{C,D,M\} \ne 2$$

PROOF. Using axiom **Rk-Lower-Dimension**, we obtain A,B,C and D such that $rk\{A,B,C,D\} = 4$.
Suppose that $rk\{A,B,M\} = 2$ and $rk\{C,D,M\} = 2$.
Using R3-ALT we have that:

$$rk\{A,B,C,D,M\} + rk\{M\} \le rk\{A,B,M\} + rk\{C,D,M\}$$

Hence $rk\{A,B,C,D,M\} \le 3$, which contradicts $rk\{A,B,C,D\} = 4$.

We can also derive a lemma which expresses concisely that for every point there exists one which is different, for every line there exists a point not on this line and for every plane there exists a point not on this plane.
We carried out this proof using an alternate axiom system for ranks. Whereas our development is based on matroid axioms R1, R2 and R3, one can prove that they are equivalent to the following set of axioms:

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R1' $rk(\emptyset) = 0$

R2' $rk(X) \leq rk(X \cup \{x\}) \leq rk(X) + 1$

R3' $rk(X \cup \{y\}) = rk(X \cup \{z\}) = rk(X) \Rightarrow rk(X) = rk(X \cup \{y, z\})$

Using this axiom system the proof is straightforward.

Lemma 8 (Construction).

$$\forall E, rk(E) \leq 3 \Rightarrow \exists P, rk(E \cup \{P\}) = rk(E) + 1$$

PROOF. Consider E such that $rk(E) \leq 3$.
Using axiom **Rk-Lower-Dimension** we obtain A, B, C and D such that $rk\{A, B, C, D\} = 4$. Using R2' we know that $rk(E) \leq rk(E \cup \{A\}) \leq rk(E) + 1$ and similarly for B, C and D. Suppose that $rk(E \cup \{A\}) = rk(E \cup \{B\}) = rk(E \cup \{C\}) = rk(E \cup \{D\}) = rk(E)$, then we would obtain $rk(E \cup \{A, B, C, D\}) = rk(E)$ by repeated applications of R3'. This is in contradiction with $rk\{A, B, C, D\} = 4$ since $rk(E) \leq 3$. Hence there exists a P such that $rk(E \cup \{P\}) = rk(E) + 1$.

Overall, this axiom system is convenient as first it only deals with points and hence the theory is dimension-independent, second ranks allow to summarize both positive and negative assumptions about sets of points homogeneously.

5.1.2. Implementation in Coq

The formalization in Coq of our axiom system is quite straightforward[3]. To increase reusability of the proofs, we define it as a module type of Coq's module system (see figure 8). This module depends on *DecPoints* which defines the type of points with a decidable equality.

On the technical side, defining our axiom system based on ranks requires a formal description of the concept of sets of points. As our development manipulates only *finite* sets, we use the development **FSets** of Filliâtre and Letouzey [11]. Since the provided set equality ($=_{\text{set}}$) differs from standard (Leibniz) Coq equality, we have to prove that rk is a morphism with respect to set equality:

---

[3]For lack of space, some technical details are omitted here.

```
Module Type RankProjectiveSpace (DecPoints:DecidableType).

Module Export FiniteSetsDefs := BuildFSets DecPoints.

Definition set_of_points := t.
Definition Point := DecPoints.t.
Parameter rk : set_of_points -> nat.
Axiom matroid1_a  : forall X, rk X >= 0.
Axiom matroid1_b : forall X, rk X <= cardinal X.
Axiom matroid2: forall X Y, Subset X Y -> rk X <= rk Y.
Axiom matroid3: forall X Y,
 rk(union X Y) + rk(inter X Y) <= rk X + rk Y.
Axiom rk_singleton_ge : forall P, rk (singleton P) >= 1.
Axiom rk_couple_ge : forall P Q,
               ~ DecPoints.eq P Q -> rk(couple P Q) >= 2.
Axiom pasch : forall A B C D, rk (quadruple A B C D) <= 3 ->
exists J, rk (triple A B J) = 2 /\ rk (triple C D J) = 2.
Axiom three_points : forall A B, exists C, rk (triple A B C) = 2 /\
                        rk (couple B C) = 2 /\ rk (couple A C) = 2.
Parameter P0 P1 P2 P3 : Point.
Axiom lower_dim : rk (quadruple P0 P1 P2 P3) >= 4.

End RankProjectiveSpace.
```

Figure 8: Definition of projective space geometry with ranks in Coq

Lemma 9. $\forall XY, X =_{set} Y \Rightarrow rk(X) = rk(Y)$

PROOF. By double inclusion using axiom R2.

Using this lemma, we can then define rk as a morphism in Coq with respect to $=_{set}$. This allows to easily replace a set by an equal one when it occurs as an argument of rk.

5.2. Desargues Theorem

In this section we describe the proof of Desargues theorem. The idea of the proof is classic: we first prove a version of the theorem where the two triangles are not coplanar, we call it Desargues 3D (see section 5.2.1) and

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then we deduce from it a version where A, B, C, A', B' and C' lie on a same plane (Desargues 2D) as shown on figure 9.

As we will see in the next section, using the concept of rank the proof of the 3D version is straightforward and special cases can be handled smoothly. In section 5.2.2, we will show how we actually build the 2D version and conclude the proof of the original theorem.

5.2.1. A 3D version of Desargues theorem

In this section, we prove Desargues 3D theorem.

Theorem 1 (Desargues 3D). *Let's consider two (non degenerate) triangles ABC and abc such that they are perspective from a given point O:*

$$rk\{A, B, C\} = rk\{a, b, c\} = 3$$
$$rk\{a, A, O\} = rk\{b, B, O\} = rk\{c, C, O\} = 2$$

We assume this forms a non planar figure:

$$rk\{A, B, C, a, b, c\} \geq 4$$

and define three points α, β, γ such that:

$$rk\{A, B, \gamma\} = rk\{a, b, \gamma\} = 2$$
$$rk\{A, C, \beta\} = rk\{a, c, \beta\} = 2$$
$$rk\{B, C, \alpha\} = rk\{b, c, \alpha\} = 2$$

Under these assumptions, $rk\{\alpha, \beta, \gamma\} \leq 2$ holds.

Lemma 10. $rk\{A, B, C, \alpha\} = 3$

PROOF. By assumption $rk\{A, B, C\} = 3$, hence using axiom R2, $rk\{A, B, C, \alpha\} \geq 3$. Moreover using R3-ALT we have:

$$
\begin{array}{ccccccccc}
rk\{A, B, C, \alpha\} & + & rk\{B, C\} & \leq & rk\{A, B, C\} & + & rk\{\alpha, B, C\} \\
rk\{A, B, C, \alpha\} & + & 2 & \leq & 3 & + & 2
\end{array}
$$

Hence, we can conclude that $rk\{A, B, C, \alpha\} = 3$. Similar proofs can be done with β and γ.

Lemma 11. $rk\{A, B, C, \alpha, \beta\} = 3$

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PROOF. First, using axiom R2 and lemma 10 we have $rk\{A, B, C, \alpha, \beta\} \geq 3$. Second, using R3-ALT we have:

$$\begin{array}{ccccccc}
rk\{A, B, C, \alpha, \beta\} & + & rk\{A, B, C\} & \leq & rk\{A, B, C, \alpha\} & + & rk\{A, B, C, \beta\} \\
rk\{A, B, C, \alpha, \beta\} & + & 3 & \leq & 3 & + & 3
\end{array}$$

Hence, we can conclude that $rk\{A, B, C, \alpha, \beta\} = 3$.

Lemma 12. $rk\{A, B, C, \alpha, \beta, \gamma\} = rk\{a, b, c, \alpha, \beta, \gamma\} = 3$

PROOF. The proof is similar to lemma 11.

Lemma 13. $rk\{A, B, C, a, b, c, \alpha, \beta, \gamma\} \geq 4$

PROOF. By assumption $rk\{A, B, C, a, b, c\} \geq 4$, hence using axiom R2, $rk\{A, B, C, a, b, c, \alpha, \beta, \gamma\} \geq 4$.

Using these lemmas we can conclude the proof:
From R3-ALT we know that:
$rk\{A, B, C, a, b, c, \alpha, \beta, \gamma\} + rk\{\alpha, \beta, \gamma\}$
$$\leq rk\{A, B, C, \alpha, \beta, \gamma\} + rk\{a, b, c, \alpha, \beta, \gamma\}$$
Hence, using lemmas 12 and 13 $rk\{\alpha, \beta, \gamma\} \leq 2$ holds.

5.2.2. Lifting from 2D to 3D
In this section, we prove the 2D version of Desargues using the 3D version.

Statement. Most assumptions are the same as in the 3D version. Let's consider two triangles ABC and $A'B'C'$ such that they are perspective from a given point O:

$$rk\{A, B, C\} = rk\{A', B', C'\} = 3$$
$$rk\{A', A, O\} = rk\{B', B, O\} = rk\{C', C, O\} = 2$$

We define three points α, β, γ such that:

$$rk\{A, B, \gamma\} = rk\{A', B', \gamma\} = 2$$
$$rk\{A, C, \beta\} = rk\{A', C', \beta\} = 2$$
$$rk\{B, C, \alpha\} = rk\{B', C', \alpha\} = 2$$

Contrary to the 3D case, we assume this forms a planar figure:

$$rk\{A, B, C, A', B', C', O\} = 3$$

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In addition to these assumptions which are closely related to those of Desargues 3D theorem, the following non-degeneracy conditions are required:

$$rk\{A, B, O\} = rk\{A, C, O\} = rk\{B, C, O\} = 3$$
$$rk\{A, A'\} = rk\{B, B'\} = rk\{C, C'\} = 2$$

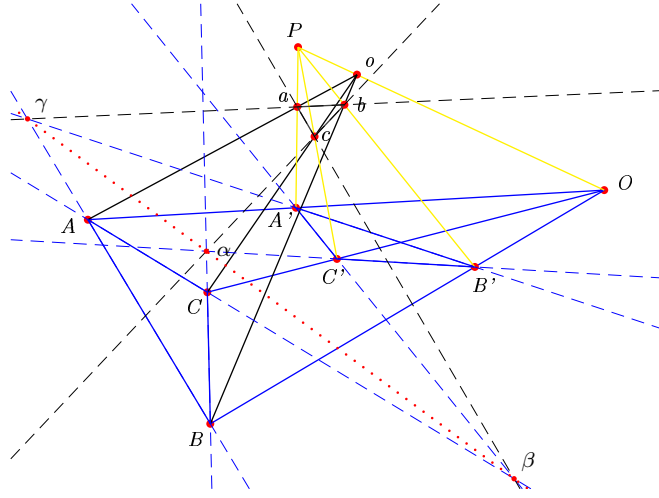Desargues theorem states that, under these assumptions, $rk\{\alpha, \beta, \gamma\} \leq 2$ holds.

Figure 9: Desargues theorem (3D extrusion) $A,B,C,A',B',C',O,\alpha,\beta$ and γ are co-planar. If P is the sun, triangle abc then casts its shadow in $A'B'C'$.

Informal Proof. We have to lift triangle $A'B'C'$ into a new triangle abc which is not coplanar with triangle ABC in order to have a configuration of points in which Desargues 3D theorem can be applied. The construction is shown in figure 9. Here are the main steps: we first construct a point P which lies outside the plane formed by A, B, C, A', B', C' and O. We know such a point P exists thanks to lemma 8. We then build a line incident to P and O (the point from which triangles ABC and $A'B'C'$ are perspective) and consider a third point o on this line (axiom **Rk-Three-Points** ensures such a point exists and is different from both O and P). We construct a new point a as the intersection of lines PA' and oA. We know these two lines intersect because of Pasch's axiom and the fact that lines AA' and Po intersect in O. We do the same to construct points b and c. Applying Desargues 3D

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theorem to ABC and abc requires to make sure we have a non-degenerate 3D figure and that abc is a non-degenerate triangle. We also have to make sure α defined as the intersection of lines BC and $B'C'$ is the same as the α of Desargues 3D theorem which is the intersection of BC and bc. This requirement can be satisfied by simply proving that α is incident to bc. The same requirement applies for β and γ. Overall, we have to prove the following statements which are requirements to apply Desargues 3D version (proofs are given below). Note that when applying the theorem, the point o plays the role of point O.

$$rk\{A, B, C\} = rk\{a, b, c\} = 3$$
$$rk\{A, B, C, a, b, c\} \geq 4$$
$$rk\{a, b, \gamma\} = rk\{a, c, \beta\} = rk\{b, c, \alpha\} = 2$$
$$rk\{A, B, \gamma\} = rk\{A, C, \beta\} = rk\{B, C, \alpha\} = 2$$

Statements $rk\{A, B, \gamma\} = rk\{A, C, \beta\} = rk\{B, C, \alpha\} = 2$ and $rk\{A, B, C\} = 3$ are assumptions of the Desargues 2D theorem, therefore their proofs are immediate.

Preliminary Lemmas. We remind the reader that A, B, C, A', B', C' and O lie in the same plane. P is a point outside this plane. o is a third point on the line OP. The point a is defined as the intersection of lines PA' and oA. Points b and c are defined in a similar way. In this setting, the following lemmas hold:

Lemma 14. $rk\{A', B', O\} = rk\{A', C', O\} = rk\{B', C', O\} = 3$

Lemma 15. $rk\{A', B', O, P\} = rk\{A', B', O, o\} = 4$

Lemma 16. $rk\{A, B, O, a\} \geq 4 \quad rk\{A, A', C, a\} \geq 4$

Lemma 17. $rk\{A, B, O, b\} \geq 4 \quad rk\{A, B, O, c\} \geq 4$

Lemma 18. $rk\{o, a\} = rk\{o, b\} = rk\{o, c\} = 2$

Lemma 19. $rk\{a, c, A, C, \beta\} = rk\{a, c, A', C', \beta\} = 3$

General Lemmas. Most proofs are fairly technical, simply using the matroid axioms of rank. However, some lemmas can be highlighted, especially for their genericity and their pervasive use throughout the proofs. Among them, some stability lemmas state that one of the points of a set characterizing a flat (e.g a plane or the whole space) can be replaced by another one belonging to this flat.

Lemma 20 (Plane representation change).

$$\left(\begin{array}{c} rk\{A,B,C\}=3 \\ rk\{A,B,C,M\}=3 \\ rk\{B,C,M\}=3 \\ rk\{A,B,C,P\}=4 \end{array} \right) \Rightarrow rk\{M,B,C,P\}=4$$

This lemma is heavily used to prove all possible statements expressing that P lies outside the plane formed by A,B,C,A',B,C',O.

Other lemmas about coplanarity and also upper bound on ranks when merging a plane and a line are convenient as well. They could form the basis of an automation procedure when doing computer-checked formal proofs.

Proving Desargues 3D assumptions.

Lemma 21. $rk\{A,B,C,a,b,c\} \geq 4$

PROOF. By lemma 17, we have $rk\{A,B,O,b\} >= 4$, hence $rk\{A,B,C,O,b\} \geq 4$. Using axiom R3-ALT, we have:

$$rk\{A,B,C,b\} \quad + \quad rk\{A,B,C,O\} \quad \geq \quad rk\{A,B,C,O,b\} \quad + \quad rk\{A,B,C\}$$
$$rk\{A,B,C,b\} \quad + \quad 3 \quad \geq \quad 4 \quad + \quad 3$$

Consequently we have $rk\{A,B,C,b\} \geq 4$ and applying axiom R2 twice, it leads to $rk\{A,B,C,a,b,c\} \geq 4$.

Lemma 22. $rk\{a,b,c\} = 3$

PROOF. By axiom R1 we have $rk\{a,b,c\} \leq 3$.
Let's prove $rk\{a,b,c\} \geq 3$. By axiom R3-ALT, we have:

$$rk\{a,b,c,o,A,B\} \quad + \quad rk\{o,C,c\} \quad \geq \quad rk\{A,B,C,o,a,b,c\} \quad + \quad rk\{o,c\}$$
$$rk\{a,b,c,o,A,B\} \quad + \quad 2 \quad \geq \quad 4 \quad + \quad 2$$

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Hence $rk\{a, b, c, o, A, B\} \geq 4$. Again, using axiom R3-ALT we have:

$$\begin{array}{ccccccc}
rk\{a, b, c, o, A\} & + & rk\{o, B, b\} & \geq & rk\{a, b, c, o, A, B\} & + & rk\{o, b\} \\
rk\{a, b, c, o, A\} & + & 2 & \geq & 4 & + & 2
\end{array}$$

Hence $rk\{a, b, c, o, A\} \geq 4$. Applying axiom R3-ALT one last time yields:

$$\begin{array}{ccccccc}
rk\{a, b, c\} & + & rk\{o, A, a\} & \geq & rk\{a, b, c, o, A\} & + & rk\{a\} \\
rk\{a, b, c\} & + & 2 & \geq & 4 & + & 1
\end{array}$$

Hence $rk\{a, b, c\} \geq 3$. Note that this proof relies on the facts that $rk\{o, b\} = 2$ and $rk\{o, c\} = 2$ which are proved as lemma 18.

Lemma 23. $rk\{a, b, \gamma\} = rk\{a, c, \beta\} = rk\{b, c, \alpha\} = 2$

PROOF. Using axiom R3-ALT, we have:
$rk\{a, c, A, C, \beta\} + rk\{a, c, A', C', \beta\} \geq rk\{a, c, A, C, A', C', \beta\} + rk\{a, c, \beta\}$
We have $rk\{a, c, A, C, A', C', \beta\} \geq 4$ using axiom R2 and lemma 16. Using lemma 19, we obtain $rk\{a, c, \beta\} \leq 2$. As $rk\{a, c\} = 2$ (because $rk\{a, b, c\} = 3$), we conclude $rk\{a, c, \beta\} = 2$. Proofs for α and γ are the same.

5.3. Formalization in Coq

Formalizing Desargues theorem in Coq is straightforward once we have the axiom system dealing with ranks and proof techniques to handle them nicely. All the above-mentionned lemmas are easily proved and then Desargues theorem can be stated as follows:

```
forall A' B' C' A B C O : Point,
rk(triple A B C)=3 -> rk(triple A' B' C')=3 ->
rk(triple A B O)=3 -> rk(triple A  C  O )=3 ->
rk(triple B C O)=3 ->
rk(triple A A' O)=2 -> rk(triple B B' O)=2 -> rk(triple C C' O)=2 ->
rk(couple A A')=2 -> rk(couple B B')=2 -> rk(couple C C')=2 ->
rk(triple A  B gamma)=2 -> rk(triple A' B' gamma)=2 ->
rk(triple A C beta)=2 ->
rk(triple A' C' beta)  =2 -> rk(triple B C alpha)  =2 ->
rk(triple B' C' alpha) =2 ->
rk(triple alpha beta gamma) <= 2.
```

Its proofs in Coq proceeds exactly the same way as the proof presented in the previous section. It is simply a bit more technical and requires a lot of computation to decide equality between sets which are equal but not syntactically equal.

6. Conclusions

We described axiom systems for both projective plane geometry and projective space geometry in Coq. We then formally proved that Desargues property can not be proved in the projective plane but it holds in Fano's plane and pappusian planes. Finally we proved Desargues theorem in the at least 3 dimensional projective space.

Proofs in the plane were performed in a traditional setting using points and lines. In the projective space, we proposed a new way to express nicely incidence relations thanks to ranks. We designed an axiom system to capture projective geometry using ranks. We then presented some proof engineering techniques which allow to have proofs of reasonable size.

Overall, the proofs consist in more than 10000 lines with about 280 lemmas and their formal proofs organized as shown in the figures below.

	2D	3D	Total
lines of Coq specs	1800	1800	3600
lines of Coq proofs	4600	5800	10400

Future work includes further formalization of Hexamys in Coq. We expect to formalize all the properties enumerated and proved by Pouzergues in [24]. Regarding at least 3-dimensional space, we plan to study how ranks can be used to automatically derive incidence properties. We believe that the genericity of the notation will help the automation process. Geometric algebra can also be an alternate mean to handle projective geometry nicely [10].

Availability The full Coq development is available in the user contributions of Coq.

References

[1] Y. Bertot and P. Castéran. *Interactive Theorem Proving and Program Development, Coq'Art: The Calculus of Inductive Constructions.* Springer, 2004.

[2] M. Bezem and D. Hendriks. On the Mechanization of the Proof of Hessenberg's Theorem in Coherent Logic. *J. of Automated Reasoning,* 40(1):61–85, 2008.

[3] J. Brandt and K. Schneider. Using three-valued logic to specify and verify algorithms of computational geometry. In *ICFEM*, volume 3785 of *LNCS*, pages 405–420. Springer-Verlag, 2005.

[4] F. Buekenhout, editor. *Handbook of Incidence Geometry*. North Holland, 1995.

[5] C. Cerroni. Non-Desarguian Geometries and the foundations of Geometry from David Hilbert to Ruth Moufang. *Historia Mathematica*, 31(3):320–336, 2004.

[6] S.-C. Chou, X.-S. Gao, and J.-Z. Zhang. *Machine Proofs in Geometry*. World Scientific, 1994.

[7] Coq development team. *The Coq Proof Assistant Reference Manual, Version 8.2*. LogiCal Project, 2008.

[8] H. S. M. Coxeter. *Projective Geometry*. Springer, 1987.

[9] C. Dehlinger, J.-F. Dufourd, and P. Schreck. Higher-Order Intuitionistic Formalization and Proofs in Hilbert's Elementary Geometry. In *ADG'00*, volume 2061 of *LNAI*, pages 306–324. Springer-Verlag, 2000.

[10] L. Dorst, D. Fontijne, and S. Mann. *Geometric Algebra for Computer Science*. Elsevier, 2007.

[11] J.-C. Filliâtre and P. Letouzey. Functors for Proofs and Programs. In *ESOP'2004*, volume 2986 of *LNCS*, pages 370–384. Springer-Verlag, 2004.

[12] F. Guilhot. Formalisation en Coq et visualisation d'un cours de géométrie pour le lycée. *TSI*, 24:1113–1138, 2005. In french.

[13] C. M. Hoffmann and R. Joan-Arinyo. *Handbook of Computer Aided Geometric Design*, chapter Parametric Modeling, pages 519–541. Elsevier, 2002.

[14] P. Janičić, J. Narboux, and P. Quaresma. The area method : a recapitulation. submitted, october 2009.

[15] C. Jermann, G. Trombettoni, B. Neveu, and P. Mathis. Decomposition of geometric constraint systems: a survey. *International J. of Computational Geometry and Application*, 16(5-6):379–414, 2006.

[16] E. Kusak. Desargues theorem in projective 3-space. *J. of Formalized Mathematics*, 2, 1990.

[17] N. Magaud, J. Narboux, and P. Schreck. Formalizing Projective Plane Geometry in Coq. proceedings of ADG'08, September 2008.

[18] L. Meikle and J. Fleuriot. Formalizing Hilbert's Grundlagen in Isabelle/Isar. In *TPHOLs'03*, volume 2758 of *LNCS*, pages 319–334, 2003.

[19] D. Michelucci, S. Foufou, L. Lamarque, and P. Schreck. Geometric constraints solving: some tracks. In *SPM '06*, pages 185–196. ACM Press, 2006.

[20] D. Michelucci and P. Schreck. Incidence Constraints: a Combinatorial Approach. *International J. of Computational Geometry and Application*, 16(5-6):443–460, 2006.

[21] F. R. Moulton. A Simple Non-Desarguesian Plane Geometry. *Transactions of the American Mathematical Society*, (3):192–195, 1902.

[22] J. Narboux. A decision procedure for geometry in Coq. In *TPHOLs'04*, volume 3223 of *LNCS*, pages 225–240. Springer-Verlag, 2004.

[23] J. Narboux. Mechanical theorem proving in Tarski's geometry. In *ADG'06*, volume 4869 of *LNAI*, pages 139–156. Springer-Verlag, 2007.

[24] R. Pouzergues. Les hexamys. Technical report, IREM Nice, 1993. http://hexamys.free.fr.

[25] P. Schreck. Robustness in CAD Geometric Construction. In *IV'01*, pages 111–116, London, 2001.