Balkan MO 1997

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1 Suppose that O is a point inside a convex quadrilateral $ABCD$ such that

$$OA^2 + OB^2 + OC^2 + OD^2 = 2\mathcal{A}[ABCD],$$

where by $\mathcal{A}[ABCD]$ we have denoted the area of $ABCD$. Prove that $ABCD$ is a square and O is its center.

Yugoslavia

2 Let $S = \{A_1, A_2, \ldots, A_k\}$ be a collection of subsets of an n-element set A. If for any two elements $x, y \in A$ there is a subset $A_i \in S$ containing exactly one of the two elements x, y, prove that $2^k \geq n$.

Yugoslavia

3 The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ touch each other externally at D, and touch a circle ω internally at B and C, respectively. Let A be an intersection point of ω and the common tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ at D. Lines AB and AC meet $\mathcal{C}_1$ and $\mathcal{C}_2$ again at K and L, respectively, and the line BC meets $\mathcal{C}_1$ again at M and $\mathcal{C}_2$ again at N. Prove that the lines AD, KM, LN are concurrent.

Greece

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) = f^2(x) + y$$

for all $x, y \in \mathbb{R}$.