Balkan MO 1990

1 The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 3$, and $a_{n+2} = (n+3)a_{n+1} - (n+2)a_n, \forall n \in \mathbb{N}$. Find all values of n for which a_n is divisible by 11.

2 The polynomial $P(X)$ is defined by $P(X) = (X + 2X^2 + \ldots + nX^n)^2 = a_0 + a_1X + \ldots + a_{2n}X^{2n}$. Prove that $a_{n+1} + a_{n+2} + \ldots + a_{2n} = \frac{n(n+1)(5n^2+5n+2)}{24}$.

3 Let ABC be an acute triangle and let A_1, B_1, C_1 be the feet of its altitudes. The incircle of the triangle $A_1B_1C_1$ touches its sides at the points A_2, B_2, C_2. Prove that the Euler lines of triangles ABC and $A_2B_2C_2$ coincide.

4 Find the least number of elements of a finite set A such that there exists a function $f : \{1, 2, 3, \ldots\} \to A$ with the property: if i and j are positive integers and $i - j$ is a prime number, then $f(i)$ and $f(j)$ are distinct elements of A.