Balkan MO 1999

– May 8th

1 Let O be the circumcenter of the triangle ABC. The segment XY is the diameter of the circumcircle perpendicular to BC and it meets BC at M. The point X is closer to M than Y and Z is the point on MY such that $MZ = MX$. The point W is the midpoint of AZ.

a) Show that W lies on the circle through the midpoints of the sides of ABC;

b) Show that MW is perpendicular to AY.

2 Let p be an odd prime congruent to 2 modulo 3. Prove that at most $p-1$ members of the set $\{m^2 - n^3 - 1 \mid 0 < m,\ n < p\}$ are divisible by p.

3 Let ABC be an acute-angled triangle of area 1. Show that the triangle whose vertices are the feet of the perpendiculars from the centroid G to AB, BC, CA has area between $\frac{4}{27}$ and $\frac{1}{4}$.

4 Let $\{a_n\}_{n\geq 0}$ be a non-decreasing, unbounded sequence of non-negative integers with $a_0 = 0$. Let the number of members of the sequence not exceeding n be b_n. Prove that

$$(a_0 + a_1 + \cdots + a_m)(b_0 + b_1 + \cdots + b_n) \geq (m+1)(n+1).$$