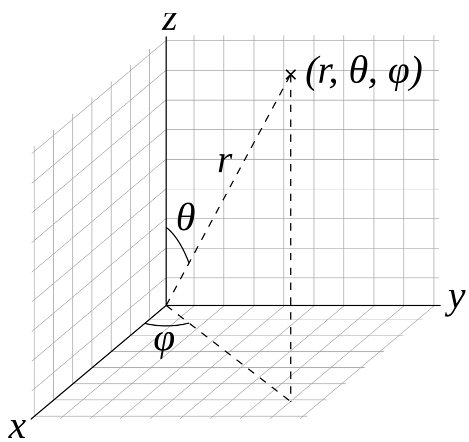
1 Coordinate Conversions

1.1 Spherical Coordinate system

In the polar coordinate system we represent the location of the target in terms of range, bearing and elevation. where range denotes the distance away from the target's position, bearing denotes the angle to the target (taken from the north axis), and elevation denotes the angle between the tanget plan and the target.



1.2 Cooridnate system representation

We denote by $\hat{e_c} = (e_e, e_n, e_n)$ the column vector that represents the the coordinate system in ENU terms. each e_e, e_n, e_n is a unit vector which represents the distance in east direction, north direction and up direction, respectively.

We denote by $\hat{e_p} = (e_r, e_\theta, e_\phi)$ the column vector that represents the the coordinate system in spherical terms. each e_r, e_θ, e_ϕ is a unit vector which represents the distance in east direction, north direction and up direction, respectively.

1.3 Conversion from one system to the other

Let $\vec{p} = (x, y, z) \cdot \hat{e_c} = xe_e + ye_n + ze_n$ denote the position of the target. We would like to convert the vector $\hat{e_c}$ to spherical coordinates $\hat{e_p}$. Clearly, for the position we have:

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \cos\theta \cos\phi \\
z &= r \sin\phi
\end{aligned} \tag{1}
$$

That is, $\vec{p} = r \sin\theta \cos\phi e_e + r \cos\theta \cos\phi e_n + r \sin\phi e_n = r(\sin\theta \cos\phi e_e + \cos\theta \cos\phi e_n + \sin\phi e_n)$

$$\vec{p} = r(\sin\theta\cos\phi e_e + \cos\theta\cos\phi e_n + \sin\phi e_n) \tag{2}$$

And we obtain:

$$e_r = \frac{\frac{\partial\vec{p}}{\partial r}}{\|\frac{\partial\vec{p}}{\partial r}\|} = \sin\theta\cos\phi e_e + \cos\theta\cos\phi e_n + \sin\phi e_n \tag{3}$$

$$\begin{aligned} e_\theta = \frac{\frac{\partial\vec{p}}{\partial\theta}}{\|\frac{\partial\vec{p}}{\partial\theta}\|} = \frac{r(\cos\theta\cos\phi e_e - \sin\theta\cos\phi e_n)}{r\|\cos\theta\cos\phi e_e - \sin\theta\cos\phi e_n\|} = \\ = \frac{r(\cos\theta\cos\phi e_e - \sin\theta\cos\phi e_n)}{r|\cos\phi|} = \cos\theta e_e - \sin\theta e_n \end{aligned} \tag{4}$$

$$\begin{aligned} e_\phi = \frac{\frac{\partial\vec{p}}{\partial\phi}}{\|\frac{\partial\vec{p}}{\partial\phi}\|} = \frac{r(-\sin\theta\sin\phi e_e - \cos\theta\sin\phi e_n + \cos\phi e_n)}{r\|-\sin\theta\sin\phi e_e - \cos\theta\sin\phi e_n + \cos\phi e_n\|} = \\ = \frac{r(-\sin\theta\sin\phi e_e - \cos\theta\sin\phi e_n + \cos\phi e_n)}{r} = \\ = -\sin\theta\sin\phi e_e - \cos\theta\sin\phi e_n + \cos\phi e_n \end{aligned} \tag{5}$$

Namely,

$$\begin{pmatrix} e_r \\ e_\theta \\ e_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & \sin\phi \\ \cos\theta & -\sin\theta & 0 \\ -\sin\theta\sin\phi & -\cos\theta\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e_e \\ e_n \\ e_n \end{pmatrix} \tag{6}$$

In Equation 4 we canceled the term $\cos\phi$ since it is always positive $\|\cos\phi\| = \cos\phi$.

Denote by Σ the conversion matrix as described above, note that Σ is unitary matrix.

$$\Sigma^{-1} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta & -\sin\theta\sin\phi \\ \cos\theta\cos\phi & -\sin\theta & -\cos\theta\sin\phi \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \tag{7}$$

1.4 Velocity conversion

Let $\vec{v}$ denote the velocity vector. We denote by $\dot{r}, \dot{\theta}, \dot{\phi}$ the terms: range-rate, bearing-rate, elevation-rate. In spherical coordinate system, the following equation holds:

$$\vec{v} = \frac{d}{dt}(r(t)e_r) \tag{8}$$

By the chain-rule: $\frac{d}{dt}(r(t)e_r) = \dot{r}e_r + r(t)\frac{d}{dt}e_r$.

For the left term we have:

$$\frac{d}{dt}e_r = \frac{d}{dt}(\sin\theta\cos\phi e_e + \cos\theta\cos\phi e_n + \sin\phi e_n) =$$
$$\dot\theta\cos\theta\cos\phi e_e - \dot\phi\sin\theta\sin\phi e_e - \dot\theta\sin\theta\cos\phi e_n - \dot\phi\cos\theta\sin\phi e_n + \dot\phi\cos\phi e_n =$$
$$= \dot\theta\cos\phi e_\theta + \dot\phi e_\phi$$

Which concludes to:

$$\vec{v} = \dot{r}e_r + r\dot\theta\cos\phi e_\theta + r\dot\phi e_\phi \tag{9}$$

1.5 Conversion from spherical to cartesian

Using Equations 6 and 9, we deduce:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \dot{r} & \dot\theta\cos\phi & \dot\phi \end{pmatrix} \qquad \begin{pmatrix} e_r \\ e_\theta \\ e_\phi \end{pmatrix} = \begin{pmatrix} \dot{r} & \dot\theta\cos\phi & \dot\phi \end{pmatrix} (\Sigma\hat{e_c}) \tag{10}$$