Cooridinate System Conversions

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1 Geographic coordinate system & ECEF

1.1 LLH to ECEF

Geodetic coordinates (latitude ϕ, longitude λ, height h) can be converted into ECEF coordinates using the following equation:

$$X = (R(\phi) + h)\cos\phi\cos\lambda$$
$$Y = (R(\phi) + h)\cos\phi\sin\lambda$$
$$Z = (\frac{b^2}{a^2}R(\phi) + h)\sin\phi$$

where:

$$R(\phi) = \frac{a^2}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \tag{1}$$

In the above equation, a and b are the equatorial radius (semi-major axis) and the polar radius (semi-minor axis), respectively.

1.2 ECEF to LLH

A number of techniques and algorithms are available but the most accurate, according to Zhu [3], is the following procedure established by Heikkinen [1], as cited by Zhu.

See also [2].

2 Coordinate Conversions From Spherical to Cartesian

2.1 Notations

2.1.1 Spherical Coordinate system

In the polar coordinate system we represent the location of the target in terms of range, bearing and elevation. where range denotes the distance away from

$$
\begin{aligned}
r &= \sqrt{X^2 + Y^2} \\
e'^2 &= \frac{a^2 - b^2}{b^2} \\
F &= 54b^2 Z^2 \\
G &= r^2 + \left(1 - e^2\right) Z^2 - e^2 \left(a^2 - b^2\right) \\
c &= \frac{e^4 F r^2}{G^3} \\
s &= \sqrt[3]{1 + c + \sqrt{c^2 + 2c}} \\
P &= \frac{F}{3\left(s + \frac{1}{s} + 1\right)^2 G^2} \\
Q &= \sqrt{1 + 2e^4 P} \\
r_0 &= \frac{-Pe^2 r}{1 + Q} + \sqrt{\frac{1}{2}a^2 \left(1 + \frac{1}{Q}\right) - \frac{P\left(1 - e^2\right) Z^2}{Q(1 + Q)} - \frac{1}{2}Pr^2} \\
U &= \sqrt{\left(r - e^2 r_0\right)^2 + Z^2} \\
V &= \sqrt{\left(r - e^2 r_0\right)^2 + \left(1 - e^2\right) Z^2} \\
z_0 &= \frac{b^2 Z}{aV} \\
h &= U\left(1 - \frac{b^2}{aV}\right) \\
\phi &= \arctan\left[\frac{Z + e'^2 z_0}{r}\right] \\
\lambda &= \operatorname{arctan2}[Y, X]
\end{aligned}
$$

Note: arctan2[Y, X] is the four-quadrant inverse tangent function.

the target's position, bearing denotes the angle to the target (taken from the north axis), and elevation denotes the angle between the tanget plan and the target.

2.1.2 Cooridnate system representation

We denote by $\hat{e_c} = (e_e, e_n, e_u)$ the column vector that represents the the coordinate system in ENU terms. each e_e, e_n, e_u is a unit vector which represents the distance in east direction, north direction and up direction, respectively.

We denote by $\hat{e_p} = (e_r, e_\theta, e_\phi)$ the column vector that represents the the coordinate system in spherical terms. each e_r, e_θ, e_ϕ is a unit vector which represents the distance in east direction, north direction and up direction, respectively.

2.2 Conversion from one system to the other

2.2.1 Position conversions

Let $\vec{p} = (x, y, z) \cdot \hat{e_c} = xe_e + ye_n + ze_u$ denote the position of the target.

Conversion of position from spherical to cartesian:

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \cos\theta \cos\phi \\
z &= r \sin\phi
\end{aligned} \tag{2}
$$

Conversion of position from cartesian to spherical:

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\frac{x}{y} \\ \phi &= \arctan\frac{z}{\sqrt{x^2+y^2}} \end{aligned} \tag{3}$$

2.2.2 Unit conversions

We would like to convert the vector $\hat{e_c}$ to spherical coordinates $\hat{e_p}$.

That is, $\vec{p} = r\sin\theta\cos\phi e_e + r\cos\theta\cos\phi e_n + r\sin\phi e_u = r(\sin\theta\cos\phi e_e + \cos\theta\cos\phi e_n + \sin\phi e_u)$

$$\vec{p} = r(\sin\theta\cos\phi e_e + \cos\theta\cos\phi e_n + \sin\phi e_u) \tag{4}$$

And we obtain:

$$e_r = \frac{\frac{\partial\vec{p}}{\partial r}}{\|\frac{\partial\vec{p}}{\partial r}\|} = \sin\theta\cos\phi e_e + \cos\theta\cos\phi e_n + \sin\phi e_u \tag{5}$$

$$\begin{aligned} e_\theta = \frac{\frac{\partial\vec{p}}{\partial\theta}}{\|\frac{\partial\vec{p}}{\partial\theta}\|} &= \frac{r(\cos\theta\cos\phi e_e - \sin\theta\cos\phi e_n)}{r\|\cos\theta\cos\phi e_e - \sin\theta\cos\phi e_n\|} = \\ = \frac{r(\cos\theta\cos\phi e_e - \sin\theta\cos\phi e_n)}{r|\cos\phi|} &= \cos\theta e_e - \sin\theta e_n \end{aligned} \tag{6}$$

$$\begin{aligned} e_\phi = \frac{\frac{\partial\vec{p}}{\partial\phi}}{\|\frac{\partial\vec{p}}{\partial\phi}\|} &= \frac{r(-\sin\theta\sin\phi e_e - \cos\theta\sin\phi e_n + \cos\phi e_u)}{r\|-\sin\theta\sin\phi e_e - \cos\theta\sin\phi e_n + \cos\phi e_u\|} = \\ &= \frac{r(-\sin\theta\sin\phi e_e - \cos\theta\sin\phi e_n + \cos\phi e_u)}{r} = \\ &= -\sin\theta\sin\phi e_e - \cos\theta\sin\phi e_n + \cos\phi e_u \end{aligned} \tag{7}$$

Namely,

$$\begin{pmatrix} e_r \\ e_\theta \\ e_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & \sin\phi \\ \cos\theta & -\sin\theta & 0 \\ -\sin\theta\sin\phi & -\cos\theta\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e_e \\ e_n \\ e_u \end{pmatrix} \tag{8}$$

In Equation 6 we canceled the term $\cos\phi$ since it is always positive $\|\cos\phi\| = \cos\phi$.

Denote by Σ the conversion matrix as described above, note that Σ is unitary matrix.

$$\Sigma^{-1} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta & -\sin\theta\sin\phi \\ \cos\theta\cos\phi & -\sin\theta & -\cos\theta\sin\phi \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \tag{9}$$

2.2.3 Velocity conversions

Let $\vec{v}$ denote the velocity vector. We denote by $\dot{r}, \dot{\theta}, \dot{\phi}$ the terms: range-rate, bearing-rate, elevation-rate. In spherical coordinate system, the following equation holds:

$$\vec{v} = \frac{d}{dt}(r(t)e_r) \tag{10}$$

By the chain-rule: $\frac{d}{dt}(r(t)e_r) = \dot{r}e_r + r(t)\frac{d}{dt}e_r$.
For the left term we have:

$$\frac{d}{dt}e_r = \frac{d}{dt}(\sin\theta\cos\phi e_e + \cos\theta\cos\phi e_n + \sin\phi e_u) =$$
$$\dot{\theta}\cos\theta\cos\phi e_e - \dot{\phi}\sin\theta\sin\phi e_e - \dot{\theta}\sin\theta\cos\phi e_n - \dot{\phi}\cos\theta\sin\phi e_n + \dot{\phi}\cos\phi e_u =$$
$$= \dot{\theta}\cos\phi e_\theta + \dot{\phi}e_\phi$$

Which concludes to:

$$\vec{v} = \dot{r}e_r + r\dot{\theta}\cos\phi e_\theta + r\dot{\phi}e_\phi \tag{11}$$

Conversion of velocity from spherical to cartesian: Using Equations 3 and standard derivation we obtain:

$$\begin{aligned}
\dot{r} &= \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2+y^2+z^2}} \\
\dot{\theta} &= \frac{\dot{x}y - x\dot{y}}{x^2+y^2} \\
\dot{\phi} &= \frac{\dot{z}(x^2+y^2) - z(x\dot{x}+y\dot{y})}{\sqrt{x^2+y^2}(x^2+y^2+z^2)}
\end{aligned} \tag{12}$$

Conversion of velocity from cartesian to spherical: Using Equations 8 and 11, we deduce:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \dot{r} & r\dot{\theta}\cos\phi & r\dot{\phi} \end{pmatrix} \quad \begin{pmatrix} e_r \\ e_\theta \\ e_\phi \end{pmatrix} = \begin{pmatrix} \dot{r} & r\dot{\theta}\cos\phi & r\dot{\phi} \end{pmatrix} (\Sigma\hat{e_c}) \tag{13}$$

Hence,

$$\begin{aligned}
v_x &= \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \\
v_y &= \dot{r}\cos\theta\cos\phi - r\dot{\theta}\sin\theta\cos\phi - r\dot{\phi}\cos\theta\sin\phi \\
v_z &= \dot{r}\sin\phi + r\dot{\phi}\cos\phi
\end{aligned} \tag{14}$$

2.2.4 Covariance Matrix Conversions

The transformation of coordinates for the covariance matrices P and Q is determined by the Jacobian matrix J as described in the formula: (cf. Appendix) $P = JQJ^T$.

Conversion of covariance from spherical to cartesian: Given a diagonal matrix (θ, r, ϕ), the jacobian of (x, y, z) with respect to (θ, r, ϕ) is:

$$\begin{pmatrix} r\cos\theta\cos\phi & \sin\theta\cos\phi & -r\sin\theta\sin\phi \\ -r\sin\theta\cos\phi & \cos\theta\cos\phi & -r\cos\theta\sin\phi \\ 0 & \sin\phi & r\cos\phi \end{pmatrix} \tag{15}$$

Given a diagonal matrix $(\theta, \dot\theta, r, \dot r, \phi, \dot\phi)$ the jacobian of $(x, \dot x, y, \dot y, z, \dot z)$ with respect to $(\theta, \dot\theta, r, \dot r, \phi, \dot\phi)$ is:

Non-zero elements of matrix **Q** are calculated as follows:

$$Q_{11} = \frac{\partial x}{\partial \alpha} = r\cos\alpha\cos\theta$$

$$Q_{13} = \frac{\partial x}{\partial r} = \sin\alpha\cos\theta$$

$$Q_{15} = \frac{\partial x}{\partial \theta} = -r\sin\alpha\sin\theta$$

$$Q_{21} = \frac{\partial \dot x}{\partial \alpha} = \dot r\cos\alpha\cos\theta - \dot\alpha r\sin\alpha\cos\theta - \dot\theta r\cos\alpha\sin\theta$$

$$Q_{22} = \frac{\partial \dot x}{\partial \dot\alpha} = r\cos\alpha\cos\theta$$

$$Q_{23} = \frac{\partial \dot x}{\partial r} = -\dot\alpha\sin\alpha\cos\theta - \dot\theta\sin\alpha\sin\theta$$

$$Q_{24} = \frac{\partial \dot x}{\partial \dot r} = \sin\alpha\cos\theta$$

$$Q_{25} = \frac{\partial \dot x}{\partial \theta} = -\dot r\sin\alpha\sin\theta - \dot\alpha r\cos\alpha\sin\theta - \dot\theta r\sin\alpha\cos\theta$$

$$Q_{26} = \frac{\partial \dot x}{\partial \dot\theta} = -r\sin\alpha\sin\theta$$

$$Q_{31} = \frac{\partial y}{\partial \alpha} = -r\sin\alpha\cos\theta$$

$$Q_{33} = \frac{\partial y}{\partial r} = \cos\alpha\cos\theta$$

$$Q_{35} = \frac{\partial y}{\partial \theta} = -r\cos\alpha\sin\theta$$

$$Q_{41} = \frac{\partial \dot y}{\partial \alpha} = -\dot r\sin\alpha\cos\theta - \dot\alpha r\cos\alpha\cos\theta + \dot\theta r\sin\alpha\sin\theta$$

$$Q_{42} = \frac{\partial \dot y}{\partial \dot\alpha} = -r\sin\alpha\cos\theta$$

$$Q_{43} = \frac{\partial \dot y}{\partial r} = -\dot\alpha\sin\alpha\cos\theta - \dot\theta\cos\alpha\sin\theta$$

$$Q_{44} = \frac{\partial \dot y}{\partial \dot r} = \cos\alpha\cos\theta$$

$$Q_{45} = \frac{\partial \dot y}{\partial \theta} = -\dot r\cos\alpha\sin\theta + \dot\alpha r\sin\alpha\sin\theta - \dot\theta r\cos\alpha\cos\theta$$

$$Q_{46} = \frac{\partial \dot y}{\partial \dot\theta} = -r\cos\alpha\sin\theta$$

$$Q_{53} = \frac{\partial z}{\partial r} = \sin\theta$$

$$Q_{55} = \frac{\partial z}{\partial \theta} = r\cos\theta$$

$$Q_{63} = \frac{\partial \dot z}{\partial r} = \dot\theta\cos\theta$$

$$Q_{64} = \frac{\partial \dot z}{\partial \dot r} = \sin\theta$$

$$Q_{65} = \frac{\partial \dot z}{\partial \theta} = \dot r\cos\theta - \dot\theta r\sin\theta$$

$$Q_{66} = \frac{\partial \dot z}{\partial \dot\theta} = r\cos\theta$$

As described in pages 14-16.

Conversion of covariance from spherical to cartesian: Given a diagonal matrix (x, y, z), the jacobian of (θ, r, ϕ) with respect to (x, y, z) is:

$$\begin{pmatrix} \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & \frac{\sqrt{x^2+y^2}}{r^2} \end{pmatrix} \tag{16}$$

Given a diagonal matrix $(x, \dot x, y, \dot y, z, \dot z)$,the jacobian of $(x, \dot x, y, \dot y, z, \dot z)$ with respect to $(\theta, \dot\theta, r, \dot r, \phi, \dot\phi)$ is:

Non-zero elements of matrix B are calculated as follows:

$$B_{11} = \frac{\partial \alpha}{\partial x} = \frac{y}{x^2 + y^2}$$

$$B_{13} = \frac{\partial \alpha}{\partial y} = -\frac{x}{x^2 + y^2}$$

$$B_{21} = \frac{\partial \dot{\alpha}}{\partial x} = \frac{(x^2 - y^2)\dot{y} - 2xy\dot{x}}{\left(x^2 + y^2\right)^2}$$

$$B_{22} = \frac{\partial \dot{\alpha}}{\partial \dot{x}} = \frac{y}{x^2 + y^2}$$

$$B_{23} = \frac{\partial \dot{\alpha}}{\partial y} = \frac{(x^2 - y^2)\dot{x} + 2xy\dot{y}}{\left(x^2 + y^2\right)^2}$$

$$B_{24} = \frac{\partial \dot{\alpha}}{\partial \dot{y}} = -\frac{x}{x^2 + y^2}$$

$$B_{31} = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{33} = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{35} = \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{41} = \frac{\partial \dot{r}}{\partial x} = \frac{\dot{x}(y^2 + z^2) - x(y\dot{y} + z\dot{z})}{(x^2 + y^2 + z^2)^{3/2}}$$

$$B_{42} = \frac{\partial \dot{r}}{\partial \dot{x}} = B_{31}$$

$$B_{43} = \frac{\partial \dot{r}}{\partial y} = \frac{\dot{y}(x^2 + z^2) - y(x\dot{x} + z\dot{z})}{(x^2 + y^2 + z^2)^{3/2}}$$

$$B_{44} = \frac{\partial \dot{r}}{\partial \dot{y}} = B_{33}$$

$$B_{45} = \frac{\partial \dot{r}}{\partial z} = \frac{\dot{z}(x^2 + y^2) - z(x\dot{x} + y\dot{y})}{(x^2 + y^2 + z^2)^{3/2}}$$

$$B_{46} = \frac{\partial \dot{r}}{\partial \dot{z}} = B_{35}$$

$$B_{51} = \frac{\partial \theta}{\partial x} = -\frac{xz}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2}}$$

$$B_{53} = \frac{\partial \theta}{\partial y} = -\frac{yz}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2}}$$

$$B_{55} = \frac{\partial \theta}{\partial z} = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2}$$

$$B_{61} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{\left[\begin{array}{l} -2zx^4\dot{x} - zx^2\dot{x}y^2 + z\dot{x}y^4 + z^3\dot{x}y^2 - \dot{z}x^3z^2 - x\dot{z}y^2z^2 - 3zy\dot{y}x^3 - \\ 3xzy^3\dot{y} - xz^3y\dot{y} + \dot{z}x^5 + 2\dot{z}x^3y^2 + x\dot{z}y^4 \end{array}\right]}{(x^2 + y^2 + z^2)^2\left(x^2 + y^2\right)^{3/2}}$$

$$B_{62} = \frac{\partial \dot{\theta}}{\partial \dot{x}} = B_{51}$$

$$B_{63} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{\left[\begin{array}{l} z\dot{y}x^4 - zy^2\dot{y}x^2 - 2zy^4\dot{y} + z^3\dot{y}x^2 - y\dot{z}x^2z^2 - \dot{z}y^3z^2 - 3yzx^3\dot{x} - \\ 3zx\dot{x}y^3 - yz^3x\dot{x} + y\dot{z}x^4 + 2\dot{z}x^2y^3 + \dot{z}y^5 \end{array}\right]}{(x^2 + y^2 + z^2)^2\left(x^2 + y^2\right)^{3/2}}$$

$$B_{64} = \frac{\partial \dot{\theta}}{\partial \dot{y}} = B_{53}$$

$$B_{65} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{x^3\dot{x} + x\dot{x}y^2 - x\dot{x}z^2 + y\dot{y}x^2 + y^3\dot{y} - y\dot{y}z^2 + 2z\dot{z}x^2 + 2z\dot{z}y^2}{(x^2 + y^2 + z^2)^2\sqrt{x^2 + y^2}}$$

$$B_{66} = \frac{\partial \dot{\theta}}{\partial \dot{z}} = B_{55}$$

As described in pages 11-13.

References

[1] M Heikkinen et al. Geschlossene formeln zur berechnung räumlicher geodätischer koordinaten aus rechtwinkligen koordinaten. 1982.

[2] Karl Osen. Accurate conversion of earth-fixed earth-centered coordinates to geodetic coordinates. 2017.

[3] J. Zhu. Conversion of earth-centered earth-fixed coordinates to geodetic coordinates. *IEEE Transactions on Aerospace and Electronic Systems*, 30(3):957–961, July 1994. http://dx.doi.org/10.1109/7.303772 doi:10.1109/7.303772.

3 Appendix

Let the LCCS origin coordinates be (Φ_s, Θ_s, H_s) in GCS.

Let the given point coordinates in LCCS be (x, y, z).

The transformation from LCCS to PCS is given by

Azimuth: $\alpha = \arctan\frac{x}{y}$

Note: the C function *atan2* or its equivalent should be used to compute the value of α, and the negative signs in the numerator and denominator should be kept in the call to *atan2*. Next, 2π should be added to the values in the interval $(-\pi,0)$ since the azimuth values have to belong to the interval $(0,2\pi]$.

Range: $r = \sqrt{x^2 + y^2 + z^2}$

Elevation: $\theta = \arctan\frac{z}{\sqrt{x^2 + y^2}}$

Range rate: $\dot{r} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{r}$

$$\dot{\alpha} = \frac{y\dot{x} - x\dot{y}}{x^2 + y^2}$$

$$\dot{\theta} = \frac{-z(x\dot{x} + y\dot{y}) + (x^2 + y^2)\dot{z}}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2}}$$

The transformation of the covariance matrix $\mathbf{P}_{LCCS}$ of vector $(x, \dot{x}, y, \dot{y}, z, \dot{z})$ into covariance matrix $\mathbf{P}_{PCS}$ of vector $(\alpha, r, \theta, \dot{r})$ is given by $\mathbf{P}_{PCS} = \mathbf{B}\mathbf{P}_{LCCS}\mathbf{B}^T$, where matrix **B** is the Jacobian.

If the data vector does not include velocity components, then $B = \left\|\frac{\partial(\alpha, r, \theta)}{\partial(x, y, z)}\right\| = \left\|B_{ij}\right\|_{i,j=1..3}$.

Non-zero elements of matrix **B** are calculated as follows:

$$B_{11} = \frac{\partial\alpha}{\partial x} = \frac{y}{x^2 + y^2}$$

$$B_{12} = \frac{\partial\alpha}{\partial y} = -\frac{x}{x^2 + y^2}$$

$$B_{21} = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{22} = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{23} = \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

11

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$$B_{31} = \frac{\partial \theta}{\partial x} = -\frac{xz}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2}}$$

$$B_{32} = \frac{\partial \theta}{\partial y} = -\frac{yz}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2}}$$

$$B_{33} = \frac{\partial \theta}{\partial z} = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2}$$

Otherwise, $B = \left\| \frac{\partial(\alpha, \dot{\alpha}, r, \dot{r}, \theta, \dot{\theta})}{\partial(x, \dot{x}, y, \dot{y}, z, \dot{z})} \right\| = \left\| B_{ij} \right\|_{i,j=1,6}$

Non-zero elements of matrix **B** are calculated as follows:

$$B_{11} = \frac{\partial \alpha}{\partial x} = \frac{y}{x^2 + y^2}$$

$$B_{13} = \frac{\partial \alpha}{\partial y} = -\frac{x}{x^2 + y^2}$$

$$B_{21} = \frac{\partial \dot{\alpha}}{\partial x} = \frac{(x^2 - y^2)\dot{y} - 2xy\dot{x}}{(x^2 + y^2)^2}$$

$$B_{22} = \frac{\partial \dot{\alpha}}{\partial \dot{x}} = \frac{y}{x^2 + y^2}$$

$$B_{23} = \frac{\partial \dot{\alpha}}{\partial y} = \frac{(x^2 - y^2)\dot{x} + 2xy\dot{y}}{(x^2 + y^2)^2}$$

$$B_{24} = \frac{\partial \dot{\alpha}}{\partial \dot{y}} = -\frac{x}{x^2 + y^2}$$

$$B_{31} = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{33} = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{35} = \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{41} = \frac{\partial \dot{r}}{\partial x} = \frac{\dot{x}(y^2 + z^2) - x(y\dot{y} + z\dot{z})}{(x^2 + y^2 + z^2)^{3/2}}$$

$$B_{42} = \frac{\partial \dot{r}}{\partial \dot{x}} = B_{31}$$

$$B_{43} = \frac{\partial \dot{r}}{\partial y} = \frac{\dot{y}(x^2 + z^2) - y(x\dot{x} + z\dot{z})}{(x^2 + y^2 + z^2)^{3/2}}$$

$$B_{44} = \frac{\partial \dot{r}}{\partial \dot{y}} = B_{33}$$

$$B_{45} = \frac{\partial \dot{r}}{\partial z} = \frac{\dot{z}(x^2 + y^2) - z(x\dot{x} + y\dot{y})}{(x^2 + y^2 + z^2)^{3/2}}$$

$$B_{46} = \frac{\partial \dot{r}}{\partial \dot{z}} = B_{35}$$

$$B_{51} = \frac{\partial \theta}{\partial x} = -\frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$B_{53} = \frac{\partial \theta}{\partial y} = -\frac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$B_{55} = \frac{\partial \theta}{\partial z} = \frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}$$

$$B_{61} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{\begin{bmatrix} -2zx^4\dot{x}-zx^2\dot{x}y^2+z\dot{x}y^4+z^3\dot{x}y^2-\dot{z}x^3z^2-x\dot{z}y^2z^2-3zy\dot{y}x^3- \\ 3xzy^3\dot{y}-xz^3y\dot{y}+\dot{z}x^5+2\dot{z}x^3y^2+x\dot{z}y^4 \end{bmatrix}}{(x^2+y^2+z^2)^2\left(x^2+y^2\right)^{3/2}}$$

$$B_{62} = \frac{\partial \dot{\theta}}{\partial \dot{x}} = B_{51}$$

$$B_{63} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{\begin{bmatrix} z\dot{y}x^4-zy^2\dot{y}x^2-2zy^4\dot{y}+z^3\dot{y}x^2-y\dot{z}x^2z^2-\dot{z}y^3z^2-3yzx^3\dot{x}- \\ 3zx\dot{x}y^3-yz^3x\dot{x}+y\dot{z}x^4+2\dot{z}x^2y^3+\dot{z}y^5 \end{bmatrix}}{(x^2+y^2+z^2)^2\left(x^2+y^2\right)^{3/2}}$$

$$B_{64} = \frac{\partial \dot{\theta}}{\partial \dot{y}} = B_{53}$$

$$B_{65} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{x^3\dot{x}+x\dot{x}y^2-x\dot{x}z^2+y\dot{y}x^2+y^3\dot{y}-y\dot{y}z^2+2z\dot{z}x^2+2z\dot{z}y^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2}}$$

$$B_{66} = \frac{\partial \dot{\theta}}{\partial \dot{z}} = B_{55}$$

Acceleration components are calculated as follows:

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{\partial \dot{r}}{\partial x}\dot{x} + \frac{\partial \dot{r}}{\partial \dot{x}}\ddot{x} + \frac{\partial \dot{r}}{\partial y}\dot{y} + \frac{\partial \dot{r}}{\partial \dot{y}}\ddot{y} + \frac{\partial \dot{r}}{\partial z}\dot{z} + \frac{\partial \dot{r}}{\partial \dot{z}}\ddot{z} =$$
$$B_{21}\dot{x} + B_{22}\ddot{x} + B_{23}\dot{y} + B_{24}\ddot{y} + B_{25}\dot{z} + B_{26}\ddot{z},$$

$$\ddot{\alpha} = \frac{d\dot{\alpha}}{dt} = \frac{\partial \dot{\alpha}}{\partial x}\dot{x} + \frac{\partial \dot{\alpha}}{\partial \dot{x}}\ddot{x} + \frac{\partial \dot{\alpha}}{\partial y}\dot{y} + \frac{\partial \dot{\alpha}}{\partial \dot{y}}\ddot{y} + \frac{\partial \dot{\alpha}}{\partial z}\dot{z} + \frac{\partial \dot{\alpha}}{\partial \dot{z}}\ddot{z} =$$
$$B_{41}\dot{x} + B_{42}\ddot{x} + B_{43}\dot{y} + B_{44}\ddot{y} + B_{45}\dot{z} + B_{46}\ddot{z},$$

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{\partial \dot{\theta}}{\partial x}\dot{x} + \frac{\partial \dot{\theta}}{\partial \dot{x}}\ddot{x} + \frac{\partial \dot{\theta}}{\partial y}\dot{y} + \frac{\partial \dot{\theta}}{\partial \dot{y}}\ddot{y} + \frac{\partial \dot{\theta}}{\partial z}\dot{z} + \frac{\partial \dot{\theta}}{\partial \dot{z}}\ddot{z} =$$
$$B_{61}\dot{x} + B_{62}\ddot{x} + B_{63}\dot{y} + B_{64}\ddot{y} + B_{65}\dot{z} + B_{66}\ddot{z},$$

where matrix B is calculated as above.

1.1.4.5. Covariance matrix of the transformation from ITCS to PCS

If P_{ITCS} is the covariance matrix of the point in ITCS, then the covariance matrix of the point in PCS is given by $\mathbf{P}_{PCS} = \mathbf{Z}\mathbf{P}_{ITCS}\,\mathbf{Z}^T$, matrix $Z = B \cdot C \cdot F \cdot D$ where $B,\ C,\ F,\ D$ are defined above in 1.1.4.1-1.1.4.4.

1.1.5. Inverse Coordinate Transformations

Goal of Capability. The capability transforms coordinates given in PCS into ITCS through LCCS, GCCS, and GCS.

$$\boxed{\text{PCS}} \xrightarrow{1.1.5.1} \boxed{\text{LCCS}} \xrightarrow{1.1.5.2} \boxed{\text{GCCS}} \xrightarrow{1.1.5.3} \boxed{\text{GCS}} \xrightarrow{1.1.5.4} \boxed{\text{ITCS}}$$

1.1.5.1. Transformation from PCS to LCCS

Let the given point coordinates in PCS be azimuth α, range r, and elevation θ.

$$x = r\sin\alpha\cos\theta$$
$$y = r\cos\alpha\cos\theta$$
$$z = r\sin\theta$$

The velocity components are given by

$$\dot{x} = \dot{r}\sin\alpha\cos\theta + \dot{\alpha}r\cos\alpha\cos\theta - \dot{\theta}r\sin\alpha\sin\theta$$
$$\dot{y} = \dot{r}\cos\alpha\cos\theta - \dot{\alpha}r\sin\alpha\cos\theta - \dot{\theta}r\cos\alpha\sin\theta$$
$$\dot{z} = \dot{r}\sin\theta + \dot{\theta}r\cos\theta$$

The transformation of the covariance matrix $\mathbf{P}_{PCS}$ of vector $(\alpha,\dot{\alpha},r,\dot{r},\theta,\dot{\theta})$ into covariance matrix $\mathbf{P}_{LCCS}$ is given by $\mathbf{P}_{LCCS} = \mathbf{Q}\ \mathbf{P}_{PCS}\ \mathbf{Q}^T$, where matrix $\mathbf{Q}$ is the Jacobian.

If the data vector does not include velocity components, then $Q = \left|\dfrac{\partial(x,y,z)}{\partial(\alpha,r,\theta)}\right| = \|Q_{ij}\|_{i,j=\overline{1,3}}$.

Non-zero elements of matrix $\mathbf{Q}$ are calculated as follows:

$$Q_{11} = \frac{\partial x}{\partial \alpha} = r\cos\alpha\cos\theta$$

$$Q_{12} = \frac{\partial x}{\partial r} = \sin\alpha\cos\theta$$

$$Q_{13} = \frac{\partial x}{\partial \theta} = -r\sin\alpha\sin\theta$$

$$Q_{21} = \frac{\partial y}{\partial \alpha} = -r\sin\alpha\cos\theta$$

$$Q_{22} = \frac{\partial y}{\partial r} = \cos\alpha\cos\theta$$

$$Q_{23} = \frac{\partial y}{\partial \theta} = -r\cos\alpha\sin\theta$$

$$Q_{32} = \frac{\partial z}{\partial r} = \sin\theta$$

$$Q_{33} = \frac{\partial z}{\partial \theta} = r\cos\theta$$

Otherwise, $Q = \left\| \frac{\partial(x,\dot{x},y,\dot{y},z,\dot{z})}{\partial(\alpha,\dot{\alpha},r,\dot{r},\theta,\dot{\theta})} \right\| = \left\| Q_{ij} \right\|_{i,j=\overline{1..6}}$.

Non-zero elements of matrix **Q** are calculated as follows:

$$Q_{11} = \frac{\partial x}{\partial \alpha} = r\cos\alpha\cos\theta$$

$$Q_{13} = \frac{\partial x}{\partial r} = \sin\alpha\cos\theta$$

$$Q_{15} = \frac{\partial x}{\partial \theta} = -r\sin\alpha\sin\theta$$

$$Q_{21} = \frac{\partial \dot{x}}{\partial \alpha} = \dot{r}\cos\alpha\cos\theta - \dot{\alpha}r\sin\alpha\cos\theta - \dot{\theta}r\cos\alpha\sin\theta$$

$$Q_{22} = \frac{\partial \dot{x}}{\partial \dot{\alpha}} = r\cos\alpha\cos\theta$$

$$Q_{23} = \frac{\partial \dot{x}}{\partial r} = -\dot{\alpha}\sin\alpha\cos\theta - \dot{\theta}\sin\alpha\sin\theta$$

$$Q_{24} = \frac{\partial \dot{x}}{\partial \dot{r}} = \sin\alpha\cos\theta$$

$$Q_{25} = \frac{\partial \dot{x}}{\partial \theta} = -\dot{r}\sin\alpha\sin\theta - \dot{\alpha}r\cos\alpha\sin\theta - \dot{\theta}r\sin\alpha\cos\theta$$

$$Q_{26} = \frac{\partial \dot{x}}{\partial \dot{\theta}} = -r\sin\alpha\sin\theta$$

$$Q_{31} = \frac{\partial y}{\partial \alpha} = -r\sin\alpha\cos\theta$$

$$Q_{33} = \frac{\partial y}{\partial r} = \cos\alpha\cos\theta$$

$$Q_{35} = \frac{\partial y}{\partial \theta} = -r\cos\alpha\sin\theta$$

$$Q_{41} = \frac{\partial \dot{y}}{\partial \alpha} = -\dot{r}\sin\alpha\cos\theta - \dot{\alpha}r\cos\alpha\cos\theta + \dot{\theta}r\sin\alpha\sin\theta$$

$$Q_{42} = \frac{\partial \dot{y}}{\partial \dot{\alpha}} = -r\sin\alpha\cos\theta$$

$$Q_{43} = \frac{\partial \dot{y}}{\partial r} = -\dot{\alpha}\sin\alpha\cos\theta - \dot{\theta}\cos\alpha\sin\theta$$

$$Q_{44} = \frac{\partial \dot{y}}{\partial \dot{r}} = \cos\alpha\cos\theta$$

$$Q_{45} = \frac{\partial \dot{y}}{\partial \theta} = -\dot{r}\cos\alpha\sin\theta + \dot{\alpha}r\sin\alpha\sin\theta - \dot{\theta}r\cos\alpha\cos\theta$$

$$Q_{46} = \frac{\partial \dot{y}}{\partial \dot{\theta}} = -r\cos\alpha\sin\theta$$

$$Q_{53} = \frac{\partial z}{\partial r} = \sin\theta$$

$$Q_{55} = \frac{\partial z}{\partial \theta} = r\cos\theta$$

$$Q_{63} = \frac{\partial \dot{z}}{\partial r} = \dot{\theta}\cos\theta$$

$$Q_{64} = \frac{\partial \dot{z}}{\partial \dot{r}} = \sin\theta$$

$$Q_{65} = \frac{\partial \dot{z}}{\partial \theta} = \dot{r}\cos\theta - \dot{\theta}r\sin\theta$$

$$Q_{66} = \frac{\partial \dot{z}}{\partial \dot{\theta}} = r\cos\theta$$

Acceleration components are calculated as follows:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{\partial \dot{x}}{\partial r}\dot{r} + \frac{\partial \dot{x}}{\partial \dot{r}}\ddot{r} + \frac{\partial \dot{x}}{\partial \alpha}\dot{\alpha} + \frac{\partial \dot{x}}{\partial \dot{\alpha}}\ddot{\alpha} + \frac{\partial \dot{x}}{\partial \theta}\dot{\theta} + \frac{\partial \dot{x}}{\partial \dot{\theta}}\ddot{\theta} =$$
$$Q_{21}\dot{r} + Q_{22}\ddot{r} + Q_{23}\dot{\alpha} + Q_{24}\ddot{\alpha} + Q_{25}\dot{\theta} + Q_{26}\ddot{\theta},$$
$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{\partial \dot{y}}{\partial r}\dot{r} + \frac{\partial \dot{y}}{\partial \dot{r}}\ddot{r} + \frac{\partial \dot{y}}{\partial \alpha}\dot{\alpha} + \frac{\partial \dot{y}}{\partial \dot{\alpha}}\ddot{\alpha} + \frac{\partial \dot{y}}{\partial \theta}\dot{\theta} + \frac{\partial \dot{y}}{\partial \dot{\theta}}\ddot{\theta} =$$
$$Q_{41}\dot{r} + Q_{42}\ddot{r} + Q_{43}\dot{\alpha} + Q_{44}\ddot{\alpha} + Q_{45}\dot{\theta} + Q_{46}\ddot{\theta},$$
$$\ddot{z} = \frac{d\dot{z}}{dt} = \frac{\partial \dot{z}}{\partial r}\dot{r} + \frac{\partial \dot{z}}{\partial \dot{r}}\ddot{r} + \frac{\partial \dot{z}}{\partial \alpha}\dot{\alpha} + \frac{\partial \dot{z}}{\partial \dot{\alpha}}\ddot{\alpha} + \frac{\partial \dot{z}}{\partial \theta}\dot{\theta} + \frac{\partial \dot{z}}{\partial \dot{\theta}}\ddot{\theta} =$$
$$Q_{61}\dot{r} + Q_{62}\ddot{r} + Q_{63}\dot{\alpha} + Q_{64}\ddot{\alpha} + Q_{65}\dot{\theta} + Q_{66}\ddot{\theta},$$

where matrix Q is calculated as above.

1.1.5.2. Transformation from LCCS to GCCS

Ψ is a LCCS orientation as defined in 1.1.2 (c).
Let the LCCS origin coordinates be (Φ_s, Θ_s, H_s) in GCS, (X_s, Y_s, Z_s) be its coordinates in GCCS, $(\dot{X}_s, \dot{Y}_s, \dot{Z}_s)$ be its velocity in GCCS.