Let the LCCS origin coordinates be (Φ_s, Θ_s, H_s) in GCS.

Let the given point coordinates in LCCS be (x, y, z).

The transformation from LCCS to PCS is given by

Azimuth: $\alpha = \arctan\dfrac{x}{y}$

Note: the C function *atan2* or its equivalent should be used to compute the value of α, and the negative signs in the numerator and denominator should be kept in the call to *atan2*. Next, 2π should be added to the values in the interval $(-\pi, 0)$ since the azimuth values have to belong to the interval $(0, 2\pi]$.

Range: $r = \sqrt{x^2 + y^2 + z^2}$

Elevation: $\theta = \arctan\dfrac{z}{\sqrt{x^2 + y^2}}$

Range rate: $\dot{r} = \dfrac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{x\dot{x} + y\dot{y} + z\dot{z}}{r}$

$$\dot{\alpha} = \frac{y\dot{x} - x\dot{y}}{x^2 + y^2}$$

$$\dot{\theta} = \frac{-z(x\dot{x} + y\dot{y}) + (x^2 + y^2)\dot{z}}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2}}$$

The transformation of the covariance matrix $\mathbf{P}_{LCCS}$ of vector $(x, \dot{x}, y, \dot{y}, z, \dot{z})$ into covariance matrix $\mathbf{P}_{PCS}$ of vector $(\alpha, r, \theta, \dot{r})$ is given by $\mathbf{P}_{PCS} = \mathbf{B}\mathbf{P}_{LCCS}\,\mathbf{B}^T$, where matrix **B** is the Jacobian.

If the data vector does not include velocity components, then $B = \left\| \dfrac{\partial(\alpha, r, \theta)}{\partial(x, y, z)} \right\| = \left\| B_{ij} \right\|_{i,j=1..3}$.

Non-zero elements of matrix **B** are calculated as follows:

$$B_{11} = \frac{\partial \alpha}{\partial x} = \frac{y}{x^2 + y^2}$$

$$B_{12} = \frac{\partial \alpha}{\partial y} = -\frac{x}{x^2 + y^2}$$

$$B_{21} = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{22} = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{23} = \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$B_{31} = \frac{\partial\theta}{\partial x} = -\frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$B_{32} = \frac{\partial\theta}{\partial y} = -\frac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$B_{33} = \frac{\partial\theta}{\partial z} = \frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}$$

Otherwise, $B = \left\| \frac{\partial(\alpha,\dot{\alpha},r,\dot{r},\theta,\dot{\theta})}{\partial(x,\dot{x},y,\dot{y},z,\dot{z})} \right\| = \left\| B_{ij} \right\|_{i,j=1,6}$

Non-zero elements of matrix B are calculated as follows:

$$B_{11} = \frac{\partial\alpha}{\partial x} = \frac{y}{x^2+y^2}$$

$$B_{13} = \frac{\partial\alpha}{\partial y} = -\frac{x}{x^2+y^2}$$

$$B_{21} = \frac{\partial\dot{\alpha}}{\partial x} = \frac{(x^2-y^2)\dot{y}-2xy\dot{x}}{\left(x^2+y^2\right)^2}$$

$$B_{22} = \frac{\partial\dot{\alpha}}{\partial\dot{x}} = \frac{y}{x^2+y^2}$$

$$B_{23} = \frac{\partial\dot{\alpha}}{\partial y} = \frac{(x^2-y^2)\dot{x}+2xy\dot{y}}{\left(x^2+y^2\right)^2}$$

$$B_{24} = \frac{\partial\dot{\alpha}}{\partial\dot{y}} = -\frac{x}{x^2+y^2}$$

$$B_{31} = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$B_{33} = \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$B_{35} = \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$B_{41} = \frac{\partial\dot{r}}{\partial x} = \frac{\dot{x}(y^2+z^2)-x(y\dot{y}+z\dot{z})}{(x^2+y^2+z^2)^{3/2}}$$

$$B_{42} = \frac{\partial\dot{r}}{\partial\dot{x}} = B_{31}$$

$$B_{43} = \frac{\partial\dot{r}}{\partial y} = \frac{\dot{y}(x^2+z^2)-y(x\dot{x}+z\dot{z})}{(x^2+y^2+z^2)^{3/2}}$$

$$B_{44} = \frac{\partial\dot{r}}{\partial\dot{y}} = B_{33}$$

$$B_{45} = \frac{\partial\dot{r}}{\partial z} = \frac{\dot{z}(x^2+y^2)-z(x\dot{x}+y\dot{y})}{(x^2+y^2+z^2)^{3/2}}$$

$$B_{46} = \frac{\partial \dot{r}}{\partial \dot{z}} = B_{35}$$

$$B_{51} = \frac{\partial \theta}{\partial x} = -\frac{xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$B_{53} = \frac{\partial \theta}{\partial y} = -\frac{yz}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}$$

$$B_{55} = \frac{\partial \theta}{\partial z} = \frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}$$

$$B_{61} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{\begin{bmatrix} -2zx^4\dot{x}-zx^2\dot{x}y^2+z\dot{x}y^4+z^3\dot{x}y^2-\dot{z}x^3z^2-x\dot{z}y^2z^2-3zy\dot{y}x^3- \\ 3xzy^3\dot{y}-xz^3y\dot{y}+\dot{z}x^5+2\dot{z}x^3y^2+x\dot{z}y^4 \end{bmatrix}}{(x^2+y^2+z^2)^2\left(x^2+y^2\right)^{3/2}}$$

$$B_{62} = \frac{\partial \dot{\theta}}{\partial \dot{x}} = B_{51}$$

$$B_{63} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{\begin{bmatrix} z\dot{y}x^4-zy^2\dot{y}x^2-2zy^4\dot{y}+z^3\dot{y}x^2-y\dot{z}x^2z^2-\dot{z}y^3z^2-3yzx^3\dot{x}- \\ 3zx\dot{x}y^3-yz^3x\dot{x}+y\dot{z}x^4+2\dot{z}x^2y^3+\dot{z}y^5 \end{bmatrix}}{(x^2+y^2+z^2)^2\left(x^2+y^2\right)^{3/2}}$$

$$B_{64} = \frac{\partial \dot{\theta}}{\partial \dot{y}} = B_{53}$$

$$B_{65} = \frac{\partial \dot{\theta}}{\partial z} = -\frac{x^3\dot{x}+x\dot{x}y^2-x\dot{x}z^2+y\dot{y}x^2+y^3\dot{y}-y\dot{y}z^2+2z\dot{z}x^2+2z\dot{z}y^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2}}$$

$$B_{66} = \frac{\partial \dot{\theta}}{\partial \dot{z}} = B_{55}$$

Acceleration components are calculated as follows:

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{\partial \dot{r}}{\partial x}\dot{x} + \frac{\partial \dot{r}}{\partial \dot{x}}\ddot{x} + \frac{\partial \dot{r}}{\partial y}\dot{y} + \frac{\partial \dot{r}}{\partial \dot{y}}\ddot{y} + \frac{\partial \dot{r}}{\partial z}\dot{z} + \frac{\partial \dot{r}}{\partial \dot{z}}\ddot{z} =$$
$$B_{21}\dot{x} + B_{22}\ddot{x} + B_{23}\dot{y} + B_{24}\ddot{y} + B_{25}\dot{z} + B_{26}\ddot{z},$$

$$\ddot{\alpha} = \frac{d\dot{\alpha}}{dt} = \frac{\partial \dot{\alpha}}{\partial x}\dot{x} + \frac{\partial \dot{\alpha}}{\partial \dot{x}}\ddot{x} + \frac{\partial \dot{\alpha}}{\partial y}\dot{y} + \frac{\partial \dot{\alpha}}{\partial \dot{y}}\ddot{y} + \frac{\partial \dot{\alpha}}{\partial z}\dot{z} + \frac{\partial \dot{\alpha}}{\partial \dot{z}}\ddot{z} =$$
$$B_{41}\dot{x} + B_{42}\ddot{x} + B_{43}\dot{y} + B_{44}\ddot{y} + B_{45}\dot{z} + B_{46}\ddot{z},$$

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{\partial \dot{\theta}}{\partial x}\dot{x} + \frac{\partial \dot{\theta}}{\partial \dot{x}}\ddot{x} + \frac{\partial \dot{\theta}}{\partial y}\dot{y} + \frac{\partial \dot{\theta}}{\partial \dot{y}}\ddot{y} + \frac{\partial \dot{\theta}}{\partial z}\dot{z} + \frac{\partial \dot{\theta}}{\partial \dot{z}}\ddot{z} =$$
$$B_{61}\dot{x} + B_{62}\ddot{x} + B_{63}\dot{y} + B_{64}\ddot{y} + B_{65}\dot{z} + B_{66}\ddot{z},$$

where matrix B is calculated as above.

1.1.4.5. Covariance matrix of the transformation from ITCS to PCS

If P_{ITCS} is the covariance matrix of the point in ITCS, then the covariance matrix of the point in PCS is given by $\mathbf{P}_{PCS} = \mathbf{Z}\mathbf{P}_{ITCS}\,\mathbf{Z}^T$, matrix $Z = B \cdot C \cdot F \cdot D$ where $B,\ C,\ F,\ D$ are defined above in 1.1.4.1-1.1.4.4.

1.1.5. Inverse Coordinate Transformations

Goal of Capability. The capability transforms coordinates given in PCS into ITCS through LCCS, GCCS, and GCS.

$$\boxed{\text{PCS}} \xrightarrow{1.1.5.1} \boxed{\text{LCCS}} \xrightarrow{1.1.5.2} \boxed{\text{GCCS}} \xrightarrow{1.1.5.3} \boxed{\text{GCS}} \xrightarrow{1.1.5.4} \boxed{\text{ITCS}}$$

1.1.5.1. Transformation from PCS to LCCS

Let the given point coordinates in PCS be azimuth α, range r, and elevation θ.

$x = r\sin\alpha\cos\theta$

$y = r\cos\alpha\cos\theta$

$z = r\sin\theta$

The velocity components are given by

$\dot{x} = \dot{r}\sin\alpha\cos\theta + \dot{\alpha}r\cos\alpha\cos\theta - \dot{\theta}r\sin\alpha\sin\theta$

$\dot{y} = \dot{r}\cos\alpha\cos\theta - \dot{\alpha}r\sin\alpha\cos\theta - \dot{\theta}r\cos\alpha\sin\theta$

$\dot{z} = \dot{r}\sin\theta + \dot{\theta}r\cos\theta$

The transformation of the covariance matrix $\mathbf{P}_{PCS}$ of vector $(\alpha, \dot{\alpha}, r, \dot{r}, \theta, \dot{\theta})$ into covariance matrix $\mathbf{P}_{LCCS}$ is given by $\mathbf{P}_{LCCS} = \mathbf{Q}\,\mathbf{P}_{PCS}\,\mathbf{Q}^T$, where matrix **Q** is the Jacobian.

If the data vector does not include velocity components, then $Q = \left\|\dfrac{\partial(x,y,z)}{\partial(\alpha,r,\theta)}\right\| = \|Q_{ij}\|_{i,j=1,3}$.

Non-zero elements of matrix Q are calculated as follows:

$$Q_{11} = \frac{\partial x}{\partial \alpha} = r\cos\alpha\cos\theta$$

$$Q_{12} = \frac{\partial x}{\partial r} = \sin\alpha\cos\theta$$

$$Q_{13} = \frac{\partial x}{\partial \theta} = -r\sin\alpha\sin\theta$$

$$Q_{21} = \frac{\partial y}{\partial \alpha} = -r\sin\alpha\cos\theta$$

$$Q_{22} = \frac{\partial y}{\partial r} = \cos\alpha\cos\theta$$

$$Q_{23} = \frac{\partial y}{\partial \theta} = -r\cos\alpha\sin\theta$$

$$Q_{32} = \frac{\partial z}{\partial r} = \sin\theta$$

$$Q_{33} = \frac{\partial z}{\partial \theta} = r\cos\theta$$

Otherwise, $Q = \left\| \frac{\partial(x,\dot{x},y,\dot{y},z,\dot{z})}{\partial(\alpha,\dot{\alpha},r,\dot{r},\theta,\dot{\theta})} \right\| = \|Q_{ij}\|_{i,j=\overline{1..6}}$.

Non-zero elements of matrix **Q** are calculated as follows:

$$Q_{11} = \frac{\partial x}{\partial \alpha} = r\cos\alpha\cos\theta$$

$$Q_{13} = \frac{\partial x}{\partial r} = \sin\alpha\cos\theta$$

$$Q_{15} = \frac{\partial x}{\partial \theta} = -r\sin\alpha\sin\theta$$

$$Q_{21} = \frac{\partial \dot{x}}{\partial \alpha} = \dot{r}\cos\alpha\cos\theta - \dot{\alpha}r\sin\alpha\cos\theta - \dot{\theta}r\cos\alpha\sin\theta$$

$$Q_{22} = \frac{\partial \dot{x}}{\partial \dot{\alpha}} = r\cos\alpha\cos\theta$$

$$Q_{23} = \frac{\partial \dot{x}}{\partial r} = -\dot{\alpha}\sin\alpha\cos\theta - \dot{\theta}\sin\alpha\sin\theta$$

$$Q_{24} = \frac{\partial \dot{x}}{\partial \dot{r}} = \sin\alpha\cos\theta$$

$$Q_{25} = \frac{\partial \dot{x}}{\partial \theta} = -\dot{r}\sin\alpha\sin\theta - \dot{\alpha}r\cos\alpha\sin\theta - \dot{\theta}r\sin\alpha\cos\theta$$

$$Q_{26} = \frac{\partial \dot{x}}{\partial \dot{\theta}} = -r\sin\alpha\sin\theta$$

$$Q_{31} = \frac{\partial y}{\partial \alpha} = -r\sin\alpha\cos\theta$$

$$Q_{33} = \frac{\partial y}{\partial r} = \cos\alpha\cos\theta$$

$$Q_{35} = \frac{\partial y}{\partial \theta} = -r\cos\alpha\sin\theta$$

$$Q_{41} = \frac{\partial \dot{y}}{\partial \alpha} = -\dot{r}\sin\alpha\cos\theta - \dot{\alpha}r\cos\alpha\cos\theta + \dot{\theta}r\sin\alpha\sin\theta$$

$$Q_{42} = \frac{\partial \dot{y}}{\partial \dot{\alpha}} = -r\sin\alpha\cos\theta$$

$$Q_{43} = \frac{\partial \dot{y}}{\partial r} = -\dot{\alpha}\sin\alpha\cos\theta - \dot{\theta}\cos\alpha\sin\theta$$

$$Q_{44} = \frac{\partial \dot{y}}{\partial \dot{r}} = \cos\alpha\cos\theta$$

$$Q_{45} = \frac{\partial \dot{y}}{\partial \theta} = -\dot{r}\cos\alpha\sin\theta + \dot{\alpha}r\sin\alpha\sin\theta - \dot{\theta}r\cos\alpha\cos\theta$$

$$Q_{46} = \frac{\partial \dot{y}}{\partial \dot{\theta}} = -r\cos\alpha\sin\theta$$

$$Q_{53} = \frac{\partial z}{\partial r} = \sin\theta$$

$$Q_{55} = \frac{\partial z}{\partial \theta} = r\cos\theta$$

$$Q_{63} = \frac{\partial \dot{z}}{\partial r} = \dot{\theta}\cos\theta$$

$$Q_{64} = \frac{\partial \dot{z}}{\partial \dot{r}} = \sin\theta$$

$$Q_{65} = \frac{\partial \dot{z}}{\partial \theta} = \dot{r}\cos\theta - \dot{\theta}r\sin\theta$$

$$Q_{66} = \frac{\partial \dot{z}}{\partial \dot{\theta}} = r\cos\theta$$

Acceleration components are calculated as follows:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{\partial \dot{x}}{\partial r}\dot{r} + \frac{\partial \dot{x}}{\partial \dot{r}}\ddot{r} + \frac{\partial \dot{x}}{\partial \alpha}\dot{\alpha} + \frac{\partial \dot{x}}{\partial \dot{\alpha}}\ddot{\alpha} + \frac{\partial \dot{x}}{\partial \theta}\dot{\theta} + \frac{\partial \dot{x}}{\partial \dot{\theta}}\ddot{\theta} =$$
$$Q_{21}\dot{r} + Q_{22}\ddot{r} + Q_{23}\dot{\alpha} + Q_{24}\ddot{\alpha} + Q_{25}\dot{\theta} + Q_{26}\ddot{\theta},$$
$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{\partial \dot{y}}{\partial r}\dot{r} + \frac{\partial \dot{y}}{\partial \dot{r}}\ddot{r} + \frac{\partial \dot{y}}{\partial \alpha}\dot{\alpha} + \frac{\partial \dot{y}}{\partial \dot{\alpha}}\ddot{\alpha} + \frac{\partial \dot{y}}{\partial \theta}\dot{\theta} + \frac{\partial \dot{y}}{\partial \dot{\theta}}\ddot{\theta} =$$
$$Q_{41}\dot{r} + Q_{42}\ddot{r} + Q_{43}\dot{\alpha} + Q_{44}\ddot{\alpha} + Q_{45}\dot{\theta} + Q_{46}\ddot{\theta},$$
$$\ddot{z} = \frac{d\dot{z}}{dt} = \frac{\partial \dot{z}}{\partial r}\dot{r} + \frac{\partial \dot{z}}{\partial \dot{r}}\ddot{r} + \frac{\partial \dot{z}}{\partial \alpha}\dot{\alpha} + \frac{\partial \dot{z}}{\partial \dot{\alpha}}\ddot{\alpha} + \frac{\partial \dot{z}}{\partial \theta}\dot{\theta} + \frac{\partial \dot{z}}{\partial \dot{\theta}}\ddot{\theta} =$$
$$Q_{61}\dot{r} + Q_{62}\ddot{r} + Q_{63}\dot{\alpha} + Q_{64}\ddot{\alpha} + Q_{65}\dot{\theta} + Q_{66}\ddot{\theta},$$

where matrix Q is calculated as above.

1.1.5.2. Transformation from LCCS to GCCS

Ψ is a LCCS orientation as defined in 1.1.2 (c).
Let the LCCS origin coordinates be (Φ_s, Θ_s, H_s) in GCS, (X_s, Y_s, Z_s) be its coordinates in GCCS, $(\dot{X}_s, \dot{Y}_s, \dot{Z}_s)$ be its velocity in GCCS.