4. Transition Path

Adv. Macro: Heterogenous Agent Models

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Introduction

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  2. Example from GEModelToolsNotebooks/HANC (except stuff on *linearized solution* and *simulation*)

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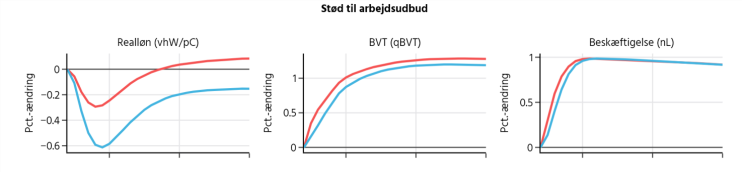
  2. Example from GEModelToolsNotebooks/HANC
     (except stuff on *linearized solution* and *simulation*)
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Documentation for GEModelTools
     (except stuff on *linearized solution* and *simulation*)
  3. Kirkby (2017)

Example I

- What do we mean by transition path?
- Permanent shock to labor supply (think increase in retirement age) in the macroeconomic model of the Ministry of Finance:

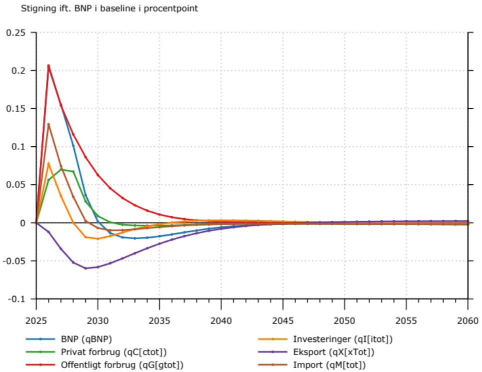


- Note: Permanent shock, so transition path *between* two different steady states

Example II

- Temporary shock to public spending (i.e. fiscal stimulus during recessions)

Figure 6.0.1: BNP og underkomponenter



- Note: Temporary shock, so model returns to the *same steady state*

Standard Ramsey model

Ramsey: Summary

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(K_{t-1}, 1) - C_t^{hh}$$

- **Production function:** $\Gamma_t K_t^{\alpha} L_t^{1-\alpha}$
- **Utility function:** $\frac{\left(C_t^{hh}\right)^{1-\sigma}}{1-\sigma}$
- **Steady state:**

$$K_{ss} = \left(\frac{\left(\frac{1}{\beta} - 1 + \delta\right)}{\Gamma_{ss}\alpha}\right)^{\frac{1}{\alpha-1}}$$
$$C_{ss}^{hh} = (1-\delta)K_{ss} + \Gamma_{ss}K_{ss}^{\alpha} - K_{ss}$$

Ramsey: As an equation system

$$\begin{bmatrix} r_t^K - \alpha\Gamma_t K_t^{\alpha-1} L_t^{1-\alpha} \\ w_t - (1-\alpha)\Gamma_t K_t^{\alpha} L_t^{-\alpha} \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t^{hh} - \left((1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\ C_t^{hh,-\sigma} - \beta(1+r_{t+1})C_{t+1}^{hh,-\sigma} \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0,1,\dots\}, \text{given } K_{-1} \end{bmatrix} = \mathbf{0}$$

Remember: Perfect foresight w.r.t aggregate variables

Unknowns: $\{r_t^K, w_t, L_t, K_t, r_t, A_t, C_t^{hh}, A_t^{hh}\}$ for $\forall t \in \{0,1,\dots\}$

Recap: Newton's method I

- Before solving the dynamic Ramsey model, consider a simpler example

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- **How to find** x? First-order Taylor approximation around current guess x^i:

$$f(x) \approx f\left(x^i\right) + f'\left(x^i\right)\left(x - x^i\right)$$

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- **How to find** x? First-order Taylor approximation around current guess x^i:

$$f\left(x\right)\approx f\left(x^i\right)+f'\left(x^i\right)\left(x-x^i\right)$$

- Set $f\left(x\right)=0$ and solve for x to get:

$$x=x^i-\frac{f\left(x^i\right)}{f'\left(x^i\right)}$$

Recap: Newton's method II

- Newton's method: Given initial guess x_0 update guess for x from i to $i+1$ as:

$$x^{i+1} = x^i - \frac{f\left(x^i\right)}{f'\left(x^i\right)}$$

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- Use numerical approximation:

$$f'\left(x^i\right) \approx \frac{f\left(x^i + h\right) - f\left(x^i\right)}{h}$$

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  - For small h.
- How well does it work?
  - If $f(x)$ is linear this update solves $f(x) = 0$ in **1 iteration**
  - If $f(x)$ is non-linear we typically need more iterations, but works well if initial guess is within basis of attraction

Recap: Multivariate Newton's method

- Generalize to vector-valued, multivariate functions $[f_1(x_{1,}x_2), f_2(x_{1,}x_2)]' = \boldsymbol{f}(\boldsymbol{x})$ with $\boldsymbol{x} = (x_{1,}x_2)'$:

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- Where $\boldsymbol{J}\left(\boldsymbol{x}^i\right)$ is the *Jacobian* of $f\left(\boldsymbol{x}\right)$ w.r.t $\boldsymbol{x}^i$:

$$\boldsymbol{J}\left(\boldsymbol{x}_i\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1^i} & \frac{\partial f_1}{\partial x_2^i} \\ \frac{\partial f_2}{\partial x_1^i} & \frac{\partial f_2}{\partial x_2^i} \end{bmatrix}$$

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- Can calculate this jacobian in the same way as $f'(x)$ in previous example, but need to so for every element in $\boldsymbol{x}$
- **Go through code**

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- Newton's method updates Jacobian $\boldsymbol{J}$ in **every iteration**

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- If $\boldsymbol{J}$ is expensive to calculate, this is a serious bottleneck
- Broyden's method solves this issue by only calculating J around some initial point.
- Then apply following (linear) update of $f'\left(x^{i+1}\right)$ at every iteration i:

$$f'\left(x^{i+1}\right) = f'\left(x^{i}\right) + \frac{\left[f\left(x^{i+1}\right) - f\left(x^{i}\right)\right] - f'\left(x^{i}\right)\left(x^{i+1} - x^{i}\right)}{x^{i+1} - x^{i}}$$

Recap: Broyden's method II

1. Guess $\boldsymbol{x}^0$ and set $i = 0$
2. Calculate the Jacobian around initial point $\boldsymbol{J_0}$
3. Calculate $\boldsymbol{f}^i = \boldsymbol{f}(\boldsymbol{x}^i)$.
4. Stop if $\left|\boldsymbol{f}^i\right|$ below tolerance ϵ
5. Calculate Jacobian by
$$\boldsymbol{J}^i = \begin{cases} \boldsymbol{J_0} & \text{if } i = 0 \\ \boldsymbol{J}^{i-1} + \frac{(\boldsymbol{f}^i - \boldsymbol{f}^{i-1}) - \boldsymbol{J}^{i-1}(\boldsymbol{x}^i - \boldsymbol{x}^{i-1})}{\left|\boldsymbol{x}^i - \boldsymbol{x}^{i-1}\right|_2} \left(\boldsymbol{x}^i - \boldsymbol{x}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
6. Update guess by $\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - \left(\boldsymbol{J}^i\right)^{-1} \boldsymbol{f}^i$
7. Increment i and return to step 3

- **Go through code**

Back to Ramsey

$$
\begin{bmatrix}
r_t^K - \alpha \Gamma_t K_t^{\alpha-1} L_t^{1-\alpha} \\
w_t - (1-\alpha)\Gamma_t K_t^{\alpha} L_t^{-\alpha} \\
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- 2 issues:
  - Many unknowns (8 eqs per period)
  - In fact, infinitely many since time is infinite, $T \to \infty$

Truncated Ramsey, reduced vector form

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, K_{-1}) = \begin{bmatrix} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

$$\begin{aligned}
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K - \delta \\
C_t^{hh} &= (\beta(1+r_{t+1}))^{-\sigma} \, C_{t+1}^{hh} \text{ (backwards)} \\
L_t^{hh} &= 1 \\
A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \text{ (forwards)}
\end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Further reduced

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1}) = \left[\boldsymbol{A} - \boldsymbol{A}^{hh}\right] = \boldsymbol{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

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for $\forall t \in \{0, 1, \ldots, T-1\}$

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$$\begin{bmatrix} C_0 & C_1 & C_2 & \dots \end{bmatrix}' = a + b \begin{bmatrix} Y_0 & Y_1 & Y_2 & \dots \end{bmatrix}'$$
$$\Leftrightarrow \boldsymbol{C} = a + b\boldsymbol{Y}$$
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- Powerfull since it also applies *non-linear*, forward-looking and backwards-looking eqs:

$$C_t = a + b_0 Y_t + b_1 \log Y_{t-4} + b_2 Y_{t+4}^2$$
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- As long as we have the sequence $\boldsymbol{Y}$ we can calculate $\boldsymbol{C}$
  - Will leverage this later when working with the HA model

Solution in sequence space

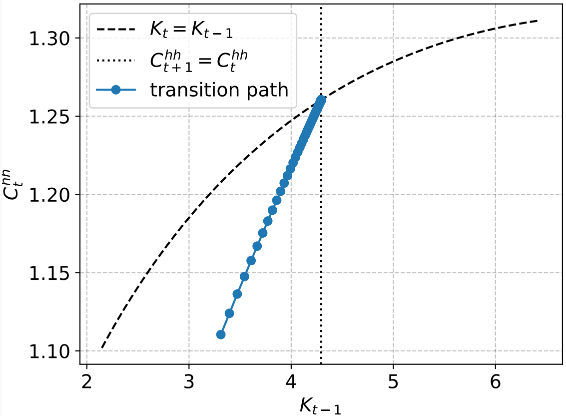
- **Truncation:** $T = 200$ (transition path should have converged to ss by then)
- **Jacobian:** Find $\boldsymbol{H}_{\boldsymbol{K}}$ by *numerical differentiation*

$$\boldsymbol{H}_{\boldsymbol{K}} = \begin{bmatrix} \frac{\partial(A_0 - A_0^{hh})}{\partial K_0} & \frac{\partial(A_0 - A_0^{hh})}{\partial K_1} & \cdots \\ \frac{\partial(A_1 - A_1^{hh})}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

- **Transition path:** Given $\boldsymbol{\Gamma}$ and K_{-1} solve $\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1})$ with non-linear equation system solver (e.g. broyden)
- **Notebook:** *Ramsey.ipynb*

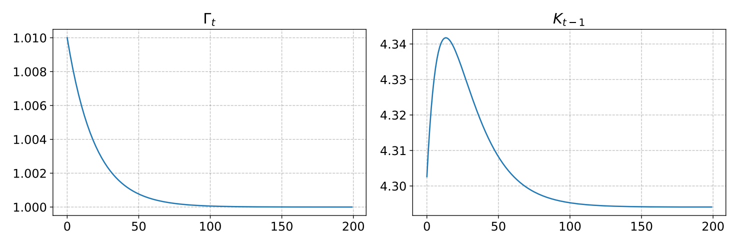
Example 1: Initially low capital

Initially away from steady state: $K_{-1} = 0.75 K_{ss}$



Example 2: Technology shock

Technology shock: $\Gamma_t = 0.01 \times \Gamma_{ss} \times 0.95^t$ (i.e AR(1) with $\rho = 0.95$) (exogenous, deterministic)



Terminology: MIT-shock

Transition path with HA

Equation system

The model can be written as an **equation system**

$$
\begin{bmatrix}
r_t^K - F_K(K_{t-1}, L_t) \\
w_t - F_L(K_{t-1}, L_t) \\
r_t - (r_t^K - \delta) \\
A_t - K_t \\
\boldsymbol{D}_t - \Pi_z \underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} - \Lambda_t \boldsymbol{D}_t \\
A_t - A_t^{hh} \\
L_t - L_t^{hh} \\
\forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0
\end{bmatrix} = \mathbf{0}
$$

where $\{\Gamma_t\}_{t\geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Remember: Policies and choice transitions depend on prices

1. Policy function: $x_t^* = x^*\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$ and $X_t^{hh} = \sum_i x_{it}^* D_{it} = \boldsymbol{x}_t^{*\prime} \boldsymbol{D}_t$
2. Choice transition: $\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau\geq t}\right)$

Transition path - close to verbal definition

For a given $\underline{\boldsymbol{D}}_0$ and a path $\{\Gamma_t\}$

1. Quantities $\{K_t\}$ and $\{L_t\}$,
2. prices $\{r_t\}$ and $\{w_t\}$,
3. the distributions $\{\boldsymbol{D}_t\}$ over β_i, z_t and a_{t-1}
4. and the policy functions $\{a_t^*\}$, $\{\ell_t^*\}$ and $\{c_t^*\}$

are such that in all periods

1. Firms maximize profits (prices)
2. Household maximize expected utility (policy functions)
3. $\boldsymbol{D}_t$ is implied by simulating the household problem forwards from $\underline{\boldsymbol{D}}_0$
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

Reduce size of equation system

- In the equation system above we have many **unknowns** and many **equations**
- Makes finding the solution with Broyden's method since **Jacobian is large**
  - With truncation T and N equations/unknowns $\boldsymbol{J}$ has size $(T \times N, T \times N,)$
  - $\Rightarrow$ Ekspensive to calculate

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  - $\Rightarrow$ Ekspensive to calculate
- We can typically **exploit model structure** to reduce size of system
  - Did this earlier for Ramsey
  - Now more formally

Truncated, reduced vector form

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0) = \left[\begin{array}{c} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{array} \right] = \boldsymbol{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ r_t &= r_t^K - \delta \\ A_t &= K_t \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t^{hh} &= \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\ &\forall t \in \{0, 1, \ldots, T-1\} \end{aligned}$$

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DAG - Directed Acyclic Graph

- **Orange square:** Shocks (exogenous)
- **Blue square:** Unknowns (endogenous)
- **Green circles:** Blocks (with variables and targets inside)

- This DAG implies: Exo. input + guess ⇒ Firm block ⇒ Mutual fund ⇒HHs ⇒ Residuals

Further reduction

$$\boldsymbol{H}\left(\boldsymbol{K}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0\right) = \left[\begin{array}{c} A_t - A_t^{hh} \\ \forall t \in \{0, 1, \dots, T-1\} \end{array}\right] = \boldsymbol{0}$$

where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned}
L_t &= 1 \\
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K - \delta \\
\boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\
A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\
&\forall t \in \{0, 1, \dots, T-1\}
\end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Solve with Broyden

- As with standard Ramsey model from before we have:
  - Equation system with T equations ($\boldsymbol{H}$)
  - And T unknowns ($\boldsymbol{K}$)
- If we can calculate the jacobian of $\boldsymbol{H}$ w.r.t $\boldsymbol{K}$ we can solve with Broyden's method as before

How to compute Jacobian?

- How do we compute the Jacobian of the residuals $\boldsymbol{H}$ w.r.t unknowns $\boldsymbol{K}$?
  - Before: Compute Jacobian of entire model using num. diff
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- *Example*. Represent model in block form:

$$\boldsymbol{w}, \boldsymbol{r}^K = Firm(\boldsymbol{K}), \quad \boldsymbol{A}, \boldsymbol{r} = MutFund\left(\boldsymbol{K}, \boldsymbol{r}^K\right)$$

$$\boldsymbol{A}^{hh} = hh(\boldsymbol{r}, \boldsymbol{w}), \quad \boldsymbol{A} - \boldsymbol{A}^{hh} = \boldsymbol{H}\left(\boldsymbol{A}, \boldsymbol{A}^{hh}\right)$$

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- Let $\mathcal{J}^{y,x}$ be Jacobian of y w.r.t x. Then:

$$\begin{aligned}\boldsymbol{H}_{\boldsymbol{K}} \equiv \mathcal{J}^{A-A^{hh},K} &= \mathcal{J}^{A-A^{hh},A}\mathcal{J}^{A,K} \\ &+ \mathcal{J}^{A-A^{hh},A^{hh}}\mathcal{J}^{A^{hh},r}\mathcal{J}^{r,r^k}\mathcal{J}^{r^k,K} \\ &+ \mathcal{J}^{A_t-A^{hh},A^{hh}}\mathcal{J}^{A^{hh},w}\mathcal{J}^{w,K}\end{aligned}$$

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- How to get individual Jacobians?
  - Some are easy: For $\mathcal{J}^{w,K}, \mathcal{J}^{r^k,K}$ we just have to diff. $r_t^K = \alpha\Gamma_t(K_{t-1}/L_t)^{\alpha-1}, w_t = (1-\alpha)\Gamma_t(K_{t-1}/L_t)^{\alpha}$
    - Cheap, and can often be vectorized
  - What about HH Jacobians $\mathcal{J}^{A_{hh},r}, \mathcal{J}^{A_{hh},w}$?

Bottleneck: How do we find the Jacobian?

- **Naive approach:** For each input i into HH block $i \in \{r, w\}$
  - For each $s \in \{0, 1, ..., T-1\}$
    1. Shock input i in period s by small amount Δ
    2. Solve household problem backwards along transition path
    3. Simulate households forward along transition path
    4. Calculate column s, row t of jacobian as $\frac{\partial \mathcal{J}_t^{A_{hh},i}}{\partial i_s} = \frac{A_t^{hh} - A_{ss}^{hh}}{\Delta}$ for all t

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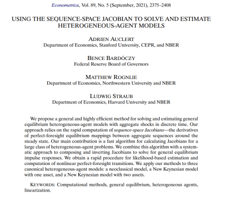
- Solution: **Fake news algorithm** - only need T steps! (later today)



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USING THE SEQUENCE-SPACE JACOBIAN TO SOLVE AND ESTIMATE HETEROGENEOUS-AGENT MODELS

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We propose a general and highly efficient method for solving and estimating general equilibrium heterogeneous-agent models with aggregate shocks in discrete time. Our approach relies on the rapid computation of *sequence-space Jacobians*—the derivatives of perfect-foresight equilibrium mappings between aggregate sequences around the steady state. Our main contribution is a fast algorithm for calculating Jacobians for a large class of heterogeneous-agent problems. We combine this algorithm with a systematic approach to composing and inverting Jacobians to solve for general equilibrium impulse responses. We obtain a rapid procedure for likelihood-based estimation and computation of nonlinear perfect-foresight transitions. We apply our methods to three canonical heterogeneous-agent models: a neoclassical model, a New Keynesian model with one asset, and a New Keynesian model with two assets.

KEYWORDS: Computational methods, general equilibrium, heterogeneous agents, linearization.

Summary

- Conditional on being able to compute HH jacobian efficiently we can compute **transition path** through following steps:
  1. Compute stationary state of model
  2. Formulate transition path as DAG
     - Reduce number of unknowns and residual equations
     - Not essential, but often good idea
  3. Compute Jacobian of residuals $\boldsymbol{H}$ w.r.t unknowns $\boldsymbol{K}$
  4. Formulate shock (i.e. TFP increases by 1% for 4 years)
  5. Use Broyden's method to solve for transition path

Assumptons and interpretation

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- **»Shock«, Γ:** A fully unexpected non-recurrent event $\equiv$ *MIT shock*
    - Unexpected before occuring at time 0
    - From time 0 and onwards agents have perfect foresight w.r.t transition dynamics
- **Transition path, $\boldsymbol{K}$:** Non-linear perfect foresight response to
    1. Initial distribution, $\underline{\boldsymbol{D}}_0 \neq \boldsymbol{D}_{ss}$or $K_0 \neq K_{ss}$ (convergence to steady state)
    2. Shock, $\Gamma_t \neq \Gamma_{ss}$ for some t (i.e. impulse-response)

The HANC example from GEModelToolsNotebooks

- **Presentation:** I go through the code

Interpreting the household Jacobians

- **Jacobian of consumption wrt. wage:** *What happens to consumption in period t when the wage (and thus income) increases in period s?*

$$\mathcal{J}^{C^{hh},w} = \begin{bmatrix} \frac{\partial C_0^{hh}}{\partial w_0} & \frac{\partial C_0^{hh}}{\partial w_1} & \cdots \\ \frac{\partial C_1^{hh}}{\partial w_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

- **Columns:** The full dynamic response to a unit shock in period s

Decomposition of GE response

- **GE transition path:** $\boldsymbol{r}^*$ and $\boldsymbol{w}^*$
- **PE response of each:**
    1. Set $(\boldsymbol{r}, \boldsymbol{w}) \in \{(\boldsymbol{r}^*, \boldsymbol{w}_{ss}), (\boldsymbol{r}_{ss}, \boldsymbol{w}^*)\}$
    2. Solve household problem backwards along transition path
    3. Simulate households forward along transition path
    4. Calculate outcomes of interest

Fake News Algorithm

Fake news algorithm

- **Household block:**

$$\boldsymbol{Y}^{hh} = hh(\boldsymbol{X}^{hh})$$

  - i.e. $\boldsymbol{Y}^{hh} = C^{hh}, A^{hh}$ and $\boldsymbol{X}^{hh} = w, r$

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  - Shock at time $s = 0$, solve + simulate HH block for T periods
  - Repeat until $s = T - 1$
  - Requires T^2 solution and simulation steps

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- **Next slides:** *Sketch of much faster approach*

Initial step

- Note that aggregate is (matrix) product of individual policy function $\boldsymbol{y}_t$ and distribution $\boldsymbol{D}_t$.
- Linearize (first-order Taylor) around ss:

$$\begin{aligned} \boldsymbol{Y}^{hh} &= (\boldsymbol{y}_t')\,\boldsymbol{D}_t \\ \Rightarrow \frac{d\boldsymbol{Y}^{hh}}{d\boldsymbol{X}^{hh}} &= \left(\frac{d\boldsymbol{y}_t}{d\boldsymbol{X}^{hh}}'\right)\boldsymbol{D}_{ss} + (\boldsymbol{y}_{ss}')\,\frac{d\boldsymbol{D}_t}{d\boldsymbol{X}^{hh}} \end{aligned}$$

- What can we say about policy function term $d\boldsymbol{y}_t$?

Pertubation of policy function

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$$\boldsymbol{y}_t^s = \begin{cases} \boldsymbol{y}_{ss} & t > s \\ \boldsymbol{y}_{t+j}^{s+j} & t \leq s \end{cases}$$

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- Verbally: The response of the policy function $\boldsymbol{y}$ at time t to a shock at s is the as the response at time $t+j$ to a shock at $s+j$
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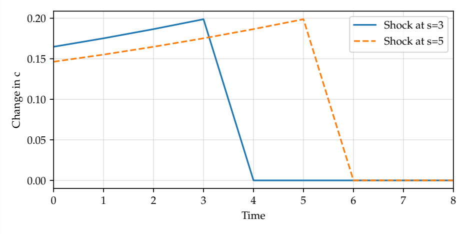
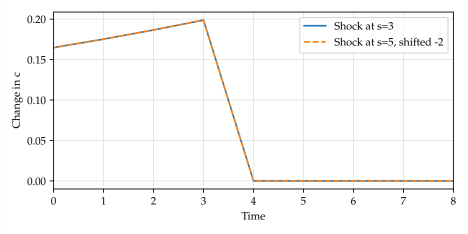
- **Implication**: We need to only do a single backwards iteration to a shock at $s = T-1$.
  - Can then construct change in policy function $d\boldsymbol{y}_t^s/d\boldsymbol{X}^{hh}$ for different s by shifting policy function around

Numerical illustration

Graphically. Response of c_t to income shock at $s = 3, 5$





Aggregate Jacobian

- Can we use same logic for aggreregate Jacobian, $\mathcal{J}_{t,s} = \mathcal{J}_{t-1,s-1}$?
  - No - the above is true for *policy* function, but not **distribution**
  - Distribution is backwards looking ($\boldsymbol{D}_t^s = (\boldsymbol{\Lambda}_t^s \Pi_{ss})' \, \boldsymbol{D}_{t-1}^s$) so number of periods t since announcement matters

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- Why »fake news«? $\mathcal{F}_{t,s}$ captures effect of announcing a date$-s$ shock at time 0, and retracting the annountment at date 1
  - Policy variables revert to steady state after period 1, but distribution changes since $d\boldsymbol{y}_0 \neq 0$

Fake News Matrix

- Can show that the fake news matrix can be computed as:

$$\mathcal{F}_{t,s} \equiv \begin{cases} \left(\frac{d\boldsymbol{y}_0^s}{d\boldsymbol{X}^{hh}}\right)' \boldsymbol{D}_{ss} & t = 0 \\ \left(\boldsymbol{y}_{ss}\right)' \left(\boldsymbol{\Lambda}'_{ss}\right)^t \frac{d\boldsymbol{D}_1^s}{d\boldsymbol{X}^{hh}} & t > 0 \end{cases}$$

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- $t = 0$ element: Easy to compute when we have $d\boldsymbol{y}_0^s/d\boldsymbol{X}^{hh}$
  - Can get this from a single backwards run (T periods) due to logic from before
- $t > 0$ elements: Only involves basic matrix multiplication once we have $d\boldsymbol{D}_1^s/d\boldsymbol{X}^{hh}$
  - Since we have derivatves of policy function for all t, s $d\boldsymbol{y}_t^s/d\boldsymbol{X}^{hh}$ can get $d\boldsymbol{D}_1^s/d\boldsymbol{X}^{hh}$ easily
  - Note: Not too expensive since histogram method for distribution is fast and efficient

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- Auclert et. al (2021) introduce an efficient alogirthm to compute aggregate jacobians for models with heterogeneous agents
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    - Can then use Quasi-Newton methods to solve dynamic GE model!
- GEModeltools does all of this »under the hood« when you compute HH Jacobians
    - You just tell GEModeltools the inputs and outputs of the household block
    - Entire algorithm is automated

Exercises

Exercises: HANCGovModel

Same model. Your choice of τ_{ss}. New questions:

1. **Define the transition path.**
2. **Plot the DAG**
3. **What do the Jacobians look like?**
4. **Find the transition path for** $G_t = G_{ss} + 0.01 G_{ss} 0.95^t$
5. **What explains household savings behavior?**
6. **What happens to consumption inequality?**

Summary

Summary and next week

- **Today:**
  1. The concept of a transition path
  2. Details of the GEModelTools package
- **Homework:** Work on completing the model extension exercise
- **Next week:** Begin working on Assignment 1