3. Stationary Equilibrium

Adv. Macro: Heterogenous Agent Models

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Introduction

Introduction

- **Last time:**
  1. Partial equilibrium
  2. No interactions
- **Today:** Interaction through markets
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Equilibrium-concept:** *Stationary equilibrium*
  1. What determines income and wealth inequality *in the long run*?
  2. What determines the real interest rate *in the long run*?
- **Code:** Based on the GEModelTools package
  1. Is in active development
  2. You can help to improve interface, find bugs and features

  Documentation: See GEModelToolsNotebooks
  - Many examples in repo, so look if you have issues
- **Literature:** Aiyagari (1994)

Ramsey-recap

The Ramsey model

- We will study the *stationary equilibrium* in the Heterogeneous Agent Neo-Classical (HANC) model
- Merges two well known models in the literature:
  - Standard Ramsey–Cass–Koopman model (*NC*)
    - What do we mean by Neo-classical?
  - One-asset Buffer-stock model (*HA*)
- Went through the Buffer-stock model over the last two lectures
- **Now**: Recap of the Ramsey model

Ramsey: Firms

- **Production function:** $Y_t = F(\Gamma_t, K_{t-1}, L_t)$ [note timing of capital] where Γ_t is technology
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1}, L_t} \Pi_t$

  1. Rental rate: $\frac{\partial \Pi_t}{\partial K_{t-1}} = 0 \Leftrightarrow r_t^K = F_K(\Gamma_t, K_{t-1}, L_t)$
  2. Real wage: $\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = F_L(\Gamma_t, K_{t-1}, L_t)$

  With CRS we get zero profits: $\Pi_t = 0 \Rightarrow Y_t = w_t L_t + r_t^K K_{t-1}$

  [functional income distribution]

Ramsey: Zero-profit mutual fund

- Introduce **mutual fund**
  - Takes savings A_{t-1} from households and investment them in available assets
  - In the Ramsey model: Only capital K_{t-1} but could also include gov. bonds, firm equity etc.
- **Capital depreciate** with rate $\delta \in (0, 1)$,

$$K_t = (1 - \delta)K_{t-1} + I_t$$

- **Deposits** (from households), A_{t-1}: The rate of return is

$$r_t = r_t^K - \delta$$

- **Balance sheet:**

$$A_{t-1} = K_{t-1}$$

Ramsey: Households

- **Utility maximization:**

$$v_0(A_{-1}^{hh}) = \max_{\{C_t^{hh}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t^{hh})$$

s.t.

$$C_t^{hh} + A_t^{hh} = (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh}$$

  Exogenous labor supply: $L_t^{hh} = 1$

- **Euler-equation** (implied by Lagrangian):

$$u'(C_t^{hh}) = \beta(1 + r_{t+1})u'(C_{t+1}^{hh})$$

Ramsey: Market Clearing

- **Capital market:** $K_t = A_t = A_t^{hh}$
- **Labor market:** $L_t = L_t^{hh} = 1$
- **Goods market:** $Y_t = C_t^{hh} + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears. Start from

$$
\begin{aligned}
C_t^{hh} + A_t^{hh} &= (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh}) \\
\Leftrightarrow C_t^{hh} + I_t &= \left[(1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - A_t^{hh})\right] + (K_t - (1 - \delta)K_{t-1}) \\
&= [(1 + r_t)K_{t-1} + w_t L_t - K_t)] + (K_t - (1 - \delta)K_{t-1}) \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

- Note: Means that we can check if we have solved the numerical model correctly by:
  - Impose two of the market clearing conditions
  - Then check the third market clearing condition (should be zero)

Ramsey: Summary

- **Simplified form:**

$$\begin{aligned} u'(C_t^{hh}) &= \beta(1 + F_K(\Gamma_t, K_t, 1) - \delta)u'(C_{t+1}^{hh}) \\ K_t &= (1-\delta)K_{t-1} + F(\Gamma_t, K_{t-1}, 1) - C_t^{hh} \end{aligned}$$

- **Extended form:**

$$\begin{aligned} r_t^K &= F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t &= F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t &= r_t^K - \delta \\ A_t &= K_t \\ A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \\ u'(C_t^{hh}) &= \beta(1+r_{t+1})u'(C_{t+1}^{hh}) \\ A_t &= A_t^{hh} \\ L_t &= L_t^{hh} \end{aligned}$$

Ramsey: As an equation system

Eqs. system with unknowns $\left\{K_t, L_t, r_t^K, w_t, r_t, A_t, A_t^{hh}, C_t^{hh}\right\}_{t=0}^{\infty}$ and eqs:

$$\begin{bmatrix} r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t^{hh} - \left((1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\ u'(C_t^{hh}) - \beta(1+r_{t+1})u'(C_{t+1}^{hh}) \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0,1,\dots\}, \text{given } K_{-1} \end{bmatrix} = \mathbf{0}$$

Ramsey: Steady state

- **Euler-equation** can be solved for K_{ss}:

$$
\begin{aligned}
u'(C_{ss}) = \beta(1 + F_K(\Gamma_{ss}, K_{ss}, 1) - \delta)u'(C_{ss}) \Leftrightarrow \\
F_K(K_{ss}, 1) = \frac{1}{\beta} - 1 + \delta
\end{aligned}
$$

- **Accumulation equation + goods mkt. clearing** then implies C_{ss}:

$$
\begin{aligned}
K_{ss} &= (1-\delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - C_{ss} \Leftrightarrow \\
C_{ss} &= (1-\delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - K_{ss}
\end{aligned}
$$

HANC

HANC model overview

- **Model blocks:**
  1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
  2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
  3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
  4. **Markets:** Perfect competition in labor, goods and capital markets
- **Add-on to Ramsey-Cass-Koopman:** *Heterogeneous households*
- **Other names:**
  1. The Aiyagari-model
  2. The Aiyagari-Bewley-Hugget-Imrohoroglu-model
  3. The Standard Incomplete Market (SIM) model

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

$$\text{s.t.}$$

$$\ell_{it} = z_{it}$$

$$a_{it} = (1 + r_t) a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1$$

$$a_{it} \geq 0$$

- **Where does heterogeneity enter?**

- **Incomplete markets due to borrowing constraint**
  (fancy words: partial self-insurrance, lack of Arrow-Debreu securities)

Recursive formulation

- **Value function** (at decision)

$$v_t(\beta_i, z_{it}, a_{it-1}) = \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(\beta_i, z_{it}, a_{it})$$

$$\text{s.t.}$$

$$\begin{aligned} \ell_{it} &= z_{it} \\ a_{it} &= (1 + r_t)a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t \\ \log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1} \\ a_{it} &\geq 0 \end{aligned}$$

- **Beginning-of-period value function** (before shock realization):

$$\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E}\left[v_t(\beta_i, z_{it}, a_{it-1}) \,|\, \beta_i, z_{it-1}, a_{it-1}\right]$$

Distributions and aggregates

- Household policy function x^* where $x \in \{a, c, \ell\}$ function of:
  - Individuals states $(\beta_i, z_{it}, a_{it-1})$
  - Aggregates (w_t, Π_t, r_t)
- Aggregate policy:

$$X_t^{hh}\left(\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}\right) = \int x_t^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}) d\boldsymbol{D}_t$$

- When aggregating we **integrate** out individual states
  - Aggregate X_t^{hh} is only a function of $\{r_\tau, w_\tau, \Pi_\tau\}_{\tau \geq t}$ in GE as long as exogenous states don't change
- $\Rightarrow$ If we know aggregates (w_t, Π_t, r_t) can calculate aggregate household behavior (consumption or savings)

Equation system

$$\begin{bmatrix} r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ A_t^{hh} - \int a_t d\boldsymbol{D}_t \\ L_t^{hh} - \int \ell_t d\boldsymbol{D}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t' \Pi_z' \underline{\boldsymbol{D}}_t \\ a_t - a_t^* \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \mathbf{0}$$

- **Note**: Much larger system compared to Ramsey due to last 2 eqs.
  - $\boldsymbol{D}_t, a_t^*$ define mass and optimal savings policy at **the individual level**
  - Standard Ramsey model: 8 eqs. per period
  - HANC with $N_\beta = 3, N_z = 7, N_a = 300$:
    $8 + 2 \times 3 \times 7 \times 300 = 12.608$ per period

Market clearing

- **Capital market:** $K_t = A_t = \int a_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t$
- **Labor market:** $L_t = \int \ell_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t = \int z_{it} d\boldsymbol{D}_t = 1$
- **Goods market:** $Y_t = \int c_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears

$$
\begin{aligned}
C_t^{hh} + I_t &= \int c_{it}^* d\boldsymbol{D}_t + [K_t - (1-\delta)K_{t-1}] \\
&= \int \left[(1+r_t)a_{it-1} + w_t z_{it} - a_{it}\right] d\boldsymbol{D}_t \\
&= [(1+r_t)K_{t-1} + w_t L_t - K_t] + [K_t - (1-\delta)K_{t-1}] \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

Stationary Equilibrium

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$\begin{bmatrix} r_{ss}^K - F_K(\Gamma_{ss}, K_{ss}, L_{ss}) \\ w_{ss} - F_L(\Gamma_{sst}, K_{ss}, L_{ss}) \\ r_{ss} - (r_{ss}^K - \delta) \\ A_{ss} - K_{ss} \\ A_{ss} - A_{ss}^{hh} \\ L_{ss} - L_{ss}^{hh} \\ A_{ss}^{hh} - \int a_{ss} d\boldsymbol{D}_{ss} \\ L_{ss}^{hh} - \int \ell_{ss} d\boldsymbol{D}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda_{ss}' \Pi_z' \underline{\boldsymbol{D}}_{ss} \\ a_{ss} - a_{ss}^* \end{bmatrix} = \mathbf{0}$$

Note : Households still move around »inside« the distribution due to idiosyncratic shocks. Does not affect aggregates due to »law of large numbers«

Stationary equilibrium - more verbal definition

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss} (always $\Pi_{ss} = 0$),
3. the distribution $\boldsymbol{D}_{ss}$ over β_i, z_{it} and a_{it-1}
4. and the policy functions a_{ss}^*, ℓ_{ss}^* and c_{ss}^*

are such that

1. Households maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. $\boldsymbol{D}_{ss}$ is the invariant distribution implied by the household problem
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

Direct implementation (K guess)

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$ **Root-finding problem** in K_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $r_{ss} = \alpha \Gamma_{ss} (K_{ss})^{\alpha - 1} - \delta$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss}$

Note: $\boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss} = \sum_i a_{i,ss}^{*} D_i$

Direct implementation (r guess)

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$ **Root-finding problem** in r_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $K_{ss} = \left(\frac{r_{ss}+\delta}{\alpha\Gamma_{ss}}\right)^{\frac{1}{\alpha-1}}$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$

Indirect implementation

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$ **Consider** Γ_{ss} **and** δ **as »free« parameters:**

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Set $K_{ss} = \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)(K_{ss})^{\alpha}}$
7. Set $r_{ss}^{K} = \alpha \Gamma_{ss} (K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^{k} - r_{ss}$

Code

Calibration

- **Preferences:** $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$
    1. Discount factors: $\beta \in \{0.965, 0.975, 0.985\}$ in equal pop. shares
    2. Relative risk aversion: $\sigma = 2$
- **Income:**
    1. AR(1): $\rho_z = 0.95$
    2. Std.: $\sigma_\psi = 0.30\sqrt{(1-\rho_z^2)}$
- **Technology:** $F(K, L) = \Gamma K^\alpha L^{1-\alpha}$
    1. Capital share: $\alpha = 0.36$
    2. TFP: $\Gamma_{ss} = 1.082$
    3. Depreciation: $\delta = 0.193$
- **Steady state:**
    1. Prices: $r_{ss} = 0.01$ and $w_{ss} = 1$
    2. Quantities: $K_{ss}/Y_{ss} = 1.776$

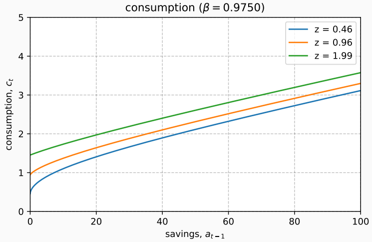
$\Rightarrow$ **Code example in repo**

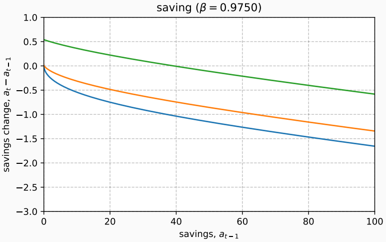
Consumption function

- **Euler-equation still necessary for $a_{it} > 0$:**

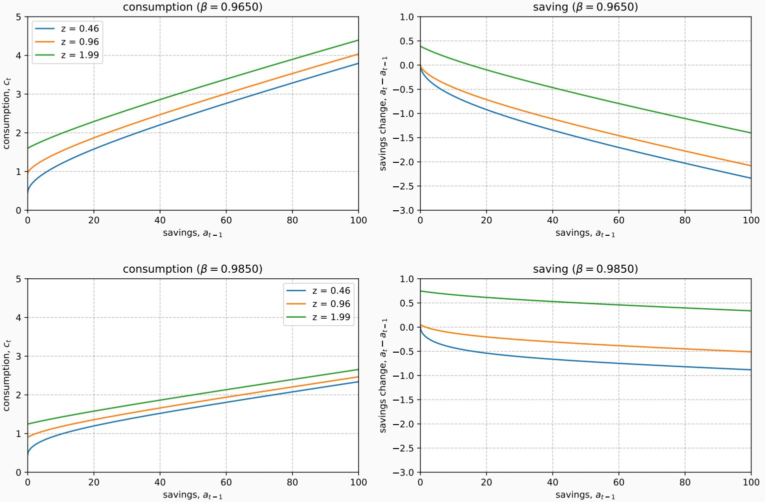
$$c_{it}^{-\sigma} = \beta_i (1 + r_{t+1}) \mathbb{E}_t \left[c_{it+1}^{-\sigma} \right]$$

- **Precautionary saving:**
  1. Low consumption for low cash-on-hand → *buffer-stock target*
  2. Steep slope for low cash-on-hand → *high MPC*



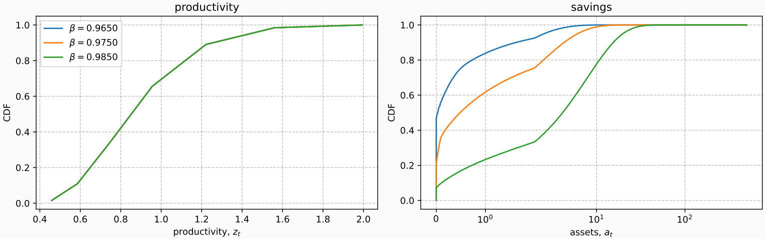


Low vs. high β_i



Distribution, D_t

- **Productivity:** Marginal distribution over only z_{it}
- **Savings:** Marginal distribution over a_{it} cond. on β_i



- **Drivers of wealth inequality:**
  1. Stochastic income
  2. Heterogeneous patience $\rightarrow$ savings behavior

Steady state interest rate

- **Representative agent / complete markets:** Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1 + r_{ss})C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1 + r_{ss}}$$

- **Heterogeneous agents:** *No such equation exists*
  1. Euler-equation replaced by asset market clearing condition
  2. Idiosyncratic income risk affects the steady state interest rate

σ_ψ	PE ($r_{ss} = 1\%$), A^{hh}	GE, r_{ss}	GE, A^{hh}
0.09	2.78	1.00%	2.78
0.14	7.39	0.12%	2.97
0.19	13.68	-1.11%	3.30

Partial

Equilibrium: Same interest rate. **General Equilibrium**: Capital+labor market clearing.

Calibration

How to choose parameters?

- **External calibration:** Set subset of parameters to the *standard values in the literature* or *directly from data estimates* (e.g. income process)
- **Internal calibration:** Set remaining parameters so the model fit a number of chosen *macro-level and/or micro-level targets* based on empirical estimates
  1. **Informal:** Roughly match targets by hand
  2. **Formal:** 2a. Solve root-finding problem 2b. Minimize a squared loss function
  3. **Estimation:** Formal with squared loss function (think GMM) or likelihood function + standard errors
- **Complication:** *We must always solve for the steady state for each guess of the parameters to be calibrated*

Exercises

Exercise: HANCGovModel

- **No production.** No physical savings instrument
- **Households:** Get stochastic endowment z_{it} of consumption good
- **Government:**
  1. Choose government spending
  2. Collect taxes, τ_t, proportional to endowment
  3. Bonds: Pays 1 consumption good next period. Price is $p_t^B < 1$

$$p_t^B B_t + \int \tau_t z_{it} d\boldsymbol{D}_t = B_{t-1} + G_t$$

$$\tau_t = \tau_{ss} + \eta_t + \varphi \left(B_{t-1} - B_{ss} \right)$$

  where η_t is a tax-shifter

- **Market clearing:**

$$B_t = A_t^{hh}$$

$$C_t^{hh} + G_t = \int z_{it} d\boldsymbol{D}_t = 1$$

Households:

$$v_t(z_{it}, a_{it-1}) = \max_{c_{it}} \frac{c_{it}^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[v_{it+1}(z_{it+1}, a_{it}) \right]$$

$$\text{s.t. } p_t^B a_{it} + c_{it} = a_{it-1} + (1-\tau_t) z_{it} \geq 0$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1} \;\; , \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \; \mathbb{E}[z_{it}] = 1$$

Euler-equation:

$$c_{it}^{-\sigma} = \beta \frac{\underline{v}_{a,t+1}(z_{it}, a_{it})}{p_t^B}$$

Envelope condition:

$$\underline{v}_{a,t}(z_{it-1}, a_{it-1}) = c_{it}^{-\sigma}$$

Exercise: Questions

1. **Define the stationary equilibrium**
2. **Solve and simulate the household problem** with $p^B_{ss} = 0.975$ and $\tau_{ss} = 0.12$.
3. **Find the stationary equilibrium** with $G_{ss} = 0.10$ and $\tau_{ss} = 0.12$.
4. **What happens for** $\tau_{ss} \in (0.11, 0.15)$?
5. **When is average household utility maximized?**

Note: Full solution in repository folder *GEModelToolsNotebooks/HANCGovModel*

Summary

Summary and next week

- **Today:**
  1. The concept of a stationary equilibrium
  2. Introduction to the GEModelTools package
- **Next week:** *Work on assignment*
- **Exercise/Homework:**
  1. Work on completing the HANCGovModel exercise
  2. Go through *Stationary-Equilibrium* notebook in repository
  3. Assignment