1. Introduction

Adv. Macro: Heterogenous Agent Models

Nicolai Waldstrøm

2024

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  1. What explains the level and dynamics of heterogeneity/inequality?
  2. What role does heterogeneity play for understanding consumption-saving dynamics in partial equilibrium?
  3. What role does heterogeneity play for understanding business-cycle fluctuations in general equilibrium?

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- **Central technical method:** Programming in Python
  Prerequisite: *Intro. to Programming and Numerical Analysis*
  Complicated: *Close to the research frontier*

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  Complicated: *Close to the research frontier*
- **Plan for today:**
  1. More about the course
  2. Consumption-saving models
  3. Numerical dynamic programming

- **Model components:**
  1. Optimizing individual agents (households $+$ firms)
  2. Idiosyncratic and aggregate risk (*uncertainty*)
  3. Information flows (who knows what when $\Rightarrow$ often everything)
  4. Market clearing

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- **HANK:** Heterogeneous Agent *New Keynesian* model
  (i.e. include price and wage setting frictions)

Teaching method

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  3. Examples of code for central mechanisms
     (you should run the notebook codes simultaneously)

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  Web: sites.google.com/view/numeconcph-advmacrohet/

  Git: github.com/numeconcopenhagen/adv-macro-het

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  Git: github.com/numeconcopenhagen/adv-macro-het
- **Code:**
  1. We provide code you will build upon
  2. Based on the GEModelTools package

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- **Exam:**
    1. Hand-in $3\times$**assignments**
    2. **36 hour take-home:** Programming of new extension
       $+$ analysis of model $+$ interpretation of results

Python

1. **Assumed knowledge:** From Introduction to Programming and Numerical Analysis you are assumed to know the basics of
   1.1 Python
   1.2 VSCode
   1.3 git
2. **Updated Python:** Install (or re-install) newest Anaconda
3. **Packages:** `pip install quantecon, EconModel, consav`
4. **GEMoodel tools:**
   4.1 Clone the `GEModelTools` repository
   4.2 Locate repository in command prompt
   4.3 Run `pip install -e .`

Course plan

See CoursePlan.pdf in repository

Knowledge

1. Account for, formulate and interpret precautionary saving models
2. Account for stochastic and non-stochastic simulation methods
3. Account for, formulate and interpret general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions
4. Discuss the difference between the stationary equilibrium, the transition path and the dynamic equilibrium
5. Discuss the relationship between various equilibrium concepts and their solution methods
6. Identify and account for methods for analyzing the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Skills

1. Solve precautionary saving problems with dynamic programming and simulate behavior with stochastic and non-stochastic techniques
2. Solve general equilibrium models with ex ante and ex post heterogeneity, idiosyncratic and aggregate risk, and with and without pricing frictions (stationary equilibrium, transition path, dynamic equilibrium)
3. Analyze dynamics of income and wealth inequality
4. Analyze transitional and permanent structural changes (e.g. inequality trends and the long-run decline in the interest rate)
5. Analyze the dynamic distributional effects of long-run policy (e.g. taxation and social security) and short-run policy (e.g. monetary and fiscal policy)

Competencies

1. Independently formulate, discuss and assess research on both the causes and effects of heterogeneity and risk for both long-run and short-run outcomes
2. Discuss and assess the importance of how heterogeneity and risk is modeled for questions about both long-run and short-run dynamics

History of heterogeneous agent macro

1. Heathcote et al. (2009), »Quantitative Macroeconomics with Heterogeneous Households«
2. Kaplan and Violante (2018), »Microeconomic Heterogeneity and Macroeconomic Shocks«
3. Cherrier et al. (2023), »Household Heterogeneity in Macroeconomic Models: A Historical Perspective«

Consumption-Saving

Generations of models

1. Permanent income hypothesis (Friedman, 1957) or life-cycle model (Modigliani and Brumburg, 1954)
2. Buffer-stock consumption model (Deaton, 1991, 1992; Carroll, 1992, 1997)
3. Multiple-asset buffer-stock consumption models (e.g. Kaplan and Violante (2014))

Consumption-saving

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

s.t.

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**
  Consumption: c_t
  Productivity: z_t
  End-of-period savings: a_t (*no debt at death*)
- **Parameters:**
  Discount factor: β
  Wage: w
  Interest rate: r (define $R \equiv 1 + r$ as interest factor)

It is a *static* problem

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- **It is a *static* problem:**
    1. **Information:** z_t is known for all t at $t = 0$
    2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
    3. **Behavior:** Choose $c_0, c_1, \ldots, c_{T-1}$ *simultaneously*
    4. **Solution:** Sequence of consumption *choices* $c_0^*, c_1^*, \ldots, c_{T-1}^*$

IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$
\begin{aligned}
a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\
&= R^2 a_{T-3} + Rwz_{T-2} - Rc_{T-2} + wz_{T-1} - c_{T-1} \\
&= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t}(wz_t - c_t)
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- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$
R^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0
$$

where $s_0 \equiv Ra_{-1}$ (after-interest assets)
and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

FOC and Euler-equation

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t u'(c_t) + \lambda (1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda \left(\beta R\right)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda \left(\beta R\right)^{-t}}{-\lambda \left(\beta R\right)^{-(t+k)}} = \left(\beta R\right)^k$$

- Equates Marginal Rate of Substitution (MRS) with relative price of postponing consumption k periods

Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned} \sum_{t=0}^{T-1} R^{-t}(\beta R)^{t/\sigma} c_0 &= s_0 + h_0 \Leftrightarrow \\ c_0^* &= \frac{1-(\beta R)^{1/\sigma} R^{-1}}{1-\left((\beta R)^{1/\sigma} R^{-1}\right)^T}(s_0 + h_0) \end{aligned}$$

- **Finite horizon** solution to the consumption-saving problem

Infinite horizon

- Infinite horizon. Assume log utility, $\sigma = 1$. For $\beta < 1$: Let $T \to \infty$ to get solution to consumer propblem at time 0 :

$$c_0^* = (1 - \beta)(s_0 + h_0)$$

- Consume a constant fraction $1 - \beta$ out of initial wealth + lifetime human capital

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$$\frac{\partial c_0}{\partial s_0} = 1 - \beta$$

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- Note from euler equation $\frac{u'(c_t)}{u'(c_{t+1})} = \beta R$ so a steady state will feature $\beta R = 1 \Leftrightarrow 1 - \beta = \frac{R-1}{R} \Leftrightarrow 1 - \beta = \frac{r}{1+r} \approx r$

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- Standard model with **no borrowing constraints or uncertainty features a small MPC**

Uncertainty and always borrowing constraint

$$
\begin{aligned}
v_0(z_0, a_{-1}) &= \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \\
&\text{s.t.} \\
a_t &= (1+r)a_{t-1} + wz_t - c_t \\
z_{t+1} &\sim \mathcal{Z}(z_t) \\
a_t &\geq \underline{a} \\
\lim_{t\to\infty} (1+r)^{-t} a_t &\geq 0 \quad \text{[No-Ponzi game]}
\end{aligned}
$$

- **Stochastic income** from 1st order Markov-process, $\mathcal{Z}$
- ***A true dynamic problem:***
  1. **Information:** z_t is revealed period-by-period
  2. **Target:** *Expected* discounted utility, $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
  3. **Behavior:** Choose c_t *sequentially* as information is revealed
  4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t\left[u'(c_{t+1})\right]$:
  Increase c_t by marginal $\Delta > 0$, and lower c_{t+1} by $R\Delta$
  1. **Feasible:** Yes, if unconstrained $a_t > \underline{a}$
  2. **Utility change:** $u'(c_t) + \beta\,(-R)\,\mathbb{E}_t\left[u'(c_{t+1})\right] > 0$

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- **Case II:** If $u'(c_t) < \beta R \mathbb{E}_t [u'(c_{t+1})]$:
  Lower c_t by marginal $\Delta > 0$, and increase c_{t+1} by $R\Delta$
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- **Conclusion:** By contradiction
  1. **Constrained:** $a_t = \underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t \left[u'(c_{t+1}) \right]$, or
  2. **Unconstrained:** $a_t > \underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t \left[u'(c_{t+1}) \right]$

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- Note: Can also derive using Lagrangian/Karush–Kuhn–Tucker conditions

Further resources

1. Lecture notes by Christopher Carroll
2. Lecture notes by Pierre-Olivier Gourinchas
3. The Economics of Consumption, Jappelli and Pistaferri (2017)
4. »Liquidity constraints and precautionary saving«
   Carroll, Holm, Kimball (JET, 2021)

Dynamic Programming

Dynamic solution: Bellman's Principle of Optimality

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- **In math:**
  1. Instead of looking at entire lifetime utility stream $\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$, use *recursive* from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta\mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

where v_t is the **value function**

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  2. **Policy function, c_t^*:** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta\mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
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Vocabulary

$$\begin{aligned} v_t(z_t, a_{t-1}) &= \max_{c_t} u(c_t) + \beta \mathbb{E}_t [v_{t+1}(z_{t+1}, a_t)] \\ \text{s.t. } a_t &= (1+r)a_{t-1} + wz_t - c_t \geq \underline{a} \end{aligned}$$

1. **State variables:** z_t and a_{t-1}
2. **Control variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t [v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r, w, and stuff in $u(\bullet)$

Note: Straightforward to extend to more goods, more assets or other states, more complex uncertainty, bounded rationality etc.

Infinite horizon: $T \to \infty$?

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  - For instance: Consume everything in final period

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- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy functions converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough T*

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- **Maximum upper limit for β:** $\frac{1}{1+r}$

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- **Maximum upper limit for β:** $\frac{1}{1+r}$
- **In practice:**
  1. Make arbitrary initial guess (e.g. $v_{t+1} = 0$)
  2. Solve backwards until value and policy functions does not change anymore (given some tolerance)

Timing of shocks

- **Realization of shocks:** First in the period before choices are made

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- **Beginning-of-period value function** (before realization)**:**

$$\underline{v}_t(z_{t-1}, a_{t-1}) = \mathbb{E}_{t-1}\left[v_t(z_t, a_{t-1})\right]$$

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- **End-of-period value function** (after realization)**:**

$$\begin{aligned} v_t(z_t, a_{t-1}) = \max_{c_t} & \; u(c_t) + \beta \underline{v}_{t+1}(z_t, a_t) \\ \text{s.t. } & \; a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a} \end{aligned}$$

Discretization

- Income z_t and savings a_t typically **continuous** variables. How to handle on computer?

Discretization

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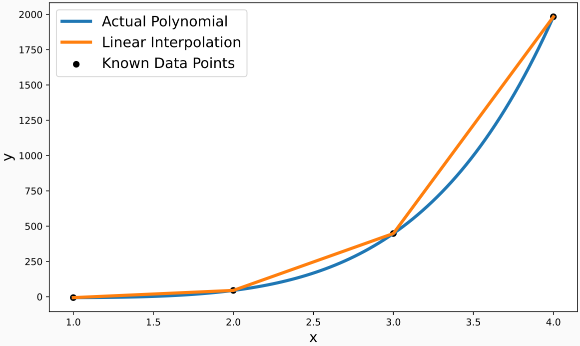
- How to compute $\underline{v}_{t+1}(z_t, a_t^*)$? $\Rightarrow$ **interpolation**

Linear interpolation

- **Linear interpolation**
- Approximate $y = f(x)$ using linear approximation between known points



Linear interpolation

- Linear interpolation in math
  1. Assume $\underline{v}_{t+1}$ is known on grids $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
  2. Want to evaluate $\underline{v}_{t+1}(z^{i_z}, a)$ for arbitrary a
  3. Find place in grid $\mathcal{G}_a$ where $a^l < a < a^{l+1}$
  4. Compute interpolation:

$$
\begin{aligned}
\underline{\breve{v}}_{t+1}(z^{i_z}, a) &= \underline{v}_{t+1}(z^{i_z}, a^l) + \omega(a - a^l) \\
\omega &\equiv \frac{v_{t+1}(z^{i_z}, a^{l+1}) - v_{t+1}(z^{i_z}, a^l)}{a^{l+1} - a^l} \\
l &\equiv \text{largest } i_a \in \{0, 1, \ldots, \#_a - 2\} \text{ such that } a^{i_a} \leq a
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Discretization of income process

- Assume that idiosyncratic income z_t follows an AR(1) process:

$$\log z_t = \rho_z \log z_{t-1} + \psi_t, \;\; \psi_t \sim \mathcal{N}(0, \sigma_\psi^2)$$

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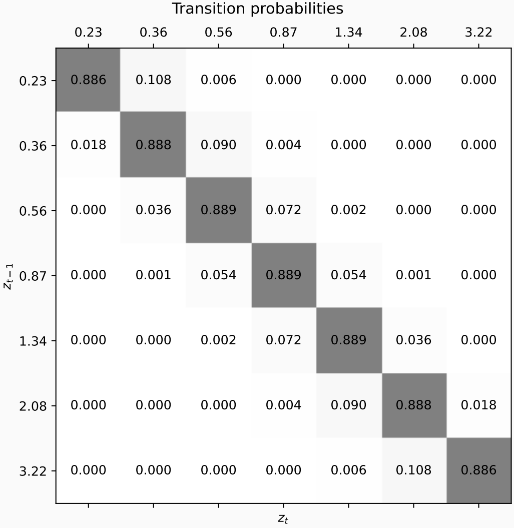
- Households move between states (points in $\mathcal{G}_z$) with transition probability $\pi_{j,i} = \Pr[z_t = z^i \mid z_{t-1} = z^j]$

Transition probability matrix



Value function iteration (VFI)

- **Beginning-of-period value function:**

$$\underline{v}_t(z^{i_{z-}}, a^{i_{a-}}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_{z-},i_z} v_t(z^{i_z}, a^{i_{a-}})$$

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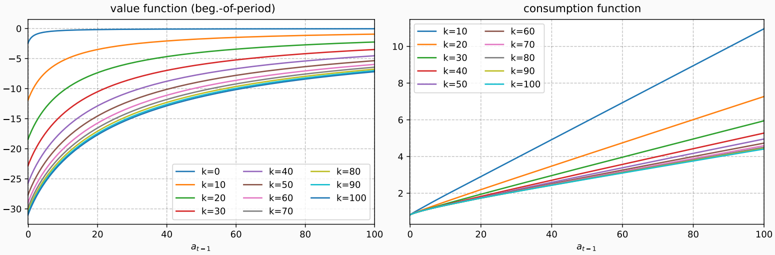
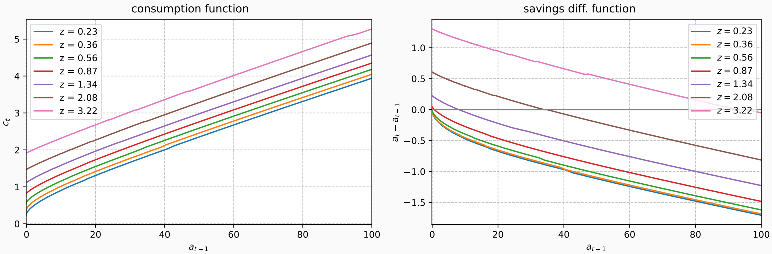
- **Outer loop:** Backwards from $t = T - 1$ (note $\underline{v}_T = 0$, or known)

Convergence $(t = T - 1 - k)$



with $z_t = 0.87$

Converged policy functions



Precautionary saving:

- Consumption function *concave*
- Savings drift imply buffer-stock target
  - Impatience vs. precautionary saving

Numerical Monte Carlo simulation

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- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness $\Rightarrow$ Need large N to avoid noise

Numerical histogram simulation - general idea

- Alternative to Monte Carlo simulation that avoids stochasticity: **histogram method**
- End goal: obtain discretized distribution directly on the grids $\mathcal{G}_z \times \mathcal{G}_a$

Numerical histogram simulation - general idea

- Alternative to Monte Carlo simulation that avoids stochasticity: **histogram method**
- End goal: obtain discretized distribution directly on the grids $\mathcal{G}_z \times \mathcal{G}_a$
- If households only make choices on grid (i.e. no continuous choice) then obtain distribution as follows:
  - **Initial distribution:** Choose $\underline{\boldsymbol{D}}_0(z_{-1}, a_{-1})$, which is defined on $\mathcal{G}_z \times \mathcal{G}_a$ and sum to $1 \equiv$ *histogram*
  - **Simulation:** Forwards in time from $t = 0$ and in each time period
    1. **Distribute stochastic mass:** For each i_z and i_{a-} calculate $\boldsymbol{D}_t(z^{i_z}, a^{i_{a-}}) = \sum_{i_{z-}=0}^{\#_z-1} \pi_{i_{z-},i_z} \underline{\boldsymbol{D}}_t(z^{i_{z-}}, a^{i_{a-}})$
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    4. Find $l \equiv i_a \in \{0, 1, \ldots, \#_a - 2\}$ such that $a_t^*(z^{i_z}, a^{i_{a-}}) = a^l$ (on grid assumption)
    5. Increment $\underline{\boldsymbol{D}}_{t+1}(z^{i_z}, a^l)$ with $\boldsymbol{D}_t(z^{i_z}, a^{i_{a-}})$

Numerical histogram simulation I

- If households may choose savings policy $a_t^*(z^{i_z}, a^{i_a})$ which is not on the grid $\mathcal{G}_a$ how do we distribute mass across the grid?
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- **Review:**
  1. **Pro:** Computationally efficient and no randomness
  2. **Con:** Introduces a non-continuous distribution

Small example

- **Grids:** $\mathcal{G}_z = \{\underline{z}, \overline{z}\}$ and $\mathcal{G}_a = \{0, 1\}$
- **Transition matrix:** $\pi_{0,0} = \pi_{1,1} = 0.5$
- **Policy function:**
  - Low income: $a^*(\underline{z}, 0) = a^*(\underline{z}, 1) = 0$
  - High income: Let $a^*(\overline{z}, 0) = 0.5$ and $a^*(\overline{z}, 1) = 1$
- **Initial distribution**: $\underline{\boldsymbol{D}}_0(z_{it}, a_{it-1}) = \begin{cases} 1 & \text{if } z_{it} = \underline{z} \text{ and } a_{it} = 0 \\ 0 & \text{else} \end{cases}$
- **Task:** Calculate by hand the transitions to

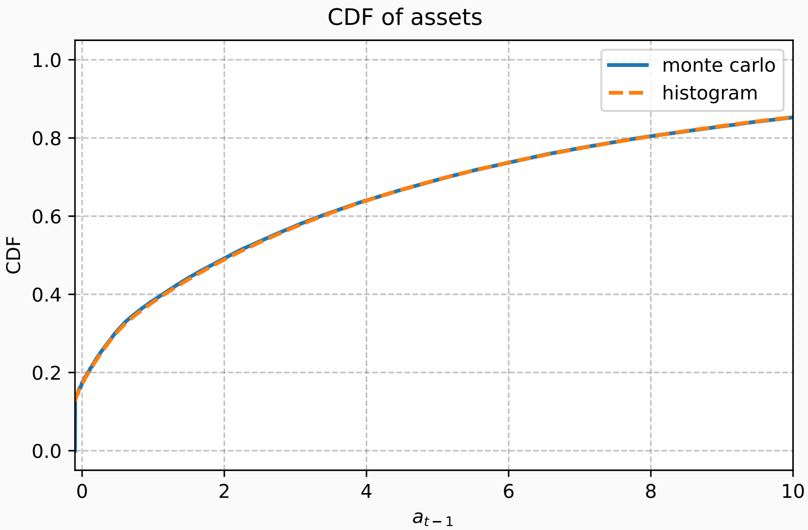
$$\boldsymbol{D}_0,\ \underline{\boldsymbol{D}}_1,\ \boldsymbol{D}_1, \ldots$$

See simple simple_histogram_simulation.xlsx

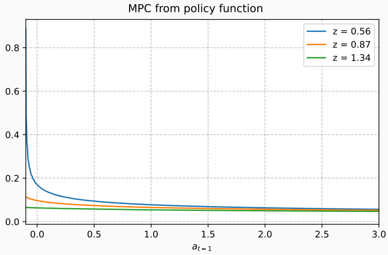
Infinite horizon: $T \to \infty$?

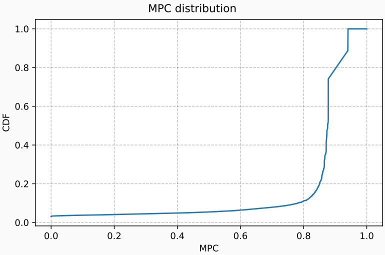
- **Initial guess:** Can be arbitrary.
  1. Everyone in one grid point, or
  2. Ergodic distribution of z_{it} and everyone has zero savings,
- **Convergence:** Simulate forward until the distribution does not change anymore (given some tolerance)

Converged CDF of savings



MPCs


MPC from policy function
z = 0.56
z = 0.87
z = 1.34
0.0
0.2
0.4
0.6
0.8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
a_{t-1}


MPC distribution
CDF
0.0
0.2
0.4
0.6
0.8
1.0
MPC

Side-note: Matrix formulation

- The histogram method can be written in **matrix form**:

$$\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$$
$$\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$$

  where

  $\underline{\boldsymbol{D}}_t$ is vector of length $\#_z \times \#_a$

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  Π_z' is derived from the π_{i_{z-},i_z}'s

  Λ_t' is derived from the l's and ω's

- **Note:** Example shown in notebook
- **Further details:** Young (2010), Tan (2020), Ocampo and Robinson (2022)

EGM

Characteristics of VFI

- Value function iteration (VFI) is the standard method for solving dynamic programs
  - Robust
  - Easy to implement for a wide class of models
  - Well known convergence properties (c.f. contraction mapping theorem)

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- Solution: **Endogenous grid-point method**

Endogenous grid-point method (EGM)

Alternative to VFI using Euler, i.e. $c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$:

1. Calculate **post-decision marginal value of cash:**

$$q(z^{i_z}, a^{i_a}) = \sum_{i_{z+}=0}^{\#_z-1} \pi_{i_z, i_{z+}} c_+(z^{i_{z+}}, a^{i_a})^{-\sigma}$$

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4. **Consumption function:** Calculate $m = (1+r)a^{i_{a-}} + wz^{i_z}$
   If $m \leq m(z^{i_z}, a^0)$ constraint binds: $c^*(z^{i_z}, a^{i_{a-}}) = m + \underline{a}$
   Else: $c^*(z^{i_z}, a^{i_{a-}}) =$ interpolate $m(z^{i_z}, :)$ to $c(z^{i_z}, :)$ at m

Practice

In practice

- **EconModel:** Go through notebook 01. Using the EconModelClass (except part on C++)
- **ConSav:** Look at the 04. Tools folder.
- **Todays notebook:** *Consumption-Saving Model* show implementation of solution and simulation methods.

Summary

Summary and next week

- **Today:**
  1. Introduction to course
  2. Consumption-saving models
  3. Numerical dynamic programming
- **Next week:** More on consumption-saving models
- **Homework:**
  1. Ensure that your Python installation is working, and that you can use ConSav, GEModelTools
  2. Familiarize your self with today's code and the basic concepts of dynamic programming