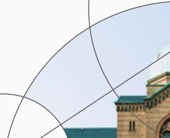
8. The New Keynesian Model

Adv. Macro: Heterogenous Agent Models

Nicolai Waldstrøm

2024



Introduction

Introduction

- **Previously:**
  1. Economic content: Long run trends and outcomes
  2. Methods: Stationary eq., Non-linear transition path and perfect foresight

Introduction

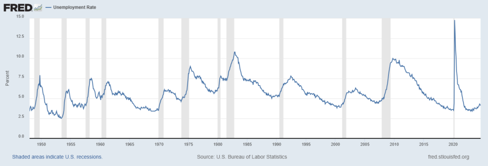
- **Previously:**
  1. Economic content: Long run trends and outcomes
  2. Methods: Stationary eq., Non-linear transition path and perfect foresight
- **Today:**
  1. Business cycles in the New Keynesian model
  2. Linearized solution in models with aggregate risk

Introduction

- **Previously:**
  1. Economic content: Long run trends and outcomes
  2. Methods: Stationary eq., Non-linear transition path and perfect foresight
- **Today:**
  1. Business cycles in the New Keynesian model
  2. Linearized solution in models with aggregate risk
- **Literature:**
  - NK:
    1. Gali textbook ch. 3-4
    2. *Macroeconomics* textbook ch. 16
  - Solution methods:
    1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
    2. Boppart et. al. (2018), »Exploiting MIT shocks in heterogeneous-agent economies: The impulse response as a numerical derivative«
    3. Documentation for GEModelTools

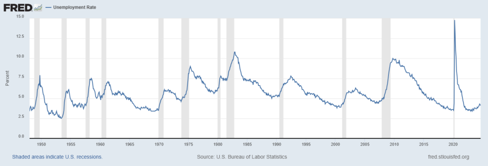
Business cycles

- Macro variables relatievly volatile around long-run trends



Business cycles

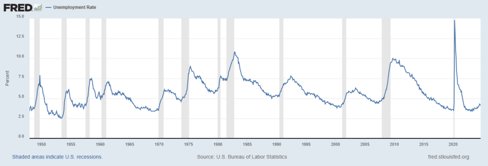
- Macro variables relatievly volatile around long-run trends



- Rest of the course:
  - Study how aggregate shocks cause business cycles
    - Does the transmission change with heterogeneous agents?
  - Implications for fiscal and monetary policy

Business cycles

- Macro variables relatievly volatile around long-run trends



- Rest of the course:
  - Study how aggregate shocks cause business cycles
    - Does the transmission change with heterogeneous agents?
  - Implications for fiscal and monetary policy
- First point on agenda: Need **role** for monetary policy
  - Models so far in the course have featured **monetary-non neutrality**
  - Monetary policy cannot affect real quantities (unemployment, GDP)

New Keynesian framework

- A proper mode of business cycles and stabalization policy require that output is partially demand determined

New Keynesian framework

- A proper mode of business cycles and stabalization policy require that output is partially demand determined
- The study of monetary policy requires **monetary-non neutrality**

New Keynesian framework

- A proper mode of business cycles and stabalization policy require that output is partially demand determined
- The study of monetary policy requires **monetary-non neutrality**
- The New Keynesian (NK) model adresses these two concerns by adding to the standard model:
  - Monopolistic competetion (price-setting)
  - Price rigidities

New Keynesian framework

- A proper mode of business cycles and stabalization policy require that output is partially demand determined
- The study of monetary policy requires **monetary-non neutrality**
- The New Keynesian (NK) model adresses these two concerns by adding to the standard model:
  - Monopolistic competetion (price-setting)
  - Price rigidities
- The basic NK model is simple (can be reduced to 3 equations) but **extremely influential**

The New Keynesian model

Model

- Several version of the NK model

Model

- Several version of the NK model
- I present the simplest version her

Model

- Several version of the NK model
- I present the simplest version her
- The model consists of the following agents:
  - A representative household who consumes, saves and supplies labor
  - Firms with market power who produce output using labor and sets prices subject to nominal rigidities
  - A central bank which conduct monetary policy

Overview

- **Households:**
  1. Representativ agent
  2. Supply labor and choose consumption

Overview

- **Households:**
  1. Representativ agent
  2. Supply labor and choose consumption
- **Intermediary goods firms** (continuum)
  1. Produce differentiated goods with labor
  2. Set price under monopolistic competition
  3. Pay profits to households

Overview

- **Households:**
  1. Representativ agent
  2. Supply labor and choose consumption
- **Intermediary goods firms** (continuum)
  1. Produce differentiated goods with labor
  2. Set price under monopolistic competition
  3. Pay profits to households
- **Final goods firms** (representative)
  1. Produce final good with intermediary goods
  2. Take price as given under perfect competition

Overview

- **Households:**
  1. Representativ agent
  2. Supply labor and choose consumption
- **Intermediary goods firms** (continuum)
  1. Produce differentiated goods with labor
  2. Set price under monopolistic competition
  3. Pay profits to households
- **Final goods firms** (representative)
  1. Produce final good with intermediary goods
  2. Take price as given under perfect competition
- **Central bank**: Sets nominal interest rate

Households

- Representative household solve the following problem:

$$\max_{C_t, A_t, L_t} E_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(C_t\right) - \nu\left(L_t^{hh}\right) \right]$$

$$s.t.$$

$$C_t + A_t = \left(1 + r_t\right) A_{t-1} + \left(w_t L_t^{hh} + \Pi_t\right)$$

- Note: Expectation taken w.r.t **aggregate shocks** (TFP, monetary policy, markup etc.)
- Standard first-order conditions:

$$u'\left(C_t\right) = E_t \beta \left(1 + r_{t+1}\right) u'\left(C_{t+1}\right)$$

$$\nu'\left(L_t^{hh}\right) = w_t u'\left(C_t\right)$$

Final goods firms

- Final goods firm buy goods y_{jt} from intermediary goods firms and assemble into aggregate output Y_t using CES technology with sub. elaticity ϵ

Final goods firms

- Final goods firm buy goods y_{jt} from intermediary goods firms and assemble into aggregate output Y_t using CES technology with sub. elaticity ϵ
- Intermediary firms/intermediary goods indexed by $j \in [0, 1]$

Final goods firms

- Final goods firm buy goods y_{jt} from intermediary goods firms and assemble into aggregate output Y_t using CES technology with sub. elaticity ϵ
- Intermediary firms/intermediary goods indexed by $j \in [0,1]$
- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

for given output price, P_t, and input prices, p_{jt}

Final goods firms

- Final goods firm buy goods y_{jt} from intermediary goods firms and assemble into aggregate output Y_t using CES technology with sub. elaticity ϵ
- Intermediary firms/intermediary goods indexed by $j \in [0,1]$
- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\,\forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

for given output price, P_t, and input prices, p_{jt}

- **Demand curve** derived from FOC wrt. y_{jt}

$$\forall j : \; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t$$

Final goods firms

- Final goods firm buy goods y_{jt} from intermediary goods firms and assemble into aggregate output Y_t using CES technology with sub. elaticity ϵ
- Intermediary firms/intermediary goods indexed by $j \in [0,1]$
- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

for given output price, P_t, and input prices, p_{jt}

- **Demand curve** derived from FOC wrt. y_{jt}

$$\forall j : y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t$$

- **Note:** Zero profits (can be used to derive price index)

Intermediary goods firms

- Intermediary goods firms produce using labor, choose price subject to quadratic adjustment cost of changing prices (Rotemberg)

Intermediary goods firms

- Intermediary goods firms produce using labor, choose price subject to quadratic adjustment cost of changing prices (Rotemberg)
- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, l_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t l_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + E_t \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t l_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t, \;\; \Omega(p_{jt}, p_{jt-1}) = \frac{\theta}{2} \left[\frac{p_{jt}}{p_{jt-1}} - 1 \right]^2$$

- Intermediary goods firms produce using labor, choose price subject to quadratic adjustment cost of changing prices (Rotemberg)
- ***Dynamic*** **problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, l_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t l_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + E_t \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t l_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t, \;\; \Omega(p_{jt}, p_{jt-1}) = \frac{\theta}{2} \left[\frac{p_{jt}}{p_{jt-1}} - 1 \right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$

Intermediary goods firms

- Intermediary goods firms produce using labor, choose price subject to quadratic adjustment cost of changing prices (Rotemberg)
- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, l_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t l_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + E_t \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t l_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\epsilon} Y_t, \;\; \Omega(p_{jt}, p_{jt-1}) = \frac{\theta}{2} \left[\frac{p_{jt}}{p_{jt-1}} - 1\right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$
- **NKPC** with slope $\kappa = \frac{\epsilon}{\theta}$ and $\mu = \frac{\epsilon}{\epsilon - 1}$ derived from FOC wrt. p_{jt} and envelope condition (note $mc_t = \frac{MC_t}{P_t} = \frac{w_t}{\Gamma_t}$):

$$\pi_t(1+\pi_t) = \kappa \left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu}\right) + E_t \frac{Y_{t+1}}{Y_t} \frac{\pi_{t+1}(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

Intermediary goods firms

- Intermediary goods firms produce using labor, choose price subject to quadratic adjustment cost of changing prices (Rotemberg)
- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, l_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t l_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + E_t \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t l_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\epsilon} Y_t, \;\; \Omega(p_{jt}, p_{jt-1}) = \frac{\theta}{2}\left[\frac{p_{jt}}{p_{jt-1}} - 1\right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$
- **NKPC** with slope $\kappa = \frac{\epsilon}{\theta}$ and $\mu = \frac{\epsilon}{\epsilon - 1}$ derived from FOC wrt. p_{jt} and envelope condition (note $mc_t = \frac{MC_t}{P_t} = \frac{w_t}{\Gamma_t}$):

$$\pi_t(1+\pi_t) = \kappa\left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu}\right) + E_t \frac{Y_{t+1}}{Y_t} \frac{\pi_{t+1}(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

- **Implied production:** $Y_t = y_{jt}$, $L_t = l_{jt}$ (from symmetry)

Intermediary goods firms

- Intermediary goods firms produce using labor, choose price subject to quadratic adjustment cost of changing prices (Rotemberg)
- ***Dynamic*** **problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, l_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t l_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + E_t \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t l_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t, \;\; \Omega(p_{jt}, p_{jt-1}) = \frac{\theta}{2} \left[\frac{p_{jt}}{p_{jt-1}} - 1 \right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$
- **NKPC** with slope $\kappa = \frac{\epsilon}{\theta}$ and $\mu = \frac{\epsilon}{\epsilon - 1}$ derived from FOC wrt. p_{jt} and envelope condition (note $mc_t = \frac{MC_t}{P_t} = \frac{w_t}{\Gamma_t}$):

$$\pi_t(1+\pi_t) = \kappa \left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu} \right) + E_t \frac{Y_{t+1}}{Y_t} \frac{\pi_{t+1}(1 + \pi_{t+1})}{1 + r_{t+1}}, \;\; \pi_t \equiv P_t / P_{t-1} - 1$$

- **Implied production:** $Y_t = y_{jt}$, $L_t = l_{jt}$ (from symmetry)
- **Implied dividends:** $\Pi_t = Y_t - w_t L_t - \frac{\theta}{2} \left[\frac{p_{jt}}{p_{jt-1}} - 1 \right]^2 Y_t$

Derivation of NKPC

- **FOC wrt.** p_{jt}**:**

$$0 = (1-\epsilon)\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon}\frac{Y_t}{P_t} + \epsilon\frac{w_t}{\Gamma_t}\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon-1}\frac{Y_t}{P_t}$$
$$-\theta\left[\frac{p_{jt}}{p_{jt-1}}-1\right]\frac{Y_t}{p_{jt-1}} + E_t\frac{J'_{t+1}(p_{jt})}{1+r_{t+1}}$$

Derivation of NKPC

- **FOC wrt. p_{jt}:**

$$
\begin{aligned}
0 = & (1-\epsilon)\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon}\frac{Y_t}{P_t} + \epsilon\frac{w_t}{\Gamma_t}\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon-1}\frac{Y_t}{P_t} \\
& -\theta\left[\frac{p_{jt}}{p_{jt-1}} - 1\right]\frac{Y_t}{p_{jt-1}} + E_t\frac{J'_{t+1}(p_{jt})}{1+r_{t+1}}
\end{aligned}
$$

- **Envelope condition:** $J'_{t+1}(p_{jt}) = -\theta\left[\frac{p_{jt+1}}{p_{jt}} - 1\right]\left(\frac{p_{jt+1}}{p_{jt}^2}\right)Y_{t+1}$

Derivation of NKPC

- **FOC wrt. p_{jt}:**

$$0 = (1-\epsilon)\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon}\frac{Y_t}{P_t} + \epsilon\frac{w_t}{\Gamma_t}\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon-1}\frac{Y_t}{P_t}$$
$$-\theta\left[\frac{p_{jt}}{p_{jt-1}}-1\right]\frac{Y_t}{p_{jt-1}} + E_t\frac{J'_{t+1}(p_{jt})}{1+r_{t+1}}$$

- **Envelope condition:** $J'_{t+1}(p_{jt}) = -\theta\left[\frac{p_{jt+1}}{p_{jt}}-1\right]\left(\frac{p_{jt+1}}{p_{jt}^2}\right)Y_{t+1}$
- **FOC + Envelope + Symmetry +** $\pi_t = P_t/P_{t-1}-1$

$$0 = \left[(1-\epsilon)+\epsilon\frac{w_t}{\Gamma_t}\right]\frac{Y_t}{P_t}$$
$$-\theta\left[\frac{P_t}{P_{t-1}}-1\right]\frac{Y_t}{P_{t-1}} - E_t\frac{\theta\left[\frac{P_{t+1}}{P_t}-1\right]\left(\frac{P_{t+1}}{P_t^2}\right)Y_{t+1}}{1+r_{t+1}}$$

Central NKPC intution

$$\pi_t(1+\pi_t) = \kappa\left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu}\right) + E_t\frac{Y_{t+1}}{Y_t}\frac{1}{1+r_{t+1}}\pi_{t+1}\left(1+\pi_{t+1}\right)$$

1. **Zero-inflation steady state:**
   $\pi_t = 0 \rightarrow w_t = \frac{\Gamma_t}{\mu} \rightarrow$ wage is mark-downed relative to MPL ($\mu > 1$)

Central NKPC intution

$$\pi_t(1+\pi_t) = \kappa\left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu}\right) + E_t \frac{Y_{t+1}}{Y_t}\frac{1}{1+r_{t+1}}\pi_{t+1}\left(1+\pi_{t+1}\right)$$

1. **Zero-inflation steady state:**
   $\pi_t = 0 \rightarrow w_t = \frac{\Gamma_t}{\mu} \rightarrow$ wage is mark-downed relative to MPL ($\mu > 1$)
2. **Larger adjustment costs**, $\kappa \downarrow$ (more sticky prices):
   Less pass-through from marginal costs, $\frac{w_t}{\Gamma_t}$, to inflation, π_t

Central NKPC intution

$$\pi_t(1+\pi_t) = \kappa\left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu}\right) + E_t \frac{Y_{t+1}}{Y_t}\frac{1}{1+r_{t+1}}\pi_{t+1}\left(1+\pi_{t+1}\right)$$

1. **Zero-inflation steady state:**
   $\pi_t = 0 \rightarrow w_t = \frac{\Gamma_t}{\mu} \rightarrow$ wage is mark-downed relative to MPL ($\mu > 1$)
2. **Larger adjustment costs**, $\kappa \downarrow$ (more sticky prices):
   Less pass-through from marginal costs, $\frac{w_t}{\Gamma_t}$, to inflation, π_t
3. **Larger (expected) future inflation, $\pi_{t+1} \uparrow$:**
   Increase price today, $\pi_t \uparrow$
   Especially in a boom, $\frac{Y_{t+1}}{Y_t} > 1$

Central NKPC intution

$$\pi_t(1+\pi_t) = \kappa\left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu}\right) + E_t \frac{Y_{t+1}}{Y_t}\frac{1}{1+r_{t+1}}\pi_{t+1}\left(1+\pi_{t+1}\right)$$

1. **Zero-inflation steady state:**
   $\pi_t = 0 \rightarrow w_t = \frac{\Gamma_t}{\mu} \rightarrow$ wage is mark-downed relative to MPL ($\mu > 1$)
2. **Larger adjustment costs**, $\kappa \downarrow$ (more sticky prices):
   Less pass-through from marginal costs, $\frac{w_t}{\Gamma_t}$, to inflation, π_t
3. **Larger (expected) future inflation, $\pi_{t+1} \uparrow$:**
   Increase price today, $\pi_t \uparrow$
   Especially in a boom, $\frac{Y_{t+1}}{Y_t} > 1$
4. Note:
   - Sometimes a β^{firm} is used instead of $\frac{1}{1+r_{t+1}}$
   - $\pi_t(1+\pi_t) \approx \pi_t$ for small π_t

- **Monetary policy:** Follow Taylor-rule:

$$i_t = i_t^* + \phi \pi_t$$

where i_t^* is a monetary policy shock (target for CB)

Government and central bank

- **Monetary policy:** Follow Taylor-rule:

$$i_t = i_t^* + \phi \pi_t$$

  where i_t^* is a monetary policy shock (target for CB)

- **Fisher relationship:**

$$r_t = (1 + i_{t-1})/(1 + \pi_t) - 1$$

Government and central bank

- **Monetary policy:** Follow Taylor-rule:

$$i_t = i_t^* + \phi \pi_t$$

where i_t^* is a monetary policy shock (target for CB)

- **Fisher relationship:**

$$r_t = (1 + i_{t-1})/(1 + \pi_t) - 1$$

- **Government:** In standard model Government simply supplies bonds that are in net-zero supply, $B = 0$
    - Note: HHs still make consumption-saving decisions (so cannot impose $A = 0$ in budget), but in equilibrium prices will adjust such that $A = B = 0$
    - Simplifying assumption, can easily incorporate more reaslitic government $\tau_t = r_t B_{ss} + G_t$ with $B_{ss} > 0$ (see HANK later)

Equilibrium

- Three markets that need to clear in the NK model:

Equilibrium

- Three markets that need to clear in the NK model:
- Goods market:
$$Y_t = C_t + \frac{\theta}{2}\pi_t^2 Y_t$$

Equilibrium

- Three markets that need to clear in the NK model:
- Goods market:

$$Y_t = C_t + \frac{\theta}{2}\pi_t^2 Y_t$$

- Labor market:

$$L_t^{hh} = L_t$$

Equilibrium

- Three markets that need to clear in the NK model:
- Goods market:

$$Y_t = C_t + \frac{\theta}{2}\pi_t^2 Y_t$$

- Labor market:

$$L_t^{hh} = L_t$$

- Asset market:

$$A = 0$$

Equilibrium

- Three markets that need to clear in the NK model:
- Goods market:

$$Y_t = C_t + \frac{\theta}{2}\pi_t^2 Y_t$$

- Labor market:

$$L_t^{hh} = L_t$$

- Asset market:

$$A = 0$$

- As usual, in practice we will only impose market clearing in two of the markets when solving the model

Aggregate shocks

- In the standard NK model business cycles arise due to fluctuations in aggregate shocks:

1. TFP (supply)

$$\ln \Gamma_t = \overline{\Gamma} + \ln \Gamma_{t-1} + \epsilon_t^{\Gamma}, \quad \epsilon_t^{\Gamma} \sim \mathcal{N}\left(0, \sigma_{\Gamma}^2\right)$$

2. Discount factor (demand)

$$\ln \beta_t = \overline{\beta} + \ln \beta_{t-1} + \epsilon_t^{\beta}, \quad \epsilon_t^{\beta} \sim \mathcal{N}\left(0, \sigma_{\beta}^2\right)$$

3. Monetary policy

$$i_t^* = \overline{i^*} + \ln i_{t-1}^* + \epsilon_t^{i^*}, \quad \epsilon_t^{i^*} \sim \mathcal{N}\left(0, \sigma_{i^*}^2\right)$$

The 3 equation NK model

- Consider the deterministic, zero-inflation steady state of the model (with TFP and prices normalized to 1):

$$\pi_{ss} = 0, \quad Y_{ss} = C_{ss} = 1$$
$$r_{ss} = i_{ss} = \frac{1}{\beta} - 1, \quad w_{ss} = \frac{1}{\mu}$$

The 3 equation NK model

- Consider the deterministic, zero-inflation steady state of the model (with TFP and prices normalized to 1):

$$\pi_{ss} = 0, \quad Y_{ss} = C_{ss} = 1$$
$$r_{ss} = i_{ss} = \frac{1}{\beta} - 1, \quad w_{ss} = \frac{1}{\mu}$$

- Linearize the model arounds this steady state with notation $\hat{x}_t = x_t - x_{ss}$ for some endo. variable x_t

The 3 equation NK model

- Consider the deterministic, zero-inflation steady state of the model (with TFP and prices normalized to 1):

$$\pi_{ss} = 0, \quad Y_{ss} = C_{ss} = 1$$
$$r_{ss} = i_{ss} = \frac{1}{\beta} - 1, \quad w_{ss} = \frac{1}{\mu}$$

- Linearize the model arounds this steady state with notation $\hat{x}_t = x_t - x_{ss}$ for some endo. variable x_t
- The model can be reduced to three equations:

$$\begin{aligned}
\hat{Y}_t &= -\sigma \left(i_t - \pi_{t+1} \right) + \hat{Y}_{t+1} + \epsilon_t^D && \text{(Euler/demand curve)} \\
\hat{\pi}_t &= \tilde{\kappa} \hat{Y}_t + \beta \hat{\pi}_{t+1} + \epsilon_t^S && \text{(NKPC/supply curve)} \\
\hat{i}_t &= \phi \hat{\pi}_t + \epsilon_t^i && \text{(Monetary policy)}
\end{aligned}$$

The 3 equation NK model

- Consider the deterministic, zero-inflation steady state of the model (with TFP and prices normalized to 1):

$$\pi_{ss} = 0, \quad Y_{ss} = C_{ss} = 1$$
$$r_{ss} = i_{ss} = \frac{1}{\beta} - 1, \quad w_{ss} = \frac{1}{\mu}$$

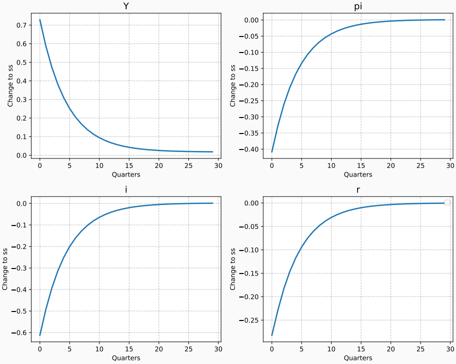
- Linearize the model arounds this steady state with notation $\hat{x}_t = x_t - x_{ss}$ for some endo. variable x_t
- The model can be reduced to three equations:

$$\begin{aligned}
\hat{Y}_t &= -\sigma\left(i_t - \pi_{t+1}\right) + \hat{Y}_{t+1} + \epsilon_t^D && \text{(Euler/demand curve)} \\
\hat{\pi}_t &= \tilde{\kappa}\hat{Y}_t + \beta\hat{\pi}_{t+1} + \epsilon_t^S && \text{(NKPC/supply curve)} \\
\hat{i}_t &= \phi\hat{\pi}_t + \epsilon_t^i && \text{(Monetary policy)}
\end{aligned}$$

- With three unknowns (per period) $\hat{Y}_t, \hat{\pi}_t, \hat{i}_t$

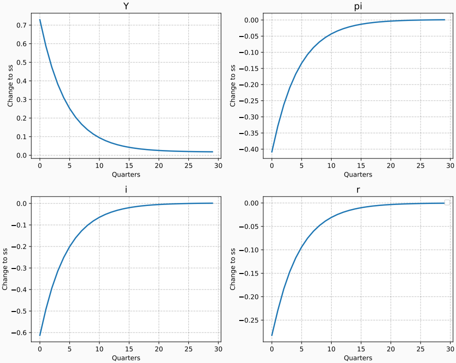
Technology shocks in the NK model

- Effects of a positive TFP shock (increase Γ_t)



Technology shocks in the NK model

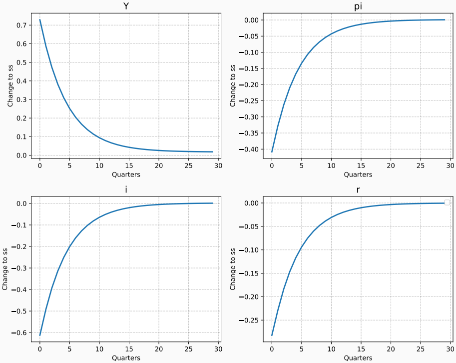
- Effects of a positive TFP shock (increase Γ_t)



- Increase in productivity decreases marginal costs w_t/Γ_t

Technology shocks in the NK model

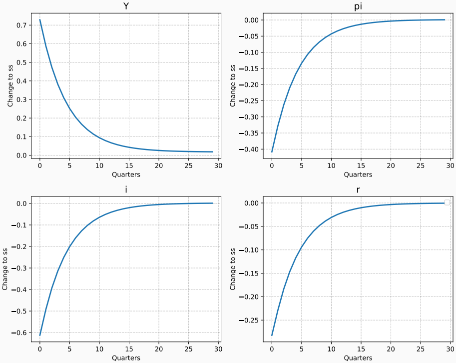
- Effects of a positive TFP shock (increase Γ_t)



- Increase in productivity decreases marginal costs w_t/Γ_t
- Firms reduce prices $\Rightarrow$ CB reduce nominal interest rate

Technology shocks in the NK model

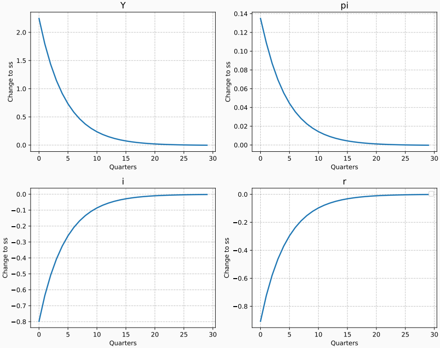
- Effects of a positive TFP shock (increase Γ_t)



- Increase in productivity decreases marginal costs w_t/Γ_t
- Firms reduce prices $\Rightarrow$ CB reduce nominal interest rate
- Intertemporal sub. $\Rightarrow C, Y \uparrow$

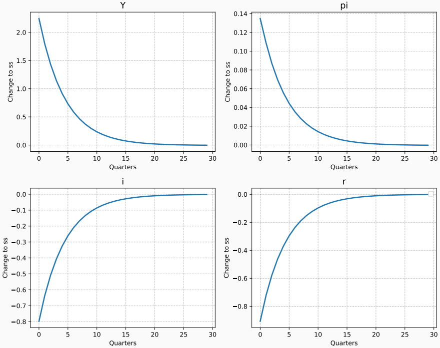
Monetary policy shocks in the NK model

- Effects of accomodating monetary policy (easing) with persistent decline in i_t^*



Monetary policy shocks in the NK model

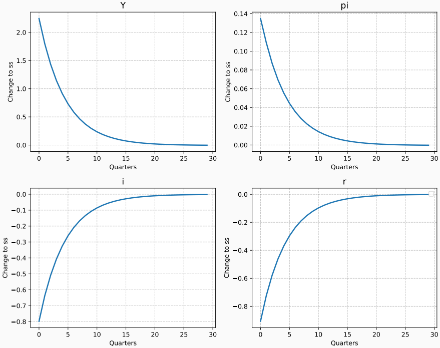
- Effects of accomodating monetary policy (easing) with persistent decline in i_t^*



- Decrease real rate r which induce intertemporal substitution, so $C, Y \uparrow$

Monetary policy shocks in the NK model

- Effects of accomodating monetary policy (easing) with persistent decline in i_t^*



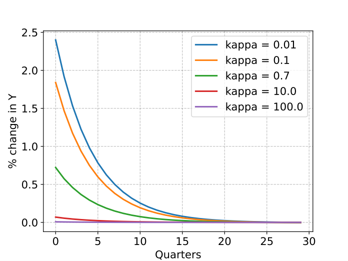
- Decrease real rate r which induce intertemporal substitution, so $C, Y \uparrow$
- Increase in employment pushes up wages (marginal costs), so inflation increases

Monetary neutrality

- Monetary policy can affect consumption, employment and output in the short run because the model features **monetary non-neutrality**
    - Comes from sticky prices of firms

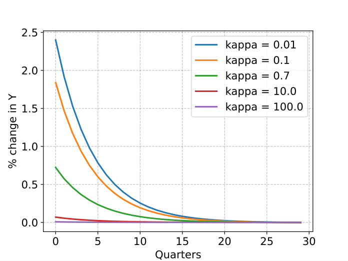
Monetary neutrality

- Monetary policy can affect consumption, employment and output in the short run because the model features **monetary non-neutrality**
  - Comes from sticky prices of firms
- Consider monetary policy shock with increasing slope of NKPC κ
  - Recall $\kappa \to \infty$ is flexible prices (think Ramsey) and constant markup



Monetary neutrality

- Monetary policy can affect consumption, employment and output in the short run because the model features **monetary non-neutrality**
  - Comes from sticky prices of firms
- Consider monetary policy shock with increasing slope of NKPC κ
  - Recall $\kappa \to \infty$ is flexible prices (think Ramsey) and constant markup



- Why? With completely flexible prices monetary policy just increases inflation 1-1 without affecting r

Review questions

- Consider the standard New Keynesian model

Review questions

- Consider the standard New Keynesian model
- Review questions
  1. How does a positive demand shock ϵ_t^β (which decrease β) affect output Y, inflation π, and interest rates i, r?
  2. Are firm markups pro-cyclical or counter-cyclical (w.r.t Y) in response to the demand shock?
  3. Consider an extension with a government that spends G and raises lumpsum taxes τ
     - What is the effect of a shock to G? Is the fiscal multiplier $\frac{dY}{dG}$ above or below one?
     - Does the effects of the shock dependent on the method of financing (debt vs taxes)?

IRFs and simulation

Aggregate uncertainty

- In business cycle model common to have *aggregate uncertainty*

Aggregate uncertainty

- In business cycle model common to have *aggregate uncertainty*
- I.e. underlying shocks (TFP, demand etc) x follow stochastic process with dist, f, $x_t \sim f$

Aggregate uncertainty

- In business cycle model common to have *aggregate uncertainty*
- I.e. underlying shocks (TFP, demand etc) x follow stochastic process with dist, f, $x_t \sim f$
- This implies that all variables which are functions of x are also random.
  - If TFP is random $\Rightarrow$ wages, interest rates, labor demand etc. are **random** until observed

Aggregate uncertainty

- In business cycle model common to have *aggregate uncertainty*
- I.e. underlying shocks (TFP, demand etc) x follow stochastic process with dist, f, $x_t \sim f$
- This implies that all variables which are functions of x are also random.
  - If TFP is random $\Rightarrow$ wages, interest rates, labor demand etc. are **random** until observed
- Implies that we need to compute expectation in Euler, NKPC and other forward looking equations:

$$u'\left(C_t\right) = \beta \mathbb{E}_t \left[R_{t+1}\left(x_{t+1}\right) u'\left(C_t\left(x_{t+1}\right)\right)\right]$$

Aggregate uncertainty

- In business cycle model common to have *aggregate uncertainty*
- I.e. underlying shocks (TFP, demand etc) x follow stochastic process with dist, f, $x_t \sim f$
- This implies that all variables which are functions of x are also random.
  - If TFP is random $\Rightarrow$ wages, interest rates, labor demand etc. are **random** until observed
- Implies that we need to compute expectation in Euler, NKPC and other forward looking equations:

$$u'\left(C_t\right) = \beta \mathbb{E}_t \left[R_{t+1}\left(x_{t+1}\right) u'\left(C_t\left(x_{t+1}\right)\right)\right]$$

- Note: So far in the course we have generally assumed **perfect foresight** w.r.t aggregate variables (w, r) so no expectation
  - Implies that aggregate shocks are not random process, but rather *MIT shocks*

Aggregate uncertainty

- In business cycle model common to have *aggregate uncertainty*
- I.e. underlying shocks (TFP, demand etc) x follow stochastic process with dist, f, $x_t \sim f$
- This implies that all variables which are functions of x are also random.
  - If TFP is random $\Rightarrow$ wages, interest rates, labor demand etc. are **random** until observed
- Implies that we need to compute expectation in Euler, NKPC and other forward looking equations:

$$u'\left(C_t\right) = \beta \mathbb{E}_t\left[R_{t+1}\left(x_{t+1}\right) u'\left(C_t\left(x_{t+1}\right)\right)\right]$$

- Note: So far in the course we have generally assumed **perfect foresight** w.r.t aggregate variables (w, r) so no expectation
  - Implies that aggregate shocks are not random process, but rather *MIT shocks*
- Interpretation of MIT shocks generally hard to reconcile with business cycles

Stochastic vs deterministic models

- To see how the **stochastic** model and deterministic model are related consider the Euler with random x:

$$u'\left(C_t\right) = R\beta\mathbb{E}_t\left[u'\left(C_t\left(x_{t+1}\right)\right)\right]$$

Stochastic vs deterministic models

- To see how the **stochastic** model and deterministic model are related consider the Euler with random x:

$$u'(C_t) = R\beta\mathbb{E}_t\left[u'(C_t(x_{t+1}))\right]$$

- First-order Taylor approx. around deterministic ss (use $R\beta = 1$):

$$du'(C_t) \approx u''(C_{ss}) \cdot C'(x_{ss}) \cdot d\mathbb{E}_t x_{t+1}$$

Stochastic vs deterministic models

- To see how the **stochastic** model and deterministic model are related consider the Euler with random x:

$$u'(C_t) = R\beta\mathbb{E}_t\left[u'(C_t(x_{t+1}))\right]$$

- First-order Taylor approx. around deterministic ss (use $R\beta = 1$):

$$du'(C_t) \approx u''(C_{ss}) \cdot C'(x_{ss}) \cdot d\mathbb{E}_t x_{t+1}$$

- Assume $x_t = \rho^x x_{t-1} + \epsilon_t^x$ with $\mathbb{E}\epsilon_t^x = 0$. Period 0 solution in deterministic/perfect foresight model:

$$du'(C_0) \approx u''(C_{ss}) \cdot C'(x_{ss}) \cdot \rho^x d\epsilon_0^x$$

Stochastic vs deterministic models

- To see how the **stochastic** model and deterministic model are related consider the Euler with random x:

$$u'\left(C_t\right) = R\beta\mathbb{E}_t\left[u'\left(C_t\left(x_{t+1}\right)\right)\right]$$

- First-order Taylor approx. around deterministic ss (use $R\beta = 1$):

$$du'\left(C_t\right) \approx u''\left(C_{ss}\right) \cdot C'\left(x_{ss}\right) \cdot d\mathbb{E}_t x_{t+1}$$

- Assume $x_t = \rho^x x_{t-1} + \epsilon_t^x$ with $\mathbb{E}\epsilon_t^x = 0$. Period 0 solution in deterministic/perfect foresight model:

$$du'\left(C_0\right) \approx u''\left(C_{ss}\right) \cdot C'\left(x_{ss}\right) \cdot \rho^x d\epsilon_0^x$$

- Stochastic model we use:

$$\begin{aligned} d\mathbb{E}_0 x_1 &= d\mathbb{E}_0\left(\rho^x x_0 + \epsilon_1^x\right) \\ &= \rho^x d\mathbb{E}_0 x_0 = \rho^x d\epsilon_0^x = dx_1 \end{aligned}$$

Stochastic vs deterministic models

- To see how the **stochastic** model and deterministic model are related consider the Euler with random x:

$$u'\left(C_t\right) = R\beta\mathbb{E}_t\left[u'\left(C_t\left(x_{t+1}\right)\right)\right]$$

- First-order Taylor approx. around deterministic ss (use $R\beta = 1$):

$$du'\left(C_t\right) \approx u''\left(C_{ss}\right) \cdot C'\left(x_{ss}\right) \cdot d\mathbb{E}_t x_{t+1}$$

- Assume $x_t = \rho^x x_{t-1} + \epsilon_t^x$ with $\mathbb{E}\epsilon_t^x = 0$. Period 0 solution in deterministic/perfect foresight model:

$$du'\left(C_0\right) \approx u''\left(C_{ss}\right) \cdot C'\left(x_{ss}\right) \cdot \rho^x d\epsilon_0^x$$

- Stochastic model we use:

$$\begin{aligned} d\mathbb{E}_0 x_1 &= d\mathbb{E}_0\left(\rho^x x_0 + \epsilon_1^x\right) \\ &= \rho^x d\mathbb{E}_0 x_0 = \rho^x d\epsilon_0^x = dx_1 \end{aligned}$$

- Same result! Aggregate uncertainty **does not matter to first-order** when linearizing w.r.t aggregate shock

When does uncertainty matter?

- **Insight:** *The IRF from an MIT shock is equivalent to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)

When does uncertainty matter?

- **Insight:** *The IRF from an MIT shock is* equivalent *to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)
- What about **high order**?

When does uncertainty matter?

- **Insight:** *The IRF from an MIT shock is <u>equivalent</u> to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)
- What about **high order**?
- Approximate Euler to **second** order:

$$
\begin{aligned}
du'\left(C_t\right) \approx & u''\left(C_{ss}\right)\cdot C'\left(x_{ss}\right)\cdot d\mathbb{E}_t x_{t+1} + \frac{1}{2}u'''\left(C_{ss}\right)C''\left(x_{ss}\right)\cdot \mathbb{E}_t\left(x_{t+1}-x_{ss}\right)^2 \\
& u''\left(C_{ss}\right)\cdot C'\left(x_{ss}\right)\cdot d\mathbb{E}_t x_{t+1} + \frac{1}{2}u'''\left(C_{ss}\right)C''\left(x_{ss}\right)\cdot \sigma_{x,t}^2
\end{aligned}
$$

When does uncertainty matter?

- **Insight:** *The IRF from an MIT shock is* <u>*equivalent*</u> *to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)
- What about **high order**?
- Approximate Euler to **second** order:

$$du'\left(C_t\right) \approx u''\left(C_{ss}\right)\cdot C'\left(x_{ss}\right)\cdot d\mathbb{E}_t x_{t+1} + \frac{1}{2}u'''\left(C_{ss}\right)C''\left(x_{ss}\right)\cdot \mathbb{E}_t\left(x_{t+1}-x_{ss}\right)^2$$
$$u''\left(C_{ss}\right)\cdot C'\left(x_{ss}\right)\cdot d\mathbb{E}_t x_{t+1} + \frac{1}{2}u'''\left(C_{ss}\right)C''\left(x_{ss}\right)\cdot \sigma^2_{x,t}$$

- In deterministic model $\sigma^2_{x,t} = 0$ - not true in stochastic model!
  - Models deviate once we go beyond 1st order approximation (linearization)

When does uncertainty matter?

- **Insight:** *The IRF from an MIT shock is equivalent to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)
- What about **high order**?
- Approximate Euler to **second** order:

$$
\begin{aligned}
du'(C_t) \approx & u''(C_{ss}) \cdot C'(x_{ss}) \cdot d\mathbb{E}_t x_{t+1} + \frac{1}{2} u'''(C_{ss}) C''(x_{ss}) \cdot \mathbb{E}_t (x_{t+1} - x_{ss})^2 \\
& u''(C_{ss}) \cdot C'(x_{ss}) \cdot d\mathbb{E}_t x_{t+1} + \frac{1}{2} u'''(C_{ss}) C''(x_{ss}) \cdot \sigma^2_{x,t}
\end{aligned}
$$

- In deterministic model $\sigma^2_{x,t} = 0$ - not true in stochastic model!
  - Models deviate once we go beyond 1st order approximation (linearization)
- Still extremely usefull though - we may solve deterministic models to first-order and interpret as models with aggregate uncertainty
  - How do we linearize models numerically?

Reminder of model class

- **Unknowns: $\boldsymbol{U}$**
- **Shock: $\boldsymbol{Z}$**
- **Additional variables: $\boldsymbol{X}$**
- **Target equation system:**

$$\boldsymbol{H(U, Z) = 0}$$

Reminder of model class

- **Unknowns:** $\boldsymbol{U}$
- **Shock:** $\boldsymbol{Z}$
- **Additional variables:** $\boldsymbol{X}$
- **Target equation system:**

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0}$$

- In deterministic, perfect foresight model, solve $\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0}$ w.r.t $\boldsymbol{U}$ by:
    1. Calculating the jacobian of $\boldsymbol{H}$w.r.t $\boldsymbol{U}$around steady state
    2. Use Newton/Broyden's method to find non-linear transition path given shocks $\boldsymbol{Z}$

Linearized IRFs

- What if just want *first order solution*?

Linearized IRFs

- What if just want *first order solution*?
    1. Solve for Impulse Response Functions (IRFs) for unknowns

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U}^{-1}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

Linearized IRFs

- What if just want *first order solution*?

    1. Solve for Impulse Response Functions (IRFs) for unknowns

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U}^{-1}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$

Linearized IRFs

- What if just want *first order solution*?

  1. Solve for Impulse Response Functions (IRFs) for unknowns

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U}^{-1}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$
- **Limitations:**

Linearized IRFs

- What if just want *first order solution*?
  1. Solve for Impulse Response Functions (IRFs) for unknowns

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U}^{-1}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$
- **Limitations:**
  1. Imprecise for *large* shocks

Linearized IRFs

- What if just want *first order solution*?

  1. Solve for Impulse Response Functions (IRFs) for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U}^{-1}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$
- **Limitations:**
  1. Imprecise for *large* shocks
  2. Imprecise in models with *aggregate non-linearities*

Linearized IRFs

- What if just want *first order solution*?

    1. Solve for Impulse Response Functions (IRFs) for unknowns

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H_U} d\boldsymbol{U} + \boldsymbol{H_Z} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H_U}^{-1}\boldsymbol{H_Z}}_{\equiv \boldsymbol{G_U}} d\boldsymbol{Z}$$

- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$
- **Limitations:**
    1. Imprecise for *large* shocks
    2. Imprecise in models with *aggregate non-linearities*
    3. No aggregate uncertainty (precautionary savings w.r.t aggregate shocks etc.)

Linearized IRFs

- What if just want *first order solution*?

    1. Solve for Impulse Response Functions (IRFs) for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z}) = 0 \Rightarrow \boldsymbol{H}_{\boldsymbol{U}} d\boldsymbol{U} + \boldsymbol{H}_{\boldsymbol{Z}} d\boldsymbol{Z} = 0 \Leftrightarrow d\boldsymbol{U} = \underbrace{-\boldsymbol{H}_{\boldsymbol{U}}^{-1}\boldsymbol{H}_{\boldsymbol{Z}}}_{\equiv \boldsymbol{G}_{\boldsymbol{U}}} d\boldsymbol{Z}$$

- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$
- **Limitations:**
    1. Imprecise for *large* shocks
    2. Imprecise in models with *aggregate non-linearities*
    3. No aggregate uncertainty (precautionary savings w.r.t aggregate shocks etc.)
- Next slide: **Can we solve model with aggregate risk globally** (i.e. to more than first-order)?

Aggregate risk (dynamic equilibrium)

- To solve models with aggregate risk we need to write them in *state-space* form instead of *sequence-space*
  - Think of HA household problem - that is always in state-space form
  - Endogenous variables c_t, a_t as function of current states a_{t-1}, z_t

Aggregate risk (dynamic equilibrium)

- To solve models with aggregate risk we need to write them in *state-space* form instead of *sequence-space*
    - Think of HA household problem - that is always in state-space form
    - Endogenous variables c_t, a_t as function of current states a_{t-1}, z_t
- **Aggregate stochastic variables:** $\boldsymbol{Z}$ follow some known process with innovations ϵ. *State space form*: RHS is what is known today

$$\begin{bmatrix} \boldsymbol{U}_t \\ \boldsymbol{Z}_t \end{bmatrix} = \mathcal{M}\left(\begin{bmatrix} \boldsymbol{U}_{t-1} \\ \boldsymbol{Z}_{t-1} \end{bmatrix}, \boldsymbol{\epsilon}_t \right)$$

$\neq$ perfect foresight wrt. future agg. variables in *sequence-space*

Aggregate risk (dynamic equilibrium)

- To solve models with aggregate risk we need to write them in *state-space* form instead of *sequence-space*
  - Think of HA household problem - that is always in state-space form
  - Endogenous variables c_t, a_t as function of current states a_{t-1}, z_t
- **Aggregate stochastic variables: $\boldsymbol{Z}$** follow some known process with innovations ϵ. *State space form*: RHS is what is known today

$$\begin{bmatrix} \boldsymbol{U}_t \\ \boldsymbol{Z}_t \end{bmatrix} = \mathcal{M}\left(\begin{bmatrix} \boldsymbol{U}_{t-1} \\ \boldsymbol{Z}_{t-1} \end{bmatrix}, \boldsymbol{\epsilon}_t\right)$$

  $\neq$ perfect foresight wrt. future agg. variables in *sequence-space*

- In standard NK model: no backward looking eqs. so number of state variables $=$ Number of shocks

Example: Krussel-Smith

- What if we add heterogeneous agents? Canonical example: The Krussel-Smith model (1998)
  - HANC with aggregate uncertainty (TFP shocks)

Example: Krussel-Smith

- What if we add heterogeneous agents? Canonical example: The Krussel-Smith model (1998)
  - HANC with aggregate uncertainty (TFP shocks)
- **Recursive formulation of household problem:**

$$v(\boldsymbol{D}_t, \Gamma_t, z_{it}, a_{it-1}) = \max_{a_{it}, c_{it}} u(c_{it}) + \beta \mathbb{E}_t \left[v(\boldsymbol{D}_{t+1}, \Gamma_{t+1}, z_{it+1}, a_{it}) \right]$$

s.t.

$$\begin{aligned}
K_{t-1} &= \int a_{it-1} d\boldsymbol{D}_t \\
r_t &= \alpha \Gamma_t K_{t-1}^{\alpha-1} - \delta \\
w_t &= (1-\alpha)\Gamma_t K_{t-1}^{\alpha} \\
a_{it} + c_{it} &= (1+r_t)a_{it-1} + w_t z_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0,
\end{aligned}$$

Example: Krussel-Smith

- What if we add heterogeneous agents? Canonical example: The Krussel-Smith model (1998)
  - HANC with aggregate uncertainty (TFP shocks)
- **Recursive formulation of household problem:**

$$v(\boldsymbol{D}_t, \Gamma_t, z_{it}, a_{it-1}) = \max_{a_{it}, c_{it}} u(c_{it}) + \beta \mathbb{E}_t \left[v(\boldsymbol{D}_{t+1}, \Gamma_{t+1}, z_{it+1}, a_{it}) \right]$$

s.t.

$$\begin{aligned}
K_{t-1} &= \int a_{it-1} d\boldsymbol{D}_t \\
r_t &= \alpha \Gamma_t K_{t-1}^{\alpha-1} - \delta \\
w_t &= (1-\alpha)\Gamma_t K_{t-1}^{\alpha} \\
a_{it} + c_{it} &= (1+r_t)a_{it-1} + w_t z_{it} \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0,
\end{aligned}$$

- $\boldsymbol{D}_t$ is a state variable $\Rightarrow$ Massive state space

Comparisons

- **State-space approach with linearization**: Ahn et al. (2018); Bayer and Luetticke (2020); Bhandari et al. (2023); Bilal (2023)
  Con:
  1. Harder to implement
  2. Valuable to be able to interpret Jacobians

  Pro**:**
  1. Easier path to 2nd and higher order approximations
- **Global solution:** The distribution of households is a state variable for each household $\Rightarrow$ *explosion in complexity*
  1. Original: Krusell and Smith (1997, 1998); Algan et al. (2014);
  2. Deep learning: Fernández-Villaverde et al. (2021); Maliar et al. (2021); Han et al. (2021); Kase et al. (2022); Azinovic et al. (2022); Gu et al. (2023); Chen et al. (2023)
- **Discrete aggregate risk:** Lin and Peruffo (2023)

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\boldsymbol{\epsilon}_{t-s}$

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\epsilon_{t-s}$
- **Linearized simulation:**

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\epsilon_{t-s}$
- **Linearized simulation:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\epsilon_{t-s}$
- **Linearized simulation:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T-1} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
     Note: $d\boldsymbol{Z}_s\tilde{\boldsymbol{\epsilon}}_{t-s}$ = effect of shock s periods ago today

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\epsilon_{t-s}$
- **Linearized simulation:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T-1} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
     Note: $d\boldsymbol{Z}_s \tilde{\epsilon}_{t-s} =$ effect of shock s periods ago today
  3. Calculate the time series of other aggregate variables as

$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T-1} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a *unit-shock* after s periods (just needs jacobian of $\boldsymbol{X}$ w.r.t shocks $\boldsymbol{Z}$)

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
  - EX: If shock $\boldsymbol{Z}$ follows an AR(1) then $d\boldsymbol{Z}_s = \rho^{s-t}\boldsymbol{\epsilon}_{t-s}$
- **Linearized simulation:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{T-1} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
     Note: $d\boldsymbol{Z}_s \tilde{\epsilon}_{t-s} =$ effect of shock s periods ago today
  3. Calculate the time series of other aggregate variables as

$$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{T-1} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a *unit-shock* after s periods (just needs jacobian of $\boldsymbol{X}$ w.r.t shocks $\boldsymbol{Z}$)
- **Intuition:** Sum of first order effects from all previous shocks

Generalized linearized simulation [advanced]

- **Generality:** Prev. slide: Aggregate variables. What about micro level household variables?

Generalized linearized simulation [advanced]

- **Generality:** Prev. slide: Aggregate variables. What about micro level household variables?
- **Full distribution:**

Generalized linearized simulation [advanced]

- **Generality:** Prev. slide: Aggregate variables. What about micro level household variables?
- **Full distribution:**
    1. The IRF for grid point i_g in a policy function can be calculated as

$$da^*_{i_g,s} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a^*_{i_g}}{\partial X^{hh}_{s'-s}} dX^{hh}_{s'}.$$

    where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)

Generalized linearized simulation [advanced]

- **Generality:** Prev. slide: Aggregate variables. What about micro level household variables?
- **Full distribution:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da_{i_g,s}^{*}=\sum_{s'=s}^{T-1}\sum_{X^{hh}\in\boldsymbol{X}^{hh}}\frac{\partial a_{i_g}^{*}}{\partial X_{s'-s}^{hh}}dX_{s'}^{hh}.$$

     where $\partial\boldsymbol{a}_{i_g}^{*}/\partial X_{k}^{hh}$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)
  2. The policy function can there be simulated as

$$\boldsymbol{a}_{i_g,t}^{*}=\sum_{s=0}^{T-1}da_{i_g,s}^{*}\tilde{\boldsymbol{\epsilon}}_{t-s}$$

Generalized linearized simulation [advanced]

- **Generality:** Prev. slide: Aggregate variables. What about micro level household variables?
- **Full distribution:**
  1. The IRF for grid point i_g in a policy function can be calculated as

     $$da^*_{i_g,s} = \sum_{s'=s}^{T-1} \sum_{X^{hh} \in \boldsymbol{X}^{hh}} \frac{\partial a^*_{i_g}}{\partial X^{hh}_{s'-s}} dX^{hh}_{s'}.$$

     where $\partial \boldsymbol{a}^*_{i_g} / \partial X^{hh}_k$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)
  2. The policy function can there be simulated as

     $$\boldsymbol{a}^*_{i_g,t} = \sum_{s=0}^{T-1} da^*_{i_g,s} \tilde{\boldsymbol{\epsilon}}_{t-s}$$

  3. Distribution can then be simulated forwards using standard method

Calculating moments - variance

- **Identical and independent distributed innovations:**

$$\mathbb{E}\left[\epsilon_t^i \epsilon_{t'}^j\right] = \begin{cases} \sigma_i^2 & \text{if } t = t' \text{ and } i = j \\ 0 & \text{else} \end{cases}$$

Calculating moments - variance

- **Identical and independent distributed innovations:**

$$\mathbb{E}\left[\epsilon_t^i \epsilon_{t'}^j\right] = \begin{cases} \sigma_i^2 & \text{if } t = t' \text{ and } i = j \\ 0 & \text{else} \end{cases}$$

- **Calculating moments such as var**(dC_t) from the IRFs:

$$\begin{aligned} \text{var}(dC_t) &= \mathbb{E}\left[\left(\sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} dC_s^i \epsilon_{t-s}^i\right)^2\right] \\ &= \sum_{i \in \mathcal{Z}} \sum_{s=0}^{T-1} \mathbb{E}\left[\epsilon_{t-s}^i \epsilon_{t-s}^i\right] \left(dC_s^i\right)^2 \\ &= \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1} \left(dC_s^i\right)^2 \end{aligned}$$

where dC_s^i is the IRF to a unit-shock to i after s periods and σ_i is the standard deviation of shock i

Calculating moments - variance

- Implications of prior slide:
  - Very easy to calculate business cycle moments

Calculating moments - variance

- Implications of prior slide:
  - Very easy to calculate business cycle moments
- Steps (variance of C) (1 shock):
  1. Formulate shock to e.g. public spending, $\{dG_t\}_{t=0}^{T} = d\boldsymbol{G}$ (could be an AR(1))
  2. Linearize and solve model to get IRF of $\{dC_t\}_{t=0}^{T} = d\boldsymbol{C}$ w.r.t $\{dG_t\}$
  3. Calculate variance $\text{var}(dC_t) = \sum_{s=0}^{T-1} (dC_s)^2$

Calculating moments - variance

- Implications of prior slide:
  - Very easy to calculate business cycle moments
- Steps (variance of C) (1 shock):
  1. Formulate shock to e.g. public spending, $\{dG_t\}_{t=0}^{T} = d\boldsymbol{G}$ (could be an AR(1))
  2. Linearize and solve model to get IRF of $\{dC_t\}_{t=0}^{T} = d\boldsymbol{C}$ w.r.t $\{dG_t\}$
  3. Calculate variance $\text{var}(dC_t) = \sum_{s=0}^{T-1} (dC_s)^2$
- Same principle with more shocks

Calculating moments - covariance

- **Covariances:**

$$\text{cov}(dC_t, dY_{t+k}) = \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i$$

Calculating moments - covariance

- **Covariances:**

$$\text{cov}(dC_t, dY_{t+k}) = \sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i$$

- **Covariance decomposition:**

$$\frac{\text{contribution from one shock}}{\text{contributions from all shocks}} = \frac{\sigma_j^2 \sum_{s=0}^{T-1-k} dC_s^j dY_{s+k}^j}{\sum_{i \in \mathcal{Z}} \sigma_i^2 \sum_{s=0}^{T-1-k} dC_s^i dY_{s+k}^i}$$

Exercise

Exercise - NK model with government

1. Familiarize yourself with the model equations in *blocks.py*. Do you understand all the equations?
2. Compute the non-linear response to a temporary increase in government spending
   2.1 Use *model.find_transition_path()* for the non-linear response (results are in *model.path*)
   2.2 Use *model.find_IRFs()* for the linear response (results are in *model.IRF*)
3. Add a zero lower bound to the model:

   $$i_t = \max\left\{i_{ss} + \phi\pi_t, 0\right\}$$

   Compute linear and non-linear responses to a β-shock of size 0.05 and compare.
4. Assume that the government tries to stabilize the economy after the demand shock. Compute linear and non-linear responses to a simultaneous shock to β ($d\beta_0 = 0.05$) and G ($dG_0 = 0.03$).
5. Is stabilization policy more or less efficient once we take the ZLB into account? Hint: Compare the multipliers $\frac{dY^{\beta,G} - dY^{\beta}}{dG}$ for the linear and non-linear responses and compare.
   MORE ON NEXT SLIDE

Exercise - NK model with government

6. Simulate a monetary policy shock of size 0.01. Calculate the variance of consumption using the analytical formula:

$$\text{var}\,(dC) = \sum_{s=0}^{T-1} (dC_s)^2$$

Check that you get the same variance if you simulate a timeseries of consumption using model.simulate(skip_hh=True), and calculate the variance as:

$$\text{var}\,(dC) = \frac{1}{N}\sum_{i=0}^{N} \left(dC_i^{sim}\right)^2$$

Hints: You can set the size of the shock for the IRFs using model.par.jump_eps_i, while the standard error of the shocks in the simulation is set using model.par.std_eps_i.

Make sure that the standard error of other shocks in the model are zero when you simulate. You can find the simulated series in model.sim.dC.

Summary

Summary and next week

- **Today:**
  1. The New Keynesian model
  2. Aggregate risk and linearized dynamics (IRF and simulation)
  3. Calculating aggregate moments (for calibration or estimation)
- **Next week:** HANK + Fiscal policy
- **Homework:**
  1. Work on exercise
  2. Skim-read Auclert et al. (2023),
     »The Intertemporal Keynesian Cross«