Termički sistemi
Modeli fizičkih sistema

Modeliranje i simulacija sistema

Termički sistemi

- sistemi gde postoji skladištenje ili prenos toplote
- modeli sa koncentrisanim parametrima
- linearizacija
- bez promene agregatnih stanja i hemijskih procesa

Promenljive

- Temperatura – θ [K]
  - smatraćemo da je θ u svim tačkama tela ista
  - θ_a - spoljašnja temp.
  - ukoliko je bitno da pojedini delovi tela imaju različite θ, onda se telo može posmatrati iz više segmenata
  - obično se θ bira za promenljive stanja

- Količina toplote – q [J/s]≡[W]

Elementi u termičkom sistemu

- Dva tipa pasivnih elemenata:
  - Termička kapacitivnost
  - Termička otpornost

- Aktivan elemenat
  - Termički izvor

Termička kapacitivnost

- Postoji algebarska zavisnost između
  - temperature tela θ i
  - akumulirane toplote u njemu Δq

- Pojednostavljeno - linearna veza

$$\dot{\theta}(t) = \frac{1}{C}\left(q_{in}(t) - q_{out}(t)\right) \qquad \theta(t) = \theta(t_0) + \int_{t_0}^{t} \frac{1}{C}\left(q_{in}(t) - q_{out}(t)\right)dt$$

- C - toplotni kapacitet tela [J/K]

$$C = m \cdot \sigma$$

σ - specifična toplota tela

Termička otpornost

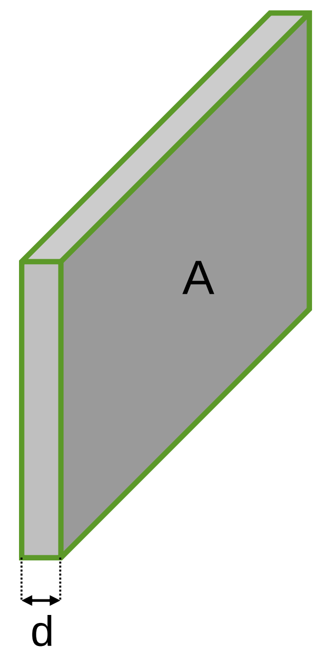
- Posmatramo samo provođenje toplote:

$$q(t) = \frac{1}{R}\left[\theta_1(t) - \theta_2(t)\right] \qquad q(t) = \frac{1}{R} \cdot \Delta\theta$$

- R $\equiv$ termička otpornost [Ks/J]

$$R = \frac{d}{A\alpha}$$

α - termička provodnost

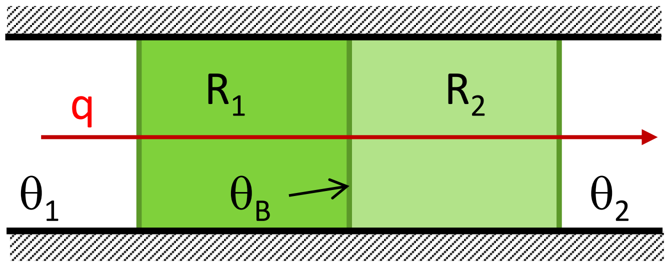


Osobine materijala

Materijal	Gustina ρ [kg/m³]	Termička provodljivost α [W/(m·K)]	Specifična toplota tela c_p [kJ/(kg·K)]
Beton	2600	1,44-1,68	0,8
Beton armirani	2200	1,55	0,84
Gipsana ploča	600-1200	0,291-0,581	1,089
Opeka, suva	1600-1800	0,38-0,52	0,84
Opeka izolaciona	550	0,1395	-
Sneg sveže napadao	200	0,1	2,09
Staklo	2400-3200	0,582-1,047	0,77
Staklena vuna	50	0,037	0,67
Stiropor	32	0,027	1,382
Šperploča	600	0,15	2,51
Drvo bor T	546	0,16	2,7
Drvo bor -	551	0,35	2,7
Drvo hrast T	825	0,21	2,4
Drvo hrast -	890	0,36	2,4
Drvo jela T	546	0,14-0,16	2,72
Drvo jela -	-	0,35-0,72	2,72
Vazduh 0°C	1,293	0,0244	1,005
Vazduh 10°C	1,247	0,0251	1,005
Vazduh 20°C	1,205	0,0259	1,005
Vazduh 30°C	1,165	0,0267	1,005
Vazduh 40°C	1,128	0,0276	1,005

Primer

- Odrediti termičku otpornost



savršena termička izolacija

$$q = \frac{1}{R_1}\left[\theta_1 - \theta_B\right]$$

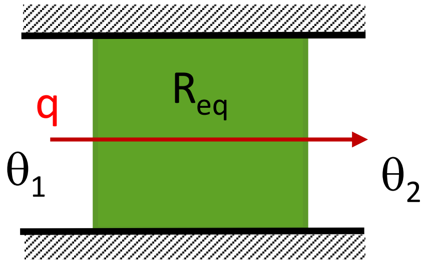
$$q = \frac{1}{R_2}\left[\theta_B - \theta_2\right]$$

$$\Downarrow$$

$$R_2 q + \theta_2 = \theta_B$$

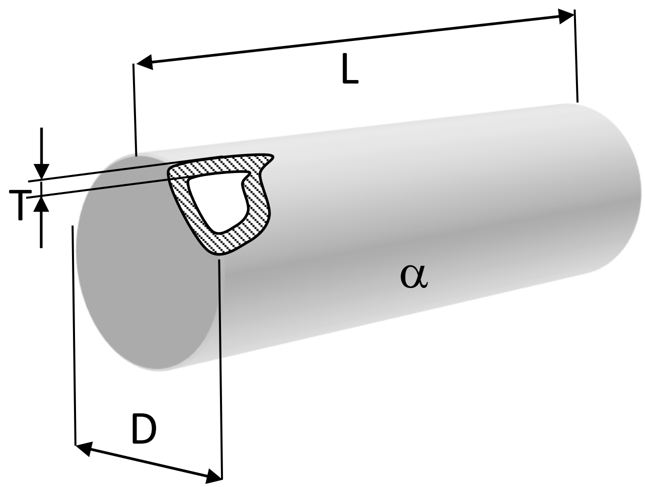
$$R_1 q = \theta_1 - \theta_2 - R_2 q \Rightarrow q = \frac{1}{R_1 + R_2}\left(\theta_1 - \theta_2\right)$$

$$R_{eq} = R_1 + R_2 \leftarrow \text{serijska veza}$$



Primer

- Odrediti termičku otpornost cilindrične posude debljine zida T i termičke provodnosti α



Term. otp. baze: $R_e = \frac{d}{A\alpha} = \frac{4T}{\pi D^2 \alpha}$

Term. otp. omotača: $R_c = \frac{T}{\pi D L \alpha}$

$$q_e = \frac{1}{R_e}\Delta\theta \qquad q_c = \frac{1}{R_c}\Delta\theta$$

$$q_{total} = 2q_e + q_c = \left(\frac{2}{R_e} + \frac{1}{R_c}\right)\Delta\theta = \frac{1}{R_{eq}} \cdot \Delta\theta$$

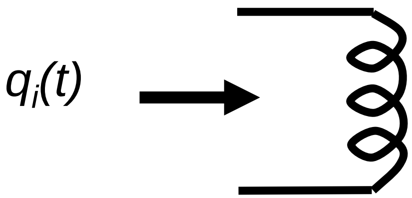
$$\frac{1}{R_{eq}} = \left(\frac{1}{R_c} + \frac{2}{R_e}\right) \qquad \Rightarrow R_{eq} = \frac{R_c \cdot R_e}{2R_c + R_e}$$

Paralelna veza

Termički izvor

- Tipovi termičkog izvora
  - izvor koji dovodi toplotu
  - izvor koji odvodi toplotu

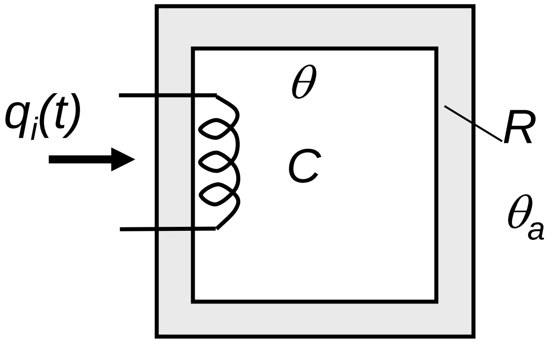
- Idealan termički izvor



Dinamički model termičkih sistema

- kao promenljive stanja temperature svakog tela sa C

- Primer



$$q_{in} = q_i(t)$$

$$q_{out} = \frac{1}{R}(\theta(t) - \theta_a)$$

$$\dot{\theta}(t) = \frac{1}{C}\left(q_{in}(t) - \frac{1}{R}(\theta - \theta_a)\right)$$

$$\dot{\theta}(t) + \frac{1}{RC}\theta = \frac{1}{C}q_i(t) + \frac{1}{RC}\theta_a$$

Primer (nastavak)

- Linearizacija modela

$$\dot{\theta}(t) + \frac{1}{RC}\overline{\theta} = \frac{1}{C}\overline{q}_i(t) + \frac{1}{RC}\theta_a$$

- U ustaljenom stanju:

$$\overline{\theta} = \theta_a + R\overline{q}_i$$

$$\hat{\theta}(t) = \theta(t) - \overline{\theta}$$

$$\hat{q}_i(t) = q_i(t) - \overline{q}_i$$

- Uvođenjem smena se dobija

$$\dot{\hat{\theta}}(t) + \frac{1}{RC}\left(\hat{\theta} + \overline{\theta}\right) = \frac{1}{C}\left[\hat{q}_i(t) + \overline{q}_i\right] + \frac{1}{RC}\theta_a$$

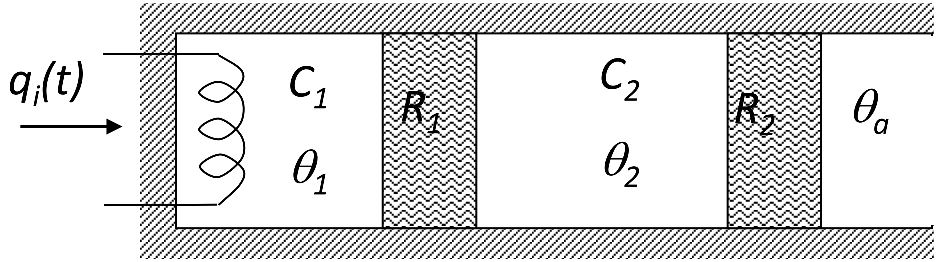
- Eliminisanja konstantnih članova na osnovu $\overline{\theta} = \theta_a + R\overline{q}_i$

$$\dot{\hat{\theta}} + \frac{1}{RC}\hat{\theta} = \frac{1}{C}\hat{q}_i(t)$$

Primer

- Odrediti

$$\hat{\theta}_2 = f\left(\hat{q}_i(t)\right) = ?$$



$$\dot{\theta}_1(t) = \frac{1}{C_1}\left(q_i(t) - \frac{1}{R_1}(\theta_1 - \theta_2)\right) \qquad \dot{\theta}_2(t) = \frac{1}{C_2}\left(\frac{1}{R_1}(\theta_1 - \theta_2) - \frac{1}{R_2}(\theta_2 - \theta_a)\right)$$

$$\left.\begin{aligned} &\bar{q}_i - \frac{1}{R_1}\left(\bar{\theta}_1 - \bar{\theta}_2\right) = 0 \\ &\frac{1}{R_1}\left(\bar{\theta}_1 - \bar{\theta}_2\right) = \frac{1}{R_2}\left(\bar{\theta}_2 - \theta_a\right) \end{aligned}\right\} \qquad \begin{aligned} \bar{\theta}_2 &= \theta_a + R_2\bar{q}_i \\ \bar{\theta}_1 &= \theta_a + (R_1 + R_2)\bar{q}_i \end{aligned}$$

$$\hat{\theta}_1(t) = \theta_1(t) - \bar{\theta}_1$$

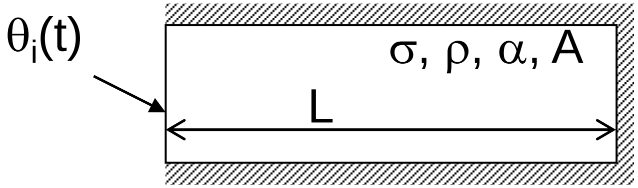
$$\hat{\theta}_2(t) = \theta_2(t) - \bar{\theta}_2$$

$$\dot{\hat{\theta}}_1 + \frac{1}{R_1C_1}\hat{\theta}_1 = \frac{1}{R_1C_1}\hat{\theta}_2 + \frac{1}{C}\hat{q}_i$$

$$\dot{\hat{\theta}}_2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2}\right)\cdot\hat{\theta}_2 = \frac{1}{R_2C_2}\hat{\theta}_1$$

Primer

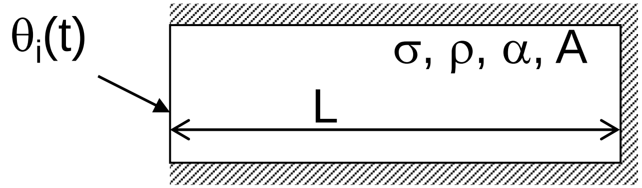
- Formirati model tela



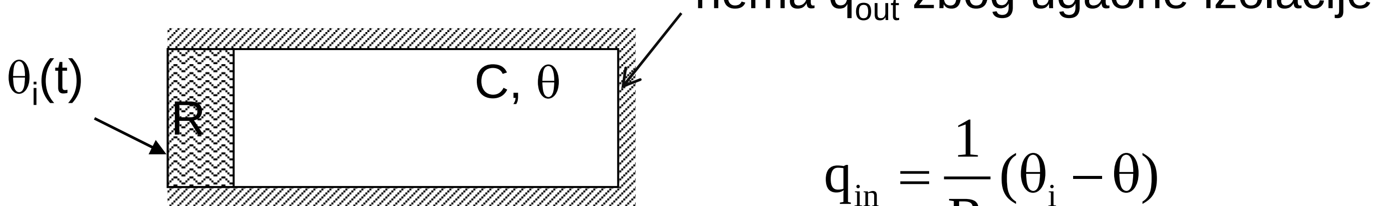
Posmatra se telo u obliku položenog valjka od poznatog materijala. Levi kraj tela (baza valjka) se greje i njegova temperatura je stalna θi. Odrediti promenu temperature u valjku ako je njegova preostala površina idealno termički izolovana. Gustina ρ, specifični toplotni kapacitet σ, termička provodnost α, površina poprečnog preseka A i dužina tela l su poznati. Pre početka grejanja leve strane tela temperatura tela je jednaka spoljašnjoj temperaturi θo.

Primer

- Formirati model tela



- <u>Aproksimacija 1</u>



$$q_{in} = \frac{1}{R}(\theta_i - \theta)$$

$$R = \frac{L}{A\alpha}$$

$$\dot{\theta} = \frac{1}{RC}\left(\theta_i - \theta\right)$$

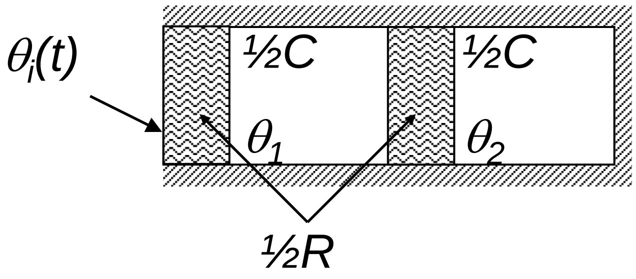
$$C = \sigma m = \sigma\rho AL$$

σ - specifična toplota tela

$$\dot{\theta} + \frac{1}{RC}\theta = \frac{1}{RC}\theta_i\left(t\right)$$

Primer (nastavak)

- Aproksimacija 2



- Leva komora

$$q_{in} = \frac{1}{0,5R}(\theta_i(t) - \theta_1)$$

$$\dot{\theta}_1(t) = \frac{4}{RC}\left(\theta_i - 2\theta_1 + \theta_2\right)$$

- Desna komora

$$q_{in} = \frac{1}{0,5R}(\theta_1 - \theta_2) \qquad q_{out} = 0$$

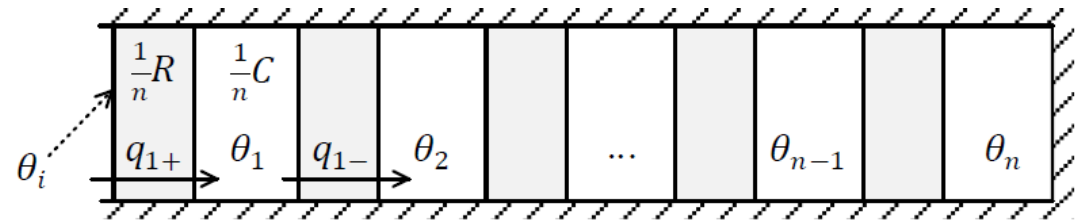
$$\dot{\theta}_2 = \frac{4}{RC}\left(\theta_1 - \theta_2\right)$$

$$\dot{\hat{\theta}}_1 + \frac{8}{RC}\hat{\theta}_1 = \frac{4}{RC}\hat{\theta}_2 + \frac{4}{RC}\hat{\theta}_i(t)$$

$$\dot{\hat{\theta}}_2 + \frac{4}{RC}\hat{\theta}_2 = \frac{4}{RC}\hat{\theta}_1$$

Primer (nastavak)

- Aproksimacija 3: *n* komora



$$\dot{\theta}_k(t) = \frac{n}{C}\left(q_{k+} - q_{k-}\right), \quad k = 1,2,\ldots,n$$

$$q_{k+} = \frac{n}{R}(\theta_{k-1}(t) - \theta_k(t)), \quad q_{k-} = \frac{n}{R}(\theta_k(t) - \theta_{k+1}(t))$$

$$\dot{\theta}_k(t) = \frac{n^2}{RC}\left(\theta_{k-1}(t) - 2\theta_k(t) + \theta_{k+1}(t)\right)$$

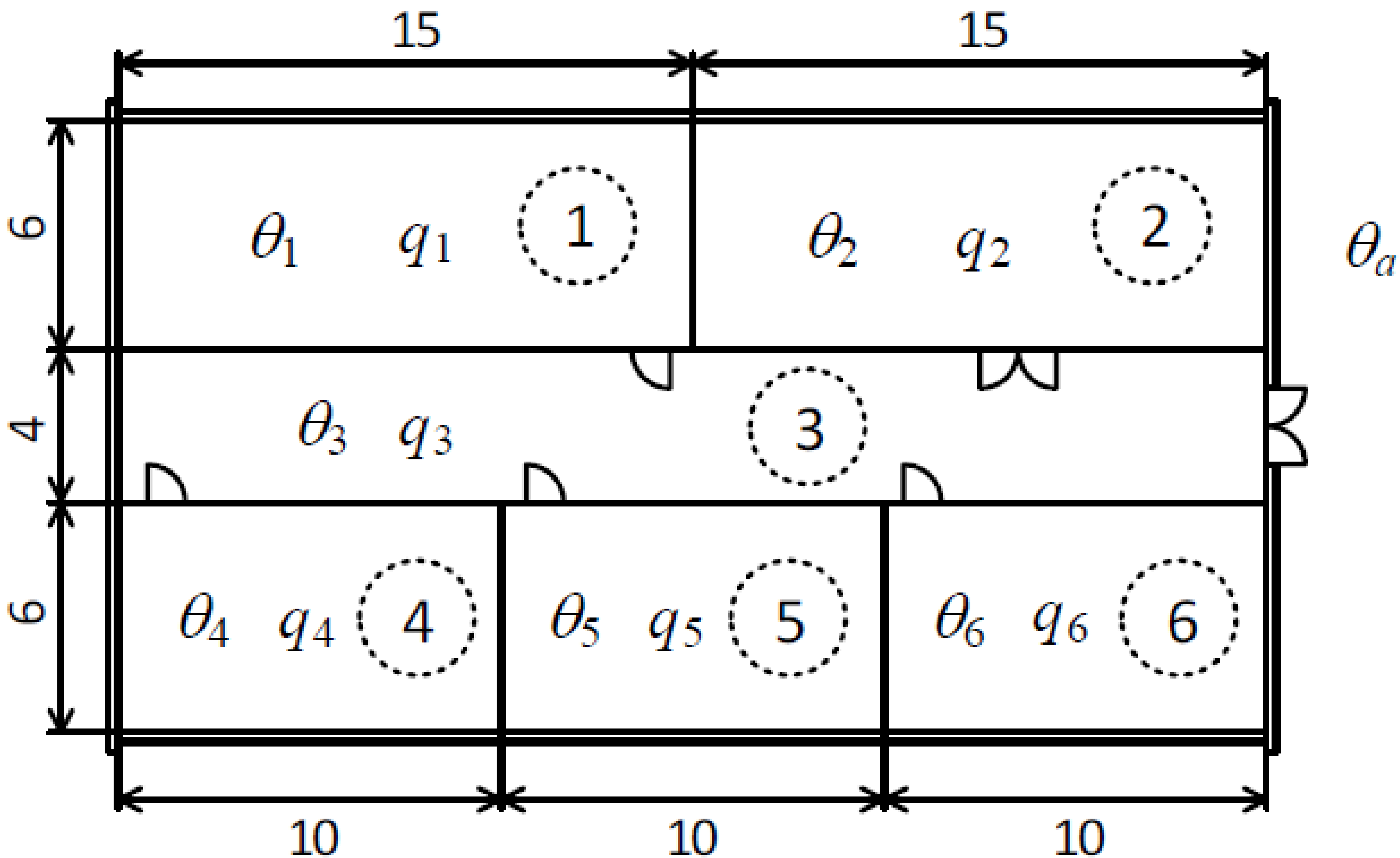
$$q_{1+} = \frac{2}{R}(\theta_i(t) - \theta_1(t)), \qquad q_{1-} = q_{2+}, \quad \dot{\theta}_1(t) = \frac{n^2}{RC}\left(\theta_i(t) - 2\theta_1(t) + \theta_2(t)\right)$$

$$q_{2+} = \frac{2}{R}(\theta_1(t) - \theta_2(t)), \qquad q_{2-} = q_{3+}, \quad \dot{\theta}_2(t) = \frac{n^2}{RC}\left(\theta_1(t) - 2\theta_2(t) + \theta_3(t)\right)$$

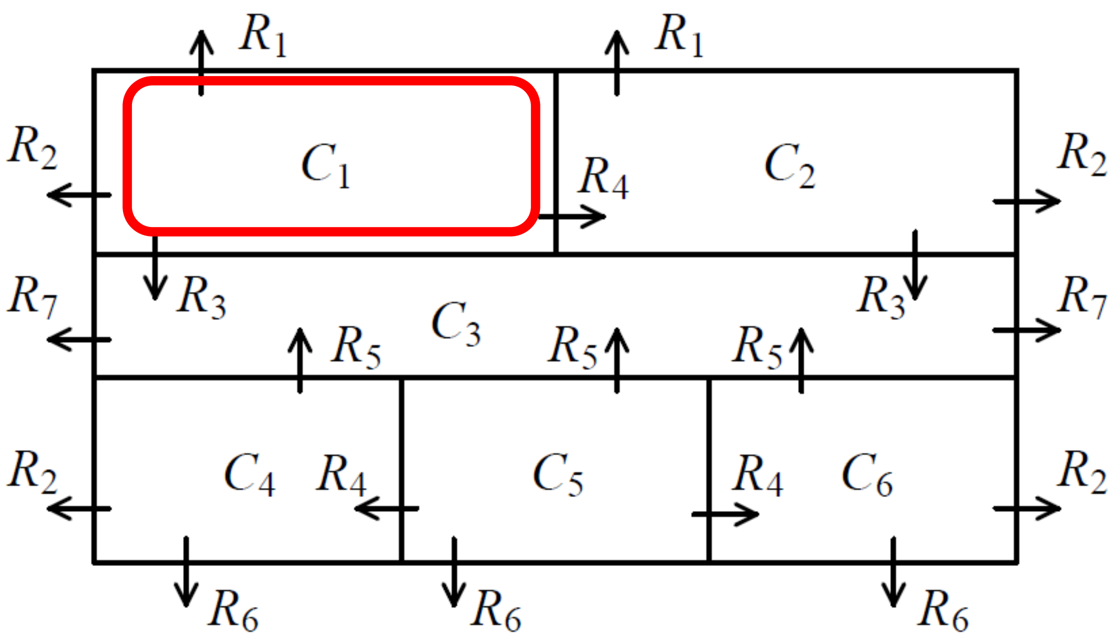
$$\ldots \qquad \ldots \qquad \ldots$$

$$q_{n+} = \frac{2}{R}(\theta_{n-1}(t) - \theta_n(t)), \quad q_{n-} = 0, \quad \dot{\theta}_n(t) = \frac{n^2}{RC}\left(\theta_{n-1}(t) - \theta_n(t)\right)$$

Primer: Grejanje zgrade



Formiranje modela



$$\dot{\theta}_1 = \frac{1}{C_1}\left(q_1 - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)(\theta_1 - \theta_a) - \frac{1}{R_3}(\theta_1 - \theta_3) - \frac{1}{\mathrm{R}_4}(\theta_1 - \theta_2)\right)$$

$$\dot{\theta}_1 = \frac{1}{C_1}\left(q_1 - \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)\theta_1 + \frac{1}{R_4}\theta_2 + \frac{1}{R_3}\theta_3 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\theta_a\right)$$

Dinamički model

$$\dot{\theta}_1 = \frac{1}{C_1}\left(q_1 - \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)\theta_1 + \frac{1}{R_4}\theta_2 + \frac{1}{R_3}\theta_3 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\theta_a\right)$$

$$\dot{\theta}_2 = \frac{1}{C_2}\left(q_2 + \frac{1}{R_4}\theta_1 - \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)\theta_2 + \frac{1}{R_3}\theta_3 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\theta_a\right)$$

$$\dot{\theta}_3 = \frac{1}{C_3}\left(q_3 + \frac{1}{R_3}\theta_1 + \frac{1}{R_3}\theta_2 - \left(\frac{2}{R_3} + \frac{3}{R_5} + \frac{2}{R_7}\right)\theta_3 + \frac{1}{R_5}\theta_4 + \frac{1}{R_5}\theta_4 + \frac{1}{R_5}\theta_6 + \frac{2}{R_7}\theta_a\right)$$

$$\dot{\theta}_4 = \frac{1}{C_4}\left(q_4 + \frac{1}{R_5}\theta_3 - \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)\theta_4 + \frac{1}{R_4}\theta_5 + \left(\frac{1}{R_2} + \frac{1}{R_6}\right)\theta_a\right)$$

$$\dot{\theta}_5 = \frac{1}{C_5}\left(q_5 + \frac{1}{R_5}\theta_3 + \frac{1}{R_4}\theta_4 - \left(\frac{2}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)\theta_5 + \frac{1}{R_4}\theta_6 + \frac{1}{R_6}\theta_a\right)$$

$$\dot{\theta}_6 = \frac{1}{C_6}\left(q_6 + \frac{1}{R_5}\theta_3 + \frac{1}{R_4}\theta_5 - \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)\theta_6 + \left(\frac{1}{R_2} + \frac{1}{R_6}\right)\theta_a\right)$$

Statički model

$$\bar{q}_1 - \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)\bar{\theta}_1 + \frac{1}{R_4}\bar{\theta}_2 + \frac{1}{R_3}\bar{\theta}_3 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\theta_a = 0$$

$$\bar{q}_2 + \frac{1}{R_4}\bar{\theta}_1 - \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)\bar{\theta}_2 + \frac{1}{R_3}\bar{\theta}_3 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\theta_a = 0$$

$$\bar{q}_3 + \frac{1}{R_3}\bar{\theta}_1 + \frac{1}{R_3}\bar{\theta}_2 - \left(\frac{2}{R_3} + \frac{3}{R_5} + \frac{2}{R_7}\right)\bar{\theta}_3 + \frac{1}{R_5}\bar{\theta}_4 + \frac{1}{R_5}\bar{\theta}_5 + \frac{1}{R_5}\bar{\theta}_6 + \frac{2}{R_7}\theta_a = 0$$

$$\bar{q}_4 + \frac{1}{R_5}\bar{\theta}_3 - \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)\bar{\theta}_4 + \frac{1}{R_4}\bar{\theta}_5 + \left(\frac{1}{R_2} + \frac{1}{R_6}\right)\theta_a = 0$$

$$\bar{q}_5 + \frac{1}{R_5}\bar{\theta}_3 + \frac{1}{R_4}\bar{\theta}_4 - \left(\frac{2}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)\bar{\theta}_5 + \frac{1}{R_4}\bar{\theta}_6 + \frac{1}{R_6}\theta_a = 0$$

$$\bar{q}_6 + \frac{1}{R_5}\bar{\theta}_3 + \frac{1}{R_4}\bar{\theta}_5 - \left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)\bar{\theta}_6 + \left(\frac{1}{R_2} + \frac{1}{R_6}\right)\theta_a = 0$$

Vektorski zapis

$$A\bar{\Theta} + \bar{Q} + b\theta_a = 0$$

$$\bar{\Theta} = \begin{bmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \\ \bar{\theta}_3 \\ \bar{\theta}_4 \\ \bar{\theta}_5 \\ \bar{\theta}_6 \end{bmatrix} \qquad \bar{Q} = \begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \\ \bar{q}_4 \\ \bar{q}_5 \\ \bar{q}_6 \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & \frac{1}{R_4} & \frac{1}{R_3} & 0 & 0 & 0 \\ \frac{1}{R_4} & a_2 & \frac{1}{R_3} & 0 & 0 & 0 \\ \frac{1}{R_3} & \frac{1}{R_3} & a_3 & \frac{1}{R_5} & \frac{1}{R_5} & \frac{1}{R_5} \\ 0 & 0 & \frac{1}{R_5} & a_4 & \frac{1}{R_4} & 0 \\ 0 & 0 & \frac{1}{R_5} & \frac{1}{R_4} & a_5 & \frac{1}{R_4} \\ 0 & 0 & \frac{1}{R_5} & 0 & \frac{1}{R_4} & a_6 \end{bmatrix}$$

$$a_1 = -\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$
$$a_2 = -\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$
$$a_3 = -\left(\frac{2}{R_3} + \frac{3}{R_5} + \frac{2}{R_7}\right)$$
$$a_4 = -\left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)$$
$$a_5 = -\left(\frac{2}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)$$
$$a_6 = -\left(\frac{1}{R_2} + \frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right)$$

$$b = \begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} \\ \frac{1}{R_1} + \frac{1}{R_2} \\ \frac{2}{R_7} \\ \frac{1}{R_2} + \frac{1}{R_6} \\ \frac{1}{R_6} \\ \frac{1}{R_2} + \frac{1}{R_6} \end{bmatrix}$$