CIMPA
Algebraic and Tropical Methods for Solving Differential Equations

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1 Tropical Geometry

1.1 Tropical Algebra

Let us define an operation for $a, b \in \mathbb{R}$:

$$a + b = \max\{a, b\}$$

So every element is idempotent, that is, $a + a = a$ for all a. Notice we cannot have a zero element, for there is no element e such that $a + e = a$ for all a. So we must work on the set $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$.

And the product will be ordinary sum:

$$a \cdot b = a + b$$

We must also establish that $a \cdot -\infty = -\infty$ and $(-\infty) \cdot (-\infty) = -\infty$. This makes $(\mathbb{T}, +, \cdot)$ a semiring called the Tropical Semiring.

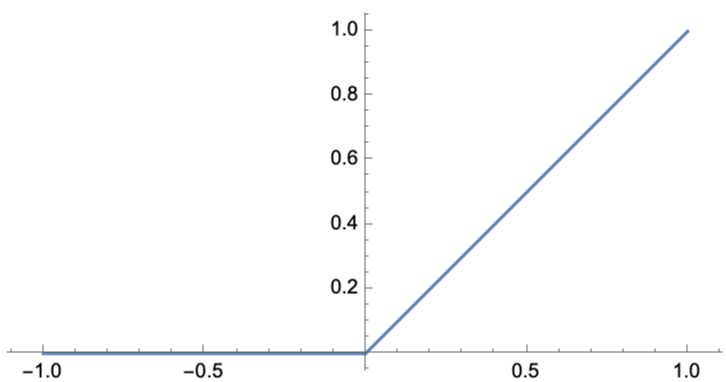
1.2 One variable polynomials

Notice these operations turn usual polynomials into linear equations:

$$\text{“}x^2\text{”} = \text{“}xx\text{”} = 2x$$
$$\text{“}x^2 + y^3\text{”} = \text{“}xxyyy\text{”} = 2x + 3y$$

We write with comas polynomials with usual operations.

Example 1. “$x + 0$” $= \max\{x, 0\}$ is a piecewise linear function with a singularity at 0.



Example 1

Definition (Root)**.** $r \in \mathbb{R}$ is a root of a polynomial f if we can write

$$f(x) = \text{“}\sum_{i=0}^{d} = c_i x^i\text{”} = \max_{i=0}^{d}\{c_i + ix\}$$

and max is reached of at least two monomials. That is, $r \in \mathbb{R}$ is a root if the function defined by f is not lineal in a neighbourhood of r.

Which corresponds to the root 0 of the polynomial in our example.

Example 2. Take the polynomial

$$\text{“}0 + (-1)x + x^2\text{”}$$

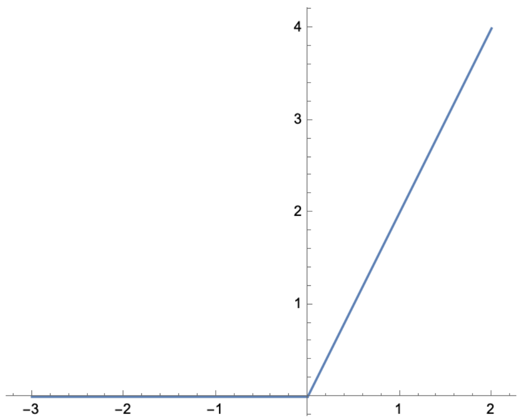


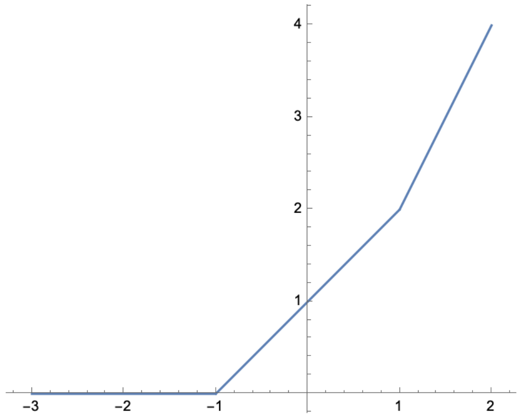
Figure 1

0 is a root of multiplicity 2, for 2 is the distance between the monomials that are not “ghosts” in the polynomial. Only -1 is a ghost coefficient, and the distance between 0 and 2 is 2.

Example 3. Take

$$\text{“}0x + 1 \cdot x + x^2\text{”} = \max\{0, x+1, 2x\}$$

Now we have two singularities, which ammount to two roots, and now both have multiplicity 1.



Example 3

Definition. For $f(x) =$ "$\sum_{i=0}^{d} a_i x^i$" and $x_0 \in \mathbb{R}$, we define

$$\mathrm{In}_{x_0} f(x) = \text{“}\sum_{i \text{ s.t. } f(x_0)=a_i x_0^i} a_i x^i\text{”}$$

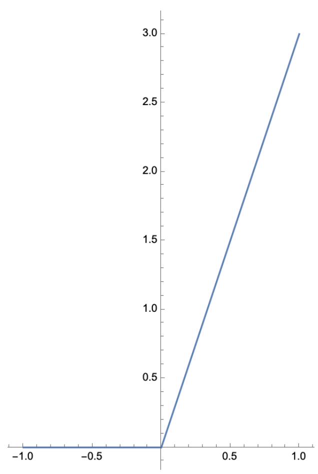
and

$$\mathrm{mult}_{\ f}(x_0) = \ell(\mathrm{conv}\ \{i : a_i x_0^i = f(x_0)\})$$

Example 4. Take

$$f(x) = \text{“}0 + x + x^2 + x^3\text{”}$$

For $x_0 < 0$ we have $\mathrm{In}_{x_0} f(x) = 0$ and $\mathrm{mult}_{\ f}(x_0) = 0$. For $x_0 > 0$ we have $\mathrm{In}_{x_0} f(x) =$ "x^3" and $\mathrm{mult}_{\ f}(x_0) = 0$. And for $x_0 = 0$ we have $\mathrm{In}_{x_0} f(x) = f(x)$ and $\mathrm{mult}_{\ f}(x_0) = 3$.



Theorem (Fundamental Theorem of Tropical Algebra)**.** For any polynomial of degree d,

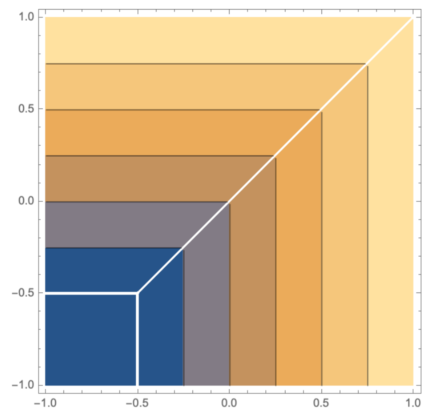
$$\sum_{r \text{ is a root of } f} \text{mult } r = d$$

Thus we now know what is the multiplicity of a root. We must agree that the multiplicity of $-\infty$ is the minimum exponent of the polynomial.

1.3 Two variables

Example 5.

$$f(x, y) = \text{“}0 + x + y\text{”} = \max\{0, x, y\}$$



Definition. For $f(x, y) = \text{“}\sum a_{ij}x^i y^j\text{”}$ define

$$\text{ex}_f = \{(i, j) : a_{ij} \neq 0\}$$

And then the Newton Polygon is

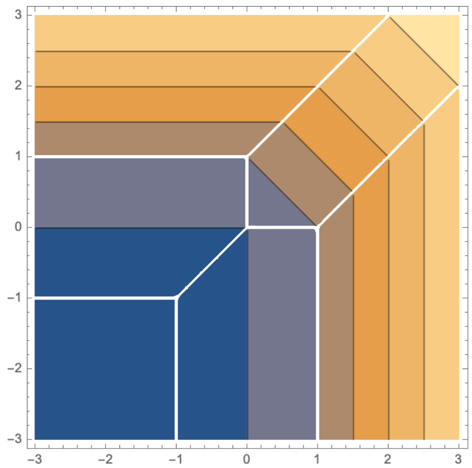
$$\text{conv } \{\text{ex}_f\}$$

So its a vertex for every monomial. And finally:

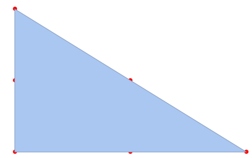
$$C_f := \{(x, y) : \text{In}_{(xy)} f \text{ is not a monomial}\}$$

Example 6. Check this one out:

$$f(x) = \text{“}0 + x + y + xy + x^2 + y^2\text{”} = \max\{0, x + 1, y + 1, x + y + 1, 2x, 2y\}$$



Algebraic curve



Newton polygon

Although the Newton polygon is the dual geometric object of the graph, it cannot be superposed with it because they live in different geometric spaces.

Anyway let's define

Definition. A graph in $\mathbb{R}^2$ is tropical if

1. Edges to rational
2. Weights in the edges
3. Balanced

Definition. Tropical hypersurface defined by f:

$$H_f = \{p : \mathrm{In}_p f \text{ is not a monomial}\}$$

And for an ideal I

$$N_I = \{p : \forall f \in I, \mathrm{In}_p f \text{ is not a monomial}\}$$

1.4 Tropicalization

We will learn how to use tropical geometry to study classical geometry. How can we go from classical polynomial to tropical ones?

Let's take a classical polynomial in $\mathbb{C}[X]$ and induce a tropical polynomial:

$$F(X) = \sum_{i=0}^{d} A_i X^i \rightsquigarrow f(x) = \text{“} \sum_{i=0, A_i \neq 0}^{d} x^i \text{”}$$

It looks to me that you basically get rid of the coefficients and then take the tropical operations.

Now let's try making the coefficients functions of t:

$$F_t(X) = \sum_{i=0}^{d} A_i(t)X^i \rightsquigarrow f(x) = \text{“}\sum_{i=0, A_i \neq 0}^{d} (-\gamma_1)x^i\text{”} := F^{\text{trop}}$$

where $A_i(t) = \gamma_j t^j$.

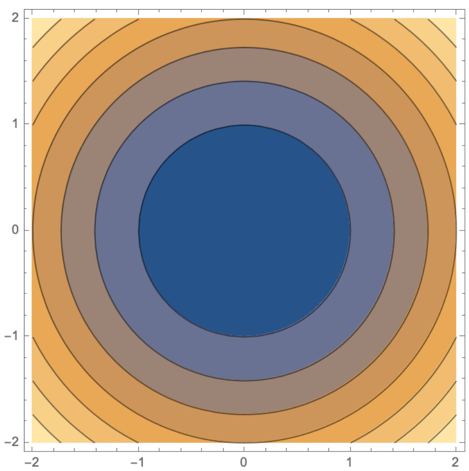
So this time you only take the first coefficient. Is there anything to correct here?

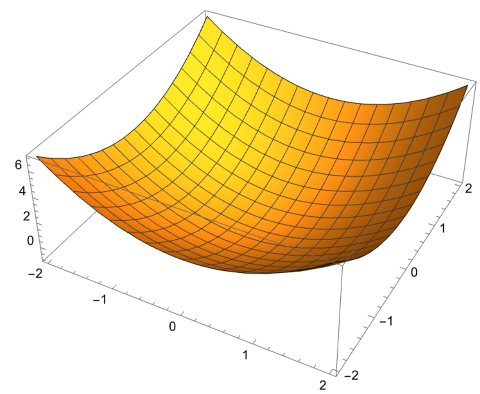
Theorem 1 (Kaparanov)**.**

$$\begin{tikzcd}
F_t \arrow[r, "zeros"] \arrow[d, "g"'] & \text{Surface of genus } g \arrow[d, "h"] \\
f \arrow[r, "k"'] & \text{Tropical curve with } g \text{ bounded regions}
\end{tikzcd}$$

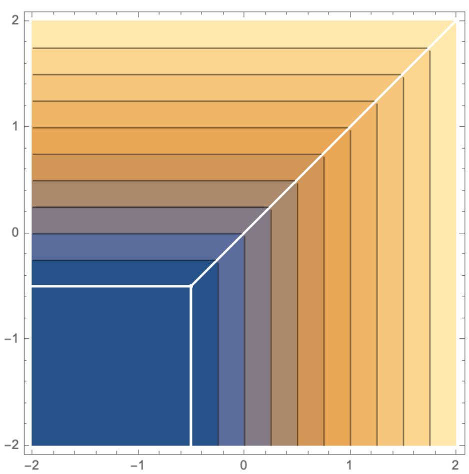
1.5 Dani's examples

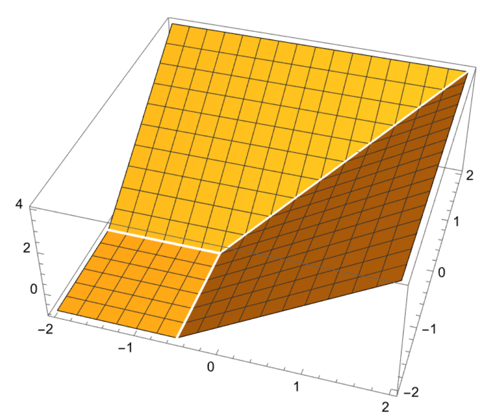
Example 7. Here are the roots of the polynomial $x^2 + y^2 - 1$ (with the usual operations in $\mathbb{R}[x, y]$) and its plot in $\mathbb{R}^3$:



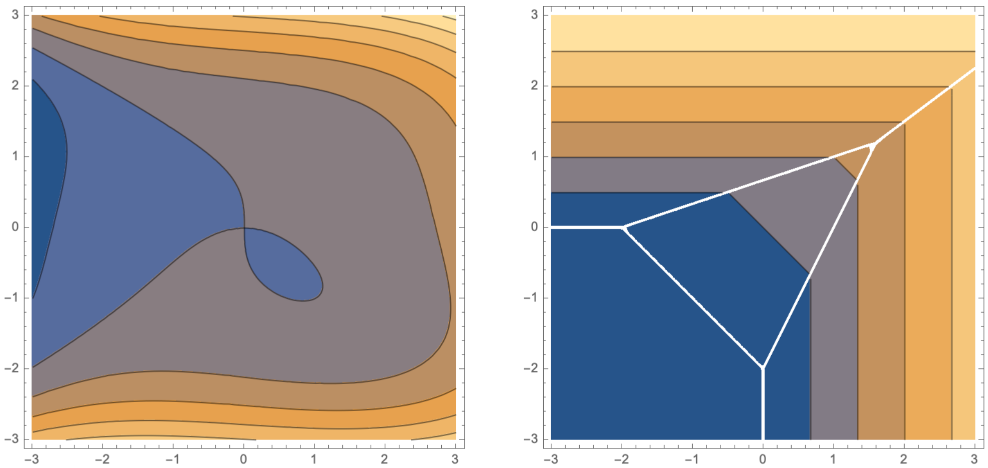


And now we do exactly the same but for the tropical polynomial "$x^2 + y^2 - 1$" $=$ $\max\{2x, 2y, 0\}$:





Example 8. Here are the roots of $x^3+2xy+y^4 \in \mathbb{R}[x, y]$ and a related tropical polynomial "$x^3 + 2xy + y^4 + 0$" $= \max\{3x, 2 + x + y, 4y, 0\}$:



Notice we must include the 0 in the tropical polynomial, because without it we do not get the triangle in the plot.

2 Computational commutative algebra

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2.1 Univariate polynomials and resultants

R a ring, like polynomials. It should be commutative and an integral domain ($a, b \in \mathbb{R}\backslash\{0\} \implies ab \neq 0$). Also take a field $\mathbb{K}$ and its algebraic closure $\bar{\mathbb{K}}$.

2.1.1 Univariate polynomials

Take the polynomial ring $k[x]$ and $f \in k[x]$ with $f = \sum_{i=0}^{d} c_i x^i$ for $c_i \in k$ and $c_d \neq 0$. We say $\deg f = d$.

Now let $k \subseteq \mathbb{K}$ and $p \in \mathbb{K}$ we define $f(p) = \sum_i c_i p^i \in \mathbb{K}$ the evaluation of f. And we say a root of f is p such that $f(p) = 0$.

Question Given $f_1, ..., f_r \in k[x]$, do they have a common root?

Definition. Given $f_1, ..., f_r \in k[x]$ we define the ideal $\langle f_1, ..., f_r \rangle = \{\sum g_i f_i : (g_1, ..., g_r) | ink[x]\}$

Prop. Let $p \in \mathbb{K}$. $\forall f \in \langle f_1, ..., f_r \rangle$, $f(p) = 0 \iff f_i(p) = 0 \forall i$.

Obs. If $g \in \langle f \rangle$, then if $p \in \mathbb{K}$ is such that $f(p) = 0$ then $g(p) = 0$.

Prop. Given $f, g \in k[x]$, there are unique $(q, r) \in k[x]^2$ such that $f = q \cdot g + r$ and $r = 0$ or $\deg r < \deg g$.

We define rem $(f, g) = r$ and write $f|g$ if and only if rem $(f, g) = 0$.

Theorem 2. $k[x]$ is a principal ideal domain. That is, for every ideal $\langle f_1, ..., f_r \rangle$ there exists one polynomial g such that $\langle f_1, ..., f_r \rangle = \langle g \rangle$. g is called GCD of $f_1, ..., f_r$.

I challenge to prove that

Prop. $GCD(f_1, ..., f_r)$ is the smallest polynomial with degree such that if $(\forall i) h | f_i$, then $h | GCD(f_1, ... f_r)$.

Theorem 3 (Euclidean algorithm)**.** Input: $f, g \in k[x]$ with $\deg f \geq \deg g$.

Output: $r \in k[x]$ such that $\langle f, g \rangle = \langle r \rangle$.

Here it goes:

$$r_{-1} = f, r_0 = g, i = 0$$
$$i = i + 1$$
$$r_i = \text{rem } (r_{i-2}, r_{i-1})$$
$$\text{Return } r_{i-1}$$

Theorem 4. Euclidean Algorithm terminates and is correct.

Proof. It terminates because $\forall i \geq 1)$ $\deg(r_i) > \deg(r_{i+1})$ Hence $\exists i_*$ such that $r_{i_*} = 0$. Observe that for each i, $\exists q_i \in k[x]$ such that $r_{i-2} = r_{i-1}q_i + r_i$. Hence $\langle r_{i-2}, r_{i-1}\rangle = \langle r_{i-1}, r_i\rangle$.
$h_1 r_{i-2} + h_2 r_{i1} \iff (h_1 q_1 + h_2) r_{i-1} + h_i r_i$.
Therefore, $\langle f, g\rangle = \langle r_{-1}, r_0 = ...)\langle r_{k-1}, r_{i_*}\rangle$ □

HW Prove that $GCD(f_1, f_2, f_3) = GCD(GCD(f_1, f_2), f_3)$.

Conclusion If $GDC(f_1, ..., f_r) = 1$, then there are no common solutions.
If $GCD(f_1, ..., f_r) \neq 1$. Then $f_1, ..., f_r$ have ea common factor if $k = \bar{k}$ and $\exists p \in k$ such that $f_1(p) = ... = f_r(p) = 0$.

Now consider a UFD ring R like $\mathbb{C}[y], \mathbb{C}[x, y]$.

Definition (Resultant)**.** Given $f(x, y) = \sum_{i=1}^m f_i(y)x^i \in \mathbb{C}[x, y]$ that is, $f \in \mathbb{C}[y]$, and $g = \sum_{i=1}^m g_i(y)x^i$. We define Sylvester matrix Sylv (f, g, x).

$$\begin{pmatrix} f_m & 0 & ... & 0 & g_m & 0 & ... & 0 \\ f_{m-1} & f_m & ... & 0 & g_{m-1} & g_m & ... & 0 \\ f_0 & f_1 & ... & 0 & g_0 & g_1 & ... & 0 \\ f_0 & f_0 & ... & f_0 & g_0 & g_0 & ... & g_0 \end{pmatrix}$$

The resultant $Res(f, g, x) = \det \text{Sylv } (f, g, x) \in \mathbb{C}$.

We may substitute $\mathbb{C}$ with any ring R.

Obs. Sylv (f, g, x) represents $\overline{\text{Sylv}}_{f,g,x}(A, B) \mapsto Af + Bg = \sum c_i x^i$ with $c_i \in \mathbb{R}$. Where $\deg_x(A) < \deg_x(g)$ and $\deg_x(B) < \deg_x(f)$. So A and B are polunomials of the form $A = \sum_{i=0}^{m-1} A_i x^i$ and $B = \sum_{i=1}^{m-1} B_i x^i$. So with two polynomials I obtain another polynomial.

Prop. Res $f, g, x = $ img Sylv $_{f,g,x}$ that is, $\exists A, B$ such that $\deg A < \deg g$, $\deg B < \deg A$ such that $Af + By = $ Res (f, g, x).

HW Prove it. Use adjugate matrix of Sylv (f, g, x).

Prop. If $f, g \in k[x]$. The Res $(f, g, x) = 0 \iff GCD(f, g) \neq 1$.

2.1.2 Solving bivariate polynomial systems

Example 9.

$$\text{Sylv } (f, g) = \begin{pmatrix} y & 0 & 2y & 0 \\ y^2 + y & y & 0 & 2y \\ 1 & y^2 + y & -y^2 + 3 & 0 \\ 0 & 1 & 0 & y^2 + 3 \end{pmatrix}$$

Then Res $(f, g, x) = \underbrace{y^2}_{\text{does not lead to a solution}} \underbrace{(2y^2 - 3y - 1)(y + 1)^3}_{\text{lead to solutions!}}$

Theorem 5. If $(p_x, p_y) \in \mathbb{C}^2$ are such that $f(p_x, p_y) = g(p_n, p_y) = 0$ then Res $(f, g, x)|_{y=p_y} = 0$.

Proof. $\exists A, B$ such that $Af + Bg = \text{Res }(f, g, x) = 0 = \text{Res }(f, g, x)|_{y=p_y}$. $\square$

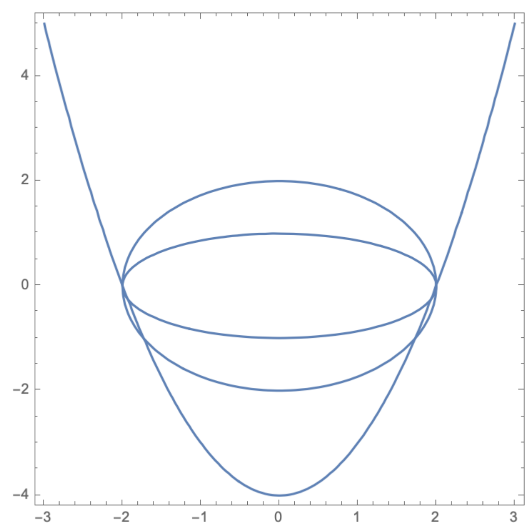
Theorem 6 (Extension theorem)**.** Given $f, g \in \mathbb{C}[x, y]$ as before, let p_y be such that Res $(f, g, x)|_{y=p_y} = 0$. Then if $f_m(p_i) \neq 0$ or $g_m(p_y) \neq 0$, then $\exists p_x$ such that (p_x, p_y) is solution of $f(x, y) = g(x, y) = 0$.

2.2 Ideals and varieties

Example 10. Consider the polynomials

$$\begin{aligned} f_1 &= x^2 + 4y^2 - 4 \\ f_2 &= x^2 + y^2 - 4 \\ f_3 &= x^2 - y - 4 \end{aligned}$$

How can we verify that f_3 vanishes at common zeros of (f_1, f_2)?



Let $R = \mathbb{C}[x_1, ..., x_m]$, $f \in \mathbb{R}$ with $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}} c_\alpha x^\alpha$ where $c_\alpha \in \mathbb{C}$ there finite α such that $c_\alpha \neq 0$, x^α is monomial , $x^\alpha = \prod_{i=1}^m x_i^{\alpha_i}$.

The degree is $\deg f = \max_{\alpha \in \mathbb{Z}_{\geq 0}}(\sum \alpha_i : c_\alpha \neq 0)$.

Ok now fiven $p \in \mathbb{C}^m$, we can evaluate f doing $f(p) = \sum c_\alpha p^\alpha$. And we say p is a zero if $f(p) = 0$.

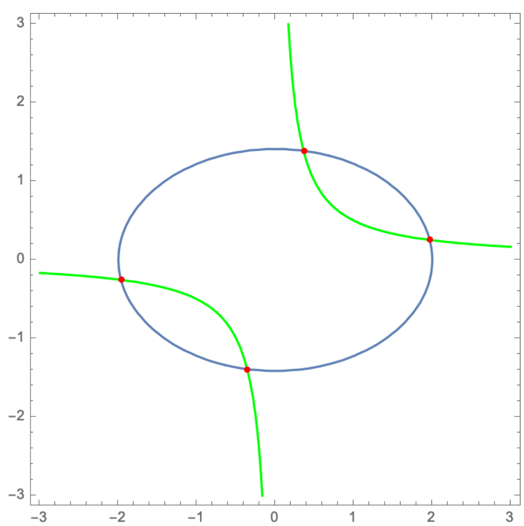
Definition. Given $f_1, ..., f_r \in \mathbb{R}$ we define the affine algebraic variety $V(f_1, ..., f_r) = \{p \in \mathbb{C}^m : (\forall i) f_i(p) = 0\}$. And in general, if $I \subseteq R$ is an ideal, we define

$$V(I) = \{p \in \mathbb{C}^m : (\forall i) f \in I = 0\}$$

.

Example 11. Take

$$f_1 = x^2 + 2y^2 - 4$$
$$f_2 = 2xy - 1$$



Definition. An ideal I is a subset of R such that $(\forall f, g \in I) f + g \in I$ and $(\forall f \in I)(\forall r \in \mathbb{R}) fr \in I$.

Prop. Given $f_1, ..., f_r \in R$, we define the generated ideal as

$$\langle f_1, .., f_r \rangle = \{\sum_{i=1}^{r} g_i f_i : g_1, ..., g_r \in R\}$$

Prop. $V(\langle f_1, ..., f_r \rangle) = V(f_1, ..., f_r)$.

Definition. The ring R is Noetherian if when $I \subseteq R$ is an ideal then $\exists f_1, ..., f_r$ such that $I = \langle f_1, ..., f_r \rangle$.

2.2.1 Operations on ideals

Prop. Fix $I, J \subseteq R$ ideals. Then:

1. $I \subseteq J \implies V(J) \subseteq V(I)$
2. $I + J = \{f + g : f \in I, g \in J\}$ is an ideal.
3. $I \cap J$ is an ideal and $V(I \cap J) = V(I) \cup V(J)$.

2.2.2 Hilbert's Nullstellensatz

Definition. Let $W \subseteq \mathbb{C}^n$, we define the ideal

$$I(W) = \{f \in R : (\forall p \in W) f(p) = 0\}$$

.

Prop. Given $W \subseteq \mathbb{C}^n$, $V(W)$ is the smallest with inclusions of the affine algebraic varieties that contain W. If $W \subseteq Z$ and Z variety, then $V(I(W)) \subseteq Z$. Also $\bar{W} = V(I(W))$.

Theorem 7 (Hilbert's Nullstellensatz)**.** Let $I \subseteq R$ be an ideal and $f \in R$. Then, $f \in I(V(I))$ if and only if $f^k \in I$ for some $k \in \mathbb{N}$.

Example 12. This is actually Example 5.

$$\begin{aligned} f_1 &= x^2 + 4y^2 - 4 \\ f_2 &= x^2 + y^2 - 4 \\ f_3 &= x^2 - y - 4 \end{aligned}$$

It turns out that $f_r \notin \langle f_1, f_2 \rangle$. But actually $f_3^2 = (x^2 + \frac{4}{3}y^2 + \frac{2}{3}y - \frac{11}{3})f_1 + (\frac{16}{3}y^2 + \frac{8}{3}y + \frac{1}{2})f_2$ is.

Definition. Given $I \subseteq R$ we define its radical ideal as

$$\sqrt{I} = \{f \in R : (\exists k \in \mathbb{N}) f^k \in I\}$$

Obs. By Hilbert's Nullstellensatz, $\sqrt{I} = I(V(I))$.

Obs. It is necesary that the field we are working on is algebraically closed for Hilbert's Nullstellensatz to be valid.

Coro. 1 (Weak version of Hilbert's Nullstellensatz)**.** $V(I) = \emptyset \iff 1 \in I$

2.2.3 From geometry to algebra

Prop. Given $v, w \in \mathbb{C}^m$ varieties,

1. $I(V \cap W) = \sqrt{I(V) + I(W)}$
2. $I(V \cup W) = I(V) \cap I(W)$
3. Given ideals $I, J \subseteq R$, $\sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}$

2.2.4 Radical membership

Question Given I ideal and f polynomial, does $f \in \sqrt{I}$? Let us introduce a new variable t to define

$$R[t] = \mathbb{C}[x_1, ..., x_n, t]$$

.

Theorem 8. Let $I = \langle f_1, ..., f_r \rangle \subseteq R$. Consider $g \in R$. Then

$$g \in \sqrt{I} \iff 1 \in \langle f_1, ..., f_r, gt - 1 \rangle \subseteq R[t]$$

.

Proof. By Hilbert's Nullstellensatz, $(\forall p \in V(I))g(p) = 0$. And by Weak Hilbert's Nullstellensatz, $V_{\mathbb{C}^{m+1}}(f_1, ..., f_r, gt - 1) =$. We will prove these two conditions are also equivalent.

We have:

$$(p_1, ..., p_m, p_0) \in V_{\mathbb{C}^{m+1}}(f_1, ..., f_r) \iff \begin{cases} f_1(p_1, ..., p_m) = 0 \\ ... \\ f_r(p_1, ..., p_m) = 0 \\ g(p_1, ..., p_m)p_0 - 1 = 0 \end{cases}$$

The first r equations are equivalent to $(p_1, ..., p_m) \in V_{\mathbb{C}^{m+1}}(f_1, ..., f_r)$. The last equation says $p_0 = \frac{1}{g(p_1,...,p_m)}$ and that $g(p_1, ..., p_m) \neq 0$.

Anyway we get

$$V_{\mathbb{C}^{m+1}}(f_1, ..., f_r, gt - 1) = \iff (\forall p \in V_{\mathbb{C}^m}(I))g(p) = 0$$

$\square$

Example 13 (The Rabinowitsch trick, 1929)**.** Use Hilbert's Weak Nullstellensatz to prove Hilbert's Nullstellensatz.

Hint If $(\forall p \in V(I))g(p) = 0$, then $\exists h_1, ..., h_r, h_0 \in R[t]$ such that $1 = \sum h_i f_i + h_0(gt-1)$ Replace t by $\frac{1}{g(x_1,...,x_m)}$ symbolically and clean denominators.

Solution. Let $g \in I(V(I))$ for some ideal $I = \langle f_1, ..., f_r \rangle$, that is, g vanishes whenever all the f_i vanish. Then the polynomials $f_1, ..., f_r, gt - 1$ cannot vanish all at the same time, so that the zeroes of ideal generated by all of them is empty. By Hilbert's Weak Nullstellensatz, this ideal must be the unit ideal (1 is in there). All this is exactly the first phrase in the **Hint**: there must $\exists h_1, ..., h_r, h_0 \in R[t]$ such that $1 = \sum h_i f_i + h_0(gt - 1)$.

When substituting t by $\frac{1}{g(x_1,...,x_m)}$ we obtain that

$$1 = \sum h_i\Big(\frac{1}{g(x_1, ..., x_m)}, x_1, ..., x_n\Big) f_i\Big(\frac{1}{g(x_1, ..., x_m)}, x_1, ..., x_n\Big)$$

Notice that the expression above is a sum that may have many coefficients of the form $\left(\frac{1}{g(x_1,...,x_m)}\right)^k$. Actually, any denominator on this sum must be of this kind. Thus we can write

$$1 = \frac{\sum h_i(x_1,...,x_n) f_i(x_1,...,x_n)}{g(x_1,...,x_n)^r}$$

for some r which makes $g(x_1,...,x_n)^r$ the common denominator. We have shown $g \in \sqrt{(V(I))}$ $\square$

2.3 Gröbner bases

2.4 Elimination theory

2.5 Bibliography

3 Real tropical geometry

Let us define a family of polynomials of degree d parametrized by t:

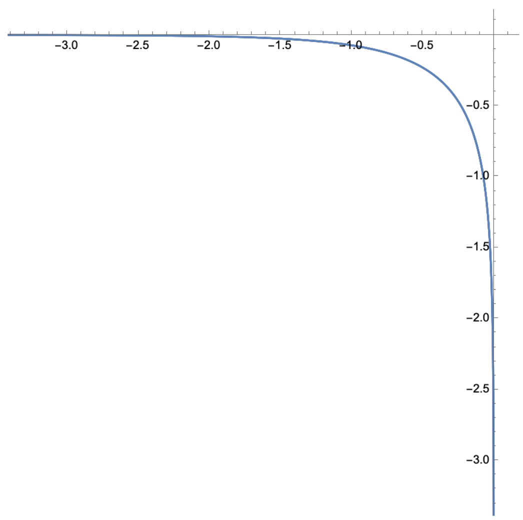
$$p_t(x,y) = \sum_{i+j\leq d} c_{i,j} t^{v_{i,j}} x^i y^j$$

Now consider de map

$$\begin{aligned} \mathrm{Log}_t : (\mathbb{C}^\times)^2 &\to \mathbb{R}^2 \\ (x,y) &\mapsto \Big(\frac{\log|x|}{\log t}, \frac{\log|y|}{\log t}\Big) \end{aligned}$$

And then define for a curve $\mathcal{C} \subset (\mathbb{C}^\times)^2$ the amoeba of $\mathcal{C}$ as $\mathrm{Log}_t(\mathcal{C})$. Notice $\mathcal{C}$ is a *complex* curve, which is a real surface. So the image of our function is a region in $\mathbb{R}^2$.

Here's Dani's incomplete attempt of an amoeba:



And now define as in Lucía's talk:

$$\mathfrak{p} = \text{“}\sum_{i+j\leq d} c_{ij} x^i y^j\text{”} = \max_{i+j\leq d}\{c_{ij} + ix + jy\}$$

Taking edges as perpendicular lines to edges in a tropical conic, and vertices as the centres of regiones determined by the conic, we obtain a subdivision of the plane that looks like the dual tiling of the conic.

Obs. The genus of the curve equals the ammount of bounded regions determined by the tropical curve.

Then he takes the amoebas of the real parts of curves and he obtains plane curves. He then 'unfolds' these curves by taking each part of the curve that is lying in each of the four quadrants of the plane, and then he pastes back each of these pieces.

And then take the limit as $t \to \infty$ and finally folding them back. This is some sort of combinatorial procedure. And it turns out that:

Theorem (Viro, 1976)**.**

$$(\mathbb{R}^2, T_{\mathbb{R}}) \cong (\mathbb{R}^2, \mathcal{C}_{\mathbb{R}})$$

where $\cong$ is homeomorphic. So they have the same Newton polygon and same degree.

Where $T_{\mathbb{R}}$ is the real part he's been working with in the last two paragraphs and $\mathcal{C}_{\mathbb{R}}$ looks like the whole complex curve (real surface).

Example 14 (Dani's attempt to use Benoit's Log function)**.** Let us try to evaluate our Log_t function in a complex algebraic curve. We must remember that the zeroes of polynomials of the form $y^2 = \prod_{k=1}^{2g+1}(x - a_k)$ are called *hyperelliptic curves* when $g > 1$ and *elliptic curves* when $g = 1$ (According to [1]).

Taking $g = 1$, we have a curve of genus 1, that is, a torus. We hope that when we evaluate Log in this torus we get exactly one bounded region in the resulting drawing. The polynomial in this case is

$$y^2 = (x - a_1)(x - a_2)(x - a_3)$$

References

[1] E. Girondo and G. González-Diez. *Introduction to Compact Riemann Surfaces and Dessins D'Enfants*. Introduction to Compact Riemann Surfaces and Dessins D'enfants. Cambridge University Press, 2012. ISBN: 9780521519632.