Exercises in algebraic geometry

If not explicity stated, exercises are from Hartshorne.

Contents

1 Chapter I

1.1 Noções básicas

- Ussando que $\mathbb{R} = k[x_1, \ldots, x_n]$ é Noetheriano (=todo ideal é finitamente generado), vimos que todo aberto de Zariski é uma união finita de conjuntos da forma.

- Se $X \subseteq \mathbb{A}^n$ é uma variedade afim (=existe $\mathfrak{a} \subset R$ tal que $X = \mathbb{V}(\mathfrak{a})$), os conjuntos fechados de X são $\mathbb{V}(\mathfrak{b}) \cap \mathbb{V}(\mathfrak{a}) = \mathbb{V}(\mathfrak{a} + \mathfrak{b})$

- Nullstellenstz fraco:

$$\text{Spm}(R) := \{\text{ideais maximais de } R\} = \{\langle x_1 - a_1, \ldots, x_n - a_n \rangle : a_i \in k\}$$

- Nullstellensatz:

$$\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \sqrt{\mathfrak{d}} \qquad \mathbb{V}(\mathbb{I}(S)) = \overline{S}$$

- Corolário:

$$\{\text{affine varieties of } \mathbb{A}^n\} \xrightarrow{\;1-1\;} \{\text{radical ideals of R}\}$$

$$\{\text{irreducible varieties of } \mathbb{A}^n\} \longleftrightarrow \text{Spec}(R) = \{\text{ideais primos}\}$$

$$\{\text{pontos de } \mathbb{A}^n\} \longleftrightarrow \text{Spm}(R)$$

- Para duas variedades $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$,

$$\text{Hom}(X, Y) = \{\varphi : X \to Y | \exists \tilde\varphi = (f_1, \ldots, f_m), f_i \in R\}$$

  e as f_i extendem a φ.

- Tem uma equivalência de categorias

$$(\text{affine alg. varieties})^{\text{op}} \to \text{reduced f.g. } k\text{-algebras}$$

  enviando cada variedade X a $R[X]$.

- Dimensão de Krull (definida na seção 3 deste documento), o supremo das alturas de cadeias de ideiais primos, height.

- Teorema de Krull.

- A imagem de um morfismo não é necessesariamente uma variedade, porém, pode pegar o fecho e tudo bem.

$$D_f = \mathbb{A}^n \setminus \mathbb{V}(f).$$

1.2   My first algebraic variety

Exercise 1.1 (My first algebraic variety)

(a) Let Y be the plane curve $y = x^2$ (ie., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

*(c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Proof.

(a) Consider the map

$$k[x,y] \to k[x]$$
$$1 \mapsto 1$$
$$x \mapsto x$$
$$y \mapsto x^2$$

Notice that $y - x^2 \in k[x,y]$ is mapped to 0, so the kernel of this map is $(y - x^2)$. It is also surjective, so we have $A(Y) = k[x,y]/(y-x^2) \cong k[x]$.

(b) In constructing a map like in the former exercise, we may fix 1 and x, and we should map y to $1/x$. However, $1/x$ is not an element of $k[x]$ so we really have an isomorphism $k[x,y]/(xy-1) \cong k[x, \frac{1}{x}] \not\cong k[x]$.

(c) (From StackExchange)

Step 1 Factorize the degree 2 homogeneous part into linear factors using that k is algebraically closed.

Step 2 If these linear factors are linearly dependent,

  i. Change coordinates to make the linear factor be the new X. The equation becomes $X^2 + aX + bY + c$.

  ii. Change coordinates to make $aX + bY + Z$ be the new Y.

  iii. We get $X^2 + Y$.

Step 3 If these linear factors are linearly independent,

  i. Change coordinates to make one of the linear factors be X and the other Y. The equation becomes $XY + aX + bY + c$ which can also be written as $(X + a)(Y + b) + d$.

  ii. Change the coordinates again to make it $XY + d$.

$\square$

1.3 The Segre embedding (extra)

Exercise 2.14 (The Segre Embedding)   Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the order pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the **Segre embedding**. Show that the image of ψ is a subvariety of $\mathbb{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} : i = 0, \ldots, r, j = 0, \ldots, s\}$ and let $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which sends z_{ij} to $x_i y_j$. Then show that $\operatorname{img} \psi = Z(\mathfrak{a})$.

Solution. First let's make sure the dimension N is correct. The easy way is found in wiki: $N = (r+1)(s+1) - 1$ which is the number of possible choices of pairs of things taking

one out $r + 1$, another out of $s + 1$, and then remember there is only one zero index so take one away.

To see that ψ is injective we follow StackExchange: Let $z = [z_{00} : z_{01} : \ldots : z_{ij} : \ldots : z_{rs}]$ be an element of the image of ψ and let $(a, b) \in \mathbb{P}^r \times \mathbb{P}^s$ be such that $\psi(a, b) = z$. WLOG we can assume $a_0 = b_0 = z_{00} = 1$. Then $b_j = z_{0j}$ for all $0 \leqslant j \leqslant s$ and $a_i = z_{i0}$ so a, b are uniquely determined and this map is bijective onto the image.

Actually, what we have done is constructed an inverse morphism of the Segre map. According to StackExchange, this makes it into an embedding.

To show that $\operatorname{img} \psi$ is a subvariety of $\mathbb{P}^N$ we need to find a set of homogeneous polynomials in $k[z_{ij}]/$

Following the hint, as before let $z \in \operatorname{img} \psi$ and f any polynomial in the kernel of

$$k[\{z_{ij}\}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s]$$

. We must show that $f(z) = 0$. Well it doesn't make much sense because if $f = \sum a_{ij} z_{ij}$ is in the kernel of that map, then its image $\sum a_{ij} x_i y_j$ is the zero polynomial, so obviously $f(z) = \sum a_{ij} z_{ij} = \sum a_{ij} x_i y_j = 0$. So this is confusing.

So what are the equations of $\operatorname{img} \psi$? A polynomial $f(z_{00}, \ldots, z_{rs})$ will vanish on $\operatorname{img} \psi$ if somehow it vanishes $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

1.4 The quadric surface in $\mathbb{P}^3$ (Bruno)

Exercise 2.15 (The Quadric Surface in $\mathbb{P}^3$) Consider the surface Q (a **surface** is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - wz = 0$.

1. Show that Q is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

2. Show that Q contains two families of lines (a **line** is a linear variety of dimension 1), $\{L_t\}, \{M_t\}$ each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$ then $L_t \cap L_u = \varnothing$ and if $M_t \neq M_u$, $M_t \cap M_u = \varnothing$, and for all t, u, $L_t \cap M_u$ is a point.

3. Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ where each $\mathbb{P}^1$ has its Zariski topology.

Solution.

1. It turns out that the image of the Segre embedding $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ equals is the algebraic variety given by the zeroes of the polynomial $f = z_{00} z_{11} - z_{10} z_{01} \in k[z_{00}, z_{01}, z_{10}, z_{11}]$. One contention is easy: if $(x, y) = ([x_0, x_1], [y_0, y_1]) \in \operatorname{img} \psi$, then clearly $f(\psi(x, y)) = x_0 y_0 x_1 y_1 - x_0 y_1 x_1 y_0$ is zero because these are numbers in the field k.

   Now for the other contention pick $z = [z_{00}, z_{01}, z_{10}, z_{11}] \in V(f)$ and let's find an element $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $\psi(x, y) = z$. $z \in V(f)$ means that $z_{00} z_{11} = z_{10} z_{01}$. If $z_{00} \neq 0$, then we can define $([z_{00}, z_{11}], [z_{01}, z_{10}])$ what?

Maybe for the other contention try to define the inverse map $\operatorname{img}\psi \to \mathbb{P}^1 \times \mathbb{P}^1$ by $z = [z_{00}, z_{01}, z_{10}, z_{11}] \mapsto ([z_{00}, z_{01}], [z_{00}, z_{10}])$ when $z_{00} \neq 0$ and $([z_{11}, z_{01}], [z_{11}, z_{10}])$ when $z_{11} \neq 0$. Is this defining a global map?

2. The lines correspond to fixing one entry and running over the other one in the Segre embedding $(x, y) \to z$.

$\square$

1.5   Summary of section 3

1.5.1   In Hartshorne

Definition (Regular function)   It is a function from a variety to the field such that at every point there is an open neighbourhood (with respect to the Zariski topology, I guess) such that the function equals the quotient of two polynomials. In the affine case the polynomials are thought as functions on the coordinates of $\mathbb{A}^n$, and in the projective case they must be homogeneous polynomials *of the same degree* (which (apparently) makes the quotient be a well-defined function on $\mathbb{P}^n$, in contrast with general homogeneous polynomials that are not functions).

Here's a very nice reminder of how homogeneous coordinates work taken from wiki:

1. Any point in the projective plane is represented by a triple (X, Y, Z), called ***homogeneous coordinates*** where X, Y and Z are not all 0.

2. The point represented by a given set of homomogeneous coordinates is unchanged if the coordinates are multiplied by a common factor.

3. Conversely, two sets of homogeneous coordinates represent the samer point if and only if one is obltained from the other by multipliying all the coordinates by the same non-zero constant.

4. When Z is not 0 the point represented it the point $(X/Z, Y/Z$ in the Euclidean plane.

5. Whan Z is 0 the point represented is the point at infinity.

6. (The triple $(0, 0, 0)$ does not represent any point. The origin of the Euclidean plane is represented by $(0, 0, 1)$.

Definition (Morphism)   A map between two varieties $\varphi : X \to Y$ is a ***morphism*** if it pulls back regular functions on Y to regular functions on X, ie. if $f : V \subseteq Y \to k$ is regular, then $f \circ \varphi$ is regular too.

An ***isomorphism*** is a morphism that admits an inverse morphism.

Remark   An isomorphism must be bijective and bicontinuous but a bijective bicontinuous morphism need not be an isomorphism.

Definition (Rings of regular functions) $\mathcal{O}(Y)$ is the ring of all regular functions on the variety Y and $\mathcal{O}_P(Y)$ is the ***local ring*** consisting on functions defined on neighbourhoods of $P \in Y$, any two identified if they coincide near P. This is in fact a local ring with maximal ideal the functions that vanish at P.

Proposition (3.5) Let X be any variety and let Y be any affine variety. Then there is a natural bijective mappingng of sets

$$\alpha : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(A(Y), \mathcal{O}(X))$$

where the left Hom means morphism of varieties and the right Hom means homomorphisms of k-algebras.

Corollary (3.8) The functor $X \mapsto A(X)$ induces an arrow-reversing equivalence of categories between athe category of affine varieties over k and the category of finitely generated integral domains over k.

1.5.2 In da Silva, Symplectic Toric Manifolds

I found in this book a great crash-course in introductory (complex) algebraic geometry.

A **Zariski closed set** on $\mathbb{C}^n$ is a set of common zeroes of a finite number of polynomials from $\mathbb{C}[z_1, \ldots, z_n]$. The fact that an infinite intersection of closed sets is a closed set follows from the stabilization property for zero sets of polynomials: any decreasing sequence of such sets $X_1 \supset X_2 \supset \ldots$ stabilizes (this is a restatement of Hilbert's basis theorem that any ideal in $\mathbb{C}[z_1, \ldots, z_n]$ is finitely generated). Notice any nonempty open set is dense so Zariski topology is not Hausdorff.

Definition An *affine variety*

1.6 Conics in $\mathbb{A}^2$ and $\mathbb{P}^2$ (Dani)

Exercise 3.1   a. Show that any conic in $\mathbb{A}^2$ is isomorphic either to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$.

  b. Show that $\mathbb{A}^1$ is not isomorphic to any proper open subset of itself.

  c. Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

  d. We will see later that any two curves are homeomorphic. But show now that $\mathbb{A}^2$ is not even homeomorphic to $\mathbb{P}^2$.

  e. If an affine variety is isomorphic to a projective variety, then it consists of only one point.

Solution. I understand a **conic** to be zero-set of an (irreducible) quadratic polynomial in either $k[x, y]$ (affine) or in $S[x, y, z]$.

  a. In exercise 1.1 it is shown that any conic in $\mathbb{A}^2$ has coordinate ring isomorphic to either $k[x]$ or $k[x, \frac{1}{x}]$. But the coordinate ring of $\mathbb{A}^1 \setminus \{0\}$ is not isomorphic to $k[x]$, so we are done.

  b. Notice that an open subset of $\mathbb{A}^1$ is $\mathbb{A}^1$ minus a finite a set of points, say $a_1, \ldots, a_m$. The coordinate ring of such a variety is $k[x]/(x - a_i)_i$, which cannot be isomorphic to $k[x]$. I have read that in such a ring $x - a_i$ is a *unit*, while I think it is just zero (the zero class=additive identity).

  c.

Step 1 Let $f \in k[x, y, z]$ be an irreducible homogeneous polynomial of degree 2. Then f can be written as $x^T M x$ with M a 3×3 symmetric matrix. But any such matrix is diagonalizable because k is closed, meaning there is another matrix Q such that $Q^T M Q = D$ with D diagonal.

Step 2 Q defines a morphism of $\mathbb{P}^n$ to itself since it is composed of linear polynomials (recall that a morphism must pull back regular functions to regular functions, and in the projective case regular functions are locally quotients of homogeneous polynomials of the same degree, so a linear transformation preserves such a structure).

Step 3 Q restricts to an isomorphism on the zeroes of the initial matrix M and the diagonal matrix D, meaning any two conics are isomorphic.

Step 4 The image of the so-called 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is also a conic, so all conics are isomorphic to $\mathbb{P}^1$.

d. Algebraic topology.

e. The ring of regular functions on a projective variety is isomorphic to k by Thm 3.4. But then $A(Y) \cong \mathcal{O}(X)$ by Prop 3.5, which is only true for the coordinate rings of maximal ideals, that correspond to points.

$\square$

1.7 Homeomorphisms that are not isomorphisms (Dani)

Exercise 3.2 A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism (of algebraic varieties). (Bicontinuous morphism need not be isomorphism.)

a. For example, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

b. For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the ***Frobenius morphism.***

Solution.

a. First notice that φ is bijective on its image simply because it is injective.
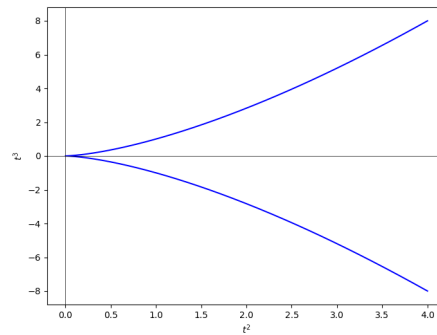
It is clear that it is a morphism.

To see if it is continuous first remember that affine space (and also projective space) is equipped with the Zariski topology, whose open sets are complements of algebraic sets (=sets of zeroes of polynomials). But $A = k[x]$ is principal so every algebraic set is the set of zeroes of one polynomial, basically because of this corollary of the Nullstellensatz:

Corollary ([Hart] 1.4] There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in $k[x_1, \ldots, x_n]$ (i.e., ideals which are equal to their own radical), given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible iff its ideal is a prime ideal.

And k is closed so every polynomial n roots, so algebraic sets are just finite sets.

This was seemed relevant at some point...

Definition The *(Krull) dimension* of a ring A is the supremum of all *heights* of all prime ideals, i.e. the largest proper chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}$.

Proposition (1.7) If Y is an affine algebraic set, then the dimension of Y is equal to the dimension of its affine coordinate ring $A(Y)$.

We only need to know that closed sets are finite points. This means that polynomials are continuous functions with respect to Zariski topology (because preimage of a closed set=finite set of points is closed because it is the root of $p - \alpha$). That makes φ continuous.

Being bicontinuous is the same as being a homeomorphism, which is equivalent to showing that φ is open or closed (since we already noted it is bijective). But, finite sets of points are closed sets in the curve (because it is put in affine space—this is not true in general!).

Finally we show that φ is not an isomorphism. Yesterday with Victor we tried to

9

write the inverse map, which should satisfy

$$(x,y) = (\varphi^{-1}(x,y)^2, \varphi^{-1}(x,y)^3)$$

$$\implies \begin{cases} \varphi^{-1}(x,y)^2 & = x = \frac{x^2}{x^3} = \frac{x^2}{y^2} = \left(\frac{x}{y}\right)^2 \\ \varphi^{-1}(x,y)^3 & = y = \frac{y^2}{y^3} = \frac{x^3}{y^3} = \left(\frac{x}{y}\right)^3 \end{cases}$$

so basically φ^{-1} must be $(x,y) \mapsto \frac{x}{y}$ but this is not defined at $y = 0$ so I thought well why don't you just put

$$(x,y) \longmapsto \begin{cases} 0 & y = 0 \\ \frac{x}{y} & y \neq 0 \end{cases}$$

but then all $(x,0)$ will map to $(0,0)$ under $\varphi \circ \varphi^{-1}$. So what happens is that this map is not well-defined so it is not an isomorphism.

b. The Frobenius morphism is in fact a ring homomorphism meaining it preserves multiplication and addition. Addition is more interesting: since p is prime, it divides the numerator and not the denominator of the binomial coefficient $\frac{p!}{k!(p-k)!}$, and this will make that number 0 unless $k = 0, n$, which are the first final terms of the binomial expansion $(a + b)^p$. Therefore we must only show that φ has trivial kernel. Now since k is a field, all elements are units, so none of them can actually satisfy $a^p = 0$, making $\ker \varphi = 0$.

That φ surjective follows from an equivalence of the field being algebraically closed and thus perfect.

That φ is continuous and bicontinuous follows from the same reasons as the previous question.

To finish we check that the induced map on coordinates rings is not an isomorphism, since its inverse must be of the form $t \mapsto t^{\frac{1}{p}}$, which is not polynomial.

$\square$

1.8 $\mathbb{P}^n$ minus a hyperplane (Alex)

Exercise 3.5 (Alex, Sept 17) By abuse of language, we will say that a variety *is affine* it is isomorphic to an affine variety. If $H \subseteq \mathbb{P}^n$ is any hypersurface, show that $\mathbb{P}^n - H$ is affine.

Solution. Suppose $H = V(f)$ with $\deg f = d$. Consider the d-uple embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$

$\square$

1.9 Products of Quasi-Projective Varieties (extra)

Exercise 3.16 (Products of Quasi-Projective Varieties) Use the Segre embedding (Ex. 2.14) to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it a structure of projective variety.

Now for any two quasi-projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ consider $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$.

(a) Show that $X \times Y$ is a quasi-projective variety.

(b) If X, Y are both projective, show that $X \times Y$ is projective.

(c) Show that $X \times Y$ is a product in the category of varieties.

Solution. content... □

1.10   Normal varieties (Victor)

Exercise 3.17 (Normal varieties)

Exercise 3.18 (Projectively Normal Varieties)

Definition

- A variety Y is *normal* ar a point $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is *normal* if it is normal at every point.

- A projective variety $Y \subseteq \mathbb{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

- Let $A \subset B$ be a subring (or a morphism $A \to B$) $b \in B$ is *integrally closed* if it is the root of a polynomial in $A[x]$. Here we mean that $S(Y)$ equals the set of integrally closed elements with respect to its field of fractions.

a. If Y is projectively normal, then Y is normal.

b. There are normal varieties in projective space which are not projectively normal. For

Solution.

a. $\forall p \in Y$ and prime ideal $\mathfrak{m}_p \in \mathrm{Spec}(Y)$ we have

$$S(Y) \subseteq S(Y)_{\mathfrak{m}_p} \subseteq F(Y)$$

How does this conclude?

b.

□

Exercise 5.4 (Intersection mulitplicity)   If $X, Y \subseteq \mathbb{A}^2$

1.11 September 10

1.11.1 Analitically isomorphic singularities (Arthur)

Two curves C_1, C_2 with points $P \in C_1$ and $Q \in C_2$, and $\mathcal{O}_P \supset \mathfrak{m}_P$, $\mathcal{O}_Q \supset \mathfrak{m}_Q$. And

$$C : f = 0, \qquad f[x, y]/(f), \quad \mathfrak{m} = (x, y).$$

We define

$$\hat{\mathcal{O}}_p = \varprojlim \mathcal{O}_P / \mathfrak{m}_P^n \cong k[[x, y]]/(y)$$

Definition *Analitical isomorphism* is a k-isomorphism with $\hat{\mathcal{O}}_P \cong \hat{\mathcal{O}}_Q$.

Exercise   a. An. isomorphic $\implies \mu_{C_1}(D) = \mu_{C_2}(Q)$.

$$f(x, y) = f_r(x, y) + f_{r+1}(x, y) + \ldots + f_n(x, y).$$

$$\mu(p) = r$$

b. Let $f = f_r + f_{r+1} + \ldots \in k[[x, y]]$. Suppose that $f_r = g_s h_t \in k[x, y]$ have no linear factors in common (which means they are coprime). Then there exist $g, h \in k[[x, y]]$ such that
$$f = gh, \quad g = g_s, \quad h = h_t + \ldots$$

c. Let $f = f_r + f_{r+1} + \ldots \in k[x, y]$. If

$$f_r = L_1 \cdot L_2 \cdot \ldots \cdot L_r$$

then $P = (0, 0)$ is an ordinary r-fold. Show that any 2-folds are analitically isomorphic.

d. Consider the curve $C : f = 0, \quad \mu = 2$. It is an ordinary 2-fold. Show that

$$\hat{\mathcal{O}}_P \cong \frac{k[[x]]}{(y^2 - x^r)}$$

Solution.

a.

b. Suppose they exist and are $g = \sum_i g_i, h = \sum_j h_j$ so that

$$f = gh = \sum_{d=r}^{\infty} \sum_{i+j=d} g_i h_j$$

which means that

$$f_{r+d} = \sum_{i+j=d} g_i h_j = g_s h_{t+d} + g_{s+d} h_t + \sum_{?} g_i h_j$$

12

and
$$f_{r+1} = g_s h_{t+1} + g_{s+1} h_t$$

Notice that since g and h don't share linear factors, we may fix the second variable and the resulting polynomials generate $k[t]$, ie.

$$(g_s(t,1), h_t(t,1)) = k[t]$$

Further computations show that the next term is well defined.

c. Consider the completion at the origin, $\hat{\mathcal{O}}_P \cong k[[x,y]]/(f)$. We shall try to show that in fact

$$k[[x,y]]/(f) \xrightarrow{\cong} k[[x,y]]/(f_r)$$

so consider the map

$$x \mapsto x + A(x,y) \qquad y \mapsto y + B(x,y)$$

so that

$$f(x + A(x,y), y + B(x,y)) = f_r(x,y).$$

Using a projective transformation, that makes the tangents be (?), we see that

$$\hat{\mathcal{O}}_P \cong k[[x,y]]/(xy)$$

Claim (Not true)   Any k-isomorphism

$$\varphi : \frac{k[[x,y]]}{(f)} \longrightarrow \frac{k[[x,y]]}{(y)}$$

have the form

$$x \mapsto x + A(x,y) \qquad y \mapsto y + B(x,y) \quad \text{when } \mu \geqslant 2$$

Suppose that

$$f(x,y) = y^2 A(y) \qquad A(0) \neq 0, \qquad f(x,y) = (y^2 + a(x)y + b(x)) \cdot u(x,y)$$

recall that

Theorem (Weierstrass preparation theorem)   Let A be a local ring, $k[[x]]$, $\mathfrak{m} = (x)$, and $f \in A[[y]]$. Then

$$f = (y^n + k_{r-1}y^{n-1} + \ldots + b_0) \cdot u(y)$$

for $u \in A[[y]]^\times$ and $b_i \in \mathfrak{m}$

13

$$f(x,y) = (y^2 + c(x))u(x,y)$$

$$\hat{\mathcal{O}}_P \cong \frac{k[[x,y]]}{(y^2 + c(x))}$$

$$c(x) = x^r V(x), \quad \text{where } V \in k[[x]]^\times$$

$$V(x) = 1 + x + \dots$$

$$\sqrt[r]{1+x} = 1 + c_1 x + \dots$$

$$V(x) = W(x)^r$$

$$x^r V(a) = (xW)^r$$

$$y^2 - x^r.$$

$\square$

1.11.2 Degree and Hilbert polynomial (Bruno)

Definition Let M be a finitely generated graded S-module, where $S = k[x_0, \dots, x_m]$. A *Hilbert function* is

$$\varphi : \mathbb{Z} \longrightarrow M$$
$$t \longmapsto \dim_k(M_t)$$

Theorem There exists a unique polynomial $P_M \in \mathbb{Q}[t]$ such that for all $t \gg 0$, $P_M(t) = \varphi_M(t)$.

Exercise 7.2 Let Y be a variety of dimension r in $\mathbb{P}^n$, with Hilbert polynomial P_Y. We define the **arithmetic genus** of Y to be

$$P_Y(t) := P_{S[Y]}(t)$$

$$= (-1)^r (P_Y(0) - 1)$$

  a. $Y = \mathbb{P}^n$.

  b. $Y \subset \mathbb{P}^n$ plane curve of degree d.

  c. $Y \subset \mathbb{P}^n$ hypersurface of degree d.

  d. $Y_a, Y_b \subset \mathbb{P}^3$ of degrees a and b, $Y = Y_a \cap Y_b$ complete intersection.

  e. $Y^r \subset \mathbb{P}^n, Z^s \subset \mathbb{P}^m : Y \times Z \subset \mathbb{P}^N$. Here

$$P_{Y \times Z}(t) = P_Y(t) P_Z(t)$$

Solution.

a. $S[Y] = k[x_0, \ldots, x_n]$,

$$\dim(k[x_0, \ldots, x_n]_t) = \binom{n+t}{n} = \frac{(n+1)(n+t-1)\ldots(t+1)}{n!} = \frac{t^n}{n!} + \ldots$$

so that

$$P_Y(0)_1 \implies P_a(\mathbb{P}^n) = 0.$$

b. Now

$$P_a(Y) = \frac{(d-1)(d-2)}{2}$$

which follows from c so let's do c.

c. Suppose $Y = Z(f)$. Then

$$(f) \hookrightarrow k[x_0, \ldots, x_n] \to k[Y]$$

$$(f) = \{f\varphi : \varphi \in k[x_0, \ldots, x_n]\}$$
$$(f)_t = \binom{n+t-d}{n}$$
$$P_Y(t) = \binom{n+t}{n} - \binom{n+t-d}{n}$$
$$P_a(Y) = (-1)^n \binom{n-d}{n}$$
$$= (-1)^n \frac{(n-d)(n-d-1)\ldots(1-d)}{n!}$$
$$= \frac{(d-1)\ldots(d-n)}{n!}$$
$$= \binom{d-1}{n}$$

d. Let's first write the formula and then we discuss it

$$P(t) = \binom{3+t}{t} - \binom{3+t-a}{t} - \binom{3+t-b}{t} + \binom{3+t-a-b}{3}$$
$$= (ab)t + P_Y(0)$$

$$P_Y(0) = 1 - f(a) - f(b) + f(a+b)$$
$$P_a(Y) = f(a) + f(b) - f(a+b)$$

Considering multiplication by f_b as map

$$\frac{k[\mathbb{P}^3]}{(f_a)} \xrightarrow{f_b} \frac{k[\mathbb{P}^3]}{(f_a)}$$

15

so we get

$$0 \longrightarrow \frac{k[\mathbb{P}^3]}{(f_a)} \longrightarrow \frac{k[\mathbb{P}^3]}{(f_a)} \longrightarrow \frac{k[\mathbb{P}^3]}{(f_a, f_b)} \longrightarrow 0$$

but we need to consider the map $[-b]$ to stop the grading from moving up from one arrow to another. Then we get

$$P_Y = P_{Y_a}(t) - P_{Y_a}(t - b)$$

so

$$P_A - P_B + P_C = 0.$$

e. O problema está em ver se

$$K[Y \times Z]_d \overset{?}{\cong} k[Y]_d \otimes_k k[Z]_d$$

Daí o Hilbert polynomial é o produto dos Hilbert polynomials e tá.

$\square$

1.12 September 13

1.12.1 Veronesse surface (Alex)

Exercise 2.13 Let Y be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This is the **Vernoses surface**. If $Z \subseteq Y$ is a closed curve (a **curve** is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbb{P}^5$ such that $V \cap Y = Z$.

Solution.

$$\rho_2 : \mathbb{P}^2 \overset{\text{Veronese}}{\longrightarrow} \mathbb{P}^5$$
$$[x_0 : x_1 : x_2] \longmapsto [x_0^2 : x_1^2 : x_2^2 : x_0 x_1 : x_1 x_2 : x_0 x_2]$$

And $Y := \rho_2(\mathbb{P}^2)$. Notice that $Y = \mathbb{V}(\ker \theta)$ where

$$\theta : k[y_0, \dots, y_5] \longrightarrow k[x_0, \dots, x_2]$$
$$y_i \longmapsto M_i = i\text{-th monomial of degree } d \text{ in } x_0, \dots, x_n$$

$$y_0 \longmapsto x_0^2$$
$$y_1 \longmapsto x_1^2$$
$$y_2 \longmapsto x_2^2$$
$$y_3 \longmapsto x_0 x_1$$
$$y_4 \longmapsto x_0 x_2$$
$$y_5 \longmapsto x_1 x_2$$

it is not immediate that kernel of this map is Y, but it is true.

By exercise 2.8 there exists a homogeneous polynomial (which is irreducible) such that $V(f) = \rho_2^{-1}(Z)$. Recall that Z is a curve in Y so $\rho^{-1}(Z)$ is (probably) a curve in $\mathbb{P}^2$.

So, we have

$$Z = \rho_2(V(f)) = \rho_2(V(f^2)),$$

which follows from the fact that θ is surjective over the set of polynomials of even degree. How to prove this? It sufficies to prove for monomials. This is a combinatorial problem. Consider

$$x_0^{a_0} x_1^{a_1} x_2^{a_2} \longmapsto (a_0, a_1, a_2) \in \mathbb{Z}^3$$

Anyway... this means that there exists $g \in k[x_0, x_1, x_2]$ such that $f^2 = \theta(g)$. And then

$$Z = Y \cap V(g)$$

which means that being the Veronese image of a zero of f is the same as being a zero of g. (Some more thought on how θ is working?) □

Explanation The Veronese map image is actually the symmetric matrices of rank 1:

$$v \longmapsto v^{\mathbf{T}} \cdot v$$

$$\begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} v_0 & v_1 & v_2 \end{pmatrix} = \begin{pmatrix} v_0 v_0 & v_0 v_1 \\ v_1 v_0 & v_1 v_1 \end{pmatrix}$$

1.12.2   Degree of Veronese and Segre embeddings (Alex)

Exercise 7.1

   a. Find the degree of the 2-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$.

   b. Find the degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$.

Solution.

   a. Recall that the degree of a map the coefficient of greatest degree of the Hilbert polynomial times the degree factorial, i.e. if the Hilbert polynomial is

$$P_X(z) = a_n z^n + \ldots + a_0,$$

the degree of X is

$$a_n n!$$

(So remember that the Hilbert polynomial is the unique polynomial in $\mathbb{Q}[t]$ that equals the Hilbert function (the Hilbert functions at n is just the dimension of the degree-n component of $S[X] = \bigoplus_n S_n$, so homogeneous polynomials of degree n) for sufficiently large numbers.)

   Claim

$$S(X)_m \cong k[x_0, \ldots, x_n]_{md}$$

where d is the d in the d-uple embedding.

If we show this, we can see that

$$P_x(m) = \binom{md+n}{n} = \prod_{i=1}^{n} \frac{(md+i)?}{n!} = \frac{d^n m^n}{n!} + \text{lower order terms}$$

Now the pullback map is

$$\frac{k[x_0, x_1, \ldots, x_N]}{I(X)} \xrightarrow{\cong} ? k[x_0, x_1, \ldots, x_n]$$

b.

$\square$

1.12.3 Curvas elípticas (Carlos)

k não neces. algebricamente fechado, $\bar{k}$ algebraic closure, V variedade afim, $V \subseteq \mathbb{A}^n = k^n$, $k[V] = k[c_1, \ldots, x_n]/I(V)$. $P \in V$, $\mathfrak{m}_p = \{g \in k[V] : g(p) = 0\}$ é um ideal maximal.

Para $g \in k[V]$,
$$\operatorname{ord}_P(g) = \sup\{d \in \mathbb{Z}_{\geqslant 0} : g \in \mathfrak{m}_p^d\}.$$

Uma *curva* é uma variedade projetiva suave de dimensão 1. *Curvas elípticas*:

$$E : x^2 = x^3 + ax + b$$
$$\downarrow \text{projetização}$$
$$zy^2 = x^3 + axz^2 + bz^3$$

Exercise   Seja E/k uma curva eliptica suave:
$$y^2 = x^3 + ax + b$$
$$= (x - e_1)(x - e_2)(x - e_3)$$

com $e_1, e_2, e_3 \in \bar{k}$. Considere

$$f_1 = (x - e_1), \quad f_2 = (x - e_2), \quad f_3 = (x - e_3)$$
$$p_1 = (e_1, 0), \quad p_2 = (e_2, 0), \quad p_3 = (e_3, 0)$$

O que é

$$\operatorname{ord}_{f_i} p_i = ?$$

Solution. Lembre que p é um ponto suave se e somente se $\dim \mathfrak{m}_p/\mathfrak{m}_p^\alpha = \dim V$.

Step 1 Considere a projetização de E :
$$y^2 z = (x - e_1 z)(x - e_2 z)(x - e_3 z).$$

Daí,
$$p_1 = [e_1 : 0 : 1], \quad p_2 = [e_2 : 0 : 1], \quad p_3 = [e_3 : 0 : 1]$$

$\square$

18

1.13 September 17

1.13.1 Dimension of affine variety $\cap$ hyperplane (Alex+Arthur)

Exercise 1.8 Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$ and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

So: intersection of variety of dimension r and hypersurface is variety dimensin $r - 1$.

Solution. Suppose $H = \mathbb{V}(f)$ and $Y = \mathbb{V}(\mathfrak{q})$. Then $Y \cap H = \mathbb{V}(\mathfrak{q} + (f))$. Pick an irreducible component in $Y \cap H$ which is associated to a prime ideal $\mathfrak{p}$. Then we have that $\mathfrak{q} + (f) \subseteq \mathfrak{p}$. Passing to the quotient, we have that $(f)/\mathfrak{q} \subseteq \mathfrak{p}/\mathfrak{q}$. Now we use the result that

Theorem (Krull's Hauptidealsatz) Let A be a noetherian ring. For an element $x \in A$ that is not a unit nor a zero divisor,

$$(x) \subseteq \mathfrak{p},$$

where $\mathfrak{p}$ is a minimal ideal, then $\mathfrak{p}$ has height 1.

and since $(f)/\mathfrak{q}$ is principal in the quotient, ie. $(f)/\mathfrak{q} = (f + \mod \mathfrak{q})$. $\qquad \square$

1.14 Sept 20

1.14.1 Scroll (Sergey)

You can find this Sec^k on a book by Harris, name?

There is an exercise from last class:

Exercise from Sergey, Sept 17 Compute Sec^k of $\operatorname{Segre}(a, b)$.

For today:

Exercise (Sergey, Sept 20)

    a. Compute $\dim \operatorname{Gr}(k, N)$. You can use that $GL(N)$ acts transitively.

    b. $\dim \operatorname{Sec}^k \operatorname{Gr}(N, 2)$. **Hint:** Use linear algebra: $\Sigma_{0,n} = \operatorname{Cone}(v_n \mathbb{P}^1, \mathbb{F}_n$ Hinzbruch surfaces. Rational Ruled Normal Surfaces (see below for definition of scroll ie. $\Sigma_{0,n}$.

Solution (Victor, September 27).

    a. Consideremos os pontos da Grassmaniana $\operatorname{Gr}(m, N)$ como classes de equivalência de bases de vetores identificadas baixo a ação de $GL(n)$. Considere uma base $(v_1, \ldots, v_m)$. Ela pode ser pensada como uma matriz de $N \times m$ de rango maximo, ie. ela tem posto m. Então, essa matriz tem uma submatriz de $m \times m$ não singular, e uma submatriz de $m(N - m)$ que é singular. Variando as entradas dessa matriz singular, obtemos coordenadas para uma variedade de dimensão $m(N - m)$. [Explicar mais]

Definition (Extra: mergulho de Plücker, Sept 27)   A Grassmaniana é uma variedade projetiva:

$$P_k : Gr(m, V) \longrightarrow \mathbb{P}\left(\Lambda^m(V)\right)$$
$$W \longmapsto [w_1 \wedge \ldots \wedge w_n]$$

onde W está generado por $w_1, \ldots, w_n$. Aqui tem que mostrar que isso está bem definido.

b. Lembre que

Definition   (Ver Wiki) Dada uma variedade projetiva $V \subset \mathbb{P}^r$, *variedade secante* de V é

$$Sec^k(V) = \overline{\{\overline{x_1, \ldots, x_k} : x_1, \ldots, x_k \in V\}}$$

Em palavras, o fecho de Zariski do conjunto de "espaços generados" por k-tuplas de pontos em V. Por exemplo,

$$Sec^2(V) = \overline{\{\overline{xy} : x, y \in V\}}$$

é o fecho de Zariski do conjunto de todas as linhas secantes a V em $\mathbb{P}^r$. Mais geralmente, a k-*ésima variedade secante* é o fecho de Zariski da união de todos os espaços lineares generados por coleções de k (ou $k + 1$?) pontos em V. (Note que os espaços lineares numa variedade projetiva são as projetivizações dos espaços lineares correspondentes no espaço afim.)

Pegue dois elementos em $Gr(2, N)$, digamos W, W' e pegue bases delas $\{w_1, w_2\}$ e $\{w'_1, w'_2\}$. Daí considere

$$\left\{ \left(tw_1 - (1 - t)w'_1, tw_2 - (1 - t)w'_2 \right) \right\}$$

$\square$

Definition   The *Segre embedding* is

$$\mathbb{P}(U) \times \mathbb{P}(V) \hookrightarrow \mathbb{P}(U \otimes W)$$
$$([u], [v]) \longmapsto [u \otimes v]$$
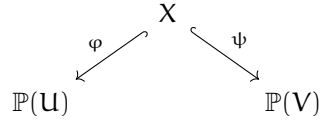
Now take $X \subset \mathbb{P}(W)$. Then

$$Sec^k(X) = \overline{Z}$$

where

$$Z = \{[w] : \exists p_1, \ldots, p_k \in X \hookrightarrow \mathbb{P}(W)\} = \bigcup_{p_1, \ldots, p_k \in X} \langle p_1, \ldots, p_k \rangle$$

where $p_i = [v_i]$ and $w = \sum_i v_i$.

Definition (Scroll)   I have a variety X and two linear systems:

$$\varphi \swarrow \overset{\text{X}}{} \searrow \psi$$

$$\mathbb{P}(U) \qquad\qquad \mathbb{P}(V)$$

Then the *scroll* is

$$\mathbb{P}(U \oplus V) \supset \Sigma_{\varphi,\psi} = \{[u \oplus v] : [u] \in \varphi(X), [v] \in \psi(X)\}$$

Exercise: Show this is a projective variety.

So the scroll is a union of projective lines. It is called scroll because lines in one direction survive (in a drawing, they are all parallel) but the other lines are curved (so it looks like a scroll).

Exercise about scroll   As abstract varieties, $\Sigma_{a,b}$ is isomorphic to $\Sigma_{a'b'}$ iff $b - a = b' - b' = n$ and either $a = a' = 0$ or $a > 0, a' > 0$.

1.14.2   Elliptic curve is not rational (Alex)

Exercise 6.3, Elliptic curve is not rational   Let Y be the curve $y^2 = x^3 - x$ in $\mathbb{A}^2$, and assume that the characteristic of the base field k is not 2. In this exercise we will show that Y is not a rational curve, and hence $K(Y)$ is not a pure trascendental extension of k.

  a. Show that Y is nonsingular, and deduce that $A = A(Y) \cong \frac{k[x,y]}{(y^2-x^3+x)}$.

  b. Let $k[x]$ be the subring of …

Exercise 6.2   Let
$$Y = V(y^2 - x^3 + x \subset \mathbb{A}^2$$

be an elliptic curve. Show it is not rational.

Definition   A curve C is *rational* if C is birationally equivalent to $\mathbb{P}^1$

  Fact   If C is a nonsingular rational but not isomorphic to $\mathbb{P}^1$, then $A(C)$ is a UFD.

Step 1   Y is nonsingular. Looks like we can just differentiate here and find that the zero locus of the derivatives is not in C.

  Definition (Tangent space)   Consider the local ring at the point and its maximal ideal. So m_p/m_p^2 is the *cotangent space*. Then the *tangent space* is the dual $\left(\frac{m}{m^2}\right)^*$. A point is *singular* if the tangent space has more dimension than it should.

  Interpretation   Functions in m_p are those which vanish at p. A function in m_p is $\sum_i f_i g_i$ so its differential is $d\left(\sum_i f_i g_i\right) = \sum_i f_i dg_i + \sum_i g_i f d_i$ which you want to mod out because it is zero at p because both f_i and g_i.

Remark For $X, Y \subset Z$, $T_p X, T_p Y \subset T_p Z$

$$T_p(X \cap Y) = T_p X \cap T_p Y$$

Step 2

Theorem (5.1) $Y \subseteq \mathbb{A}^n$ affine variety and $P \in Y$. Then Y is nonsingular at $P \iff$ local ring $\mathcal{O}_{P,Y}$ is a regular local ring

Theorem (6.2A) A noetherian local domain of sdimension one with maximal ideal $\mathfrak{m}$. Then the following are equivalent

(i) A is a regular local ring.

(ii) A is integrally closed (over the fraction field).

In other words, normality in dimension 1 is the same as normality.

Step 3 Here's the *hyperelliptic involution*

$$\sigma : k[Y] \longrightarrow K[Y]$$
$$x \longmapsto x$$
$$y \longmapsto -y$$

and then there's the *norm*

$$N : K[Y] \longrightarrow K[Y]$$
$$a \longmapsto a\sigma(a)$$

Notice that
$$N(1) = 1 \qquad N(a,b) = N(a)N(b)$$

Explanation: we are killing y, so the norm takes us to a simpler space which is Y/σ.

This means X is irreducible, Y is irreducible. Then a discussion on the invertibility of the norm, units on these rings...

1.14.3 Tangent map (Bruno)

Exercise 5.10 X afine variety and $p \in X$.

a. $\dim T_p X \geqslant \dim X$, $\dim T_p X \iff p$ is nonsingular. Here $\dim X = \dim \mathcal{O}_{P,X}$ which is just the Krull dimension, so maximum height of chain of prime ideals.

b. Define the tangent map.

c. Project the parabola $x = y^2$ in $\mathbb{A}^2$ to the x axis and compute its derivative at 0.

Solution.

- $\dim T_p X = \dim \mathfrak{m}_p/\mathfrak{m}_p^2$, *as a vector space!* . Now that inequality is a result in homological algebra:

Theorem A noetherian local domain and $\mathfrak{m}$ maximal ideal then the dimension of the quotient of ideals as a vector space is greater or equal to the dimension of A as a ring.

-

Definition $X \xrightarrow{\varphi} Y$, $\varphi^* : \mathscr{O}_Y \to \mathscr{O}_X$ which is precomposition. Then since $\varphi(p) = q$ we see that $\varphi^*(\mathfrak{m}_q) \subset \mathfrak{m}_p \implies \varphi^*(\mathfrak{m}_q^k) \subset \mathfrak{m}_p^k$. And then take quotient by $\mathfrak{m}_p$ and $\mathfrak{m}_q^2$. These local rings are the functions that vanish at the point. Also they are localization of polynomials by maximal ideal.

-

Claim $(T\varphi)_0 = 0$

look at the pullback

$$\varphi^* : k[\mathbb{A}^1] \longrightarrow k[Z] = k[x,y]/(x - y^2)$$
$$t \longmapsto x$$

which follows from definition.

Then

$$\varphi^*(\mathfrak{m}_0) \subset \mathfrak{m}_{(0,0)}^2 \implies \varphi^* : \mathfrak{m}_0/\mathfrak{m}_0^2 \to \mathfrak{m}_{0,0}/\mathfrak{m}_{0,0}^2$$

is zero because

$$\varphi^*(t) = x = y^2, \qquad y \in \mathfrak{m}_{(0,0)}$$

$\square$

Exercise Sergey Is the variety $\sum_{i=0}^{s} x_i^3 = 0$ rational in $\mathbb{P}^5$.

2 Chapter II

2.1 September 27

2.2 Sheaves (Bruno)

2.2.1 Background

Let X be a topological space and $\mathscr{F}$ a sheaf of rings, which is a presheaf + gluing condition.

Sheafification

$$\text{Presheaf} \longrightarrow \text{Sheaf}$$
$$\mathscr{F} \longmapsto \mathscr{F}^+$$

This $\mathscr{F}^+$ has two properties:

1. Universal

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & G \\ \downarrow & \nearrow & \\ \mathscr{F}^+ & & \end{array}$$

2. For any $p \in X$, the stalks $\mathscr{F}_p = \mathscr{F}_p^+$.

3. (Extra)

$$\text{Sheafs} \xrightarrow{\text{forget}} \text{Presheaf}$$
$$\xleftarrow[\text{Sheafify}]{}$$

Remark Also kernel and image of sheafs. Must sheafify the image.

2.2.2 Exercise 1.8

Let $U \subset X$ be an open subset of X. Consider

$$\Gamma(U, \cdot) : \text{Sheafs} \longrightarrow \text{Rings}$$
$$\mathscr{F} \longmapsto \Gamma(U, \mathscr{F})$$

Show that $\Gamma(U, \cdot)$ is a left exact functor, that is,

$$0 \longrightarrow \mathscr{A} \xrightarrow{a} \mathscr{B} \xrightarrow{b} \mathscr{B}$$

exact implies

$$0 \longrightarrow \Gamma(U, \mathscr{A}) \xrightarrow{a_U} \Gamma(U, \mathscr{B}) \xrightarrow{b_U} \Gamma(U, C)$$

Solution.

- a_V is injective is clear.

- $\text{img } a_U = \ker b_U$. Suppose $s \in \Gamma(U, \mathscr{B})$, so that $s =$

$\square$

2.3 September 24

2.3.1 Generically finite morphisms (Arthur)

Exercise 2.37

Solution. **First part.** We will show the hint: the field extension $K(X) \subseteq K(Y)$ is finite.

Let $V = \operatorname{Spec} A \subseteq Y$ and $U = \operatorname{Spec} B \subseteq f^{-1}(V) \subseteq X$ be affine open subsets. Write C for the coordinate ring of $f^{-1}(\eta) \cap U$.

Since f is of finite type, C is finitely generated as a $K(Y)$-algebra. Explain more. By Noether's normalization lemma, there exist $y_1, \ldots, y_d \in C$ such that C is finite over

$K(Y)[y_1, \ldots, y_d]$, where $d = \dim C$. But $\dim C = \dim(f^{-1}(\eta) \cap U) = 0$ because $f^{-1}(\eta) \cap U$ is a finite set. Hence C is finite over $K(Y)$, i.e., C is a finite $K(Y)$-vector space. In particular, $K(X) \subseteq C$ is finite over $K(Y)$.

Second part. We will show that there is an open affine $V \subseteq Y$ and an affine open cover $\{U_i\}$ of $f^{-1}(V)$ such that each restriction $U_i \to V$ is finite.

Let $V = \operatorname{Spec} A \subseteq Y$ be an open affine. Since f is of finite type, there is a finite affine open cover $U_i = \operatorname{Spec} B_i$ of $f^{-1}(V)$ such that B_i is an A-algebra of finite type. Let $\{x_{ij}\}$ be a set of generators for B_i. Now $B_i \subseteq K(X)$ and $K(X)$ is a finite $K(Y)$-module, so each x_{ij} is integral over $K(Y)$, i.e., there are polynomials $P_{ij}(T) \in K(Y)[T]$ such that $P_{ij}(x_{ij}) = 0$.

There a is distinguished open subset $V_g \subseteq V$ such that $P_{ij}(T) \in \mathcal{O}_Y(V')[T]$ for all i, j. Replace U_i by $(U_i)_g$ and V by V_g. Then B_i is finite over A because it is finitely generated by integral elements. Hence $f : U_i \to V$ is a finite morphism (U_i and V are affine) for all i.

Third part. Find an affine $V \subseteq Y$ such that $f^{-1}(V)$ is affine.

Now let $U \subseteq \bigcap_i U_i$ be an affine open. Then $U \subseteq f^{-1}(V)$ and $f : U \to V$ is affine. We will show that we can choose an affine open $V' \subseteq V$ such that $f^{-1}(V') \subset U$. Then we will have $f : f^{-1}(V') \to V'$ affine, concluding the exercise.

To do this, let $Z = f^{-1}(V) \setminus U$. If $\eta \in \overline{f(Z)}$ then actually $\eta \in f(Z)$ because it is the generic point of V. Looking at the affine $U_i = \operatorname{Spec} B_i$, we see that any $z \in U_i \cap Z$ such that $\eta = f(z)$ corresponds to a prime $\mathfrak{p} \subseteq B_i$ such that $\mathfrak{p} \cap A = (0)$. Since $\mathfrak{p}$ is above the prime $(0) \subseteq A$, the following theorem tells us that $\mathfrak{p} = (0)$:

Theorem (Cohen)

But $(0) \in U$ and $Z \cap U = \varnothing$. Hence $\eta \notin \overline{f(Z)}$. Therefore there is some neighborhood V' of η such that $V' \cap f(Z) = \varnothing$, which is what we had to show.

Alternatively, let $Z = f^{-1}(V) \setminus U_1$. The same theorem shows that $f(Z)$ is closed. Since $U_i \to V$ is finite for all i, we have $\dim f^{-1}(V) = \dim V$. Moreover

$$\dim V = \dim f^{-1}(V) > \dim Z \geqslant \dim f(Z),$$

so $f(Z) \neq V$ and there is an open affine V' such that $V' \cap f(Z) = \varnothing$. $\qquad \square$

2.3.2 Closed subschemes of $\operatorname{Proj}(S)$ (Dani)

Review of Proj

- **Set.** Let S be a graded ring. $\operatorname{Proj}(S)$ is the set of homogeneous prime ideals $\mathfrak{p}$ that do not conatin $S_+ = \bigoplus_{d>0} S_d$.

- **Topology.** The closed subsets of $\operatorname{Proj}(S)$ are $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj}(S) : \mathfrak{p} \supseteq \mathfrak{a}\}$.

- **Sheaf.** For each $\mathfrak{p} \in \operatorname{Proj}(S)$ consider $S_{(\mathfrak{p})}$, the localized ring $T^{-1}S$ where T is the multiplicative system of homogeneous elements of S that are not in $\mathfrak{p}$. For any open subset $U \subseteq \operatorname{Proj}(S)$ define $\mathcal{O}(U)$ to be the set of functions (s for section?) $s : U \to \coprod S_{(\mathfrak{p})}$ such that for $p \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ (perhaps like a vector field: a

point maps to a vector attached to the tangent space at that point), and s is locally a quotient of elements of S, ie. $s(\mathfrak{q}) = a/f$ in a neighbourhood of $\mathfrak{p}$.

Motivation (Görtz & Wedhorn, p. 375)   If $\varphi : A \to B$ is a homomorphism of graded R-algebras, the inverse image of a relevant prime ideal (=an ideal not containing the homogeneous ideal of positive degree groups $\bigoplus_{d \geqslant 1} S_d$) of B may not be a relevant prime ideal of A. Thus $\mathrm{Proj}(A)$ is not functorial in A with respect to arbitrary homomorphisms of graded R-algebras. But there is a unique morphism of R-schemes

$$\mathrm{Proj}\,\varphi : G(\varphi) \longrightarrow \mathrm{Proj}\,A,$$

where $G(\varphi) \subseteq \mathrm{Proj}(B)$ is the open subscheme

$$G(\varphi) = \mathfrak{q} \in \mathrm{Proj}(B) : \varphi^{-1}(\mathfrak{q}) \not\supseteq A_+\}$$

such that ...

Exercise 2.14

    b. Let $\varphi : S \to T$ be agraded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \mathrm{Proj}(T) : \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of $\mathrm{Proj}(T)$ and show that φ determines a natural morphism $f : U \to \mathrm{Proj}(S)$.

    c. The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \mathrm{Proj}\,T$ and the induced morphism $f : \mathrm{Proj}\,T \to \mathrm{Proj}\,S$ is an isomorphism.

So basically the following exercise is about studying a case when this induced morphism behaves nicely, namely when φ is surjective.

Exercise 3.12

    a. Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U from exercise 2.14 is equal to $\mathrm{Proj}\,T$ and the morphism f is a closed immersion.

    b. If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subsceme of $\mathrm{Proj}\,S$ defined as the image of the closed immersion $\mathrm{Proj}(S/I) \to \mathrm{Proj}\,S$. Show that different homogeneous ideals can give rise to the same closed subscheme.

Definition (3.41 Gortz)   A morphism $i : Z \to X$ of schemes is called a **_closed immersion_** if the underlying continuous map is a homeomorphism between Z and a closed subset of X and the sheaf homomorphism $i^\flat : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ is surjective.

Solution.

    a. The fact that $U = \mathrm{Proj}\,T$ is immediate from definition and sujectivity since $\varphi(S_+) = T_+$. The induced map is

$$f : U = \mathrm{Proj}\,T \longrightarrow \mathrm{Proj}\,S$$
$$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$$

Now let's see that f is a closed immersion.

(f is injective.) Follows from surjectivity: if $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} T$ are such that $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$, then $\varphi(\varphi^{-1}(\mathfrak{p})) = \mathfrak{p} = \mathfrak{q} = \varphi(\varphi^{-1}(\mathfrak{q}))$.

($f\,(\operatorname{Proj}(T))$ is closed.) The easiest way is to notice that $S/\ker \varphi \cong T$, and that homogeneous prime ideals of $S/\ker \varphi$ (=elements of $\operatorname{Proj}(T)$) correspond to homogeneous ideals of S which contain $\ker \varphi$, so $f(\operatorname{Proj}(T)) = V(\ker \varphi)$. (But I should confirm this works...)

A more explicit computation is as follows:

$$f(\operatorname{Proj} T) = V \left(\bigcap_{\mathfrak{p} \in \operatorname{Proj}(T)} \varphi^{-1}(\mathfrak{p}) \right)$$

$$= \left\{ \mathfrak{q} \in \operatorname{Proj} S : \mathfrak{q} \supseteq \bigcap_{\mathfrak{p} \in \operatorname{Proj} T} \varphi^{-1}(\mathfrak{p}) \right\}$$

The contention $\subseteq$ is clear since $\varphi^{-1}(\mathfrak{q}) \in f(\operatorname{Proj} T) \implies \varphi^{-1}(\mathfrak{q}) \supseteq \bigcap_{\mathfrak{q} \in \operatorname{Proj} T} \varphi^{-1}(\mathfrak{p})$.

For $\supseteq$ we want to see that $\mathfrak{q} = \varphi^{-1}(\varphi(\mathfrak{q}))$. Suppose that there exists $x \in \varphi^{-1}(\varphi(\mathfrak{q})) \setminus \mathfrak{q}$. Then there exists $y \in \mathfrak{q}$ such that $\varphi(x) = \varphi(y) \implies \varphi(x - y) = 0$. But $0 = (x-y) \subset \bigcap_{\mathfrak{p} \in \operatorname{Proj} T} \varphi(\mathfrak{p})$, meaning $x - y \in \mathfrak{q}$, but that is not possible since $x \notin \mathfrak{q}$.

(f is a homeomorphism.) It preserves inclusions of ideals?

(Induced map on stalks is surjective.) What is going on?

Remark (Gortz, p. 57) A morphism $(f, f^\flat) : X \to Y$ of ringed spaces induces morphisms on the stalks as follows. Let $x \in X$. Let $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ be the morphism corresponding to $f^\flat$ by adjointness. Using the identification $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)}$ established in (2.8.2) we get

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}.$$

So I'm not really sure how this is defined but we are looking at a map

$$f_x^\sharp : \mathcal{O}_{f(\operatorname{Proj} T)} \longrightarrow \mathcal{O}_{\operatorname{Proj} S}$$

Ah! Now I see:

Proposition (2.5)

    a. For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathcal{O}_\mathfrak{p}$ is isomorphic to the local sing $S_{(\mathfrak{p})}$.

So in fact we have some map $f_\mathfrak{p}^\sharp : S_{(\varphi^{-1}(\mathfrak{p}))} \longrightarrow T_{(\mathfrak{p})}$. I don't really know how this is defined, but in a broad sense we know this is a map induced by φ, which was surjective, acting on the localizations of S and T, which are really S and T made larger by adding some fractions.

b. Let d_0 be an integer and define $I' = \bigoplus_{d \geqslant d_0} I_d$. Basically this is just exercise 2.14(c). We just notice that I' just I taking away the first d_0 degrees, so we have a projection map

$$\varphi : S/I' \to S/I$$

which is just the identity after degree d_0. That is the hypothesis of exercise 2.14(c), which yield an isomorphism $\mathrm{Proj}(S/I) \cong \mathrm{Proj}(S/I')$. And that clearly means that the induced subschemes of $\mathrm{Proj}\, S$ are isomorphic.

<div align="right">□</div>

3 Chapter III

3.1 Notes: injective resolutions, derived functors and cohomology

Definition (wiki) An object Q is *injective* if, given a monomorphism $f : X \to Y$, any $g : X \to Q$ can be extended to Y.

$$
\begin{array}{ccc}
X & \overset{f}{\hookrightarrow} & Y \\
{\scriptstyle g}\downarrow & \swarrow {\scriptstyle h} & \\
Q & &
\end{array}
$$

(wiki) The crucial assumption we need to make about our abelian category (=morphisms and objects can be added and kernels and cokernels exist and have desiarable properties) is that it has *enough injectives*, meaning that for every object A in **A** there exists a monomorphism $A \to I$ where I is an injective object in **A**.

The right derived functors of the covariant left-exact functor $F : A \to B$ are then defined as follows. Start with an object X of **A**. Because there are enough injectives, we can construct a long exact sequence of the form

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

where I^i are injective. (This is known as an *injective resolution* of X). Applying the functor F to this sequence, and choppingng off the first term, we obtain the chain complex

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots$$

This is in general not an exact sequence anymore. But we can compute its cohomology at the i-th spot (the kernel of the map from $F(I^i)$ modulo the image of the map to $F(I^i)$); we call the result $R^i F(X)$.

Of course various things have to be checked: the result does not depend on the given injective resolution of X, and any morphism $X \to Y$ naturally yields a morphism $R^i F(X) \to R^i F(Y)$ so that we indeed obtain a functor. Note that left exactness means that

$$0 \longrightarrow F(X) \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow 0$$

is exact, so $R^0 F(X) = F(X)$, so we only get something interesting for $i > 0$. (In Hartshorn Thm 1.1A: $F \cong R^0 F$.)

Explanation   A *derived functor* of a given functor maps an object to the i-th cohomology of the image of an injective resolution of the object.

The above-mentioned (where?) property of turning short exact sequences into long exact sequences is a consequence of the snake lemma. In Hartshorne Thm 1.1A: For each short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and for each $i \geqslant 0$ there is a natural morphism $\delta^i : R^i F(A'') \to R^{i+1}F(A')$ such that we obtain a long exact sequence

$$\cdots \longrightarrow R^i F(A') \longrightarrow R^i F(A) \longrightarrow R^i F(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \longrightarrow R^{i+1}F(A) \longrightarrow \cdots$$

Proposition (2.1A, Cateogory of A-modules has enough injectives)   If A is a ring, then every A-module is isomorphic to a sub-module of injective A-module.

Proof. Hartshorne cites other resources. □

Proposition   Let $(X, \mathscr{O}_X)$ be a ringed space. Then the category $\mathfrak{Mod}(X)$ of sheaves of $\mathscr{O}_X$-modules has enough injectives.

Corollary   If X is any topological space, then the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.

Proof. Just because abelian groups are $\mathbb{Z}$-modules so it's just a particular case of the last theorem. □

Definition   The *cohomology functors* of a topological space X are the derived functors of the global section functor $\Gamma(X, \cdot)$. (What is this global sections functor for any topological space?) For any sheaf $\mathscr{F}$, the groups $H^i(X, \mathscr{F})$ are the *cohomology groups* of $\mathscr{F}$.

Remark   This is equivalent to sheaf cohomology construction via Čech cohomology.

4   Chapter IV

Exercise 1.2 (I like this one)   Again let X be a curve, and let $P_1, \ldots, P_r$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i and regular elsewhere.

Exercise 1.7 (no one)   A curve X is called *hyperelliptic*...

Exercise 1.8 (Alex)   Very useful to know, I think this is done in that book by Bosch of modules,

Exercise 1.9 (Victor)   Riemann-Roch for singular curves.

Exercise 2.3(h)   28 bitangents. Remind Sergey.

Exercise 2.5 Prove the theorem of Hurwitz that a curve X of genus $g \geqslant 2$ over a field of characteristic 0 has at most $84(g-1)$.

Exercise 3.1 If X is a curve of genus 2, show that a divisor D is very ample $\iff$ $\deg D \geqslant 5$. This strengthens (3,3.4).

Exercise 3.12 For each value of $d = 2, 3, 4, 5$ and r satisfying $0 \leqslant r \leqslant \frac{1}{2}(d-1)(d-2)$, show that there exists an irreducible plane curve of degree d with r nodes and no other singularities.

Exercise 4.10 If X is an elliptic curve (Sergey: for abelian varieties is also true), show that there is an exact sequence... Picard groups.

Exercise 5.3 Moduli of Curves of Genus 4. The hyperelliptic curves of genus 4 form an irreducible family of dimension 7. The nonhyperelliptic ones form an irreducible family of dimension 9. The subset of those having only one g_3^1 is an irreducible family of dimension 8. [Hint: Use (5.2.2) to count how many complete intersections $Q \cap F_3$ there are.]

Exercise 6.2 A rational curve of degree 5 in $\mathbb{P}^3$ is always contained in a cubic surface, but there are such curves which are not contained in any quadric surface.

5   Chapter V

Exercise 1.8 Divisor cohomology, neron severi

Exercise 2.8 Locally free sheaves.

Exercise 3.5 5 points in the field, hyperelliptic curve, point at infinity is singular.

Exercise 4.5

Exercise 4.16 27 lines on Fermat cubic

Exercise 5.1

Exercise 5.4

Exercise 5.5

Exercise 6.2 (Arthur)   Beautiful exercise.