# Exercises and notes in Algebraic Geometry

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December 10, 2024

# Contents

# 1. Chapter I

If not explicity stated, exercises are from Hartshorne. For Dani's exercises I have used this solution pdf and

## 1.1 Definitions and results

### 1.1.1 Lecture 1 by Lucas

- Ussando que $\mathbb{R} = k[x_1, \ldots, x_n]$ é Noetheriano (=todo ideal é finitamente generado), vimos que todo aberto de Zariski é uma união finita de conjuntos da forma.

- Se $X \subseteq \mathbb{A}^n$ é uma variedade afim (=existe $\mathfrak{a} \subset R$ tal que $X = \mathbb{V}(\mathfrak{a})$), os conjuntos fechados de $X$ são $\mathbb{V}(\mathfrak{b}) \cap \mathbb{V}(\mathfrak{a}) = \mathbb{V}(\mathfrak{a} + \mathfrak{b})$

- Nullstellenstz fraco:

$$\mathrm{Spm}(R) := \{\text{thingis maximais de } R\} = \{\langle x_1 - a_1, \ldots, x_n - a_n \rangle : a_i \in k\}$$

- Nullstellensatz:
$$\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \sqrt{\mathfrak{d}} \qquad \mathbb{V}(\mathbb{I}(S)) = \overline{S}$$

- Corolário:

$$\{\text{affine varieties of } \mathbb{A}^n\} \xrightarrow{\ \ 1-1\ \ } \{\text{radical ideals of } R\}$$

$$\{\text{irreducible varieties of } \mathbb{A}^n\} \longleftrightarrow \mathrm{Spec}(R) = \{\text{thingis primos}\}$$

$$\{\text{pontos de } \mathbb{A}^n\} \longleftrightarrow \mathrm{Spm}(R)$$

- Para duas variedades $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$,

$$\mathrm{Hom}(X, Y) = \{\varphi : X \to Y | \exists \tilde{\varphi} = (f_1, \ldots, f_m), f_i \in R\}$$

  e as $f_i$ extendem a $\varphi$.

- Tem uma equivalência de categorias

$$(\text{affine alg. varieties})^{\mathrm{op}} \to \text{reduced f.g. k-algebras}$$

  enviando cada variedade $X$ a $R[X]$.

- Dimensão de Krull (definida na seção 3 deste documento), o supremo das alturas de cadeias de ideiais primos, height.

- Teorema de Krull.

- A imagem de um morfismo não é necessesariamente uma variedade, porém, pode pegar o fecho e tudo bem.

$$D_f = \mathbb{A}^n \setminus \mathbb{V}(f).$$

### 1.1.2   Summary of Hartshorne section 3

**Definition** (Regular function)    It is a function from a variety to the field such that at every point there is an open neighbourhood (with respect to the Zariski topology, I guess) such that the function equals the quotient of two polynomials. In the affine case the polynomials are thought as functions on the coordinates of $\mathbb{A}^n$, and in the projective case they must be homogeneous polynomials *of the same degree* (which (apparently) makes the quotient be a well-defined function on $\mathbb{P}^n$, in contrast with general homogeneous polynomials that are not functions).

Here's a very nice reminder of how homogeneous coordinates work taken from wiki:

1. Any point in the projective plane is represented by a triple $(X, Y, Z)$, called ***homogeneous coordinates*** where $X$, $Y$ and $Z$ are not all 0.

2. The point represented by a given set of homomogeneous coordinates is unchanged if the coordinates are multiplied by a common factor.

3. Conversely, two sets of homogeneous coordinates represent the samer point if and only if one is obltained from the other by multipliying all the coordinates by the same non-zero constant.

4. When $Z$ is not 0 the point represented it the point $(X/Z, Y/Z$ in the Euclidean plane.

5. Whan $Z$ is 0 the point represented is the point at infinity.

6. (The triple $(0, 0, 0)$ does not represent any point. The origin of the Euclidean plane is represented by $(0, 0, 1)$.

**Definition** (Morphism)  A map between two varieties $\varphi : X \to Y$ is a *morphism* if it pulls back regular functions on Y to regular functions on X, ie. if $f : V \subseteq Y \to k$ is regular, then $f \circ \varphi$ is regular too.

An *isomorphism* is a morphism that admits an inverse morphism.

**Remark**  An isomorphism must be bijective and bicontinuous but a bijective bicontinuous morphism need not be an isomorphism.

**Definition** (Rings of regular functions)  $\mathcal{O}(Y)$ is the ring of all regular functions on the variety Y and $\mathcal{O}_P(Y)$ is the *local ring* consisting on functions defined on neighbourhoods of $P \in Y$, any two identified if they coincide near P. This is in fact a local ring with maximal ideal the functions that vanish at P.

**Proposition** (3.5)  Let X be any variety and let Y be any affine variety. Then there is a natural bijective mappingng of sets

$$\alpha : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(A(Y), \mathcal{O}(X))$$

where the left Hom means morphism of varieties and the right Hom means homomorphisms of $k$-algebras.

**Corollary** (3.8)  The functor $X \mapsto A(X)$ induces an arrow-reversing equivalence of categories between athe category of affine varieties over $k$ and the category of finitely generated integral domains over $k$.

### 1.1.3   Summary from da Silva, Symplectic Toric Manifolds

I found in this book a great crash-course in introductory (complex) algebraic geometry.

A *Zariski closed set* on $\mathbb{C}^n$ is a set of common zeroes of a finite number of polynomials from $\mathbb{C}[z_1, \ldots, z_n]$. The fact that an infinite intersection of closed sets is a closed set follows from the stabilization property for zero sets of polynomials: any decreasing sequence of such sets $X_1 \supset X_2 \supset \ldots$ stabilizes (this is a restatement of Hilbert's basis theorem that any ideal in $\mathbb{C}[z_1, \ldots, z_n]$ is finitely generated). Notice any nonempty open set is dense so Zariski topology is not Hausdorff.

**Definition**  An *affine variety* is a nonempty closed set in $\mathbb{C}^n$. An affine variety is *irreducible* if it cannot be written as the union of two proper closed subsets.

**Exercise**  An affine variety defined as the zero locus of the polynomials $p_1, \ldots, p_r \in \mathbb{C}[z_1, \ldots, z_n]$ is irreducible if and only if the ideal generated by $p_1, \ldots, p_r$ is prime (an ideal $I \subset \mathbb{C}[z_1, \ldots, z_n]$ is *prime* if $\forall u, v \in \mathbb{C}[z_1, \ldots, z_n]$ $uv \in I \implies u \in I$ or $v \in I$.

Let X be the zero locus of $p_1, \ldots, p_r \in \mathbb{C}[z_1, \ldots, z_n]$.

**Definition**  A *regular function* on X is a function $X \to \mathbb{C}$ that is the restriction to X of a polynomial function in $\mathbb{C}^n$. The ring of regular functions on X is denoted $\mathbb{C}[X]$.

**Exercise**  $\mathbb{C}[X]$ is isomorphic to $\mathbb{C}[z_1, \ldots, z_n]/(p_1, \ldots, p_r)$.

Let $X \subset \mathbb{C}^n$ and $X' \subset \mathbb{C}^m$ be affine varieties.

**Definition**   A *regular map* from $X$ to $X'$ is a map $\varphi : X \to X'$ that is the restriction of a polynomial map $\mathbb{C}^n \longrightarrow \mathbb{C}^m$ (entries are polynomial functions?).

**Exercise**   A map $\varphi : X \longrightarrow X'$ is regular if and only if it pulls back regular functions on $X'$ to regular functions on $X$.

**Definition**   An *isomorphism* from $X$ to $X'$ is a regular map $X \to X'$ which is invertible by a regular map.

**Exercise**   $X$ and $X'$ are isomorphic if and only if the associated rings of regular functions, $\mathbb{C}[X]$ and $\mathbb{C}[X']$ are isomorphic.

**Example**   Consider the variety $X$ in $\mathbb{C}^{2n}$ given by the zero-set of the polynomials $z_i z_{n+i}$ for $i = 1, \ldots, n$. For $n = 1$ it is just a hyperbola. Since the functions $z_i$ are invertible in the quotient ring $\mathbb{C}[X]$, we see that $\mathbb{C}[X] = \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$.

The projection $\mathbb{C}^{2n} \longrightarrow \mathbb{C}^n$ $(z_1, \ldots, z_{2n}) \longmapsto (z_1, \ldots, z_n)$ maps $X$ isomorphically onto the *n-dimensional algebraic torus*

$$(C^*)^n := (\mathbb{C} \setminus \{0\})^n = \mathbb{C}^n \setminus \text{hyperplanes } z_i, \qquad i = 1, \ldots, n.$$

So we see that this product of the multiplicative group of $\mathbb{C}$ has been endowed with the structure of an affine variety.

**Algebraic torus in Wiki**   Let $k$ be a field with algebraic closue $\bar{k}$. A $k$-*torus* is an algebraic group (=group that is a variety) defined over $k$ which is isomorphic over $\bar{k}$ to a finite product of copies of the multiplicative group. In other words, a $k$-group $\mathbb{T}$ is a torus iff $\mathbf{T}(\bar{k}) \cong (\bar{j}^\times)^r$ for some $r \geqslant 1$.

If $X \subset \mathbb{C}^n$ and $X' \subset \mathbb{C}^m$ are affine varieties, the symbol $A \longrightarrow B$

## 1.2   My first algebraic variety

**Exercise 1.1** (My first algebraic variety)

(a) Let $Y$ be the plane curve $y = x^2$ (ie., $Y$ is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$.

(b) Let $Z$ be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over $k$.

*(c) Let $f$ be any irreducible quadratic polynomial in $k[x, y]$, and let $W$ be the conic defined by $f$. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

*Proof.*

(a) Consider the map

$$k[x, y] \to k[x]$$
$$1 \mapsto 1$$
$$x \mapsto x$$
$$y \mapsto x^2$$

Notice that $y - x^2 \in k[x, y]$ is mapped to 0, so the kernel of this map is $(y - x^2)$. It is also surjective, so we have $A(Y) = k[x, y]/(y - x^2) \cong k[x]$.

(b) In constructing a map like in the former exercise, we may fix 1 and $x$, and we should map $y$ to $1/x$. However, $1/x$ is not an element of $k[x]$ so we really have an isomorphism $k[x, y]/(xy - 1) \cong k[x, \frac{1}{x}] \not\cong k[x]$.

(c) (From StackExchange)

**Step 1** Factorize the degree 2 homogeneous part into linear factors using that $k$ is algebraically closed.

**Step 2** If these linear factors are linearly dependent,

    i. Change coordinates to make the linear factor be the new $X$. The equation becomes $X^2 + aX + bY + c$.

    ii. Change coordinates to make $aX + bY + Z$ be the new $Y$.

    iii. We get $X^2 + Y$.

**Step 3** If these linear factors are linearly independent,

    i. Change coordinates to make one of the linear factors be $X$ and the other $Y$. The equation becomes $XY + aX + bY + c$ which can also be written as $(X + a)(Y + b) + d$.

    ii. Change the coordinates again to make it $XY + d$.

$\square$

## 1.3 The Segre embedding (extra)

Here's a summary written by ChatGPT after we discussed Segre embedding:

The Segre embedding provides a way to map the product of two projective spaces into a higher-dimensional projective space. Let $V$ and $W$ be two vector spaces of dimension $n + 1$ and $m + 1$, respectively. The Segre embedding embeds the product $\mathbb{P}(V) \times \mathbb{P}(W)$ into $\mathbb{P}(V \otimes W)$.

### The Homomorphism

The Segre map can be defined via a homomorphism of polynomial rings:

$$\varphi : k[z_{ij}] \to k[x_0, \ldots, x_n, y_0, \ldots, y_m],$$

where $z_{ij}$ are coordinates in a polynomial ring corresponding to the homogeneous co-ordinates of $\mathbb{P}(V) \times \mathbb{P}(W)$, and it sends the variables $z_{ij}$ to products of coordinates as follows:

$$\varphi(z_{ij}) = x_i y_j,$$

for $0 \leqslant i \leqslant n, 0 \leqslant j \leqslant m$.

## The Defining Ideal

The image of the Segre embedding is defined by a vanishing condition given by 2x2 minors of the corresponding matrix $(z_{ij})$. Specifically, the Segre variety is defined by the vanishing of all $2 \times 2$ minors of the matrix:

$$\begin{bmatrix} z_{00} & z_{01} & \dots & z_{0m} \\ z_{10} & z_{11} & \dots & z_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n0} & z_{n1} & \dots & z_{nm} \end{bmatrix}.$$

These minors represent the algebraic constraints that ensure the image lies in the Segre variety within the projectivization $\mathbb{P}(V \otimes W)$.

The vanishing of a specific 2x2 minor is given by:

$$M = z_{00} z_{11} - z_{01} z_{10}.$$

Substituting $z_{ij} = x_i y_j$ into this gives:

$$M = (x_0 y_0)(x_1 y_1) - (x_0 y_1)(x_1 y_0) = x_0 x_1 y_0 y_1 - x_0 x_1 y_0 y_1 = 0.$$

This confirms that $M$ vanishes under the Segre embedding.

## The Kernel of the Homomorphism

The kernel of the homomorphism $\varphi$ encodes the Segre variety. That is:

$$\ker(\varphi) = \langle z_{00} z_{11} - z_{01} z_{10} \rangle,$$

where $\langle z_{00} z_{11} - z_{01} z_{10} \rangle$ denotes the ideal generated by the determinant of any $2 \times 2$ minors of the matrix formed by $z_{ij}$. These correspond precisely to the algebraic constraints defining the Segre embedding.

## Rank-1 Condition and Algebraic Geometry

Geometrically, the Segre embedding corresponds to the condition that a tensor formed by $x_i y_j$ must have rank at most 1. This rank-1 condition translates algebraically into the vanishing of all 2x2 minors of the matrix $(z_{ij})$.

The algebraic Segre variety thus corresponds to the zero locus of these polynomials in $\mathbb{P}(V \otimes W)$.

### 1.3.1 Dani's earlier attempt

**Exercise 2.14** (The Segre Embedding)   Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the order pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Note that $\psi$ is well-defined and injective. It is called the *Segre embedding*. Show that the image of $\psi$ is a subvariety of $\mathbb{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} : i = 0, \ldots, r, j = 0, \ldots, s\}$ and let $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which sends $z_{ij}$ to $x_i y_j$. Then show that $\operatorname{img} \psi = Z(\mathfrak{a})$.

*Solution.*  First let's make sure the dimension $N$ is correct. The easy way is found in wiki: $N = (r+1)(s+1) - 1$ which is the number of possible choices of pairs of things taking one out $r + 1$, another out of $s + 1$, and then remember there is only one zero index so take one away.

To see that $\psi$ is injective we follow StackExchange: Let $z = [z_{00} : z_{01} : \ldots : z_{ij} : \ldots : z_{rs}]$ be an element of the image of $\psi$ and let $(a, b) \in \mathbb{P}^r \times \mathbb{P}^s$ be such that $\psi(a, b) = z$. WLOG we can assume $a_0 = b_0 = z_{00} = 1$. Then $b_j = z_{0j}$ for all $0 \leqslant j \leqslant s$ and $a_i = z_{i0}$ so $a, b$ are uniquely determined and this map is bijective onto the image.

Actually, what we have done is constructed an inverse morphism of the Segre map. According to StackExchange, this makes it into an embedding.

To show that $\operatorname{img} \psi$ is a subvariety of $\mathbb{P}^N$ we need to find a set of homogeneous polynomials in $k[z_{ij}]/$

Following the hint, as before let $z \in \operatorname{img} \psi$ and $f$ any polynomial in the kernel of

$$k[\{z_{ij}\}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s]$$

. We must show that $f(z) = 0$. Well it doesn't make much sense because if $f = \sum a_{ij} z_{ij}$ is in the kernel of that map, then its image $\sum a_{ij} x_i y_j$ is the zero polynomial, so obviously $f(z) = \sum a_{ij} z_{ij} = \sum a_{ij} x_i y_j = 0$. So this is confusing.

So what are the equations of $\operatorname{img} \psi$? A polynomial $f(z_{00}, \ldots, z_{rs})$ will vanish on $\operatorname{img} \psi$ if somehow it vanishes $\qquad\square$

## 1.4   The quadric surface in $\mathbb{P}^3$ (Bruno)

**Exercise 2.15** (The Quadric Surface in $\mathbb{P}^3$)   Consider the surface Q (a *surface* is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - wz = 0$.

1. Show that Q is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

2. Show that Q contains two families of lines (a *line* is a linear variety of dimension 1), $\{L_t\}, \{M_t\}$ each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$ then $L_t \cap L_u = \varnothing$ and if $M_t \neq M_u$, $M_t \cap M_u = \varnothing$, and for all $t, u$, $L_t \cap M_u$ is a point.

3. Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via $\psi$ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ where each $\mathbb{P}^1$ has its Zariski topology.

*Solution.*

1. It turns out that the image of the Segre embedding $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ equals is the algebraic variety given by the zeroes of the polynomial $f = z_{00}z_{11} - z_{10}z_{01} \in k[z_{00}, z_{01}, z_{10}, z_{11}]$. One contention is easy: if $(x, y) = ([x_0, x_1], [y_0, y_1]) \in \operatorname{img} \psi$, then clearly $f(\psi(x, y)) = x_0 y_0 x_1 y_1 - x_0 y_1 x_1 y_0$ is zero because these are numbers in the field $k$.

   Now for the other contention pick $z = [z_{00}, z_{01}, z_{10}, z_{11}] \in V(f)$ and let's find an element $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $\psi(x, y) = z$. $z \in V(f)$ means that $z_{00}z_{11} = z_{10}z_{01}$. If $z_{00} \neq 0$, then we can define $([z_{00}, z_{11}], [z_{01}, z_{10}])$ what?

   Maybe for the other contention try to define the inverse map $\operatorname{img} \psi \to \mathbb{P}^1 \times \mathbb{P}^1$ by $z = [z_{00}, z_{01}, z_{10}, z_{11}] \mapsto ([z_{00}, z_{01}], [z_{00}, z_{10}])$ when $z_{00} \neq 0$ and $([z_{11}, z_{01}], [z_{11}, z_{10}])$ when $z_{11} \neq 0$. Is this defining a global map?

2. The lines correspond to fixing one entry and running over the other one in the Segre embedding $(x, y) \to z$.

$\square$

## 1.5 Blow-up of points in $\mathbb{A}^2$

Let $\pi : B \to \mathbb{A}^2$ denote the blow-up of $O = (0, 0)$ in $\mathbb{A}^2$. One can show that

$$B = \{(x, y) \times (t : u) \in \mathbb{A}^2 \times \mathbb{P}^1 : ux = ty\}.$$

The morphism $\pi$ is an isomorphism away from $O$.

Let $Y \subseteq \mathbb{A}^2$ be a curve through $O$. Then $\pi^{-1}(Y)$ is an algebraic set which is isomorphic to $Y$ away from $O$, and which contains $E = \pi^{-1}(O)$. Indeed, $u/t = y/x$ means that, away from $E$, the $\mathbb{P}^1$-coordinate of a point $P$ in $B$ is the slope of the line from $O$ to $\pi(P)$. This allows us to construct an inverse morphism $\mathbb{A}^2 - \{O\} \to B - E$.

**The exercise:** The pre-image $\pi^{-1}(Y)$ is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by $y^2 = x^3$ and $xu = ty$. Let's look at $\pi^{-1}(Y) \cap D(t)$. We may put $t = 1$ then, so $B \cap D(t)$ is cut out by $y = ux$. Hence $\pi^{-1}(Y) \cap D(t)$ is cut out by $u^2x^2 = x^3$ in $B \cap D(t)$. Given that

$$(u^2x^2 - x^3) = (u^2 - x)(x)^2$$

in the ring of $B \cap D(t)$, it follows that

$$\pi^{-1}(Y) \cap D(t) = \left[ V(u^2 - x) \cup E \right] \cap D(t).$$

Now, $\pi^{-1}(Y) \cap V(t)$ is given by $y^2 = x^3$ and $ux = 0$. Also, since $t = 0$, we have $u \neq 0$. Hence $x = 0$ and so $y = 0$, i.e.,

$$\pi^{-1}(Y) \cap V(t) = \{O \times (0:1)\} = E \cap V(t).$$

Let $\tilde{Y}$ be the curve cut out by $u^2 = t^2 x$ in $B$. Then $\tilde{Y}$ is irreducible. Moreover, $\tilde{Y} \cap E = \{O \times (1:0)\}$ because $x = y = 0$ implies $u = 0$. What we see is that $\pi^{-1}(Y)$ is the union of the irreducible curves $\tilde{Y}$ and $E$. Therefore $\tilde{Y}$ is the strict transform of $Y$.

Consider the morphism

$$\varphi : \mathbb{A}^1 \longrightarrow \tilde{Y}, \quad u \longmapsto (u^2, u^3) \times (1:u).$$

It's inverse is give by $(x, y) \times (1:u) \mapsto u$. Therefore $\tilde{Y} \cong \mathbb{A}^1$.

Since $\pi : \tilde{Y} \to Y$ is an isomorphism away from $O$ and $\tilde{Y}$ meets $E$ at a single point, we deduce that $\pi$ is bijective. Since $\pi$ maps closed sets (finite sets) to closed sets (finite sets), we find that $\pi$ is bicontinuous. On the other hand, $\pi$ is not an isomorphism because $Y$ is not isomorphic to $\mathbb{A}^1$.

**Hartshorne's example:** Take $Y = V(y^2 - x^2(x + 1)) \subseteq \mathbb{A}^2$. We can do manipulations like above to find that $\pi^{-1}(Y)$ is the union of $\tilde{Y} = V(u^2 - t^2(x + 1))$ and $E$. So, $\tilde{Y}$ meets $E$ at the points $u = \pm 1$, which correspond to the slopes of $y^2 = x^2(x + 1)$ at $O$.

## 1.6 Conics in $\mathbb{A}^2$ and $\mathbb{P}^2$ (Dani)

**Exercise 3.1**     a. Show that any conic in $\mathbb{A}^2$ is isomorphic either to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$.

    b. Show that $\mathbb{A}^1$ is not isomorphic to any proper open subset of itself.

    c. Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

    d. We will see later that any two curves are homeomorphic. But show now that $\mathbb{A}^2$ is not even homeomorphic to $\mathbb{P}^2$.

    e. If an affine variety is isomorphic to a projective variety, then it consists of only one point.

*Solution.* I understand a **conic** to be zero-set of an (irreducible) quadratic polynomial in either $k[x, y]$ (affine) or in $S[x, y, z]$.

    a. In exercise 1.1 it is shown that any conic in $\mathbb{A}^2$ has coordinate ring isomorphic to either $k[x]$ or $k[x, \frac{1}{x}]$. But the coordinate ring of $\mathbb{A}^1 \setminus \{0\}$ is not isomorphic to $k[x]$, so we are done.

    b. Notice that an open subset of $\mathbb{A}^1$ is $\mathbb{A}^1$ minus a finite a set of points, say $a_1, \ldots, a_m$. The coordinate ring of such a variety is $k[x]/(x - a_i)_i$, which cannot be isomorphic to $k[x]$. <span style="color:magenta">I have read that in such a ring $x - a_i$ is a *unit*, while I think it is just zero (the zero class=additive identity).</span>

    c.

**Step 1** Let $f \in k[x, y, z]$ be an irreducible homogeneous polynomial of degree 2. Then $f$ can be written as $x^{\mathsf{T}}Mx$ with $M$ a $3 \times 3$ symmetric matrix. But any such matrix is diagonalizable because $k$ is closed, meaning there is another matrix $Q$ such that $Q^{\mathsf{T}}MQ = D$ with $D$ diagonal.

**Step 2** $Q$ defines a morphism of $\mathbb{P}^n$ to itself since it is composed of linear polynomials (recall that a morphism must pull back regular functions to regular functions, and in the projective case regular functions are locally quotients of homogeneous polynomials of the same degree, so a linear transformation preserves such a structure).

**Step 3** $Q$ restricts to an isomorphism on the zeroes of the initial matrix $M$ and the diagonal matrix $D$, meaning any two conics are isomorphic.

**Step 4** The image of the so-called 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is also a conic, so all conics are isomorphic to $\mathbb{P}^1$.

d. Algebraic topology.

e. The ring of regular functions on a projective variety is isomorphic to $k$ by Thm 3.4. But then $A(Y) \cong \mathcal{O}(X)$ by Prop 3.5, which is only true for the coordinate rings of maximal ideals, that correspond to points.

$\square$

## 1.7   Homeomorphisms that are not isomorphisms (Dani)

**Exercise 3.2**   A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism (of algebraic varieties). (Bicontinuous morphism need not be isomorphism.)

a. For example, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that $\varphi$ defines a bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that $\varphi$ is not an isomorphism.

b. For another example, let the characteristic of the base field $k$ be $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that $\varphi$ is bijective and bicontinuous but not an isomorphism. This is called the **Frobenius morphism.**

*Solution.*

a. First notice that $\varphi$ is bijective on its image simply because it is injective.
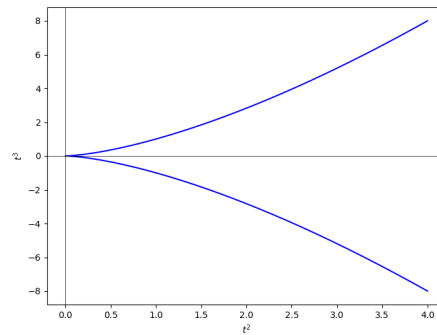
It is clear that it is a morphism.

To see if it is continuous first remember that affine space (and also projective space) is equipped with the Zariski topology, whose open sets are complements of algebraic sets (=sets of zeroes of polynomials). But $A = k[x]$ is principal so every algebraic set is the set of zeroes of one polynomial, basically because of this corollary of the Nullstellensatz:

**Corollary** ([Hart] 1.4] There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in $k[x_1, \ldots, x_n]$ (i.e., ideals which are equal to their own radical), given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible iff its ideal is a prime ideal.

And $k$ is closed so every polynomial $n$ roots, so algebraic sets are just finite sets.

This was seemed relevant at some point. . .

**Definition** The *(Krull) dimension* of a ring $A$ is the supremum of all *heights* of all prime ideals, i.e. the largest proper chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}$.

**Proposition 1.7** If $Y$ is an affine algebraic set, then the dimension of $Y$ is equal to the dimension of its affine coordinate ring $A(Y)$.

We only need to know that closed sets are finite points. This means that polynomials are continuous functions with respect to Zariski topology (because preimage of a closed set=finite set of points is closed because it is the root of $p - \alpha$). That makes $\varphi$ continuous.

Being bicontinuous is the same as being a homeomorphism, which is equivalent to showing that $\varphi$ is open or closed (since we already noted it is bijective). But, finite sets of points are closed sets in the curve (because it is put in affine space—this is not true in general!).

Finally we show that $\varphi$ is not an isomorphism. Yesterday with Victor we tried to

write the inverse map, which should satisfy

$$(x, y) = (\varphi^{-1}(x, y)^2, \varphi^{-1}(x, y)^3)$$

$$\implies \begin{cases} \varphi^{-1}(x, y)^2 &= x = \frac{x^2}{x^3} = \frac{x^2}{y^2} = \left(\frac{x}{y}\right)^2 \\ \varphi^{-1}(x, y)^3 &= y = \frac{y^2}{y^3} = \frac{x^3}{y^3} = \left(\frac{x}{y}\right)^3 \end{cases}$$

so basically $\varphi^{-1}$ must be $(x, y) \mapsto \frac{x}{y}$ but this is not defined at $y = 0$ so I thought well why don't you just put

$$(x, y) \longmapsto \begin{cases} 0 & y = 0 \\ \frac{x}{y} & y \neq 0 \end{cases}$$

but then all $(x, 0)$ will map to $(0, 0)$ under $\varphi \circ \varphi^{-1}$. So what happens is that this map is not well-defined so it is not an isomorphism.

b. The Frobenius morphism is in fact a ring homomorphism meaining it preserves multiplication and addition. Addition is more interesting: since $p$ is prime, it divides the numerator and not the denominator of the binomial coefficient $\frac{p!}{k!(p-k)!}$, and this will make that number 0 unless $k = 0, n$, which are the first final terms of the binomial expansion $(a + b)^p$. Therefore we must only show that $\varphi$ has trivial kernel. Now since $k$ is a field, all elements are units, so none of them can actually satisfy $a^p = 0$, making $\ker \varphi = 0$.

That $\varphi$ surjective follows from an equivalence of the field being algebraically closed and thus perfect.

That $\varphi$ is continuous and bicontinuous follows from the same reasons as the previous question.

To finish we check that the induced map on coordinates rings is not an isomorphism, since its inverse must be of the form $t \mapsto t^{\frac{1}{p}}$, which is not polynomial.

$\square$

## 1.8   $\mathbb{P}^n$ minus a hyperplane (Alex)

**Exercise 3.5** (Alex, Sept 17)   By abuse of language, we will say that a variety *is affine* it is isomorphic to an affine variety. If $H \subseteq \mathbb{P}^n$ is any hypersurface, show that $\mathbb{P}^n - H$ is affine.

*Solution.* Suppose $H = V(f)$ with $\deg f = d$. Consider the d-uple embedding $v_d : \mathbb{P}^n \to \mathbb{P}^N$
$\square$

## 1.9 Products of Quasi-Projective Varieties (extra)

**Exercise 3.16** (Products of Quasi-Projective Varieties)   Use the Segre embedding (Ex. 2.14) to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it a structure of projective variety. Now for any two quasi-projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ consider $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$.

  (a) Show that $X \times Y$ is a quasi-projective variety.

  (b) If $X, Y$ are both projective, show that $X \times Y$ is projective.

  (c) Show that $X \times Y$ is a product in the category of varieties.

*Solution.*  content...  □

## 1.10 Normal varieties (Victor)

**Exercise 3.17** (Normal varieties)

**Exercise 3.18** (Projectively Normal Varieties)

**Definition**

  • A variety $Y$ is *normal* ar a point $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. $Y$ is *normal* if it is normal at every point.

  • A projective variety $Y \subseteq \mathbb{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

  • Let $A \subset B$ be a subring (or a morphism $A \to B$) $b \in B$ is *integrally closed* if it is the root of a polynomial in $A[x]$. Here we mean that $S(Y)$ equals the set of integrally closed elements with respect to its field of fractions.

  a. If $Y$ is projectively normal, then $Y$ is normal.

  b. There are normal varieties in projective space which are not projectively normal. For

*Solution.*

  a. $\forall p \in Y$ and prime ideal $\mathfrak{m}_p \in \mathrm{Spec}(Y)$ we have

$$S(Y) \subseteq S(Y)_{\mathfrak{m}_p} \subseteq F(Y)$$

  How does this conclude?

  b.

  □

**Exercise 5.4** (Intersection mulitplicity)   If $X, Y \subseteq \mathbb{A}^2$

## 1.11   September 10

### 1.11.1   Analitycally isomorphic singularities (Arthur)

**Arthur's notes**

**Exercise 5.14**

(a) **Problem:** Show that if two points of two plane curves are analytically isomorphic then they have the same multiplicity.

Let $A = k[x, y]/(f)$ and $m = (x, y)A$. Then $A/m^n \simeq \hat{A}/m^n\hat{A}$ for all $n \geqslant 0$. Setting $r = \mu(f)$, we have $(f) \subseteq m^r$ and $(f) \subsetneq m^{r+1}$. Hence

$$\dim \hat{A}/m^n\hat{A} = \dim A/m^n = \dim \frac{k[x, y]}{(x, y)^n} \quad \text{for all } n \leqslant r.$$

[Dimensions of $k$-vector spaces.] However

$$\dim \hat{A}/m^{r+1}\hat{A} = \dim A/m^{r+1} = \dim \frac{k[x, y]}{(f) + (x, y)^{r+1}} < \dim \frac{k[x, y]}{(x, y)^{r+1}}.$$

Therefore we have a characterization of $r = \mu(f)$ as the first $n \geqslant 0$ such that $\dim \hat{A}/m^{n+1}\hat{A} \neq \dim k[x, y]/(x, y)^{n+1}$. Since this characterization depends only on the completion of $A$, we have obtained a solution to the the problem.

(b) Let $f, g \in k[[x, y]]$. Write $f = f_r + f_{r+1} + \cdots$, where the $f_d \in k[x, y]$ are homogeneous of degree $d$. Suppose

$$f_r = g_s h_t,$$

where $g_s$ and $h_t$ homogeneous of degrees $s$ and $t$, and have no common linear factor.

**Problem:** Find power series

$$g = g_s + g_{s+1} + \cdots,$$
$$h = h_t + h_{t+1} + \cdots$$

such that $f = gh$.

Expand the product to find

$$f = \sum_{d=0}^{\infty} \left( \sum_{i+j=d} g_{s+j} h_{t+i} \right).$$

Looking at homogeneous parts of degree $r + d$, we see that our problem is solved as soon as we have

$$f_{r+d} = \sum_{i+j=d} g_{s+j} h_{t+i} = (g_{s+d} h_t + g_s h_{t+d}) + \sum_{\substack{i+j=d \\ i,j \neq 0}} g_{s+j} h_{t+i}. \qquad (1.1)$$

Hence, one way of solving the problem is to inductively find the polynomials $g_{s+d}, h_{t+d}$.

Since $g_s$ and $h_t$ don't have linear factors in common in $k[x, y]$, the polynomials $g_s(t, 1)$ and $h_t(t, 1)$ don't have common linear factors in $k[t]$ and so generate they $k[t]$. Given a homogeneous $a(x, y) \in k[x, y]$, we can therefore find $p(t), q(t) \in k[t]$ such that

$$g_s(t, 1)p(t) + h_t(t, 1)q(t) = a(t, 1).$$

Taking $t = x/y$ and multiplying the expression above by $y^{\deg a}$, we obtain

$$g_s(x, y)\tilde{p}(x, y) + h_t(x, y)\tilde{q}(x, y) = a(x, y),$$

where $\tilde{p}(x, y) = y^{\deg a - s}p(x/y), \tilde{q}(x, y) = y^{\deg a - t}q(x/y) \in k[x, y]$ are homogeneous.

In particular, we can find $g_{s+1}, h_{t+1} \in k[x, y]$ homogeneous such that

$$g_s h_{t+1} + g_{s+1}h_t = f_{r+1};$$

and homogeneous polynomials $g_{s+2}, h_{t+2}$ such that

$$g_s h_{t+2} + g_{s+2}h_t = f_{r+2} - g_{s+1}h_{s+1}.$$

More generally, we can find $g_{s+i}, h_{t+i} \in k[x, y]$ such that

$$g_s h_{t+i} + g_{s+i}h_t = f_{r+i} - \sum_{\substack{i+j=d \\ i,j \neq 0}} g_{s+j}h_{t+i}$$

Then equation (1.1) is satisfied and the problem is solved.

(c) Coming soon.

(d) Write $f(x, y) = f_2(x, y) + f_3(x, y) + \cdots + f_n(x, y)$. Using the change of coordinates $x \mapsto x + y$, we have that $f_2(x, y) = ay^2 + bxy + cx^2$ with $a \neq 0$. Then $f(x, y) = y^2 g(y)$ for some $g(y) \in k[y]$ such that $g(0) \neq 0$.

The Weierstrass preparation theorem says that we may write

$$f(x, y) = U(x, y)(y^2 + a(x)y + b(x))$$

for *unique* $U(x, y) \in k[[x, y]]$ and $a(x), b(x) \in k[[x]]$ such that $U$ is invertible and $a(0) = b(0) = 0$. [We won't use this last condition.] Since $U$ is a unit,

$$\widehat{\mathcal{O}_P} \simeq \frac{k[[x, y]]}{(f(x, y))} = \frac{k[[x, y]]}{(y^2 + a(x)y + b(x))},$$

i.e., the curve is given near the origin by $y^2 + a(x)y + b(x) = 0$.

Because char $k \neq 2$, we can eliminate the linear term by completing the square and changing $y \mapsto y - a(x)/2$:

$$y^2 + a(x)y + b(x) = \left(y + \frac{a(x)}{2}\right)^2 - c(x)$$

Now the equation is $y^2 - c(x) = 0$. But $c(x) = x^r V(x)$ for some $V(x) \in k[[x]]$ such that $V(0) \neq 0$. Since char $k \neq 2$, we can then find a power series $U(x)$ such that $U(x)^2 = V(x)$. This is just solving the linear equations arising from

$$\sum a_n x^n = \left( \sum b_n x^n \right)^2 = \sum \left( \sum_{\substack{i+j=n \\ i,j \neq n}} b_i b_j + b_0 b_n \right) x^n.$$

Now we substitute $y \mapsto y U(x)$ to transform the ideal $(y^2 - x^r U(x)^2)$ into $(y^2 - x^r)$.

Finally, uniqueness of $r$ follows from the uniqueness part of the Weierstrass preparation theorem: Given $f(x, y) \in k[x, y]$, we did $x \mapsto x + y$ and considered the polynomial $y^2 + a(x)y + b(x)$ given by the preparation theorem. There was no choice involved in these steps. After completing the square we saw that $r$ is the largest integer such that $x^r$ divides $c(x) = -b(x) + a(x)^2/4$; hence $r$ is uniquely determined by $f(x, y)$.

**Notes from class**

Two curves $C_1, C_2$ with points $P \in C_1$ and $Q \in C_2$, and $\mathcal{O}_P \supset \mathfrak{m}_P$, $\mathcal{O}_Q \supset \mathfrak{m}_Q$. And

$$C : f = 0, \qquad f[x, y]/(f), \qquad \mathfrak{m} = (x, y).$$

We define

$$\hat{\mathcal{O}}_P = \varprojlim \mathcal{O}_P / \mathfrak{m}_P^n \cong k[[x, y]]/(y)$$

**Definition**  *Analitical isomorphism* is a $k$-isomorphism with $\hat{\mathcal{O}}_P \cong \hat{\mathcal{O}}_Q$.

**Exercise**  a. An. isomorphic $\implies \mu_{C_1}(D) = \mu_{C_2}(Q)$.

$$f(x, y) = f_r(x, y) + f_{r+1}(x, y) + \ldots + f_n(x, y).$$

$$\mu(p) = r$$

b. Let $f = f_r + f_{r+1} + \ldots \in k[[x, y]]$. Suppose that $f_r = g_s h_t \in k[x, y]$ have no linear factors in common (which means they are coprime). Then there exist $g, h \in k[[x, y]]$ such that
$$f = gh, \quad g = g_s, \quad h = h_t + \ldots$$

c. Let $f = f_r + f_{r+1} + \ldots \in k[x, y]$. If
$$f_r = L_1 \cdot L_2 \cdot \ldots \cdot L_r$$

then $P = (0, 0)$ is an ordinary $r$-fold. Show that any 2-folds are analitically isomorphic.

d. Consider the curve $C : f = 0, \quad \mu = 2$. It is an ordinary 2-fold. Show that

$$\hat{\mathcal{O}}_P \cong \frac{k[[x]]}{(y^2 - x^r)}$$

*Solution.*

  a.

  b. Suppose they exist and are $g = \sum_i g_i$, $h = \sum_j h_j$ so that

$$f = gh = \sum_{d=r}^{\infty} \sum_{i+j=d} g_i h_j$$

which means that

$$f_{r+d} = \sum_{i+j=d} g_i h_j = g_s h_{t+d} + g_{s+d} h_t + \sum_{?} g_i h_j$$

and

$$f_{r+1} = g_s h_{t+1} + g_{s+1} h_t$$

Notice that since $g$ and $h$ don't share linear factors, we may fix the second variable and the resulting polynomials generate $k[t]$, ie.

$$(g_s(t,1), h_t(t,1)) = k[t]$$

Further computations show that the next term is well defined.

  c. Consider the completion at the origin, $\hat{\mathcal{O}}_P \cong k[[x,y]]/(f)$. We shall try to show that in fact

$$k[[x,y]]/(f) \xrightarrow{\cong} k[[x,y]]/(f_r)$$

so consider the map

$$x \mapsto x + A(x,y) \qquad y \mapsto y + B(x,y)$$

so that

$$f(x + A(x,y), y + B(x,y)) = f_r(x,y).$$

Using a projective transformation, that makes the tangents be (?), we see that

$$\hat{\mathcal{O}}_P \cong k[[x,y]]/(xy)$$

**Claim** (Not true)  Any k-isomorphism

$$\varphi : \frac{k[[x,y]]}{(f)} \longrightarrow \frac{k[[x,y]]}{(y)}$$

have the form

$$x \mapsto x + A(x,y) \qquad y \mapsto y + B(x,y) \quad \text{when } \mu \geqslant 2$$

Suppose that

$$f(x,y) = y^2 A(y) \qquad A(0) \neq 0, \qquad f(x,y) = (y^2 + a(x)y + b(x)) \cdot u(x,y)$$

recall that

**Theorem** (Weierstrass preparation theorem)    Let $A$ be a local ring, $k[[x]]$, $\mathfrak{m} = (x)$, and $f \in A[[y]]$. Then

$$f = (y^n + k_{r-1}y^{n-1} + \ldots + b_0) \cdot u(y)$$

for $u \in A[[y]]^{\times}$ and $b_i \in \mathfrak{m}$

$$f(x,y) = (y^2 + c(x))u(x,y)$$
$$\hat{\mathcal{O}}_P \cong \frac{k[[x,y]]}{(y^2 + c(x))}$$
$$c(x) = x^r V(x), \quad \text{where } V \in k[[x]]^{\times}$$
$$V(x) = 1 + x + \ldots$$
$$\sqrt[r]{1 + x} = 1 + c_1 x + \ldots$$
$$V(x) = W(x)^r$$
$$x^r V(a) = (xW)^r$$
$$y^2 - x^r.$$

$\square$

## 1.11.2   Degree and Hilbert polynomial (Bruno)

**Definition**    Let $M = \bigoplus_t M_t$ be a finitely generated graded $S$-module, where $S = k[x_0, \ldots, x_m]$. A ***Hilbert function*** is

$$\varphi : \mathbb{Z} \longrightarrow M$$
$$t \longmapsto \dim_k(M_t)$$

**Theorem** (Definition)    There exists a unique polynomial $P_M \in \mathbb{Q}[t]$ called the ***Hilbert polynomial*** such that for all $t \gg 0$, $P_M(t) = \varphi_M(t)$.

**Repetition**    Let me put a quote from wiki so that I never forget what is the Hilbert polynomial:

> [...] This shows that there exist a unique polynomial $HP_X(n)$ with rational coefficients which is equal to $HF_S(n)$ for $n$ large enough. This polynomial is the ***Hilbert polynomial,*** and has the form
>
> $$HP_S(n) = \frac{P(1)}{(\delta - 1)} n^{\delta - 1} + \text{ terms of lower degree in } n$$

**Exercise 7.2**    Let $Y$ be a variety of dimension $r$ in $\mathbb{P}^n$, with Hilbert polynomial $P_Y$. We define the ***arithmetic genus*** of $Y$ to be

$$P_Y(t) := P_{S[Y]}(t)$$

$$= (-1)^r (P_Y(0) - 1)$$

a. $Y = \mathbb{P}^n$.

b. $Y \subset \mathbb{P}^n$ plane curve of degree d.

c. $Y \subset \mathbb{P}^n$ hypersurface of degree d.

d. $Y_a, Y_b \subset \mathbb{P}^3$ of degrees a and b, $Y = Y_a \cap Y_b$ complete intersection.

e. $Y^r \subset \mathbb{P}^n, Z^s \subset \mathbb{P}^m : Y \times Z \subset \mathbb{P}^N$. Here

$$P_{Y \times Z}(t) = P_Y(t) P_Z(t)$$

*Solution.*

a. $S[Y] = k[x_0, \ldots, x_n]$,

$$\dim(k[x_0, \ldots, x_n]_t) = \binom{n+t}{n} = \frac{(n+1)(n+t-1)\ldots(t+1)}{n!} = \frac{t^n}{n!} + \ldots$$

so that

$$P_Y(0)_1 \implies P_a(\mathbb{P}^n) = 0.$$

b. Now

$$P_a(Y) = \frac{(d-1)(d-2)}{2}$$

which follows from c so let's do c.

c. Suppose $Y = Z(f)$. Then

$$(f) \hookrightarrow k[x_0, \ldots, x_n] \to k[Y]$$

$$(f) = \{f\varphi : \varphi \in k[x_0, \ldots, x_n]\}$$
$$(f)_t = \binom{n+t-d}{n}$$
$$P_Y(t) = \binom{n+t}{n} - \binom{n+t-d}{n}$$
$$P_a(Y) = (-1)^n \binom{n-d}{n}$$
$$= (-1)^n \frac{(n-d)(n-d-1)\ldots(1-d)}{n!}$$
$$= \frac{(d-1)\ldots(d-n)}{n!}$$
$$= \binom{d-1}{n}$$

d. Let's first write the formula and then we discuss it

$$P(t) = \binom{3+t}{t} - \binom{3+t-a}{t} - \binom{3+t-b}{t} + \binom{3+t-a-b}{3}$$
$$= (ab)t + P_Y(0)$$

$$P_Y(0) = 1 - f(a) - f(b) + f(a+b)$$
$$P_a(Y) = f(a) + f(b) - f(a+b)$$

Considering multiplication by $f_b$ as map

$$\frac{k[\mathbb{P}^3]}{(f_a)} \xrightarrow{f_b} \frac{k[\mathbb{P}^3]}{(f_a)}$$

so we get

$$0 \longrightarrow \frac{k[\mathbb{P}^3]}{(f_a)} \longrightarrow \frac{k[\mathbb{P}^3]}{(f_a)} \longrightarrow \frac{k[\mathbb{P}^3]}{(f_a, f_b)} \longrightarrow 0$$

but we need to consider the map $[-b]$ to stop the grading from moving up from one arrow to another. Then we get

$$P_Y = P_{Y_a}(t) - P_{Y_a}(t-b)$$

so

$$P_A - P_B + P_C = 0.$$

e. O problema está em ver se

$$K[Y \times Z]_d \overset{?}{\cong} k[Y]_d \otimes_k k[Z]_d$$

Daí o Hilbert polynomial é o produto dos Hilbert polynomials e tá.

$\square$

## 1.12  September 13

### 1.12.1  Veronesse surface (Alex)

**Exercise 2.13**   Let Y be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This is the **Vernoses surface**. If $Z \subseteq Y$ is a closed curve (a **curve** is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbb{P}^5$ such that $V \cap Y = Z$.

*Solution.*

$$\rho_2 : \mathbb{P}^2 \overset{\text{Veronese}}{\longrightarrow} \mathbb{P}^5$$
$$[x_0 : x_1 : x_2] \longmapsto [x_0^2 : x_1^2 : x_2^2 : x_0 x_1 : x_1 x_2 : x_0 x_2]$$

And $Y := \rho_2(\mathbb{P}^2)$. Notice that $Y = \mathbb{V}(\ker \theta)$ where

$$\theta : k[y_0, \ldots, y_5] \longrightarrow k[x_0, \ldots, x_2]$$
$$y_i \longmapsto M_i = i\text{-th monomial of degree } d \text{ in } x_0, \ldots, x_n$$

$$y_0 \longmapsto x_0^2$$
$$y_1 \longmapsto x_1^2$$
$$y_2 \longmapsto x_2^2$$
$$y_3 \longmapsto x_0 x_1$$
$$y_4 \longmapsto x_0 x_2$$
$$y_5 \longmapsto x_1 x_2$$

it is not immediate that kernel of this map is $Y$, but it is true.

By exercise 2.8 there exists a homogeneous polynomial (which is irreducible) such that $V(f) = \rho_2^{-1}(Z)$. Recall that $Z$ is a curve in $Y$ so $\rho^{-1}(Z)$ is (probably) a curve in $\mathbb{P}^2$.

So, we have

$$Z = \rho_2(V(f)) = \rho_2(V(f^2)),$$

which follows from the fact that $\theta$ is surjective over the set of polynomials of even degree. How to prove this? It suffices to prove for monomials. This is a combinatorial problem. Consider

$$x_0^{a_0} x_1^{a_1} x_2^{a_2} \longmapsto (a_0, a_1, a_2) \in \mathbb{Z}^3$$

Anyway... this means that there exists $g \in k[x_0, x_1, x_2]$ such that $f^2 = \theta(g)$. And then

$$Z = Y \cap V(g)$$

which means that being the Veronese image of a zero of $f$ is the same as being a zero of $g$. (Some more thought on how $\theta$ is working?) □

**Explanation** The Veronese map image is actually the symmetric matrices of rank 1:

$$v \longmapsto v^{\mathrm{T}} \cdot v$$

$$\begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} v_0 & v_1 & v_2 \end{pmatrix} = \begin{pmatrix} v_0 v_0 & v_0 v_1 \\ v_1 v_0 & v_1 v_1 \end{pmatrix}$$

## 1.12.2 Degree of Veronese and Segre embeddings (Alex)

**Exercise 7.1**

    a. Find the degree of the 2-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$.

    b. Find the degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$.

*Solution.*

a. Recall that the degree of a map the coefficient of greatest degree of the Hilbert polynomial times the degree factorial, i.e. if the Hilbert polynomial is

$$P_X(z) = a_n z^n + \ldots + a_0,$$

the degree of X is

$$a_n n!$$

(So remember that the Hilbert polynomial is the unique polynomial in $\mathbb{Q}[t]$ that equals the Hilbert function (the Hilbert functions at $n$ is just the dimension of the degree-$n$ component of $S[X] = \bigoplus_n S_n$, so homogeneous polynomials of degree $n$) for sufficiently large numbers.)

**Claim**

$$S(X)_m \cong k[x_0, \ldots, x_n]_{md}$$

where $d$ is the $d$ in the $d$-uple embedding.

If we show this, we can see that

$$P_x(m) = \binom{md + n}{n} = \prod_{i=1}^n \frac{(md + i)?}{n!} = \frac{d^n m^n}{n!} + \text{lower order terms}$$

Now the pullback map is

$$\frac{k[x_0, x_1, \ldots, x_N]}{I(X)} \xrightarrow{\cong} ?k[x_0, x_1, \ldots, x_n]$$

b.

$\square$

### 1.12.3 Curvas elípticas (Carlos)

k não neces. algebricamente fechado, $\bar{k}$ algebraic closure, V variedade afim, $V \subseteq \mathbb{A}^n = k^n$, $k[V] = k[c_1, \ldots, x_n]/I(V)$. $P \in V$, $\mathfrak{m}_p = \{g \in k[V] : g(p) = 0\}$ é um ideal maximal.

Para $g \in k[V]$,

$$\text{ord}_P(g) = \sup\{d \in \mathbb{Z}_{\geqslant 0} : g \in \mathfrak{m}_p^d\}.$$

Uma *curva* é uma variedade projetiva suave de dimensão 1. *Curvas elípticas*:

$$E : x^2 = x^3 + ax + b$$

$$\downarrow \text{projetização}$$

$$zy^2 = x^3 + axz^2 + bz^3$$

**Exercise**  Seja $E/k$ uma curva eliptica suave:

$$y^2 = x^3 + ax + b$$
$$= (x - e_1)(x - e_2)(x - e_3)$$

com $e_1, e_2, e_3 \in \bar{k}$. Considere

$$f_1 = (x - e_1), \quad f_2 = (x - e_2), \quad f_3 = (x - e_3)$$

$$p_1 = (e_1, 0), \quad p_2 = (e_2, 0), \quad p_3 = (e_3, 0)$$

O que é

$$\operatorname{ord}_{f_i} p_i =?$$

*Solution.* Lembre que $p$ é um ponto suave se e somente se $\dim \mathfrak{m}_p / \mathfrak{m}_p^\alpha = \dim V$.

**Step 1** Considere a projetização de $E$ :

$$y^2 z = (x - e_1 z)(x - e_2 z)(x - e_3 z).$$

Daí,

$$p_1 = [e_1 : 0 : 1], \quad p_2 = [e_2 : 0 : 1], \quad p_3 = [e_3 : 0 : 1]$$

$\square$

## 1.13 September 17

### 1.13.1 Dimension of affine variety $\cap$ hyperplane (Alex+Arthur)

**Exercise 1.8** Let $Y$ be an affine variety of dimension $r$ in $\mathbb{A}^n$. Let $H$ be a hypersurface in $\mathbb{A}^n$ and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

So: intersection of variety of dimension $r$ and hypersurface is variety dimensin $r - 1$.

*Solution.* Suppose $H = \mathbb{V}(f)$ and $Y = \mathbb{V}(\mathfrak{q})$. Then $Y \cap H = \mathbb{V}(\mathfrak{q} + (f))$. Pick an irreducible component in $Y \cap H$ which is associated to a prime ideal $\mathfrak{p}$. Then we have that $\mathfrak{q} + (f) \subseteq \mathfrak{p}$. Passing to the quotient, we have that $(f)/\mathfrak{q} \subseteq \mathfrak{p}/\mathfrak{q}$. Now we use the result that

**Theorem** (Krull's Hauptidealsatz) Let $A$ be a noetherian ring. For an element $x \in A$ that is not a unit nor a zero divisor,

$$(x) \subseteq \mathfrak{p},$$

where $\mathfrak{p}$ is a minimal ideal, then $\mathfrak{p}$ has height 1.

and since $(f)/\mathfrak{q}$ is principal in the quotient, ie. $(f)/\mathfrak{q} = (f + \mod \mathfrak{q})$. $\square$

## 1.14 Sept 20

### 1.14.1 Scroll (Sergey)

You can find this $\operatorname{Sec}^k$ on a book by Harris, name?

There is an exercise from last class:

**Exercise from Sergey, Sept 17**    Compute $\text{Sec}^k$ of $\text{Segre}(a, b)$.

For today:

**Exercise (Sergey, Sept 20)**

a. Compute $\dim \text{Gr}(k, N)$. You can use that $\text{GL}(N)$ acts transitively.

b. $\dim \text{Sec}^k \text{Gr}(N, 2)$.  **Hint:** Use linear algebra: $\Sigma_{0,n} = \text{Cone}(v_n \mathbb{P}^1, \mathbb{F}_n$ Hinzbruch surfaces. Rational Ruled Normal Surfaces (see below for definition of scroll ie. $\Sigma_{0,n}$.

*Solution (Victor, September 27).*

a. Consideremos os pontos da Grassmaniana $\text{Gr}(m, N)$ como classes de equivalência de bases de vetores identificadas baixo a ação de $\text{GL}(n)$. Considere uma base $(v_1, \ldots, v_m)$. Ela pode ser pensada como uma matriz de $N \times m$ de rango maximo, ie. ela tem posto $m$. Então, essa matriz tem uma submatriz de $m \times m$ não singular, e uma submatriz de $m(N - m)$ que é singular. Variando as entradas dessa matriz singular, obtemos coordenadas para uma variedade de dimensão $m(N - m)$. [Explicar mais]

**Definition** (Extra: mergulho de Plücker, Sept 27)    A Grassmaniana é uma variedade projetiva:

$$P_k : \text{Gr}(m, V) \longrightarrow \mathbb{P}(\Lambda^m(V))$$
$$W \longmapsto [w_1 \wedge \ldots \wedge w_n]$$

onde $W$ está generado por $w_1, \ldots, w_n$. Aqui tem que mostrar que isso está bem definido.

b. Lembre que

**Definition**    (Ver Wiki) Dada uma variedade projetiva $V \subset \mathbb{P}^r$, ***variedade secante*** de $V$ é
$$\text{Sec}^k(V) = \overline{\{\overline{x_1, \ldots, x_k} : x_1, \ldots, x_k \in V\}}$$

Em palavras, o fecho de Zariski do conjunto de "espaços generados" por $k$-tuplas de pontos em $V$. Por exemplo,

$$\text{Sec}^2(V) = \overline{\{\overline{xy} : x, y \in V\}}$$

é o fecho de Zariski do conjunto de todas as linhas secantes a $V$ em $\mathbb{P}^r$. Mais geralmente, a $k$-***ésima variedade secante*** é o fecho de Zariski da união de todos os espaços lineares generados por coleções de $k$ (ou $k + 1$?) pontos em $V$. (Note que os espaços lineares numa variedade projetiva são as projetivizações dos espaços lineares correspondentes no espaço afim.)

Pegue dois elementos em $\text{Gr}(2, N)$, digamos $W, W'$ e pegue bases delas $\{w_1, w_2\}$ e $\{w'_1, w'_2\}$. Daí considere

$$\left\{ \left( t w_1 - (1 - t) w'_1, t w_2 - (1 - t) w'_2 \right) \right\}$$

$\square$

**Definition** The *Segre embedding* is

$$\mathbb{P}(U) \times \mathbb{P}(V) \hookrightarrow \mathbb{P}(U \otimes W)$$
$$([u], [v]) \longmapsto [u \otimes v]$$

Now take $X \subset \mathbb{P}(W)$. Then

$$\operatorname{Sec}^k(X) = \overline{Z}$$

where

$$Z = \{[w] : \exists p_1, \ldots, p_k \in X \hookrightarrow \mathbb{P}(W)\} = \bigcup_{p_1, \ldots, p_k \in X} \langle p_1, \ldots, p_k \rangle$$

where $p_i = [v_i]$ and $w = \sum_i v_i$.

**Definition** (Scroll)  I have a variety $X$ and two linear systems:



Then the *scroll* is

$$\mathbb{P}(U \oplus V) \supset \Sigma_{\varphi, \psi} = \{[u \oplus v] : [u] \in \varphi(X), [v] \in \psi(X)\}$$

**Exercise:** Show this is a projective variety.

So the scroll is a union of projective lines. It is called scroll because lines in one direction survive (in a drawing, they are all parallel) but the other lines are curved (so it looks like a scroll).

**Exercise about scroll**  As abstract varieties, $\Sigma_{a,b}$ is isomorphic to $\Sigma_{a'b'}$ iff $b - a = b' - b' = n$ and either $a = a' = 0$ or $a > 0, a' > 0$.

### 1.14.2  Elliptic curve is not rational (Alex)

**Exercise 6.3, Elliptic curve is not rational**  Let Y be the curve $y^2 = x^3 - x$ in $\mathbb{A}^2$, and assume that the characteristic of the base field $k$ is not 2. In this exercise we will show that Y is not a rational curve, and hence $K(Y)$ is not a pure trascendental extension of $k$.

  a. Show that Y is nonsingular, and deduce that $A = A(Y) \cong \frac{k[x,y]}{(y^2 - x^3 + x)}$.

  b. Let $k[x]$ be the subring of ...

**Exercise 6.2**  Let

$$Y = V(y^2 - x^3 + x \subset \mathbb{A}^2$$

be an elliptic curve. Show it is not rational.

**Definition**  A curve C is *rational* if C is birationally equivalent to $\mathbb{P}^1$

**Fact**   If C is a nonsingular rational but not isomorphic to $\mathbb{P}^1$, then $A(C)$ is a UFD.

**Step 1** Y is nonsingular. Looks like we can just differentiate here and find that the zero locus of the derivatives is not in C.

> **Definition** (Tangent space)   Consider the local ring at the point and its maximal ideal. So $\mathfrak{m}_p/\mathfrak{m}_p^2$ is the *cotangent space*. Then the *tangent space* is the dual $\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right)^*$. A point is *singular* if the tangent space has more dimension than it should.

> **Interpretation**   Functions in $\mathfrak{m}_p$ are those which vanish at p. A function in $\mathfrak{m}_p^2$ is $\sum_i f_i g_i$ so its differential is $d\left(\sum_i f_i g_i\right) = \sum_i f_i dg_i + \sum_i g_i fd_i$ which you want to mod out because it is zero at p because both $f_i$ and $g_i$.

> **Remark**   For $X, Y \subset Z$, $T_p X, T_p Y \subset T_p Z$

$$T_p(X \cap Y) = T_p X \cap T_p Y$$

**Step 2**

> **Theorem 5.1**   $Y \subseteq \mathbb{A}^n$ affine variety and $P \in Y$. Then Y is nonsingular at P $\iff$ local ring $\mathcal{O}_{P,Y}$ is a regular local ring

> **Theorem 6.2A**   A noetherian local domain of sdimension one with maximal ideal $\mathfrak{m}$. Then the following are equivalent

> (i)  A is a regular local ring.

> (ii)  A is integrally closed (over the fraction field).

> In other words, normality in dimension 1 is the same as normality.

**Step 3** Here's the *hyperelliptic involution*

$$\sigma : k[Y] \longrightarrow K[Y]$$
$$x \longmapsto x$$
$$y \longmapsto -y$$

and then there's the *norm*

$$N : K[Y] \longrightarrow K[Y]$$
$$a \longmapsto a\sigma(a)$$

Notice that

$$N(1) = 1 \qquad N(a, b) = N(a)N(b)$$

**Explanation:** we are killing y, so the norm takes us to a simpler space which is $Y/\sigma$.

This means X is irreducible, Y is irreducible. Then a discussion on the invertibility of the norm, units on these rings. . .

### 1.14.3 Tangent map (Bruno)

**Exercise 5.10**   X afine variety and $p \in X$.

  a. $\dim T_p X \geqslant \dim X$, $\dim T_p X \iff p$ is nonsingular. Here $\dim X = \dim \mathcal{O}_{P,X}$ which is just the Krull dimension, so maximum height of chain of prime ideals.

  b. Define the tangent map.

  c. Project the parabola $x = y^2$ in $\mathbb{A}^2$ to the $x$ axis and compute its derivative at $0$.

*Solution.*

- $\dim T_p X = \dim \mathfrak{m}_p / \mathfrak{m}_p^2$, *as a vector space!* . Now that inequality is a result in homological algebra:

  **Theorem**   A noetherian local domain and $\mathfrak{m}$ maximal ideal then the dimension of the quotient of ideals as a vector space is greater or equal to the dimension of $A$ as a ring.

- 

  **Definition**   $X \xrightarrow{\varphi} Y$, $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X$ which is precomposition. Then since $\varphi(p) = q$ we see that $\varphi^*(\mathfrak{m}_q) \subset \mathfrak{m}_p \implies \varphi^*(\mathfrak{m}_q^k) \subset \mathfrak{m}_p^k$. And then take quotient by $\mathfrak{m}_p$ and $\mathfrak{m}_q^2$. These local rings are the functions that vanish at the point. Also they are localization of polynomials by maximal ideal.

- 

  **Claim**   $(T\varphi)_0 = 0$

  look at the pullback

  $$\varphi^* : k[\mathbb{A}^1] \longrightarrow k[Z] = k[x,y]/(x - y^2)$$
  $$t \longmapsto x$$

  which follows from definition.

  Then

  $$\varphi^*(\mathfrak{m}_0) \subset \mathfrak{m}_{(0,0)}^2 \implies \varphi^* : \mathfrak{m}_0/\mathfrak{m}_0^2 \to \mathfrak{m}_{0,0}/\mathfrak{m}_{0,0}^2$$

  is zero because

  $$\varphi^*(t) = x = y^2, \qquad y \in \mathfrak{m}_{(0,0)}$$

  $\square$

**Exercise Sergey**   Is the variety $\sum_{i=0}^{s} x_i^3 = 0$ rational in $\mathbb{P}^5$.

## 1.15   November 26: Fano 3-folds (Sergey)

Here's what happened today.

## 1.16   November 29

### 1.16.1   Blowing up the cuspidal cubic curve (Dani)

**Exercise 4.10**   Let Y be the cuspidal cubic curve $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the point $O = (0,0)$, let E be the exceptional curve, and let $\tilde{Y}$ be the strict transform of Y. Show that E meets $\tilde{Y}$ in one point, and that $\tilde{Y} \cong \mathbb{A}^1$. In this case the morphism $\varphi : \tilde{Y} \longrightarrow Y$ is bijective and bicontinuous, but it is not an isomorphism.

*Solution.* First recall that the blow-up of $\mathbb{A}^2$ at the origin is the variety $X \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{t,u}$ defined by the equation $xu = ty$. We denote $\varphi : X \longrightarrow \mathbb{A}^2$ the projection of X onto the first factor.

Now put the cuspidal cubic curve Y given by $y^2 = x^3$ inside $\mathbb{A}^2$ and consider its strict transform $\tilde{Y}$, which is the closure of $\varphi^{-1}(Y - O)$. It turns out to be useful to consider the inverse image of Y in X because it will contain $\tilde{Y}$ and E; it is given by the equations

$$xu = ty, \qquad y^2 = x^3.$$

Take an affine chart $t = 1$. Substituting, we get $x^2u^2 = x^3$, that is, $x^2(x - u^2) = 0$. So there are two irreducible components, the first given by $x = 0 = y$ and u free, and the second by $x = u^2$. The first component is the exeptional curve E, and the second is $\tilde{Y}$. They intersect when $u = 0$. Is this enough to show that $\tilde{Y}$ and E intersect at one point only?

Now in the other affine chart $u = 1$, we get $x^2 = t^2x^3$, so, $x^2(t^2x - 1)$. The component $t^2x = 1$ does not intersect E.

It remains to show that $\tilde{Y}$ has no singularities, since Y being singular would prevent them from being isomorphic. What are the defining equations for $\tilde{Y}$? By the argument above it is clear that a point $\tilde{Y}$ satisfies $x = u^2$, $xu = y$ and $y^2 = x^3$ when $t = 1$. The curve $y = u^3$ has no singularities, as $y^2 = x^3$ when $x, y \neq 0$.

Likeways, when $u = 1$, points in $\tilde{Y}$ satisfy $t^2x = 1$, $x = ty$ and $y^2 = x^3$, all of which are nonsingular. To make sure why, we may differentiate with respect to t, which yields $2tx = 0\ldots$

But all points satisfying these equations are in $\tilde{Y}$? I would like to know what is the ideal defining $\tilde{Y}$. So maybe a projective ideal of $\mathbb{A}^2 \times \mathbb{P}^1$?                                              $\square$

**After exposition**   Here's a more algebraic way to show the restriction of the projection to the strict transform of Y cannot be an isomorphism. The coordinate ring of $Y \subset \mathbb{A}^2$ is $\dfrac{k[x,y]}{\langle y^2 = x^3 \rangle}$, and the coordinate ring of $\tilde{Y}$ is $k[u]$, which means that we are taking the chart $u \neq 0$. Then we have a map

$$\psi : k[x,y] \longrightarrow k[u]$$
$$x \longmapsto u^2$$
$$y \longmapsto u^3$$

and of course we also have the projection $k[x,y] \twoheadrightarrow \frac{k[x,y]}{\langle y^2, x^3 \rangle}$. We have argued by the universal property of (?) that

$$
\begin{array}{ccc}
k[x,y] & \longrightarrow & k[u] \\
\downarrow & \nearrow & \\
\frac{k[x,y]}{\langle y^2, x^3 \rangle} & \exists? f &
\end{array}
$$

We have made two remarks:

1. That $f$ is injective

2. That $\operatorname{img} f = k[u^2, u^3]$ which is a subring of $k[u]$ but not equal to it. So it cannot be an isomorphism.

### 1.16.2 There exists a unique generic point (Bruno)

$X$ scheme, $Z \subset X$ irreducible closed subset, then there exists a unique $\xi \in X$ such that $\overline{\{\xi\}} = Z$, a *generic point of* $X$.

First suppose that $X = \operatorname{Spec}(A)$. Choose $\xi \in X$. Then $\overline{\{\xi\}} = V(\xi)$. Because $\xi$ is a maximal ideal! $V(\xi) = \{a \in \operatorname{Spec}(A) : a > \xi\}$.

And uniqueness: $V(\xi_1) = V(\xi_2) \implies \xi_1 = \xi_2$.

And in general we just use the affine cover of a scheme. And then the intersections might not be affine, but their intersections have affine covers.

### 1.16.3 Some more Fano (Sergey)

There exists only two proofs that an abstract fano $F$ has $\pi_1(F) = 0$.

1. Uses DG/RG (diff/riemannian geometry?). Uses Yau (existence of a nice metric), compactness. This was already explained in this, last semester.

$$
\mathbb{F}_p \longleftrightarrow \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p \subset \overline{\mathbb{Q}}_p \subset ?
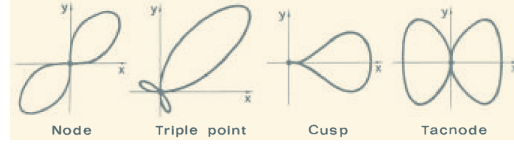$$

2. Uses $\mathscr{F}_p$, 1980s, Mori. Mori emphasized the role of rational curves.

## 1.17 December 3: some more blow-ups (Dani)

I'll start by doing the same I did last time, that way we can see how well understood it is.

**Exercise 5.6** (Blowing up curve singularities)

(a) Let $Y$ be the cusp of node of the following figure:

Node          Triple point          Cusp          Tacnode

The node is $xy = x^6 + y^6$ and the cusp is $x^3 = y^2 + x^4 + y^4$
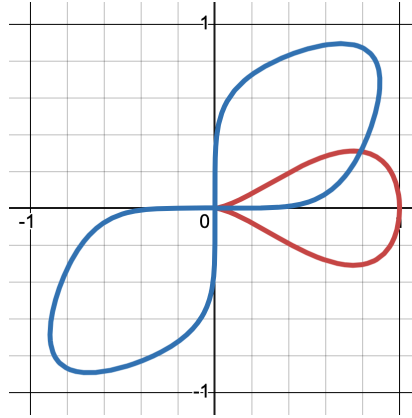


Enough of that. Show the strict transform of either of them is nonsingular.

*Solution.*

(i) Consider first the node $xy = x^6 + y^6$. Before you do anything ask yourself why is this curve singular (this is exercise 5.1). The definition of a curve being **nonsingular** at a point is that the rank of the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)\Big|_p$ is $n - r$ where $Y \subset \mathbb{A}^n$ is of dimension $r$ and its ideal is generetad by $f_i$. For our nodal curve at zero such a matrix is zero.

Now we mimic last week's exercise. The inverse image of $Y$ is given by that and $xU = yY$ in the blow-up (which is contained in $\mathbb{A}^2 \times \mathbb{P}^1$. Take the affine piece $T \to 1, U \to u$. We get $xu = y$, and then $x^2(u - x^4 + x^6 u^6) = 0$. Now if $x = 0$, also $y = 0$. This means the component corresponding to $x^2$ is again E. Now at the points of $\varphi^{-1}(Y)$ where $x = y = 0$ we must also have $u = 0$, meaining that, like before, they intersect at the point $[0, T]$. This could be the only place where $\tilde{Y}$ is singular. Now look for the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)\Big|_p$ and check its rank.

The equations defining $\tilde{Y}$ in this affine chart are $f_1 = xu - y$ and $f_2 = u - x^4 + x^6 u^6$. Then the Jacobian matrix of $\tilde{Y}$ is

$$\begin{pmatrix} \partial_x f_1 & \partial_y f_1 & \partial_u f_1 \\ \partial_x f_2 & \partial_y f_2 & \partial_u f_2 \end{pmatrix}_{(0,0)} = \begin{pmatrix} u & -1 & x \\ -4x^3 + 6x^5 u^6 & 0 & 1 + x^6 \end{pmatrix}_{(0,0)}$$

which has rank 2 at $(0, 0)$.

By the way, according to ChatGPT, in this affine chart the strict transform looks as follow:



Intersection of the Surfaces
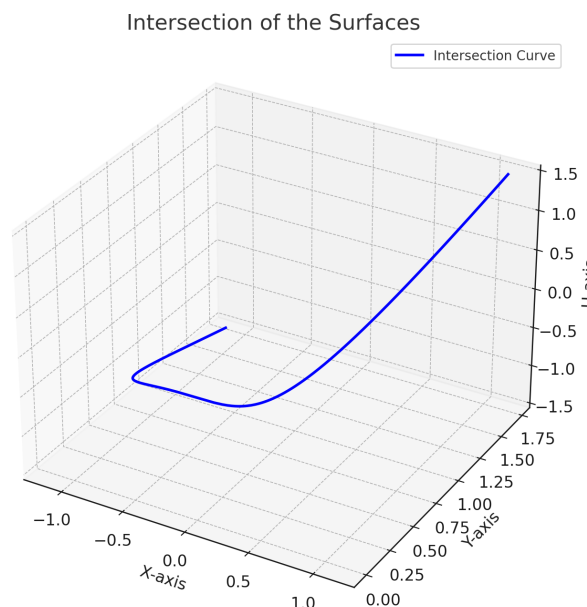
(ii) The case for the cusp $x^3 = y^2 + x^4 + y^4$ is analogous.

$\square$

(b) Now for a curve $Y \subseteq \mathbb{A}^2$ given by $f(x, y) = 0$ define the ***multiplicity of*** $P \in Y$ to be the least $r$ such that $f_r \neq 0$ for a decomposition $f = f_0 + f_1 + \ldots + f_d$ into homogeneous polynomials $f_i$ of degree $i$ which are called ***tangent directions*** after making a change of coordinates that sends $P$ to zero. And define a ***node*** to be a point of multiplicity 2.

If $P$ is a node on a plane curve $Y$, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on $\tilde{Y}$.

*Solution.* Suppose $Y$ is given by $f = f_2 + f_3 + \ldots + f_d$. *We may suppose* that $f_2 = xy$ under a suitable change of coordinates. Also suppose that $P = (0, 0)$. And there it goes again: the inverse image of $Y$ is given by $xU - yT = 0$ and $xy - f'(x, y) = 0$. Do $T \to 1$ and $U \to t$. We get $xu = y$ and $x^2 u = f'(x, xu)$, so $x^2(u - g(x, xu))$. We get two irreducible components... $\tilde{Y}$ is given by $u = -g(x, xu)$. This means $\tilde{Y}$ meets $E$ at zero, so we look for a singularity here. So the defin ing equations for $\tilde{Y}$ are $xu - y$ and $u - g(x, ux)$. The Jacobian matrix is of rank 2 because $\partial_x g = 0$ since it has degree more than 2. $\square$

(c) Let $P \in Y$ be the tacnode, which is the middle one here:

$$x^2 = x^4 + y^2 \qquad\qquad x^2 = x^4 + y^4 \qquad\qquad x^2 = x^4 + y^6$$
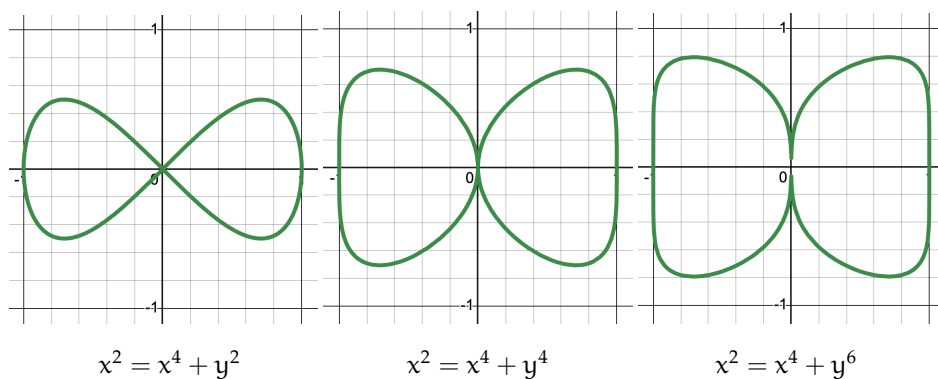
Show it is a node.

*Solution.* We just do everything again. By $T \to 1$ and $U \to u$ we get $\tilde{Y}$ given by $1 - x^2 - x^2 u^4$ so actually when $x = y = 0$ these curves don't meet. So take the other affine chart. Now you get $\tilde{Y}$ given by $t^2 - y^2 t^4 - y^2$ so they do meet at zero. (We can see the point in this chart.) Evaluating at the origin we see that there remains a term $t^2$. □

(d) Finally show that



$$y^3 = x^5$$
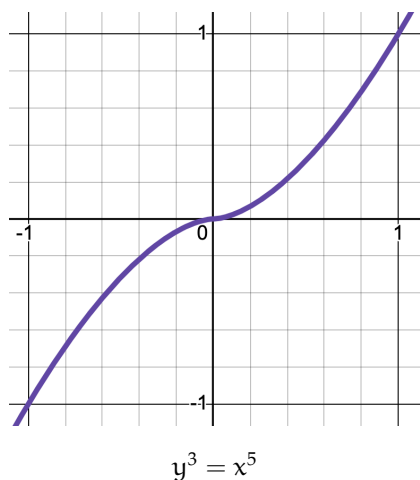
has a singularity that requires three blow-ups to be resolved. (Which is strange it looks quite smooth everywhere.)

*Solution.* Blowing up gives $x^3(u^3 - u^2)$ at the affine chart $T \to 1$, everything working nicely. Then $\tilde{Y}$ is a node, so as in item (b) we conclude that one more blow-up resolves the singularity. □

Let us finish with

**Proposition V.3.8**  Let C be an irreducible curve in the surface X. Then there exsits a finite sequence of monoidal transformations (=blow-ups) (with suitable centers) $X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$ such that the strict transform $C_n$ of C on $X_n$ is nonsingular.

**Definition on page 388**  Let C be an effective Cartier divisor on the surface X, and let f be a local equation for C at the point P. Then we define the ***multiplicity*** of C at P, denoted by $\mu_P(C)$ to be the largest integer r such that $f \in \mathfrak{m}_P^r$ where $\mathfrak{m}_P \subset \mathcal{O}_{P,X}$ is the maximal ideal.

Remember that

**Theorem I.5.1**  Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Let $P \in Y$ be a point. Then Y is nonsingular at P (Jacobian definition) iff the local ring $\mathcal{O}_{P,Y}$ is a regular local ring (=$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}_{P,Y}$, $\mathfrak{m}$ maximal ideal).

So that's algebra but the point is that multiplicity says how serious is the singularity $\leftrightsquigarrow$ how many blow-ups you will need to resolve it.

**Remark V.3.5.2**  $\mu_P(C) = 1$ iff C is nonsingular at P, because in that case $\mathfrak{m}_{P,C}$ will be a principal ideal, so $f \notin \mathfrak{m}_{P,X}^2$.

**Proposition V.3.6**  Let C be an effective divisor on X, let P be a point of multiplicity r on C, and let $\pi \colon \tilde{X} \to X$ be the monoidal transformation with center P. Then

$$\pi^* C = \tilde{C} + rE.$$

**Corollary V.3.7**  With the same hypotheses, we have $\tilde{C}.E = r$, and $p_a(\tilde{C}) = p_a(C) - \frac{1}{2}r(r-1)$.

So, I think that the fact the sequence of blow-ups must end is basically because the arithmetic genus $p_a$ is nonnegative by an exercise of part III.

# 2. Chapter II

## 2.1 September 27

## 2.2 Sheaves (Bruno)

### 2.2.1 Background

Let X be a topological space and $\mathscr{F}$ a sheaf of rings, which is a presheaf + gluing condition.

**Sheafification**

$$\text{Presheaf} \longrightarrow \text{Sheaf}$$
$$\mathscr{F} \longmapsto \mathscr{F}^+$$

This $\mathscr{F}^+$ has two properties:

1. Universal

$$
\begin{array}{ccc}
\mathscr{F} & \longrightarrow & G \\
\downarrow & \nearrow & \\
\mathscr{F}^+ & &
\end{array}
$$

2. For any $p \in X$, the stalks $\mathscr{F}_p = \mathscr{F}_p^+$.

3. (Extra)

$$\text{Sheafs} \xrightarrow{\text{forget}} \text{Presheaf}$$
$$\xleftarrow{\text{Sheafify}}$$

**Remark**    Also kernel and image of sheafs. Must sheafify the image.

### 2.2.2 Exercise 1.8

Let $U \subset X$ be an open subset of $X$. Consider

$$\Gamma(U, \cdot) : \text{Sheafs} \longrightarrow \text{Rings}$$
$$\mathcal{F} \longmapsto \Gamma(U, \mathcal{F})$$

Show that $\Gamma(U, \cdot)$ is a left exact functor, that is,

$$0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{B}$$

exact implies

$$0 \longrightarrow \Gamma(U, \mathcal{A}) \xrightarrow{a_U} \Gamma(U, \mathcal{B}) \xrightarrow{b_U} \Gamma(U, C)$$

*Solution.*

- $a_V$ is injective is clear.

- $\text{img } a_U = \ker b_U$. Suppose $s \in \Gamma(U, \mathcal{B})$, so that $s =$

$\square$

## 2.3 September 24

### 2.3.1 Generically finite morphisms (Arthur)

Exercise 3.7

*Solution.* **First part.** We will show the hint: the field extension $K(X) \subseteq K(Y)$ is finite.

Let $V = \operatorname{Spec} A \subseteq Y$ and $U = \operatorname{Spec} B \subseteq f^{-1}(V) \subseteq X$ be affine open subsets. Write $C$ for the coordinate ring of $f^{-1}(\eta) \cap U$.

Since $f$ is of finite type, $C$ is finitely generated as a $K(Y)$-algebra. Explain more. By Noether's normalization lemma, there exist $y_1, \ldots, y_d \in C$ such that $C$ is finite over $K(Y)[y_1, \ldots, y_d]$, where $d = \dim C$. But $\dim C = \dim(f^{-1}(\eta) \cap U) = 0$ because $f^{-1}(\eta) \cap U$ is a finite set. Hence $C$ is finite over $K(Y)$, i.e., $C$ is a finite $K(Y)$-vector space. In particular, $K(X) \subseteq C$ is finite over $K(Y)$.

**Second part.** We will show that there is an open affine $V \subseteq Y$ and an affine open cover $\{U_i\}$ of $f^{-1}(V)$ such that each restriction $U_i \to V$ is finite.

Let $V = \operatorname{Spec} A \subseteq Y$ be an open affine. Since $f$ is of finite type, there is a finite affine open cover $U_i = \operatorname{Spec} B_i$ of $f^{-1}(V)$ such that $B_i$ is an $A$-algebra of finite type. Let $\{x_{ij}\}$ be a set of generators for $B_i$. Now $B_i \subseteq K(X)$ and $K(X)$ is a finite $K(Y)$-module, so each $x_{ij}$ is integral over $K(Y)$, i.e., there are polynomials $P_{ij}(T) \in K(Y)[T]$ such that $P_{ij}(x_{ij}) = 0$.

There a is distinguished open subset $V_g \subseteq V$ such that $P_{ij}(T) \in \mathcal{O}_Y(V')[T]$ for all $i, j$. Replace $U_i$ by $(U_i)_g$ and $V$ by $V_g$. Then $B_i$ is finite over $A$ because it is finitely generated

by integral elements. Hence $f : U_i \to V$ is a finite morphism ($U_i$ and $V$ are affine) for all $i$.

**Third part.** Find an affine $V \subseteq Y$ such that $f^{-1}(V)$ is affine.

Now let $U \subseteq \bigcap_i U_i$ be an affine open. Then $U \subseteq f^{-1}(V)$ and $f : U \to V$ is affine. We will show that we can choose an affine open $V' \subseteq V$ such that $f^{-1}(V') \subset U$. Then we will have $f : f^{-1}(V') \to V'$ affine, concluding the exercise.

To do this, let $Z = f^{-1}(V) \setminus U$. If $\eta \in \overline{f(Z)}$ then actually $\eta \in f(Z)$ because it is the generic point of $V$. Looking at the affine $U_i = \operatorname{Spec} B_i$, we see that any $z \in U_i \cap Z$ such that $\eta = f(z)$ corresponds to a prime $\mathfrak{p} \subseteq B_i$ such that $\mathfrak{p} \cap A = (0)$. Since $\mathfrak{p}$ is above the prime $(0) \subseteq A$, the following theorem tells us that $\mathfrak{p} = (0)$:

**Theorem** (Cohen)

But $(0) \in U$ and $Z \cap U = \varnothing$. Hence $\eta \notin \overline{f(Z)}$. Therefore there is some neighborhood $V'$ of $\eta$ such that $V' \cap f(Z) = \varnothing$, which is what we had to show.

Alternatively, let $Z = f^{-1}(V) \setminus U_1$. The same theorem shows that $f(Z)$ is closed. Since $U_i \to V$ is finite for all $i$, we have $\dim f^{-1}(V) = \dim V$. Moreover

$$\dim V = \dim f^{-1}(V) > \dim Z \geqslant \dim f(Z),$$

so $f(Z) \neq V$ and there is an open affine $V'$ such that $V' \cap f(Z) = \varnothing$.  $\qquad\square$

## 2.4   October 1st

### 2.4.1   Closed subschemes of $\operatorname{Proj}(S)$ **(Dani)**

**Review of** Proj

- **Set.** Let $S$ be a graded ring. $\operatorname{Proj}(S)$ is the set of homogeneous prime ideals $\mathfrak{p}$ that do not conatin $S_+ = \bigoplus_{d>0} S_d$.

- **Topology.** The closed subsets of $\operatorname{Proj}(S)$ are $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj}(S) : \mathfrak{p} \supseteq \mathfrak{a}\}$.

- **Sheaf.** For each $\mathfrak{p} \in \operatorname{Proj}(S)$ consider $S_{(\mathfrak{p})}$, the localized ring $T^{-1}S$ where $T$ is the multiplicative system of homogeneous elements of $S$ that are not in $\mathfrak{p}$. For any open subset $U \subseteq \operatorname{Proj}(S)$ define $\mathcal{O}(U)$ to be the set of functions (s for section?) $s : U \to \coprod S_{(\mathfrak{p})}$ such that for $p \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ (perhaps like a vector field: a point maps to a vector attached to the tangent space at that point), and $s$ is locally a quotient of elements of $S$, ie. $s(\mathfrak{q}) = a/f$ in a neighbourhood of $\mathfrak{p}$.

**Motivation** (Görtz & Wedhorn, p. 375)   If $\varphi : A \to B$ is a homomorphism of graded $R$-algebras, the inverse image of a relevant prime ideal (=an ideal not containing the homogeneous ideal of positive degree groups $\bigoplus_{d \geqslant 1} S_d$) of $B$ may not be a relevant prime ideal of $A$. Thus $\operatorname{Proj}(A)$ is not functorial in $A$ with respect to arbitrary homomorphisms of graded $R$-algebras. But there is a unique morphism of $R$-schemes

$$\operatorname{Proj}\varphi : G(\varphi) \longrightarrow \operatorname{Proj} A,$$

where $G(\varphi) \subseteq \text{Proj}(B)$ is the open subscheme

$$G(\varphi) = \mathfrak{q} \in \text{Proj}(B) : \varphi^{-1}(\mathfrak{q}) \not\supseteq A_+\}$$

such that [some conditions, but let's proceed to our exercise]

**Exercise 2.14**

  b. Let $\varphi : S \to T$ be agraded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \text{Proj}(T) : \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that $U$ is an open subset of $\text{Proj}(T)$ and show that $\varphi$ determines a natural morphism $f : U \to \text{Proj}(S)$.

  c. The morphism $f$ can be an isomorphism even when $\varphi$ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where $d_0$ is an integer. Then show that $U = \text{Proj}\,T$ and the induced morphism $f : \text{Proj}\,T \to \text{Proj}\,S$ is an isomorphism.

So basically the following exercise is about studying a case when this induced morphism behaves nicely, namely when $\varphi$ is surjective.

**Exercise 3.12**

  a. Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set $U$ from exercise 2.14 is equal to $\text{Proj}\,T$ and the morphism $f$ is a closed immersion.

  b. If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let $Y$ be the closed subsceme of $\text{Proj}\,S$ defined as the image of the closed immersion $\text{Proj}(S/I) \to \text{Proj}\,S$. Show that different homogeneous ideals can give rise to the same closed subscheme.

**Definition** (3.41 Gortz)    A morphism $i : Z \to X$ of schemes is called a ***closed immersion*** if the underlying continuous map is a homeomorphism between $Z$ and a closed subset of $X$ and the sheaf homomorphism $i^\flat : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ is surjective.

*Solution.*

  a. The fact that $U = \text{Proj}\,T$ is immediate from definition and sujectivity since $\varphi(S_+) = T_+$. The induced map is

$$f : U = \text{Proj}\,T \longrightarrow \text{Proj}\,S$$
$$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$$

Now let's see that $f$ is a closed immersion.

(**f is injective.**)    Follows from surjectivity: if $\mathfrak{p}, \mathfrak{q} \in \text{Proj}\,T$ are such that $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$, then $\varphi(\varphi^{-1}(\mathfrak{p})) = \mathfrak{p} = \mathfrak{q} = \varphi(\varphi^{-1}(\mathfrak{q}))$.

(**f $(\text{Proj}(T))$ is closed.**)    The easiest way is to notice that $S/\ker\varphi \cong T$, and that homogeneous prime ideals of $S/\ker\varphi$ (=elements of $\text{Proj}(T)$) correspond to homogeneous ideals of $S$ which contain $\ker\varphi$, so $f(\text{Proj}(T)) = V(\ker\varphi)$. (But I should confirm this works...)

A more explicit computation is as follows:

$$f(\operatorname{Proj} T) = V\left(\bigcap_{\mathfrak{p} \in \operatorname{Proj}(T)} \varphi^{-1}(\mathfrak{p})\right)$$

$$= \left\{\mathfrak{q} \in \operatorname{Proj} S : \mathfrak{q} \supseteq \bigcap_{\mathfrak{p} \in \operatorname{Proj} T} \varphi^{-1}(\mathfrak{p})\right\}$$

The contention $\subseteq$ is clear since $\varphi^{-1}(\mathfrak{q}) \in f(\operatorname{Proj} T) \implies \varphi^{-1}(\mathfrak{q}) \supseteq \bigcap_{\mathfrak{q} \in \operatorname{Proj} T} \varphi^{-1}(\mathfrak{p})$.

For $\supseteq$ we want to see that $\mathfrak{q} = \varphi^{-1}(\varphi(\mathfrak{q}))$. Suppose that there exists $x \in \varphi^{-1}(\varphi(\mathfrak{q})) \setminus \mathfrak{q}$. Then there exists $y \in \mathfrak{q}$ such that $\varphi(x) = \varphi(y) \implies \varphi(x - y) = 0$. But $0 = (x - y) \subset \bigcap_{\mathfrak{p} \in \operatorname{Proj} T} \varphi(\mathfrak{p})$, meaning $x - y \in \mathfrak{q}$, but that is not possible since $x \notin \mathfrak{q}$.

**(f is a homeomorphism.)**   It preserves inclusions of ideals?

**(Induced map on stalks is surjective.)**   What is going on?

**Remark** (Gortz, p. 57)   A morphism $(f, f^\flat) : X \to Y$ of ringed spaces induces morphisms on the stalks as follows. Let $x \in X$. Let $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ be the morphism corresponding to $f^\flat$ by adjointness. Using the identification $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)}$ established in (2.8.2) we get

$$f^\sharp_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}.$$

So I'm not really sure how this is defined but we are looking at a map

$$f^\sharp_x : \mathcal{O}_{f(\operatorname{Proj} T)} \longrightarrow \mathcal{O}_{\operatorname{Proj} S}$$

Ah! Now I see:

**Proposition 2.5**

a.  For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathcal{O}_\mathfrak{p}$ is isomorphic to the local sing $S_{(\mathfrak{p})}$.

So in fact we have some map $f^\sharp_\mathfrak{p} : S_{(\varphi^{-1}(\mathfrak{p}))} \longrightarrow T_{(\mathfrak{p})}$. How is this is defined? but in a broad sense we know this is a map induced by $\varphi$, which was surjective, acting on the localizations of $S$ and $T$, which are really $S$ and $T$ made larger by adding some fractions.

b.  Let $d_0$ be an integer and define $I' = \bigoplus_{d \geqslant d_0} I_d$. Basically this is just exercise 2.14(c). We just notice that $I'$ is really just $I$ taking away the first $d_0$ degrees, so we have a projection map

$$\varphi : S/I' \to S/I$$

which is just the identity after degree $d_0$. That is the hypothesis of exercise 2.14(c), which yield an isomorphism $\operatorname{Proj}(S/I) \cong \operatorname{Proj}(S/I')$. And that clearly means that the induced subschemes of $\operatorname{Proj} S$ are isomorphic.

$\square$

**The open sets: another proof for closed immersion** (Lucas)   So, what are the open sets? Let $S$ be a graded ring. Define

$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(S) : f \notin \mathfrak{p}\}$$

And very importantly, we can take problems about Proj to the world of schemes noticing that

$$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \mathrm{Spec}\, R_f.$$

We can even get another proof that the induced map $f$ on the exercise is a closed immersion as follows. Notice that $\varphi : S_{(f)} \twoheadrightarrow T_{(\varphi(f))}$ gives

$$
\begin{array}{ccc}
\mathrm{Proj}(T) & \longrightarrow & \mathrm{Proj}(S) \\
\uparrow & & \uparrow \\
D_+(\varphi(f)) & \longrightarrow & D_+(f)
\end{array}
$$

which is just the same as a surjection

$$S_{(f)} \twoheadrightarrow R_{(\varphi(f))}$$

### 2.4.2   Skyscraper sheaves (Alex)

**Exercise 1.17**   Let $X$ be a topological space, let $P$ be a point, and let $A$ be an abelian group. Define a sheaf $i_P(A)$ on $X$ as follows:

$$i_P(A)(U) = \begin{cases} A & \text{if } P \in U \\ 0 & \text{otherwise} \end{cases}$$

Verify that the stalk of $i_P(A)$ is $A$ at every point $Q \in \overline{\{P\}}$ (closure of $\{P\}$) and 0 otherwise. Hence the name *skyscraper sheaf.* Show that this sheaf could also be described as $i_*(A)$ where $A$ denotes the constant sheaf $A$ on the closed subspace $\overline{\{P\}}$ and $i : \overline{\{P\}} \hookrightarrow X$ is the inclusion.

*Solution.* [Done in class]   $\square$

### 2.4.3   Cremona involution (Bruno, Sergey)

It is an automorphism of rational functions in say two variables. $\sigma \in \mathrm{Aut}(\mathbb{C}(x,y))$.

$$\sigma(x) = 1/x \qquad \sigma(y) = 1/y$$

It has an extension to a map

$$*\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$
$$[x : y : z] \longmapsto [yz : xz : xy]$$

**Claim**  Let $S = \mathrm{Bl}_3\,\mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $[1:0:0], [0:1:0]$ and $[0:0:1]$. Then there exists a regular extension $\sigma : S \longrightarrow S$.

**Explanation** (Sergey)  When you have a $\varphi : X \dashrightarrow Y$ you can consider the closure of its graph, $\overline{\Gamma_\varphi} \subset X \times Y$.

For our map $\varphi : \mathbb{P}^2_s \dashrightarrow \mathbb{P}^2_t$, What is $\overline{\Gamma_\varphi}$?

**Exercise**  Prove Danilov's lemma that if $Y \subset E$ is codimension 1 and $X \subset Z$ is codimension 2 then $f : Y \longrightarrow X$ is a blow-up.

**Exercise**  How many orbits are there in the $\mathrm{PGL}(3)^2 \curvearrowright \mathrm{Gr}(9,2)$ action? Or is it the action of $\mathrm{PGL}(U) \times \mathrm{PGL}(V)$ on $\mathbb{P}(U \otimes V)$, where $\mathbb{P}(U) \times \mathbb{P}(V) \hookrightarrow \mathbb{P}(U \otimes V)$. Can you classify the orbits? This classification will be a problem of linear algebra.

## 2.5  October 4

### 2.5.1  Schemes and locally ringed spaces (Victor)

**Exercise 2.1**  Let $A$ be a ring, let $X = \mathrm{Spec}(A)$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\mathrm{Spec}(A_f)$.

**Review of Spec**  **As a set.**  It's just the prime ideals of a ring $A$.

**Topology.**  The closed sets are, for an ideal $I \subseteq A$, $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{a} \subseteq \mathfrak{p}\}$. A basis for the open sets is given by $D(f)$, the complement of $V((f))$ for $f \in A$.

**Sheaf.**  (Note the similarity with the definition of the regular functions on a variety. The difference is that we consider functions into the various local rings, instead of to a field.)  For each prime ideal $\mathfrak{p} \subseteq A$ let $A_\mathfrak{p}$ be the localization of $A$ at $\mathfrak{p}$. For an open set $U \subseteq \mathrm{Spec}(A)$ degine $\mathcal{O}(U)$ to be the set of functions $s : U \longrightarrow \coprod_{\mathfrak{p} \in U} A_\mathfrak{p}$ such that $s(\mathfrak{p}) \in A_\mathfrak{p}$ for each $\mathfrak{p} \in U$, and such that $s$ is locally a quotient of elements of $A$ : to be precise (…)

**Review of ringed spaces**  A *ringed space* is a pair $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of rings.

This is a category, so we have *ringed spaces maps*, which are maps $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ where $f : X \longrightarrow Y$ is a continuos map and $f^\sharp : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ is a sheaf morphism (a ring homomorphism for every open set) that is also natural in the sense that

$$f^*(U) : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$f^\sharp(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$$

commutes.

A *locally ringed space* is a ringed space where $\mathcal{O}_P$ is a local ring for all $P \in \mathcal{O}_X$, and X is locally a Spec.

**Remark**   $D(f)$ is open since

$$V((f)) = \{\mathfrak{p} \in \operatorname{Spec}(A) : (f) \subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec}(A) : t \in \mathfrak{p}\}$$

so $D(f) = \operatorname{Spec}(A) \setminus V(f)$.

*Solution.*  Recall that $D(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) : f \notin \mathfrak{p}\}$. Consider the inclusion

$$\varphi : A \longrightarrow A_f$$
$$a \longmapsto \frac{a}{1}$$

and its induced

$$\varphi^* : \operatorname{Spec}(A_f) \longrightarrow \operatorname{Spec}(A)$$
$$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p}) = \mathfrak{p} \cap A \in D(f)$$

But such a function is in fact a bijection with inverse given by $D(f) \ni \mathfrak{p} \mapsto S^{-1}\mathfrak{p}$. In fact it is also an homeomorphism since it is an open map, since

$$\varphi^*(D(g)) = D(g) \cap D(f).$$

$\square$

## 2.5.2   Locally free sheaves (Arthur)

**Exercise 8.2**   Let X be a variety of dimension $n$ over $k$. Let $\mathcal{E}$ be a locally free sheaf of rank $r > n$ on X and let $V \subseteq \Gamma(X, \mathcal{E})$ be a vector space of global sections which generate $\mathcal{E}$. Show that there is a section $s \in V$ such that $s_p \notin \mathfrak{m}_p \mathcal{E}_p$ for all $p \in X$.

**Definition**   Let E be an $\mathcal{O}_x$-module. A vector space $V \subseteq \Gamma(X, \mathcal{E})$ generates $\mathcal{E}$ at $p \in X$ if $\{s_p : s \in V\}$ generates de the stalk $\mathcal{E}_p$ as an $\mathcal{O}_{X,p}$-module. We say V is a *generator* if it generates at all points of X.

**Definition**   A sheaf $\mathcal{E}$ on X is *locally free* when there is a covering $\{U\}$ of X such that

$$\mathcal{E}|_U \cong \bigoplus_{=1}^{n} \mathcal{O}_X|_U$$

for some $n$ and all $U$. (Recall that the restriction sheaf is taking open sets in $U$.)

## 2.6   October 8

### 2.6.1   The Grothendieck Group (Dani)

**Intuition**   (See wiki or Hatcher VBKT.)   The idea behind Grothendieck group is very much like introducing fractions to $\mathbb{Z}$ to construct $\mathbb{Q}$. Starting from a monoid (which is a structure similar to a group but lacking inverses) we consider the free abelian group generated by the monoid and quotient by an appropiate equivalence relation.

The construction in this exercise of Hartshorne introduces a group structure to the set of coherent sheaves (which are $\mathcal{O}_X$-modules) by making the direct sum of modules into a group operation. Recall that for every two modules $F_1$ and $F_3$ there is a (split) short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_1 \oplus F_3 \longrightarrow F_3 \longrightarrow 0$$

So we will take the free module (generated by the coherent sheaves), which is equipped with an operation $+$, and take take the quotient identifiying for every short exact sequence $0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$ by the relation $F_2 = F_1 + F_3$. So for the case of the trivial split sequence we see that $F_1 \oplus F_3 = F_1 + F_3$, but now we have inverses!

**Exercise 6.10** (The Grothendieck Group $K(X)$)   Let $X$ be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on $X$, by the subgroup generated by all expressions $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$ whenever there is an exact sequence $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ of coherent sheaves on $X$. If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.

   a.  If $X = \mathbb{A}_k^1$, then $K(X) \cong \mathbb{Z}$.

   b.  If $X$ is any integral scheme, and $\mathscr{F}$ a coherent sheaf, we define the **rank** of $\mathscr{F}$ to be $\dim_k \mathscr{F}_\xi$, where $\xi$ is the generic point of $X$, and $K = \mathcal{O}_\xi$ if the function field of $X$. Show that the rank function defined a surjective homomorphism rank: $K(X) \longrightarrow \mathbb{Z}$.

   c.  If $Y$ is a closed subscheme of $X$, there is an exact sequence

   $$K(Y) \longrightarrow K(X) \longrightarrow K(X - Y) \longrightarrow 0$$

   where the first map is extension by zero, and the second map is restriction. [*Hint*: For exactness in the middle, show that if $\mathscr{F}$ is a coherent sheaf on $X$ whose support is contained in $Y$, then there is a finite filtration $\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \ldots \supseteq \mathscr{F}_n = 0$, such that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is an $\mathcal{O}_Y$-module. To show surjectivity on the right, use Exercise 5.15.

   For further information about $K(X)$, and its applications to the generalized Riemann-Roch theorem, see ...

**Reminder on coherent sheaves**   A *coherent sheaf* on a ringed space $X$ is an $\mathcal{O}_X$-module that is locally finitely generated and locally finitely presented. These two conditions are expressible with exact sequences.

Here's the definition of coherent sheaves from Complex Geometry 2024-1:

**Definition**   A sheaf of $A$-modules $\mathscr{F}$ over a sheaf of rings $A$ (on a topological space $X$) is called

- ***locally finitely generated*** or ***of finite type*** if there is a covering of $X$ by $U_i$, such that modules $F(U_i)$ are finitely generated over rings $A(U_i)$, i.e. over open subset $U_i$ there is a finite number $n_i$ and a surjective homomorphisms (over $U_i$) of $A$-modules:

$$A \otimes \mathbb{C}^{n_i} \twoheadrightarrow F \to 0,$$

  in principle for different $U_i$ they can be unrelated to each other. For brevity I denote $A \otimes C^{n_i}$ by $A^n$.

  **Interpretation:**   If $F$ was a free group, we'd ask for an isomorphism $0 \to A^n \to F \to 0$, but since we only want the module to be *finitely generated*, it could be a quotient of a free group.

- ***locally finitely presented*** if there is a covering as above such that the respective modules are finitely presented over respective rings, that is locally there is an exact sequence

$$A^m \to A^n \twoheadrightarrow F \to 0$$

  where $m$ and $n$ may depend on $i$, in other words locally $F$ is isomorphic to a cokernel of a homomorphism between free finitely generated $A$-modules.

  **More explicitly:**   The cokernel of the first arrow, say $\varphi$, is $A^n / \mathrm{img}\,\varphi$. But $\mathrm{img}\,\varphi = \ker\psi$ of the second arrow $\psi$. And $F \cong A^n / \ker\psi$. But how is this related to the usual notion of presentation? Well, we simply have that $F \cong A^n / \mathrm{img}\,\varphi$ where $\varphi : A^m \to A^n$.

  **Remark** (Every $A$-module $E$ admits a presentation, Bourbaki, Commutative Algebra p.20.)   Let $A$ be a ring. An exact sequence $L_1 \to L_0 \to E \to 0$ of left $A$-modules, where $L_0$ and $L_1$ are free, is called a ***presentation*** of a left $A$-module $E$. Every $A$-module $E$ admits a presentation. We know in fact ( *Algebra*, Chapter II, prop. 20) that there exists a surjective $u : L_0 \to E$, where $L_0$ is free; if $R$ is the kernel of $u$, there exists similarly a surjective homomorphism $v : L_1 \to R$, where $L_1$ is free. If $v$ is considered as a homomorphism of $L_1$ to $L_0$, the sequence $L_1 \xrightarrow{v} L_0 \xrightarrow{u} 0$ is exact by deinition, whence our assertion.

- ***coherent*** if it is locally finitely generated, and also for any finite $n$ and a local homomorphism $h : A^n \to F$ its kernel $\ker h$ is locally finitely generated sheaf of $A$-modules. Note that coherence obviously implies locally finitely presentedness, consider $h_i : A^{n_i} \to F$ to be the surjective local (over $U_i$) homomorphisms that guarantee local finited generatedness of $F$, coherence implies their kernels are finitely generated by $m_i$ elements i.e. $A^{m_i} \twoheadrightarrow \ker h_i$, so take the composition $A^{m_i} \twoheadrightarrow \ker h_i \to A^{n_i}$ and notice it gives a local presentation of $F$, i.e. an exact sequence $A^{m_i} \to A^{n_i} \twoheadrightarrow F \to 0$.

- *locally noetherian* if for any $U$ and any increasing sequence of sheaves of submodules $F_k$ of $F$, there is a covering of $U$ by $U_i$ such that on each $U_i$ the sequence $F_k$ stabilizes.

*Solution.*

a. Idea: We want to see that the group of coherent sheaves on the field is isomorphic to the integers. Maybe if $\mathcal{F}$ was locally free it would be determined by how many copies of $k$ it takes. But $\mathcal{F}$ is not necessarily a free $\mathcal{O}_X$-module, so we have to use the definitions of finitely generated and finitely presented.

I'll provide two different proofs.

*Proof 1, StEx.* Use invariant factors theorem:

**Invariant Factors Theorem**    Let $R$ be a principal ideal domain and let $M$ be a finitely generated $R$-module. Then

$$M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \ldots \oplus R/(a_m)$$

for some integer $r \geqslant 0$ and nonzero elements $a_1, a_2, \ldots, a_m$ of $R$ which are not units in $R$ and which satisfy the divisibility relation $a_1 | a_2 | \ldots | a_m$.

Another way to write this is just $M \cong \bigoplus_i R/(a_i)$ allowing some $a_i$ to be zero accounting for the free part. The elements $a_i$ are called *invariant factors*. "Any two modules sharing the same set of invariants are isomorphic" ( wiki). So a module is determined by the choice of invariant factors... but this is not exactly an integer... so how to construct the map to $\mathbb{Z}$ (or $\mathbb{N}$)?

In StackExchange we see the alternative:

**Via Quillen-Sulsin.** Show that a coherent sheaf is locally a finitely generated *projective* module. Then use Quillen-Sulsin theorem that *every finitely generated projective module is free*. Map a module to its rank. This gives an homomorphism to $(\mathbb{N}, +)$. Its "group closure" (=Grothendieck group) is $(\mathbb{Z}, +)$. But how do we get projective module?                                                                          □

*Proof 2, Solution manual.* Notice $\mathcal{O}_X \cong k[t]$. Locally finitely presentedness means locally there is an exact sequence $k[t]^{\oplus n} \to k[t]^{\oplus m} \to M \to 0$. But since k[t] is a PID, the first morphism is injective.

Then we have an exact sequence $0 \longrightarrow \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m} \to \mathcal{F} \to 0$. Which means that $\gamma(\mathcal{F}) \cong (m - n)\gamma(\mathcal{O}_X)$.

Then the map $\mathbb{Z} \longrightarrow K(X)$, $n \mapsto n\gamma(\mathcal{O}_X)$ is surjective.

Finally, we need to use part (b).                                                                          □

b. Here we need a few definitions:

**Definition**

(a) A scheme $X$ is *integral* if for every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

(b) ([Wiki](#) or exericse 2.9) A *generic point* of a topological space $X$ is a point whose closure is all of $X$. If $X$ is a scheme, show that every non-empty irreducible closed subset has a unique generic point.

(c) (Exercise 3.6) Let $X$ be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point of $X$ is a field called the *function field* of $X$ denoted $K(X)$.

Now we check that the rank map $\mathcal{F} \mapsto \dim_K \mathcal{F}_\xi$ where $K = \mathcal{O}_\xi$ is well defined as a map $K(X) \longrightarrow \mathbb{Z}$.

This just means that the rank of equivalent coherent sheaves is the same. So suppose that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact so that $\mathcal{F}$ is equivalent to $\mathcal{F}' + \mathcal{F}''$. Since talking stalks preserves exactness, the sequence $0 \to \mathcal{F}'_\xi \to \mathcal{F}_\xi \to \mathcal{F}''_\xi \to 0$ is also exact. These stalks are $\mathcal{O}_\xi$-modules, i.e. vector spaces over $K$. Then the sequence splits, we get that $\mathcal{F}_\xi = \mathcal{F}'_\xi \oplus \mathcal{F}''_\xi$ and finally that $\dim_K \mathcal{F}_\xi = \dim_K \mathcal{F}'_\xi + \dim_K \mathcal{F}''_\xi$.

For surjectivity simply note that $[\mathcal{O}_X]$ has rank 1, that is, $\dim_{\mathcal{O}_{X,\xi}} \mathcal{O}_{X,\xi} = 1$.

c. Surjectivity on the right is immediate from exercise 5.15 since every sheaf can be extended.

For exactness in the middle I can't provide many details since I'm out of time. A sketch is as follows:

**Step 1** Let $\mathcal{F}$ be a sheaf with support in $Y$. We want to see that it is in the image of sheaves extended by zero from $Y$ to $X$. (Why isn't $\mathcal{F}$ not already an extension by zero of a sheaf in $Y$? It is zero outside $Y$. I think the problem is related to its equivalence class...) Construct a filtration $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \ldots \supseteq \mathcal{F}_n = 0$ such that each $\mathcal{F}_i/\mathcal{F}_{i+1}$ is an extension by zero. Looking at their representatives in Grothendieck Group, we see that $[\mathcal{F}] = \sum_{i=0}^{n-1} [\mathcal{F}_i/\mathcal{F}_{i+1}]$ which makes $\mathcal{F}$ be an extension by zero. We need to construct such a filtration.

**Step 2** Let $i : Y \hookrightarrow X$. Pullback and pushforward functors are adjoint (meaning that $\mathrm{Hom}(i^*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$, which gives us a map $\eta : \mathcal{F} \longrightarrow i_* i^* \mathcal{F}$. Next step is confusing, but the idea is that $\eta$ is surjective.

**Step 3** See Prop. 5.4: an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent iff for every affine open subset $U = \mathrm{Spec}(A)$ of $X$ there is an $A$-module $M$ such that $\mathcal{F}|_U = \tilde{M}$. And Coro. 5.10: there is a 1-1 correspondence between ideals in $A$ and closed subschemes of $\mathrm{Spec}(A)$ given by $\mathfrak{a} \mapsto \mathrm{Spec}(A)/\mathfrak{a}$.

So there is an open set $U \subset X$ that $U = \mathrm{Spec}(A)$ and also $\mathcal{F}|_U = \tilde{M}$. (Perhaps this means that $F|_U = \tilde{M}(U) = M$.) Then restrict the above map $\eta$ to $U$: we get a map $\eta|_U : \mathrm{Spec}(A) \longrightarrow Y \cap \mathrm{Spec}(A) \cong \mathrm{Spec}(A)/I$ for some ideal $I$. That means that $\eta$ is given (locally?) by $M \to M/MI$, which makes it surjective.

**Step 4** Define $\mathcal{F} = \mathcal{F}_0$ and $\mathcal{F}_j$ as the kernel of $\mathcal{F}_{j-1} \longrightarrow i_* i^* \mathcal{F}_j$. By definition, I guess of $i_* i^*$ we see that each quotient is an extension. For example, $\mathcal{F}_1$ is the kernel

of $\mathcal{F}_0 \longrightarrow i_* i^* \mathcal{F}_0$, so $\mathcal{F}_0/\mathcal{F}_1 = i_* i^* \mathcal{F}_0$.

**Step 5** Finiteness of the sequence uses noetherianess of $A$ and at least another exercise.

$\square$

**Comment from Sergey** So, yesterday (Oct 8) we mostly discussed 3 points on the exercise on Grothendieck group with Daniel leading, and others helping him.

Some topics discussed or to consider (for simlper understanding):

- definition of a Grothendieck group K(A) of an abelian category A

- definition of a Grothendieck group K(T) of a triangulated category T

- there is a natural isomorphism from K(A) to K(D(A)), here D is bounded derived category of T

- This simplifies study of K(A) — class of an object [M] can be easier to compute if M is replaced by some quasi-isomorphic complex N.

E.g. if a sheaf has a resolution by vector bundles (on smooth projective varieties such resolutions can be constructed with help of Theorems A,B of Serre), then its class is a linear combination in a subgroup of K, generated by vector bundles. Then given some invariant I (e.g. rank, degree) of a vector bundle, it can be extended to K as follows: — first check that invariant is additive for vector bundles, i.e. I(B) = I(A) + I(C) for any short exact sequence of vector bundles $0 \to A \to B \to C \to 0$ E.g. for rank you can do it by localizing to sufficiently small open where A,B,C trivialize, then localizing further to field of fractions, so the statement becomes additivity of dimension of finitely dimensional vector spaces. For degree (first Chern class) that would be Whittney-Grothendieck formula, for Chern character it is also true (can be part of definition, depending on formalism).

Then to define I for any coherent sheaf do the following:

- replace M by quasi-equivalent complex of vector bundles N

- define I(M) as sum $(-1)^i I(N_i)$ –

- prove it does not depend on a choice — i.e. for any pair of quasi-equivalent complexes of v.b. these numbers agree. (this is just the general K(A) = K(D(A)) above)

In Daniel's exercise (a) he has Coh Spec A where A is a PID (A=k[x]), and we saw that in this case any f.g. module is a quotient of a free module $A^n$ by free submodule $A^m$ (in fact any submodule of f.g. torsion-free is torsion-free and over PID is free), so its class is (n-m)[A], and moreover there is a surjective homomorphism from K(A) to Z given by rank, so this homomorphism for PIDs is an isomorphism. So in this situation classes of all torsion sheaves are zero (unlike the case of a projective curve).

Decomposition of a coherent sheaf on a curve into direct sum of torsion and torsion-free holds for all smooth curves, see e.g. answer there (in mo).

**Exercise** (Sergey)  Consider the map $F : X \subset \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ $x, y \mapsto x^d, y^d$. This map puts a curve $X$ such as $x^2 + y^2 = 1$ (circle) or $x + y = 1$ (line) in $\mathbb{P}^2$. What is the degree of the image? Is it singular?

**Claim** (Sergey)  If f algebraic then f(X) algebrraic.

How to find equations?

## 2.7  November 12: finding equations for f(X) **(Dani)**

My search shows that here might be the answer.

**Exercise** (Sergey)  Consider the map $F : X \subset \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ $x, y \mapsto x^d, y^d$. This map puts a curve $X$ such as $x^2 + y^2 = 1$ (circle) or $x + y = 1$ (line) in $\mathbb{P}^2$. What is the degree of the image? Is it singular?

*Solution.*  Consider the case when $d = 3$. Define

$$f : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$
$$(x : y : z) \longmapsto (x^3 : y^3 : z^3)$$

Let $X = V(x^2 + y^2 = z^2)$. We claim that $f(X) = V\Big((x^2 + y^2 - z^2)^3 - 27(xyz)^2\Big)$. One contention is immediate: if $(x : y : z) \in X$ then $f(x : y : z) = (x^2 : y^2 : z^2)$ satisfies　　☐

OK since that is very hard I will proceed to the abstract proof for schemes.

*Solution by Sergey.*  Consider roots of unity $\rho^d$. Let $z = x^d$ and $w = y^d$. Then we have that $z^{2/d} + w^{2/d} - 1 = 0$. The solution reads

$$\prod_{a=0}^{d-1} \prod_{b=0}^{d-1} \Big(\rho^a z^{2/d} + \rho^b w^{2/d} - 1\Big) = 0$$

Compute this in a computer. Looks like for $d = 3$ we get

$$w^6 + 3w^4(z^2 - 1) + 3w^2(z^4 + 7z^2 + 1) + (z^2 - 1)^3.$$

This was computed with the following line on WolframAlpha:

```
Product[Product[e^(a * 2 * pi * i / 3) * z^(2/3) + e^(b
* 2 * pi * i / 3) * w^(2/3) - 1, b, 0, 2], a, 0, 2]
```

☐

**Claim** (Sergey)  If f algebraic then f(X) algebrraic.

How to find equations?

Recall that in a past exercise by Dani (later also done by Victor) we studied the case of how a homomorphism of rings $\varphi : A \to B$ may or may not induce a map $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$—we saw that the natural definition taking a prime homogeneous ideal $\mathfrak{p} \subset B$ to $\varphi^{-1}(\mathfrak{p})$ doesn't always work because such a preimage may not be a relevant ideal.

We proved that there is a certain open set where the induced map may be defined, which coincides with $\mathrm{Proj}(B)$ if the ring homomorphism is surjective. Also in the case when $\varphi$ is surjective, we showed that the induced map in Proj is a closed immersion. The following proposition states the analogue result for Spec:

**Proposition II.2.3**

  (b) If $\varphi : A \to B$ is a homomorphism of rings, then $\varphi$ induces a natural morphism of locally ringed spaces

$$(f, f^{\#}) : \Big( \mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)} \Big) \longrightarrow \Big( \mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)} \Big).$$

  (c) If $A$ and $B$ are rings, then any morphism of locally ringed spaces from $\mathrm{Spec}(B)$ to $\mathrm{Spec}(A)$ is induced by a homomorphism of rings $\varphi : A \to B$ as in (b).

  (d) Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^{\#} : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective, then $\varphi$ is surjective.

Right, so we consider a map of schemes locally, that is, a map $f : Y := \mathrm{Spec}(B) \to X := \mathrm{Spec}(A)$, so that we have a map

$$\phi : A \to B.$$

This gives a well-defined map

$$A \longrightarrow A/\ker \phi \longrightarrow B$$

which in turn induces a morphism from $Y = \mathrm{Spec}(B)$ to a subset of $X = \mathrm{Spec}(A)$, $\mathrm{Spec}(A/I)$. In fact, $\mathrm{Spec}(A/I) = V(I)$ since the prime ideals of $A/I$ are the ideals of $A$ that contain $I$.

Now if we suppose $f$ is algebraic we get that $f(X)$ is closed and onto, so that by item (d) we get that and that the map of rings is surjective. This shows that it is enough to compute $I$.

**Remarks**

  • Looks like the construction may be taken to Proj, I only didn't find the statement that a map of Proj induces a map of rings. Indeed,

  • how is the map of rings defined? And,

  • how to take this to the practical difficulty of finding the equations for the above projective map? In StackExchange we are suggested to compute the ideal *locally*. All we have to do is find the generators! So take the affine chart $\{z = 1\}$, what is $B$ if $\mathrm{Spec}(B) = X = V(x^2 + y^2 - 1)$? Is it now just $k[x, y]/(x^2 + y^2 - 1)$

  • Another question is how to compute the degree of the image. This remains rather mysterious.

## 2.8 October 11

### 2.8.1 $k$-structure, $S$-points, adjoint functors (Sergey)

**Definition** A $k$-*structure,* on … is a pair $(V, L)$ with

1. $V'L$

2. An isomorphism $V \cong V' \times_? \mathrm{Spec}(?)$

$$
\begin{array}{ccc}
V & \longrightarrow & V' \\
\downarrow & & \downarrow \\
\mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(k)
\end{array}
$$

**Definition** $S$-*points* of $V$ are

**Exercise** Look at any points on the curve $(t, t^2, t^3, \ldots, t^d)$. Then their convex hull is a convex polytope called the **cyclic polytope**.

**Exercise** For any complex algebraic variety construct a real algebraic variety such that the real points of the real algebraic variety are the complex points of the complex algebraic variety.

For $X/\mathbb{C}$ construct $Y/\mathbb{R}$ such that $X(\mathbb{C}) = Y(\mathbb{R})$.

### 2.8.2 Reduced schemes (Bruno)

**Definition** Ring is **reduced** if there's no nilpotents. Scheme is reduced if $\mathcal{O}_X(U)$ is reduced for any open $U \subset X$.

**Exercise 2.3**

a. $X$ reduced iff $\mathcal{O}_{X,P}$ reduced for all $P \in X$.

b. If $X$ is a locally ringed space then also $X_{\mathrm{red}}$ is.

*Solution.*

1. Pretty straighforward.

2. First check that the reduction of the structure sheaf is a sheaf so that $X_{\mathrm{red}}$ is a ringed space. Now to make it a scheme check that (it is locally a spec) if $X = \mathrm{Spec}(A)$ then $X_{\mathrm{red}} \cong \mathrm{Spec}(A_{\mathrm{red}})$. So a homeomorphism from topological spaces and isomorphism of sheaves. For homeomorphism use that $\bigcap_{\mathfrak{a} \in \mathrm{Spec}(A)} \mathfrak{a} = \mathrm{nil}_A$ (fact from homological algebra). Then check isomorphism

3.

$\square$

See StackExchange for an answer on that question by Sergey.

## 2.9   October 18

### 2.9.1   (Victor)

**II 3.12**

   a. Let $\varphi : S \to T$ be a surjection homomorphism of graded rings (preserving degrees). Show that the open

## 2.10   November 8: extension of coherent sheaves (Victor)

**Exercise II 5.15**   Let $X$ be a Noetherian scheme (has an open affine cover $U_i$ such that $\mathcal{O}_X(U_i)$ is Noetherian). Let $U$ be an open subset and $\mathcal{F}$ a coherent sheaf on $U$. Then there is a coherent sheaf $\mathcal{F}'$ on $X$ such that $\mathcal{F}'|_U \cong \mathcal{F}$.

## 2.11   December 3

### 2.11.1   Blowing up a nonsingular subvariety (Victor)

**Question** (Dani)   Why should we be interested in blowing up a nonsingular subvariety?

**Exercise**   Let $X$ be a nonsingular variety, let $Y$ be a singular subvariety of codimension $r \geqslant 2$, let $\pi : \tilde{X} \to X$ be the blowing-up of $X$ along $Y$, and let $Y' = \pi^{-1}(Y)$.

**Question** (dani)   What is the definition of blowing-up along a subvariety?

In Chapter II we find the definition of blowing up with respect to an arbitrary sheaf of ideals. To get there, first we must define de Proj of a sheaf of graded algebras $\mathcal{S}$ over a scheme $X$ (so far we have only defined Proj for a graded ring). And then we must define a projective space bundle associated to a locally free sheaf $\mathcal{E}$.

### 2.11.2   Grassmanians

Take a linear section of a Grassmanian

$$\underbrace{\mathbb{P}^{2-c} \cap \mathrm{Gr}(k, N)}_{X_C} \hookrightarrow \mathbb{P}^{\binom{N}{k}-1}$$

for some $C \geqslant 0$. Compute its canonical bundle $\omega_{X_C}$

Easier question: for which $k, N$ is $\mathrm{Gr}(k, \mathbb{R}^N)$ orientable? So we want to know what is its canonical bundle $\omega_{\mathrm{Gr}(k,N)}$.

Let's review ***Plücker embedding***. It is

$$Gr(k, V) \longrightarrow \mathbb{P}(\Lambda^k V)$$
$$U := \langle u_1, \ldots, u_k \rangle \subset V \longmapsto [u_1 \wedge \ldots \wedge u_k]$$
$$U \longmapsto \Lambda^k(U) = [\det U]$$

**Exercise** (Raffaele's paper)   Now take vector space $\dim W = n + 2$, classify all tuples $A^\alpha \in \Lambda^2(W)$, $\alpha = 1, \ldots, n$, such that (Einsetin convention) $\phi_{\beta\alpha} A^\beta \wedge A^\alpha = 0$ for some nondegenerate symmetric $\phi$s, meaning $\phi_{\beta\alpha} = \phi_{\alpha\beta}$ and $\det \phi_{\alpha\beta} = 0$.

We have taken the conditions on the $\phi$s to the following form. Consider

$$\mathbb{C} \xrightarrow{\phi} S^2 U \xrightarrow{S^2 A} S^2 \Lambda^2 W$$
$$\downarrow \pi$$
$$\Lambda^4 W$$

where $\pi$ is a map between two representations, one irreducible and the other one not. Then the conditions on the $\phi$s become

$$\pi \circ S^2 A \circ \phi = 0.$$

I'm not sure wether this solved the problem.

We have concluded that

$$\omega_{Gr(k,N)} = \mathcal{O}(-N), \qquad \text{Pic}(Gr(k, N)) = \mathbb{Z}\mathcal{O}(1).$$

Which come from Plücker embedding. So, telling if $Gr(k, \mathbb{R}^N)$ is orientable reduces to checking that Plücker is a nontrivial bundle in Grassmanian.

## 2.12   December 10: Kähler Differentials and the Canonical Sheaf

Here's ChatGPT's today discussion summary:

Let $X = \text{Spec}(A)$ be an affine algebraic variety over a field $k$, and let $A = \mathcal{O}(X)$ denote its coordinate ring (the ring of global sections of its structure sheaf). The sheaf of Kähler differentials $\Omega_X$ can be defined using the concept of Kähler differentials of $A$ over $k$.

### 2.12.1   Definition of the Kähler Differential Module

The module of Kähler differentials of $A$ over $k$ is denoted by $\Omega_{A/k}$. It is defined by a universal property: there exists a map

$$d : A \to \Omega_{A/k}$$

that satisfies the Leibniz rule:

$$d(ab) = a \cdot d(b) + b \cdot d(a) \quad \text{for all } a, b \in A.$$

This map $d$ is $k$-linear and describes how functions in $A$ vary infinitesimally.

### 2.12.2   Formal Partial Derivatives

If $A = k[x_1, \ldots, x_n]$ is a polynomial ring, then $\Omega_{A/k}$ can be explicitly generated by $dx_1, \ldots, dx_n$ with relations given by:

$$d(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i,$$

where $\frac{\partial f}{\partial x_i}$ are the formal partial derivatives of $f$ with respect to $x_i$. These derivatives are computed in the polynomial sense, without requiring analytic smoothness.

### 2.12.3   The Sheaf of Kähler Differentials

When working with a general algebraic variety $X$ (not just affine), $\Omega_X$ is defined by gluing these local constructions. Specifically:

$$\Omega_X = \Omega_{A/k}$$

locally on affine open subsets of $X$.

### 2.12.4   Geometric Meaning

The sheaf $\Omega_X$ encapsulates the infinitesimal variation of functions on $X$. It serves as the algebraic geometric analog of the sheaf of holomorphic differential forms in complex geometry.

### 2.12.5   Summary

The sheaf of Kähler differentials $\Omega_X$ is the sheafification of $\Omega_{A/k}$ over $X$ and provides insight into the local differential structure of the variety $X$.

# 3. Chapter III

## 3.1 Notes: injective resolutions, derived functors and cohomology

**Definition** (wiki) An object $Q$ is *injective* if, given a monomorphism $f : X \to Y$, any $g : X \to Q$ can be extended to $Y$.

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\
Q & &
\end{array}
$$

(wiki) The crucial assumption we need to make about our abelian category (=morphisms and objects can be added and kernels and cokernels exist and have desiarable properties) is that it has *enough injectives*, meaning that for every object $A$ in **A** there exists a monomorphism $A \to I$ where $I$ is an injective object in **A**.

The right derived functors of the covariant left-exact functor $F : A \to B$ are then defined as follows. Start with an object $X$ of **A**. Because there are enough injectives, we can construct a long exact sequence of the form

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

where $I^i$ are injective. (This is known as an *injective resolution* of $X$). Applying the functor $F$ to this sequence, and choppingng off the first term, we obtain the chain complex

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots$$

This is in general not an exact sequence anymore. But we can compute its cohomology at the $i$-th spot (the kernel of the map from $F(I^i)$ modulo the image of the map to $F(I^i)$); we call the result $R^i F(X)$.

Of course various things have to be checked: the result does not depend on the given injective resolution of X, and any morphism $X \to Y$ naturally yields a morphism $R^i F(X) \to R^i F(Y)$ so that we indeed obtain a functor. Note that left exactness means that

$$0 \longrightarrow F(X) \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow 0$$

is exact, so $R^0 F(X) = F(X)$, so we only get something interesting for $i > 0$. (In Hartshorn Thm 1.1A: $F \cong R^0 F$.)

**Explanation**   A *derived functor* of a given functor maps an object to the $i$-th cohomology of the image of an injective resolution of the object.

The above-mentioned (where?) property of turning short exact sequences into long exact sequences is a consequence of the snake lemma. In Hartshorne Thm 1.1A: For each short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and for each $i \geqslant 0$ there is a natural morphism $\delta^i : R^i F(A'') \to R^{i+1} F(A')$ such that we obtain a long exact sequence

$$\cdots \longrightarrow R^i F(A') \longrightarrow R^i F(A) \longrightarrow R^i F(A'') \xrightarrow{\delta^i} R^{i+1} F(A') \longrightarrow R^{i+1} F(A) \longrightarrow \cdots$$

**Proposition** (2.1A, Cateogory of $A$-modules has enough injectives)   If $A$ is a ring, then every $A$-module is isomorphic to a sub-module of injective $A$-module.

*Proof.* Hartshorne cites other resources.                                                              □

**Proposition**   Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives.

**Corollary**   If $X$ is any topological space, then the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on $X$ has enough injectives.

*Proof.* Just because abelian groups are $\mathbb{Z}$-modules so it's just a particular case of the last theorem.                                                                                                □

**Definition**   The *cohomology functors* of a topological space $X$ are the derived functors of the global section functor $\Gamma(X, \cdot)$. (What is this global sections functor for any topological space?) For any sheaf $\mathcal{F}$, the groups $H^i(X, \mathcal{F})$ are the *cohomology groups* of $\mathcal{F}$.

**Remark**   This is equivalent to sheaf cohomology construction via Čech cohomology.

# 4. Chapter IV

## 4.1 October 25: $f_*$ for divisors (Arthur)

**Exercise 2.6** Let $f : X \to Y$ a non-contstant map of curves (projective, smooth, over $k = \bar{k}$. Then from Chpt II $f$ is finite and from Chpt III $f$ is flat. Then by Stacks Project 00NX $f$ is finite locally free.

**Definition** (Finite locally free)  For any $U \subset Y$ affine, $f^{-1}(U)$ is affine and $\mathcal{O}_X(f^{-1}(U)) \cong \bigoplus_{i=1}^n \mathcal{O}_Y(Y)$ where $n = \deg f := [K(X) : f^\# K(Y)]$

Let $\Sigma$ be a locally free sheaf on $Y$ of rank $r$. Then $\Lambda^i E$ is rank 1.

$$f_\# : \mathrm{Div}(X) \longrightarrow \mathrm{Div}(Y)$$

$$f_* D = f_* \left( \sum_{p \in X} n_p p \right) = \sum n_p f(p)$$

    a. $f_*$ only depends on the equivalence class of

$$f_*(\mathrm{div}(\varphi)) = \mathrm{div}(N_{K(X)/K(Y)})$$

$$\det(f_* \mathcal{L}(D)) \cong (\det f_* \mathcal{O}_X) \otimes \mathcal{L}(f_* D) \qquad \forall D \in \mathrm{Div}(X)$$

    $M \in \mathrm{Pic}(Y)$, $f_* M$ locally free on $Y$.

    Then $\Lambda^n(f_* M) = \det f_* M$

*Proof.* Let $D \geqslant 0$ be an effective divisor on $X$.

$$0 \longrightarrow \mathcal{L}(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

$$0 \longrightarrow f_* \mathcal{L}(-D) \longrightarrow \mathcal{L}\mathcal{O}_X \longrightarrow f_* \mathcal{O}_D \longrightarrow 0$$

so that second one is exact because the first ones are quasi-coherent sheaves. There is an isomorphism

$$\det(f_*\mathcal{O}_X) \cong \det(f_*\mathcal{L}(-D)) \otimes \det(f_*\mathcal{O}_D)$$

$$\mathcal{O}_D = \bigoplus_{j=1}^{m} k(P_j)^{\oplus n_j}$$

skyscaper.

$$f_*\mathcal{O}_P = \bigoplus_{j=1}^{m} k(f(P_j))^{\oplus n_j} = \mathcal{O}_{f_*D}$$

Now

$$\det f_*\mathcal{O}_X \cong (\det f_*\mathcal{L}(-D)) \otimes (\det \mathcal{O}_{f_*D}$$

Then

$$\det \mathcal{O}_{f_*D} \cong \mathcal{L}(f_*D)$$

$$\det f_*\mathcal{O}_X \cong \det f_*\mathcal{L}(-D)) \otimes \mathcal{L}(f_*D)$$

$$\det f_*\mathcal{L}(-D) \cong (\det f_*\mathcal{O}_X) \otimes \mathcal{L}(-f_*D)$$

Let $D = \underbrace{D_1}_{\geqslant 0} - \underbrace{D_2}_{\geqslant 0}$ arbitrary.

$$0 \longrightarrow \mathcal{L}(-D_1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0$$

$$\otimes \mathcal{L}(D)$$

$$0 \longrightarrow \mathcal{L}(-D_2) \longrightarrow \mathcal{L}(D) \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0$$

$$k(P) \otimes \mathbb{L}(D) \cong k(P) \otimes \mathcal{L}(D)_P \cong k(P)$$

$$\det f_*\mathcal{L}(D) \cong (\det f_*\mathcal{L}(-D_2)) \otimes \underbrace{\det \mathcal{O}_{f_*D}}_{\mathcal{L}(f_*D_1)}$$

$$\det f_*\mathcal{L}(D) \cong \det f_*\mathcal{O}_X \otimes \mathcal{L}(-f_*D_2) \otimes \mathcal{L}(f_*D_1) = (\det f_*\mathcal{O}_X) \otimes \mathcal{L}(f_*D)$$

$$\square$$

b. Conclude that $f_*D$ depends only on the linear equivalence class of D, so there is an induced homomorphism

$$f_* : \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(Y).$$

Show that

$$f_* \circ f^* : \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(Y)$$

is just bultiplication by $n = \deg f$.

*Proof.*

$$f^*Q = \sum_{p \in p^{-1}(Q)} e_p(f)P$$

$$f_*f^*Q = \sum_{p \in f^{-1}(Q)} e_p(f)\underbrace{f(p)}_{Q} = \left(\sum_{p \in f^{-1}(Q)} e_p(f)\right)Q = nQ$$

recall

**Projection formula** (Lucas)

$$f_*(L \otimes f^*L') = f_*L \otimes L'.$$

and this gives

$$Nm_*(L) = \det(f_*L) \otimes \det(f_*\mathcal{O})^{-1}$$
$$Nm(f^*L) = \det(f_*f^*L) \otimes \det(f_*\mathcal{O})^{-1}$$
$$= L^n \otimes \det(f_*\mathcal{O}) \otimes \det(f_*\mathcal{O})$$
$$= L^n.$$

$\square$

   c. Use duality for a finite flat morphism (III 6.10) and III.7.2 to show that

$$\det f_*\Omega_X \cong (\det f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y^{\otimes n}$$

*Proof.*

$$f_*\Omega_X \cong f_* \operatorname{Hom}_X(\mathcal{O}_X, f^!\Omega_Y)$$
$$\cong \operatorname{Hom}_Y(f_*\mathcal{O}_X, \Omega_Y)$$

**Remark** (Lucas)    finite and flat $\implies$

$$f_* \operatorname{Hom}(\mathcal{F}, f^!\mathcal{G}) \cong \operatorname{Hom}(f_*\mathcal{F}, \mathcal{G})$$

(But looks like that's not so relevant.)

(But we can take it as a blackbox.

$$\varphi : (f_*\mathcal{O}_X)^{-1} \xrightarrow{\cong} \mathcal{O}_Y^{\oplus n}$$
$$\varphi \otimes \operatorname{id} : (f_*\mathcal{O}_X)^\vee \otimes \Omega_Y \xrightarrow{\cong} \mathcal{O}_Y^{\oplus n}$$
$$\det(\varphi \otimes \operatorname{id}) : \det(f_*\mathcal{O}_X)^{-1} \otimes \Omega_Y \xrightarrow{\sim}$$
$$\det(f_*\Omega_X) \cong \Omega_Y^{\otimes n}$$
$$\det f_*\Omega_X \cong \underbrace{\det(\operatorname{Hom}(f_*\mathcal{O}_X, \mathcal{O}_Y))}_{=(\det f_*\mathcal{O}_X)^{-1}} \otimes \Omega_Y^{\otimes n}$$

$\square$

d. Now assuma that f is separable, so we have the ramification divisor R, which is

$$R = \sum_{p \in X} \text{len} \left( \Omega_{X/Y} \right)_p P$$

We define the **branch divisor** to be divisor $f_* R$ on Y. Show that

$$(\det f_* \mathcal{O}_X)^2 \cong \mathcal{L}(-B)$$

**On the board:**

$$K_X \sim f^* K_Y + R$$

*Proof.* Start with

$$f_* K_X \sim \underbrace{f_* f^* K_Y}_{=nK_Y} + \underbrace{f_* R}_{:=B}$$

By a.,

$$\det f_* \Omega_X \cong (\det f_* \mathcal{O}_X) \otimes \mathcal{L}(nK_Y + B)$$

$$\cong (\det f_* \mathcal{O}_X) \otimes (\Omega_Y^{\otimes n} \otimes \mathcal{L}(B)$$

By c.,

$$\det f_* \mathcal{O}_X \cong (\det f_* \mathcal{O}_X)^{-1} \otimes \Omega_Y^{\otimes n}$$

and the terms on the left and right cancel.

$\square$

## 4.2 November 1st: duality for a finite flat morphism (Arthur)

This is III 6.10.

**Definition** $f : X \to Y$ of schemes is **affine** if for every $U \subseteq Y$ affine, $f^{-1}(U)$ is affine i.e. it is a finite union of specs of some commutative rings with unity.

Let $f : X \to Y$ be an affine morphism of Noetherian schemes. Let G be a quasi-coherent sheaf on Y. Then

$$\mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, G)$$

It will be easier to work on the affine sets of Y.

## 4.3 An exercise on divisors by Lucas

**Exercise** Let C be a complete smooth curve over $k = \bar{k}$ of genus g.

(a) If $D \in \text{Div}(C)$ is effective, show that $\text{div} |D| \leqslant d = \deg D$ and equality iff $D = 0$ or $g = 0$.

(b) If C has $g = 2$, show that D is very ample iff $d \geqslant 5$.

(c) If $C \hookrightarrow \mathbb{P}^n$ with degree $D \leqslant n$ nondegenerate, show that $D = n$, $g = 0$ and $C$ differs from the rational normal curve of degree $D$ by an automorphism of $\mathbb{P}^n$.

(d) Chacterize curves of degrees 2, 3 and 4 in $\mathbb{P}^n$.

*Solution.*

(a) by Riemann Roch we have that

$$h^0(D) - h^1(D) = d - g + 1$$

We know that $\dim |D| = h^0(D) - 1$ because $|D| = \mathbb{P}(C, \mathcal{O}_X(D))$ so $\dim |D| = h^1(D) + d - g$. $h^1(D) = h^0(-D+K) \leqslant h^0(K) = g$ using Serre duality. This gives $\dim |D| \leqslant d$.

□

**Exercise 1.2** (I like this one)   Again let $X$ be a curve, and let $P_1, \ldots, P_r$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the $P_i$ and regular elsewhere.

**Exercise 1.7** (no one)   A curve $X$ is called ***hyperelliptic***...

**Exercise 1.8** (Alex)   Very useful to know, I think this is done in that book by Bosch of modules,

**Exercise 1.9** (Victor)   Riemann-Roch for singular curves.

**Exercise 2.3(h)**   28 bitangents. Remind Sergey.

**Exercise 2.5**   Prove the theorem of Hurwitz that a curve $X$ of genus $g \geqslant 2$ over a field of characteristic 0 has at most $84(g-1)$.

**Exercise 3.1**   If $X$ is a curve of genus 2, show that a divisor $D$ is very ample $\iff$ $\deg D \geqslant 5$. This strengthens (3,3.4).

**Exercise 3.12**   For each value of $d = 2, 3, 4, 5$ and $r$ satisfying $0 \leqslant r \leqslant \frac{1}{2}(d-1)(d-2)$, show that there exists an irreducible plane curve of degree $d$ with $r$ nodes and no other singularities.

**Exercise 4.10**   If $X$ is an elliptic curve (Sergey: for abelian varieties is also true), show that there is an exact sequence... Picard groups.

**Exercise 5.3**   Moduli of Curves of Genus 4. The hyperelliptic curves of genus 4 form an irreducible family of dimension 7. The nonhyperelliptic ones form an irreducible family of dimension 9. The subset of those having only one $g_3^1$ is an irreducible family of dimension 8. [Hint: Use (5.2.2) to count how many complete intersections $Q \cap F_3$ there are.]

**Exercise 6.2**   A rational curve of degree 5 in $\mathbb{P}^3$ is always contained in a cubic surface, but there are such curves which are not contained in any quadric surface.

# 5. Chapter V

## 5.1 October 22: K3 surfaces and quartics (Dani)

(Communicated by Daniela Paiva)

In Lecture 10 of K3 Surfaces course we proved the following proposition:

**Proposition** A K3 surface $M$ is isomorphic to a quartic if and only if $\mathrm{Pic}(M)$ contains a very ample bundle $L$ with $(L, L) = 4$

And that leads to

**Corollary** Let $M$ be a K3 surface such that $\mathrm{Pic}(M) = \mathbb{Z}$ and $L$ the line bundle generating $\mathrm{Pic}(M)$. Assuma that $(L, L) = 4$. Then $M$ is isomorphic to a quartic.

In this document I will prove the corollary.

**Plan**

1. $L$ ample implies that the map associated to the linear system $|L|$ is an embedding
   $$\varphi_{|L|} : M \hookrightarrow S \subset \mathbb{P}^n$$

2. The fact that $L^2 = 4$ implies that $n = 3$ and that the degree of $S$ is 4.

First recall that there is an isomorphism

$$\mathrm{Pic}(M) \cong \mathrm{Div}(M)/\mathrm{PDiv}(M)$$

where $\mathrm{Pic}(M)$ is the group of isomorphism classes of line bundles on $M$ and

$$\mathrm{Div}(M) := \left\{ D = \sum a_i D_i : a_i \in \mathbb{Z}, D_i \text{ irreducible subvariety of codimension } 1 \right\}$$

and

$$\mathrm{PDiv}(M) = \{\mathrm{div}(f) : f \in m_M\}$$

where $m_M$ is the space of meromorphic functions on M.  Recall that the (principal) divisor associated to a meromorphic function is a formal combination of the subvarieties where its zeroes and poles lie counted with multiplicity.

Note that the quotient by $\mathrm{PDiv}(M)$ ammounts to *linear equivalence* which is given by

$$D \sim D' \iff \exists f \in m_M \text{ s.t. } D - D' = \mathrm{div}(f).$$

Now recall that a divisor E is called *effective* (denoted $E \geqslant 0$) if all its coefficients are greater or equal than zero, and that the *linear system* associated to a divisor D is

$$|D| = \{E \in \mathrm{Div}(M) : E \geqslant 0, E \sim D\}$$

which is a finitely generated vector space over the field of meromorphic functions with the product defined by

$$f \cdot E = \mathrm{div}(f) + E.$$

Indeed, $f \cdot E \in |D|$ because it is linearly equivalent to D:

$$f \cdot E - D = \mathrm{div}(f) + E - D = \mathrm{div}(f) - \mathrm{div}(g) \in m_M$$

and also it is effective:

$$?$$

First observation after my talk is here.

**Definition** (Linear system=Linear series)   Bruno:

$$|D| = H^0(M, \mathcal{O}_M(D))$$

Misha:

$$\text{zero divisors of holomorphic sections of D}$$

Stacks Project: k field, X proper scheme over k, $d, r \geqslant 0$. A *linear series of degree* d *and dimension* r is

a pair $(\mathcal{L}, V)$, $\mathcal{L}$ invertible $\mathcal{O}_M$-module and $V \subset H^0(M, \mathcal{O}_M(D))$ k-subvector space of dim $r + 1$

The linear system defines a rational map

$$\varphi_{|D|} : M \longrightarrow \mathbb{P}^n$$
$$x \longmapsto [f_0(x) : \ldots : f_n(x)]$$

where $f_i$ are generators of $|D|$.  I'd like to have another look at how this map is constructed. See wiki to make sure it's not obvious.

Here's another look at this map:

$$\varphi_{|D|} : M \longrightarrow \mathbb{P}\big((H^0(M, \mathcal{O}_M(D)))^*\big)$$
$$x \longmapsto [\mathrm{ev}_x]$$

where

$$\mathrm{ev}_x : H^0(M, \mathcal{O}_M(D)) \longrightarrow \mathcal{L}_x$$
$$s \longmapsto s(x) \in \mathcal{L}_x$$

meaning that

$$\mathrm{ev}_x \in \mathrm{Hom}_{\mathbb{C}}(H^0(M, \mathcal{O}_M(D)), \mathcal{L}_x)$$

Notice that this map could be undefined at points where all the $f_i$ vanish. These conform the **base locus** of $|D|$ and each such point is called a **base point**. If there are no base points we say $|D|$ is **base-point free**.

Now we turn to our exercise, where $L$ is the generator of $\mathrm{Pic}(M)$. Our objectives are

- If $|L|$ is base-point free, the above map $\varphi_{|L|}$ is a morphism (not only a rational map).

- If $L$ is very ample then $\varphi_{|L|}$ is an embedding.

- The dimension of the projective space is $h^0(M, \mathcal{O}_M(L))$.

Knowing these three things allows to see $M$ as a projective variety, and then we only need to compute its degree, which should be 4. But first let's address the three points above. We shall use

**Theorem 5** (Taken from Mori)  Let $X$ be a K3 surface defined over an algebraically closed field of characteristic $\neq 2$. Let $H$ be a numerically effective divisor on $X$. Then one has

1. $H$ is not base point free if and only if there exit irreducible curves $E, \Gamma$, and an integer $k \geqslant 2$ such that $H \sim kE + \Gamma$, $E^2 = 0$, $\Gamma^2 = -2$ and $E \cdot \Gamma = 1$. In this case [...]

2. Let $H^2 \geqslant 4$. Then $H$ is very ample if and only if

   (a) there is no irreducible curve $E$ such that $E^2 = 0$ and $E \cdot H = 1, 2$,

   (b) there is no irreducible curve $E$ such that $E^2 = 2$, $H \sim 2E$, and

   (c) there is no irreducible curve $E$ such that $E^2 = -2$, $E \cdot H = 0$.

I think that the base-point free condition does not follow from this statement since, supposing that such $E$ and $\Gamma$ exisy we can barely show that $k = 3$. So something might be missing.

However to show that $L$ is ample we barely notice that the second and third case are impossible supposing that $E = kL$ since we get, in the second case that $k^2 \cdot 4 = 2$ and in the third that $k^2 \cdot 4 = -2$. The first case also can't happen since $E^2 = 0$ implies $k = 0$ which implies $E \cdot L = 0$.

**Remark**

In Misha's course we actually showed this in the following proposition and corollary from Lecture 10 (just after definition of **line system**):

**Theorem** Let M be a K3 surface such that $\text{Pic}(M) = \mathbb{Z}$, and L the line bundle generating $\text{Pic}(M)$. Assume that $(L, L) > 2$. Then L or $L^*$ is ample, base point free and the map $\psi : M \to \mathbb{P}H^0(M, L)^*$ is an embedding or a 2-sheeted ramified covering.

The second cas is when $|L|$ contains at least one hyperelliptic curve (and then all curves in $|L|$ are hyperelliptic). In fact, it is shown later that the hyperelliptic case corresponds to $(L, L) = 2$. This gives us the embedding we needed.

Also there's

**Corollary 2** Let M be a K3 surface such that $\text{Pic}(M) = \mathbb{Z}$ and L the line bundle generating $\text{Pic}(M)$. Assume that $(L, L) > 0$. Then L is very ample.

Then we wish to calculate the dimension of the image of the projective embedding. It turns out that the projective space in the image of $\varphi_{|L|}$ is $\mathbb{P}(H^0(M, \mathcal{O}_M(L)) \cong |L|$, which means that we can compute its dimension via

$$\dim |L| = \dim_{\mathbb{C}} H^0(M, \mathcal{O}_M(L)) - 1.$$

And then we cite Saint-Donat, where he shows that by Riemann-Roch and Serre duality we have for any invertible sheaf L on a K3 that

$$h^0(\mathcal{O}_M(L)) = \dim |L| + 1 = 2 + \frac{L^2}{2} + h^1(\mathcal{O}_M(L))$$

and since L is ample, $h^1(M, \mathcal{O}_M(L)) = 0$ and we get

$$1 + \dim |L| = 2 + \frac{L^2}{2} = 4 \implies \dim |L| = 3$$

So that the codomain of $\varphi_{|L|}$ is $\mathbb{P}^3$.

Finally we compute the degree of M. This part is a little sketchy but here goes.

**Definition** Let $X \subset \mathbb{P}^n$, its *degree* is the intersection number of X with $\dim X$ hyperplanes.

We have constructed an embedding $\varphi_{|L|} : M \hookrightarrow S \subseteq \mathbb{P}^3$. Its image S is a codimension-1 hypersurface in $\mathbb{P}^3$, which means that it is an element of $\text{Pic}(\mathbb{P}^3) = \mathbb{Z}$. So $S \sim kH$, and this k is the degree. So I'm not sure how this is used.

Now to compute the degree we must intersect with two hyperplanes because we are in a 3-dimensional thing (so intersection of $n - 1$ hyperplanes in an $n$-dimensional thing gives a line). For some reason it's enough to intersect S with H. Here H is the generator of $\text{Pic}(\mathbb{P}^3)$ which is in fact pulled back to L in $\text{Pic}(M)$ (by construction of the map $\varphi_{|L|}$.

$$\deg(S) = k = S \cdot H \cdot H = H^2 = L^2 = 4$$

Since we may pull back divisors by an isomorphism and the intersection number is preserved (see Beauville prop. 1.8)

**Definition**    Let $X \hookrightarrow \mathbb{P}^n$, its *degree* is the intersection number of $X$ with $\dim X$ hyperplanes.

**Exercise** (Sergey)

$$\deg X \geqslant \operatorname{codim} X + 1$$

if $X$ is not degenerate (not contained in hyperplane)

*Solution.* Intersect $X$ with a bunch of linear hyperplanes, this gives a finite number of points, as many as the degree of $X$.                                    □

## 5.2    November 5: intersection form

### 5.2.1    A short exercise (Dani)

(Chapter II, sec. 6) Really quickly recall that

1. $\operatorname{Div}(X)$ (for, say, $X$ a noetherian integral separated scheme which is regular in codimension one, see II.6) is the abelian group generated by closed integral subschemes of codimension 1, which are called *prime divisors*.

2. A divisor $D = \sum n_i Y_i$ with $n_i \geqslant 0$ is called *effective*.

3. If $f \in K^*$ is a rational function, a *principal divisor* is $\operatorname{div}(f) := \sum v_Y(f) \cdot Y$ where the sum ranges over all prime divisors of $X$ and $v_Y$ is the valuation of the local ring $\mathcal{O}_{y,X}$ where $y \in Y$ is the generic point of $Y$ (valuation is an integer, and the sum is finite). How to show this is just the zeroes (positive valuation) and poles (negative valuation)?

4. Two divisors $D$ and $D'$ are *linearly equivalent* if $D - D'$ is a principal divisor.

5. The group of divisors of $X$ modulo linear equivalence is isomorphic to $\operatorname{Pic}(X)$, the group of invertible sheaves up to isomorphism.

**Proposition 6.4**    Let $X$ be the projective space $\mathbb{P}^n_k$ over a field $k$. For any divisor $D = \sum n_i Y_i$, define the degree of $D$ by $\deg D = \sum n_i \deg Y_i$, where $Y_i$ is the degree of the hypersurface $Y_i$. Let $H$ be the hyperplane $x_0 = 0$. Then

(a) if $D$ is any divisor of degree $d$, then $D \sim dH$;

(b) for any $f \in K^*$, $\deg f = 0$;

(c) the degree function gives an isomorfism $\operatorname{Cl} X \to \mathbb{Z}$, where $\operatorname{Cl} X$ is $\operatorname{Div}(X)$ divided by the subgroup of principal divisors.

(Chapter V, sec 1) Let $X$ be a *surface* (=nonsingular projective surface over an algebraically closed field $k$).

**Theorem 1.1**    There is a unique pairing $\operatorname{Div}(X) \times \operatorname{Div}(X) \to \mathbb{Z}$ denoted by $\cdot$ such that

1. If $C$ and $D$ are nonsingular curves meeting transversally, then $C \cdot D = \#(C \cap D)$, the number of points of $C \cap D$,

2. it is symmetric: $C \cdot D = D \cdot C$,

3. it is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$, and

4. it depends only on the linear equivalence classes: if $C_1 \sim C_2$ then $C_1 \cdot D = C_2 \cdot D$.

**What's the definition of that map?** (Lucas)   Let $X$ be an algebraic $k$-surface (separated, $k$-scheme, finite type, dimension 2).

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$$

$$L_1 \cdot L_2 = \chi(\mathcal{O})X) - \chi(L_1^*) - \chi(L_2^*) + \chi(L_1^* \otimes L_2^*)$$

where the **euler characteristic** for any coherent sheaf $\mathcal{F}$ is just $\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) + h^2(\mathcal{F})$.

**Exercise**   Consider $\mathbb{P}^2_{x,y,z}$ and the divisor

$$D = -3D_1 + 2D_2$$

where

$$D_1 = V(x^2 + y^2 + 3xy), \qquad D_2 = \{x = 0\}$$

a. $D \cdot D = ?$

b. If $f = x^3$, what is $\mathrm{div}(f)$?

c. $f \in \mathcal{O}(p)$? what?

*Solution.*

(a) In Example 1.4.2 it is argued that the intersection form on $\mathbb{P}^2$ is the product of degrees. This is beacause $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$, so we can think it is generated by a line, which has self-intersection 1 (because any two lines intersect in one point, and self-intersection is defined using linear equivalence; so here I'm just assuming that any two lines are linearly equivalent which sounds reasonable). This determines the intersection form on $\mathrm{Pic}(\mathbb{P}^1)$.

Now it is clear that $D_1$ is of degree 2 and $D_2$ of degree 1 because they are roots of polynomials of respective degrees. This just means that

$$\begin{aligned}
D \cdot D &= (-3D_1 + 2D_2, -3D_1 + 2D_2) \\
&= 9D_1^2 - 12D_1 \cdot D_2 + 4D_2^2 \\
&= 36 - 24 + 4 = 16.
\end{aligned}$$

Or we can just recall that the degree of a divisor $D = \sum n_i Y_i$ is $\sum n_i \deg Y_i$, so degree of our D is -4 and $(-4)^2 = 16$.

(b) Here's a guess: it's $3 \in \mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^2)$. The idea is that it has a zero of multiplicity 3 on the generating line $x = 0$, but it should be justified that the definition above is consistent with this intuiton.

$\square$

### 5.2.2   More on intersection form

**Theorem** (Nakai-Moishezon Criterion for surfaces)   X proper $k$-surface. Let $L \in \text{Pic}(X)$. Then L is ample if and only if $L.L > 0$ (easy to prove) and $L.C > 0$ for all curves C (tricky).

**Theorem** (Nakai-Moishezon Criterion)   Now for $\dim X > 0$. So L is ample if (and only if? or the other way around?) $L^{\dim X} > 0$ and for all $Z \subset X \int_Z L^k > 0$.

**Theorem** (Kleiman-Mori critetion)   (if the restriction to the cone of cuves is positive)

$$N_1(X) \times N^1(X) \longrightarrow \mathbb{Z}, \overline{\text{ConeOfCurves}(X)} \subset N_1(X), \text{EffCurves}(X) \to N_1(X) \otimes \mathbb{R} = \mathbb{R}^{\rho(X)}.$$

$$L \cdot \alpha > 0 \qquad \forall \alpha \in \text{ConeOfCurves}$$

**Exercise** (Lucas)   X normal integral projective $k$-scheme, $\dim X \geqslant 2$. D ample effective Cartier divisor. Show that $\text{supp}(D)$ is connected.

**Definition** (Victor)

$$\text{supp}(D) = \{x \in X : \mathcal{O}(D)_x \neq 0\}$$

**Exercise** (Lucas)   C smooth curve in X, $D \in \text{Div}(X)$. Show that

$$(C.D) = (\mathcal{O}_X(C).\mathcal{O}_X(D)) = \deg(\mathcal{O}_X(D)|_C)$$

## 5.3   An exercise on K3

Now the actual exercise that I was trying to do when all this started is this:

**Exercise**   Show that there exists a K3 surface X with a line bundle L such that $L^2 = 4$ but X is not isomorphic to a quartic.

And while Dani Paiva has already given a solution let me put the spoiler that Sergey gave along with the question:

**Spoiler** (Sergey)   consider a general elliptic K3 surface with a section. It has $\text{Pic} = \mathbb{Z}^2$ with generators F - fiber and S - section. Describe intersection matrix/lattice of Pic and find in it a vector of squared length 4, show that linear system associated to any such vector in this lattice does not give an isomorphism to quadric.

**Idea**   Looks like the K3 surface should contain an elliptic curve (genus 1). There must be a theorem saying that this cannot happen when the surface is diffeomorphic to a quartic (this has to do with non-very ampleness of a line bundle (prop. 1.1 Huybrechts). Showing that there exists an elliptic curve is probably equivalent to showing that there is a vector $E \in \text{Pic}$ with $E^2 = 0$ and $L.E = 1$.

We start considering a general elliptic K3 surface.

**Definition 1.1** (Huybrechts, ch. 11)  An *elliptic K3 surface* is a K3 surface X together with a surjective morphism $\pi : X \to \mathbb{P}^1$ such that geometric generic fibre is a smooth integral curve of genus one or, equivalently, if there exists a closed point $t \in \mathbb{P}^1$ such that $X_t$ is a smooth integral curve of genus one.

**Proposition 1.3** (Huybrechts, ch. 11)  Let X be a K3 surface over an algebraically closed field k with char(k) $\neq 2, 3$. Then X admits an elliptic fibration if and only if there exists a non-trivial line bundle E with $E^2 = 0$.

This gives me $E^2$ for free. Now I should find L with $L^2 = 4$.

That $E.L = 0$ may be prop. 10.1.6 Huybrechts.

## 5.4 December 6

### 5.4.1 Riemann hypothesis for curves (Arthur)

Riemann hypothesis is a statement related to the *zeta function*

$$(s) = \sum_{n \geqslant 1} \frac{1}{n^{-s}}, \qquad \mathrm{Re}\, s > 1.$$

Such a function has a an analogue in finite fields. We shall prove the analogue of the Riemann hypothesis for curves, which is the statement that

$$|\#C(\mathbb{F}_q) - \#\mathbb{P}'(\mathbb{F}_q)| \leqslant 2g\sqrt{q}$$

where C is a smooth curve of genus g over the field $\mathfrak{F}_q$, and $C(\mathbb{F}_q)$ is the number of points of C rational over $\mathbb{F}_q$.

The proof relies of on the computation of some intersection numbers. For the first one, we have shown that two varieties intersect transversally, so that their intersection number is exactly the size of their intersection as a set. Indeed,

**Theorem V1.1**  There is a unique pairing $\mathrm{Div}\, X \times \mathrm{Div}\, X \to \mathbb{Z}$ denoted by C.D for any two divisors C, D such that

1. if C and D are nonsingular curves meeting transversally, then $C.D = \#(C \cap D)$, the numebr of points of $C \cap D$,

2-4 symmetric, additive, depends only on linear equivalence.

So we should say that

**Definition**  $D, E \subset S$ *intersect transversally* at $P \in D \cap E$ if

$$\mathfrak{m}_P = \langle \mathfrak{m}_{P,D}, \mathfrak{m}_{P,E} \rangle$$

which reminds us of the fact that transversality means that $T_P S = T_P D + T_P E$. The condition above is equivalent: recall that $T_P S = \left( \frac{\mathfrak{m}_P}{\mathfrak{m}_P^2} \right)^\vee$. The equality $T_P S = T_P D + T_P E$ ammounts to showing that a linear functional on $T_P S$ vanishes iff it vanishes on both $T_P D$ and $T_P E$. This is clear from the fact that $\mathfrak{m}_P = \langle \mathfrak{m}_{P,D}, \mathfrak{m}_{P,E} \rangle$.

**Exercise 1.8**   Divisor cohomology, neron severi

**Exercise 2.8**   Locally free sheaves.

**Exercise 3.5**   5 points in the field, hyperelliptic curve, point at infinity is singular.

**Exercise 4.5**

**Exercise 4.16**   27 lines on Fermat cubic

**Exercise 5.1**

**Exercise 5.4**

**Exercise 5.5**

**Exercise 6.2** (Arthur)   Beautiful exercise.

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