Notes on complex geometry

Contents

1   Basic complex geometry

Definition of ∂ and $\bar{\partial}$...

2   Sheaves

$\mathfrak{m}_x \subset C^\infty(\mathbb{R}^n)$ is the ideal of smooth functions vanishing at $x \in \mathbb{R}^n$. In [?] definition of *local ring of* P *in* Y, which is the ring of germs of regular functions of the variety Y at the point P, that it is a local ring with maximal ideal $\mathfrak{m}$, the set of germs of regular functions which vanish at P. The residue field is k.

Exercise   Show that indeed $\mathfrak{m}_x$, the ideal of functions vanishing at x, is maximal.

Solution. Suppose that $\mathfrak{n} \supseteq \mathfrak{m}_x$ is another ideal contained in $\mathcal{O}_{X,x}$. If $[f] \in \mathfrak{n}$ does not vanish at x, then we can do $[1/f]$ very near x, giving $[f][1/f] \in \mathfrak{n}$, so that $\mathfrak{n} = \mathcal{O}_{X,x}$. And if all $[f] \in \mathfrak{n}$ vanish at x, we get $\mathfrak{n} = \mathfrak{m}_x$. $\qquad\square$

3   Exponential exact sequence

It's a short exact sequence of sheaves:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

where X is a complex manifold, $\mathcal{O}(X)$ the sheaf of holomorphic functions and $\mathcal{O}^*(X)$ the sheaf of nowhere-vanishing holomorphic functions. Neither of these two sheaves have cohomologies that match either of $H^\bullet(X, \mathbb{C})$ nor $H^\bullet(X, \mathbb{R})$; however the constant sheaf $\underline{\mathbb{Z}}$ does give the same cohomology as singular integer cohomology.

As exact sequences often do, this ones gives a cohomology long exact sequence

$$\cdots \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X^*) \longrightarrow$$

$$H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow$$

$$\xrightarrow{\ c_1\ } H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow \cdots$$

To see it probably you'd have to delve back into Čech, but $H^1(X, \mathcal{O}_X^*)$ is the same as $\mathrm{Pic}(X)$. So in the end a line bundle gets assigned the cohomology class of its curvature w.r.t the Chern connection, or what

And not only that but actually the intersection form is cup product of Chern classes

4   Riemann-Roch

4.1   Genus

Definition ([?], p.180)   X nonsingular projective variety, *geometric genus* of X is $p_g := \dim_k \Gamma(X, \omega_X)$, where $\omega_X = \Lambda^n(\Omega_{X/k})$ is the canonical sheaf/bundle.

Remark I think these sections $\Gamma(X, \omega_X)$ can also be written as $H^1(X, \mathcal{O}_X)$. (To me it's not obvious why.)

Definition (p. 54) The hilbert polynomial P_Y of a projective variety Y is the polynomial whose coefficients are the dimensions of every summand in the graded decomposition $\bigoplus S^i$ of $\mathcal{O}_X$ (using that Y is projective).

The *arithmetic genus* of Y is $p_a(Y) := (-1)^r (P_Y(0) - 1)$.

Remark In the case of a projective nonsingular *curve*, the arithmetic genus and the geometric genus coincide (by Serre duality). This may not be true in dimension $\geqslant 2$.

Proposition IV.1.1 If X is a curve, then
$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X),$$
so we call this number simply the *genus* of $\mathfrak{X}$ and denote it by g.

4.2 Euler characteristic

Definition ([?], p. 360) For any coherent sheaf $\mathcal{F}$,
$$\chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Remark Once upon a time, after a long discussion in Donaldson I convinced myself that $H^0(X, \mathcal{F})$ is the set of sections of the sheaf $\mathcal{F}$.

4.3 For curves

According to wikipedia, $\ell(D)$ for a divisor D on a Riemann surface (D is a sum of points with some coefficients) is the set of all meromorphic functions h such that the coefficients of $(h) + D$ are non-negative.

Theorem IV.1.3 (Riemann-Roch) Let D be a divisor on a curve X of genus g. Them
$$\ell(D) - \ell(K - D) = \deg D - g + 1.$$

4.4 For surfaces

Theorem (K3 course) L line bundle on a surface and $K_X = \Omega^2(X)$ its canonical bundle. Then
$$\chi(L) = \chi(\mathcal{O}_X) + \frac{(L - K_X, L)}{2}.$$

5 Ampleness

Here's the **upshot** about ampleness:

From II.7, subsection *Ample Invertible Sheaves*, p. 153:

Now that we have seen that a morphism of a scheme X to a projective space can be characterized by giving an invertible sheaf on X and a suitable set of its global sections, [...]

Recall that in §5 we defined a sheaf $\mathcal{L}$ on X to be ***very ample relative to*** Y if there is an immersion $i : X \to \mathbb{P}^n_Y$ for some n such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. In case $Y = \mathsf{Spec}\, A$, this is the same thing as saying that $\mathcal{L}$ admists a set of global sections $s_0, \ldots, s_n$ such that the corresponding morphism $X \to \mathbb{P}^n_A$ is an immersion.

We have also seen (5.17) that if $\mathcal{L}$ is a very ample invertible sheaf on a projective scheme X over a noetherian ring A, then for any coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ such that for all $n \geqslant n-$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. [...]

Definition   An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is said to be ***ample*** if for every coherent sheaf $\mathcal{F}$ on X there is an integer $n_0 > 0$ (depending on $\mathcal{F}$) such that for every $n \geqslant n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections. (Here $\mathcal{L}^n := \mathcal{L}^{\otimes n}$.)

[...]

Remark II.7.4.3   [...] we will see below (7.6) that if $\mathcal{L}$ is ample, then some tensor power $\mathcal{L}^m$ of $\mathcal{L}$ is very ample.

So, being very ample is just a term to hide the possibility of embedding the variety in a projective space and pulling back the hyperplane bundle to that line bundle. But it turns out that this is equivalent to being generated by global sections in some sense (maybe taking tensor product).

Example II.6.4   We will see later (IV, 3.3) that if D is a divisor on a complete nonsingular curve X, them $\mathcal{L}(D)$ is ample iff $\mathsf{deg}\, D > 0$. This is a consequence of the Rimeann-Roch theorem.

Question   What's up with the base-points?

To answer we move along to subsection *Linear systems* in Hartshorne.

[...] global sections of an invertible sheaf correspond to effective divisors on a variety. Thus giving an invertibnle sheaf and a set of its global sections (which is related to finding the embedding of the variety into projective space pulling back the hyperplane bundle!) is the same as giving a certain set of effective divisors, all linearly equivalent to each other.

Idea (dani)   That we can associate divisors to sections. Then we consider all linearly equivalent divisors to a given one (a linear system), which corresponds to a set of sections $V \subseteq \Gamma(X, \mathcal{L})$.

And then

Lemma II.7.8   [...] In particular, [a linear system] $\mathfrak{d}$ is base-point-free iff $\mathcal{L}$ is generated by the global sections in V.

5.1 Line bundles that are the pullback of the hyperplane bundle

6 Adjunction formula

This is really an application of Riemann-Roch for surfaces:

Proposition V.1.5 ([?]) If C is a nonsingular curve of genus g on the surface X and K is the canonical divisor on X, then

$$2g - 2 = C.(C + K).$$

6.1 Kodaira ampleness criterion

In K3 lecture 9 we have

Theorem (Kodaira) A bundle L is very ample iff $c_1(L)$ is a Kähler class.

Recall that a line bundle is *prequantizable* if its curvature is symplectic and integral.

7 Positive $(1, 1)$-forms

Any positive $(1, 1)$ form looks like this: $\sum \alpha I x_i \wedge x_i$ for some positive functions $\alpha_i \geqslant 0$.

8 Hodge Index Theorem

Theorem V.1.9 (Hodge Index Theorem) Let

9 Volume form

dani: It's

$$\text{constant} \prod \partial z_i \wedge \bar{\partial} z_i = \text{const.} \sum dx_i \wedge dy_i$$

voi:It's

$$\frac{\omega^n}{n!} = \prod dz_i \wedge d\bar{z}_i$$

where $dz_i = dx_i + \sqrt{-1} dy_i$. It is the volume form of the hermitian manifold, i.e. the unique nowhere-vanishing section of the determinant bundle that gives 1 to the volume of the real unit cube $e_1 \wedge I e_1 \wedge \ldots \wedge e_n \wedge I e_n$ obtained from an h-orthonormal complex basis $\{e_i\}$.

10 Adjunction formula

11 Kähler metric

The Kähler form is the differential of a plurisubharmonic function ψ. that is $\omega = dd^c\psi = \sqrt{-1}\partial\bar\partial\psi$.

12 Picard group, Neron-Severi group

Definition (dani) *Picard group* is the group of line bundles with tensor product.

Remark See [**?**] p. 151 for the quick comment "We have seen (6.17) that $\operatorname{Pic}\mathbb{P}^n_k \cong \mathbb{Z}$ and is generated by $\mathcal{O}(1)$".

Definition ([Huy16], p.5) *Néron-Severy group* of an algebraic surface X is the quotient

$$\operatorname{NS}(X) := \operatorname{Pic}(X)/\operatorname{Pic}_0(X)$$

by the connected component of the Picard variety $\operatorname{Pic}(X)$, i.e. by the subgroup of line bundles that are *algebraically* equivalent to zero (?).

Definition ([Huy16], p.5)
$$\operatorname{Num}(X) := \operatorname{Pic}(X)/\operatorname{Pic}^\tau(X)$$

where $\operatorname{Pic}^\tau(X)$ is the subgroup of *numerically trivial* line bundles, i.e. line bundles L such that $(L.L') = 0$ for all line bundles L'. (E.g. any $L \in \operatorname{Pic}^0$ is numerically trivial.)

13 K3 surfaces

Definition (dani) A *K3* surface is a complex surface with trivial canonical bundle and vanishing first (co)homology. (It's dimension 2 so 1st cohomology is first homology by Poincaré.)

Definition (K3 course) A *K3 surface* is a complex surface M with $b_1 = 0$ (Betti number is dimension of homology) and $c_1(M, \mathbb{Z}) = 0$. Recall that the Chern class coming from the exponential sequence is $\operatorname{Pic}(M) \xrightarrow{c_1} H^2(M, \mathbb{Z})$, so in particular it means that the first Chern class of the canonical bundle $K_X \in \operatorname{Pic}(M)$ is trivial, which in turn makes it trivial via Hodge theory.

Remark [Huy16] shows that K3 surfaces have (algebraic, what is that?) fundamental group in remark 2.3.

Definition ([Huy16]) A *K3 surface* over k is a complete non-singular variety X of dimension two such that

$$\Omega^2_{X/k} \cong \mathcal{O}_X \qquad \text{and} \qquad H^1(X, \mathcal{O}_X) = 0$$

Proposition 1.2.1 ([Huy16])   NS(X) and its quotient Num(X) are finitely generated.

The rank of NS(X) is called the **Picard number** $\rho(X) := \text{rk NS}(X)$.

Proposition 1.2.4 ([Huy16])   For a K3 surface X the natural surjections are isomorphisms (**warning:** the second isomorphism might not hold for general complex K3 surfaces):

$$\text{Pic}(X) \xrightarrow{\sim} \text{NS}(X) \xrightarrow{\sim} \text{Num}(X)$$

and the intersection pairing on Pic(X) is even, non-degenerate, and of signature $(1, \rho(X) - 1)$.

14   Fubini-Study

It's a metric, it's a symplectic form. **Fubini-Study (symplectic) form** is a closed 2-form defined on $\mathbb{C}P^n$ as the exterior differential of the logarithm of the length functions $\ell = \sum |z_i|^2$, i.e. $\omega = dd^c \log \ell$.

This also has a local expression in coordinates $(z_1, \ldots, z_n)$ that might be interesting.

The **Fubini-Study metric** is $g(\cdot, \cdot) = \omega(\cdot, I\cdot)$.

15   Hypercomplex manifolds

Definition   A manifold M is **hypercomplex** if it has three integrable almost complex structures I, J, K satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\text{Id}$ and $IJ = -JI = K$.

Remark (Obata Connection, GPT)   Given a hypercomplex manifold (M, I, J, K), there exists a unique torsion-free connection ∇^{ob} such that

$$\nabla^{ob} I = \nabla^{ob} J = \nabla^{ob} K = 0.$$

This is called the *Obata connection*. Unlike the Levi-Civita connection, it is not necessarily compatible with a metric. Instead, it preserves the entire hypercomplex structure and serves as the natural connection in hypercomplex geometry.

References

[Huy16] D. Huybrechts. *Lectures on K3 Surfaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.