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Notes on complex geometry

Contents

1 basic math

Definition (Tensor product) If V and W are R-modules and R is a ring, I think of the ***tensor product*** as

$$V \otimes W := \{v \otimes w, v \in V, w \in W\} \Big/ \begin{matrix} v \otimes (w_1 + w_2) \sim v \otimes w_1 + v \otimes w_2 \\ (v_1 + v_2) \otimes w \sim (v_1 \otimes w) + (v_2 \otimes w) \\ \lambda(v \otimes w) \sim (\lambda v) \otimes w \sim v \otimes (\lambda v) \end{matrix}$$

But I think it is quotient by finite linear combinations of that?

And of course there's the construction using universal property. (Probably the definition above still needs a proof to make sure this object exists!)

14.3.6, [?] (Tensor algebra constructions) Let M be an A-module. The ***tensor algebra*** $T^\bullet(M)$ is just (the direct sum, probably) of $T^0(M) := A$, $T^n(M) := \underbrace{M \otimes_A \ldots \otimes_A M}_{n \text{ times}}$.

The ***symmetric algebra*** $\mathsf{Sym}^\bullet(M)$ is the quotient of $T^\bullet(M)$ by the (two-sided) ideal generated by all the elements of the form $x \otimes y - y \otimes x$. So,

$$\mathsf{Sym}^n(M) = M \otimes \ldots \otimes M \Big/ m_1 \otimes \ldots \otimes m_n - m_1' \otimes \ldots \otimes m_n'$$

where $(m_1', \ldots, m_n')$ is a rearrangement of $(m_1, \ldots, m_n)$.

Finally the ***exterior algebra*** $\Lambda^\bullet(M)$ is defined to be the quotient of $T^\bullet M$ by the (two-sided) ideal generated by all the elements of the form $x \otimes x$ for all $x \in M$. Which implies that $a \otimes b = -b \otimes a$ (Pf. do $(a+b) \otimes (a+b)$, but only if $\mathsf{char} \neq 2$ right?) Apparently this gives you that

$$\Lambda^n M = \text{quotient of no sé que with a permutation}$$

Anyway I can finally say that if $\mathcal{F}$ is a locally free rank-m sheaf, I can define $T^n\mathcal{F}$, $\mathsf{Sym}^n \mathcal{F}$ and $\Lambda^n\mathcal{F}$, that will be locally free (exercise, and find their ranks, exercise). And just to conclude for today, the ***determinant (line) bundle*** is $\Lambda^{\mathsf{rk}\mathcal{F}}\mathcal{F} := \det \mathcal{F}$.

So why all this. Because adjunction formula gives you that the canonical bundle of the submanifold is $K_{\text{ambient mfd}} \otimes_{\mathbb{C}?} \det \mathcal{N}$ where $\mathcal{N}$ is the normal bundle of the submanifold which is the cokernel sheaf/bundle of the inclusion. So that's that.

2 Basic complex geometry

Definition of ∂ and $\bar{\partial}$...

3 Sheaves

$\mathfrak{m}_x \subset C^\infty(\mathbb{R}^n)$ is the ideal of smooth functions vanishing at $x \in \mathbb{R}^n$. In [Har77] definition of ***local ring of*** P ***in*** Y, which is the ring of germs of regular functions of the variety Y at the point P, that it is a local ring with maximal ideal $\mathfrak{m}$, the set of germs of regular functions which vanish at P. The residue field is k.

Exercise Show that indeed $\mathfrak{m}_x$, the ideal of functions vanishing at x, is maximal.

Solution. Suppose that $\mathfrak{n} \supseteq \mathfrak{m}_x$ is another ideal contained in $\mathcal{O}_{X,x}$. If $[f] \in \mathfrak{n}$ does not vanish at x, then we can do $[1/f]$ very near x, giving $[f][1/f] \in \mathfrak{n}$, so that $\mathfrak{n} = \mathcal{O}_{X,x}$. And if all $[f] \in \mathfrak{n}$ vanish at x, we get $\mathfrak{n} = \mathfrak{m}_x$. ☐

4 Exponential exact sequence

It's a short exact sequence of sheaves:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

where X is a complex manifold, $\mathcal{O}(X)$ the sheaf of holomorphic functions and $\mathcal{O}^*(X)$ the sheaf of nowhere-vanishing holomorphic functions. Neither of these two sheaves have cohomologies that match either of $H^\bullet(X, \mathbb{C})$ nor $H^\bullet(X, \mathbb{R})$; however the constant sheaf $\underline{\mathbb{Z}}$ does give the same cohomology as singular integer cohomology.

As exact sequences often do, this ones gives a cohomology long exact sequence

$$\cdots \longrightarrow H^0(X,\mathbb{Z}) \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(X,\mathcal{O}_X^*) \longrightarrow$$

$$H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow$$

$$\xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow \cdots$$

To see it probably you'd have to delve back into Čech, but $H^1(X, \mathcal{O}_X^*)$ is the same as $\mathsf{Pic}(X)$. So in the end a line bundle gets assigned the cohomology class of its curvature w.r.t the Chern connection, or what

And not only that but actually the intersection form is cup product of Chern classes

5 Riemann-Roch

5.1 Genus

Definition ([Har77], p.180) X nonsingular projective variety, ***geometric genus*** of X is $p_g := \dim_k \Gamma(X, \omega_X)$, where $\omega_X = \Lambda^n(\Omega_{X/k})$ is the canonical sheaf/bundle.

Remark I think these sections $\Gamma(X, \omega_X)$ can also be written as $H^1(X, \mathcal{O}_X)$. (To me it's not obvious why.)

Definition (p. 54) The hilbert polynomial P_Y of a projective variety Y is the polynomial whose coefficients are the dimensions of every summand in the graded decomposition $\bigoplus S^i$ of $\mathcal{O}_X$ (using that Y is projective).

The ***arithmetic genus*** of Y is $p_a(Y) := (-1)^r(P_Y(0) - 1)$.

Remark In the case of a projective nonsingular *curve*, the arithmetic genus and the geometric genus coincide (by Serre duality). This may not be true in dimension $\geqslant 2$.

Proposition IV.1.1 If X is a curve, then

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X),$$

so we call this number simply the ***genus*** of $\mathfrak{X}$ and denote it by g.

5.2 Euler characteristic

Definition ([Har77], p. 360) For any coherent sheaf $\mathcal{F}$,

$$\chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Remark Once upon a time, after a long discussion with Donaldson I convinced myself that $H^0(X, \mathcal{F})$ is the set of sections of the sheaf $\mathcal{F}$. (Not in person.)

5.3 For curves

According to wikipedia, $\ell(D)$ for a divisor D on a Riemann surface (D is a sum of points with some coefficients) is the set of all meromorphic functions h such that the coefficients of $(h) + D$ are non-negative.

Theorem IV.1.3 (Riemann-Roch) Let D be a divisor on a curve X of genus g. Them

$$\ell(D) - \ell(K - D) = \deg D - g + 1.$$

5.4 For surfaces

Theorem (K3 course) L line bundle on a surface and $K_X = \Omega^2(X)$ its canonical bundle. Then

$$\chi(L) = \chi(\mathcal{O}_X) + \frac{(L - K_X, L)}{2}.$$

6 Ampleness

> "In $\mathbb{C}^n$ many complex hypersurfaces can be written as regular level sets of globally defined holomorphic functions. But in a compact complex man-

> ifold, this is never possible, because all globall holomorphic functions are constants. Instead, we can use sections of line bundles."

6.1 The hyperplane bundle

Upshot (About the hyperplane bundle) Because the dual bundle is, of course, the bundle whose fiber at each point is the dual vector space of the original one. So hyperplane is because every functional of the dual space (an element of the fiber of the dual) is just a hyperplane.

And if you take a lot of tensor powers? You'll end up with the homogeneous degree d functions (which you should *not* think as hyperplanes, that's only for $d = 1$). (But most likely they are irreducible varieties... so is $\mathcal{O}(d)$ the "degree-d-variety bundle"?)

And that's why [Lee24] Example 3.37 works: a homogeneous polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ gives me a section of $\mathcal{O}(1)$: at each point $\xi \in \mathbb{C}P^n$ I have the homogeneous degree d function that f is when restricted to ξ. (This is also very tautological.) (I would call this the ***tautological section***.)

Here's what I wrote when I first understood the hyperplane bundle:

In lem. 3.30 [Lee24] we see that the d-th tensor power of the dual bundle of a line bundle L, a thing denoted by $(L^*)^d$, is naturally isomorphic to the bundle whose fiber at a point $p \in M$ is the space of functions $\varphi : L_p \to \mathbb{C}$ that are ***homogeneous of degree*** d, meaning that $\varphi(\lambda v) = \lambda^d \varphi(v)$ for all $\lambda \in \mathbb{C}$ and $v \in L_p$.

> which basically makes me understand that $\mathcal{O}(1) = \mathcal{O}(-1)^* =$ dual of the tautological bundle, is naturally isomorphic to the bundle whose fiber is the space of homogeneous functionals of degree 1. Which are hyperplanes. So that's why $\mathcal{O}(1)$ is called the ***hyperplane bundle***.

6.2 Line bundle associated with a hypersurface?=?normal bundle

Theorem 3.39, [Lee24] (Line bundle associated with a hypersurface) Save techincalities, we have that if $S \subseteq M$ is a closed complex hypersurface, there exists a line bundle $L_S \to M$ and a section $\sigma \in \mathcal{O}(M; L_S)$ that vanishes (simply) on S and nowhere else.

Nice example So the associated line bundle of a hypersurface determined by a homogeneous degree d polynomial is... the tautological section of this polynomial! (and the bundle is $\mathcal{O}(d)$).

6.3 Ampleness

Objective: when do global sections appear?

Story goes, the space of global sections of a holomorphic vector bundle on a compact complex manifold is finite dimensional. This is essentialy due to complex analysis. Montel's theorem is used to prove this in Lee. For a down-to-earth-illustrative point of view consider the line bundle of $\mathcal{O}_X$ of holomorphic functions on the the manifold (which is a bundle since it is a free rank-1 $\mathcal{O}_X$ module—the trivial bundle), which, if we remember, it's only constant functions because M is compact, so it's actually $\mathbb{C}$, which is finite-dimensional over $\mathbb{C}$. (But the corresponding thing in smooth function world is very infinite-dimensional.)

Anyway you can choose a basis $(s_0, \dots, s_m)$ of $\mathcal{O}(L)$. And then. you. construct. a. map. to. $\mathbb{P}^m$. As follows: choose a point $x \in M$, and take a local trivializaing open set of the bundle, where you have a local frame consisting of one local section (because it's a line bundle) called $s : U \to E(U)$. Then each of the global sections s_i is represented locally as $f_i s = s_i$. So you get this $\mathbb{C}$-valued functions, right? And then you get the map $p \mapsto (f_0(p), \dots, f_n(p))$. But that would depend on the trivializing open set. But if you think projectively you can show that it does not depend on the choice of frame.

Exercise You have 13 minutes to prove that.

Solution. The section s_i is a map $X \to W \times \mathbb{C}$. The transition function is an endomorphism of W, giving $s_i(p) = (p, v_i) \mapsto (p, \tau v_i$. Now if $s_i(p) = f_i(p)s(p)$ I get $s_i(p) \mapsto \tau(p)$, substitute...

Now I'm at home and I prove it like this. Just notice that $s_i = f_i s : W \to \pi^{-1}(W)$ (where of course $W \stackrel{\text{def}}{=} U \cap V$ is the intersection of two different trivializing charts and f_i is the function corresponding to the base $s \stackrel{\text{def}}{=} s_U$ of the chart U). Right so there is a transition function mapping $\pi^{-1}(W) \to \pi^{-1}(W)$ that is the identity on the base and a linear map on the fibers. Now, as I said above, since s_i is a function from W to the bundle, the transition function maps the vector part of the fiber w.r.t U to the vector part w.r.t. V by a linear transformation. Which is a number. So the equation we all want is

$$s = \xi s'$$

which gives

$$s_i = f_i s, \qquad s_i = g_i s' = g_i \xi s' \implies f_i = \xi g_i$$

and then the coordinates of this map we are defining become

$$[f_0, \dots, f_n] = [\xi g_0, \dots, \xi g_n] = [g_0, \dots, g_n]$$

so that the thing when defined projectively does not depend on the charts. $\square$

Nice! So we see, that whenever $H^0(X, L)$ is not empty, we can construct a map from X to projective space. And the next question is whether this map is an embedding. And the answer is ampleness. L is ***very ample*** when it is an embedding.

6.4 Ampleness in Hartshorne

Here's the **upshot** about ampleness:

From II.7, subsection *Ample Invertible Sheaves*, p. 153:

> Now that we have seen that a morphism of a scheme X to a projective space can be characterized by giving an invertible sheaf on X and a suitable set of its global sections, [...]
>
> Recall that in §5 we defined a sheaf $\mathcal{L}$ on X to be ***very ample relative to*** Y if there is an immersion $i : X \to \mathbb{P}^n_Y$ for some n such that $\mathcal{L} \cong i^*\mathcal{O}(1)$. In case $Y = \operatorname{Spec} A$, this is the same thing as saying that $\mathcal{L}$ admists a set of global sections $s_0, \dots, s_n$ such that the corresponding morphism $X \to \mathbb{P}^n_A$ is an immersion.
>
> We have also seen (5.17) that if $\mathcal{L}$ is a very ample invertible sheaf on a projective scheme X over a noetherian ring A, then for any coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ such that for all $n \geqslant n-$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. [...]
>
> **Definition** An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is said to be ***ample*** if for every coherent sheaf $\mathcal{F}$ on X there is an integer $n_0 > 0$ (depending on $\mathcal{F}$) such that for every $n \geqslant n_0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections. (Here $\mathcal{L}^n := \mathcal{L}^{\otimes n}$.)
>
> [...]
>
> **Remark II.7.4.3** [...] we will see below (7.6) that if $\mathcal{L}$ is ample, then some tensor power $\mathcal{L}^m$ of $\mathcal{L}$ is very ample.

So, being very ample is just a term to hide the possibility of embedding the variety in a projective space and pulling back the hyperplane bundle to that line bundle. But it turns out that this is equivalent to being generated by global sections in some sense (maybe taking tensor product).

Example II.6.4 We will see later (IV, 3.3) that if D is a divisor on a complete nonsingular curve X, them $\mathcal{L}(D)$ is ample iff $\deg D > 0$. This is a consequence of the Rimeann-Roch theorem.

Question What's up with the base-points?

To answer we move along to subsection *Linear systems* in Hartshorne.

> [...] global sections of an invertible sheaf correspond to effective divisors on a variety. Thus giving an invertibnle sheaf and a set of its global sections (which is related to finding the embedding of the variety into projective space pulling back the hyperplane bundle!) is the same as giving a certain set of effective divisors, all linearly equivalent to each other.

Idea (dani) That we can associate divisors to sections. Then we consider all linearly equivalent divisors to a given one (a linear system), which corresponds to a set of sections $V \subseteq \Gamma(X, \mathcal{L})$.

And then

Lemma II.7.8 [...] In particular, [a linear system] $\mathfrak{d}$ is base-point-free iff $\mathcal{L}$ is generated by the global sections in V.

7 Adjunction formula

Definition 2.2.16 ([?]) Let $Y \subset X$ be a complex submanifold. The ***normal bundle*** of Y in X is the holomorphic vector bundle $\mathcal{N}_{Y/X}$ on Y is the cokernel of the natural injection $\mathcal{T}_Y \subset \mathcal{T}_X|_Y$. Thus there exists a short exact sequence of holomorphic vector bundles, the ***normal bundle sequence***:

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0$$

Explanation (Example 2.23, [Lee24]) The ***holomorphic normal bundle*** of Y in X is the bundle $NY \to Y$ defined by $NY := (T'X|_Y/T'Y$ where the apostrophe means holomorphic tangent bundle for Lee. So probably this is the metric-free definition of the normal bundle. "(It is important to observe that the ***geometric normal bundle*** that can be defined as the set of tangent vectors that are orthogonal to S with respect to some Riemannian metric on M will not in general have holomorphic transition functions.)" So maybe metric normal bundle may not be holomorphic, and instead metric-free normal bundle is holomorphic always?

Dream It'd still be nice to understand the construction geometrically: take the tangent bundle of the ambient variety, and quotient by the tangent bundle of the subvariety. At a point, we have collapsed the tangent bundle of the subvariety, and all we are left with is the vectors that are not tangent to the subvariety. I guess it kind of makes sense: take for example a curve on $\mathbb{R}^3$, collapse the tangent line at a point of the curve, and you get $\mathbb{R}^2$.

Proposition 2.2.17, [?] (Adjunction formula) So what is the determinant bundle?

7.1 a formula from Hartshorne

This is really an application of Riemann-Roch for surfaces:

Proposition V.1.5 ([Har77]) If C is a nonsingular curve of genus g on the surface X and K is the canonical divisor on X, then

$$2g - 2 = C.(C + K).$$

7.2 Kodaira ampleness criterion

In K3 lecture 9 we have

Theorem (Kodaira) A bundle L is very ample iff $c_1(L)$ is a Kähler class.

Recall that a line bundle is ***prequantizable*** if its curvature is symplectic and integral.

8 Positive $(1,1)$-forms

Any positive $(1,1)$ form looks like this: $\sum \alpha I x_i \wedge x_i$ for some positive functions $\alpha_i \geqslant 0$.

9 Hodge Index Theorem

Theorem V.1.9 (Hodge Index Theorem) Let

10 Volume form

dani: It's
$$\text{constant} \prod \partial z_i \wedge \bar{\partial} z_i = \text{const.} \sum dx_i \wedge dy_i$$

voi:It's
$$\frac{\omega^n}{n!} = \prod dz_i \wedge d\bar{z}_i$$

where $dz_i = dx_i + \sqrt{-1}dy_i$. It is the volume form of the hermitian manifold, i.e. the unique nowhere-vanishing section of the determinant bundle that gives 1 to the volume of the real unit cube $e_1 \wedge Ie_1 \wedge \ldots \wedge e_n \wedge Ie_n$ obtained from an h-orthonormal complex basis $\{e_i\}$.

11 Kähler metric

The Kähler form is the differential of a plurisubharmonic function ψ. that is $\omega = dd^c\psi = \sqrt{-1}\partial\partial\psi$.

12 Picard group, Neron-Severi group

Definition (dani) ***Picard group*** is the group of line bundles with tensor product.

Remark See [Har77] p. 151 for the quick comment "We have seen (6.17) that $\mathsf{Pic}\, \mathbb{P}^n_k \cong \mathbb{Z}$ and is generated by $\mathcal{O}(1)$".

Definition ([Huy16], p.5) ***Néron-Severy group*** of an algebraic surface X is the quotient
$$\mathsf{NS}(X) := \mathsf{Pic}(X)/\mathsf{Pic}_0(X)$$
by the connected component of the Picard variety $\mathsf{Pic}(X)$, i.e. by the subgroup of line bundles that are *algebraically* equivalent to zero (?).

Definition ([Huy16], p.5)
$$\mathsf{Num}(X) := \mathsf{Pic}(X)/\mathsf{Pic}^\tau(X)$$
where $\mathsf{Pic}^\tau(X)$ is the subgroup of *numerically trivial* line bundles, i.e. line bundles L such that $(L.L') = 0$ for all line bundles L'. (E.g. any $L \in \mathsf{Pic}^0$ is numerically trivial.)

13 K3 surfaces

Definition (dani) A ***K3*** surface is a complex surface with trivial canonical bundle and vanishing first (co)homology. (It's dimension 2 so 1st cohomology is first homology by Poincaré.)

Definition (K3 course) A ***K3 surface*** is a complex surface M with $b_1 = 0$ (Betti number is dimension of homology) and $c_1(M, \mathbb{Z}) = 0$. Recall that the Chern class coming from the exponential sequence is $\mathsf{Pic}(M) \xrightarrow{c_1} H^2(M, \mathbb{Z})$, so in particular it means that the first Chern class of the canonical bundle $K_X \in \mathsf{Pic}(M)$ is trivial, which in turn makes it trivial via Hodge theory.

Remark [Huy16] shows that K3 surfaces have trivial (algebraic) fundamental group (what is that?) in remark 2.3.

Definition ([Huy16]) A ***K3 surface*** over k is a complete non-singular variety X of dimension two such that

$$\Omega^2_{X/k} \cong \mathcal{O}_X \qquad \text{and} \qquad H^1(X, \mathcal{O}_X) = 0$$

Proposition 1.2.1 ([Huy16]) $\mathsf{NS}(X)$ and its quotient $\mathsf{Num}(X)$ are finitely generated.

The rank of $\mathsf{NS}(X)$ is called the ***Picard number*** $\rho(X) := \mathsf{rk}\,\mathsf{NS}(X)$.

Proposition 1.2.4 ([Huy16]) For a K3 surface X the natural surjections are isomorphisms (**warning:** the second isomorphism might not hold for general complex K3 surfaces):

$$\mathsf{Pic}(X) \xrightarrow{\sim} \mathsf{NS}(X) \xrightarrow{\sim} \mathsf{Num}(X)$$

and the intersection pairing on $\mathsf{Pic}(X)$ is even, non-degenerate, and of signature $(1, \rho(X) - 1)$.

14 Fubini-Study

It's a metric, it's a symplectic form. ***Fubini-Study (symplectic) form*** is a closed 2-form defined on $\mathbb{C}P^n$ as the exterior differential of the logarithm of the length functions $\ell = \sum |z_i|^2$, i.e. $\omega = dd^c \log \ell$.

This also has a local expression in coordinates $(z_1, \dots, z_n)$ that might be interesting.

The ***Fubini-Study metric*** is $g(\cdot, \cdot) = \omega(\cdot, I\cdot)$.

15 Hypercomplex manifolds

Definition A manifold M is ***hypercomplex*** if it has three integrable almost complex structures I, J, K satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\mathsf{Id}$ and $IJ = -JI = K$.

Remark (Obata Connection, GPT) Given a hypercomplex manifold (M, I, J, K), there exists a unique torsion-free connection ∇^{ob} such that

$$\nabla^{ob} I = \nabla^{ob} J = \nabla^{ob} K = 0.$$

This is called the *Obata connection*. Unlike the Levi-Civita connection, it is not necessarily compatible with a metric. Instead, it preserves the entire hypercomplex structure and serves as the natural connection in hypercomplex geometry.

16 Fano manifolds

Exercise By Kodaira vanishing theorem, you can show that the cohomology $H^i(X, L)$ of a Fano variety X vanishes. You just have to put $L = \mathcal{O}(k)$ with $k \geqslant -r$ for the ***Fano index***

$$r(X) = \min\{r : \frac{c_1(X)}{r} \in H^2(X, \mathbb{Z})\}.$$

Anyway the point is that $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$ for Fano.

References

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.

[Huy16] D. Huybrechts. *Lectures on K3 Surfaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.

[Lee24] J.M. Lee. *Introduction to Complex Manifolds*. Graduate Studies in Mathematics. American Mathematical Society, 2024.

[Vak25] R. Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. Princeton University Press, 2025.