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Notes on complex geometry

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1 Basic complex geometry

Definition of ∂ and $\bar{\partial}$...

2 Volume form

dani: It's

$$\text{constant} \prod \partial z_i \wedge \bar{\partial} z_i = \text{const.} \sum dx_i \wedge dy_i$$

voi:It's

$$\frac{\omega^n}{n!} = \prod dz_i \wedge d\bar{z}_i$$

where $dz_i = dx_i + \sqrt{-1} dy_i$. It is the volume form of the hermitian manifold, i.e. the unique nowhere-vanishing section of the determinant bundle that gives 1 to the volume of the real unit cube $e_1 \wedge Ie_1 \wedge \ldots \wedge e_n \wedge Ie_n$ obtained from an h-orthonormal complex basis $\{e_i\}$.

3 Adjunction formula

4 Fubini-Study

It's a metric, it's a symplectic form. ***Fubini-Study (symplectic) form*** is a closed 2-form defined on $\mathbb{C}P^n$ as the exterior differential of the logarithm of the length functions $\ell = \sum |z_i|^2$, i.e. $\omega = dd^c \log \ell$.

This also has a local expression in coordinates $(z_1, \ldots, z_n)$ that might be interesting.

The ***Fubini-Study metric*** is $g(\cdot,\cdot) = \omega(\cdot, I\cdot)$.

5 Hypercomplex manifolds

Definição A manifold M is ***hypercomplex*** if it has three integrable almost complex structures I, J, K satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\operatorname{Id}$ and $IJ = -JI = K$.

Observação (Obata Connection, GPT) Given a hypercomplex manifold (M, I, J, K), there exists a unique torsion-free connection ∇^{ob} such that

$$\nabla^{ob} I = \nabla^{ob} J = \nabla^{ob} K = 0.$$

This is called the *Obata connection*. Unlike the Levi-Civita connection, it is not necessarily compatible with a metric. Instead, it preserves the entire hypercomplex structure and serves as the natural connection in hypercomplex geometry.