GEOMETRIC STABILITY CONDITIONS AND GROUP ACTIONS

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Hannah's notes at hannahdell.me/cimpa2025

These notes at github.com/danimalabares/cimpa-floripa

Abstract. Group actions on categories arise naturally from symmetries of varieties and quivers, but how does this interact with Bridgeland stability? In the frist half of this course we will introduce equivariant categories, which generalises the category of equivariant sheaves. Then we will show there is a correspondence between stability conditions on a category with a fniite group action, and stability conditions on the equivariant category – this will also play a role in Xiaolei Zhao's course. We will use this to produce stability conditions on quotient varieties (and stacks).

In the second half of the course, we will apply this to study open questions about the geometry of the stability manifold: in particular, we will discuss "geometric stability conditions" – those for which all skyscraper sheaves of points are stable. In practice, these are constructed using slope stability for sheaves. Some varieties have only geometric stability conditions, whereas in other cases, there are more (for example if there is an equivalence with quiver representations). Lie Fu, Chunyi Li, and Xiaolei Zhao were the frist to provide a general result explaining this phenomenon. In particular, they showed that if a variety has a fniite map to an abelian variety, then all stability conditions are geometric. We will test the converse on free quotients of abelian varieties by fniite groups, including Beauville-type and bielliptic surfaces. This is based on joint work with Edmund Heng and Anthony Licata.

Contents

Motivation

The Bridgeland stability machine:

[*diagram*]

Question 1. How does the geometry of $\mathrm{Stab}(X)$ relate to X? **Question 2.** How to invariants (of X or Ü) behave under (finite) group actions?

Goal of these lectures. Answer both by studying "free quotients" (and see lots of examples along the way!)

1. Geometric stability conditions on surfaces

Setup. X smooth projective surface over $\mathbb{C}$.

$$\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X) \qquad \text{Neron-Severi group}$$
$$\mathrm{NS}_{\mathbb{R}}(X) := \mathrm{NS}(X) \otimes \mathbb{R}$$
$$\mathrm{Amp}_{\mathbb{R}}(X) := \text{ ample cone, i.e. } H \in \mathrm{NS}_{\mathbb{R}}(X) \text{ such that } H^2 > 0$$

Definition 1.1. Let $0 \neq E \in \mathrm{Coh}(X)$ and $H \in \mathrm{Amp}_{\mathbb{R}}(X)$,

(1) The H*-slope* of E is
$$\mu_H(E) := \begin{cases} +\infty & \mathrm{ch}_0(E) = 0 \\ \frac{H \cdot \mathrm{ch}_1(E)}{H^2 \mathrm{ch}_0(E)} & \text{otherwise} \end{cases}$$
(2) E is H*-(semi)stable* if $0 \neq F \subset E$ implies
$$\mu_H(F) < \mu_H(E/F) \qquad (\mu_H \leq \mu_H(E/F)$$

Example 1.2.

Problem. [missing]

Definition 1.3. A path $\sigma = (\mathcal{A}, Z)$ is a *(numerical) Bridgeland stability condition* for $D^b(X)$ if

(1) $\mathcal{A} \subset D^b(X)$ is the heart of an bounded t-structure.
(2) $Z : K_0(X) \to \mathbb{C}$ homomorphism (the *central charge*) such that
   - $0 \neq E \in \mathcal{A} \implies Z(E) \in \mathbb{H}$, where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$.
   - HN filtrations exist.
   - Factors via $\mathrm{Knum}(X)$.
(3) Support property.

Let X be a surface, and fix $H \in \mathrm{Amp}_{\mathbb{R}}(X)$ and $\beta \in \mathbb{R}$. Define

$$\tau_{M,\beta} := \{E \in \mathrm{Coh}(X) : E \overset{\text{surj.}}{\to} Q \neq 0 \implies \mu_M(Q) > \beta\}$$
$$\mathcal{F}_{H,\beta} := \{E \in \mathrm{Coh}(X) : 0 \neq F \subset E \implies \mu_M(F) \leq \beta\}$$

Exercise 1.4. $(\tau_{M,\beta}, \mathcal{F}_{H,\beta})$ is a torsion pair in $\mathrm{Coh}(X)$.

Definition 1.5.
$$\mathrm{Coh}^{H,\beta}(X) := \left\{ E \in D^b(X) : \substack{H^0(E) \in \tau_{H,\beta} \\ H^{-1}(E) \in \mathcal{F}_{H,\beta} \\ H^i(E) = 0} \right\}$$

Exercise 1.6.

There are notes available about the following:

Theorem 1.7 (Bridgeland '08, Arcara-Bertam, Macri-Schmidt)**.** *For $\alpha \gg 0$,*

$$\mathbb{C} \cdot \{\sigma_{H,D,\alpha,\beta} = (Coh^{H,B}(X), Z_{H,D,\alpha,\beta})\}$$

is a continuous family of Bridgeland stability conditions.

Question 3. Is this all of $\mathrm{Stab}(X)$?

Exercise 1.8. $\mathcal{O}_x$ is $\sigma_{H,D,\alpha,\beta}$-stable for all $x \in X$.

Definition 1.9. X smooth projective variety, $\sigma \in \mathrm{Stab}(X)$ is *geometric* if for all $x \in X$, $\mathcal{O}_x$ is σ-stable. Write $\mathrm{Stab}^{\mathrm{geo}}(X) =$ all geometric stability conditions.

Now we can refine Question 3:

Question 3.1. Does Theorem 1.7 describe all geometric stability conditions?

Question 3.2. Do there exist nongeometric stability conditions?

Answer 3.2.

- $\dim X = 1$ no! Unless $X = \mathbb{P}^1$.
- $\dim X \leq 3$? Yes if X has a full exceptional collection.
- $\dim X = 2$?
  1. If X is an abelian surface, no!
  2. Yes for $\mathbb{P}^2$, and more generally for rational surfaces.
  3. Yes for K3 surfaces.
  4. $X \supset C \cong \mathbb{P}^1$ such that $C^2 < 0$.

Takeaway. $\mathrm{Stab}(X)$ sees something about how (1) is different to (2)-(4) but what is it?

The following theorem is valid in any dimension:

Theorem 1.10 (Lie Fu-Chunyi Li-Xiaolei Zhao '21)**.** *X has a finite Albanese morphism ($\iff$ there exist a finite morphism to an abelian variety), then*

$$Stab(X) = Stab^{geo}(X)$$

Sketch of proof.

- $\sigma \in \mathrm{Stab}(X)$.
- Let $E_1, \ldots, E_k$ be the *Jordan-Hölder factors* of $\mathcal{O}_X$ w.r.t. σ (i.e. E_i σ-stable connected component containing the identity).
- $\mathcal{L} \in \mathrm{Pic}^0(X)$.

By [Polishchuk '07], $\otimes\mathcal{L}$ does not change stability. Also $\mathcal{O}_x \otimes \mathcal{L} \cong \mathcal{O}_x$ so HN filtration and Jordan-Hölder factors are preserved $\implies E_i \otimes \mathcal{L} \cong E_i$.

- $\mathcal{L} \in \mathrm{Pic}^0(\mathrm{Alb}(X))$, $E_i \otimes \mathrm{alb}^\vee(\mathcal{L}) \cong E_i$.

$$\begin{aligned}
&\implies R\mathrm{alb}_*(E_i) \otimes \mathcal{L} \cong R\mathrm{alb}_*(E_i) \qquad \text{(projection formula)}\\
&\implies R\mathrm{alb}_*(E_i) \text{ has finite support} \qquad \text{([Polishucuk '03])}\\
&\implies E_i \text{ has finite support} \qquad (\mathrm{alb}_* \text{ finite})\\
&\overset{\text{claim}}{\implies} \mathcal{O}_x \text{ is } \sigma\text{-stable}
\end{aligned}$$

$\square$

Question 4. Albanese morphism of X is not finite implies that there exist nongeometric stability conditions?

The idea is to investigate this for examples arising as free quotients. If $G \curvearrowright X$ is a free action of a finite group, then alb_X is finite, and for $Y := X/G$ we have alb_Y is not finite. In sections 2 and 3 we will compate $\mathrm{Stab}(X)$ and $\mathrm{Stab}(X/G)$.

2. Equivariant categories

Let G be a finite group and $\mathcal{D}$ a k-linear category (i.e. all Hom are k-vector spaces, in particular it is additive), where $k = \bar{k}$ and $(|G|, \mathrm{char} k) = 1$.

The following definition is by [Deligne '97].

Definition 2.1 (Group action)**.** A *(right) action* of G on $\mathcal{D}$ is the data of

(1) $\forall g \in G$, $\phi_g : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$.
(2) $\forall g, h \in G$, a natural isomorphism $\mathcal{E}_{g,h} : \phi_g \circ \phi_h \xrightarrow{\sim} \phi_{hg}$ such that

$$\begin{tikzcd} \phi_f \circ \phi_g \circ \phi_h \arrow[r, "\mathcal{E}_{g,h}"] \arrow[d, "\mathcal{E}_{f,g}"'] & \phi_f \circ \phi_{hg} \arrow[d, "\mathcal{E}_{f \circ hg}"] \\ \phi_{gf} \circ \phi_h \arrow[r, "\mathcal{E}_{gf,h}"] & \phi_{hgf} \end{tikzcd}$$

Remark 2.2. This is more than a group a homomorphism $G \to \mathrm{Aut}(\mathcal{D})$. Indeed, given this, $\forall E \in \mathcal{D}$,

$$\phi_g \circ \phi_h(E) \cong \phi_{gh}(E)$$

but may not come from a natural isomorphism of functors $\phi_g \circ \phi_h \cong \phi_{hg}$.

Example 2.3. (1) X: scheme, $G \leq \mathrm{Aut}(X)$. For every $g \in G$ define $\phi_g := g^* : \mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(X)$. Then for every g, h there are canonical isomorphisms

$$\phi_g \circ \phi_h = g^* \circ h^* \xrightarrow{\sim} (hg)^* = \phi_{hg}.$$

$G \curvearrowright \mathrm{Coh}(X)$ lifts to $G \curvearrowright D^b(X)$.

Eg. an Enriques surface, i.e. $Y = X/\mathbb{Z}/2\mathbb{Z}$, X a K3 surface, $\mathbb{Z}/2\mathbb{Z} = \langle i \rangle$, for $i : X \to X$ involution, $x \in X$, $i^*(\mathcal{O}_x) = \mathcal{O}_{i^{-1}(x)}$.

(2) Q: acyclic quiver, $G \leq \mathrm{Aut}^+(Q) =$ automorphisms of the graph, orientation preserving. Then $G \curvearrowright \mathrm{Rep}Q$. Via $(M_i, \varphi_\alpha) \mapsto (M_{g(i)}, \varphi_{g(x)})$.

Eg. [Missing]

Equivariantization.

Definition 2.4. Let $G \curvearrowright \mathcal{D}$. A *$G$-equivariant* object is a pair $(E, \{\lambda_g\}_G$,

- $E \in \mathcal{D}$,
- $\lambda_g : E \xrightarrow{\sim} \phi_g(E)$, a **choice** of isomorphism for all $g \in G$, such that

$$\begin{tikzcd} E \arrow[r, "\lambda_g"] \arrow[d, "\lambda_h g{,}\sim"'] & \phi_g(E) \\ \phi_{hg}(E) & \phi_g(\phi_n(E)) \arrow[l, "\mathcal{E}_{gh}"'] \end{tikzcd}$$

A *morphism of G-equivariant objects* is

$$(E, \{\lambda_g\}_G) \to (E'_g, \{\lambda'_g\}_G)$$

is $F \in \mathrm{Hom}_{\mathcal{D}}(E, E')$ such that for all $g \in G$,

$$\begin{tikzcd} E \arrow[r, "\lambda_g"] \arrow[d, "F"'] & \phi_g(E) \arrow[d, "\phi_g(F)"] \\ E' \arrow[r, "\lambda'_g"] & \phi_g(E') \end{tikzcd}$$

Together these form a category, $\mathcal{D}_G$, called *G-equivariant category.*

Remark 2.5. This uses the isomorphisms $\mathcal{E}_{g,h}$.

Example 2.6. (1) $G \curvearrowright X$, G-equivariant objects are G-equivariant coherent sheaves.

$$(\mathrm{Coh}(X))_G =: \mathrm{Coh}_G(X)$$

In fact, $\mathrm{Coh}_G(X) \cong \mathrm{Coh}([X/G])$.

If G acts freely, then

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & X/G & \end{tikzcd}$$

and $\mathrm{Coh}(X/G) \cong \mathrm{Coh}_G(X)$ via $E \mapsto (\pi^* E, \{\lambda_g\}_{g \in G}$ where

$$\begin{tikzcd} \lambda_g : \pi^* E \arrow[r] \arrow[dr, "\lambda_g"'] & (\pi \circ g)^* E \arrow[d, "="] \\ & g^*(\pi^* E) \end{tikzcd}$$

Eg. $Y = X/(\mathbb{Z}/2\mathbb{Z})$ Enriques surface, $y \in Y$, $\mathrm{supp}\pi^{-1}(y) = \{x, x' = i^{-1}(x)\}$

$$\pi^*(\mathcal{O}_x) = \begin{matrix} \mathcal{O}_x & \xrightarrow{\mathrm{id}} & \mathcal{O}_x \\ \oplus & & \oplus \\ \mathcal{O}_{x'} & \xrightarrow{\mathrm{id}} & \mathcal{O}_{x'} \end{matrix} = \mathrm{id}^*(\pi^*(\mathcal{O}_y))$$

$$\pi^*(\mathcal{O}_y) = \begin{tikzcd} \mathcal{O}_x \arrow[r, "i^*"] \arrow[dr] & \mathcal{O}_x \\ \oplus & \oplus \\ \mathcal{O}_{x'} \arrow[r, "i^*"] \arrow[ur] & \mathcal{O}_{x'} \end{tikzcd} = i^*(\pi^*(\mathcal{O}_y))$$

In general,

$$(D^b(X))_G \cong D^b(\mathrm{Coh}_G(X)) =: D^b_G(X)$$

(2) **Theorem (Demonet '10).** $G \curvearrowright Q$ then $\exists Q_G$ such that $\mathrm{Rep}(Q_G) \cong \mathrm{Rep}(Q)_G$.
Eg. [Missing.]

(3) $G \curvearrowright \mathcal{D}$ for G abelian, $\hat{G} := \mathrm{Hom}(G, K^*) \curvearrowright \mathcal{D}_G$ by

$$\phi_{\underbrace{x}_{\in G}}((E, \{\lambda_g\}_G)) := (E, \{\lambda_g\}_G) \otimes x = (E, \{\lambda_g \cdot x(g)\}_{g \in G})$$

Eg. Y Enriques surface ... [Missing]

Theorem 2.7 (Elagin '15)**.** *G abelian, $G \curvearrowright \mathcal{D}$*

$$(\mathcal{D}_G)_{\hat{G}} \cong \mathcal{D}$$

3. Exercise lecture

On a surface X, to have a Bridgeland stability condition we wanted a map

$$\begin{tikzcd} Z : K_0(X) \arrow[rr] \arrow[dr] & & \mathbb{C} \\ & \Lambda \arrow[ur] & \end{tikzcd}$$

For a numerical B.s.c., we ask

$$\begin{tikzcd} Z : K_0(X) \arrow[rr] \arrow[dr] & & \mathbb{C} \\ & K_{num} \arrow[ur] & \end{tikzcd}$$

Recall that the *Grothendieck group* K_0 can be defined to be the free group generated by isomorphism calsess of objects in $D^b(X)$ quotient by the *Euler pairing*, i.e.

$$[F] = [E] + [G] \iff E \to F \to G \to E[i]$$

Now let

$$\chi(E,F) = \sum_i (-1)^i \dim \underbrace{\mathrm{Ext}^i(E,F)}_{=\mathrm{Hom}(E,F[i])}$$

then

$$K_{num} = K_0(X) / \underbrace{\mathrm{null}(\chi)}_{=\{E : \chi(E,F)=0 \forall F \in K_0(X)\}}$$

To obtain a lattice consider

$$\begin{tikzcd} K_0(X) \arrow[r, "ch"] \arrow[d] & \mathrm{CH}^*(X) \arrow[d] \\ K_{num}(X) \arrow[r, "ch"] & \underbrace{\mathrm{CH}^*_{num}(X)}_{\simeq \underbrace{\mathbb{Z}}_{ch_0} \oplus \mathrm{NS}(X) \oplus \frac{1}{2} \underbrace{\mathbb{Z}}_{ch_2}} \end{tikzcd}$$

Fix $B \in \mathrm{NS}(X)_{\mathbb{R}}$.

$$\mathrm{NS}(X) \ni A \mapsto \deg(A \cdot B) \in c \cdot \mathbb{Z}$$

for $c \in \mathbb{R}$. For a surface,

$$\Lambda = \mathbb{Z} \oplus \mathbb{Z}^{\oplus \mathrm{rk}(\mathrm{NS}(X))} \oplus \frac{1}{2}\mathbb{Z}$$

$$Z_{H,D,\alpha,\beta} = (\alpha + i\beta) H^2 ch_0 + (D + iH) ch_1 - ch_2.$$

Exercise 3.1. $\mathcal{O}_X$ is simple (i.e. no subobjects) in $\mathrm{Coh}^{H,\beta}(X)$.