KUMMER TYPE HYPERKÄHLER VARIETIES

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Notes at github.com/danimalabares/cimpa-floripa

Abstract. Examples of noncommutative K3 surfaces arise from semiorthogonal decompositions of the bounded derived category of certain Fano varieties. The most interesting cases are those of cubic fourfolds and Gushel-Mukai varieties of even dimension. Using the deep theory of families of stability conditions, locally complete families of hyperkähler manifolds deformation equivalent to Hilbert schemes of points on a K3 surface have been constructed from moduli spaces of stable objects in these noncommutative K3 surfaces. On the other hand, an explicit description of a locally complete family of hyperkähler manifolds deformation equivalent to a generalized Kummer variety is not available from classical geometry. In this lecture series, we will construct families of noncommutative abelian surfaces as equivariant categories of the derived category of K3 surfaces which specialize to Kummer K3 surfaces. Then we will explain how to induce stability conditions on them and produce examples of locally complete families of hyperkähler manifolds of Kummer type. Based on joint work with Arend Bayer, Alex Perry and Laura Pertusi.

Contents

1. K3 surfaces and Hyperkähler varieties

Definition 1.1. A *K3 surface* (over $\mathbb{C}$) is a smooth projective surface S such that $\omega_S \cong \mathcal{O}_S$ and $\pi_1(S) = 1$.

Example 1.2 (K3 surfaces)**.**
1. $S \subset \mathbb{P}^3$ of degree 4. A straightforward computation shows the conditions of the definition are verified.
2. $S \to \mathbb{P}^2$ double cover ramified along a degree 6 curve.

Definition 1.3. A *hyperkähler* variety is a smooth projective variety X such that
1. $\pi_1(X) = 1$
2. $H^0(X, \Omega^2_X) = \mathbb{C}\omega$ (i.e. the space of global holomorphic 2-forms is one dimensional) where ω is holomorphically symplectic.

Remark 1.4. If X is hyperkähler, it must be even dimensional and $\omega_X = \mathcal{O}_X$.

Remark 1.5. We just mention the name of Beauville-Bogomolov theorem.

For a while people were looking for examples of hyperkähler varieties.

Example 1.6 (Hyperkähler varieties)**.** (1) K3 surface S.

(2) $X = S^{[n]}$ Hilbert scheme of n points on K3 surface. (Moduli space of stable sheaves on K3 S of rank 1, $c_1 = 0$, $c_2 = 0$.)
(3) $\mathcal{M}_H(s, v) =$ moduli of H-stable sheaves on S of class v. (v primitive, H is v-generic.) (Recall the example in Cristina's course where we studied how the moduli changes under changes in the polarization H.) Let $[E] \in \mathcal{M}_H(s, v)$. We have the Yoneda map
$$T_{[E]}\mathcal{M} \cong \mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^2(E, E) \overset{\substack{S \text{ is} \\ \text{a K3}}}{\cong} \mathrm{Hom}(E, E)^* \cong \mathbb{C}$$
where the first factor product of $\mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E)$ is associated to first arrow, the second factor to the second arrow, of the following diagram:
$$E \to E[1] \to E[2]$$
This moduli space is always of Picard rank 2.
(4) $Y \subset \mathbb{P}^5$ a smooth cubic fourfold.
$$F(Y) = \{[\ell] \in \mathrm{Gr}(2, 6) | \ell \subseteq Y\}$$
is a hyperkähler 4-fold. For a specific choice Y_0 of hyperkähler 4-fold in the moduli of smooth cubic 4-folds (which is 20-dimensional) we find $F(Y_0)$ is deformation equivalent to $S^{[2]}$ where S is a K3 surface.

Here is a general picture:

Let X, Y be projective smooth varieties and $f : X \to Y$. Consider the functors
$$Rf_* : D^b(X) \to D^b(Y) \qquad Lf^* : D^b(Y) \to D^b(X)$$
for $F, G \in D^b(X)$, $F \overset{L}{\otimes} G \in D^b(X)$. **Convention:** we drop the R and L.

Definition 1.7. Let $K \in D^b(X \times Y)$. The *Fourier-Mulai functor* is
$$\begin{aligned} \Phi_K : D^b(X) &\longrightarrow D^b(Y) \\ F &\longmapsto \mathrm{pr}_{Y,*}(\mathrm{pr}_X^* F \otimes K) \end{aligned}$$
where we are using our convention — all functors here are derived. Here the maps are
$$\begin{tikzcd} & X \times Y \arrow[dl, "\mathrm{pr}_X"'] \arrow[dr, "\mathrm{pr}_Y"] & \\ X & & Y \end{tikzcd}$$

Example 1.8. $f : X \to Y$ and $\Gamma_f \equiv X \times Y$ graph. Then $\Phi_{\mathcal{O}_{p_f}} = f_*$.

Theorem 1.9 (Orlov)**.** *If $F : D^b(X) \to D^b(Y)$ is an equivalence, then $\exists! K \in D^b(X \times Y)$ such that $F \cong \Phi_K$.*

Theorem 1.10. *The following are preserved by derived equivalence:*
(1) *(Bondal-Orlov.)* $\dim X$.
(2) *(Bondal-Orlov.)* $\bigoplus_{m \geq 0} H^0(X, \pm mK_X)$.
(3) $H^*(X, \mathbb{Q})$*, and* $\bigoplus_{p-q=i} H^{p,q}(X)$ *for any* $i \in \mathbb{Z}$.

Non-trivial equivalences. S a K3 surface. $H^2(S,\mathbb{Z})$.

- (Hodge decomposition.) $H^2(S,\mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.
- Pairing on $H^2(S,\mathbb{Z})$ given by cup product.

Theorem 1.11 (Torelli for K3)**.** *Two K3 surfaces S, S' are isomorphic if and only if there exists a Hodge isometry $\varphi : H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ (i.e. an isomorphism that preserves the Hodge decomposition and the pairing).*

This was the classical content.

1.12. **Mukai lattice.**

$$\tilde{H}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$$

- Weight 2 Hodge structure
$$\tilde{H}^{2,0} = H^{2,0} \qquad \tilde{H}^{1,1} = H^0 \oplus H^{1,1} \oplus H^2 \qquad \tilde{H}^{0,2} \cong H^{0,2}.$$
- Pairing:
$$\langle (a,b,c),(a',b',c')\rangle = bb' - ac' - a'c$$

Theorem 1.13 (Mukai, Orlov)**.** *If S, S' are K3 surfaces, then $D^b(S) \cong D^b(S') \iff \tilde{H}(S,\mathbb{Z}) \underset{\varphi}{\cong} \tilde{H}(S',\mathbb{Z})$ Hodge isometry.*

1.14. **Mukai vector.** Let $K_0(S)$ be the free abelian group generated by $\mathrm{Ob}(D^b(S))$. Consider

$$v : K_0(S) \to \tilde{H}(S;\mathbb{Z})$$

Where $[F] = [E] + [G]$ if $E \to F \to G \xrightarrow{+1}$ is a distinguished triangle.

$$v : [E] \to \mathrm{ch}(E) \cdot \sqrt{\det(S)}$$

where $\mathrm{ch}(E)$ is the Chern character.

$$\langle v(E), v(F)\rangle = -\chi(E,F),$$

where $\chi(E,F) := \sum (-1)^i \dim \mathrm{Ext}^i(E,F) := \mathrm{Hom}_{D^b(S)}(E,F[i])$.

Proof of the backward implication of Mukai-Orlov theorem. Let

$$\varphi : \tilde{H}(S,\mathbb{Z}) \longrightarrow \tilde{H}(S',\mathbb{Z})$$
$$(0,0,1) \longmapsto v$$
$$(-1,0,0) \longmapsto v'$$

The intersection matrix of v and v' is

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For simplicity assume that v is of positive rank. ($V = (a,b,c)$, where $a \in H^0$ is positive.)

The heart of the proof uses the following result by Mukai. There exists a nonempty moduli space $\mathcal{M}$ of stable sheaves on S' of class $v \implies S'$ is a K3 surface!

$$0 = \chi(v,v) = \underbrace{\mathrm{Hom}(E,E)}_{=1} - \underbrace{\mathrm{ext}^1(E,E)}_{=2} + \underbrace{\mathrm{ext}^2(E,E)}_{=1}$$

$v \cdot v' = 1 \implies \mathcal{M}$ is a fine moduli space. There exists a universal family $D^b(S \times M) \ni \mathcal{E} \to S' \times M$.

Claim. $\Phi_{\mathcal{E}} : D^b(S') \xrightarrow{\cong} D^b(\mathcal{M})$ (general criteria).
Now $\tilde{H}(S,\mathbb{Z}) \xrightarrow{\varphi,\cong} \tilde{H}(S',\mathbb{Z}$

$$\begin{tikzcd}
\tilde{H}(S,\mathbb{Z}) \arrow[r, "\varphi,\cong"] & \tilde{H}(S',\mathbb{Z}) \arrow[r, "\Phi_{\mathcal{E}}"] & \overset{\vee}{H}(M,\mathbb{Z}) \\
(0,0,1) \arrow[r, mapsto] & v \arrow[r, mapsto] & (0,0,1) \\
(1,0,0) \arrow[rr, mapsto] & & (1,0,0)
\end{tikzcd}$$

$\square$

2. Semiorthogonal decomposition and Calabi-Yau categories

Situation 2.1.

$$\begin{tikzcd}
D^b(Y), {}^{Y\ \text{a cubic}}_{\text{4-fold}} \arrow[dr, "\substack{\text{semiorthogonal}\\ \text{decomposition}}"'] & & \substack{\text{moduli}\\ \text{on K3}\\ \text{as HK}} \\
& \mathrm{Ku}_Y{}^{\text{K3}}_{\text{category}} \arrow[ur, "\text{stability conditions}"'] &
\end{tikzcd}$$

Definition 2.2. Let X be a projective smooth variety over $\mathbb{C}$. A *SOD* $D^b(X) = \langle A_1, \dots, A_n \rangle$ is a sequence A_i full triangulated subcategory such that

(1) $\mathrm{Hom}(F, G) = 0$ for all $F \in A_i$, $G \in A_j$.
(2) $\forall F \in D^b(X)$ $\exists 0 = F_n \to F_{n-1} \to \dots \to F_0 = F$ such that $\mathrm{Cone}(F_i \to F_{i-1}) \in A_i$.

Exercise 2.3. (1) $D^b(\mathrm{Spec}\mathbb{C}) \ni V \simeq \bigoplus_i H^i(V)[-i]$.

(2) For $E \in D^b(X)$ define

$$\phi_E : D^b(\mathrm{Spec}\mathbb{C}) \longrightarrow D^b(X)$$

$$V \longmapsto V \otimes E = \bigoplus_i H^i(V) \otimes E[-i]$$

Then ϕ_E fully faithful $\iff$ E is exceptional

$$\left(\mathrm{Ext}^p(E,E) = \begin{cases} \mathbb{C} & p = 0 \\ 0 & p \neq 0 \end{cases} \right)$$

Definition 2.4. $\langle E \rangle := \phi_E(D^b(\mathrm{Spec}\mathbb{C})) \subset D^b(X)$ for exceptional E.

(3) **Example.** $\mathcal{O}(i)$ exceptional on $\mathbb{P}^n$. (Beilinson.) $D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$

Definition 2.5. A triangulated category $A \subset D^b(X)$ is *right admissible* if $\alpha : A \to D^b(X)$ admits a right adjoint. $\alpha^! : D^b(X) \to A$. (I.e., $\mathrm{Hom}_{D^b(X)}(\alpha(E), F) \simeq \mathrm{Hom}_A(E, \alpha^!(F))$)

Exercise 2.6. Let E be exceptional. $\langle E \rangle \subset D^b(X)$ is right admissible.

$$\alpha^! = R\,\mathrm{Hom}(E, F) \in D^b(\mathrm{Spec}\mathbb{C})$$

Let

$$A^\perp := \{F \in D^b(X) : \mathrm{Hom}(E, F) = 0, \forall E \in ?\}$$

Lemma 2.7. *$A \subset D^b(X)$ is right admissible* $\implies$ $D^b(X) = \langle A^\perp, A \rangle$.

Proof. (1) Is clear.

(2) $0 = F_2 \to F_1 \to F_0 = F$, $F_1 \in A$, $\mathrm{Cone}(F_1 \to F) \in A^\perp$. We use adjunction as follows:

$$\mathrm{id} \in \mathrm{Hom}(\alpha^!(F), \alpha^!(F)) = \mathrm{Hom}(\alpha\alpha^! F, F)$$

We have the following exact triangle:

$$\alpha\alpha^! F \xrightarrow{\text{counit}} F \longrightarrow B$$

Claim. $B \in A^\perp$. Let $E \in A$

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & \mathrm{Hom}(\alpha(E), \alpha\alpha^!(F)) \arrow[rr] \arrow[dr, "\cong"] & & \mathrm{Hom}(\alpha(E), F) \arrow[r] \arrow[dl, "\cong"'] & \mathrm{Hom}(\alpha(E), B) \\ & & \mathrm{Hom}(E, \alpha^!(F)) & & \end{tikzcd}$$

which implies that $\mathrm{Hom}(\alpha(E), B) = 0$.

□

Definition 2.8. *Left mutation* $\mathbb{L}_A$ is defined as

$$\alpha\alpha^! F \to F \to \mathbb{L}_A F.$$

Exercise 2.9. $A = \langle E \rangle$.

$$R\,\mathrm{Hom}(E, F) \otimes E \xrightarrow{\text{ev}} F \to \mathbb{L}_{\langle E \rangle} F$$

Corollary. $E_1, \ldots, E_n \in D^b(X)$ exceptional objects with $\mathrm{Ext}^\bullet(E_i, E_j) = 0$ for $i > j$ (we say this is an *exceptional collection*). Then

$$D^b(X) = \langle R_x, E_1, \ldots, E_n \rangle, \qquad R_X = \langle E_1, \ldots, E_n \rangle^\perp$$

Example 2.10. X Fano, $-K_X = rH$, H ample, $r > 0$, $\mathcal{O}_X, \mathcal{O}_X(H), \ldots, \mathcal{O}_X((r-1)H)$ an exceptional collection.

$(\mathrm{Ext}^\bullet(\mathcal{O}(iH), \mathcal{O}(jH)) = H^\bullet(X, \mathcal{O}(j-i)H) = 0)$ if $-r < j - i < 0$. $\implies$

$$D^b(X) = \langle R_X, \mathcal{O}_X \ldots \mathcal{O}_X(r-i)H \rangle$$

Definition 2.11. A *Serre functor* for a Δ-category $\mathcal{D}$ is an autoeq $S_{\mathcal{D}}$ such that

$$\mathrm{Hom}_{\mathcal{D}}(E, F)^\vee \cong \mathrm{Hom}(F, S_{\mathcal{D}}(E)).$$

functorially in $E, F \in \mathcal{D}$.

Remark 2.12. It is unique if it exists.

The following example explains why this is called the Serre functor — it is a generalization of Serre duality.

Example 2.13. X smooth projective variety of dimension n. Consider

$$S_{D^b(X)} = (- \otimes \omega_X)[n]$$

eg. E locally free on X.

$$\begin{aligned} H^i(X, E) &= \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, E[i]) \\ &= \mathrm{Hom}_{D^b(X)}(E[i], \omega_X(n])^* \\ &= \mathrm{Hom}(\mathcal{O}, E^\vee \otimes \omega_X[n-i]) \\ &= H^{n-i}(E^\vee \otimes \omega_X)^* \end{aligned}$$

Definition 2.14. $\mathcal{D}$ is

- *Calabi-Yau category* of dimension n if $S_{\mathcal{D}} \cong [n]$. (I.e. the Serre functor is just shifting by n.)
- *Fractional Calabi-Yau category* of dimension p/q if $S_{\mathcal{D}}^q \cong [p]$ (where the exponent q just means composing the functor $S_{\mathcal{D}}$ $q \in \mathbb{Z}$ times).

Theorem 2.15 (Kuznetsov)**.** *$X \subset \mathbb{P}^n$ smooth Fano hypersurface of degree $d \leq n$.*

$$D^b(X) = \langle Ku_X, \mathcal{O}_X, \dots, \mathcal{O}_X(n-d)\rangle$$

Then Ku_X has a Serre functor S with $S^{d/c} \cong \left[\frac{(n+1)(d-2)}{c}\right]$ where $c = gcd(d, n+1)$.

Example 2.16. $d = 3$, a cubic 3-fold, $n = 4$, $S^3 = [5]$, cubic 4-fold $n = 5$, $S = [2]$.

Theorem 2.17 (Kuznetsov)**.** *X cubic 4-fold,*

(1) *$\exists$ special X such that $Ku_X \simeq D^b(S)$, where S is a K3 surface.*
(2) *X very general $\implies$ $Ku_X \not\simeq D^b(var)$.*

We shall not prove this theorems. However, by the fourth lecture we may give the idea of their proofs.

Conjecture (Kuznetsov). Cubic 4-fold X is rational if and only if $\mathrm{Ku}_X \simeq D^b$(K3 surface).

[KKPY] very general X is not rational.

Let's go back and show how to prove the statement in Example 2.16 about the cubic 4-fold.

Lemma 2.18. *Assume that $\mathcal{D} = \left\langle A^\perp, A\right\rangle$, $S_{\mathcal{D}}\exists$. Then $S_{A^\perp}^{-1} = \mathbb{L}_A \circ S_{\mathcal{D}}^{-1}|_{A^\perp}$.*

Proof. $F, G \in A^\perp$,

$$\begin{aligned} \mathrm{Hom}_{A^\perp}(F,G) &\cong \mathrm{Hom}_{\mathcal{D}}(G, S_{\mathcal{D}}(F))^\vee \\ &\cong \mathrm{Hom}_{\mathcal{D}}(S_{\mathcal{D}}^{-1}(G), F)^\vee \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\mathbb{L}_A(\underbrace{S_{\mathcal{D}}^{-1}(G))}_{\in A^\perp}, F)^\vee \end{aligned}$$

□

Now: cubic 4-fold $X \subset \mathbb{P}^5$,

$$D^b(X) = \langle \mathrm{Ku}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$$

The lemma tells us that

$$S_{\mathrm{Ku}_X}^{-1} = \mathbb{L}_{\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle} \circ (- \otimes \mathcal{O}(3)[-4])$$

Key observation 1. $X \overset{i}{\hookrightarrow} \mathbb{P}^5$, for $F \in D^b(X)$ there exists an exact Δ:

$$i^* i_* F \to F \to F \otimes \mathcal{O}_X(-3)[2]$$

Assume $F \in \mathrm{Ku}_X$, apply S_{Ku}^{-1}.

$$\mathbb{L}_{\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle} \circ (i^* i_* F \otimes \mathcal{O}(3)[-4]) \to S_{\mathrm{Ku}}^{-1}(F) \to \underbrace{\mathbb{L}_{\langle \dots\rangle} \circ (F[-2])}_{\cong F[-2], \text{ as } F \in \mathrm{Ku}_X}$$

Key observation 2. $i^*i_*F \otimes \mathcal{O}(3) \in \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$, so the first term vanishes. (It's enough to show that $i_*F \otimes \mathcal{O}(3) \in \langle \mathcal{O}_{\mathbb{P}^5}, \mathcal{O}_{\mathbb{P}^5}(1), \mathcal{O}_{\mathbb{P}^5}(2)\rangle \iff i_*F \in \underbrace{\langle \mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1)\rangle}_{=\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle^{\perp}} \subset D^b(\mathbb{P}^5)$.)

$$R\operatorname{Hom}\begin{pmatrix} \mathcal{O} \\ \mathcal{O}(1), i_*F \\ \mathcal{O}(2) \end{pmatrix} = 0$$

$$R\operatorname{Hom}\begin{pmatrix} \mathcal{O} \\ i^*\mathcal{O}(1), F \\ \mathcal{O}(2) \end{pmatrix} = 0 \qquad \text{since } F \in \mathrm{Ku}_X$$

and the left-hand-sides on the last two equations are $\cong$.

Proof of Key observation 1 assuming F is a sheaf. Take $i^*i_*F \to F \to G$ and consider its pushforward

$$i_*i^*i_*F \to i_*F \to i_*G$$

Note that

$$\begin{aligned} i_*i^*i_*F \simeq i_*F \overset{L}{\otimes}_{\mathcal{O}_{\mathbb{P}^5}} &\cong \left[\underbrace{i_*F \otimes \mathcal{O}_{\mathbb{P}^5}(-x)}_{\deg -1} \xrightarrow{\vee} i_*F \otimes \mathcal{O}_{\mathbb{P}^5} \right] \\ &\simeq i_*F \oplus i_*F \otimes \mathcal{O}_{\mathbb{P}^5}(-x)[1] \\ &\xrightarrow[\text{1^{st} factor}]{\text{proj. to}} i_*F \\ &\implies i_*G \simeq i_*F \otimes \mathbb{P}_{\mathbb{P}^5(-x)[2]} \\ &\implies G \simeq F \otimes \mathcal{O}_X(-x)[2] \qquad i_* \text{ f.f.} \end{aligned}$$

$\square$