BRIDGELAND STABILITY: THE GENERAL THEORY

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Notes at github.com/danimalabares/cimpa-floripa

Abstract. Bridgeland stability is a powerful tool for extracting geometry from homological algebra. In particular, it gives a framework for studying moduli spaces of objects in a triangulated category, such as the derived category of an algebraic variety. The subject was born as a mathematical interpretation of work in string theory, but has since impacted many areas, including classical algebraic geometry, derived categories of coherent sheaves, enumerative geometry, homological mirror symmetry, and symplectic geometry. The goal of this course is to develop the foundations of Bridgeland stability, covering the following topics: 1) The theory of t-structures on triangulated categories, including examples via tilting. 2) The defniition of stability conditions and the stability manifold, as well as Bridgeland's deformation theorem. 3) Constructions of stability conditions. 4) Moduli spaces of stable objects.

Contents

Plan.

(1) Lecture 1: What is a stability condition.
(2) Lecture 2: Main constructions in dimensions 2 and 3.
(3) Lecture 3: Reaching the Geiseker chamber.
(4) Lecture 4: Some applications on structures.

1. Recall

Recall that for μ_H-stability we have the following property.

- (Schur's lemma.) If $A, B \in \mathrm{Coh}$ are μ_H semistable with $\mu_H(A) > \mu_H(B)$ then $\mathrm{Hom}(A, B) = 0$. Exercise.

- (Harder-Narasimhan filtrations.) $E \in \mathrm{Coh}(X)$ then is a filtration

$$\begin{tikzcd} 0 \arrow[r] & E_0 \arrow[r, "i_0"] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{k-1} \arrow[r, "i_k"] & E_k = E \arrow[dl] \\ & F_1 & & & F_k & \end{tikzcd}$$

with E_0 is the torsion subsheaf of E, F_j are μ_H-semistable with

$$\mu_H(F_1) > \mu_H(F_2) > \ldots > \mu_H(F_k)$$

Exercise: (1) $\implies$ (2).

These two results basically say that $\mathrm{Coh}(X)$ is generated by μ_H-semistable sheaves (via extensions) and torsion sheaves.

$$0 \to E_{j-1} \hookrightarrow E_j \to F_j \to_0$$

$E_0 := F_0$.

Can we do this on $D^b(X)$?

We look for a subcategory with a fixed slope to generate the whole category...

The following definition is intended to "impose" Schur's lemma on our objects:

Definition 1.1. A *slicing* $\mathcal{P}$ of $D^b(X)$ consists of full subcategories $\mathcal{P}(\phi)$ for every $\phi \in \mathbb{R}$ such that

(1) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$ /
(2) If $\phi_1 > \phi_2$ then $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$.
(3) (HN property.) For every $E \in D^b(X)$ there is a diagram (filtration)

$$\phi_1 > \phi_2 > \cdots > \phi_k$$

$$\begin{tikzcd} 0 = E_0 \arrow[r, "i_0"] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{k-1} \arrow[r, "i_k"] & E_k = E \arrow[dl] \\ F_1 & & & F_k & \end{tikzcd}$$

where $F_j = \mathrm{Cone}(i_j) \in \mathcal{P}(d_j)$. **Exercise:** this is unique because of (2).
(4) $F \in Q(\phi)$ is called *semistable* of phase ϕ. The simple objects (those which have no subobjects) in $\mathcal{P}(\phi)$ are called *stable of phase ϕ*.
(5) $\phi_1 := \phi^+(E)$, $\phi_k := \phi^-(E)$.

The following definitions is related to the former, and to Schur's lemma:

Definition 1.2. The *heart of a bounded t-structure* on $D^b(X)$ is a full additive subcategory $\mathcal{A} \subset D^b(X)$ such that

(1) For $i > j$, $\mathrm{Hom}(A[i], B[j]) = 0$ for $A, B \in \mathcal{A}$.
(2) For every $E \in D^b(X)$ there are integers $k_1 > k_2 > \ldots > k_m$ and a filtration

(1.2.1)

$$\begin{tikzcd} 0 = E_0 \arrow[r, "i_0"] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{k-1} \arrow[r, "i_k"] & E_k = E \arrow[dl] \\ A_1[k_1] & & & A_k[k_m] & \end{tikzcd}$$

exact triangles, with $A_j \in \mathcal{A}$.

Exercise 1.3. $\mathcal{A}$ is abelian.

Exercise 1.4. If $\mathcal{P}$ is a slicing of $D^b(X)$ then $\mathcal{P}(0,]$ (the extension closure of $\{\mathcal{P}(\phi) : 0 < \phi \leq 1\}$) is the heart of a bounded t-structure.

Example 1.5. $\operatorname{Coh}(X) \subset D^b(X)$ is the heart of a bounded structure.

$$\begin{tikzcd}
\sigma_{\leq i}(E^\bullet): \arrow[d] & \cdots \arrow[r] & E^{j-2} \arrow[r] \arrow[d, "="'] & E^{j-1} \arrow[r, "\delta"] \arrow[d, "="'] & \operatorname{Ker}\delta_j \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\sigma_{\leq j+1}(E^\bullet): \arrow[d] & \cdots \arrow[r] & E^{j-2} \arrow[r] \arrow[d] & E^{j-1} \arrow[r, "d_j"] \arrow[d] & E^j \arrow[r] \arrow[d] & \operatorname{Ker}(\delta_{j+1}) \arrow[d] & \\
{} & & 0 & 0 & E_2k/\operatorname{Ker}\delta_j & \operatorname{Ker}\delta_{j+1} &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \sigma_{\leq j}(E^\bullet) \arrow[rr] & & \sigma_{\leq j+1}(E^\bullet) \arrow[r] \arrow[dl] & \cdots \\
& & H^{i+j}(E^\bullet)[-j-1] & &
\end{tikzcd}$$

Definition 1.6. The objects A_j appearing in Eq. 1.2.1 are called the *cohomologies* of E with respect to the t-structure $\mathcal{A}$, $A_j := \mathcal{H}_{\mathcal{A}}^{-k_j}(E)$.

2. Constructing new hearts

How can we construct other t-structures? Suppose that $\mathcal{A}$ is an abelian category. A *torsion pair* on $\mathcal{A}$ consists of two full subcategories $(\mathcal{T}, \mathcal{F})$ such that

(1) $\operatorname{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
(2) For every $A \in \mathcal{A}$ there is an exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Again: with the first property we may prove that the exact sequence in the second property is unique (Exercise).

Exercise 2.1. $\mathcal{A} = \operatorname{Coh}(X)$, $\mathcal{T} =$ torsion sheaves, $\mathcal{F} =$ torsion free sheaves.

Proposition 2.2. *If $\mathcal{A}$ is a hear of a bounded t-structure on $D^b(X)$ and $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\mathcal{A}$, then*

$$\begin{aligned}\mathcal{A}^\sharp &= \langle \mathcal{F}[1], \mathcal{T} \rangle \\ &= \left\{ E \in D^b(X) : \substack{\mathcal{H}^j_{\mathcal{A}}(E)=0 \\ for\ j \neq -1, 0 \\ \mathcal{H}^{-1}_{\mathcal{A}}(E)=\mathcal{F} \\ \mathcal{H}^{\bullet}_{\mathcal{A}}(E) in \mathcal{T}} \right\}\end{aligned}$$

which is called the tilted heart category.

Note. We can think of those as complexes $E^\bullet$ that fit in an exact triangle

$$0 \longrightarrow F[1] \longrightarrow E^\bullet \longrightarrow T \longrightarrow 0$$

(which a prori is just an exact triangle, but it is an exact sequence in our case).

The first proof of this fact was done by Bridgeland [confirm]:

Idea of proof. [Picture.] □

We show an example to understand how the proof works.

Example 2.3. Suppose that the $\mathcal{A}$-filtration of an object E is of the form

$$\begin{tikzcd} 0 \arrow[r] & F_1[1] \arrow[r] & E \arrow[r] & F_2 \arrow[r] & 0 \end{tikzcd}$$

Let's comute the cohomologies:

$$\begin{tikzcd} 0 = E_0 \arrow[rr, "i_0"] & & E_1 \arrow[dl] \arrow[rr, "i_2"] & & E_2 = E \arrow[dl] \\ & F_1[1] & & F_2 & \end{tikzcd}$$

$F_1, F_2 \in \mathcal{A}$. So, the cohomologies, by definition are $\mathcal{H}^{-1}_{\mathcal{A}}(E) = F_1$, $\mathcal{H}^0_{\mathcal{A}}(E) = F_2$. Now consider the following diagram:

$$\begin{tikzcd} & F[1] \arrow[d] & & T \arrow[d] & \\ 0 \arrow[r] & F_1[1] \arrow[r] \arrow[d] & E \arrow[r] \arrow[dr, "f"'] & F_2 \arrow[r] \arrow[d] & 0 \\ & F[1] & & F & \end{tikzcd}$$

so wer consider

$$\begin{tikzcd} & & F[1] \arrow[dl] \arrow[d] & & \\ 0 \arrow[r] & C = \mathrm{Cone}(f)[-1] \arrow[r] & E \arrow[r, "f"] & F \arrow[r] & 0 \end{tikzcd}$$

Recall that in a triangulated category,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X[1] \arrow[r] \arrow[d, dotted] & X[1] \arrow[d] \\ Y \arrow[r] & Z \arrow[r] & X^1 \arrow[r] & Y[1] \end{tikzcd}$$

where the leftmost square commutes.

$$\begin{tikzcd} 0 \arrow[rr] & & F[1] \arrow[rr, "g"] \arrow[dl] & & C \arrow[rr] \arrow[dl] & & E \arrow[dl] \\ & \underbrace{F[1]}_{\mathcal{A}^\sharp[1]} & & \underbrace{\mathrm{Cone}(g)}_{\mathcal{A}^\sharp} & & \underbrace{F}_{\mathcal{A}^\sharp[-1]} & \end{tikzcd}$$

3. Stability conditions

Definition 3.1 (Stability condition, definition 1). A *stability condition* is a pair $\sigma = (\mathcal{P}, Z)$ where $\mathcal{P}$ is a slicing of $D^b(X)$, and a morphism Z called the *central change* of the form

$$\begin{tikzcd} Z : K_0(D^b(X)) \arrow[rr] \arrow[dr, "\nu"'] & & \mathbb{C} \\ & \Gamma \arrow[ur, "z"'] & \end{tikzcd}$$

where Γ is a finite rank lattice, satisfying that

(1) $Z(E) \in \mathcal{E}_{>0}e^{i\pi\phi}$ for any nonzero object $E \in \mathcal{P}(\phi)$.
(2) (Support property.) For a fixed norm $\|\cdot\|$ on $\Gamma \otimes \mathbb{R}$ we have

$$C_\sigma = \inf\left\{ \frac{|Z(E)|}{\|\nu(E)\|} : {}^{E \text{ non-zero}}_{E\in\mathcal{P}(\phi),\phi\in\mathbb{R}} \right\} > 0$$

Definition 3.2 (Stability condition, definition 2). A *stability condition* is a pair $\sigma = (\mathcal{A}, Z)$, where $\mathcal{A}$ is the heart of a bounded t-structure on $D^b(X)$, and a morphism Z of the form

$$\begin{tikzcd} Z : K_0(D^b(X)) \arrow[rr] \arrow[dr, "\nu"'] & & \mathbb{C} \\ & \Gamma \arrow[ur, "Z"'] & \end{tikzcd}$$

where Γ is a finite rank lattice, satisfying that

(1) $Z(E) \in \mathbb{R}_{>0}e^{i\pi\phi}$ for $E \in \mathcal{A}$,

$$\left[\frac{\mathrm{Im}Z(E) = 0}{\mathrm{Im}Z(E) \geq 0} \implies \mathrm{Re}Z(E) < 0\right]$$

(2) If $\mu_Z = -\frac{\mathrm{Re}Z}{\mathrm{Im}Z}$ then objects in $\mathcal{A}$ have the HN property w.r.t. μ_Z and we have the support property.

Exercise 3.3. Prove that the two definitions of stability are equivalent.

- Prove Schur's Lemma for μ_Z-stability.
- $\mathcal{P}(\phi) = \mu_Z$-semistable objects of phase ϕ, i.e., $Z(E) \in \mathbb{R}_{>0}e^{i\pi\phi}$ for $\phi \in (0, 1]$ and set $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
- Take $\mathcal{A}(0, 1]$ =generated by $\mathcal{P}(\phi)$ with $0 < \phi \leq 1$.

Exercise 3.4. If $\dim X = 1$, $\mathcal{A} = \mathrm{Coh}(X)$, $Z = -\deg + i\mathrm{rank}$, $v = (\deg, \mathrm{rank})$.

Rephrasing of support property. There exists a quadratic form Q on $\Gamma \otimes \mathbb{R}$ such that $Q(v(E)) \geq 0$ for E μ_Z-semistable, If $Z(V) = 0$ then $Q(V) < 0$ for nonzero V. This is like imposing Bogomolov inequality. (See moduli-spaces-of-sheaves.)

Let $\mathrm{Stab}^\Gamma(X)$ be the set of stability conditions with central charge factoring through $v : K_0(\mathcal{A}) \to \Gamma$.

Theorem 3.5 (Bridgeland). *$Stab^\Gamma(X)$ is a complex manifold. Moreover,*

$$\begin{aligned} Stab^\Gamma(X) &\longrightarrow \mathrm{Hom}(\Gamma, \mathbb{C}) \\ \sigma = (\mathcal{A}, Z) &\longmapsto Z \end{aligned}$$

is a local homomorphism.

This is known as Bridgeland deformation result. In fact, it is a consequence of the existence of the quadratic form [reference?].

For every object in $\mathcal{A}$ where $\sigma = (\mathcal{A}, Z)$ is a stability condition there is a HN filtration

$$\begin{tikzcd} 0 \arrow[r] & E_0 \arrow[r, "i_0"] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[r, "i_m"] & E_m = E \arrow[dl] \\ & F_1 & & & F_m & \end{tikzcd}$$

$F_1 \in \mathcal{P}(\phi_1)$, $\phi_1 := \phi^+_\sigma(E)$, $F_m \in \mathcal{P}(\phi_m)$, $\phi_m := \phi^-_\sigma(E)$.

The topology on $\mathrm{Stab}^\Gamma(X)$ is the coarsest topology that makes the functions

$$\sigma \mapsto \phi^+_\sigma, \qquad \sigma \mapsto \phi^-_\sigma, \qquad \sigma \mapsto Z$$

[continuous].

Now let $\dim X = 2$. Let $H, B \in \mathrm{NS}(X)_\mathbb{R}$, H ample.

$$\mathrm{ch}^B(E) = e^{-B}\mathrm{ch}(E) = \text{“}\mathrm{ch}(E \otimes (-B))\text{”}$$

When computing the Chern characters we obtain

$$\mathrm{ch}^B_0 = \mathrm{ch}_0, \qquad \mathrm{ch}^B_1 = \mathrm{ch}_1 - B\mathrm{ch}_0$$
$$\mathrm{ch}^B_2 = \mathrm{ch}_2 - B\mathrm{ch}_1 + \frac{B^2}{2}\mathrm{ch}_0$$

B-Twisted Mumford slope.

$$\mu_{B,H} = \begin{cases} \frac{H\mathrm{ch}^B_1}{H^2\mathrm{ch}_0} & \text{if } \mathrm{ch}_0 \neq 0 \\ +\infty & \text{if } \mathrm{ch}_0 = 0 \end{cases}$$

$$\begin{aligned} \mu_{B,H}(E) &= \frac{H(\mathrm{ch}_1(E) - B\mathrm{ch}_0(E)}{H^2\mathrm{ch}_0(E)} \\ &= \mu_H(E) - BH \end{aligned}$$

$$\mathcal{T}_{B,H} = \{E \in \mathrm{Coh}(X) : \mu_H\text{-HN factors have } \mu_{B,H} > 0\}$$
$$\mathcal{F}_{B,H} = \{E \in \mathrm{Coh}(X) : \mu_H\text{-HN factors have } \mu_{B,H} \leq 0\}$$

- $\mathrm{Hom}(T, F) = 0$ for $T \in \mathcal{T}_{B,H}$, $F \in \mathcal{F}_{B,H}$ (Schur's lemma).
- If $\mathcal{E} \in \mathrm{Coh}(X)$, take its HN-filtration

$$\begin{tikzcd} 0 \arrow[r] & E_0 \arrow[r, "i_0"] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[r, "i_m"] & E_m = \mathcal{E} \arrow[dl] \\ & F_1 & & & F_m & \end{tikzcd}$$

$$\mu_{B,H}(E_0) > \mu_{B,H}(F_1) > \ldots > \mu_{B,H}(F_m)$$

Let j such that $\mu_{B,H}(F_j) > 0$ and $\mu_{B,H} \leq 0$.

$$0 \to \underbrace{E_j}_{\mathcal{T}_{B,H}} \to \mathcal{E} \to \underbrace{\mathcal{E}/E_j}_{\in \mathcal{F}_{B,H}} \to 0$$

Set a tilted heart $\mathrm{Coh}^{B,H}(X) = \langle \mathcal{F}_{B,H}[1], \mathcal{T}_{B,H} \rangle$.

Generators:

- T torsion sheaves $\to H\mathrm{ch}_1^B(T) \geq 0$.
- $\mathcal{E}$ $\mu_{B,H}$-semistable sheaves with $\mathfrak{z}_{B,H} > 0 \to H\mathrm{ch}_1^B(\mathcal{E}) > 0$.
- $F[1]$, F is $\mu_{B,H}$-semistable with $\mathfrak{z}_{B,H} \leq 0$.

Then

$$H\mathrm{ch}_1^B(F) \leq 0, \qquad H\mathrm{ch}_1^B(F[1]) \geq 0$$

since shifting by 1 changes the sign of the Chern character. [Comment about imaginary part of stability condition.]

Is there a central charge Z such that $\sigma_{B,H} = (\mathrm{Coh}^{B.H}(X), Z)$ is a stability condition?

$$Z_{B,H} = \mathrm{Re} Z_{B,H} + iH\mathrm{ch}_1^B$$

We need: $H\mathrm{ch}_1^B(E) = 0$ for some $E \in \mathrm{Coh}^{B,H}(X)$ then $\mathrm{Re}Z_{B,H}(E) < 0$.

Ingredients:

(1) (Hodge index theorem.) $(DH)^2 \geq D^2H^2$.
(2) (Bogomolov inequality.) If E is $\mu_{B,H}$-semistable then

$$\begin{aligned}\Delta(E) &= [\mathrm{ch}_1(E)]^2 - 2\mathrm{ch}_0(E) - \mathrm{ch}_2(E) \geq 0 \\ &= \mathrm{ch}_1^B(E)^2 - 2\mathrm{ch}_0^B\mathrm{ch}_2^B(E) \geq 0\end{aligned}$$

We obtain the *BH-discriminant*

$$\Delta_{B,H}(E) := (H\mathrm{ch}_1^B(E))^2 - 2\mathrm{ch}_0(E)H^2\mathrm{ch}_2^B(C) \geq H^2\Delta(E) \geq 0$$

If E is μ_H-semistable and $\mathrm{Im}Z_{B,H}(E) = 0$, $H\mathrm{ch}_1^B(E) = 0 \implies \mathrm{ch}_2^B(E) \leq 0$, $\mathrm{ch}_2^B(E) - a\mathrm{ch}_0(E) < 0$ for any $a > 0$.

Theorem 3.6 (Bridgeland, Areni-Bestum)**.**

$$Z_{B,H} = -(ch_2^B - a\,ch_0) + iH\,ch_1^B$$

is the central charge of a stability condition on $Coh^{B,H}(X)$ for any $a > 0$.

Theorem 3.7. $\Delta_{B,H} \geq 0$ *for any $v_{B,H,a}$-semistable object, where* $v_{B,H,a} = \frac{ch_2^B - a\,ch_0}{H\,ch_1^B}$.

For $n := \dim X \geq 2$,

$$\mu_{B,H} = \begin{cases} \frac{H^{n-1}\mathrm{ch}_1^B}{\mathrm{ch}_0} & \text{if } \mathrm{ch}_0 \neq 0 \\ +\infty & \text{if } \mathrm{ch}_0 = 0 \end{cases}$$

$\mathrm{Coh}^{B,H}(X)$, $\Delta_{B,H}(E) = \dim n$ analogy. $Z_{B,H,a}$ does not satisfy the positivity property of a central charge since

$$Z_{B,H,a} = -(H^{n-2}\mathrm{ch}_2^B - a\mathrm{ch}_0 + iH^{n-1}\mathrm{ch}_1^B)$$

[skyscraper sheaf argument]

Exercise 3.8. Let $E \in \mathrm{Coh}^{B,H}(X)$. If $\mathrm{ch}_0(E) > 0$ and E is $\nu_{B,H,a}$-semistable for all $a \gg 0$, then E is a sheaf. Moreover, E is $\mu_{B,H}$-semistable.

Proposition 3.9 (Lo,—)**.** *If $a > \mu_{B,H}(E)\Delta_{B,H}(E)$, then if E is twisted semistable sheaf then E is $\nu_{B,H,a}$-semistable.*

If $\dim X = 3$ consider the subcategories of $\mathrm{Coh}^{B,H}(X)$

$$T_{B,H,a} = \{E \in \mathrm{Coh}^{B,H}(X) : \text{HN factors with } \nu_{B,H,a} > 0\}$$

$$\mathcal{F}_{B,H,a} = \{E \in \mathrm{Coh}^{B,H}(X) : \text{HN factors with } \nu_{B,H,a} \leq 0\}$$

$$\mathcal{A} = \langle \mathcal{F}_{B,H,a}[1], \mathcal{T}_{B,H,a} \rangle$$

supports a stability condition.

4. GBG inequality conjecture

Here's the construction so far:

$$Coh(X) \xrightarrow[\text{tilt}]{\text{first}} Coh^{B,H}(Z) \xrightarrow[\text{tilt}]{\text{second}} \mathcal{A}^{B,H,\alpha}$$

$$\underbrace{\mu_{B,H}}_{\text{Mumford stability}} \qquad \underbrace{\nu_{B,H,\alpha}}_{\text{Tilt}} \qquad \underbrace{\lambda_{B,H,\alpha,s}}_{\text{Bridgeland stability when possible}}$$

$$\Delta \geq 0 \qquad \Delta_{B,H} \geq 0 \qquad \text{Conjecture generalized BG inequality}$$

And

$$Z_{\beta,H,\alpha,s} = -\left(ch_3^B - \left(s + \frac{1}{6}\right)\alpha^2 H^2 ch_1^B)\right) \qquad s > 0$$
$$+ i\left(Hch_2^B - \frac{\alpha^2}{2}H^3 ch_0\right)$$

GBG-inequality. If E is $\nu_{B,H,\alpha}$-semistable,

$$\begin{aligned} &Hch_2^B(E) - \frac{\alpha^2}{2}H^3 ch_0(E) = 0 \\ \Longrightarrow\ &ch_3^B - \frac{1}{6}\alpha^2 H^2 ch_1^B(E) \leq 0 \end{aligned} \tag{4.0.1}$$

Bayer-Maci-Toda. If Eq. 4.0.1 is true then $(\mathcal{A}^{B,H,\alpha}, Z_{B,H,\alpha,s})$ is a strability condition.

5. Remarks on $\nu_{B,H,\alpha}$-stability

Keep in mind that

$$\nu_{B,H,\alpha} = \frac{Hch_2^B - \frac{\alpha^2}{2}ch_0 H^3}{H^2 ch_1^B}$$

and

$$L \in Coh^{B,H}(X) \qquad (L[1])$$

as long as

$$H^2 ch_1^B(L) > 0 \qquad (H^2 ch_1^B(L[1]) \leq 0.$$

If E is $\nu_{B,H,\alpha}$-semistable for $\alpha \gg 0$ then $E \in Coh(X)$ and is semistable with respect to the stability given by

$$q(t) = t\frac{H^2 ch_1}{ch_0} + \frac{Hch_2}{ch_0}, \qquad t \gg 0$$

and if E is $q(t)$-semistable then E is $\nu_{B,H,\alpha}$-semistable for $\frac{\alpha^2}{2}H^2 > \mu_{B,H}(E)\Delta_{B,H}(E)$.

If X has Picard rank $\rho(X) = 1$ then

$$\Delta_{B,H} = H^3\Delta$$

If L is a line bundle $\Delta_{B,H}(L) = H^3\Delta(L) = 0$ (as $L[1]$), then L is $\nu_{B,H,\alpha}$-semistable as long as $L \in Coh^{B,H}(X)$ (as $L[1]$)

6. Search for counterexamples

Idea. Look for counterexamples with $ch_0 = 0$. When does $\mathcal{O}_D$ for $D \geq 0$ divisor on X (3-fold) produces a counterexample?

Fix H ample, $B = \beta H$. We need:

(1) $\mathcal{O}_D$ is $\nu_{\beta,\alpha}$-semistable.
(2) $\nu_{\beta,\alpha}(\mathcal{O}_D) = 0$.
(3) $ch_3^B(\mathcal{O}_D) - \frac{\alpha^2}{6}H^2 ch_1^B(\mathcal{O}_D) > 0$.

Lemma 6.1. *If $D \geq 0$ satisfies*

$$D^3 > \frac{(DH^2)^3}{4(H^3)^2} + \frac{3}{4}\frac{(D^3H)^2}{DH^2}$$

then $\mathcal{O}_D$ is a counterexample for the GBG inequality 4.0.1.

Example 6.2. Let Y be any smooth projective 3-fold, $X = Bl_pY$, D the exceptional divisor, $L = \pi^*A$ with A ample, $H = mL - D$ ample on X.

[Slides]

7. Concrete constructions

How is a wall equation? Suppose that $\dim X = 2$, or even better that $X = \mathbb{P}^2$. Also let $\Gamma = \supset \oplus \supset \oplus \frac{1}{2}\mathbb{Z}$ and $\nu = (ch_0, ch_1, ch_2) := (r, c, d)$.

$$\nu_{\beta,\alpha} = \frac{ch_2^\beta - \frac{\alpha^2}{2}ch_0 H^2}{Hch_1^\beta}$$
$$= \frac{ch_2 - \beta ch_1 + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)ch_0}{ch_1 - \beta ch_0}$$

A wall in the (β, α)-plane is produced by an exact sequence in $Coh^\beta(\mathbb{P}^2)$

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

with $\nu_{\beta,\alpha}(A) = \nu_{\beta,\alpha}(E)$ and $ch(E) = V$.

First: numerics: $w = (r', c', d')$ and solve $\nu_{\beta,\alpha}(w) = \nu_{\beta,\alpha}(v)$.

$$\beta^2 - 2\beta\left(\frac{rd' - r'd}{rc' - r'c}\right) + \alpha^2 = \frac{dc' - d'c}{rc' - r'c}$$

$$c = \frac{rd' - r'd}{rc' - r'c}, \qquad R = \sqrt{c^2 + \frac{dc' - d'c}{rc' - r'c}}$$

[Picture.] Look for wall for 1-dimensional sheaves ($r = 0$, $c > 0$). $c = \frac{d}{c}$ same center.

Remark 7.1. Walls either coincide or they don't intersect. [Picture.]

Lemma 7.2 (Bertan's lemma)**.** *If an object A destabilizes E at some point then it destabilizes E at every point of the corresponding wall.*

Take $\beta = \frac{d}{c}$. If $A \hookrightarrow E \xrightarrow{surj.} B$ is short exact in $Coh^{\beta}(\mathbb{P}^2)$.

- If $ch(E) = (0, c, d)$ and E is a sheaf then A is a sheaf, just not a subsheaf.

$$0 \longrightarrow \underbrace{\mathcal{H}^{-1}A}_{\implies 0} \longrightarrow \underbrace{\mathcal{H}^{-1}E}_{=0} \longrightarrow \mathcal{H}^{-1}B \longrightarrow \mathcal{H}^0 A \longrightarrow \mathcal{H}^0 E \longrightarrow \mathcal{H}^0 B \longrightarrow 0$$

$\implies a = \mathcal{H}^0 A$ sheaf.

(1) A produces a semi-circular wall only when $ch_0(A) > 0$.
(2) $0 \leq ch_1^{\beta}(A) \leq ch_1^{\beta}(E) = c$. This says that there are finitely many possibilities for $c_1^{\beta}(A)$.
(3) At the wall $W_{A,E}$ we have A is $\nu_{\beta,\alpha}$-semistable.

$$\Delta_{\beta,\alpha}(A) \geq 0$$
$$ch_1^{\beta}(A)^2 - 2ch_0(A)ch_2^{\beta}(A) \geq 0$$
$$ch_1^{\beta}(A)^2 \geq 2ch_0(A)ch_2^{\beta}(A)$$

(4) Wall equation

$$\begin{aligned}\nu_{\frac{1}{c}\alpha}(A) &= \nu_{\frac{1}{c},\alpha}(E)\\ &= \frac{ch_2(E) - \beta ch_1(E)}{ch_1(E)}\\ &= \frac{d - \frac{d}{c}c}{c} = 0\end{aligned}$$

$$ch_2^{\beta}(A) - \frac{\alpha^2}{2}ch_0^{\beta}(A) = 0$$
$$ch_2^{\beta}(A) = \frac{\alpha^2}{2}ch_2^{\beta}(A) \geq 0$$

$\implies$ **walls are finite**.

Moduli of $\nu_{\beta,\alpha}$-semistable objects are projective for $X = \mathbb{P}^2$. [Picture of walls as nested circles in a plane.] $\nu_{\beta\alpha}$-semistability $\implies$ King's Θ-semistability.

Example 7.3. $v = (0, 4, -4)$, $M_H(v) \leftarrow$ sheaves can be found in one of the following short exact sequences in $Coh^{-1}(\mathbb{P}^2)$.

[Duzét-Muican]

Codimension 3:

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}(-2) \longrightarrow 0$$

Codimension 1:

$$0 \longrightarrow \mathcal{I}_P(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_q^{\vee}(-3)[q] \longrightarrow 0$$

Codimension 0:

$$0 \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}(-2)[1] \longrightarrow 0$$

The first wall exchanges

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}(-3)[1] \longrightarrow 0$$

for (upon wall crossing)

$$0 \longrightarrow \mathcal{O}(-3)[1] \longrightarrow E^* \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

$$\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-3)[1]) = \mathrm{Ext}^2(\mathcal{O}(1), \mathcal{O}(-3)) \cong \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))^\vee$$

Theorem 7.4 (Li-Zhao). *IF v is primitive and σ, σ' are generic geometric with respect to v then $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_{\sigma'}(v)$ are birational.*

We finish this minicourse with another application of Schur's lemma:

Theorem 7.5 (Koseki). *Let X be a smooth projective surface over $k = \overline{k}$ with char$(k) > 0$, then there is a constant $C_{[X]} \geq 0$ and dependending only on the birational equivalence clas of X such that*

$$\tilde{\Delta}(E) = \Delta(E) + C_{[X]} ch_0^2(E) \geq 0$$

for any μ_H-semistable sheaf.

Morever

(1) If X is minimal and of general type,

$$C_{[X]} = 2 + 5K_X^2 - \chi(\mathcal{O}_X)$$

(2) If $K(X) = 1$ (K is the Kodaira dimension) and X is quasi-elliptic then

$$C_{[X]} = 2 - \chi(\mathcal{O}_X), \qquad \text{[char 2+3]}$$

(3) $C_{[X]} = 0$

Corollary.

$$Z_{B,H,\alpha}^{C_{[X]}} = -\left(\cong_2^B - \left(\frac{C_{[X]}}{2H^2} + \frac{\alpha^2}{2}\right) ch_0 H^2\right) + i ch_1^B(X)$$

is a stability condition on $Coh^{B,H}(X)$.

Remark 7.6.

$$\tilde{\Delta}_{B,H} = \Delta_{B,H} + ch_0^2 C_{[X]} \leq 2$$

on any $\nu_{B,H,\alpha}^{C_{[X]}}$-semistable object.

Lemma 7.7 (Low key lemma).

- *For $\alpha \gg 0$, ν-semistable are the Gieseker-semistable sheaves.*
- *IF $\alpha^2 H^2 > \mu_{B,H}(E)\tilde{\Delta}_{B,H}(E)$ and if E is Gieseker-semistable $\Longrightarrow$ E is Bridgeland semistable.*

Theorem 7.8. $H^2 > 6C_{[X]}$ *then* $H^1(H \otimes K_X) = 0$*. (If char$(k) = 0$ we can take $C_{[X]} = 0$.)*

Proof idea. Find a stability condition $\sigma = (Z, s)$ such that

(1) H, $\mathcal{O}[1] \in \mathcal{A}$ are σ-semistable.

(2) $\mu_Z(H) > \mu_Z(\mathcal{O}[1])$

$$\implies H^1(H \otimes K_X) \cong \operatorname{Ext}^1(H, \mathcal{O})^\vee = \operatorname{Hom}(H_1 \mathcal{O}[1])$$

$\square$