DERIVED CATEGORIES OF SHEAVES

Minicourse by Lyalya Guseva, CIMPA school Florianópolis 2025.
Notes at github.com/danimalabares/cimpa-floripa

Abstract. Derived category of coherent sheaves is a convenient environment of investigating algebraic geometry of a variety. It provides useful techniques and gives a perspective point of view. On the level of derived categories one can see unexpected connections which are not visible on the classical level. I will try to give an introduction into the techniques of derived categories and semiorthogonal decomposition and (hopefully) will present many examples of applications of derived categories in algebraic geometry.

Contents

Upshot. The derived category of an abelian category is the localization of its homotopy category with respect to the class of quasi-isomorphisms.

- The objects of $D(\mathcal{A})$ are complexes in $\mathcal{A}$ and the arrows are equivalence classes of roofs.
- $D(\mathcal{A})$ is additive but not abelian unless $\mathcal{A}$ is semisimple.
- $D(\mathrm{Vect}_k)$ = graded vector spaces.
- For all $A, B \in \mathcal{A}$,

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B[i]) = \begin{cases} 0 & i < 0 \\ \mathrm{Hom}_{\mathcal{A}}(A, B) & i = 0 \\ \mathrm{Ext}^i(A, B) & i > 0 \end{cases}$$

- A sequence

$$A \xrightarrow{f} B \xrightarrow{b} \mathrm{Cone}(f) \xrightarrow{a} A[I] \longrightarrow 0$$

gives long exact sequences $\mathrm{Hom}^{\bullet}_{D(\mathcal{A})}(C, \cdot)$ and $\mathrm{Hom}^{\bullet}_{D(\mathcal{A})}(\cdot, C)$, which are the distinguished triangles.

1. Basic concepts

Definition 1.1. An *abelian category* $\mathcal{A}$ is a category such that

(1) $\forall X, Y \in \mathcal{A}$, Hom$\mathcal{A}(X,Y)$ is an abelian group such that $\mathrm{Hom}_{\mathcal{A}}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Y)$ is bilinear.
(2) There is a zero object $0_{\mathcal{A}}$ such that for every $A \in \mathcal{A}$, $\mathrm{Hom}(0_{\mathcal{A}}, A) = \mathrm{Hom}(A, 0) = 0$.
(3) There exist finite products.
(4) For every $f \in \mathrm{Hom}(A,B)$ there exist kernel and cokernel such that

$$\operatorname{Ker} f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{Coker} f$$

Definition 1.2. $\mathcal{A}$ satisfying 1-3 is called *additive category.*

Examples: abelian groups, R-modules, $\mathrm{Coh}(X)$. Non-example: vector bundles over X, it is an additive category but not abelian.

Now consider the category $\mathrm{Com}(\mathcal{A})$, the category of complexes in $\mathcal{A}$, with quasi-isomorphisms:

Definition 1.3. $f \in \mathrm{Hom}_{\mathrm{Com}(\mathcal{A})}(A', B')$ is a *quasi-isomorphism* if it induces isomorphisms $H^i(f) : H^i(A^\bullet) \cong H^i(B^\bullet)$ for all i.

Definition 1.4. Let $\mathcal{A}$ be an abelian category. The *derived category* of $\mathcal{A}$ is the localization $\mathrm{Com}(\mathcal{A})[QIS]^{-1}$, where QIS is the class of quasi-isomorphisms.

But what is localization? Given a class of morphisms S inside the class of morphisms of a given category, the localization will allow us to invert these morphisms.

Definition 1.5. Let $\mathcal{C}$ be a category, S a class of morphisms in $\mathcal{C}$ and $\mathrm{Iso}(\mathcal{C})$ the class of all isomorphisms of $\mathcal{C}$. The *localization* of $\mathcal{C}$ with respect to S is the category $\mathcal{C}[S]^{-1}$ with an isomorphism

$$Q : \mathcal{C} \to \mathcal{C}[S]^{i-\mathcal{A}}$$

that satisfy

(1) $Q(S) \subset \mathrm{Iso}(\mathcal{C}[S]^{-1})$
(2) (Universality.) For any $F : \mathcal{C} \to \tilde{\mathcal{C}}$ and $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ there exists a unique $G : \mathcal{C}[S^{-1}] \to \tilde{\mathcal{C}}$ making the diagram commute, i.e. $F = GQ$.

Exercise 1.6. Show that if $\mathcal{C}[S^{-1}]$ exists, it must be unique up to isomorphism.

Definition 1.7. If instead of $\mathrm{Com}(\mathcal{A})$ we consider

$$\mathrm{Com}^b(\mathcal{A}) := \{A^\bullet \in \mathrm{Com}(\mathcal{A}) : A^i = 0 \forall |i| \gg 0\}$$

Then we obtain the *bounded derived category* $D^b(\mathcal{A})$.

2. Motivation

Let X be a smooth projective variety over $\mathbb{C}$. Then we can associate $X \rightsquigarrow \mathrm{Coh}(X)$.

Theorem 2.1 (Gabriel)**.** *Let X and Y be smooth projective varieties. If $Coh(X) \cong Coh(Y)$ then $X \cong Y$.*

This means we may reconstruct X from its coherent sheaf category. But it is not such an interesting invariant. It's more interesting the derived category. So the motivation for studying derived categories are:

(1) (Beilinson 1979.) $\mathrm{Coh}(\mathbb{P}^n) \hookrightarrow D^b(\mathbb{P}^n)$ gives good information about coherent sheaves on $\mathbb{P}^n$.
(2) (Mukai.) If $\mathcal{A}$ is an abelian variety and $\hat{\mathcal{A}}$ is the dual abelian variety, $D^b(\mathcal{A}) \cong D^b(\hat{\mathcal{A}})$.
(3) (Bondal, Orlov, 1990's.) Let X be a smooth projective variety with an ample or anti-ample canonical bundle ω_X. Suppose that $D^b(X) \cong D^b(Y)$ for a smooth projective variety Y. Then $Y \cong X$.

Remark 2.2. If you consider not only the bounded derived category $D^b(X)$ but consider it with the tensor product $D^b \times D^b(X) \xrightarrow{\otimes} D^b(X)$ allow to reconstruct X.

I hope you are now convinced this is interesting.

3. Explicit description of localization under some assumption

If S satisfies some properties then $\mathcal{C}[S^{-1}]$ can be explicitly described. The conditions are:

(1) All identities should belong to S and if any two morphisms from f, g and $f \circ g$ (whenever composition is defined) belong to S, then all three belong to S.
(2) For any diagram

$$\begin{tikzcd} X \arrow[d, "f"] & W \arrow[l, dotted, "\tilde{s}\in S"'] \arrow[d, dotted] \\ Z & Y \arrow[l, "s\in S"] \end{tikzcd}$$

can be completed to a commutative square.
(3) If for $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $u : B \to U$ such athat $u \circ f = u \circ g$ there exists $t : C \to A$ such that $ft = gt$, that is,

$$C \xrightarrow{t} A \xrightarrow{f,g} B \xrightarrow{u} U$$

The quasi-isomorphisms don't satisfy condition 2.

Proposition 3.1. *Let $\mathcal{C}$ be a category and $S \subset Mor(\mathcal{C})$ satisfy right Ore conditions. Then $\mathcal{C}[S^{-1}]$ can be described as follows:*

- *$Ob(\mathcal{C}[S^{-1}] = Ob(\mathcal{C})$.*
- *$Mor_{\mathcal{C}[S^{-1}]}(A, B) = \{$equivalence classes of roofs$\}$, where a roof is*

$$\begin{tikzcd} & C \arrow[dl, "s\in S"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

where $fs^{-1} \sim gt^{-1}$ if there exists a commutative diagram

$$\begin{tikzcd}
& & \tilde{W} \arrow[dl, "a \in S"'] \arrow[dr, "b \in S"] & & \\
& C \arrow[dl, "s \in S"'] \arrow[rr, "f"] & & \tilde{C} \arrow[dr, "t"] & \\
A & & & & B
\end{tikzcd}$$

and composition is defined as follows:

$$\begin{tikzcd}
& & \tilde{W} \arrow[dl, "a \in S"'] \arrow[dr, "b \in S"] & & \\
& Z \arrow[dl, "s \in S"'] \arrow[dr, "f"] & & \tilde{Z} \arrow[dl, "t"'] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}$$

where

$$(fs^{-1}) \circ (gt^{-1}) = (sa^{-1}) \circ (gb)$$

$\mathcal{C}[S^{-1}]$ always exists but morphisms are difficult to describe. The proof of the proposition is an exercise:

Proof. (1) This is an equivalence relation.
(2) Composition doesn't depend on a and b.
(3) Composition doesn't depend on the choice of the roof in equivalence classes.
(4) Composition is associative.
(5) $\mathrm{Id}_X \times \mathrm{Id}_X^{-1}$ is identity morphism.

□

Definition of $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$. (Missing.)
Universal property.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[dr, "Q"'] & \tilde{\mathcal{C}} \\
& \mathcal{C}[S^{-1}] \arrow[u, dotted]
\end{tikzcd}$$

defines a universal property.

Proposition 3.2. *Let $\mathcal{C}$ be an additive category and S satisfy right Ore conditions. Then $\mathcal{C}[S^{-1}]$ is also additive and $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is additive.*

Proof. Some indications, then exercise. □

The point is that with Ore conditions you can really define what is addition of fractions.

4. Ore conditions for category of complexes

Let $\mathcal{C} = \mathrm{Com}(\mathcal{A})$ and $S = \mathrm{QIS}$. Then QIS **does not** satisfy Ore conditions: second and third properties fail. But we can consider instead the homotopy category $\mathcal{H}(\mathcal{A})$. But what is that.

Definition 4.1. (1) A morphism $f \in \mathrm{Hom}_{\mathrm{Com}\mathcal{A}}(A^\bullet, B^\bullet)$ is *homotopically equivalent to zero*, i.e. $f \sim_{\mathrm{hom}} 0$ if for all $i \in \mathbb{Z}$, $h^i : A^i \to B^i$ such that $f^i = d_B^{-1} \circ h^i \circ d_A^i$.

$$\begin{tikzcd} A^{i-1} \arrow[r] & A^i \arrow[r] \arrow[d, "f^i"] \arrow[dl, dotted, "h^i"'] & A^{i+1} \arrow[dl, dotted, "h^{i+1}"] \\ B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \end{tikzcd}$$

(2) $f \sim_{\mathrm{hom}} g$ are *homotopically equivalent* if $f - g \sim_{\mathrm{hom}} 0$.
(3) The *homotopy category* $\mathcal{H}(\mathcal{A})$ of $\mathcal{A}$ has objects $\mathrm{Ob}(\mathcal{H}(\mathcal{A}) = \mathrm{Com}(\mathcal{A})$ and morphisms

$$\mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Com}\mathcal{A}}(A, B) \Big/ \{f \sim_{\mathrm{hom}} 0\}.$$

Exercise 4.2. (1) $f \sim_{\mathrm{hom}} 0$, $H^i(f) : H^i(A) \xrightarrow{0} H^i(B)$.
(2) Morphisms homotopic to zero form an ideal in $\mathrm{Com}(\mathcal{A})$, that is, abelian subgroups w.r.t. compositions.
(3) $\mathcal{H}(\mathcal{A})$ is an additive category and $\mathrm{Com}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$ is additive.

Objectives.

(1) QIS satisfy the Ore conditions in $\mathcal{H}(\mathcal{A})$.
(2) $D(\mathcal{A}) = \mathrm{Com}(\mathcal{A})[\mathrm{QIS}^{-1}] \cong \mathcal{H}(\mathcal{A})[\mathrm{QIS}^{-1}]$. This will allow to describe $D(\mathcal{A})$ explicitly as an additive category.
(3) $\mathcal{H}(\mathcal{A})$ and $D(\mathcal{A})$ are not abelian, but triangulated!

When we lose the property of an abelian category, we lose the notion of exact sequence.

Goal. Now we shall prove that QIS satisfy right Ore conditions on $\mathfrak{X}(f)$ and $D(\mathcal{A}) \cong \mathcal{H}[\mathrm{QIS}]^{-1}$.

Definition 4.3. The *shift functor* is given by

$$\begin{aligned} [1] : \mathrm{Com}(\mathcal{A}) &\longrightarrow \mathrm{Com}(\mathcal{A}) \\ A^\bullet &\longmapsto (A^\bullet[1]) \end{aligned}$$

where

$$(A^\bullet[1])^i = A^{i+1}, \qquad d_{A[1]} = -d_A$$

Definition 4.4. For a morphsm $f \in \mathrm{Hom}_{\mathrm{Com}(\mathcal{A})}(A^\bullet, B^\bullet)$, its *mapping cone* is the complex

$$\mathrm{Cone}(f) := B^\bullet \oplus A^\bullet[1]$$

with

$$\begin{aligned} d^i_{\mathrm{Cone}(f)} : A^{i+1} \oplus B^i &\longrightarrow A^{i+2} \oplus B^{i+1} \\ (a^{i+1}, b^i) &\longmapsto (-d_A a^{i+1}, d_B b^i + f(a^{i+1})) \end{aligned}$$

Exercise 4.5. $\mathrm{Cone}(f) \in \mathrm{Com}(\mathcal{A})$.

Exact sequence in $\mathrm{Com}(\mathcal{A})$:

$$0 \longrightarrow B \longrightarrow \mathrm{Cone}(f) \longrightarrow A^\bullet[1] \longrightarrow 0$$

Definition 4.6. $A^\bullet \in \mathcal{H}(\mathcal{A})$ is *contractible* if $\mathcal{A} \cong 0$ in $\mathcal{H}(\mathcal{A})$, that is, $\mathrm{Id}_{A^\bullet} \sim_{\mathrm{hom}} 0$.

Exercise 4.7. $f \in \mathrm{Hom}(A^\bullet, B^\bullet)$ is isomorphism in $\mathcal{H}(\mathcal{A})$ if and only if $\mathrm{Cone}(f)$ is contractible.

Definition 4.8. Let $A^\bullet, B^\bullet \in \mathrm{Com}(\mathcal{A})$. Then we define $\underline{\mathrm{Hom}}(A^\bullet, B^\bullet) \in \mathrm{Com}(\mathrm{Ab})$,

$$\mathrm{Hom}^i(A, B) = \prod_{n \in \mathbb{Z}} \mathrm{Hom}(A^n, B^{n+i})$$

$$d^i_{\underline{\mathrm{Hom}}} : \underline{\mathrm{Hom}}(A, B)^i \longrightarrow \underline{\mathrm{Hom}}(A, B)^{i+1}$$
$$f = \prod f^n \longmapsto g$$

and $g^n = d_B f^n - (-1)^i f^{n+1} d_A$.

Exercise 4.9. (1) Check that $\underline{\mathrm{Hom}}$ is a complex.
(2) $\mathrm{Ker}\, d^i_{\underline{\mathrm{Hom}}} = \mathrm{Hom}_{\mathrm{Com}(\mathcal{A})}(A^\bullet, B^\bullet[i])$.
(3) $\mathrm{Im} d^{i-1}_{\mathrm{Hom}} = \{f sim_{\mathrm{hom}} 0\}$

Notation: $\mathrm{Hom}^i(A^\bullet, B^\bullet) = \mathrm{Hom}(A^\bullet, B^\bullet[i])$.

Lemma 4.10. *Consider the following composition*

$$A^\bullet \xrightarrow[f]{} B^\bullet \xrightarrow{b} Cone(f) \xrightarrow{a} A^\bullet[1]$$

(which is your first distinguished triangle in life) induces the following long exact sequence for all $C^{\bullet \in Com(\mathcal{A})}$*:*

$$\cdots \longrightarrow Hom^i_{\mathcal{H}(\mathcal{A})}(C^\bullet, A^\bullet) \longrightarrow Hom_{\mathcal{H}}(\mathcal{A})^i(C^\bullet, B^\bullet) \longrightarrow$$

$$\longrightarrow Hom^i_{\mathcal{H}(\mathcal{A})}(C, Cone(f)) \longrightarrow Hom^{i+1}_{\mathcal{H}(\mathcal{A})}(C^\bullet, A^\bullet) \longrightarrow \cdots$$

and the same happens for $Hom^i_{\mathcal{H}(\mathcal{A})}(\cdot, C^\bullet)$*.*

It is very important that we went in $\mathcal{H}(\mathcal{A})$!

Idea of proof. (The long exact sequence is the cohomology exact sequence associated to a short exact sequence. Also some exercise from the past is hidden in the details.)

The short exact sequence

$$0 \longrightarrow B^\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow A[1] \longrightarrow 0$$

splits. This gives

$$0 \longrightarrow B^i \longrightarrow B^i \oplus A^{i+1} \longrightarrow A^{i+1} \longrightarrow 0$$

Which in turn gives

$$0 \longrightarrow \underline{\mathrm{Hom}}(C^\bullet, B^\bullet) \longrightarrow \underline{\mathrm{Hom}}(C^\bullet, \mathrm{Cone}(f)) \longrightarrow \underline{\mathrm{Hom}}(C^\bullet, A[1] \longrightarrow 0$$

and *then* we take the cohomology long exact sequence. □

Exercise 4.11. $bf = ab = fa[-1] = 0$ in $\mathcal{H}(\mathcal{A})$.

Exercise 4.12. Prove that the right Ore conditions are not satisfied in $\mathrm{Com}(\mathcal{A})$.

Proposition 4.13. *QIS satisfy the right Ore conditions in $\mathcal{H}(\mathcal{A})$.*

Proof. In lecture we proved condition 1 and part of condition 2 of the right Ore conditions, see Subsection **??**. Finishing the proof was left as exercise, including Ore condition 3. □

Now we shall show that

$$D(\mathcal{A}) \cong \mathcal{H}(\mathcal{A})[\mathrm{QIS}]$$

$$H : \mathrm{Com}(\mathcal{A} \to \mathcal{H}(\mathcal{A})$$

Proposition 4.14. *The localization functor*

$$\begin{tikzcd} Q : Com(\mathcal{A}) \arrow[r] \arrow[d, "H"'] & D(\mathcal{A}) \cong \mathcal{H}(\mathcal{A})[QIS]^{-1} \\ \mathcal{H}(\mathcal{A}) \arrow[ur, "Q'"'] & \end{tikzcd}$$

can be be decomposed as $Q' \circ H$. Moreover $Q' \cong \mathcal{H}(\mathcal{A})$, where $Q_{\mathcal{H}(\mathcal{A})} : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A})[QIS]^{-1}$.

Proof. Done in class, with some exercises for us to complete. □

This shows that the derived category is equivalent to the homotopy category with localization. We have the following corollary:

Lemma 4.15. *We can specifically describe*

$$D(\mathcal{A}) : Ob(D(\mathcal{A}) = Com(\mathcal{A})$$

$$Hom_{D(\mathcal{A})}(A^\bullet, B^\bullet) = \{equivelnce\ classes\ of\ roofs\}$$

Recall that a roof is

$$\begin{tikzcd} & C \arrow[dl, "QIS \ni S"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

Now we shall show that $D(\mathcal{A})$ is additive. (Or have we already proved this? ie. from the corollary?)

Why $\mathcal{H}(\mathcal{A})$ and $D(\mathcal{A})$ are not abelian: there are not so many injective and surjective morphisms.

Exercise 4.16. $f \in \mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(A^\bullet, B^\bullet)$ is injective if and only if there exists $g : B \to A^\bullet \in \mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(B^\bullet, A^\bullet)$ such that $g \circ f = \mathrm{id}_A$. Formulate the same for surjections.

5. Some properties of the derived category

For every $i \in \mathbb{Z}$ define

$$\mathcal{A} \longrightarrow D(\mathcal{A})$$

$$A \longmapsto A[i], \quad \{0 \to \overset{i}{A} \to 0\}$$

So $A[i]$ is *that* complex. It is "concentrated in degree i"

Goal: understand morphisms between $M, N[i]$.

Definition 5.1. Let $A \in \mathrm{Com}(\mathcal{A})$. For all n,

$$\tau_{\leq n}(A^\bullet)^i := \begin{cases} A^i & i < n \\ \operatorname{Ker} d^n & i = n \\ 0, & i > n \end{cases}$$

then

$$\tau_{\geq n}(A^\bullet)^i = \begin{cases} 0 & i < n \\ A^n/\mathrm{Im} d^{n-1} & i = n \\ A^i, & i > n \end{cases}$$

and $A^\bullet \to \tau_{\geq n} A^\bullet$.

Then

$$\tau_{\leq n} : \mathrm{Com}(\mathcal{A}) \to \mathrm{Com}(\mathcal{A})$$

defines a functor $\tau_{\leq n}(\mathrm{QIS}) \subset \mathrm{QIS}$ where

$$\tau_{\leq}(f \sim_{\mathrm{hom}} 0) \subset \sim_{\mathrm{hom}} 0 \implies$$

$$\tau_{\leq n} : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A})$$
$$D(\mathcal{A}) \longmapsto D(\mathcal{A})$$

the same for $\tau_{\geq n}$.

The following lemma allows us to understand how are morphisms behaved.

Lemma 5.2. *Let $A, B \in \mathcal{A}$. Then*

1. *$Hom_{D(\mathcal{A})}(A, B[-i]) = 0$ for all $i > 0$.*
2. *$Hom_{\mathcal{A}}(A, B) = Hom_{D(\mathcal{A})}(A, B)$.*

In particular, $\mathcal{A} \hookrightarrow A(\mathcal{A})$ is fully-faithful.

Now we introduce a notion of Ext, which will recover the usual Ext (derived functors).

Definition 5.3. For $i > 0$,

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]).$$

Definition 5.4 (Yoneda)**.** For $i > 0$ let

$$\mathrm{Ext}^i_Y(A, B) := \left\{ {\text{equivalence classes of exact sequences} \atop B \to K^{i-1} \to K^{i-2} \to \ldots \to K^0 \to A} \right\}$$

and say that two such sequences are equivalent if there is a commutative diagram

$$\begin{tikzcd} B \arrow[r] & K^{i-1} \arrow[r] \arrow[d] & K^{i-2} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & A \\ B \arrow[r] & \tilde{K}^{i-1} \arrow[r] & \tilde{K}^{i-2} \arrow[r] & \cdots \arrow[r] & \tilde{K}^0 \arrow[r] & A \end{tikzcd}$$

and are equivalent if there exists a sequence of equivalent elements

$$K \sim K_0 \sim \ldots \sim \tilde{K}$$

Proposition 5.5.

$$Ext^i_{\mathcal{A}}(A, B) \cong Ext^i_Y(A, B)$$

Proof. Sketched in class. ☐

Exercise 5.6. $D(\mathcal{A})$ and $\mathcal{H}(\mathcal{A})$ are abelian if and only if $\mathcal{A}$ is semisimple, that is, all exact sequences in $\mathcal{A}$ split. If $\mathcal{A}$ is semisimple, then $D(\mathcal{A}) \cong \bigoplus_{iinn\mathbb{Z}} \mathcal{A}[-i]$.

6. Triangulated categories

Definition 6.1. An additive category $\mathcal{T}$ is a *triangulated category* if it has the following structures:

- $[1] : \mathcal{T} \to \mathcal{T}$ autoequiv.
- Class of sequences of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

  which are called *distinguished triangles*, and satisfy the following axioms:
  - (Axiom 1.)
    * $A \xrightarrow{\text{id}} A \longrightarrow 0 \longrightarrow A[1]$ is a distinguished triangle.
    * Any triangle $A \to B \to C \to A[I]$ isomorphic to a distinguished triangle is a distinguished triangle.
    * Any morphism $(A \xrightarrow{f} B) \in \mathcal{T}$ can be completed to a distinguished triangle $A \xrightarrow{f} B \to \mathcal{K} \to A[I]$, and $\mathcal{K}$ is called the *cone* of f.
  - (Axiom 2.) A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

    is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{g} A[1] \xrightarrow{f[1]} B[1]$$

    is distinguihsed.
  - (Axiom 3.) For any two distinguished triangles

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, dotted, "u"] & A[1] \arrow[d, "a[1]"] \\ A' \arrow[r, "f'"'] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

    and any a, b such that $bf = f'a$ there exists u that makes the above diagram commute (it is not unique!)

– (Octahedral axiom.) Any two distinguished triangles with a common vertex of the form

$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] & X[1] \\ & Y' \arrow[d] & & \\ & W \arrow[d] & & \\ & Y[1] & & \end{tikzcd}$$

can be completed to the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "="'] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] & X[1] \arrow[d, "="] \\ X \arrow[r] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X[1] \arrow[d, "{u[1]}"] \\ & W \arrow[r, "="] \arrow[d, "h"'] & W \arrow[r, "h"] \arrow[d] & Y[1] \\ & Y[1] \arrow[r, "{v[1]}"'] & Z[1] & \end{tikzcd}$$

Properties. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle.

Lemma 6.2. (1) $gf = hg = f[1]h = 0$.

(2) *It induces the following sequence*

$$\cdots \longrightarrow Z[-1] \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] \longrightarrow Y[1] \longrightarrow Z[1] \longrightarrow \cdots$$

such that any sequence of length h (contained in it?) is a distinguished triangle.

Proof.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "="'] & Z \arrow[r, "h"] \arrow[d, "="'] & C[1] \arrow[r, "{f[1]}"] \arrow[d] & Y[1] \arrow[d, "{g[1]}"] \\ Z \arrow[r, "="] & Z \arrow[r] & 0 \arrow[r] & Z[1] \end{tikzcd}$$

and by axiom 3 we get $gf = 0$. By Axiom 2 we get the second statement. □

Lemma 6.3. *If $X \to Y \to Z \to X[1]$ is a distinguished triangle, then for every $U \in \mathcal{T}$ we have the following long exact sequences:*

$$\cdots \longrightarrow Hom(U, X[i]) \longrightarrow Hom(U, Y[i] \longrightarrow Hom(U, Z[i]) \longrightarrow Hom(U, X[i+1] \longrightarrow \cdots$$

and the same happens for $Hom(\cdot, U)$

Proof. Done in lecture, uses previous lemma and Axiom 3. ☐

We have the following corollary:

Lemma 6.4. *If in Axiom 3 two morphisms are isomorphisms, then the constructed one is also an isomorphism.*

Proof. Exercise. **Hint.** Use Yoneda lemma. ☐

Exercise 6.5. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is a distinguished triangle, then $h = 0$ if and only if f has a left inverse, if and only if g has a right inverse.

As a corollary,

Lemma 6.6. *The cone of a morphism is unique up to isomorphism.*

However, the cone is not functorial, which is a problem.

Now we want to show that the homotopy category $\mathcal{H}(\mathcal{A})$ of an abelian category $\mathcal{A}$ has a triangulated structure. We need to say which are the distinguished triangles: but we already did this for the derived category: they are

$$A \xrightarrow{f} B \xrightarrow{b} \mathrm{Cone}(f) \xrightarrow{a} A[I] \longrightarrow 0$$

Theorem 6.7. $\mathcal{H}(\mathcal{A})$ *is triangulated.*

Proof. It looks like Axiom 1 should be immediate from our construction. Also Axiom 2 (indeed, we have a notion of $\mathrm{Cone}(f)$ which we obviously expect to satisfy the property of the cone $\mathcal{K}$. Axiom 3 uses an exercise using the notion of cylinder. Proving the octahedral axiom is also an exercise with a big hint. ☐

$D(\mathcal{A})$ **is triangulated.** Let $\mathcal{C}$ be triangulated, and S a class of morphisms satisfying the right Ore conditions. We know that $\mathcal{C}[S]^{-1}$ is an additive category, but when is it triangulated?

Definition 6.8. S is *compatible* with a triangulated structure if

(1) $s \in S$, $S[1] \in S$,
(2) In Axiom 3, if $a, b \in S$ then we can choose $u \in S$.

Proposition 6.9. *It $\mathcal{T}$ is triangulated an S satisfies the right Ore conditions and is compatible with the triangulated structure of $\mathcal{T}$ then $\mathcal{T}[S]^{-1}$ is triangulated.*

Proof. Axioms 1-3 proved, Octahedral axiom is exercise with hint. ☐

We finally obtain:

Lemma 6.10. $D(\mathcal{A})$ *is triangulated.*

Proof. Two or three lines, uses 5 lemma, uses long exact sequences of Hom. ☐

7. Derived functors

Definition 7.1. An additive functor $F : \mathcal{T} \to \mathcal{T}'$ between triangulated categories is *exact* if

(1) $F \circ [1]_{\mathcal{T}} = [1]_{\mathcal{T}'} \circ F$.
(2) Maps distinguished triangles to distinguished triangles.

Here are some bounded notions:

$$\begin{aligned}
&\mathrm{Com}^\bullet(\mathcal{A})\\
&\mathrm{Com}^+(\mathcal{A}) = \{A \in \mathrm{Com}(\mathcal{A}) | A^i = 0, \quad i \ll 0\}\\
&\mathrm{Com}^-(\mathcal{A}) = \{A \in \mathrm{Com}(\mathcal{A}) | i \gg 0\}\\
&\mathrm{Com}^b(\mathcal{A}) = \{A \in \mathrm{Com}(\mathcal{A}) ||i| \gg 0\}
\end{aligned}$$

each of which gives a derived category D^* for $* = b, +$ or $-$.

Now consider an exact functor of abelian categories $F : \mathcal{A} \to \mathcal{B}$. We would like to obtain a functor $D^*(\mathcal{A}) \to D^*(\mathcal{B})$.

In general functors are not exact. Thus our goal now is:

- From a left exact functor $F : \mathcal{A} \to \mathcal{B}$ obtain $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$,
- From a right exact functor $G : \mathcal{A} \to \mathcal{B}$ obtain $LG : D^-(\mathcal{A}) \to D^-(\mathcal{B})$.

Given an additive category $\mathcal{I} \subset \mathcal{A}$ of injective objects in $\mathcal{A}$, we have $\mathcal{H}^*(\mathcal{I}) \subset \mathcal{H}^*(\mathcal{A}) \xrightarrow{G_{\mathcal{A}}} D^*(\mathcal{A})$.

Lemma 7.2. *Suppose that $\mathcal{A}$ has enough injective objects, $\forall A^\bullet \in \mathcal{A}$ can be embedded $A \hookrightarrow I \in \mathcal{I}$. Then $i : \mathcal{H}^+(\mathcal{I}) \to D^+(\mathcal{A})$ is an equivalence.*

Proof. Not proved. □

8. Semiorthogonal decompositions

Idea. To split $D^*(\mathcal{A})$ in smaller pieces that are easier.

Definition 8.1. $\mathcal{A} \subset \mathcal{T}$ is a *strictly full triangulated subcategory* if

- $\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{T}}(A, B)$ for all $A, B \in \mathcal{A}$.
- If $B \in \mathcal{T}$ is isomorphic to an object from $\mathcal{A}$, then $B \in \mathcal{A}$.

Definition 8.2. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}, \mathcal{B}$ strictly full trinagulated subcategories in $\mathcal{T}$. Then $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a *semiorthogonal decomposition* if

- $\mathrm{Hom}_{\mathcal{T}}(B, A) = 0$ for all $B \in \mathcal{B}$ and $A \in \mathcal{A}$.
- For all $T \in \mathcal{T}$ there exists a distinguished triangle
$$T_B \to T \to T_{\mathcal{A}} \to T_{\mathcal{B}}[1]$$
where $T_B \in \mathcal{B}$ and $T_{\mathcal{A}} \in \mathcal{A}$.

Proposition 8.3 (Properties of semiorthogonal decomposition)**.** (1) *Functoriality: for any $f \in T \to T'$,*

$$\begin{tikzcd}
T_{\mathcal{B}} \arrow[r] \arrow[d, dotted, "\exists f_{\mathcal{B}}"'] & T \arrow[r] \arrow[d, "f"] & T_{\mathcal{A}} \arrow[r] \arrow[d, dotted, "f_{\mathcal{A}}"] & T_{\mathcal{B}}[1] \\
T'_{\mathcal{B}} \arrow[r] & T' \arrow[r] & T'_{\mathcal{A}} \arrow[r] & T'_{\mathcal{B}}[1]
\end{tikzcd}$$

there exist $f_{\mathcal{B}}$ and $f_{\mathcal{A}}$ as in the diagram.

Proof. For $f_{\mathcal{A}}$,apply $\mathrm{Hom}_{\mathcal{T}}(\cdot, T'_{\mathcal{A}})$ to the upper distinguished triangle, obtain a long exact sequence... □

Functoriality: defnote by $\alpha : \mathcal{A} \hookrightarrow \mathcal{T}$ the inclusion, then we have a functor $\alpha^* : \mathcal{T} \to \mathcal{A}$ given by $T \to T_{\mathcal{A}}$, $f \mapsto f_{\mathcal{A}}$. Similarly for $\beta : \mathcal{B} \hookrightarrow \mathcal{T}$ we have a functor $\beta^* : \mathcal{T} \to \mathcal{B}$ by $T \to T_{\mathcal{B}}$, $f \mapsto f_{\mathcal{B}}$.

Lemma 8.4. *α^* is left adjoint to α and β^* is right adjoint to β.*

Proof. We have for any $A \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(\alpha^* T, A) = \mathrm{Hom}_{\mathcal{T}}(T, \alpha_* A)$. But that is equal to $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}_{\mathcal{A}}, A)$. □

Exercise 8.5. If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$, then

$$\mathcal{B} = \mathcal{A}^{\perp} = \langle B \in \mathcal{T} | \mathrm{Hom}(B, A), \forall A \in \mathcal{A} \rangle := \mathrm{Ker}\, \alpha^*$$

(the $\perp$ sign should be on the left of $\mathcal{A}$ but I don't know how to write this) and

$$\mathcal{A} = \mathcal{B}^{\perp} = \langle A \in \mathcal{T} | \mathrm{Hom}(B, A) = 0, \forall B \in \mathcal{B} \rangle := \mathrm{Ker}\, B$$

(here the $\perp$ is correct on the right side of $\mathcal{B}$).

This construction can be reversed.

Definition 8.6. A strictly full subcategory $\mathcal{A} \hookrightarrow \mathcal{T}$ is called

- *left admissible* is there exists $\alpha^* : \mathcal{T} \to \mathcal{A}$ left adjoint,
- *right admissible* is there exists $\alpha^! : \mathcal{T} \to \mathcal{A}$ right adjoint.

Lemma 8.7. *If $\mathcal{A} \hookrightarrow \mathcal{T}$ is left admissible, then $\mathcal{T} = \left\langle \mathcal{A}, \mathcal{A}^{\perp} \right\rangle$ is semiorthogonal decomposition. If $\mathcal{B} \hookrightarrow \mathcal{T}$ is right admissible then $\mathcal{T} = \left\langle \mathcal{B}^{\perp}, \mathcal{B} \right\rangle$ is semiorthogonal decomposition.*

Proof. Uses a remark, Yoneda lemma. □

Proposition 8.8 (Bondall, Van der Borgh)**.** *If $\mathcal{T}$ is smooth and proper (if $\mathcal{T} = D^b(X)$, where X is smooth and proper, we call $\mathcal{T}$ smooth and proper) then $\mathcal{A} \subset \mathcal{T}$ is left admissible $\iff$ $\mathcal{A}$ is right admissible $\iff$ $\mathcal{A}$ is smooth proper. If $\mathcal{A}$ is smooth and proper and $\mathcal{A} \hookrightarrow \mathcal{T}$ then $\mathcal{A}$ is left and right admissible.*

If $D^b(C) \hookrightarrow \mathcal{T}$ is fully faithful, then it is left and right admissible.

Question. When can we construct a full faithful embedding

$$D^b(\mathrm{pt}) \cong D^b(\mathrm{mod} k) \hookrightarrow \mathcal{T}?$$

Let

$$\begin{aligned} D^b(\mathrm{mod} k) &\longrightarrow \mathcal{T} \\ k &\longmapsto E \end{aligned}$$

and define

$$\varphi_E : V^\bullet \to V^\bullet \otimes E$$

where $V^\bullet$ is a graded vector space.

When is φ_E full faithful? Some computations show that φ_E is fully faithful if and only if $\mathrm{Hom}(E, E) = k$. Recall from `categories.tex` that

Definition 8.9. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) We say F is *faithful* if for any objects $x, y \in \mathrm{Ob}(\mathcal{A})$ the map

$$F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$$

is injective.

(2) If these maps are all bijective then F is called *fully faithful*.

Definition 8.10. $E \in \mathcal{T}$ is exceptional if

$$\operatorname{Hom}(E, E[i]) = \begin{cases} k & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Lemma 8.11. *If E is exceptional then $\varphi_E : D(pt) \to \mathcal{T}$ is a fully faithful embedding. Then*

$$\mathcal{T} = \left\langle \varphi_E(D^b(pt)^{\perp}, E \right\rangle = \left\langle E^{\perp}, \varphi_E(D^b(pt)) \right\rangle$$

Notation. In this case, $\mathcal{T} = \left\langle E, E^{\perp} \right\rangle = \left\langle E^{\perp}, E \right\rangle$, where on the first bracket the $\perp$ should be on the left side of E.

Example 8.12. On $\mathbb{P}^n$, $\mathcal{O}(i)$ is exceptional. More general, if X is Fano (i.e. smooth projective with $-K_X$ ample), then any line bundle on X is ample.

Proof. $\operatorname{Ext}^i(\mathcal{L}, \mathcal{L}) = H^i(X, \mathcal{O}_X)$, $H^0(X, \mathcal{O}) = k$ and $H^{>0}(X, \mathcal{O}) = 0$ by Kodaira Vanishing. □

For Fano varieties we have nontrivial semiorthogonal decomposition.

Remark 8.13. If $K_X \cong \mathcal{O}_X$ then there is no semiorthogonal decomposition of $D^b(X)$ (for K3 or abelian).

Back to Fano,

Definition 8.14. Let $\mathcal{A}_1, \dots, \mathcal{A}_n \in \mathcal{T}$ be strictly full subcategories and $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ a *semiorthogonal decomposition*

- $\operatorname{Hom}(\mathcal{A}_j, \mathcal{A}_i) = 0$ for all $j > i$.
- For all T there is a chain of morphisms

  $$0 = T_n \to T_{n-1} \to \dots \to T_1 \to T_0 = T$$

  such that $\operatorname{Cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$.

Exercise 8.15. Prove functoriality of $T \to \operatorname{Cone}(T_i \to T_j)$ for $i > j$.

Definition 8.16. A collection of objects $E_1, \dots, E_n$ is *exceptional* if

(1) For all i, E_i is exceptional.
(2) $\operatorname{Hom}(\langle E_i \rangle, \langle E_j \rangle) = 0$ for all $i > j$, where $\langle E_j \rangle$ is a minimal triangulated category generated by E_i.

We have a corollary:

Lemma 8.17. *If $E_1, \dots, E_n$ is an exceptional collection, then*

$$\mathcal{T} = \left\langle E_1, \dots, E_n, \langle E_1, \dots, E_n \rangle^{\perp} \right\rangle = \left\langle \langle E_1, \dots, E_n \rangle^{\perp}, E_1, \dots, E_n \right\rangle$$

(where $\perp$ should be on the left of the bracket in the first appearence) is a semiorthogonal decomposition.

Definition 8.18. If $\langle E_1, \dots, E_n \rangle^{\perp} = 0$ (equivalently with $\perp$ on the left).

Exercise 8.19. Let X be a Fano variety with $-K_X = mH$ for H ample (m is the index of X). Then for every line bundle $\mathcal{L}$,

$$(\mathcal{L}(-m+1)H, \dots, \mathcal{L}(-H)\mathcal{L})$$

is an exceptional collection.

Proof. Use Serre duality and Kodaira Vanishing. □

Here is the promised theorem from the beginning of this minicourse:

Theorem 8.20 (Beilinson). $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \dots, \mathcal{O} \rangle$ *is a full exceptional collection.*

This means that we can classify all objects of $D^b(\mathbb{P}^n)$. Because whenever we have a semiorthogonal decomposition we get these chains of morphisms, i.e. for all $D^b(\mathbb{P}^n)$ there exists a chain

$$0 = T_n \to \dots \to T_0 = T$$

such that $\mathrm{Cone}(T_i \to T_{i-1}) \in \langle \mathcal{O}(-i) \rangle \cong D^b(\mathrm{pt})$.

Probably this is the content of [Bei84].

Proof of Beilinson theorem. Consider the Koszul complex

$$0 \to \mathcal{O}(-n-1) \to \dots \to \Lambda^2 V^\vee \otimes \mathcal{O}(-2) \to V^\vee \otimes \mathcal{O}(-1) to \mathcal{O} \to 0$$

This implies that

$$\mathcal{O}(n-1) \in \langle \mathcal{O}(-n), \dots, \mathcal{O} \rangle$$

... proof completed in class. □

References

[Bei84] A. A. Beilinson, *The derived category of coherent sheaves on* $\mathbf{P}^n$, Selecta Math. Soviet. **3** (1983/84), no. 3, 233–237, Selected translations.