MODULI SPACES OF SHEAVES

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Notes at github.com/danimalabares/cimpa-floripa

Abstract. This course will be an introduction to Moduli Spaces of vector bundles. A moduli space of stable vector bundles on a smooth, algebraic variety X is a scheme whose points are in "natural bijection" to isomorphic classes of stable vector bundles on X. Using Geometric Invariant Theory the moduli space can be constructed as a quotient of certain Quot-scheme by a natural group action. We introduce the crucial concept of stability of vector bundles over smooth projective varieties and we give a cohomological characterization of the (semi)stability. The notion of (semi)stability is needed to ensure that the set of vector bundles one wants to parameterize is small enough to be parameterized by a scheme of fniite type. We introduce the formal defniition of moduli functor, fnie moduli space and coarse moduli space and we recall some generalities on moduli spaces of vector bundles. Then we focus on vector bundles on algebraic surfaces. Quite a lot is known in this case and we will review the main results, some of which will illustrate how the geometry of the surface is refelcted in the geometry of the moduli space. Going beyond surfaces, we introduce the notion of monad, which allow the classifciation of vector bundles on, for instance, $\mathbb{P}^3$. Monads appeared in a wide variety of contexts within algebraic geometry, and they are very useful when we want to construct vector bundles with prescribed invariants like rank, determinant, Chern classes, etc. Finally, we will study moduli spaces of vector bundles on higher dimensional varieties. As we will stress, the situation drastically difefrs and results like the smoothness and irreducibility of moduli spaces of stable vector bundles on algebraic surfaces turn to be false for moduli spaces of stable vector bundles on higher dimensional algebraic varieties.

Contents

1. Introduction

The goal is the following. X smooth projective variety over $\mathbb{C}$. We want to define a moduli space of sheaves up to some kind of isomorphism. Also give some discrete invariants. And give structure of scheme which "respects limits".

Functor of points. For a given scheme X, the set $\mathrm{Hom}(\mathrm{Spec}\mathbb{C}, X)$ is the set of points of X. This motivates defining a functor $h_X : \mathrm{Sch} \to \mathrm{Sets}$ by $h_X(S) = \mathrm{Hom}(S, X)$, and morphisms are just mapped via composition.

And there's more: given a morphism of schemes $\varphi : X \to Y$ we can produce $h_\varphi : h_X \to h_Y$.

Definition 1.1. A functor F is called *representable (by a scheme)* if there is a scheme X such that $F \cong h_X$ and we say X is a *fine moduli space for X*.

Remark 1.2. If X exists it is unique.

Example 1.3. X scheme,

$$\mathrm{Quot}_X : (\mathrm{Sch}) \to (\mathrm{Sets})$$
$$\mathrm{Quot}_X(S) = \{\mathcal{O}_{X\times S}^{\oplus n} \to \mathcal{F} : \mathcal{F} \text{ is a flat sheaf}\}\Big/\sim$$

and morphisms are mapped by pullback.

Here, "flat" accounts for a continuously varying family of copies of X over S, i.e. the product $X \times S$.

Theorem 1.4. *$Quot_X$ is represented by a projective scheme.*

Almost all examples of moduli spaces are quotients.

The one example of a moduli space everybody knows is...

Example 1.5 (Projective space is a quotient)**.** Take $X =$ point, and then

$$F(S) = \{\mathcal{O}_S^{\oplus n+1} \to \mathcal{L}\} \cong h_{\mathbb{P}^n}$$

where $\mathcal{L}$ is a line bundle on S.

Definition 1.6. If F is represented by a scheme X and $\eta : F(S) \to h_X(S)$ in part we have a bijection

$$\begin{aligned} F(X) &\longrightarrow h_X(X) := \mathrm{Hom}(X, X) \\ U &\longmapsto \mathrm{id} \end{aligned}$$

Again, flat means continuously varying. Consider first

$$\mathcal{M}(S) = \{\mathcal{F}\text{a fheaf on } X \times S \text{ flat over } S\}/\sim$$

and

$$\mathcal{M}(f : T \to S) = f^*$$

and equivalence is given by: $\mathcal{F} \sim \mathcal{G}$ iff there exists a line bundle $\mathcal{L}$ on S such that $\mathcal{F} \cong \mathcal{G} \otimes \pi^* S$.

Problem. $X := \mathbb{P}^1$ and rank 2 sheaves on $\mathbb{P}^1$. In this case [...]

Fact. Fine moduli spaces are rare.

Definition 1.7. A *coarse moduli space* of X is a scheme X with a natural transformation $\eta : F \to h_X$ such that

- $\eta_{\eta\times\mathbb{C}} : F(hec\mathbb{C}) \to h_X(hec\mathbb{C}$ is hij.
- For all Y and $F \to h_T$ we have $[F \to h_Y,\, F \to h_X,\, \exists! h_X \to h_Y$.

Summary.

(1) Schemes are functors.
(2) The best moduli spaces are represented functors, they are called fine. These have universal families, and all are pullbacks of from the universal family.

2. Discrete data

X projective and H an ample divisor on X. The *Hilbert polynomial* is

$$P(\mathcal{F}, m) := \chi(\mathcal{F} \otimes \mathcal{O}(m))$$

Fact. $P(\mathcal{F}, m) = \sum_{i=0}^{\dim X} \alpha_i(\mathcal{F}) \frac{m^i}{i!}$.

The *rank of* $\mathcal{F}$ is $\mathrm{rk}(\mathcal{F}) := \frac{\alpha_d(\mathcal{F})}{\alpha_d(\mathcal{O}_X)}$. And $\alpha_d(\mathcal{O}_X)$ is the *degree* of $\mathfrak{X}$ w.r.t $\mathcal{O}(1)$.

The *reduced Hilbert polynomial* is $P(\mathcal{F}, m) := \frac{P(\mathcal{F},m)}{\alpha_d(\mathcal{F})}$. The *slope* of $\mathcal{F}$ is $\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F})H^{d-1}}{\mathrm{rk}\mathcal{F}}$.

Example 2.1 (Curve of genus g using Riemann-Roch)**.** (Missing.)

Here begin the contents of the second lecture.

Definition 2.2. $\mathcal{F}$ is *Gieseker/μ-semistable* w.r.t. H if

- $P_\varepsilon(\mathfrak{m}) \le P_{\mathcal{F}}(\mathfrak{m})$ for all $0 \ne \varepsilon \subset \mathcal{F}$ proper subsheaf.
- $\mu_M(\varepsilon) \le \mu_M(\mathcal{F})$ of rank $0 < r \le r - 1$.

If the inequality is strict then $\mathcal{F}$ is called *stable*.

Example 2.3. X a curve, $d = \deg\mathcal{F}$, $r = \mathrm{rk}\mathcal{F}$, $p = \deg Hm + \frac{d}{r} + 1 - g$. Then μ-stability is the same as G-stability.

In general:

$$\mu\text{-stability} \implies \text{G-stability} \implies \text{G semi-stability} \implies \mu \text{ semi-stability}$$

Remark 2.4. Stability depends on the choice of H.

Example 2.5 (Stability depends on choice of ample divisor)**.** $X = \mathbb{P}^1 \times \mathbb{P}^1$, then $\mathrm{Pic}(X) = \mathbb{Z}[\ell] \oplus \mathbb{Z}[m]$ where $\ell^2 = m^2 = 0$ and $\ell m = 1$. Consider $\mathcal{F}$ given by

$$0 \longrightarrow \mathcal{O}_X(\ell - 3m) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X(3m) \longrightarrow 0 \tag{2.5.1}$$

Then $c_1(\mathcal{F}) = \ell$ and $c_2(\mathcal{F}) = (\ell - 3m)(3m) = 3$. Then $L = \ell + sm$; $L' = \ell + 7m$. We will show that $\mathcal{F}$ is semistable w.r.t. L but not w.r.t. L'. We do have non-trivial extensions as in the sequence 2.5.1. These are parametrized by

$$\mathrm{Ext}^1(\mathcal{O}_X(3m)), \mathcal{O}_X(-3m) \cong H^1(X, \mathcal{O}_X(\ell - 6m)) \cong \ldots \cong \mathbb{C}^{10}$$

So it is semi-stable w.r.t. L.

Suppose $\mathcal{O}(s) \subset \mathcal{F}$. We want to show that $\mu(\mathcal{O}(s)) \le \mu(\mathcal{F})$.

We claim that we have either $\mathcal{O}(s) \hookrightarrow \mathcal{O}(3m)$ or $\mathcal{O}(s) \hookrightarrow \mathcal{O}(\ell - 3m)$. Claim was proved and further computations showed the required inequality, so $\mathcal{F}$ is μ-semistable w.r.t. L. On the other side, $\mathcal{F}$ is not semistable w.r.t. L' since $\mathcal{O}(\ell - 3m) \hookrightarrow \mathcal{F}$ (so here I realise that probably taking Chern class will be monotonous with respect to this inclusion. . .) which gives $\mu_{L'}(\mathcal{O}(\ell - 3m)) = (\ell - 3m)(\ell + 7m) = 4 \not\le \frac{7}{2}$.

Example 2.6 (Results on μ-stability)**.**
- All line bundles are stable.
- If $0 \to L_0 \to \mathcal{F} \to L_1 \to 0$ is a non-trivial extension with degree $L_0 = 0$, $\deg(L_1) = 1$, L_i line bundles, then $\mathcal{F}$ is stable.
- E_i semi-stable shows $E_1 \oplus E_2$ is semistable if and only if $\mu(E_i) = \mu(E_2)$.
- E is μ-semistable if and only if for all L line bundles $E \otimes L$ is semistable.
- E is semistable then $E^\vee$ is semi-stable.
- E_1, E_2 semistable then $E_1 \otimes E_2$ is semistable.

Now we explain some **easy criteria.** (We shall find a condition on cohomology that is equivalent to stability.)

Definition 2.7. $\mathcal{F}$ a reflexive sheaf on $\mathbb{P}^n$ of rank r. We define $\mathcal{F}_{\mathrm{norm}} := \mathcal{F}(k)$ where k is the unique integer such that $c_1(\mathcal{F}(k)) \in \{-r+1, \ldots, c\}$.

Proposition 2.8 (Cohomological characterization of stability)**.** *$\mathcal{F}$ reflexive sheaf of rank 2, then $\mathcal{F}$ is μ-stable if and only if $H^0(\mathbb{P}^n, \mathcal{F}_{norm}) = 0$.*

Proof. $\mathcal{F}$ vector bundle of rank r on a smooth projective variety with $\mathrm{Pic} = \mathbb{Z}$.
- If $H^0(X, (\Lambda^z \mathcal{F})_{\mathrm{norm}}) = 0$ for all $z \in \{1, \ldots, r-1\}$ then $\mathcal{F}$ is μ-stable.
- If $H^0(X, \Lambda^z \mathcal{F})_{\mathrm{norm}}(-1)) = 0$ for all $z \in \{1, \ldots, r-1\}$, then $\mathcal{F}$ is μ-semi-stable.

□

Example 2.9. $\Omega_{\mathbb{P}^n}$ is stable, and also $T_{\mathbb{P}^1}$. The cohomologies vanish using Bott.

Example 2.10.

Filtrations. H ample, $\mathcal{F}$ torsion free sheaf, then there exists a unique filtration $0 \hookrightarrow \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$ with $E_i = \mathcal{F}_i/\mathcal{F}_{i+1}$ μ-semi-stable w.r.t. H and $\mu(E_1) > \mu(E_2) > \ldots > \mu(E_m)$. This is called *Harder-Narorimnan filtration.*

Given any Chern classes, can we guarantee the existence of a sheaf with these Chern classes? No.

Theorem 2.11 (Bogomolov inez)**.** *X smooth, projective variety of dimension ≥ 2, H ample. If $\mathcal{F}$ is torsion-free, μ semi-stable sheaf w.r.t. all*

$$\Delta(\mathcal{F}) := (2rc_2(\mathcal{F}) \underbrace{-}_{discriminant} (r-1)c_1(\mathcal{F})^2)H^{n-2} \geq 0$$

where $r = rk(\mathcal{F})$ and $\dim X = n$.

3. Boundedness

We need further preparation before we can define moduli spaces.

Definition 3.1. A family of isomorphism classes of coherent sheaves on X is called *bounded* if there exists S of **finite type** and an $\mathcal{O}_{S\times X}$ sheaf F such that the family is contained in $\{\mathcal{F}_{S\times X} | s \text{ closed in } S\}$.

Definition 3.2. A coherent sheaf is called *m-regular* if $H^i(X, \mathcal{F}(m-i)) = 0$ for all $i > 0$.

Proposition 3.3. *If $\mathcal{F}$ is m-regular,*
- *F is m' regular for all $m' \geq m$.*
- *$F(m)$ is globally generated.*

Definition 3.4. $\text{reg}\mathcal{F} := \inf\{m \in \mathbb{Z} : \mathcal{F} \text{ is } m\text{-regular}\}$

Lemma 3.5. *The following are equivalent:*

- $\{\mathcal{F}_i\}_{i\in I}$ *is bounded*
- *The set of Hilbert polynomials* $\{P(\mathcal{F}_i)\}_{i\in I}$ *is finite and there exists a bound* ρ *such that* $reg\mathcal{F}_i \leq \rho$ *for all* $i \in I$.

The following is a difficult result that we shall only state and use.

Theorem 3.6. *X projective, H ample, P a fixed polynomial of degree d. The family of torsion-free sheaves with Hilbert polynomial P is bounded.*

Proposition 3.7. *Let X be a curve of genus g with fixed Hilbert polynomial. The semi-stable sheaves on X are bounded.*

Proof. Let E be a semis-stable sheaf. We want m such that $H^1(X, E(m-1))$. We apply Serre Duality as follows:

$$H^1(X, E(m-1)) \overset{\text{S.D.}}{=} \text{Hom}(E, \omega(1-m))^\vee$$

Then $E \to \omega(m+1) \implies \omega^\vee(m-1) \to E^\vee$. E is semi-stable then $-\text{deg}(\omega^\vee) \leq \frac{\text{deg}(E^\vee)}{r}$ which implies that $-2g+2+m-1 \leq -\frac{d}{r}$ for $d = \text{deg}E$, and then $m \leq 2g-2-\frac{d}{r}+1$. If we take $m > 2g-2\frac{d}{r}+1$. $\square$

Proposition 3.8. *If E is a stable sheaf, then $Hom(E, E) = \mathbb{C}$.*

Proof. If $f : E \to E$ factors through $E \to F \to F$ and is not surjective, then $\mu(E) < \mu(F) < \mu(E) \implies f$ is injective. As an exercise, prove that E is stable if and only if for all $E \to F$ surjective we have that $\mu(E) < \mu(F)$. $\square$

4. Construction of the moduli space of semi-stable sheaves

$$\mathcal{M}^p(s) = \left\{ \begin{smallmatrix} \mathcal{F} \text{ on } X \times S, \text{ pet over } S \\ \text{with Hilbert polynomial } P \end{smallmatrix} \right\} \Big/ \sim$$

where $\mathcal{F} \sim \mathcal{G} \iff$ there exists L on S such that $\mathcal{F} \otimes \pi^* L \cong \mathcal{G}$ where $X \times S \xrightarrow{\pi} \mathcal{G}$.

$\mathcal{F}$ is a semi-stable sheaf is $\exists\ m$ such that $\mathcal{F}$ is m-regular, which implies that $\mathcal{F}(m)$ is globally generated.

Warning. Following are several formulas I don't understand and just copied from the board as they were.

$$\dim H^0(X, \mathcal{F}(m)) = P(m)$$

$$H^i(X, \mathcal{F}(m-1)) = 0 \qquad \forall i > 0$$

$$V = \mathcal{O}_X^{\oplus P(m)} \to \mathcal{F}(m) \to 0$$

$$\implies \left[\mathcal{O}_X^{\oplus P(m)}(-m) \to \mathcal{F} \to 0\right] \in \text{Quot}(\mathcal{H}, P)$$

Let R be open locus in $\text{Quot}(\mathcal{H}, P)$ such that

- $\mathcal{F}$ is semi-stable.
- $H^0(\mathcal{H}(m)) \cong H^0(\mathcal{F}(m))$.

$\mathrm{GL}(V) \curvearrowright R$ by $g[\rho] = \pi \circ g$ for $\rho \in R$.

$$\begin{tikzcd} \mathcal{O}^{\oplus p(m)} \arrow[r, "\rho"] \arrow[d, "g,\cong"'] & \mathcal{F} \\ \mathcal{O}_X^{\oplus p(m)} \arrow[ur] & \end{tikzcd}$$

We want "$R/\mathrm{GL}(V)$".

Theorem 4.1. *There exists a coarse moduli space $M_{X,P}$ of semi-stable sheaves with fixed Hilbert polynomial P, $\mathcal{M}_{X,P}$ is projective and $\mathcal{M}^s_{X,P}$ of stable sheaves with Hilbert polynomial P.*

Fact. The GIT (semi-)stability coincides with the Gieseker (semi-)stability (see [HL97]).

Remarks on working with stacks.

- Instead of constructing sheaves which are coarse/fine moduli spaces we may find more general geometric objects called "stacks".
- Advantage: Quotients always exist: $G \curvearrowright X \rightsquigarrow [X/G]$.
- Stacks have universal families.
- Disadvantage: we need to work with more general moduli functors. Instead of $\mathcal{M} : (\mathrm{Sch}) \to (\mathrm{Sets})$, we substitute (Sets) with a 2-category.
- Disadvantage: not all algebraic stacks have coarse moduli spaces but if the number of automorphisms of points [is finite?].
- Disadvantage: may not be projective (even if proper/compact).

In sum, we may not be able to produce a moduli space, and even if we can, it may not be projective.

Even in the easiest situation we can have, we don't have universal family:

Example 4.2 (The line bundles of degree d on $\mathbb{P}^1$ do not have universal family)**.** Line bundles on $\mathbb{P}^1$ of degree d are $\mathcal{O}(d)$. Them $\mathcal{M} = \mathrm{Spec}\mathbb{C}$. It does not have a universal family:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{P}^1 \arrow[r] & \mathrm{Spec}\mathbb{C} \end{tikzcd}$$

and $\mathcal{O}(d) \neq \mathcal{O}_{\mathbb{P}^1}$, $d \neq 0$.

5. Remarks on stacks

The following was not defined properly. As a stack,

$$\mathrm{Pic}_{\mathbb{P}^1,d} = [\cdot/\mathbb{G}_m] = B\mathbb{G}_m$$

parametrized by $\mathbb{G}_m$-bundles.

$$\begin{tikzcd} \mathcal{L} \setminus \{\text{zero section}\} \arrow[r] \arrow[d] & \Gamma^A[?] \arrow[d] \\ \mathbb{P}^1 \arrow[r, "\varphi"'] & B\mathbb{G}_m \end{tikzcd}$$

Tangent space. The tangent space of $\mathcal{M}_{X,P}$, in $\ni E$, is $\mathrm{Ext}^1(E,E)$ if $\mathrm{Ext}^2(E,E) = 0 \implies \mathcal{M}_{X,P}$ is smooth. (Ext groups are defined in Stacks Project, Algebra; this result also appears to be a theorem in deformation theory.)

Example 5.1. Let X be a curve. E is a vector bundle of rank n, then

$$\mathrm{Ext}^2(E,E) = H^2(X, E \otimes E^\vee) = 0$$

then the moduli space is smooth.

The dimension of the stack is $\dim T_{\mathcal{M},E} - \dim \mathrm{Aut}(E)$, which is just $\dim(\mathrm{Ext}^1(E,E) - \dim \mathrm{Ext}^0(E,E)$, which in turn says that $\mathcal{X}(X, E \otimes E^\vee) = V^2(f-g)$.

Question 2. Once we have a moduli space we want so study its geometry. What can we say about $\mathcal{M}_{X,P,H}$?

Let X be a projective surface and E a vector bundle. Let $r = \mathrm{rk}E$, $L = \Lambda^n E$. Consider $\mathcal{M}_{X,P,H}(r,L,n)$. $n = c_2(E)$

Theorem 5.2. *If $\Delta = 2rm - (r-1)L^2 \gg 0$ then $\mathcal{M}_{X,H}(r,L,n)$ is a normal, generically smooth irreducible quasi-projective variety. (If Δ is small then it is not true.)*

Theorem 5.3. *Fix H, H' ample. For $\Delta \gg 0$, $\mathcal{M}_{X,H}(r,L,n)$ and $\mathcal{M}_{X,H}(r,L,n)$ are birrational.*

Intuitive idea of what follows: there are ample cones, and walls between them, and when we cross the wall, we get to an object birational to the one we were previously at.

Definition 5.4. Let C_X be the ample cone $\mathbb{R} \otimes \mathrm{Num}(X)$. For $\xi \in \mathrm{Num}(X)$, $W^\xi := C_X \cap \{x \in \mathrm{Num}(X) \otimes \mathbb{R} : x \cdot \xi = 0\}$. W^ξ is called a *wall* of type $(c_1, c_2) \iff \exists G \in \mathrm{Pic}(X)$ with $\xi = G$ such that

- $G + c_1$ is divisible by 2 in $\mathrm{Pic}(X)$
- $c_1^2 - 4c_2 \leq G^2 < 0$.

Remark 5.5. $W^\xi \neq 0$ if there exists an ample line bundle L with $L\xi = 0$, $W(c_1,c_2) = \bigcup_\xi W^1$.

Definition 5.6. A *chamber* is a connected component of $C_X \setminus W(c_1,c_2)$. (Perhaps this is the same as $C_X \setminus \bigcup_\xi W\xi$.)

Theorem 5.7 (Qin)**.** *The moduli space $M_{X,H}(2c_1,c_2)$ of rank 2 locally free sheaves only depends on the chamber of H.*

Remark 5.8.
- $c_1^2 - 4c_2 < 0$ is Bogomolov inequality.
- $G^2 < 0$ is Hodge index theorem
- $F \subset E$ destabilizing line bundle $F \iff H\ c_1(F) > \frac{c_1(E)H}{2} \iff (\underbrace{2c_1(F) - c_1(E)}_{=G})H > 0 \implies 2c_1(F) = c_1(G) + c_1(E) \implies c_1(G) + c_1(E)$ is divisible by 2.
- Suppose we have $0 \to F \to E \to M \otimes |_Z \to 0$ with M a line bundle, Z codimension 2 in X.

[Lots of computations I missed.]

Exercise 5.9. $X = \mathbb{P}^1 \times \mathbb{P}^1$, E of rank 2, $c_1 = \ell$, $c_2 = 3$. $G = a\ell + bm$, $G^2 = 2ab \implies ab < 0$, $\Delta = c_1^2 - 4c_2 \leq G^2 = ab \implies -12 \leq 2ab < 0$, $(a+1)\ell + bm$ divisible by 2, then $a+1$ is even.

I understand the walls and chambers were computed: $L_0 = (\ell + 7m) \in C_0$, $L_1 = (\ell + 5m) \in C_1$ and $L_2 = (\ell + 3m) \in C_2$.

We have

- For L in C_0, $\mathcal{M}(2, \ell, 3) = \emptyset$. (Proved.)
- For L in C_1, $\mathcal{M}(2, \ell, 3) \cong \mathbb{P}^5$.
- For L in C_2, $\mathcal{M}(2, \ell, 3)$ is an open in $\mathbb{P}^3$.

For more general rank:

Theorem 5.10. *$F \subset E$ a subsheaf of rank $r' < r$, $\mu_H(F) = \mu_H(E)$, then $\xi = rc_1(F) - r'c_1(E)$ satisfies $\xi H = 0$ and $\frac{r^2}{4}\Delta \leq \xi^2 < 0$, with $e_2 \iff \xi = 0$.*

Theorem 5.11. *X smooth irreducible projective surface, H, H' ample and $\Delta \gg 0$. Then $\mathcal{M}_{X,H}$ and $\mathcal{M}_{X,H'}$ are birational (any rank).*

For dimension ≥ 3, the moduli space of sheaves is in general smooth or irreducible.

Theorem 5.12 (Ein)**.** *Let C be a smooth projective 3-fold, $c_1, H \in Pic(X)$, H ample. Assume there exists a, b such that $ac_1 = bH$. $\mathcal{M}_{X,H}$ rank 2 vector bundle on X, μ stable with respect to $d := c_2 H$. $\mu(d) = F$ irreducible component of $\mathcal{M}_{X,H}(c_1, c_2)$, $\implies \lim_{d\to\infty} m(d) = \infty$.*

Example 5.13. In DT (Donaldson-Thomas) theory, X 3-fold,

$$I(X, P) = \left\{Z \subset X : {}^{Z \text{ a curve with}}_{\text{fixed Hilbert polynomial}}\right\}$$

$P(u) = 3u + 1$ twisted cubics have $P(u) = 3u + 1$. But genus 1 curves with a point also have $P(u) = 3u + 1$. $I(\mathbb{P}^3, 3n + 1)$ has several components.

References

[HL97] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.