NONCOMMUTATIVE ABELIAN SURFACES AND KUMMER TYPE HYPERKÄHLER VARIETIES

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Notes at github.com/danimalabares/cimpa-floripa

Abstract. Examples of noncommutative K3 surfaces arise from semiorthogonal decompositions of the bounded derived category of certain Fano varieties. The most interesting cases are those of cubic fourfolds and Gushel-Mukai varieties of even dimension. Using the deep theory of families of stability conditions, locally complete families of hyperkähler manifolds deformation equivalent to Hilbert schemes of points on a K3 surface have been constructed from moduli spaces of stable objects in these noncommutative K3 surfaces. On the other hand, an explicit description of a locally complete family of hyperkähler manifolds deformation equivalent to a generalized Kummer variety is not available from classical geometry. In this lecture series, we will construct families of noncommutative abelian surfaces as equivariant categories of the derived category of K3 surfaces which specialize to Kummer K3 surfaces. Then we will explain how to induce stability conditions on them and produce examples of locally complete families of hyperkähler manifolds of Kummer type. Based on joint work with Arend Bayer, Alex Perry and Laura Pertusi.

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1. K3 surfaces and Hyperkähler varieties

Definition 1.1. A *K3 surface* (over $\mathbb{C}$) is a smooth projective surface S such that $\omega_S \cong \mathcal{O}_S$ and $\pi_1(S) = 1$.

Example 1.2 (K3 surfaces)**.**
(1) $S \subset \mathbb{P}^3$ of degree 4. A straightforward computation shows the conditions of the definition are verified.
(2) $S \to \mathbb{P}^2$ double cover ramified along a degree 6 curve.

Definition 1.3. A *hyperkähler* variety is a smooth projective variety X such that
(1) $\pi_1(X) = 1$
(2) $H^0(X, \Omega_X^2) = \mathbb{C}\omega$ (i.e. the space of global holomorphic 2-forms is one dimensional) where ω is holomorphically symplectic.

Remark 1.4. If X is hyperkähler, it must be even dimensional and $\omega_X = \mathcal{O}_X$.

Remark 1.5. We just mention the name of Beauville-Bogomolov theorem.

For a while people were looking for examples of hyperkähler varieties.

Example 1.6 (Hyperkähler varieties)**.**
(1) K3 surface S.
(2) $X = S^{[n]}$ Hilbert scheme of n points on K3 surface. (Moduli space of stable sheaves on K3 S of rank 1, $c_1 = 0$, $c_2 = 0$.)
(3) $\mathcal{M}_H(s, v) =$ moduli of H-stable sheaves on S of class v. (v primitive, H is v-generic.) (Recall the example in Cristina's course where we studied how the moduli changes under changes in the polarization H.) Let $[E] \in \mathcal{M}_H(s, v)$. We have the Yoneda map

$$T_{[E]}\mathcal{M} \cong \mathrm{Ext}^1(E,E) \times \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2(E,E) \overset{\substack{S \text{ is} \\ \text{a K3}}}{\cong} \mathrm{Hom}(E,E)^* \cong \mathbb{C}$$

where the first factor product of $\mathrm{Ext}^1(E,E) \times \mathrm{Ext}^1(E,E)$ is associated to first arrow, the second factor to the second arrow, of the following diagram:

$$E \to E[1] \to E[2]$$

This moduli space is always of Picard rank 2.
(4) $Y \subset \mathbb{P}^5$ a smooth cubic fourfold.

$$F(Y) = \{[\ell] \in \mathrm{Gr}(2,6) | \ell \subseteq Y\}$$

is a hyperkähler 4-fold. For a specific choice Y_0 of hyperkähler 4-fold in the moduli of smooth cubic 4-folds (which is 20-dimensional) we find $F(Y_0)$ is deformation equivalent to $S^{[2]}$ where S is a K3 surface.

Here is a general picture:

Let X, Y be projective smooth varieties and $f : X \to Y$. Consider the functors

$$Rf_* : D^b(X) \to D^b(Y) \qquad Lf^* : D^b(Y) \to D^b(X)$$

for $F, G \in D^b(X)$, $F \overset{L}{\otimes} G \in D^b(X)$. **Convention:** we drop the R and L.

Definition 1.7. Let $K \in D^b(X \times Y)$. The *Fourier-Mulai functor* is

$$\begin{aligned} \Phi_K : D^b(X) &\longrightarrow D^b(Y) \\ F &\longmapsto \mathrm{pr}_{Y,*}(\mathrm{pr}_X^* F \otimes K) \end{aligned}$$

where we are using our convention — all functors here are derived. Here the maps are

$$\begin{tikzcd} & X \times Y \arrow[dl, "\mathrm{pr}_X"'] \arrow[dr, "\mathrm{pr}_Y"] & \\ X & & Y \end{tikzcd}$$

Example 1.8. $f : X \to Y$ and $\Gamma_f \equiv X \times Y$ graph. Then $\Phi_{\mathcal{O}_{\Gamma_f}} = f_*$.

Theorem 1.9 (Orlov)**.** *If $F : D^b(X) \to D^b(Y)$ is an equivalence, then $\exists! K \in D^b(X \times Y)$ such that $F \cong \Phi_K$.*

Theorem 1.10. *The following are preserved by derived equivalence:*

1. *(Bondal-Orlov.)* $\dim X$.
2. *(Bondal-Orlov.)* $\bigoplus_{m \geq 0} H^0(X, \pm m K_X)$.
3. $H^*(X, \mathbb{Q})$*, and* $\bigoplus_{p-q=i} H^{p,q}(X)$ *for any* $i \in \mathbb{Z}$.

Non-trivial equivalences. S a K3 surface. $H^2(S, \mathbb{Z})$.

- (Hodge decomposition.) $H^2(S, \mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.
- Pairing on $H^2(S, \mathbb{Z})$ given by cup product.

Theorem 1.11 (Torelli for K3)**.** *Two K3 surfaces S, S' are isomorphic if and only if there exists a Hodge isometry $\varphi : H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ (i.e. an isomorphism that preserves the Hodge decomposition and the pairing).*

This was the classical content.

1.12. **Mukai lattice.** "The Mukai lattice is some sort of topological invariant that behaves better with derived category."

$$\tilde{H}(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

- Weight 2 Hodge structure
$$\tilde{H}^{2,0} = H^{2,0} \qquad \tilde{H}^{1,1} = H^0 \oplus H^{1,1} \oplus H^2 \qquad \tilde{H}^{0,2} \cong H^{0,2}.$$
- Pairing:
$$\langle (a, b, c), (a', b', c') \rangle = bb' - ac' - a'c$$

Theorem 1.13 (Mukai, Orlov)**.** *If S, S' are K3 surfaces, then $D^b(S) \cong D^b(S') \iff \tilde{H}(S, \mathbb{Z}) \underset{\varphi}{\cong} \tilde{H}(S', \mathbb{Z})$ Hodge isometry.*

1.14. **Mukai vector.** Let $K_0(S)$ be the free abelian group generated by $\mathrm{Ob}(D^b(S))$. Consider
$$v : K_0(S) \to \tilde{H}(S; \mathbb{Z})$$
Where $[F] = [E] + [G]$ if $E \to F \to G \xrightarrow{+1}$ is a distinguished triangle.
$$v : [E] \to \mathrm{ch}(E) \cdot \sqrt{\det(S)}$$
where $\mathrm{ch}(E)$ is the Chern character.
$$\langle v(E), v(F) \rangle = -\chi(E, F),$$
where $\chi(E, F) := \sum (-1)^i \dim \mathrm{Ext}^i(E, F) := \mathrm{Hom}_{D^b(S)}(E, F[i])$.

Proof of the backward implication of Mukai-Orlov theorem. Let

$$\begin{aligned} \varphi : \tilde{H}(S,\mathbb{Z}) &\longrightarrow \tilde{H}(S',\mathbb{Z}) \\ (0,0,1) &\longmapsto v \\ (-1,0,0) &\longmapsto v' \end{aligned}$$

The intersection matrix of v and v' is

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For simplicity assume that v is of positive rank. ($V = (a,b,c)$, where $a \in H^0$ is positive.)

The heart of the proof uses the following result by Mukai. There exists a nonempty moduli space $\mathcal{M}$ of stable sheaves on S' of class $v \implies S'$ is a K3 surface!

$$0 = \chi(v,v) = \underbrace{\mathrm{Hom}(E,E)}_{=1} - \underbrace{\mathrm{ext}^1(E,E)}_{=2} + \underbrace{\mathrm{ext}^2(E,E)}_{=1}$$

$v \cdot v' = 1 \implies \mathcal{M}$ is a fine moduli space. There exists a universal family $D^b(S \times M) \ni \mathcal{E} \to S' \times M$.

Claim. $\Phi_{\mathcal{E}} : D^b(S') \xrightarrow{\cong} D^b(\mathcal{M})$ (general criteria).

Now $\tilde{H}(S,\mathbb{Z}) \xrightarrow{\varphi,\cong} \tilde{H}(S',\mathbb{Z}$

$$\begin{tikzcd}
\tilde{H}(S,\mathbb{Z}) \arrow[r, "\varphi,\cong"] & \tilde{H}(S',\mathbb{Z}) \arrow[r, "\Phi_{\mathcal{E}}"] & \overset{\vee}{H}(M,\mathbb{Z}) \\
(0,0,1) \arrow[r, mapsto] & v \arrow[r, mapsto] & (0,0,1) \\
(1,0,0) \arrow[rr, mapsto] & & (1,0,0)
\end{tikzcd}$$

□

2. Semiorthogonal decomposition and Calabi-Yau categories

Situation 2.1.

$$\begin{tikzcd}
D^b(Y), {}^{Y \text{ a cubic}}_{\text{4-fold}} \arrow[dr, "\substack{\text{semiorthogonal} \\ \text{decomposition}}"'] & & \substack{\text{moduli} \\ \text{on K3} \\ \text{as HK}} \\
& \mathrm{Ku}_Y \, {}^{\text{K3}}_{\text{category}} \arrow[ur, "\text{stability conditions}"'] &
\end{tikzcd}$$

Definition 2.2. Let X be a projective smooth variety over $\mathbb{C}$. A *SOD* $D^b(X) = \langle A_1, \dots, A_n \rangle$ is a sequence A_i full triangulated subcategory such that

(1) $\mathrm{Hom}(F,G) = 0$ for all $F \in A_i$, $G \in A_j$.
(2) $\forall F \in D^b(X)$ $\exists 0 = F_n \to F_{n-1} \to \dots \to F_0 = F$ such that $\mathrm{Cone}(F_i \to F_{i-1}) \in A_i$.

Exercise 2.3. (1) $D^b(\mathrm{Spec}\mathbb{C}) \ni V \simeq \bigoplus_i H^i(V)[-i]$.

(2) For $E \in D^b(X)$ define

$$\begin{aligned}\phi_E : D^b(\mathrm{Spec}\mathbb{C}) &\longrightarrow D^b(X)\\ V &\longmapsto V \otimes E = \bigoplus_i H^i(V) \otimes E[-i]\end{aligned}$$

Then ϕ_E fully faithful $\iff$ E is exceptional

$$\left(\mathrm{Ext}^p(E,E) = \begin{cases}\mathbb{C} & p = 0\\ 0 & p \neq 0\end{cases}\right)$$

Definition 2.4. $\langle E\rangle := \phi_E(D^b(\mathrm{Spec}\mathbb{C})) \subset D^b(X)$ for exceptional E.

(3) **Example.** $\mathcal{O}(i)$ exceptional on $\mathbb{P}^n$. (Beilinson.) $D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\rangle$

Definition 2.5. A triangulated category $A \subset D^b(X)$ is *right admissible* if $\alpha : A \to D^b(X)$ admits a right adjoint. $\alpha^! : D^b(X) \to A$. (I.e., $\mathrm{Hom}_{D^b(X)}(\alpha(E), F) \simeq \mathrm{Hom}_A(E, \alpha^!(F))$

Exercise 2.6. Let E be exceptional. $\langle E\rangle \subset D^b(X)$ is right admissible.

$$\alpha^! = R\,\mathrm{Hom}(E,F) \in D^b(\mathrm{Spec}\mathbb{C})$$

Let

$$A^\perp := \{F \in D^b(X) : \mathrm{Hom}(E,F) = 0, \forall E \in ?\}$$

Lemma 2.7. *$A \subset D^b(X)$ is right admissible* $\implies$ $D^b(X) = \langle A^\perp, A\rangle$.

Proof. (1) Is clear.

(2) $0 = F_2 \to F_1 \to F_0 = F$, $F_1 \in A$, $\mathrm{Cone}(F_1 \to F) \in A^\perp$. We use adjunction as follows:

$$\mathrm{id} \in \mathrm{Hom}(\alpha^!(F), \alpha^!(F)) = \mathrm{Hom}(\alpha\alpha^! F, F)$$

We have the following exact triangle:

$$\alpha\alpha^! F \xrightarrow{\text{counit}} F \longrightarrow B$$

Claim. $B \in A^\perp$. Let $E \in A$

$$\begin{tikzcd}\cdots \arrow[r] & \mathrm{Hom}(\alpha(E), \alpha\alpha^!(F)) \arrow[rr] \arrow[dr, "\cong"] & & \mathrm{Hom}(\alpha(E), F) \arrow[r] \arrow[dl, "\cong"'] & \mathrm{Hom}(\alpha(E), B)\\ & & \mathrm{Hom}(E, \alpha^!(F)) & & \end{tikzcd}$$

which implies that $\mathrm{Hom}(\alpha(E), B) = 0$.

□

Definition 2.8. *Left mutation* $\mathbb{L}_A$ is defined as

$$\alpha\alpha^! F \to F \to \mathbb{L}_A F.$$

Exercise 2.9. $A = \langle E\rangle$.

$$R\,\mathrm{Hom}(E,F) \otimes E \xrightarrow{\text{ev}} F \to \mathbb{L}_{\langle E\rangle} F$$

Corollary. $E_1, \ldots, E_n \in D^b(X)$ exceptional objects with $\mathrm{Ext}^\bullet(E_i, E_j) = 0$ for $i > j$ (we say this is an *exceptional collection*). Then

$$D^b(X) = \langle R_x, E_1, \ldots, E_n\rangle, \qquad R_X = \langle E_1, \ldots, E_n\rangle^\perp$$

Example 2.10. X Fano, $-K_X = rH$, H ample, $r > 0$, $\mathcal{O}_X, \mathcal{O}_X(H), \ldots, \mathcal{O}_X((r-1)H)$ an exceptional collection.

($\mathrm{Ext}^\bullet(\mathcal{O}(iH), \mathcal{O}(jH)) = H^\bullet(X, \mathcal{O}(j-i)H) = 0$) if $-r < j - i < 0$. $\implies$

$$D^b(X) = \langle R_X, \mathcal{O}_X \ldots \mathcal{O}_X(r-i)H\rangle$$

Definition 2.11. A *Serre functor* for a Δ-category $\mathcal{D}$ is an autoeq $S_{\mathcal{D}}$ such that

$$\mathrm{Hom}_{\mathcal{D}}(E, F)^\vee \cong \mathrm{Hom}(F, S_{\mathcal{D}}(E)).$$

functorially in $E, F \in \mathcal{D}$.

Remark 2.12. It is unique if it exists.

The following example explains why this is called the Serre functor — it is a generalization of Serre duality.

Example 2.13. X smooth projective variety of dimension n. Consider

$$S_{D^b(X)} = (- \otimes \omega_X)[n]$$

eg. E locally free on X.

$$\begin{aligned} H^i(X, E) &= \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, E[i]) \\ &= \mathrm{Hom}_{D^b(X)}(E[i], \omega_X(n])^* \\ &= \mathrm{Hom}(\mathcal{O}, E^\vee \otimes \omega_X[n-i]) \\ &= H^{n-i}(E^\vee \otimes \omega_X)^* \end{aligned}$$

Definition 2.14. $\mathcal{D}$ is

- *Calabi-Yau category* of dimension n if $S_{\mathcal{D}} \cong [n]$. (I.e. the Serre functor is just shifting by n.)
- *Fractional Calabi-Yau category* of dimension p/q if $S^q_{\mathcal{D}} \cong [p]$ (where the exponent q just means composing the functor $S_{\mathcal{D}}$ $q \in \mathbb{Z}$ times).

Theorem 2.15 (Kuznetsov)**.** *$X \subset \mathbb{P}^n$ smooth Fano hypersurface of degree $d \leq n$.*

$$D^b(X) = \langle Ku_X, \mathcal{O}_X, \ldots, \mathcal{O}_X(n-d)\rangle$$

Then Ku_X has a Serre functor S with $S^{d/c} \cong \left[\frac{(n+1)(d-2)}{c}\right]$ where $c = gcd(d, n+1)$.

Example 2.16. $d = 3$, a cubic 3-fold, $n = 4$, $S^3 = [5]$, cubic 4-fold $n = 5$, $S = [2]$.

Theorem 2.17 (Kuznetsov)**.** *X cubic 4-fold,*

1. *$\exists$ special X such that $Ku_X \simeq D^b(S)$, where S is a K3 surface.*
2. *X very general $\implies$ $Ku_X \not\simeq D^b(var)$.*

We shall not prove this theorems. However, by the fourth lecture we may give the idea of their proofs.

Conjecture (Kuznetsov). Cubic 4-fold X is rational if and only if $\mathrm{Ku}_X \simeq D^b(\text{K3 surface})$.

[KKPY] very general X is not rational.

Let's go back and show how to prove the statement in Example 2.16 about the cubic 4-fold.

Lemma 2.18. *Assume that $\mathcal{D} = \langle A^\perp, A\rangle$, $S_{\mathcal{D}}\exists$. Then $S^{-1}_{A^\perp} = \mathbb{L}_A \circ S^{-1}_{\mathcal{D}}|_{A^\perp}$.*

Proof. $F, G \in A^\perp$,

$$\begin{aligned}\operatorname{Hom}_{A^\perp}(F,G) &\cong \operatorname{Hom}_{\mathcal{D}}(G, S_{\mathcal{D}}(F))^\vee \\ &\cong \operatorname{Hom}_{\mathcal{D}}(S_{\mathcal{D}}^{-1}(G), F)^\vee \\ &\cong \operatorname{Hom}_{\mathcal{D}}(\mathbb{L}_A(\underbrace{S_{\mathcal{D}}^{-1}(G))}_{\in A^\perp}, F)^\vee\end{aligned}$$

□

Now: cubic 4-fold $X \subset \mathbb{P}^5$,

$$D^b(X) = \langle \mathrm{Ku}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$$

The lemma tells us that

$$S_{\mathrm{Ku}_X}^{-1} = \mathbb{L}_{\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle} \circ (- \otimes \mathcal{O}(3)[-4])$$

Key observation 1. $X \overset{i}{\hookrightarrow} \mathbb{P}^5$, for $F \in D^b(X)$ there exists an exact Δ:

$$i^* i_* F \to F \to F \otimes \mathcal{O}_X(-3)[2]$$

Assume $F \in \mathrm{Ku}_X$, apply S_{Ku}^{-1}.

$$\mathbb{L}_{\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle} \circ (i^* i_* F \otimes \mathcal{O}(3)[-4]) \to S_{\mathrm{Ku}}^{-1}(F) \to \underbrace{\mathbb{L}_{\langle \dots \rangle} \circ (F[-2])}_{\cong F[-2], \text{ as } F \in \mathrm{Ku}_X}$$

Key observation 2. $i^* i_* F \otimes \mathcal{O}(3) \in \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$, so the first term vanishes. (It's enough to show that $i_* F \otimes \mathcal{O}(3) \in \langle \mathcal{O}_{\mathbb{P}^5}, \mathcal{O}_{\mathbb{P}^5}(1), \mathcal{O}_{\mathbb{P}^5}(2)\rangle \iff i_* F \in \underbrace{\langle \mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1)\rangle}_{= \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle^\perp} \subset D^b(\mathbb{P}^5)$.)

$$R\operatorname{Hom}\begin{pmatrix}\mathcal{O} \\ \mathcal{O}(1), i_* F \\ \mathcal{O}(2)\end{pmatrix} = 0$$

$$R\operatorname{Hom}\begin{pmatrix}\mathcal{O} \\ i^*\mathcal{O}(1), F \\ \mathcal{O}(2)\end{pmatrix} = 0 \qquad \text{since } F \in \mathrm{Ku}_X$$

and the left-hand-sides on the last two equations are $\cong$.

Proof of Key observation 1 assuming F is a sheaf. Take $i^* i_* F \to F \to G$ and consider its pushforward

$$i_* i^* i_* F \to i_* F \to i_* G$$

Note that

$$\begin{aligned}
i_*i^*i_*F \simeq i_*F\overset{L}{\otimes}_{\mathcal{O}_{\mathbb{P}^5}} &\cong \left[\underbrace{i_*F\otimes\mathcal{O}_{\mathbb{P}^5}(-x)}_{\deg -1}\xrightarrow{\vee} i_*F\otimes\mathcal{O}_{\mathbb{P}^5}\right]\\
&\simeq i_*F\oplus i_*F\otimes\mathcal{O}_{\mathbb{P}^5}(-x)[1]\\
&\xrightarrow[\text{1}^{\text{st}}\text{ factor}]{\text{proj. to}} i_*F\\
&\implies i_*G\simeq i_*F\otimes\mathbb{P}_{\mathbb{P}^5(-x)[2]}\\
&\implies G\simeq F\otimes\mathcal{O}_X(-x)[2] \qquad i_* \text{ f.f.}
\end{aligned}$$

□

3. Inducing t-structure and stability conditions

Example 3.1. X cubic 3-fold, $D^b(X)=\langle \mathrm{Ku}_X,\mathcal{O},\mathcal{O}(1)\rangle$.

Goal. Use t-structure/(weak) stability conditions on $D^b(X)$ to induce a t-structure/stability conditions on Ku_X.

Proposition 3.2 (Key proposition)**.** *Let $\mathcal{D}=\langle R,E_1,\dots,E_n\rangle$ be a semiorthogonal decomposition. Assume that $\mathcal{A}\subset\mathcal{D}$ is the heart of a bonded t-structure such that*

(1) *$\forall i$, $E_i\in\mathcal{A}$.*
(2) *$\forall i$, $S_{\mathcal{D}}(E_i)\in\mathcal{A}[i]$.*

Then $\mathcal{A}:=\mathcal{A}\cap R$ is the heart of a bounded t-structure on R. (Concretely, this means for $F\in R$, view it as an object in $\mathcal{D}$ and take cohomology with respect to the t-structure, we need $H^q_{\mathcal{A}}(F)\in R$ $\forall q$.)

[BLMS] using spectral sequence.

Example 3.3. For X cubic 3-fold and $D^b(X)=\langle \mathrm{Ku}_X,\mathcal{O},\mathcal{O}(1)\rangle$,

$$\left(Coh^{-\frac{1}{2}}(X), Z_{\alpha,-\frac{1}{2}}=\frac{1}{2}\alpha^2H^2ch_0-ch_2^{-\frac{1}{2}}+iHch_1^{-\frac{1}{2}}\right)$$

Exercise 3.4.
- For all $\alpha>0$, $\mathcal{O}$, $\mathcal{O}(1)$, $\mathcal{O}(-2)[1]$, $\mathcal{O}(-1)[1]\in Coh^{-\frac{1}{2}}$.
- For all $\alpha\sim 0$,

$$\begin{aligned}\mu_{\alpha,-\frac{1}{2}}(\mathcal{O}(-2)[1])<\mu_{\alpha,-\frac{1}{2}}(\mathcal{O}(-1)[1])<0\\
<\mu_{\alpha,-\frac{1}{2}}(\mathcal{O})<\mu_{\alpha,-\frac{1}{2}}(\mathcal{O}(1)).\end{aligned}$$

So tilt $Coh^{-\frac{1}{2}}(X)$ one more time w.r.t. $\mu_{\alpha,-\frac{1}{2}}=0$. We can apply Key Proposition 3.2.

Example 3.5. How about cubic 4-fold Y? Idea: embed $Ku_X\hookrightarrow\underbrace{D^b(\mathbb{P}^3,\mathcal{D}_0)}_{\text{conic fibration}}$.

Theorem 3.6. *There exist stability conditions on Ku_Y for a cubic 4-fold Y.*

Now we shall give the idea of why Key Proposition 3.2 holds.

General question modeling Key Proposition 3.2: $F:\mathcal{C}\to\mathcal{D}$ exact functor between triangulated categories. Say $\mathcal{D}^{heart}$ is the heart of a t-structure on $\mathcal{D}$. Under what condition is

$$\mathcal{C}^{heart}=\{E\in\mathcal{C}|F(E)\in\mathcal{D}^{heart}\}$$

the heart of a t-structure on $\mathcal{C}$?

Example 3.7. Let X, Y be smooth projective and $f : X \to Y$ a finite a morphism.

$$\begin{tikzcd} D^b(X) \arrow[r, "f_*"'] \arrow[d, hook] & D^b(Y) \arrow[l, bend right, "f^*"'] \arrow[d, hook] \\ D(QCoh(X)) = D_{qc}(X) \arrow[r, "f_*"'] & \underbrace{D_{qc}(Y)}_{\substack{\text{admits small} \\ \text{direct sum}}} \arrow[l, bend right, "f^*"'] \end{tikzcd}$$

Situation 3.8. $\tilde{\mathcal{C}}, \tilde{\mathcal{D}}$ triangulated categories admitting direct sums. $\mathcal{C} \subset \tilde{\mathcal{C}}$, $\mathcal{D} \subset \tilde{\mathcal{D}}$ full essentially small triangulated subcategories. $F : \tilde{\mathcal{C}} \to \tilde{\mathcal{D}}$ exact functor such that

- F commutes with direct sums.
- F has left adjoint $\mathcal{G} : \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$.
- $F(\mathcal{C}) \subseteq \mathcal{D}$ and if $E \in \tilde{\mathcal{C}}$ is such that $F(E) \in \mathcal{D}$ then $E \in \mathcal{C}$.
- $G(\mathcal{D}) \subseteq \mathcal{C}$.
- For $E \in \mathcal{C}$, if $F(E) = 0 \implies E = 0$.

Definition 3.9. Let $\mathcal{D}^{heart}$ be the heart of a t-structure on $\mathcal{D}$. Define

$$\mathcal{D}^{\leq 0} = \{E \in \mathcal{D} | H^i(E) = 0, i > 0\} \mathcal{D}^{\geq 0} \qquad = \{E \in \mathcal{D} | H^i(E) = 0, i < 0\}$$

Theorem 3.10 (Polischuk)**.** *Let $\mathcal{D}^{heart}$ be the heart of a bounded t-structure on $\mathcal{D}$. If $FG : \mathcal{D} \to \mathcal{D}$ is right t-exact (i.e. $FG(\mathcal{D}^{\leq 0}) \subseteq \mathcal{D}^{\leq 0}$) then*

$$\mathcal{C}^{heart} = \{E \in \mathcal{C} | F(E) \in \mathcal{D}^{heart}\}$$

is the heart of a bounded t-structure on $\mathcal{C}$.

Example 3.11. $f_* : D^b(X) \to D^b(Y)$, $f_* f^*(E) = E \otimes f_* \mathcal{O}_x$. If this is right t-exact, then can induce t-structures, e.g. $f : H \hookrightarrow Y$ hypersurface.

$$E \to E \otimes f_* \mathcal{O}_H \to E \otimes \mathcal{O}_T(-H)[1]$$

See recent paper by Chengi Li "Real reduction ..."

Example 3.12. $D^b(X) = \langle R, E \rangle$, $i : R \to D^b(X)$. See [Kuznetsov]. There exists a semiorthogonal decomposition of $D_{qc}(X) \supset \tilde{R}$, $i^* \to i$,

$$\mathbb{L}_{\langle E \rangle} = ii^* : D^b(X) \to D^b(X) \qquad \text{right exact?}$$

The reader is really encouraged to solve the following exercise.

Exercise 3.13. Let $E \in \mathcal{D}^{heart}$, $S(E) \in \mathcal{D}^{heart}[1]$, then $\mathbb{L}_E$ is right exact.

Key homological algebra ingredient. Let $\tilde{\mathcal{C}}^{\leq 0}$ be the smallest full subcategory of $\tilde{\mathcal{C}}$ that

- contains $G(\mathcal{D}^{\leq 0})$,
- is closed under direct sum, extension, positive shift $[n]$, $n > 0$.

$$\tilde{\mathcal{C}}^{\geq 0} := \{E \in \tilde{\mathcal{C}} | \operatorname{Hom}(\tilde{\mathcal{C}}^{\leq 0}[1], E) = 0\}.$$

Then this is a t-structure on $\tilde{\mathcal{C}}$.

Example 3.14. Let X be a smooth projective curve and $p \in X$ a point. Then $D_{qc}(X)^{\leq 0}$ contains $\mathcal{O}_p$ and is closed under . . . .

$$D_{qc}(X)^{\geq 0} j_*\mathcal{O}_U/\mathcal{O}_X[-1] \to \mathcal{O}_X \to j_*\mathcal{O}_U$$

4. Moduli of objects in derived category

4.1. **Moduli on a variety.** Let X be a scheme of finite type over $\mathbb{C}$.

Definition 4.2. An object $E \in D^b(X)$ is a *perfect complex* if locally on X, E is isomorphic to a bounded cx of locally free sheaves $D_{\mathrm{perf}}(X) \subset D^b(X)$ subcategory of perfect cx. [?]

Remark 4.3.
(1) $D_{perf}(-)$ is preserved under derived pullback.
(2) X smooth $\implies D_{\mathrm{perf}}(X) = D^b(X)$.

Definition 4.4. Given T scheme of finite type over $\mathbb{C}$ and $E \in D_{\mathrm{perf}}(X \times T)$, say E is *universally gluable* over T if for all $t \in T(\mathbb{C})$, $\mathrm{Ext}^i(E_t, E_t) = 0$ if $i < 0$, where E_t is the derived pullback.

Example 4.5. Locally free on $X \times T$.

We will consider the following functor to formulate the moduli problem:

Definition 4.6. Let X be a smooth projective variety over $\mathbb{C}$. The *moduli stack of perfectly universally gluable objects* is the functor

$$\begin{aligned} \mathcal{M}_{\mathrm{proj}}(X) : (\mathit{Sch}^{ft}/\mathbb{C})^{op} &\longrightarrow \mathit{Gpds} \\ T &\longmapsto \{E \in D_{\mathrm{perf}}(X \times T) \text{ uniersally gluable}/T\} \end{aligned}$$

Theorem 4.7 (Lieblich)**.** *$\mathcal{M}_{proj}(X)$ is an algebraic stack locally of finite type over $\mathbb{C}$.*

4.8. **Moduli of σ-semistable objects.** Let X be a smooth projective variety over $\mathbb{C}$. Fix $\sigma = (A, Z) \in \mathrm{Stab}_\Lambda(X)$, $v : K_0(X) \to \Lambda$. Fix $w \in \Lambda$, define

$$\begin{aligned} \mathcal{M}_\sigma(w) : (\mathit{Sch}^{ft}/\mathbb{C})^{op} &\longrightarrow \mathit{Gpds} \\ T &\longmapsto \left\{ \begin{matrix} \text{perfect complex } E \text{ on } X \times T \text{ s.t.} \\ \forall t \in T(\mathbb{C}),\ E_t \text{ is in } \mathcal{A} \\ \sigma\text{-ss},\ V(E_t) = w \end{matrix} \right\} \end{aligned}$$

Hard problems.
(1) Are $\mathcal{M}^s_\sigma(w) \subset \mathcal{M}_\sigma(w) \subset \mathcal{M}_{\mathrm{proj}}(X)$.
  (a) Open.
  (b) Finite type?
  Known for X surface, σ via tilting (Toda).
  - [Halper-Leistner-Robotis '25].
  - Proper SC.
(2) Do there exist maps to schemes

$$\begin{tikzcd} \mathcal{M}_\sigma(w) & M_\sigma(w) \\ \mathcal{M}^s_\sigma(w) \arrow[u, hook] \arrow[r] & M^s_\sigma(w) \arrow[u, hook] \end{tikzcd}$$

where $M_\sigma(w)$ is projective and paramatrizes s-equivalent classes.

Definition 4.9. Let $\mathcal{M}$ be an algebraic stack, M algebraic space. $\pi : \mathcal{M} \to M$ is a *good moduli space* if

(1) $\pi_* : \mathrm{QCoh}(\mathcal{M}) \to \mathrm{QCoh}(M)$ is exact.
(2) $\mathcal{O}_M \xrightarrow{\simeq} \pi_* \mathcal{O}_{\mathcal{M}}$

Example 4.10. X affine, G linear reductive $G \curvearrowright X$, $[X/G] \to X//G$ is a good moduli space. **Exercise:** think this through!

Theorem 4.11 (Alper-Halpern-Leistor-Heisloth)**.** *Good moduli space exists if the stack satisfies Θ-reductivity and S-completeness.*

Where, roughly,

- Θ-reductivity means "HN filtration specializes",
- S-completeness means: two specializations of a family of objects differ by "elementary modification".

Example 4.12 (Elementary modification I)**.** $\mathbb{P}_0 = \mathbb{P}^1 \times \{0\} = \mathbb{P}^1 \times \mathbb{A}^1$,

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & \mathcal{O}^{\oplus 2}_{\mathbb{P}^1 \times \mathbb{A}^1} \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^0}(1) \arrow[r] & 0 \\ & & & \mathcal{O}_{\mathbb{P}_0^{\oplus 2}} \arrow[ur] & & \end{tikzcd}$$

Example 4.13 (Elementary modification II)**.** $C \times \mathbb{A}^1$, E rank 2 vector bundle such that

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow E|_{co} \longrightarrow \mathcal{L} \longrightarrow 0$$

for $\deg \mathcal{L}_1 = \deg \mathcal{L}^2$.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & E \arrow[rr] \arrow[dr] & & \mathcal{L}_2 \arrow[r] & 0 \\ & & & E|_{c_0} \arrow[ur] & & \end{tikzcd}$$

Upshot. $\mathcal{M}_\sigma(w) \to M_\sigma(w)$ good moduli exist.

- Bayer-Macri: there exists a (numerical) divisor class on $M_\sigma(w)$ that intersects every curve positively.

4.14. **Mukai's theorem for K3 category.**

Proposition 4.15. *X cubic 4-fold. Then there exists a Mukai lattice (cf Subsection 1.12) $\tilde{H}(Ku'_X; \mathbb{Z})$ with*

- *Weight 2 Hodge structure,*
- *Pairing compatible with $-\chi$.*

Theorem 4.16 (BLMNPS)**.** *Let $0 \neq v \in \tilde{H}_{alg}(Ku_X; \mathbb{Z})$ primitive, σ generic w.r.t. v. Then $\mathcal{M}_\sigma(v)$ has a good moduli space $M_\sigma(v) \neq \emptyset$ smooth projective hyperkähler variety of dimension $v^2 + 2$, deformation equivalent to $K3^{[\frac{v+1}{2}]}$.*

(The deepest part of this theorem is showing that $M_\sigma(v)$ is not empty!)

There exists a rank 2 lattice contained in $\tilde{H}_{alg}(Ku_X; \mathbb{Z})$ for any $X \rightsquigarrow M_\sigma(v)$ deforms in 20-dimensional family.

5. Noncommutative abelian surfaces and Hyperkähler of Kummer type

Joint work with Boyer, Perry, Petrusi ('25?)

5.1. **Polarized hyperkähler.** Recall Definition 1.3. In the past lectures we have seen that the spaces $K3^{[n]}$ are examples.

Let A be an abelian surface ($A \cong \mathbb{C}^2/\Lambda \hookrightarrow \mathbb{P}^N$). The fact that abelian surfaces are not abelian makes $M_H(v)$, the moduli of H-semistable sheaves of topological class V on A, not abelian. So we consider

$$\begin{tikzcd} M_H(v) \arrow[d, "\mathrm{alb}"'] \\ \hat{A} \times A \end{tikzcd}$$

where we must assume that v is primitive, $v^2 > 0$ and σ/H is v-gen. Then $M := \mathrm{alb}^{-1}(0)$ is a hyperkähler variety called *generalized Kummer*.

Now consider the even cohomology $H^{ev}(Z, \mathbb{Z})$. Let

$$H^{ev}(A;\mathbb{Z}) \supset V^{\perp} \xrightarrow{\theta, \simeq} H^2(M)$$

(Torelly theorem, [..., Verbitsky])

Remark 5.2.
$$\Lambda = \left\langle v, \theta^{-1}(H) \right\rangle \subset H^{ev}_{alg}(A, \mathbb{Z})$$
positive definite rank 2 lattice.

Goal. Fix Λ. Look for: Λ-polarized deformation of $D^b(A)$ (i.e. Λ remains alg.).

5.3. **Deformations via equivariant categories.** Consider an involution on an abelian surface, $\mathbb{Z}/2 \curvearrowright A$. This induces an action $\mathbb{Z}/2 \curvearrowright D^b(A)$.

There are 16 singular points in the quotient $A/(\mathbb{Z}/2)$. The resolution $\widetilde{A/(\mathbb{Z}/2)} = S$ is the so-called *Kummer K3*. (Another way to see this is considering the quotient stack A by $\mathbb{Z}/2$, the result is "basically" equivalent to S.)

[BKR] $D^b(S) \cong D^b(A)^{\mathbb{Z}/2}$.

[Elagin] $D^b(A) \cong D^b(S)^{\widehat{\mathbb{Z}/2}}$ where $\widehat{\mathbb{Z}/2}$ is an involution acting on the derived category of S, and we explain it next: there are 16 exceptional divisors $E_1, \ldots, E_{16} \subset S$. The involution is

$$(- \otimes \mathcal{L}) \cdot \prod_{i=1}^{16} ST_{\mathcal{O}_{E_i}(-1)}.$$

Key 1. In order to deform $D^b(A)$, we only need to deform $D^b(S)$ together with its involution.

Definition/Proposition.

- The derived Kummer lattice $\tilde{K} \subset \tilde{H}^*(S;\mathbb{Z})$ is the (-1)-eigenspace of $\widehat{\mathbb{Z}/2}$ action on $\tilde{H}(S;\mathbb{Z})$.
- Spanned by $(0, 2E_i, -1)$.

General theory (for deforming K3 surfaces). Due to Huybrechts, Toda, Addington-Thomas. To deform $D^b(S)$ and its $\widehat{\mathbb{Z}/2}$-action is equivalent to deforming $D^b(S)$ such that $\tilde{K}$ remains algebraic.

What need to remain algebraic under deformation?

- $\tilde{K}^{16}$ (so that the involution deforms).
- Λ^2.

[The dimension of the moduli space must then be $22 - 16 = 4$.]

Lemma 5.4.

$$\left\langle \tilde{K}, \Lambda \right\rangle \supset U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\underbrace{\left\langle \tilde{K}, \Lambda \right\rangle}_{(2,16)} = \underbrace{U}_{(1,1)} \oplus \underbrace{\Lambda_T}_{(1,15)}$$

Key 2. There exists

$$\begin{array}{l} \Phi : D^b(S) \longrightarrow D^b(T) \\ \qquad \tilde{H}(S)\tilde{H}(T) \left\langle \tilde{K}, U \right\rangle \quad U \qquad\qquad \longmapsto H^0 \oplus H^4 \\ U \oplus \Lambda_T \quad \Lambda_T \longmapsto \Lambda'_T \subset H^2 \end{array}$$

where T is a K3 surface and $\tilde{H}(S) \supset \left\langle \tilde{K}, \Lambda \right\rangle = U \oplus \Lambda_T$. Deform T as a Λ_T-polarized K3 surface.

Cine $\tilde{K}$ remains algebraic, invariant deformations w/ $D^b(T)$ take the equiv. category.

Theorem 5.5. *There exists a quasi-finite dominant map*

$$U \longrightarrow Mod^{K3}_{\Lambda_T}$$

such that $u \in U$. $D^b(T_u)^{\widehat{\mathbb{Z}/2}} =$ *4-dimensional family of categories deforming.*

Theorem 5.6. *For each* $u \in U$, *there exists an involution* $\widehat{\mathbb{Z}/2}$ *stability condition on* $D^b(T_u)$.

From these two theorems we obtain

$$\begin{tikzcd} \substack{\text{Kummer} \\ \text{type} \\ \text{HK}} \rightarrow \mathcal{M}_\sigma(D^b(T_u)^{\widehat{\mathbb{Z}/2}}, v) \arrow[d, "alb"] \\ \text{abelian 4-fold} \end{tikzcd}$$

Definition 5.7. An abelian 4-fold W is of *Weil type* if $\mathbb{R}(\sqrt{-d}) \hookrightarrow \mathrm{End}_{\mathbb{Q}}(W)$ for $d \in \mathbb{Z}_+$ along with some mild condition.

Let $\Lambda = \langle v, w \rangle$. Consider the abelian category $\mathcal{A}$ nc abelian surface. [In fact, $\mathrm{Aut}^0(\mathcal{A})$ is an abelian 4-fold.]

$$\begin{tikzcd} M_\sigma(v) \arrow[dr, "alb_v"'] & \mathrm{Aut}^0(\mathcal{A}) \arrow[l] \arrow[r] \arrow[d, bend right, "c_v"'] \arrow[d, bend left, "c_w"] & M_\sigma(w) \arrow[dl, "alb_w"] \\ & \mathrm{Alb}(\mathcal{A}) & \end{tikzcd}$$

Theorem 5.8. $c_w^{-1} \circ c_v$ *is an endomorphism of Weil type.*