# Blown-up toric surfaces with non-polyhedral effective cone

Jenia Tevelev
University of Massachussets, Amherst
(Joint with with Castravet, Laface, Ugagluia, Crelle, 2023)

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**Motivation** Understand birational geometry of $\overline{m}_{0,n}$, the moduli space of stable rational curves with $n$ markings.

$$M_{0,n} = \{\text{*some diagram involving } \mathbb{P}^{1}\text{*}\}/\operatorname{PGL}(2)$$

is a smooth affine variety of dimension $n = 3$.

$\overline{M}_{0,n} \supset M_{0,n}$ smooth projective variety.

**Universal families**

$$\begin{array}{ccc} U_n & \text{something in } \mathbb{P}^1 & \text{drawing} \\ \downarrow & \downarrow & \downarrow \\ \overline{M}_{0,n} & M_{0,n} & \partial\overline{M}_{0,n} \end{array}$$

- simple normal crossings divisor.
- Stable rational curve = tree of $\mathbb{P}^1$ with $n$ different & smooth points.
- Nodal singularities $(xy = 0) \subset \mathbb{C}^2$.
- Every compact component should have at least 3 "special" points.

**Lemma.** $U_n \cong \overline{M}_{0,n+1}$,

$$\overline{M}_{0,n+1} \longrightarrow \overline{M}_{0,n}$$

is a forgetful map + stabilizer.

Let's talk a bit about $M_{0,4}$.

$$M_{0,4} = \{0, 1, \infty, x\} = \mathbb{P}^1 - \{0, 1, \infty\}$$

There's only one compactification of this: $\overline{M}_{0,4} = \mathbb{P}^1$. What are the fibers? Drawings of fibers at 0, 1, $\infty$.

$\mathbb{P}^1$ = pencil of conics through $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$. So in this case

$$\begin{array}{c} U_q = \mathrm{Bl}_4\, \mathbb{P}^2 \text{del Pezzo surface of degree ?} \\ \downarrow \\ \overline{M}_{0,5} \end{array}$$

# 1 A normal $\mathbb{Q}$-factorial projective variety

Let X be a normal $\mathbb{Q}$-factorial projective variety.

$$\begin{array}{c} \mathrm{Pic}(X) \subset \mathrm{Cl}(X) \\ \text{Cartier divisorsWeil divisors} \end{array}$$

**Definition.** $\mathbb{Q}$-factorial: every $D \in \mathrm{Cl}(X)$ is $\mathbb{Q}$-Cartier (in $D \in \mathrm{Pic}(X)$ for some $m > 0$.

$$D \in \mathrm{Cl}(X) \qquad C \subset X \text{ integral curve}, \qquad D \cdot C \in \mathbb{Q}$$

## 2 Neron-Severi spaces

$$N'(X) = \{\sum_{a_i \in \mathbb{R}} a_i D_i\}/\equiv \qquad D \equiv 0 \text{ if } D \cdot C = 0 \forall C \subset X$$

$$N_1(X) = \left\{\sum_{a_i \in \mathbb{R}} a_i C_i\right\}/\equiv \qquad C \equiv 0 \text{if } D \cdot C = 0 \forall D \subset X$$

- $N'(X)$ and $N_1(X)$ are dual (intersection pairing) finite-dimensional vector spaces.
- ***Pseudo-effective cone*** Eff $\subset N^1(X)$ closure of the cone spanned by numerical classes of effective divisors.
- ***Cone of curves (Mori cone)***: $NE(X) \subset N_1(X)$ closure of the cone spanned by numerical classes of effective curves.

## 3 Nef cone

$$\begin{aligned} \mathrm{Nef}(X) &= NE(X)^\vee \\ &= \{D : D \cdot C \geqslant 0 \forall C \in NE(X)\} \end{aligned}$$

## 4 Linear system

$$D \in \mathrm{Cl}(X) \qquad |D| = \mathbb{P}H^0(X, \mathcal{O}_X(D)) = \{D \geqslant 0 : D \sim D\}$$

where $\mathcal{O}_X(D)$ is the divisorial sheaf. Notice thar this is nonempty iff D is effective.

Now consider a rational map

$$\begin{aligned} \varphi_D : X &\xrightarrow{\text{rat}} |D|^\vee \\ x &\longmapsto \{D^1 \in |D| : X \in D^1\} \end{aligned}$$

Base locus: $\mathrm{BS}\,|D| = \bigcap_{D^1 \in |D|} D^1$

$$D = \mathrm{Fix}(D) + M$$

on the left, divisorial part of the base locus, on the right mobile part.

$$\varphi_D = \varphi_M$$

$\mathrm{BS}(D) =$ empty then D is called ***free*** for globally generated.

D is a pullback by a hyperplane:

$$\begin{aligned} \varphi_D : X &\longrightarrow \mathbb{P}^r = |D| \\ \mathcal{O}(D) &= \varphi_D^* \mathcal{O}(1) \implies D \text{ is Cartier} \end{aligned}$$

## 5 Stable base locus

$$\mathbb{B}(D) = \bigcap_{m>0} \operatorname{Bs}|mD|$$

D is called ***semiample*** if $\mathbb{B}(D)$ =empty iff $mD$ is free for some $m > 0$.

D semiample implies D is nef.

## 6 Semi-ample Fibration Theorem

**Theorem.** D semiample $\implies \varphi_{|mD|} : X \to Y$ does not depend on $m \gg 0$ and divisible. Connected fibers, Y normal (not necessarily $\mathbb{Q}$-factorial).

**Theorem** (Zariski)**.** $D \in \operatorname{Pic}(X), \operatorname{Bs}|D|$ is finite $\implies$ D is semiample.

**Corollary.** $\dim X = 2$, $D \in \operatorname{Pic}(X)$ effective, $\operatorname{Fix}(D) = 0 \implies$ D is semiample.

- If $\varphi_{|D|}$ is a closed embedding, them D is called ***very ample***.
- $D \in \operatorname{Cl}(X)$ is called ample if $mD$ is very ample for some $m > 0$.
- $D \in \operatorname{Cl}(X)$ is ***big*** if $\dim \varphi_{|mD|}(X) = \dim X$ for some $m > 0 \iff h^0(X, mD) \sim_m \dim X$ if $m \gg 0$ and divisible, $\overset{\text{Itaka}}{\iff} \varphi_{|mD|}$ is birational.

## 7 Kleimen Criterion

Even though amppleness and bigness are defined using linear system, they are numerical properies.

**Theorem** (Kleimen Criterion)**.** $D \in \operatorname{Cl}(X)$ is ample iff $D \in \text{Interior } \operatorname{Nef}(X)$. This implies that $\operatorname{Nef}(X) \subset \operatorname{Eff}(X)$

$$\begin{aligned} D \in \operatorname{Cl}(X) &= D \in \text{Interior} \operatorname{Eff}(X) \\ &\iff mD = \text{Effective and ample for some } m > 0 \end{aligned}$$

And what you need for that is

**Lemma** (Kodaira's lemma)**.** D big, A effective, then $mD - A$ is also effective for some $m > 0$

$$0 \longrightarrow \mathcal{O}(mD - A) \longrightarrow \mathcal{O}(mD) \longrightarrow \mathcal{O}(mD)|_A \longrightarrow 0$$

this implies $mD - A$ is effective.

This is the end of the review.

# 8 Some questions asked by Fulton

$\partial\overline{M}_{0,n}$ components. There's some stratification of this space by their divisors. There are some things called ***F-curves***.

**Example.** In the surface case, $\overline{M}_{0,5} = \mathrm{Bl}_4\,\mathbb{P}^2$ there are five $\binom{5}{10} = 10$ of them. They are called ***(-1)-curves***. If you look at the effective cone of this blow up, it is generated by these 10 curves.

**Conjecture** (Fulton-Fuber)**.** $\mathrm{NE}(\overline{M}_{0,n}$ is generated by F-curves. Still poen

**Conjecture** (Fulton, it is wrong)**.** $\mathrm{Eff}(\overline{M}_{0,n}$ is generated by boundary divisors.

There are many other extremal rays (Gutrvet-Jenia 2013), also because it is not a rational polyhedral cone (not finitely generated) (2023 paper).

# 9 Some strategies for proving these sort of things

LEt's go back to the setting where $X$ is a normal $\mathbb{Q}$-factorial projective variety. Look a birational (and $\mathbb{Q}$-factorial) maps $f : X \xrightarrow{\text{bir}} Y$ and $f$ is regular in codimension 1. Then $D \subset X$ is called $f$***-exceptional*** if $\dim f(D) < \dim D$, equivalently, $f$ is not an isomorphism at a generic point $\eta \in D$. $f$ (or $Y$ )is called a ***birational contraction*** if there are no $f^{-1}$-exceptional divisors.

**Question** (Misha)**.** Is contraction always regular? No.

$f$ (or $Y$ )is called ***small*** $\mathbb{Q}$***-factorial modification*** if there are no $f$ or $f^{-1}$ exceptional divisors. So basically this means that $f$ is an isomorphism in codimension 1.

And finally, a rational map $f : X \xrightarrow{\text{rat}} Y$ of $\mathbb{Q}$-factorial varieties is called ***rational contraction*** if $f$ is a composition of birational contractions and morphisms with connected fibers (between $\mathbb{Q}$-factorial varieties).

**Principle** The birational geometry of $X$ is the study of brational contractions of $X$.

# 10 Application of this to the study of effective cones

If $f : X \xrightarrow{\text{bir}} Y$ is a birational contraction, then its exceptional divisors are extremal rays of $\mathrm{Eff}(X)$.

# 11 Some examples

## 11.1 Hassett spaces

These are the simplest examples of birational contractions.

(I talk about genus zero but most of this can be extended).

Choose some positive numers $a_1, \dots a_n > 0$ with $\sum_i a_i > 2$. They are called ***rational weights***. The birational contraction is

$$\overline{M}_{0,n} \longrightarrow \overline{M}_{0,n;\bar{a}} = \text{ moduli space of } \bar{a}\text{-stable rational curves}$$

where the bar over $a$ should be an arrow like a vector.

$$\begin{tikzcd} \overline{M}_{0,n} \arrow[rr] & & \overline{M}_{0,n;\bar{a}} \\ & M_{0,n} \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

- Semi -log canonical.
- $\omega_c\left(\sum_i a_i p_i\right)$ ample $\iff$ $p_i$ are not at the nodes.
- # of nodes on $R$ + $\sum_{p_i \in R} a_i > 2$.
- $I \subset \{1, \dots, n\}$, $p_i = p_j$, $i, j \in I \implies \sum_{i \in I} a_i \leqslant 1$

**Example.**

- Choose

  $$(1, \dots, 1) \longrightarrow \overline{M}_{0,n}$$

  where $(1, \underbrace{x, \dots, x}_{n-1})$ so that $1 + (n-1)x = 2 + \varepsilon$ with $\varepsilon \ll 1$. So there is a heavy point, the first one.

  So for example stable curve $\mathbb{P}^1$, $p_1, \dots, p_n \in \mathbb{P}^1$ with $a = 1$, $p_i \neq p_1$. So not all $p_i$ with $i > 1$ are equal. Now do

  $$p_1 \to \infty$$
  $$p_2 \to 0$$

  and the rest of the points $p_3, \dots, p_n \in \mathbb{C}$. What happens is that not all of them are zero. So what is the moduli. It's very simple: $\overline{M}_{0,n;\bar{a}} = \mathbb{P}^{n-3}$. The map $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ is called the ***Kapranov map***.

  Then there's a drawing of to curves that intersect in a curve, and we map them to $\overline{M}_{0,n;\bar{a}}$, which is only one of the original lines (the one where $p_1$ lives) and the second one has been contracted to a point. So there is a contraction:

  $$\begin{aligned} |I^c| \geqslant 3 &\implies \Delta_I \text{ is contracted} \\ &\implies \text{Every } \Delta_I \text{is contracted} \end{aligned}$$

**Exercise.**

- Take a bunch of triangles and see how many vertices they cover. Show that $\left|\bigcup_{i \in I} \Gamma_i\right| \geqslant |I| + 2$. If $|I| = 1$ or $|I| = n - 2$ we have stric equality. Here $|I|$ is the number of triangles that you chose and $\Gamma_i$ is a triangle which is just three vertices. In fact, there are $n - 2$ black triangles when you have a black-and-white triangulation of $n$ vertices. This is the ***hypertree condition***

- Strict inequality for $1 < |I| < n-2$ unless the triangulation is a *connected sum* . This is the ***irreducible hypertree condition***.

So what does that have to do. So an ***irreducible hypertree*** is given by those inequalitites. We have: $\Gamma_1, \ldots, \Gamma_n$ irreducible hypersurve. and then: hypertree curve

$$\Gamma_i = \{a, b, c\}$$
$$\Gamma_j = \{a, x, y\}$$

If it is not an octahedron there will surely be vertices with more than two black triangles.

**Question.** Is this curve the union of the two lines that pass through $\{a, b, c\}$ and $\{a, x, y\}$

So the curve is locally the union of coordinate axes.

$$M_{0,n} \subset \mathrm{Morphisims}(\mathcal{C}_\Gamma, \mathbb{P}^1)$$

which is linear on every component of $\mathcal{C}_\Gamma$.

$$\begin{array}{c}\text{drawing of 4 lines}\\ \text{the components of the curve}\\ \text{with intersections numbered}\end{array} \longrightarrow \mathbb{P}^1$$

So the intersection points go to some 6 points in $\mathbb{P}^1$. The choice of these 6 points is (the moduli space?).

And then we do

$$M_{0,n} \subset \mathrm{Morph}(\mathcal{C}_\Gamma, \mathbb{P}^1) \to \mathrm{Pic}(\mathcal{C}_\Gamma)$$

We have done

$$\mathcal{C}_\Gamma \xrightarrow{\varphi} \mathbb{P}^1 \longrightarrow \varphi^*\mathcal{O}(1)$$

So we have

$$M_{0,n} \longrightarrow (\mathbb{C}^*)^{n-3}$$

which is birational by Riemann-Roch.

So what is the exceptional locus $D_\Gamma$? A general point of $D_\Gamma$.

We do a map $\mathcal{C}_\Gamma \to \mathbb{P}^2$. Choose a point $x \in \mathbb{P}^2$ and project from $X$ and get $n$ points on $\mathbb{P}^1$.

**Theorem** (Castoret-Jenia)**.** $\Gamma$ is irreducible hypertree $\implies$ $D_\Gamma \subset M_{0,n}$ is an irreducible divisor. $\overline{D}_\Gamma \subset \overline{M}_{0,n}$ is an exceptional divisor

$$\begin{tikzcd}
D_\Gamma \arrow[r] \arrow[d, hook] & \text{contracted} \arrow[d, hook] \\
M_{0,n} \arrow[r] \arrow[d, hook] & (\mathbb{C}^*)^{n-3} \arrow[d, hook] \\
\overline{M}_{0,n} \arrow[r, "\text{bir. contraction!}"] & \text{compactified Jacobian} \\
\overline{D}_\Gamma \arrow[u, hook, "\text{contained}"] &
\end{tikzcd}$$

So $D_\Gamma$ is an exceptional ray of $\mathrm{Eff}(\overline{M}_{0,n}$.

**Theorem** (Castravet, Laface, Jenia, Ugaglic)**.** $\mathrm{Eff}(\overline{M}_{0,n}$ is infinitely generated for $n \geqslant 10$.

**Lemma.** Let $f : X \xrightarrow{\text{rat}}$ be a rational contraction. Then

$$\mathrm{Eff}(X) \text{ f. g.} \implies \mathrm{Eff}(Y) \text{ f.g.}$$

Other properties preserved by rational contractions are: being a MDS, having a S$\mathbb{Q}$S with nef but not semiample divisor, etc.

*Proof.* **Case 1** When $f$ is a birational contraction. In this case you just notice that the effective cone of $Y$ is going to be a pushforward of the effective cone of $X$ :

$$\mathrm{Eff}(Y) = f_* \,\mathrm{Eff}(X)$$

for

$$f_* : N^1(X) \to N^1(Y)$$

**Case 2** $f$ is a morphism. There is no pushforward of divisors! But we still have a pushforward, but for cycles:

$$f_* : N_1(X) \to N_1(Y)$$

And then there is the $\mathrm{Mov}_1(X) =$ cone spanned by ***movable*** curves on $X$, where movable means that the curve moves in a family that covers $X$. □

**Theorem** (Bookson Demaria PP)**.**

$$\mathrm{Eff}(X) = \mathrm{Mov}_1(X)$$

*Proof.* If the effective cone is polyhedral (finitely generated) then $\mathrm{Mov}_1(X)$ is polyhedral. Then if you look at

$$f_* : \mathrm{Mov}_1(X) \to \mathrm{Mov}_1(Y)$$

is surjective, which implies that $\mathrm{Mov}_1(Y)$ is also polyhedral. And then using duality we conclude that $\mathrm{Eff}(Y)$ is polyhedral. □

**Goal** To find a rational contraction of $\overline{M}_{0,1}$ with a non-polyhedral effective cone.

**Step 1** This requires going back to Hassett spaces. So take the weights $(1 = 0, 1 = \infty, \varepsilon, \ldots, \varepsilon$ with $\varepsilon \ll 1$. This means that the stable curves are chains of $\mathbb{P}^1$.

Drawing of many curves (little arcs) one after the other with 0 in the left most

and $\infty$ in the rightmost.

So Permutahedron=$\mathbb{P} = \overline{M}_{0,n;\bar{a}} =$ LM= Losev-Manin Space.

And the permutahedron is the convex hull $\{\sigma(1), \sigma(2), \ldots, \sigma(k)\}_{\sigma \in S_k} \subset \mathbb{R}^k$. So for $k = 3$ it is a hexagon.

**Universality property** (This is their lemma with Anna Maria) Every projective toric variety $\mathbb{P}(\Delta)$ is a rational contraction of the toric variety associated to the permutahedron, and therefore, $\overline{M}_{0,n}$ for $n \gg 0$.

Unfortunately, this is not what you want because $\mathrm{Eff}(\mathbb{P}(\Delta))$ is polyhedral (generated by toric boundary divisors).

**Step 2** Now choose another Hassett space (the last one of this talk). Now choose $(1, 1, \underbrace{x, \ldots, x}_{k})$ with $kx = 1 + \varepsilon$ $\varepsilon \ll 1$. So it looks like this

$$\overline{M}_{0,n;a} = \mathrm{Bl}_e \mathrm{LM}$$

and there is a drawing of how the exepctional divisor $E$ looks likein this blow-up.

**Theorem** (Universality theorem 2)**.** (and therefore the blow up of a toric variety at only one point is…) Every $\mathrm{Bl}_e \mathbb{P}(\Delta)$ is a rational contraction of $\mathrm{Bl}_e \mathbb{P}$ (permutahedron) (and also $\overline{M}_{0,n}$.)

**Remark.** $\Delta \subset \mathbb{R}^2$ lattice polygon. Then $\mathrm{Bl}_e \mathbb{P}(\Delta)$ can be wild!

There is a drawing of a polygon made up from joining some specific points on a $6 \times 6$ square lattice. In this example $\mathrm{Bl}_e \mathbb{P}_\Delta$ has a non-polyhedral effective cone. (Misha: it is singular because it contains integer points inside.) There is an elliptic curve inside this surface $C \subset \mathrm{Bl}_e \mathbb{P}_\Delta$ given by $C : y^2 + y = x^3 - x^2 - 24x + 54$. And then

$$\mathrm{Nef}(X) \subset \mathrm{LC}\, X = \{D : S^2 \geqslant 0 \text{ is very ample}\} \subset \mathrm{Eff}\, X$$

and $C$ is away from singularities and has intersection 0, that is, $C^2 = 0$. Also $\mathcal{O}(C)|_C = \mathrm{Pic}^0(C)$ and in fact $\mathcal{O}(C)|_C = (1, 5)$ has infinite order, so $h^0(mC) = 1 \forall m > 0$. And what this means is that $C$ is not the fiber of an elliptic curve. But any multiple of $C$ is just $C$. So $C$ is not on a facet of $\mathrm{Eff}(X)$.

**Theorem** (Nikulin)**.** Eff(Surface) is polyhedral $\implies$ Eff is generated by hesf? curves. So Eff is polyhedral $\implies$ $C$ is on the facet.

$$mC \sim xA + yB \text{ for some } x \text{ and } y$$

$$\implies h^0(mC) > 1$$

contradiction.

## 11.2 Two more anomalies

**Theorem** (Goto-Nishida-Watamabe)**.** There exists $\mathbb{P}(a, b, c)$ such that $X = \mathrm{Bl}_e \mathbb{P}(a, b, c)$ (in characteristic 0) has a nef, big, not semi-ample divisor.

**Conjecture** (Conjectural anomaly)**.** If you take $\mathrm{Bl}_e \mathbb{P}(9, 10, 13)$ then the effective cone looks like this: drawing of to lines intersecting at a point. One of th elines is $E$ and the other is $D^2 = 0$. All you have to prove is that nothing oustide of the shaded area (acute angle region) is effective. This is equivalent to $\mathbb{P}(9, 10, 13)$ has an irrational seshodri constant.

The conjecture is that almost every surface has a point with an irradional Seshadri constant, but no example is known.

This would imply Nagata conjecture for $9 \cdot 10 \cdot 13$ points on $\mathbb{P}^2$.

There is this world of blown up toric surfaces whose geometry is very