Quasi-homogeneous cones

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Abstract. A cone V in a Euclidean space is quasi-homogeneous if it contains a compact subset K such that every point of V can be mapped into K by a linear automorphism of V. I will discuss a theorem of Kac-Vinberg, which says that the projectivisation of a quasi-homogeneous cone in $\mathbb{R}^3$ with piecewise C^2 boundary is either an ellipse or a triangle.

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Definition $V \subset \mathbb{R}^3$ is a ***cone*** if

1. V is open.
2. $\forall v \in V, \lambda > 0, \lambda v \in V$.
3. Closure of the convex hull of V contains no lines.

Its automorphism group is composed of linear automorphisms of $\mathbb{R}^3$ that preserve V. V is ***homogeneous*** if $\mathrm{Aut}(V)$ acts transitively on V, and ***quasi-homogeneous*** if there exists a compact subset $K \subset V$ such that $\mathrm{Aut}(V) \cdot K = V$. So it's "generated" by a compact subset? "The intuiton is that the quotient is compact, though we don't know if it is Hausdorff."

Remark (Misha) Most of the work on cones was done by Vinberg.

The projectivization of a cone is contained in projective plane, $PV \subset \mathbb{R}P^2$, and it is a "convex subset". Its boundary $\Gamma = \partial(PV)$ is a Jordan curve.

Theorem (Kac-Vinberg, 1965?) Let V be a quasi-homogeneous cone. If Γ is piecewise C^2, then Γ is either an ellipse or a triangle.

We shall unsuccefully try to prove this theorem.

Lemma 1 V quasi-homogeneous cone $\implies$ V is convex.

Proof. Let V' be the convex hull of V. Then V' is a convex cone and $\mathrm{Aut}(V)$ acts on V'.

Now condier its projectivization PV'. There is a distance in PV':

$$d(a,b) = \log \frac{|ay| \cdot |bx|}{|ax| \cdot |by|}$$

So $d(PK, PV' \backslash \overline{PV}) > \varepsilon$, so there exists $x \in PV' \backslash \overline{PV}$. Then there exists a point $y \in PV$ very close to ε. I think here convexity was used, perhaps we can take y in the boundary.

$g \in \mathrm{Aut}(V)$ such that $gy \in K$. But then $\underbrace{d(x,y)}_{<\varepsilon} = \underbrace{d(gx,gy)}_{<\varepsilon}$, which means $gx \in PV' \backslash PV$.

□

Now let V be a quasi-homogeneous cone, $G \subset \mathrm{Aut}(V)$ such that $\exists K \subset V$ with $GK = V$.

If G' is a finite index subgroup of G, then G' also satisfies the property above.

Lemma 2 $g \in G$. If g has a 2-dimensional eigenspace, then $g^2 = \mathrm{Id}$.

Proof. Let ℓ be the projectivization of the eigenspace, g fixes ℓ pointwise. Then $\forall x \in \ell \backslash PV$, g^2 fixes tangents from x to PV. Hence $g^2 = \mathrm{Id}$. □

Lemma 3 $g \in G$ cannot have exactly two different eigenvalues.

Proof.

$$g = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda 0 & \\ 0 & 0 & \mu \end{pmatrix}, \qquad \lambda, \mu \in \mathbb{R}$$

$x \in \mathbb{R}P^2$, then for a basis v_1, v_2, v_3,

$$\lim_{n \to \infty} g^n(x) = [v_1] \qquad \text{if } x_{\neq} v_3$$

$$\lim_{n \to \infty} g^{-n} = [v_3], \qquad \text{if } x \notin (v_1, v_2)$$

then x can be on $\Gamma \implies [v_1], [v_3] \in \Gamma$.

Then g preserves $[v_1]$ and $[v_3]$. There is a curve Γ preserved by g. Then g also preserves the tangents. And g^2 preserves tangents. But the tangents, like tangents to a circle, intersect, and that gives a third point fixed by g^2, but that's impossible because there are only two eigenvalues (g^2 has the same form as g). But the end of this proof was sketchy. □

Lemma 4 G cannot be unipotent (=exponent of Lie algebra, bracket several times gives zero, and also that it cannot be strictly upper triangular that is, it has not the following form:

$$G \subset \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Proof. Suppose it has that form. If $[G, G] \ni g \neq 1$. So if g is

$$g = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which contradicts lemma 2.

If

$$g = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \subset G$$

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 01 & \end{pmatrix} \right\}$$

Remark (Misha) This is a 2-dimensional commutative subgroup of Heisenberg.

Now notice that

$$\{g^n v : n \in \mathbb{Z}\}$$

lies on parabolas. Is it true no? Concluding this is proof left as a (non-trivial I guess) exercise. □

Lemma 5 If the G is diagonal then Γ is a triangle.

Proof. This is easy. We already have three points, they are limit points; they are sitting in the boundary... also an exercise. No let's prove it.

$G \subset \{\text{diagonal matrices}\}$ so take an alement there

$$g = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

And three eigenpoints $[v_1], [v_2], [v_3]$. We can order the eigenvalues $|\lambda_1| > |\lambda_2| > |\lambda_3|$. Looks like there is some recurrent argument: $[v_1]$ and $[v_3]$ are in Γ and the lines $[v_1][v_2]$ and $[v_3][v_2]$ are tangent to the curve and of course intersect at $[v_2]$.

And then the proof of lemma 1 comes back. Because the group preserves the triangle and a cocompact convex figure inside. But the distance between a point outside the convex region and the compact set is bounded (T is triangle):

$$d(T \backslash PV, K) > \varepsilon$$

but you can take $x \in PV$ and $y \in T \backslash PV$ but you can map y into K with a distance preserving map. (The very same argument.) □

Lemma 6 If G is upper-triangular then Γ is either an ellipse or a triangle.

Proof. If G is commutative take some non-unipotent element, If it was exactly 2 eigenvalues it's bad, if it has 1, the whole group will be diagonal out of commutativity and apply lemma 5. So take $g \in [G, G]$, it must be like this:

$$g = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

Now note that $[G, G]$ is commutative, since

$$[[G, G], [G, G]] \in \begin{pmatrix} 1 & b & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

hgh^{-1} must commute with g.

Then

$$\begin{pmatrix} \lambda_1 & * & * \\ 0 & \lambda_1 & * \\ 0 & 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} 0 & a & * \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & * & * \\ 0 & \lambda_2^{-1} & * \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\lambda_2}{\lambda_3} a & * \\ & 0 & \frac{\lambda_2}{\lambda_3} a \\ & & 0 \end{pmatrix}$$

where

$$hgh^{-1} = \begin{pmatrix} 1 & \lambda a & * \\ 0 & 1 & \lambda a \\ 0 & 0 & 1 \end{pmatrix}$$

Now do this several times to obtain

$$h^n g h^{-n} = \begin{pmatrix} 1 & \lambda^n a & * \\ 0 & 1 & \lambda^n a \\ 0 & 0 & 1 \end{pmatrix}$$

and you can see perhaps that it is not discrete. And so what? We have a 1-parametric subgroup. Why? Because of the closed subgroup theorem (closed subgroup of a Lie group is Lie subgroup). OK but then you need to show that the automorphism group of the cone is closed. (Shown—because G is closed, that has to be written somewhere.)

OK so we have a 1-parametric subgroup, namely

$$\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

so the boundary *is* an ellipse. □

Now let's review what happened:

Lemma 1. V is convex.

Lemma 2. If $g \in G$ has 2-dimensional eigenspaces then $g^2 = \mathrm{Id}$.

Lemma 3. $g \in G$ cannot have 2 eigenvalues.

Lemma 6. If G is upper triangular, then we are done.

Now let's do another lemma:

Lemma 7 If the whole automorphism fixes a point of the boundary, then the curve is either an ellipsoid or a triangle (we are done).

Proof. Follows from Lemma 6, since we have a fixed flag. There exists a subgroup of index 2 that fixes a point and a line—it means it is upper tiangular. Another way to put this: we have a fixed flag, what fixes a flag in projective space is upper triangular. □

Now they (Vinberg and Kac) say that we can assume that the curve is smooth everywhere.

Lemma 8 We can assume that the curve is smooth everywhere.

Proof. There exists a finite index subgroup $G' \subset G$ preserving all non-smooth points of Γ. Then apply lemma 7. □

Lemma 9 If you have an element of infinite order that fixes a point inside ($v \in PV$) then the curve Γ is an ellipse.

Proof. Since distance is preserved, $\{g^n\}$ is not discrete. This gives a 1-parametric subgroup. □

Lemma 10 If G has an element of arbitrarily large finite order, then Γ is an ellipse.

Proof. If $g \in G$ with $g^n = \mathrm{Id}$, $v \in V$, then the cone

$$v + gv + \ldots + g^{n-1}v = v' \qquad g(v') = v'$$

$$g_n^{k_n} = \mathrm{Id} \qquad k_n \to \infty \qquad g_n(v_n) = v_n$$

somehow we map them to K, $v_n \in K$ and then we can assume they converge $v_n \to v$. So there is a subsequence $g_n^{m_n}$ that will converge to an irrational rotation around v. Then we use that the group is closed and apply the previous lemma. □

Lemma 11 If $g \in G$ preserves $x \in \Gamma$, then eigenvalues of g are real.

Proof. We already assume that the curve is smooth, so there is a unique tangent, so g preserves the unique tangent, so preserves the flag. (Also there's a way to show this without smoothness but OK) □

Lemma 12 If every $g \in G$ is either unipotent or of finite order then the boundary is an ellipse.

Proof. By lemma 10 there exists N such that every $g \in G$ of finite order satisfy $g^N = 1$.

$$(g^N - 1)^3 = 0$$

So there is an algebraic group... let $\overline{G}$ be the Zariski closure of G and G_0 connected component of $1 \in \overline{G}$. Now recall that

Theorem Real algebraic variety has finitely many connected components.

which implies that the index $[\overline{G} : G_0] < \infty$ is finite. So $G_0 \cap G$ is unipotent. Apply lemma 6. □

From all this it should supposedly be clear that there is an element

$$g = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \in G$$