Quasi-isometric embeddings in higher rank groups

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Contents

1 Groups as geometric/dynamical objects

$\Gamma < G$, G is a Lie group,, we can think $G = SL(d, \mathbb{R}), SL(d, \mathbb{C})$. Think Γ is finitely generated.

2 Algebra questions

Burnside Problem Γ is finitely generated in G and every element is torsion, then Γ is finite.

Selberg Lema

Tits alternative

Now suppose Γ is also discrete and quasi-isometric. This makes it more geometric. We want to look at deformations of Γ within G.

Let's explain what quasi-isometric means. Γ is a finitely generated group. Let $F = \{f_i\}$ be finite and symmetric ($f \in F \iff f^{-1} \in F$) generators. So $\Gamma = \langle f_i | r_j \rangle$ where r_j are relations. Consider $\rho : \Gamma \to G$ where $\rho(f_i)$ verify the relations r_j.

We are interested in $\mathrm{Hom}(\Gamma, G) \subseteq G^{|F|}$

We can define a distance in this group

$$d_F(\gamma, \eta) = |\eta^{-1}\gamma|_F = \inf\{n : \eta^{-1}\gamma = f_{i_1} \dots f_{i_n} \text{ with } f_{i_k} \in F\}$$

Definition A ***quasi-isometric embedding*** is $q : (X, d_x) \to (Y, d_y)$ such that there exist two numbers $(0,1) \ni a, b > 0$ such that

$$a d_x(\mathfrak{x}, \mathfrak{x}') - b < d_y(\mathfrak{x}, q(\mathfrak{x}')) < a^{-1} d_x(\mathfrak{x}, \mathfrak{x}' + b$$

Now we put $\rho : \Gamma \to \mathrm{Isom}(X)$ so thinking of G as $\mathrm{Isom}(X)$

Definition The ***orbit map*** for $\mathfrak{x} \in X$ is $\Phi : \gamma \mapsto \rho(\gamma)(\mathfrak{x})$. And ρ is quasi-isometric if the orbit map is a ***quasi-isometric embedding***

Example In $G = SL(2,\mathbb{R})$ equivalent to $\exists k > 0$ so that $\forall \gamma \in \Gamma \; \|\rho(\gamma)\| > e^{k|\gamma|_F}$

Example (Teichmüler space)

$$\Gamma = \pi_1(S_g) = \left\langle a_1, b_1, \dots, a_g, b_g : \prod_i [a_i, b_i] = \mathrm{id} \right\rangle$$

These representations are very well studied:

$$\mathrm{Isom}(\mathbb{H})\, \mathrm{Hom}(\Gamma, PSL(2,\mathbb{R}))/\, PSL(2,\mathbb{R}) \supseteq \mathrm{Teich}(S_g) \cong \mathbb{R}^{6g-6} \cong \mathrm{Hom}_{fd}(\Gamma, PSL(2,\mathbb{R}))/\, PSL(2,\mathbb{R})$$

where $\mathrm{Teich}\, S_g$ is the space of hyperbolic metrics in S_g modulo isotopy. (See Svarc-Milnor).

Example (Hyperbolic space) Recall that $SO(1,2)$ are the isometries of a quadratic form of signature $(1,2)$ acting on $\mathbb{R}^3$. These preserve the cone $Q = 0$, and its interior. Restrict Q to the hyperboloid $Q = 1$ inside the cone $Q = 0$ to obtain a Riemannian manifold. Also you can intersect the cone at the plane $z = 0$ to obtain the Klein model. Its metric is a logarithm of another metric.

Now do

$$\begin{tikzcd} \pi_1(S_g) \arrow[r, "\text{faithful (=injective) discrete}"] \arrow[rr, bend left, "\rho_0"] & SO(1,2) \arrow[r, hook] & SL(3,\mathbb{R})] \end{tikzcd}$$

See Hitchin. Using Higgs bundles and so on, the topology of (?) $\mathbb{R}^{12g-12}$ was understood.

Theorem (Labourie) In all the connected components of the deformation space the embedding is quasi-isometric.

Question (Misha) Are there examples other than SL? Consider

$$\begin{tikzcd} \pi_1(S) \arrow[r] \arrow[rr, bend left] & PSL(2,\mathbb{R}) \arrow[r, hook] & PSL(d,\mathbb{R}) \end{tikzcd}$$

And also

$$\pi_1(S) \xrightarrow{\rho_0} PSL(2,\mathbb{R}) \hookrightarrow SL(3,\mathbb{R})$$

Labourie introduced the notion of Anosov representation to show that sometimes being qi is an open property.

This is related to IMPA:

Theorem (Mañe, Bonatti-Diaz-Pujals) Robust "things" $\implies$ dominated splitting, which is an open condition.

Looks like there is a dynamical system related to the space of "geodesics" insde $\Gamma = \pi_1(S) = \{F : \prod[a_i, b_i] = \mathrm{id}\}$. This notion of geodeics, as I understand, is given by how far a word is from another word.

I think these geodesics are $\{f_{i_k}\} \subset F^{\mathbb{Z}}$ with $F = \{a_1^{\pm 1}, b_1^{\pm 1}, \dots, a_g^{\pm 1}, b_g^{\pm 1}\}$ with $|f_{i_k} \dots f_{i_{k+\ell}}| = \ell$. Being QI implies

$$\|\rho_0(f_{i_k} \dots f_{i_{k+\ell}}\| > e^{k\ell}$$

Definition i-***domination***. $F : \Lambda \to \Lambda$, $\Phi : \Lambda \to SL(d, \mathbb{R})$ cocycle, $\Phi^{(n)} = \Phi(T^{n-1}(x)) \dots \Phi(x)$. Φ has i-***dominated splitting*** if the i-th singular value is bigger than the $(i+1)$-th singular value, ie.

$$\exists c > 0, \lambda > 1 \text{ st } \forall x \in \Lambda \frac{\sigma_i(\Phi^{(n)}(x)}{\sigma_{i+1}(\Phi^{(n)}} > c\lambda^n$$

(Actually I think this is an equivalence by Bochi-Gourmelon).

Theorem (KLP,BPS) i-dom $\implies$ Γ is word-hyperbolic

Definition (Rafael and collaborators=BPS) $\rho : \Gamma \to SL(d, \mathbb{R})$ is i-***Anosov*** if the cocycle

$$\phi : \Lambda \to SL(d, \mathbb{R}) / \Phi(\{f_i\}) = \rho(f_0)$$

where Λ is the space of geoesics, has an i-dominated splitting.

Question Is being anosov a consequence of being faithful discrete (representation?)?

Question Take the free group on three generators inside $SL(3, \mathbb{R})$. Its robust quasi-isometric. Is it anosov?

3 Morse lemma in $\mathbb{H}^2$

Definition A sequence of points $\{a_n\}$ in $\mathbb{H}^2$ is an (a, b)-***quasi geodesic*** if

$$a|n - m|b < d_{\mathbb{H}^2}(a_n, a_m) < a^{-1}|n - m|b$$

Lemma (Morse) $\forall(a, b) \exists k := k(a, b)$ so that $\forall(a, b)$-quasi geodesic $\{a_n\}$ there exists $\gamma : \mathbb{R} \to \mathbb{H}^2$ geodesic such tat $d(\gamma(n), a_n) < k$.

We wish to understand what happens in higher dimension.