

# Practice exercises on smooth manifolds

Second meeting, 20 of December

Plan for today: selection of exercises (I skipped those with \*) from sections

- Hausdorff spaces (3)
- Compact spaces (8)
- Topological manifolds (2)
- Smooth manifolds (4)

## 1 Hausdorff spaces

**Exercise 1.12 (Points are closed in Hausdorff)** Let  $M$  be a Hausdorff topological space. Prove that all points in  $M$  are closed subsets.

*Solution.* Let  $x \in M$  and let's see that  $M \setminus \{x\}$  is open. Choose a point  $y \in M \setminus \{x\}$ . Then there are open sets  $U \ni x$  and  $V \ni y$  such that  $U \cap V = \emptyset$ . Then  $V \subset M \setminus \{x\}$ .  $\square$

**Definition 1.10** Let  $M, N$  be topological spaces. **Product topology** is a topology on  $M \times N$ , with open sets obtained as unions  $\bigcup_{\alpha} U_{\alpha} \times V_{\alpha}$ , where  $U_{\alpha}$  is open in  $M$  and  $V_{\alpha}$  is open in  $N$ .

**Exercise 1.16** Prove that a topology on  $X$  is Hausdorff if and only if the diagonal  $\Delta := \{(x, y) \in X \times X \mid x = y\}$  is closed in the product topology.

*Solution.* (  $\implies$  ) Suppose that  $X$  is Hausdorff. To check that  $\Delta$  is closed suppose that  $(x, y) \in X \times X$  is a limit point of  $\Delta$ . We need to show that  $(x, y) \in \Delta$ , i.e. that  $x = y$ . If  $x \neq y$  we can separate  $x$  and  $y$  by disjoint open subsets  $U \ni x$  and  $V \ni y$ . Then the open set  $U \times V$  contains  $(x, y)$ , and since  $(x, y)$  is a limit point of  $\Delta$  there must be a point  $(z, z) \in U \times V$ . Then  $z \in U$  and  $z \in V$ , which is a contradiction.

(  $\impliedby$  ) Suppose  $\Delta$  is closed in the product topology and choose two different points  $x \neq y$  in  $X$ . Then  $(x, y) \in (X \times X) \setminus \Delta$ , which is an open set by hypothesis. Then by definition of product topology there must be two open sets in  $X$ ,  $U \ni x$  and  $V \ni y$ . Suppose there is a point in the intersection  $z \in U \cap V$ . Then  $(z, z) \in (U \times V) \cap \Delta$ , a contradiction.  $\square$

**Exercise 1.18\*** Let  $\sim$  be an equivalence relation on a topological space  $M$ , and  $\Gamma \subset M \times M$  its *graph*, that is, the set  $\{(x, y) \in M \times M \mid x \sim y\}$ . Suppose that the map  $M \rightarrow M/\sim$  is open, and that  $\Gamma$  is closed in  $M \times M$ . Show that  $M/\sim$  is Hausdorff.

**Hint.** Prove that diagonal is closed in  $M \times M$ .

*Solution.* Notice that any open surjective map is closed: let  $f : X \rightarrow Y$  be an open surjective map and  $F \subset X$  closed, then  $f(X \setminus F) = f(X) \setminus f(F) = Y \setminus f(F)$ .

Our objective is to show that the diagonal  $\tilde{\Delta}$  in  $(M/\sim) \times (M/\sim)$  is closed. The projection of the graph  $\Gamma$  is  $\tilde{\Delta}$ . Since  $\Gamma$  is closed, by the remark above it follows that  $\tilde{\Delta}$  is closed in  $(M/\sim) \times (M/\sim)$  as we needed.  $\square$

## 2 Compact spaces

**Definition 1.12** A *cover* of a topological space  $M$  is a collection of open subsets  $\{U_\alpha \in 2^M\}$  such that  $\bigcup U_\alpha = M$ . A *subcover* of a cover  $\{U_\alpha\}$  is a subset  $\{U_\beta\} \subset \{U_\alpha\}$ . A topological space is called *compact* if any cover of this space has a finite subcover.

**Exercise 1.22 (Closed subset of compact is compact)** Let  $M$  be a compact topological space, and  $Z \subset M$  a closed subset. Show that  $Z$  is also compact.

*Solution.* Choose a cover  $\{U_\alpha\}$  of  $Z$ . Complete to a cover  $\{U_\alpha\} \cup (M \setminus Z)$  of  $M$  since  $M \setminus Z$  is open by hypothesis. Since  $M$  is compact then there is a finite subcover  $\{U_\beta\}$  of  $M$ . This is also a finite subcover of  $Z$ .  $\square$

**Exercise 1.23 (Countable metrizable  $\implies$  contains convergent subseq. or is discrete)** Let  $M$  be a countable, metrizable topological space. Show that either  $M$  contains a converging sequence of pairwise different elements, or  $M$  is discrete.

*Solution.* Suppose  $M$  is not discrete. Then there is a point  $z_0$  such that  $\{z_0\}$  is not an open set. Then every open set containing  $z_0$  contains another point. Choose for every  $n \in \mathbb{N}$  a point  $z_n$  different from  $z_0$  inside the ball  $B_{1/n}(z_0)$ . Taking a subsequence if necessary, we obtain a sequence of pairwise different elements  $\{z_i\}$  converging to  $z_0$ .

If  $M$  is discrete, it's clear that it cannot have a convergent sequence of pairwise disjoint elements: if the limit point  $\{z_0\}$  was open,  $M \setminus \{z_0\}$  would be closed and thus it would contain all its limit points!  $\square$

**Definition 1.13** A topological space is called *sequentially compact* if any sequence  $\{z_i\}$  of points of  $M$  has a converging subsequence.

**Exercise 1.24 (Metrizable compact  $\implies$  sequentially compact)** Let  $M$  be a metrizable compact topological space. Show that  $M$  is sequentially compact.

*Solution.* Let  $\{z_i\}$  be a sequence. Since the restriction of a metric to a subset is also a metric, we may use Exercise 1.23 on the countable metric subspace  $\{z_i\}$ . Suppose by contradiction that  $\{z_i\}$  has no limit point in  $M$ . In particular it has no limit point in  $\{z_i\}$ , so by Exercise 1.23 it is discrete. Then there are neighbourhoods  $U_i \ni z_i$  such that  $U_i \cap \{z_j\}_{j \neq i} = \emptyset$ . Then  $\{U_i\} \cup (M \setminus \{U_i\})$  is an open cover of  $M$ , which has a finite subcover. By the pigeon principle, at least one of the  $U_i$  contains an infinite number of points in  $\{z_i\}$ , which is not possible.  $\square$

**Exercise 1.28 (Continuous function maps compact to compact)** Let  $f : X \rightarrow Y$  be a continuous map of topological spaces with  $X$  compact. Prove that  $f(X)$  is also compact.

*Solution.* Choose an open cover  $\{U_\alpha\}$  of  $f(X)$ . Then  $\{f^{-1}(U_\alpha)\}$  is an open cover of  $X$  since  $f$  is continuous, and thus it has an open subcover  $\{f^{-1}(U_\beta)\}$ . I claim that  $\{U_\beta\}$  is a cover of  $f(X)$ : if there was a point  $f(x) \notin \bigcup U_\beta$ , then  $x$  couldn't be in any of the  $f^{-1}(U_\beta)$ , which cover  $X$ .  $\square$

**Exercise 1.29 (Compact subset of Hausdorff is closed)** Let  $Z \subset Y$  be a compact subset of a Hausdorff topological space. Prove that it is closed.

*Solution.* Recall that a set is closed if it contains all its limit points (any point that is not a limit point has a neighbourhood not intersecting the set, making the complement open).

Let  $z_0$  be a limit point of  $Z$ . Choose for every point  $z \in Z$  neighbourhoods  $U_z \ni z$  and  $V_z \ni z_0$  such that  $U_z \cap V_z = \emptyset$ . If  $z_0 \notin Z$ , then  $\{U_z\}$  is an open cover of  $Z$ , so there exists a finite subcover  $U_{z_1}, \dots, U_{z_n}$ . The set  $\bigcup_{i=1}^n V_{z_i}$  is an open neighbourhood of  $z_0$  that does not intersect  $Z$ , a contradiction.  $\square$

**Exercise 1.30** Let  $f : X \rightarrow Y$  be a continuous, bijective map of topological spaces, with  $X$  compact and  $Y$  Hausdorff. Prove that it is a homeomorphism.

*Solution.* We need to see that  $f^{-1}$  is continuous, i.e. that  $(f^{-1})^{-1}(U)$  is open for any  $U \subset Y$  open. Since  $f$  is bijective,  $(f^{-1})^{-1}(U) = f(U)$ ; so we must check  $f$  is open. Equivalently, we can check  $f$  is closed: if  $f(F)$  is closed for any closed  $F \subset X$ , then for any open set  $U \subset X$ , we see  $f(X \setminus U) = Y \setminus f(U)$  is closed.

To see  $f$  is closed note that since  $X$  is compact and  $f$  is bijective,  $f(X) = Y$  is also compact by Exercise 1.28. By Exercise 1.22 a closed subset  $F$  of  $X$  is compact. Again by continuity,  $f(F)$  is compact in  $Y$ . Finally by Exercise 1.29, since  $Y$  is Hausdorff and  $f(F)$  is compact, it must be closed.  $\square$

**Definition 1.15** A topological space  $M$  is called *pseudocompact* if any continuous function  $f : M \rightarrow \mathbb{R}$  is bounded.

**Exercise 1.31** Prove that any compact topological space is pseudocompact.

*Solution.* We must show that any continuous function  $f : M \rightarrow \mathbb{R}$  is bounded, in the sense that its image is contained in a ball of finite radius (c.f. Exercise 1.33). The image of any such function is compact by Exercise 1.28. But compact sets of  $\mathbb{R}$  are bounded: if for every  $r > 0$ , the image  $f(X)$  is not contained in the ball of radius  $r$  centered at zero,  $B_r(0)$ , then  $\{B_r(0)\}$  is an open cover of  $f(X)$  (since its union is all of  $\mathbb{R}$ ) without a finite subcover.  $\square$

**Exercise 1.32** Show that for any continuous function  $f : M \rightarrow \mathbb{R}$  on a compact space there exists  $x \in M$  such that  $f(x) = \sup_{z \in M} f(z)$ .

*Solution.* As in Exercise 1.31, the image of  $f$  is a bounded set of  $\mathbb{R}$ , which means the supremum is well-defined. To see it is attained at a point in  $M$  notice that  $f(M)$  is compact and thus closed. This means that all its limit points belong to  $f(M)$ : if a limit point  $z_0$  of the closed set  $f(M)$  is not in  $f(M)$ , then  $z_0$  has no neighbourhood contained in  $\mathbb{R} \setminus f(M)$ , but  $\mathbb{R} \setminus f(M)$  is open. In particular  $\sup_{z \in M} f(z)$  is in  $f(M)$ .  $\square$

**Exercise 1.33** Consider  $\mathbb{R}^n$  as a metric space, with the standard (Euclidean) metric. Let  $Z \subset \mathbb{R}^n$  be a closed, bounded set (*bounded* means contained in a ball of finite radius). Prove that  $Z$  is sequentially compact.

*Solution.* First consider the case  $n = 1$ . If  $\{z_n\}$  is a sequence contained in a closed and bounded set  $Z$ , then  $\sup_{n \in \mathbb{N}} z_n$  is a well-defined element in  $Z$ . Taking balls of radius  $1/m$  we construct a subsequence of  $\{z_n\}$  converging to the sup.

Now let's show that  $[0, 1]^n \subset \mathbb{R}^n$  is also sequentially compact. Choose a sequence  $\{z_n\} \subset [0, 1]^n$ . This gives a sequence in every coordinate, each of which must have a convergent subsequence, *but this doesn't give a convergent subsequence in the product...*  $\square$

### 3 Topological manifolds

**Exercise 2.3** Consider the quotient of  $\mathbb{R}^2$  by the action of  $\{\pm 1\}$  that maps  $x$  to  $-x$ . Is the quotient space a topological manifold?

*Solution.* Yes (it's not *smooth* but it is topological). A homeomorphism between  $\mathbb{R}^2/\{\pm 1\}$  and  $\mathbb{R}^2$  is  $re^{i\theta} \mapsto re^{i2\theta}$  where the angle  $\theta$  in the domain is in the interval  $[0, \pi)$  and  $r \in [0, \infty)$ . This map is clearly bijective (any equivalence class has a unique representative of the given form) and its continuous inverse is  $re^{i\varphi} \mapsto re^{i\varphi/2}$  for  $\varphi \in [0, 2\pi)$ .  $\square$

**Exercise 2.7** Show that  $\mathbb{R}^2/\mathbb{Z}^2$  is a manifold.

*Solution.* Let  $\bar{z}$  be a point in  $\mathbb{R}^2/\mathbb{Z}^2$ . Its preimage is the lattice  $\{z + (a, b) : a, b \in \mathbb{Z}\}$ . A ball of radius  $\frac{1}{2}$  centered at any representative of  $\bar{z}$  contains only one representative of any other class, so that the restriction of the projection is bijective (and continuous by definition of quotient topology). The inverse map is also continuous by definition of quotient topology: an open set in the ball on  $\mathbb{R}^2$  is mapped to an open set in the quotient because its preimage is open.  $\square$

## 4 Smooth manifolds

**Definition 2.2** A *cover* of a topological space  $X$  is a family of open sets  $\{U_i\}$  such that  $\bigcup_i U_i = X$ . A cover  $\{V_i\}$  is a *refinement* of a cover  $\{U_i\}$  if every  $V_i$  is contained in some  $U_i$ .

**Exercise 2.11** Show that any two covers of a topological space admit a common refinement.

*Solution.* Let  $\{U_i\}$  and  $\{U'_j\}$  be covers of a topological space  $X$ . Then  $\{V_{ij} := U_i \cap U'_j\}$  is a common refinement. It is obvious that  $V_{ij}$  is contained in  $U_i$  and  $U'_j$ , so it is a subcover of both covers. And it is also a cover: if  $x \in X$  then  $x$  must be in some  $U_i$  and some  $U'_j$ , so that it is in  $V_{ij}$ .  $\square$

**Definition 2.3** A cover  $\{U_i\}$  is an *atlas* if for every  $U_i$  we have a map  $\varphi_i : U_i \rightarrow \mathbb{R}^n$  giving a homeomorphism of  $U_i$  with an open subset in  $\mathbb{R}^n$ . The *transition maps*

$$\phi_{ij} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$$

are induced by the above homeomorphisms. An atlas is *smooth* if all transition maps are smooth (of class  $C^\infty$ , i.e., infinitely differentiable), *smooth of class  $C^i$*  if all transition functions are of differentiability class  $C^i$  and *real analytic* if all transition maps admit a Taylor expansion at each point.

**Definition 2.4** A *refinement of an atlas* is a refinement of the corresponding cover  $V_i \subset U_i$  equipped with the maps  $\varphi_i : V_i \rightarrow \mathbb{R}^n$  that are the restrictions of  $\varphi_i : U_i \rightarrow \mathbb{R}^n$ . Two atlases  $(U_i, \varphi_i)$  and  $(U_i, \psi_i)$  of class  $C^\infty$  or  $C^i$  (with the same cover) are *equivalent* in this class if, for all  $i$ , the map  $\psi_i \circ \varphi_i^{-1}$  defined on the corresponding open subset in  $\mathbb{R}^n$  belongs to the mentioned class. Two arbitrary atlases are *equivalent* if the corresponding covers possess a common refinement giving equivalent atlases.

**Definition 2.5** A *smooth structure* on a manifold (of class  $C^\infty$  or  $C^i$ ) is an atlas of class  $C^\infty$  or  $C^i$  considered up to the above equivalence. A *smooth manifold* is a topological manifold equipped with a smooth structure.

**Remark 2.3** Terrible, isn't it?

**Definition 2.6** A *smooth function* on a manifold  $M$  is a function  $f$  whose restriction to the chart  $(U_i, \varphi_i)$  gives a smooth function  $f \circ \varphi_i^{-1} : \varphi_i(U_i) \rightarrow \mathbb{R}$  for each open subset  $\varphi_i(U_i) \subset \mathbb{R}^n$ .

**Remark 2.4** There are several ways to define a smooth manifold. The above way is most standard. It is not the most convenient one but you should know it. Two other ways (via sheaves of functions and via Whitney's theorem) are presented further in these handouts.

**Definition 2.7** A *presheaf of functions* on a topological space  $M$  is a collection of subrings  $\mathcal{F}(U) \subset C(U)$  in the ring  $C(U)$  of all functions on  $U$ , for each open subset  $U \subset M$ , such that the restriction of every  $\gamma \in \mathcal{F}(U)$  to an open subset  $U_1 \subset U$  belongs to  $\mathcal{F}(U_1)$ .

**Definition 2.8** A presheaf of functions  $\mathcal{F}$  is called a *sheaf of functions* if these subrings satisfy the following condition. Let  $\{U_i\}$  be a cover of an open subset  $U \subset M$  (possibly infinite) and  $f_i \in \mathcal{F}(U_i)$  a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. Then there exists  $f \in \mathcal{F}(U)$  such that  $f_i$  is the restriction of  $f$  to  $U_i$  for all  $i$ .

**Remark 2.5** A *presheaf of functions* is a collection of subrings of functions on open subsets, compatible with restrictions. A *sheaf of functions* is a presheaf allowing "gluing" of a function on a bigger open set if its restriction to smaller open sets lies in the presheaf.

**Definition 2.9** A sequence  $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots$  of homomorphisms of abelian groups or vector spaces is called *exact* if the image of each map is the kernel of the next one.

**Exercise 2.13** Let  $\mathcal{F}$  be a presheaf of functions. Show that  $\mathcal{F}$  is a sheaf if and only if for every open cover  $\{U_i\}$  of an open subset  $U \subset M$  the sequence of restriction maps

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi_1} \prod_i \mathcal{F}(U_i) \xrightarrow{\varphi_2} \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with  $\eta \in \mathcal{F}(U_i)$  mapped to  $\eta|_{U_i \cap U_j}$  and  $-\eta|_{U_j \cap U_i}$ .

*Solution.* The key observation is that elements of  $\ker \varphi_2$  are collections of functions  $f_i \in \mathcal{F}(U_i)$  satisfying compatibility in pairwise intersections, i.e.,

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

**To achieve this** I think we must define  $\varphi_2$  by

$$(\dots, f_i, \dots, f_j, \dots) \mapsto (\dots, f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}, \dots).$$

Then elements in  $\ker \varphi_2$  satisfy the desired compatibility condition.

( $\implies$ ) Suppose the sequence above is exact. To show  $\mathcal{F}$  is a sheaf fix  $f_i \in \mathcal{F}(U_i)$  for every  $i$  satisfying  $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$  for every  $i, j$ . This is equivalent to choosing an element in  $\ker \varphi_2$ . By exactness this element is in  $\text{img } \varphi_1$ . This means that  $f_i$  is the restriction of some  $f \in \mathcal{F}(U)$  for every  $i$  as desired.

( $\impliedby$ ) Suppose  $\mathcal{F}$  is a sheaf. Injectivity of  $\varphi_1$  is immediate: if  $f \in \mathcal{F}(U)$  is mapped to zero under  $\varphi_1$ , meaning  $f_i|_{U_i} = 0$  for all  $i$ , it must be zero since  $\{U_i\}$  is a cover. Exactness in the second ring is equivalent to the definition of sheaf by the remarks above.  $\square$

**Exercise 2.14** Show that the following spaces of functions on  $\mathbb{R}^n$  define sheaves of functions.

- (a) Space of continuous functions.
- (b) Space of smooth functions.
- (c) Space of functions of differentiability class  $C^1$ .
- (d) (\*) Space of functions which are pointwise limits of sequences of continuous functions.
- (e) Space of functions vanishing outside a set of measure 0.

*Solution.* Injectivity of  $\varphi_1$  is immediate in all cases: a function that vanishes on every subset of an open cover vanishes identically.

- (a) Define a global function  $f$  on  $U$  by  $x \mapsto f_i(x)$  for any  $f_i \in \mathcal{F}(U_i)$  such that  $x \in U_i$ . Continuity follows from continuity of  $f_i$ , and the fact that  $f$  is well-defined follows from the gluing condition of  $\mathcal{F}$ .
- (b) Like above: smoothness follows from smoothness of  $f_i$ .
- (c) Like above.
- (d)
- (e) **Not sure** (uncountable union of measure-zero sets may have positive measure).

□

**Exercise 2.15** Show that the following spaces of functions on  $\mathbb{R}^n$  are presheaves, but not sheaves

- (a) Space of constant functions.
- (b) Space of bounded functions.
- (c) Space of functions vanishing outside of a bounded set.
- (d) Space of continuous functions with finite  $\int |f|$ .

*Solution.* The presheaf condition, that the restriction of a function to

- (a) Open sets with two connected components may not glue to a global constant function.
- (b) Unbounded functions may be bounded in open subsets! Take the open set  $(0, \infty) \subset \mathbb{R}$  and the cover  $U_i = (1/i, \infty)$ . Define the bounded function  $f_i(x) = 1/x$  in every  $U_i$ .

In  $\mathbb{R}^n$  we may do the same trick using the half-spaces with bounded last coordinate  $U_i = \{(x_1, \dots, x_n) : x_n \geq 1/i\}$  and taking  $f_i(x) = 1/\|x\|$ .

- (c) Let the open set  $U$  be all of  $\mathbb{R}^n$ . An open cover is given by balls  $B_i$  of radius  $2/3$  with center in  $i \in \mathbb{Z}^n$ . For every  $i \in \mathbb{Z}^n$  define functions  $f_i$  that vanish only outside a ball of very small radius, say  $1/6$ , with center in  $i$ . These functions coincide (they vanish) in the intersections of the cover, but the function obtained by gluing cannot vanish outside any bounded set: it is non-zero in the union of balls of radius  $1/6$  with centers in  $\mathbb{Z}^n$ .
- (d) Item (c) works if we manage to make the functions continuous. This can be done using a **partition of unity**. Also we must require that the values of the integrals in the smaller balls of radius  $1/6$  do not tend to zero (this way the global integral is an infinite sum of numbers that do not tend to zero, so it cannot be finite).

□