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Lista 3

Problema 1 Seja $f : X \to Y$ un difeomorfismo entre duas variedades orientadas conexas. Prove que df_x preserva orientação para um ponto $x \in X$ se, e somente se, df_x preserva orientação para todo ponto $x \in X$.

Demostração. Considere

$$\begin{aligned} D : X &\longrightarrow \mathbb{R} \\ x &\longmapsto \det d_x f \end{aligned}$$

É uma funcão contínua que nunca pode ser zero. Como é positiva em x, deve ser positiva sempre. □

Problem 2 Seja X uma variedade orientável. Prove que a orientação induzida em $X \times X$ é independente da orientação de X.

Demostração. A orientação de $X \times X$ está dada como segue: uma base (β_1, β_2) do espaço tangente $T_{(x,y)}X \times X$ é orientada se β_1 e β_2 são bases orientadas de X.

Agora considere a mesma construção usando $-X$. A base $(\tilde{\beta_1}, \tilde{\beta_2})$ de $-X \times -X$ é orientada se $\tilde{\beta_1}$ e $\tilde{\beta_2}$ são bases orientadas de $-X$.

Porém, é equivalente que (β_1, β_2) seja orientada em $X \times X$ e que $(\tilde{\beta_1}, \tilde{\beta_2})$ seja orientada em $-X \times -X$: tanto a transformação que manda $\beta_1 \mapsto \tilde{\beta_1}$ quanto a transformação que manda $\beta_2 \mapsto \tilde{\beta_2}$ tem determinante negativo, de modo que a transformação que manda $(\beta_1, \beta_2) \mapsto (\tilde{\beta_1}, \tilde{\beta_2})$ tem determinante positivo! □

Problem 3 Prove que $\mathsf{SO}(n)$ é uma variedade orientável e calcule a sua dimensão. Usando teoria da interseção prove que $\chi(\mathsf{SO}(n)) = 0$.

Demostração. First notice that $\mathsf{SO}(n)$ is one of the connected components of $\mathsf{O}(n)$. Indeed, $\mathsf{SO}(n) = \det^{-1}(1)$ for the submersion $\det : \mathsf{O}(n) \to \{\pm 1\}$, making into a codimension-0 submanifold of $\mathsf{O}(n)$ since $\dim\{\pm 1\} = 0$. This means that computing the dimension of $\mathsf{SO}(n)$ is the same as computing the dimension of $\mathsf{O}(n)$.

Now observe that a matrix in $\mathsf{O}(n)$ is the same as an orthonormal frame of $\mathbb{R}^n$: the column vectors of any $A \in \mathsf{O}(n)$ unitary and mutually orthogonal since $AA^T = \mathrm{Id}$ says $\sum_k a_{ik} a_{jk} = \delta_{ij}$ for every i, j.

We can compute the dimension of $\mathsf{O}(n)$ as follows. Take a vector v_1 in S^{n-1}, then a unitary vector in the orthogonal complement of v_1, i.e. a vector in S^{n-2}, and so on until we choose either of the two vectors in S^0. This means that we are choosing points in

$S^{n-1} \times S^{n-2} \times \ldots \times S^0$, which gives $\dim \mathsf{O}(n) = \sum_{i=0}^{n-1} i$. Gauss could tell at very early age that this number is $\frac{n(n-1)}{2}$.

To orient $\mathsf{SO}(n)$ just notice that it acts on itself homogeneously (by orientation-preserving diffeomorphisms). Taking a basis at the identity matrix and moving it around our manifold using this action generates a smooth global choice of local orientations; i.e. a global orientation.

The fact that $\chi(\mathsf{SO}(n)) = 0$ is immediate from the fact that its tangent bundle is trivial: there is a nowhere vanishing vector field (the orbit of any nonzero vector), giving the result by Poincaré-Hopf theorem. □

Problem 4 Seja Σ uma superfície de gênero g. Construa um campo vetorial em Σ com um único zero de índice $2-2g$.

Solution.

□

Problem 5 Seja A uma matriz de $n \times n$ com coeficientes inteiros e seja $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ tal que $f(x) = Ax$. Calcule o grau de f.

Demostração. It's the determinant! It suffices to show that there's $\det A$ points with integer coordinates within the parellelepiped determined by A...

First consider the case for $n = 1$. Take the class $[0] \in \mathbb{R}/\mathbb{Z}$ and let's check how many preimages it has in the fundamental domain $[0,1) \subset \mathbb{R}$. Our matrix A is only a number, say a. So we have $ax \in [0] = \{0 + n : n \in \mathbb{Z}\}$. This just says $ax \in \mathbb{Z}$ which happens when x is a rational number with denominator a, and there's $|a|$ such numbers in $[0,1)$.

Now for the case $n = 2$ suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let's look for points $\vec{x} = (x,y) \in [0,1)^2$ such that $A\vec{x} \in [\vec{0}] = \mathbb{Z}^2$. This means that

$$A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \in \mathbb{Z}^2.$$

Substracting these two conditions (that $ax + by \in \mathbb{Z}$ and that $cx + dy \in \mathbb{Z}$) we obtain that

$$x(a-c) + y(b-d) \in \mathbb{Z}$$

Suppose for a second that $y = 0$: we go back to a case similar to the 1-dimensional, namely there are $|a-c|$ solutions. The same works whenever $y(b-d) \in \mathbb{Z}$, for which there are $|b-d|$ choices. So there's $|a-c||b-d|$ solutions.

□

Problem 6 Prove que $\mathbb{R}P^{2n+1}$ é orientável e que $\mathbb{R}P^{2n}$ não é orientável.

Demostração. First notice that $-\operatorname{Id}$ preserves orientation iff n is odd. This map is a composition of n reflections, one about every axis of $\mathbb{R}^{n+1} \supset S^n$. Each of these reflections is orientation-reversing, and composing a map with an orientation-reversing map reverses orientation by the chain rule.

Now recall that $\mathbb{R}P^n = S^n / -\operatorname{Id}$. Suppose $\mathbb{R}P^n$ is orientable, so that the quotient map is orientation-preserving since it is a submersion: the determinant of its differential is a nowhere-zero continuous function on a connected manifold, so it cannot be positive somewhere and negative elsewhere.

Choose an oriented basis of the tangent space of $\mathbb{R}P^n$ at $[e_1]$. Pull back the basis using quotient map, this produces a basis at each of the preimages, namely e_1 and $-e_1$. These two bases must be in the same orientation of S^n since the quotient map is orientation-preserving and they are mapped to the same basis in the quotient. However, this only happens when $-\operatorname{Id}$ is orientation-preserving. $\square$