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Lista 2: solution of 4(b)

Problem 4 Let M be a non-compact manifold.

(a) Prove that multiplication by scalar $\mathbb{R} \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ is not continuous in the C^∞ topology.

(b) Prove that addition and multiplication of functions are continuous in the C^∞ topology.

Solution.

(a) How to prove (a)? I can't see why the argument below couldn't be used here.

(b) First notice (thanks to ChatGPT) that addition map

$$A : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), \qquad (f, g) \mapsto f + g$$

induces a bundle map

$$\tilde{A} : J^k(M, \mathbb{R}) \times J^k(M, \mathbb{R}) \longrightarrow J^k(M, \mathbb{R})$$

which is smooth. Indeed, addition of two sections $j^k f$ and $j^k g$ is smooth since addition of real numbers is smooth. (To define this map formally we fix a point of M and map two k-jets at x to their sum at x, which in coordinates gives a polynomial whose coefficients are sums of the coefficients of the original polynomials.)

Then we show that addition of smooth functions is C^∞-continuous like this: Let $U \subset C^\infty(M, \mathbb{R})$ open, then $A^{-1}(U)$ is some set in $C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R})$. Take a point (f, g) in there. Since $f + g$ is in the open set U, there is a k such that $f + g \in M(V)$ for some V open in $J^k(M, \mathbb{R})$. The preimage $\tilde{A}^{-1}(V) := V_1 \times V_2$ is open in $J^k(M, \mathbb{R}) \times J^k(M, \mathbb{R})$. Consider the C^∞-open set $M(V_1) \times M(V_2) \ni (f, g)$.

To conclude we must check that $M(V_1) \times M(V_2) \subset A^{-1}(M(V))$. This means that the sum of any pair of smooth functions in $M(V_1) \times M(V_2)$ remains in $M(V)$. But any two smooth induce a sum of k-jets that remains in V by construction.

Multiplication of functions is analogous; this time we should use a bundle map $\tilde{M}$ which is also continuous since multiplication of k-jets is locally multiplication of polyonimals.

□

References