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Practice exercises on smooth manifolds

A pdf file with the questions may also be found here.

1 Remedial topology

1.1 Topological spaces

Definition 1.1 A set of all subsets of M is denoted 2^M. ***Topology*** on M is a collection of subsets $S \subset 2^M$ called ***open subsets***, and satifsying the following conditions:

1. Empty set and M are open
2. A union of any number of open sets is open
3. An intersection of a finite number of open subsets is open.

A complement of an open set is called ***closed***. A set with topology on it is called a ***toplogical space***. An ***open neighbourhood*** of a point is an open set containig this point.

Definition 1.2 A map $\phi : M \to M'$ of topological spaces is called ***continuous*** if a preimage of each open set $U \subset M'$ is open in M. A bijective cintunuous map is called a ***homeomorphism*** if its inverse is also continuous.

Exercise 1.1 Let M be a set, and S a set of all subsets of M. Prove that S defines a topology on M. This topology is called ***discrete***. Describe the set of all continuous maps form M to a given topological space.

Solution. Since all sets are open, topology axioms are satisfied by S. All maps from M to a given topological space are continuous. □

Exercise 1.2 Let M be a set, and $S \subset 2^M$ a set of two subsets: empty set and M. Prove that S defines a topology on M. This topology is called ***codiscrete***. Describe the set of all continuous maps from M to a space with discrete topology.

Solution. It's trivial that S satisfies the axioms of topology. A map from M to a space N with discrete topology is always continuous: the preimage of N is always M by definition of function: every point of M corresponds to some point of N. Obviously the preimage of $\varnothing$ is $\varnothing$. □

Definition 1.3 Let M be a topological space, and $Z \subset M$ its subset. ***Open subsets*** of Z are subsets obtained as $Z \cap U$, where U is open in M. This topology is called ***induced topology***.

Definition 1.4 A ***metric space*** is a set M equipped with a ***distance function*** $d : M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying the following acioms.

1. $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. (triangle inequality) $d(x, y) + d(y, z) \geqslant d(x, z)$.

An ***open ball*** of raduis r with center in xx is $\{y \in M : d(x, y) < r\}$.

Definition 1.5 Let M be a metric space. A subset $U \subset M$ is called ***open*** if it is obtained as a union of open balls. This topology is called ***induced by the metric***.

Definition 1.6 A topological space is called ***metrizable*** if its topology can be induced by a metric.

Exercise 1.3 Show that discrete topology can be induced by a metric, and codiscrete cannot.

Solution. To induce the discrete metric define the distance between any two distinct points to be 1. This clearly satisfies the three axioms of metric, and the ball of radius $1/2$ is an open set that contains only its center, making any point and thus any subset an open set.

If a metric space contains at least two points at distance d, the ball with radius $d/2$ at any of these points is an open set distinct from the empty set and the total, so the topology induced by the metric cannot be discrete. □

Exercise 1.4 Prove that an intersection of any collection of closed subsets of a topological space is closed.

Solution. As I recall this is due to de Morgan laws stating that for any collection F_α of subsets

$$\left(\bigcap_\alpha F_\alpha \right)^c = \bigcup_\alpha F^c \tag{1}$$

where superscript c means set complement. If this is true then we are done because if F_α are closed, we see that the intersection is also closed as its complement is open.

Let's try to show eq. (1) ... □

Definition 1.7 An intersection of all closed supersets $Z \subset M$ is called ***closure*** of Z

Definition 1.8 A ***limit point*** of a set $Z \subset M$ is a point $x \in M$ such that any neighbourhood of M contains a point of Z other than x. A ***limit*** of a sequence $\{x_i\}$ of points in M is a point $x \in M$ such that any neighbourhood of $x \in M$ contains all x_i for ll i except a finite number. A sequence which has a limit is called ***convergent***.

Exercise 1.5 Show that a closure of a set $Z \subset M$ is a union of Z and all its limit points.

Solution. It's enough to show that the union of Z and all its limits points W is a closed set and that it is contained in any closed set containing Z.

(The idea is to use the fact that a set is open iff every point has a neighbourhood completely contained in the set. I could show one implication but got stuck in the other...)

To see W is closed chose a point in its complement $p \in W^c$. Since p is not a limit point of Z nor a point of Z, there is a neighbourhood of p not intersecting Z. This means that such neighbourhood is contatined in W^c. We can do this for all points in W^c, thus obtaining a W^c as a union of open sets, which is open, and then W is closed.

To see W is contained in any closed set containing Z, suppose F contains Z but not W. Then there must be a limit point of Z that is not in F. But then F cannot be closed because there is no neighbourhood of such a limit point contained in F^c, which should be open. Indeed, if F^c is open then every point contains a neighbourhood contained in F^c. □

Exercise 1.7 Let $f : M \to M'$ be a map of metrizable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous.

Solution. It is equivalent that the preimage of every open set is open (definition of f being continuous) with the preimage of every closed subset is closed: for any closed set $M' \backslash U$ with U open, $f^{-1}(M \backslash U) = f^{-1}(M') \backslash f^{-1}(U)$ is closed.

Consider the closed set $F \subset M'$ and let's check that its preimage is also closed. By the same reasoning as in Exercise 1.5, to show closedeness it's enough to show the set contains all its limit points. Take a limit point $p \in f^{-1}(F)$. We construct a convergent sequence $\{x_n\}$ taking balls of radius $\frac{1}{n}$ around p, each of which must contain a point in $f^{-1}(F)$. This gives a sequence in F, which by hypothesis must converge to a limit point $\lim_i f(x_i) = f(\lim_i x_i) \in F$. This means $p = \lim_i x_i$ is in the inverse image of F. □

Exercise 1.8* Find a counterexample to the previous problem for non-metrizable, Hausdorff topological spaces (see the next subsection of a definition of Hausdorff).

Sketch of solution. Probably Sørgenfrey line is a counter-example? I should look for its definition to make sure it is Hausdorff (and how is it defined exactly—I think open sets are positive rays). □

Exercise 1.9** Let $f : M \longrightarrow M'$ be a map of countable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous, or find a counterexample.

Sketch of solution. Is a ***countable space*** a space whose cardinality is $\mathbb{N}$? What are the possible topologies on $\mathbb{N}$? Discrete topology gives that every map is continuous. Other topologies are maybe, again, rays. □

Exercise 1.10* Let $f : M \longrightarrow N$ be a bijective map inducing homeomorphisms on all countable subsets of M. Show that it is a homeomorphism, or find a counterexample.

Sketch of solution. If we suppose that M and M' are metrizable, we can use Exercise 1.7 as follows. Choose any convergent sequence $\{x_i \in M\}$. Then the countable set $\{x_i\} \cup \{\lim_i f(x_i)\}$ is mapped homeomorphically to $\{f(x_i)\} \cup \{f(\lim_i x_i)\}$. This implies that $f(\lim_i f(x_i)) = f(\lim_i x_i)$, so f is continuous. The same holds for f^{-1}, so f is a homeomorphism.

Probably the statement isn't true in general, so let's look for a counter-example. □

1.2 Hausdorff spaces

Definition 1.9 Let M be a topological space. It is called ***Hausdorff*** or ***separable***, if any two distinct points $x \neq y \in M$ can be ***separated*** by open subsets, that is, there exist open neighbourhoods $U \ni x$ and $V \ni y$ wuch that $U \cap V = \emptyset$.

Remark 1.1 In topology, the Hausdorff axiom is usually assumed by default. In subsequent handouts, it will be always assumed (unless stated otherwise).

Exercise 1.11 Let M be a Hausdorff topological space. Prove that all points in M are closed subsets.

Solution. Fix a point $x \in M$. For every $y \in M$ distinct from x we have the neighbourhoods $U_y \ni x$ and $V_y \ni y$ with $U_y \cap V_y = \emptyset$. Then $X \backslash \{x\} = \bigcup_{y \neq x} V_y$, which is open. □

Exercise 1.13 Let M be a topological space, with all points of M closed. Prove that M is Hausdorff, or find a counterexample.

Solution. No solution yet... □

Exercise 1.14 Count the number of non-isomorphic topologies on a finite set of 4 elements. How many of these topologies are Hausdorff.

Solution. For any set S of subsets of $\{1, 2, 3, 4\}$ we can consider the ***topology generated by*** S, which consists of all unions and intersections of elements in S, along with the total space and the empty set.

For the following choices of S we get non-isomorphic topologies:

1. $S = \emptyset$ (codiscrete topology).
2. $S = \Big\{\{1\}\Big\}$
3. $S = \Big\{\{1\}, \{2\}\Big\}$.
4. $S = \Big\{\{1\}, \{2\}, \{3\}\Big\}$.
5. $S = \Big\{\{1\}, \{2\}, \{3\}, \{4\}\Big\}$ (discrete topology).

6. $S = \Big\{\{1,2\}\Big\}$.
7. $S = \Big\{\{1,2\},\{3\}\Big\}$
8. $S = \Big\{\{1,2\},\{3,4\}\Big\}$.
9. $S = \Big\{\{1,2,3\}\Big\}$.
10. $S = \Big\{\{1,2,3\},\{4\}\Big\}$.

1-5 are Hausdorff while 6-10 are not Hausdorff. $\square$

Exercise 1.5 (!) Let Z_1, Z_2 be nonintersecting closed subsets of a metrizable space M. Find open subsets $U \supset Z_1, V \supset Z_2$ which do not intersect.

Solution. Consider the distance between Z_1 and Z_2:

$$d(Z_1, Z_2) := \inf\{d(z_1, z_2) : z_1 \in Z_1, z_2 \in Z_2\}.$$

We must argue that $d(Z_1, Z_2) \neq 0$. Suppose by contradiction that $d(Z_1, Z_2) = 0$ Then for every $n \in \mathbb{N}$ there is a pair of points z_1^n and z_2^n such that $d(z_1^n, z_2^n) < 1/n$. $\square$

Definition 1.10 Let M, N be topological spaces. ***Product topology*** is a topology on $M \times N$, with open sets obtained as unions $\bigcup_\alpha U_\alpha \times V_\alpha$, where U_α is open in M and V_α is open in N.

Exercise 1.16 Prove that a topology on X is Hausdorff if and only if the diagonal $\Delta := \{(x,y) \in X \times X | x = y\}$ is closed in the product topology.

Solution. ($\implies$) Suppose that X is Hausdorff. To check that Δ is closed suppose there is a sequence (x_n, x_n) that converges to some point $(a,b) \in X \times X$. We need to show that $(a,b) \in \Delta$, i.e. that $a = b$. For every neighbourhood $W = U \times V$ of (a,b) we know that all but a finite number of (x_n, x_n) belong to U. This means that x_n converges to a and also to b. But limits are unique in Hausdorff spaces: if $a \neq b$ we can separate a and b by disjoint open subsets A and B, and then it cannot hold simultaneously that all but a finite number of points in the sequence x_n are in A and in B.

($\impliedby$) Suppose Δ is closed in the product topology. $\square$

Definition 1.11 Let $\sim$ be an equivalence relation on a topological space M. ***Factor-topology*** (or ***quotient topology***) is a topology on the set $M/\sim$ of equivalence classes such that a subset $U \subset M/\sim$ is open whenever its preimage in M is open.

Exercise 1.17 Let G be a finite group acting (continuously) on a Hausdorff topological space M.Prove that the quotient map is closed (i.e. puts closed subsets to closed subsets).

Solution. The quotient map is $\pi : M \to M/\sim$ where $x \sim y$ if $y = gx$ for some $g \in G$. To show π is closed pick $F \subset M$ closed. We need to show that $\pi(F)$ is closed, so we may show its complement is open. According to the definition of factor topology we want to show that

$$\pi^{-1}\Big((M/\sim)\backslash\pi(F)\Big) = M\backslash\pi^{-1}(\pi(F))$$

is open. Now $\pi^{-1}(\pi(F))$ is the set of points that are G-related to points in F, namely $\bigcup_{g\in G} gF$. Since G is finite and acts by homeomorphisms, this set is a finite union of closed sets, which is closed. Looks like I didn't use the Hausdorff hypothesis; and that the statement holds for countable G. □

Exercise 1.18* Let $\sim$ be an equivalence relation on a topological space M, and $\Gamma \subset M \times M$ its ***graph***, that is, the set $\{(x, y) \in M \times M | x \sim y\}$. Suppose that the map $M \longrightarrow M/\sim$ is open, and that Γ is closed in $M \times M$. Show that $M/\sim$ is Hausdorff.

Hint . Prove that diagonal is closed in $M \times M$.

Solution. Our objective is to show that the diagonal $\tilde{\Delta}$ in $(M/\sim) \times (M/\sim)$ is closed. Following the hint, if we show that the diagonal Δ in $M \times M$ is closed, we can project to $(M/\sim) \times (M/\sim)$ and we obtain that the diagonal is closed in the latter space; this is because the projection is surjective, i.e. any open surjective map is closed (let $f : X \twoheadrightarrow Y$ be an open map and $F \subset X$ closed, then $f(X\backslash F) = f(X)\backslash f(F) = Y\backslash f(F)$).

It appears that it's not necessary to prove that Δ is closed: notice that the projection of the graph Γ is $\tilde{\Delta}$. Since Γ is closed, by the remark above it follows that $\tilde{\Delta}$ is closed in $(M/\sim) \times (M/\sim)$ as we needed. □

Exercise 1.19 Let G be a finit group acting on a Hausdorff topological space M. Prove that M/G with the quotient topology is Hausdorff,

(a) (!) when M is compact.

(b) (*) for abitrary M.

Hint. Use the previous exercise.

Sketch of solution. To use the previous exercise first notice that the action of G induces an equivalence relation on M; this follows from group axioms. Then it's enough to show that the projection is open and that the graph Γ of the equivalence relation is closed in $M \times M$. But by Exercise 1.17 we already know that the projection is closed, so it's enough to show that Γ is closed.

Notice that $\Gamma = \bigcup_{x\in X}(Gx) \times (Gx)$, that is, the union of cartesian products of every orbit with itself. Each of these cartesian products is a finite set because G is finite. If M is compact, then . . . □

Exercise 1.20** Let $M = \mathbb{R}$, and $\sim$ an equivalence relation with at most two elements in each equivalence class. Prove that $\mathbb{R}/\sim$ is Hausdorff, or find a counterexample.

Solution. By Exercise 1.19, if this equivalence relation is induced by a finite-group action, we know the quotient space is Hausdorff. Let's try to show that there always exists a group inducing this equivalence relation. Since every orbit has at most two elements, we can produce a function

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} y & \exists y \sim x, y \neq x \\ x & \text{else} \end{cases} \end{aligned}$$

This function satisfies $g^2 = \mathrm{id}$. So the group $G = \{\mathrm{id}, g\}$ acts on $\mathbb{R}$ producing the equivalence relation we began with. ☐

Exercise 1.21* (Gluing of closed subsets) Let M be a metrizable topological space, and $Z_i \subset M$ a finite number of closed subsets which do not intersect, grouped into pairs of homeomorphic $Z_i \sim Z_i'$. Let $\sim$ be an equivalence relation generated by these homeomorphisms. Show that $M/\sim$ is Hausdorff.

Solution. ? ☐

2 Compact spaces

Definition 1.12 A ***cover*** of a topological space M is a collection of open subsets $\{U_\alpha \in 2^M\}$ such that $\bigcup U_\alpha = M$. A ***subcover*** of a cover $\{U_\alpha\}$ is a subset $\{U_\beta\} \subset \{U_\alpha\}$. A topological space is called ***compact*** if any cover of this space has a finite subcover.

Exercise 1.22 (Closed subset of compact is compact) Let M be a compact topological space, and $Z \subset M$ a closed subset. Show that Z is also compact.

Solution. Choose a cover $\{U_\alpha\}$ of Z. Complete to a cover $\{U_\alpha\} \cup (M \backslash Z)$ of M since $M \backslash Z$ is open by hypothesis. Since M is compact then there is a finite subcover $\{U_\beta\}$ of M. This is also a finite subcover of Z. ☐

Exercise 1.23 (Countable metrizable $\implies$ contains convergent subseq. or is discrete) Let M be a countable, metrizable topological space. Show that either M contains a converging sequence of pairwise different elements, or M is discrete.

Solution. Suppose M is not discrete. Then there is a point z_0 such that $\{z_0\}$ is not an open set. Then every open set containing z_0 contains another point. Choose for every $n \in \mathbb{N}$ a point z_n different from z_0 inside the a ball $B_{1/n}(z_0)$. Taking a subsequence if necessary, we obtain a sequence of pairwise different elements $\{z_i\}$ converging to z_0.

If M is discrete, it's clear that it cannot have a convergent sequence of pairwise disjoint elements: if the limit point $\{z_0\}$ was open, $M \backslash \{z_0\}$ would be closed and thus it would contain all its limit points! ☐

Definition 1.13 A topological space is called ***sequentially compact*** if any sequence $\{z_i\}$ of points of M has a converging subsequence.

Exercise 1.24 (Metrizable compact $\implies$ sequentially compact) Let M be a metrizable compact topological space. Show that M is sequentially compact.

Solution. Let $\{z_i\}$ be a sequence. Since the restricition of a metric to a subset is also a metric, we may use Exercise 1.23 on the countable metric subspace $\{z_i\}$. Suppose by contradiction that $\{z_i\}$ has no limit point in M. In particular it has no limit point in $\{z_i\}$, so by Exercise 1.23 it is discrete. Then there are neighbourhoods $U_i \ni z_i$ such that $U_i \cap \{z_j\}_{j \neq i} = \varnothing$. Then $\{U_i\} \cup (M \backslash \{U_i\})$ is an open cover of M, which has a finite subcover. By the pigeon principle, at least one of the U_i contains an infinite number of points in $\{z_i\}$, which is not possible. $\square$

Definition (Folland, *Real Analysis*, p. 14-15) A sequence $\{x_n\}$ in a metric space (X, ρ) is called ***Cauchy*** if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$. A subset E of X is called ***complete*** if every Cauchy sequence in E converges and its limit is in E. E is called ***totally bounded*** if for every $\varepsilon > 0$, E can be covered by finitely many balls of radius ε.

Theorem 1.6.5 (Burago-Burago-Ivanov, *A course in metric geometry*) Let X be a metric space. Then the following statements are equivalent:

1. X is compact.
2. Any sequence in X has a converging subsequence.
3. Any infinite subset of X has an accumulation point.
4. X is complete and totally bounded.

[No proof]

Theorem 0.25 (Folland, *Real Analysis*) If E is a subset of the metric space (X, ρ), the following are equivalent:

(a) E is complete and totally bounded.

(b) (**Bolzano-Weierstrass property**) Every sequence in E has a subsequence that converges to a point of E.

(c) (**The Heine-Borel Property**) If $\{V_\alpha\}_{\alpha \in A}$ is a cover of E by open sets, there is a finite set $F \subset A$ such that $\{V_\alpha\}_{\alpha \in F}$ covers E.

Plan of proof. (a) and (b) are equivalent, and (a) and (b) together imply (c). $\square$

Exercise 1.28 (Continuous function maps compact to compact) Let $f : X \to Y$ be a continuous map of topological spaces with X compact. Prove that $f(X)$ is also compact.

Solution. Choose an open cover $\{U_\alpha\}$ of $f(X)$. Then $\left\{f^{-1}(U_\alpha)\right\}$ is an open cover of X since f is continuous, and thus it has an open subcover $\{f^{-1}(U_\beta)\}$. I claim that $\{U_\beta\}$ is a cover of $f(X)$: if there was a point $f(x) \notin \bigcup U_\beta$, then x couldn't be in any of the $f^{-1}(U_\beta)$, which cover X. $\square$

Exercise 1.29 (Compact subset of Hausdorff is closed) Let $Z \subset Y$ be a compact subset of a Hausdorff topological space. Prove that it is closed.

Solution. Recall that a set is closed if it contains all its limit points (if a limit point z_0 of the closed set F is not in F, then z_0 has no neighbourhood contained in F^c, but F^c is open).

Let z_0 be a limit point of Z. $\square$

Exercise 1.30 Let $f : X \to Y$ be a continuous, bijective map of topological spaces, with X compact and Y Hausdorff. Prove that it is a homeomorphism.