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Practice exercises on smooth manifolds

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Plan for today: some exercises on partition of unity and discussion of homework 1 from complex surfaces course.

Definition 2.15 A cover $\{U_\alpha\}$ of a topological space M is called ***locally finite*** if every point in M possesses a neighbourhood that intersects only a finite number of U_α.

Exercise 2.27 Let $\{U_\alpha\}$ be a locally finite atlas on M, and $U_\alpha \xrightarrow{\phi_\alpha} \mathbb{R}^n$ homeomorphisms. Consider a cover $\{V_i\}$ of $\mathbb{R}^n$ given by open balls of radius n centered in integer points, and let $\{W_\beta\}$ be a cover of M obtained as union of $\phi_\alpha^{-1}(V_i)$. Show that $\{W_\beta\}$ is locally finite.

Solution. The result follows from the local finiteness of both $\{U_\alpha\}$ in M and $\{V_i\}$ in $\mathbb{R}^n$ as follows. (Local finiteness of $\{V_i\}$ follows from definition of $\{V_i\}$.)

Since $\{U_\alpha\}$ is locally finite, for a given point x of M there is a neighbourhood U_0 which intersects only a finite number of the U_α. Moreover, since $\{V_i\}$ is locally finite, each $\phi_\alpha(x)$ has a neighbourhood intersecting only finitely many V_i. Then there's only finitely many of the W_β intersecting U_0 (for any α and i).

□

Exercise 2.28 Let $\{U_\alpha\}$ be an atlas on a manifold M.

(a) Construct a refinement $\{W_\beta\}$ of $\{U_\alpha\}$ such that a closure of each W_β is compact in M.

(b) Prove that such a refinement can be chosen locally finite if $\{U_\alpha\}$ is locally finite.

Hint. Use the previos exercise.

Solution.

(a) The refinement is the cover $\{W_\beta\}$ from Exercise 2.27. The closure of $W_\beta = \phi_\alpha^{-1}(V_i)$ is mapped by ϕ_α to the closure of its image, $\phi_\alpha(U_\alpha) \cap V_i$. (This is because ϕ_α is a homeomorphism; by Exercise 1.6 limit points of the domain map to limit points of the image.) The closure of $\phi_\alpha(U_\alpha) \cap V_i$ is compact (since it is closed and bounded), and thus its image under ϕ^{-1} is also compact.

(b) This is immediate from Exercise 2.27.

□

Exercise 2.29 Let K_1, K_2 be non-intersecting compact subsets of a Hausdorff topological space M. Show that there exist a pair of open subsets $U_1 \supset K_1$, $U_2 \supset K_2$ satisfying $U_1 \cap U_2 = \varnothing$.

Solution. (With some help from ChatGPT). Fix a point $y \in K_2$. Since M is Hausdorff, for every $x \in K_1$ there are disjoint neighbourhoods $U_{xy} \ni x$ and $V_{xy} \ni y$. This means that $\{U_{xy}\}_{x\in X}$ is an open cover of K_1, which must have a finite subcover $U_{x_1 y}, \dots, U_{x_{n_y} y}$. These open sets correspond to open sets $V_{x_1 y}, \dots, V_{x_{n_y} y}$, the intersection of which is a neighbourhood of y disjoint from $\bigcup_{i=1}^{n_y} U_{x_i y}$.

Denote this intersection by $V_y := \bigcap_{i=1}^{n_y} V_{x_i y}$. Then $\{V_y\}_{y\in Y}$ is an open cover of Y, which must have a finite subcover $V_{y_1}, \dots, V_{y_m}$. Each V_{y_j} is associated to an open cover of K_1, from which it is disjoint. The intersection of (the unions of) these m covers of K_1 is an open set containing K_1, and it is disjoint from $\bigcup_{j=1}^{m} V_{y_j} \supset K_2$.

Upshot You have pairs of disjoint sets. The intersection of one family is disjoint from the union of the other.

□

Exercise 2.30 (!) Let $U \subset M$ be an open subset with compact closure, and $V \supset M \backslash U$ another open subset. Prove that there exists $U' \subset U$ such that the closure of U' is contained in U, and $V \cup U' = M$.

Hint. Use the previous exercise.

Solution. (Using ChatGPT.) Define the ***boundary*** ∂A of a set A in a topological space X to be the set of points $x \in X$ such that every neighbourhood of x contains a point of A and a point of $X \backslash A$.

The boundary ∂U of our open set with compact closure U is compact: it is contained in the closure of U (since all its points are limit points of U), and it is closed: every point in its complement has a neighbourhood that stays inside its complement; whether it is in U, or in $M \backslash \bar{U}$.

Now let's use Exercise 2.29. We can separate $K_1 := \partial U$ and $K_2 := U \backslash V$. Both are compact, and they are disjoint because the boundary of U is disjoint from U. Then there are disjoint neighbourhoods U_1 and U_2 of K_1 and K_2.

Now let's show that $U_2 \cap U := U'$ is the open set we are looking for, that is, that its closure is contained in U and $V \cup U' = M$. If a point in the closure of U' was outside U, then such a limit point would be in the boundary of U: any open neighbourhood must contain a point of U since it is a limit point of U, and also a point outside it, the limit point itself! But the boundary of U is disjoint from U_2.This shows that the closure of U' is inside U.

To show that $V \cup U' = M$ pick a point in $M \backslash V$. Then $U' := U_2 \cap U \supset K_2 := U \backslash V$ contains it. □

Exercise 2.31 (!) Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological space, such that a closure of each U_α is compact. Prove that there exists another cover $\{V_\alpha\}$ indexed by the same set, such that $V_\alpha \Subset U_\alpha$.

Hint. Use induction and the previous exercise.

Solution. In order to use Exercise 2.30 consider for every α the set $W_\alpha = \bigcup_{\beta \neq \alpha} U_\beta$. Then $W_\alpha \supset M \backslash U_\alpha$, so that there exists $U'_\alpha \Subset U_\alpha$ and $W_\alpha \cup U'_\alpha = M$. It remains to show that $\{U'_\alpha\}$ is a cover. Let $x \in M$ be any point. but how?

That's why the hint says use induction. We go one by one: consider U_1, an open set. The rest of the cover yields an open set like V from the last exercise, which contains the complement of U. Then that exercise yields a set $U'_1 \Subset U$ st $V \cup U'_1 = M$.

Now take $n = 2$. But don't use the original open cover: *substitute* U_1 *by* U'_1. Obviously. (It works basically because of the second condition, explaining why we went through so much hustle to construct the set U', anyway moving on.) The point is that now we get a set $U'_2 \Subset U_2$ which covers M along with U'_1 and the rest of the U_α.

This works for all α: there is $U'_\alpha \Subset U_\alpha$ such that $U'_\alpha \cup U'_{\alpha-1} \cup \ldots \cup U'_1 \cup \bigcup_{i>\alpha} U_i$ covers M.

Let's show that $\{U'_\alpha\}$ is a cover. Suppose there's a point x outside U'_α for all α. Then it is in $\bigcup_{i>\alpha} U_i$ for all α, meaning x is in a infinite ammount of open sets of the locally finite cover U_i. $\square$