Metric spaces 1: Remedial topology

Rules: It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

1.1 Topological spaces

Definition 1.1. A set of all subsets of M is denoted 2^M. **Topology** on M is a collection of subsets $S \subset 2^M$ called **open subsets**, and satisfying the following conditions.

1. Empty set and M are open
2. A union of any number of open sets is open
3. An intersection of a finite number of open subsets is open.

A complement of an open set is called **closed**. A set with topology on it is called **a topological space**. **An open neighbourhood** of a point is an open set containing this point.

Definition 1.2. A map $\phi: M \longrightarrow M'$ of topological spaces is called **continuous** if a preimage of each open set $U \subset M'$ is open in M. A bijective continuous map is called **a homeomorphism** if its inverse is also continuos.

Exercise 1.1. Let M be a set, and S a set of all subsets of M. Prove that S defines topology on M. This topology is called **discrete**. Describe the set of all continuous maps from M to a given topological space.

Exercise 1.2. Let M be a set, and $S \subset 2^M$ a set of two subsets: empty set and M. Prove that S defines topology on M. This topology is called **codiscrete**. Describe the set of all continuous maps from M to a space with discrete topology.

Definition 1.3. Let M be a topological space, and $Z \subset M$ its subset. **Open subsets** of Z are subsets obtained as $Z \cap U$, where U is open in M. This topology is called **induced topology**.

Definition 1.4. A metric space is a set M equipped with a **distance function** $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying the following axioms.

1. $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. (triangle inequality) $d(x, y) + d(y, z) \geqslant d(x, z)$.

An **open ball** of radius r with center in x is $\{y \in M \mid d(x, y) < r\}$.

Definition 1.5. Let M be a metric space. A subset $U \subset M$ is called **open** if it is obtained as a union of open balls. This topology is called **induced by the metric**.

Definition 1.6. A topological space is called **metrizable** if its topology can be induced by a metric.

Exercise 1.3. Show that discrete topology can be induced by a metric, and codiscrete cannot.

Exercise 1.4. Prove that an intersection of any collection of closed subsets of a topological space is closed.

Definition 1.7. An intersection of all closed supersets of $Z \subset M$ is called **closure** of Z.

Definition 1.8. A limit point of a set $Z \subset M$ is a point $x \in M$ such that any neighbourhood of M contains a point of Z other than x. **A limit** of a sequence $\{x_i\}$ of points in M is a point $x \in M$ such that any neighbourhood of $x \in M$ contains all x_i for all i except a finite number. A sequence which has a limit is called **convergent**.

Exercise 1.5. Show that a closure of a set $Z \subset M$ is a union of Z and all its limit points.

Exercise 1.6. Let $f: M \longrightarrow M'$ be a continuous map of topological spaces. Prove that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$.

Exercise 1.7. Let $f: M \longrightarrow M'$ be a map of metrizable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous.

Exercise 1.8 (*). Find a counterexample to the previous problem for non-metrizable, Hausdorff topological spaces (see the next subsection for a definition of Hausdorff).

Exercise 1.9 (**). Let $f: M \longrightarrow M'$ be a map of countable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous, or find a counterexample.

Exercise 1.10 (*). Let $f: M \longrightarrow N$ be a bijective map inducing homeomorphisms on all countable subsets of M. Show that it is a homeomorphism, or find a counterexample.

1.2 Hausdorff spaces

Definition 1.9. Let M be a topological space. It is called **Hausdorff**, or **separable**, if any two distinct points $x \neq y \in M$ can be **separated** by open subsets, that is, there exist open neighbourhoods $U \ni x$ and $V \in y$ such that $U \cap V = \emptyset$.

Remark 1.1. In topology, the Hausdorff axiom is usually assumed by default. In subsequent handouts, it will be always assumed (unless stated otherwise).

Exercise 1.11. Prove that any subspace of a Hausdorff space with induced topology is Hausdorff.

Exercise 1.12. Let M be a Hausdorff topological space. Prove that all points in M are closed subsets.

Exercise 1.13. Let M be a topological space, with all points of M closed. Prove that M is Hausdorff, or find a counterexample.

Exercise 1.14. Count the number of non-isomorphic topologies on a finite set of 4 elements. How many of these topologies are Hausdorff?

Exercise 1.15 (!). Let Z_1, Z_2 be non-intersecting closed subsets of a metrizable space M. Find open subsets $U \supset Z_1$, $V \supset Z_2$ which do not intersect.

Definition 1.10. Let M, N be topological spaces. **Product topology** is a topology on $M \times N$, with open sets obtained as unions $\bigcup_\alpha U_\alpha \times V_\alpha$, where U_α is open in M and V_α is open in N.

Exercise 1.16. Prove that a topology on X is Hausdorff if and only if the diagonal $\{(x,y) \in X \times X \ \mid \ x = y\}$ is closed in the product topology.

Definition 1.11. Let $\sim$ be an equivalence relation on a topological space M. **Factor-topology** (or **quotient topology**) is a topology on the set $M/\sim$ of equivalence classes such that a subset $U \subset M/\sim$ is open whenever its preimage in M is open.

Exercise 1.17. Let G be a finite group acting on a Hausdorff topological space M.[1] Prove that the quotient map is closed.[2]

Exercise 1.18 (*). Let $\sim$ be an equivalence relation on a topological space M, and $\Gamma \subset M \times M$ **its graph**, that is, the set $\{(x,y) \in M \times M \ \mid \ x \sim y\}$. Suppose that the map $M \longrightarrow M/\sim$ is open, and the Γ is closed in $M \times M$. Show that $M/\sim$ is Hausdorff.

Hint. Prove that diagonal is closed in $M \times M$.

Exercise 1.19 (!). Let G be a finite group acting on a Hausdorff topological space M. Prove that M/G with the quotient topology is Hausdorf,

a. (!) when M is compact

b. (*) for arbitrary M.

Hint. Use the previous exercise.

Exercise 1.20 (**). Let $M = \mathbb{R}$, and $\sim$ an equivalence relation with at most 2 elements in each equivalence class. Prove that $\mathbb{R}/\sim$ is Hausdorff, or find a counterexample.

Exercise 1.21 (*). ("gluing of closed subsets") Let M be a metrizable topological space, and $Z_i \subset M$ a finite number closed subsets which do not intersect, grouped into pairs of homeomorphic $Z_i \sim Z_i'$. Let $\sim$ an equivalence relation generated by these homeomorphisms. Show that $M/\sim$ is Hausdorff.

[1]Speaking of a group acting on a topological space, one always means continuous action.
[2]a **closed map** is a map which puts closed subsets to closed subsets.

1.3 Compact spaces

Definition 1.12. **A cover** of a topological space M is a collection of open subsets $\{U_\alpha \in 2^M\}$ such that $\bigcup U_\alpha = M$. **A subcover** of a cover $\{U_\alpha\}$ is a subset $\{U_\beta\} \subset \{U_\alpha\}$. A topological space is called **compact** if any cover of this space has a finite subcover.

Exercise 1.22. Let M be a compact topological space, and $Z \subset M$ a closed subset. Show that Z is also compact.

Exercise 1.23. Let M be a countable, metrizable topological space. Show that either M contains a converging sequence of pairwise different elements, or M is discrete.

Definition 1.13. A topological space is called **sequentially compact** if any sequence $\{z_i\}$ of points of M has a converging subsequence.

Exercise 1.24. Let M be metrizable a compact topological space. Show that M is sequentially compact.

Hint. Use the previous exercise.

Remark 1.2. **Heine-Borel theorem** says that the converse is also true: any metric space which is sequentially compact, is also compact. Its proof is moderately difficult (please check Wikipedia or any textbook on point-set topology, metric geometry or analysis; "Metric geometry" by Burago-Burago-Ivanov is probably the best place).

In subsequent handouts, you are allowed to use this theorem without a proof.

Exercise 1.25 (*). Construct an example of a Hausdorff topological space which is sequentially compact, but not compact.

Exercise 1.26 (*). Construct an example of a Hausdorff topological space which is compact, but not sequentially compact.

Definition 1.14. A **topological group** is a topological space with group operations $G \times G \longrightarrow G$, $x, y \mapsto xy$ and $G \longrightarrow G$, $x \mapsto x^{-1}$ which are continuous. In a similar way, one defines **topological vector spaces**, **topological rings** and so on.

Exercise 1.27 (*). Let G be a compact topological group, acting on a topological space M in such a way that the map $M \times G \longrightarrow M$ is continuous. Prove that the quotient space is Hausdorff.

Exercise 1.28. Let $f: X \longrightarrow Y$ be a continuous map of topological spaces, with X compact. Prove that $f(X)$ is also compact.

Exercise 1.29. Let $Z \subset Y$ be a compact subset of a Hausdorff topological space. Prove that it is closed.

Exercise 1.30. Let $f: X \longrightarrow Y$ be a continuous, bijective map of topological spaces, with X compact and Y Hausdorff. Prove that it is a homeomorphism.

Definition 1.15. A topological space M is called **pseudocompact** if any continuous function $f: M \longrightarrow \mathbb{R}$ is bounded.

Exercise 1.31. Prove that any compact topological space is pseudocompact.

Hint. Use the previous exercise.

Exercise 1.32. Show that for any continuous function $f: M \longrightarrow \mathbb{R}$ on a compact space there exists $x \in M$ such that $f(x) = \sup_{z \in M} f(z)$.

Exercise 1.33. Consider $\mathbb{R}^n$ as a metric space, with the standard (Euclidean) metric. Let $Z \subset \mathbb{R}^n$ be a closed, bounded set ("bounded" means "contained in a ball of finite radius"). Prove that Z is sequentially compact.

Exercise 1.34 (**). Find a pseudocompact Hausdorff topological space which is not compact.

Definition 1.16. A map of topological spaces is called **proper** if a preimage of any compact subset is always compact.

Exercise 1.35 (*). Let $f: X \longrightarrow Y$ be a continuous, proper, bijective map of metrizable topological spaces. Prove that f is a homeomorphism, or find a counterexample.

Exercise 1.36 (*). Let $f: X \longrightarrow Y$ be a continuous, proper map of metrizable topological spaces. Show that f is closed, or find a counterexample.

Geometry 2: Manifolds and sheaves

Rules: Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

The original English translation of this handout was done by Sasha Anan′in (UNICAMP) in 2010.

2.1 Topological manifolds

Remark 2.1. Manifolds can be smooth (of a given "class of smoothness"), real analytic, or topological (continuous). **Topological manifold** is easiest to define, it is a topological space which is locally homeomorphic to an open ball in $\mathbb{R}^n$.

Definition 2.1. **An action** of a group on a manifold is silently assumed to be continuous. Let G be a group acting on a set M. The **stabilizer** of $x \in M$ is the subgroup of all elements in G that fix x. An action is **free** if the stabilizer of every point is trivial. The **quotient space** M/G is the space of orbits, equipped with the following topology: an open set $U \subset M/G$ is open if its preimage in M is open.

Exercise 2.1 (!). Let G be a finite group acting freely on a Hausdorff manifold M. Show that the quotient space M/G is a topological manifold.

Exercise 2.2. Construct an example of a finite group G acting non-freely on a topological manifold M such that M/G is not a topological manifold.

Exercise 2.3. Consider the quotient of $\mathbb{R}^2$ by the action of $\{\pm 1\}$ that maps x to $-x$. Is the quotient space a topological manifold?

Exercise 2.4 (*). Let M be a path connected, Hausdorff topological manifold, and G a group of all its homeomorphisms. Prove that G acts on M transitively.

Exercise 2.5 (**). Prove that any closed subgroup $G \subset GL(n)$ of a matrix group is homeomorphic to a manifold, or find a counterexample.

Remark 2.2. In the above definition of a manifold, it is not required to be Hausdorff. Nevertheless, in most cases, manifolds are tacitly assumed to be Hausdorff.

Exercise 2.6. Construct an example of a non-Hausdorff manifold.

Exercise 2.7. Show that $\mathbb{R}^2/\mathbb{Z}^2$ is a manifold.

Exercise 2.8. Let α be an irrational number. The group $\mathbb{Z}^2$ acts on $\mathbb{R}$ by the formula $t \mapsto t + m + n\alpha$. Show that this action is free, but the quotient $\mathbb{R}/\mathbb{Z}^2$ is not a manifold.

Exercise 2.9 (**). Construct an example of a (non-Hausdorff) manifold of positive dimension such that the closures of two arbitrary nonempty open sets always intersect, or show that such a manifold does not exist.

Exercise 2.10 (**). Let $G \subset GL(n, \mathbb{R})$ be a compact subgroup. Show that the quotient space $GL(n, \mathbb{R})/G$ is also a manifold.

2.2 Smooth manifolds

Definition 2.2. A **cover** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a **refinement** of a cover $\{U_i\}$ if every V_i is contained in some U_i.

Exercise 2.11. Show that any two covers of a topological space admit a common refinement.

Definition 2.3. A cover $\{U_i\}$ is an **atlas** if for every U_i, we have a map $\varphi_i : U_i \to \mathbb{R}^n$ giving a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. The **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are induced by the above homeomorphisms. An atlas is **smooth** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), **smooth of class** C^i if all transition functions are of differentiability class C^i, and **real analytic** if all transition maps admit a Taylor expansion at each point.

Definition 2.4. A **refinement** of an **atlas** is a refinement of the corresponding cover $V_i \subset U_i$ equipped with the maps $\varphi_i : V_i \to \mathbb{R}^n$ that are the restrictions of $\varphi_i : U_i \to \mathbb{R}^n$. Two atlases (U_i, φ_i) and (U_i, ψ_i) of class C^∞ or C^i (with the same cover) are **equivalent** in this class if, for all i, the map $\psi_i \circ \varphi_i^{-1}$ defined on the corresponding open subset in $\mathbb{R}^n$ belongs to the mentioned class. Two arbitrary atlases are **equivalent** if the corresponding covers possess a common refinement giving equivalent atlases.

Definition 2.5. A **smooth structure** on a manifold (of class C^∞ or C^i) is an atlas of class C^∞ or C^i considered up to the above equivalence. A **smooth manifold** is a topological manifold equipped with a smooth structure.

Remark 2.3. Terrible, isn't it?

Exercise 2.12 (*). Construct an example of two nonequivalent smooth structures on $\mathbb{R}^n$.

Definition 2.6. A **smooth function** on a manifold M is a function f whose restriction to the chart (U_i, φ_i) gives a smooth function $f \circ \varphi_i^{-1} : \varphi_i(U_i) \longrightarrow \mathbb{R}$ for each open subset $\varphi_i(U_i) \subset \mathbb{R}^n$.

Remark 2.4. There are several ways to define a smooth manifold. The above way is most standard. It is not the most convenient one but you should know it. Two other ways (via sheaves of functions and via Whitney's theorem) are presented further in these handouts.

Definition 2.7. A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

Definition 2.8. A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.

Remark 2.5. A presheaf of functions is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions** is a presheaf allowing "gluing" a function on a bigger open set if its restriction to smaller open sets lies in the presheaf.

Definition 2.9. A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

Exercise 2.13. Let $\mathcal{F}$ be a presheaf of functions. Show that $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, the sequence of restriction maps

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\Big|_{U_i \cap U_j}$ and $-\eta\Big|_{U_j \cap U_i}$.

Exercise 2.14. Show that the following spaces of functions on $\mathbb{R}^n$ define sheaves of functions.

a. Space of continuous functions.

b. Space of smooth functions.

c. Space of functions of differentiability class C^i.

d. (*) Space of functions which are pointwise limits of sequences of continuous functions.

e. Space of functions vanishing outside a set of measure 0.

Exercise 2.15. Show that the following spaces of functions on $\mathbb{R}^n$ are presheaves, but not sheaves

a. Space of constant functions.

b. Space of bounded functions.

c. Space of functions vanishing outside of a bounded set.

d. Space of continuous functions with finite $\int |f|$.

Definition 2.10. A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Remark 2.6. Usually the term "ringed space" stands for a more general concept, where the "sheaf of functions" is an abstract "sheaf of rings," not necessarily a subsheaf in the sheaf of all functions on M. The above definition is simpler, but less standard standard.

Exercise 2.16. Let M, N be open subsets in $\mathbb{R}^n$ and let $\Psi : M \to N$ be a smooth map. Show that Ψ defines a morphism of spaces ringed by smooth functions.

Exercise 2.17. Let M be a smooth manifold of some class and let $\mathcal{F}$ be the space of functions of this class. Show that $\mathcal{F}$ is a sheaf.

Exercise 2.18 (!). Let M be a topological manifold, and let (U_i, φ_i) and (V_j, ψ_j) be smooth structures on M. Show that these structures are equivalent if and only if the corresponding sheaves of smooth functions coincide.

Remark 2.7. This exercise implies that the following definition is equivalent to the one stated earlier.

Definition 2.11. Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold** of **class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{R}^n$ of this class.

Definition 2.12. A **coordinate system** on an open subset U of a manifold $(M, \mathcal{F})$ is an isomorphism between $(U, \mathcal{F})$ and an open subset in $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions of the same class on $\mathbb{R}^n$.

Remark 2.8. In order to avoid complicated notation, from now on we assume that all manifolds are Hausdorff and smooth (of class C^∞). The case of other differentiability classes can be considered in the same manner.

Exercise 2.19 (!). Let $(M, \mathcal{F})$ and $(N, \mathcal{F}')$ be manifolds and let $\Psi : M \to N$ be a continuous map. Show that the following conditions are equivalent.
(i) In local coordinates, Ψ is given by a smooth map
(ii) Ψ is a morphism of ringed spaces.

Remark 2.9. An isomorphism of smooth manifolds is called a **diffeomorphism**. As follows from this exercise, a diffeomorphism is a homeomorphism that maps smooth functions onto smooth ones.

Exercise 2.20 (*). Let $\mathcal{F}$ be a presheaf of functions on $\mathbb{R}^n$. Figure out a minimal sheaf that contains $\mathcal{F}$ in the following cases.

a. Constant functions.

b. Functions vanishing outside a bounded subset.

c. Bounded functions.

Exercise 2.21 (*). Describe all morphisms of ringed spaces from $(\mathbb{R}^n, C^{i+1})$ to $(\mathbb{R}^n, C^i)$.

2.3 Embedded manifolds

Definition 2.13. A **closed embedding** $\phi : N \hookrightarrow M$ of topological spaces is an injective map from N to a closed subset $\phi(N)$ inducing a homeomorphism of N and $\phi(N)$. **An open embedding** $\phi : N \hookrightarrow M$ is a homeomorphism of N and an open subset of M. is an image of a closed embedding.

Definition 2.14. Let M be a smooth manifold. $N \subset M$ is called **smoothly embedded submanifold of dimension** m if for every point $x \in N$, there is a neighborhood $U \subset M$ diffeomorphic to an open ball $B \subset \mathbb{R}^n$, such that this diffeomorphism maps $U \cap N$ onto a linear subspace of B dimension m.

Exercise 2.22. Let $(M, \mathcal{F})$ be a smooth manifold and let $N \subset M$ be a smoothly embedded submanifold. Consider the space $\mathcal{F}'(U)$ of smooth functions on $U \subset N$ that are extendable to functions on M defined on some neighborhood of U.

a. Show that $\mathcal{F}'$ is a sheaf.

b. Show that this sheaf defines a smooth structure on N.

c. Show that the natural embedding $(N, \mathcal{F}') \to (M, \mathcal{F})$ is a morphism of manifolds.

Hint. To prove that $\mathcal{F}$ is a sheaf, you might need partition of unity introduced below. Sorry.

Exercise 2.23. Let N_1, N_2 be two manifolds and let $\varphi_i : N_i \to M$ be smooth embeddings. Suppose that the image of N_1 coincides with that of N_2. Show that N_1 and N_2 are isomorphic.

Remark 2.10. By the above problem, in order to define a smooth structure on N, it suffices to embed N into $\mathbb{R}^n$. As it will be clear in the next handout, every manifold is embeddable into $\mathbb{R}^n$ (assuming it admits partition of unity). Therefore, in place of a smooth manifold, we can use "manifolds that are smoothly embedded into $\mathbb{R}^n$."

Exercise 2.24. Construct a smooth embedding of $\mathbb{R}^2/\mathbb{Z}^2$ into $\mathbb{R}^3$.

Exercise 2.25 (**). Show that the projective space $\mathbb{R}P^n$ does not admit a smooth embedding into $\mathbb{R}^{n+1}$ for $n > 1$.

2.4 Partition of unity

Exercise 2.26. Show that an open ball $\mathbb{B}^n \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

Definition 2.15. A cover $\{U_\alpha\}$ of a topological space M is called **locally finite** if every point in M possesses a neighborhood that intersects only a finite number of U_α.

Exercise 2.27. Let $\{U_\alpha\}$ be a locally finite atlas on M, and $U_\alpha \xrightarrow{\phi_\alpha} \mathbb{R}^n$ homeomorphisms. Consider a cover $\{V_i\}$ of $\mathbb{R}^n$ given by open balls of radius n centered in integer points, and let $\{W_\beta\}$ be a cover of M obtained as union of $\phi_\alpha^{-1}(V_i)$. Show that $\{W_\beta\}$ is locally finite.

Exercise 2.28. Let $\{U_\alpha\}$ be an atlas on a manifold M.

a. Construct a refinement $\{V_\beta\}$ of $\{U_\alpha\}$ such that a closure of each V_β is compact in M.

b. Prove that such a refinement can be chosen locally finite if $\{U_\alpha\}$ is locally finite

Hint. Use the previous exercise.

Exercise 2.29. Let K_1, K_2 be non-intersecting compact subsets of a Hausdorff topological space M. Show that there exist a pair of open subsets $U_1 \supset K_1$, $U_2 \supset K_2$ satisfying $U_1 \cap U_2 = \emptyset$.

Exercise 2.30 (!). Let $U \subset M$ be an open subset with compact closure, and $V \supset M \backslash U$ another open subset. Prove that there exists $U' \subset U$ such that the closure of U' is contained in U, and $V \cup U' = M$.

Hint. Use the previous exercise

Definition 2.16. Let $U \subset V$ be two open subsets of M such that the closure of U is contained in V. In this case we write $U \Subset V$.

Exercise 2.31 (!). Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological space, such that a closure of each U_α is compact. Prove that there exists another cover $\{V_\alpha\}$ indexed by the same set, such that $V_\alpha \Subset U_\alpha$

Hint. Use induction and the previous exercise.

Exercise 2.32 (*). Solve the previous exercise when $\{U_\alpha\}$ is not necessarily countable.

Hint. Some form of transfinite induction is required.

Exercise 2.33 (!). Denote by $\mathbb{B} \subset \mathbb{R}^n$ an open ball of radius 1. Let $\{U_i\}$ be a locally finite countable atlas on a manifold M. Prove that there exists a refinement $\{V_i, \phi_i : \ V_i \xrightarrow{\sim} \mathbb{R}^n\}$ of $\{U_i\}$ which is also locally finite, and such that $\bigcup_i \phi_i^{-1}(\mathbb{B}) = M$.

Hint. Use Exercise 2.31 and Exercise 2.28.

Definition 2.17. A function with compact support is a function which vanishes outside of a compact set.

Definition 2.18. Let M be a smooth manifold and let $\{U_\alpha\}$ be a locally finite cover of M. A **partition of unity** subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $f_i : M \to [0,1]$ with compact support indexed by the same indices as the U_i's and satisfying the following conditions.
(a) Every function f_i vanishes outside U_i
(b) $\sum_i f_i = 1$

Remark 2.11. Note that the sum $\sum_i f_i = 1$ makes sense only when $\{U_\alpha\}$ is locally finite.

Exercise 2.34. Show that all derivatives of $e^{-\frac{1}{x^2}}$ at 0 vanish.

Exercise 2.35. Define the following function λ on $\mathbb{R}^n$

$$\lambda(x) := \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Show that λ is smooth and that all its derivatives vanish at the points of the unit sphere.

Exercise 2.36. Let $\{U_i,\ \varphi_i : U_i \xrightarrow{\sim} \mathbb{R}^n\}$ be an atlas on a smooth manifold M. Consider the following function $\lambda_i : M \to [0,1]$

$$\lambda_i(m) := \begin{cases} \lambda\big(\varphi_i(m)\big) & \text{if } m \in U_i \\ 0 & \text{if } m \notin U_i \end{cases}$$

Show that λ_i is smooth.

Exercise 2.37 (!). (existence of partitions of unity)
Let $\{U_i,\ \varphi_i : U_i \to \mathbb{R}^n\}$ be a locally finite atlas on a manifold M such that $\varphi_i^{-1}(B_1)$ cover M as well (such an atlas was constructed in Exercise 2.33). Consider the functions λ_i's constructed in Exercise 2.36. Show that $\sum_j \lambda_j$ is well defined, vanishes nowhere, and that the family of functions $\left\{f_i := \frac{\lambda_i}{\sum_j \lambda_j}\right\}$ provides a partition of unity on M.

Remark 2.12. From this exercise it follows that any manifold with locally finite countable atlas admits a partition of unity.

Exercise 2.38 (*). Let M be a manifold admitting a countable atlas. Prove that M admits a countable, locally finite atlas, or find a counterexample.

Exercise 2.39 (**). Show that any Hausdorff, connected manifold admits a countable, locally finite atlas, or find a counterexample.

Exercise 2.40. Let M be a compact manifold, $\{V_i, \phi_i :\ V_i \longrightarrow \mathbb{R}^n, i = 1, 2, ..., m\}$ an atlas (which can be chosen finite), and $\nu_i :\ M \longrightarrow [0, 1]$ the subordinate partition of unity.

a. (!) Consider the map $\Phi_i :\ M \longrightarrow \mathbb{R}^{n+1}$, with

$$\Phi_i(z) := \frac{(\nu_i\phi_i(z), 1)}{|\nu_i\phi_i(z)|^2 + 1}$$

Show that Φ_i is smooth, and its image lies in the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$.

b. (*) Show that $\Phi_i :\ M \longrightarrow S^n$ is surjective.

c. (!) Let $U_i \subset V_i$ be the set where $\nu_i \neq 0$. Show that the restriction $\Phi_i\Big|_{V_i} :\ V_1 \longrightarrow S^n$ is an open embedding.

d. (!) Show that $\prod_{i=1}^m :\ \Phi_i :\ M \longrightarrow \underbrace{S^n \times S^n \times ... \times S^n}_{m \text{ times}}$ is a closed embedding.

Remark 2.13. We have just proved a weaker form of Whitney's theorem: each compact manifold admits a smooth embedding to $\mathbb{R}^N$.

Geometry 3: Vector fields and derivations

Rules: Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

3.1 Derivations of a ring

Remark 3.1. All rings in these handouts are assumed to be commutative and with unit. Algebras are associative, but not necessarily commutative (such as the matrix algebra). **Rings over a field** k are rings containing a field k.

Definition 3.1. Let R be a ring over a field k. A k-linear map $D: R \longrightarrow R$ is called **a derivation** if it satisfies **the Leibnitz equation** $D(fg) = D(f)g + gD(f)$. The space of derivations is denoted as $\operatorname{Der}_k(R)$.

Exercise 3.1. Let $D \in \operatorname{Der}_k(R)$. Prove that $D\big|_k = 0$.

Exercise 3.2. Let D_1, D_2 be derivations. Prove that the commutator $[D_1, D_2] := D_1D_2 - D_2D_1$ is also a derivation.

Exercise 3.3 (!). Let $K \supset k$ be a field which contains a field k of characteristic 0, and is finite-dimensional over k (such fields K are called **finite extensions** of k). Find the space $\operatorname{Der}_k(K)$.

Exercise 3.4 (*). Is it true if $\operatorname{\mathsf{char}} k = p$?

Exercise 3.5. Consider a ring $k[\varepsilon]$, given by a relation $\varepsilon^2 = 0$. Find $\operatorname{Der}_k(k[\varepsilon])$.

Exercise 3.6 (*). Find all rings R over $\mathbb{C}$ such that R is finite-dimensional over $\mathbb{C}$, and $\operatorname{Der}_{\mathbb{C}}(R) = 0$.

Exercise 3.7 (**). Let $D \in \operatorname{Der}_k(K)$ be a derivation of a field K over k, $\operatorname{\mathsf{char}} k = 0$, and $[K' : K]$ a finite field extension. Prove that D can be extended to a derivation $D' \in \operatorname{Der}_k(K')$.

Exercise 3.8. Let $D \in \operatorname{Der}_k(R)$ be a derivation, and $I \subset R$ an ideal. Prove that $D(I^k) \subset I^{k-1}$.

3.2 Modules over a ring

Definition 3.2. Let R be a ring over a field k. **An R-module** is a vector space V over k, equipped with an algebra homomorphism $R \longrightarrow \operatorname{End}(V)$, where $\operatorname{End}(V)$ denotes the endomorphism algebra of V, that is, the matrix algebra.

Exercise 3.9. Let R be a field. Prove that R-modules are the same as vector spaces over R.

Remark 3.2. An R-module is a group, equipped with an operation of "multiplication by elements of R", and satisfying the same axioms of distributivity and associativity as in the definition of a vector space.

Remark 3.3. Homomorphisms, isomorphisms, submodules, quotient modules, direct sums of modules are defined in the same way as for the vector spaces. A ring R is itself an R-module. A direct sum of n copies of R is denoted R^n. Such R-module is called **a free R-module**.

Remark 3.4. R-submodules in R are the same as ideals in R.

Definition 3.3. A ring R is called **a principal ideal ring** if all non-zero submodules of R are isomorphic to R.

Exercise 3.10. Prove that R is a principal ideal ring iff R has no zero divisors, and all ideals in R are **principal**, that is, are of form Rx, for some non-invertible $x \in R$.

Exercise 3.11. Are these rings principal ideal rings?

a. $R = \mathbb{C}[t]$

b. (!) $R = \mathbb{C}[t_1, t_2]$

c. (*) $R := \mathbb{R}[x, y]/(x^2 + y^2 = -1)$.

Definition 3.4. Finitely generated R-module is a quotient module of R^n.

Exercise 3.12. Find a finitely generated, non-free R-module for $R = \mathbb{C}[t]$.

Definition 3.5. A Noetherian ring is a ring R with all ideals finitely generated as R-modules.

Exercise 3.13 (*). Let R be a Noetherian ring. Prove that any submodule of a finitely generated R-module is finitely generated.

Exercise 3.14. Consider a ring R of germs of smooth functions in a point $x \in \mathbb{R}^n$, and let K be an ideal of all functions with all derivatives of all orders vanishing. Show that this ideal is not principal.

Exercise 3.15 (*). Prove that $R = \mathbb{C}^\infty\mathbb{R}^n$ is not finitely generated.

3.3 Vector fields

Remark 3.5. Let R be a ring over k. The space $\operatorname{Der}_k(R)$ of derivations is also an R-module, with multiplicative action of R given by $rD(f) = rD(f)$.

Exercise 3.16. Let $R = k[t_1, .., t_k]$ be a polynomial ring. Prove that $\operatorname{Der}_k(R)$ is a free R-module isomorphic to R^n, with generators $\frac{d}{dt_1}, \frac{d}{dt_2}, ..., \frac{d}{dt_n}$.

Hint. Construct a map $\operatorname{Der}_k(R) \longrightarrow R^n$,

$$D \longrightarrow (D(t_1), D(t_2), ..., D(t_n))$$

and prove that it is an isomorphism of R-modules.

Exercise 3.17 (*). Let $R = k(t_1, .., t_k)$ be a ring of rational functions, that is, the ring of functions $\frac{P}{Q}$, where P and $Q \in k(t_1, .., t_k)$ are arbitrary polynomials, $Q \neq 0$. Prove that $\operatorname{Der}_k(R)$ is a free R-module, isomorphic to R^n.

Exercise 3.18 (!). Prove the **Hadamard's lemma**: Let f be a smooth function f on $\mathbb{R}^n$, and x_i the coordinate functions. Then $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, for some smooth $g_i \in C^\infty\mathbb{R}^n$.

Hint. Consider a function $h(t) \in C^\infty\mathbb{R}^n$, $h(t) = f(tx)$. Then $\frac{dh}{dt} = \sum \frac{df(tx)}{dx_i}(tx)x_i$. Integrating this expression over t, obtain $f(x) - f(0) = \sum_i x_i \int_0^1 \frac{df(tx)}{dx_i}(tx)dt$.

Definition 3.6. Consider coordinates $t_1, ..., t_n$ on $\mathbb{R}^n$, and let

$$\operatorname{Der}(C^\infty\mathbb{R}^n) \xrightarrow{\Pi} (C^\infty\mathbb{R}^n)^n,$$

map D to $(D(t_1), D(t_2), ..., D(t_n))$.

Exercise 3.19. Prove that Π is surjective.

Exercise 3.20. Prove that $\Pi(D) = 0 \Leftrightarrow D(P) = 0$ for each $P(t_1, ..., t_n)$.

Exercise 3.21. Let $\mathfrak{m}_x \subset C^\infty\mathbb{R}^n$ be an ideal of all smooth functions vanishing at $x \in \mathbb{R}^n$. Prove that it is maximal.

Exercise 3.22. Let f be a smooth function on $\mathbb{R}^n$ satisfying $f(x) = 0$ and $f'(x) = 0$. Prove that $f \in \mathfrak{m}_x^2$.

Hint. Use the Hadamard's Lemma.

Exercise 3.23 (!). Let $D \in \operatorname{Der}_{\mathbb{R}}(C^\infty\mathbb{R}^n)$ be a derivation, satisfying $D \in \ker\Pi$ (that is, vanishing on coordinate functions). Prove that for all $f \in C^\infty\mathbb{R}^n$, and all $x \in \mathbb{R}^n$, one has $D(f) \in \mathfrak{m}_x$.

Hint. Use the previous exercise and Exercise 3.8.

Exercise 3.24 (!). Prove that the map

$$\operatorname{Der}(C^\infty\mathbb{R}^n) \xrightarrow{\Pi} (C^\infty\mathbb{R}^n)^n$$

is an isomorphism

Hint. Use the previous exercise.

Exercise 3.25 (**). Find a non-trivial element $\gamma \in \operatorname{Der}_{\mathbb{R}}(C^0\mathbb{R})$ in the space of derivations of continuous functions, or prove that it is empty.

Exercise 3.26 (**). Find a non-trivial element $\gamma \in \operatorname{Der}_{\mathbb{R}}(C^1\mathbb{R})$ in the space of derivations of the ring of differentiable functions of class C^1, or prove that it is empty.

Geometry 4: Germs and sheaves

Rules: Exam problems would be similar to ones marked with ! sign. It is recommended to solve all unmarked and !-problems or to find the solution online. It's better to do it in order starting from the beginning, because the solutions are often contained in previous problems. The problems with * are harder, and ** are very hard; don't be disappointed if you can't solve them, but feel free to try. Have fun!

4.1 Direct limit

Definition 4.1. Commutative diagram of vector spaces is given by the following data. First, there is a directed graph (graph with arrows). For each vertex of this graph (also called a diagram) one gives a vector space, and each arrow corresponds to a homomorphism of the associated vector spaces. These homomorphism are compatible, in the following way. Whenever there exist two ways of going from one vertex to another, the compositions of the corresponding arrows are equal.

Remark 4.1. A **neighbourhood** of a subset $X \subset M$ is an open subset containing X.

Exercise 4.1. Let $(M, \mathcal{F})$ be a space ringed by a sheaf of functions, $x \in M$ a point, $\{U_i\}$ the set of all neighbourhoods of x. Consider a diagram with the set of vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. Prove that the space of sections $\mathcal{F}(U_i)$ with homomorphisms given by restrictions form a commutative diagram.

Definition 4.2. Let $\mathcal{C}$ be a commutative diagram of vector spaces A, B – vector spaces corresponding to two vertices of a diagram, and $a \in A, b \in B$ elements of these vector spaces. Write $a \sim b$ if a and b are mapped to the same element $d \in D$ by a composition of arrows from $\mathcal{C}$. Let $\sim$ be an equivalence relation generated by such $a \sim b$.

Exercise 4.2. a. Let $A \xrightarrow{\phi} B$ be a diagram of two spaces and one arrow. Prove that $b \sim b'$ is equivalent to $b = b'$ for each $b, b' \in B$.

b. Let $A \xrightarrow{\phi} B$, $A \longrightarrow 0$ be a diagram of three spaces, with ϕ injective. Prove that for each $b, b' \in B$, $b \sim b'$ is equivalent to $b - b' \in \operatorname{im}\phi$.

Definition 4.3. Let $\{C_i\}$ be a set of vector spaces associated with the vertices of a commutative diagram $\mathcal{C}$, and $E \subset \bigoplus_i C_i$ a subspace generated by the vectors $(x - y)$, where $x \sim y$. A quotient $\bigoplus_i C_i/E$ is called **a direct limit** of a diagram $\{C_i\}$. The same notion is also called **colimit** and **inductive limit**. Direct limit is denoted $\lim\limits_{\to}$.

Exercise 4.3. Let $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ be a diagram with all arrows injective. Prove that $\lim\limits_{\to} C_i$ is a union of all C_i.

Exercise 4.4. Let $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ... \longrightarrow C_n$ be a diagram. Prove that $\lim\limits_{\to} C_i = C_n$.

Exercise 4.5. Find an example of a diagram $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ where all spaces C_i are non-zero, and the colimit $\lim\limits_{\to} C_i$ vanishes.

Exercise 4.6 (*). Find an example of a diagram $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow ...$ where all spaces C_i are non-zero, all arrows are also non-zero, and the colimit $\lim\limits_{\to} C_i$ vanishes.

Definition 4.4. A diagram $\mathcal{C}$ is called **filtered** if for any two vertices C_i, C_j, there exists a third vertex C_k, and sequences of arrows leading from C_i to C_k and from C_j to C_k.

Exercise 4.7. Let $\mathcal{C}$ be a commutative diagram of vector spaces C_i, with all C_i equipped with a ring structure, and all arrows ring homomorphisms. Suppose that the diagram $\mathcal{C}$ is filtered. Prove that $\lim\limits_{\to} C_i$ is a ring, equipped with natural ring homomorphisms $C_i \longrightarrow \lim\limits_{\to} C_i$.

4.2 A ring of germs of a sheaf of functions

Definition 4.5. Let $M, \mathcal{F}$ be a ringed space, $x \in M$ its point, and $\{U_i\}$ the set of all its neighbourhoods. Consider a commutative diagram with vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. For each vertex U_i we take a vector space of sections $\mathcal{F}(U_i)$, and for each arrow the corresponding restriction map. The direct limit of this diagram is called **the ring of germs of the sheaf $\mathcal{F}$ in** x.

Remark 4.2. This limit is indeed a ring, as follows from the previous exercise.

Remark 4.3. As a special case of this definition, we obtain rings of germs of smooth functions, real analytic functions, continuous, C^i and so on.

Exercise 4.8. Let $\mathcal{F}$ be a sheaf of functions on a manifold such that all its germs are zero. Prove that $\mathcal{F}$ is a zero sheaf.

Definition 4.6. A constant sheaf $\mathbb{R}_M$ is a sheaf of functions which are constant on each connected $U \subset M$.

Exercise 4.9. Prove that a ring of germs of a constant sheaf at each point is $\mathbb{R}$.

Exercise 4.10 (*). Let $\mathcal{F}$ be a sheaf of $\mathbb{R}$-valued functions on M, such that all its germs are isomorphic to $\mathbb{R}$. Prove that it is constant.

Definition 4.7. **An ideal** in a ring R is an abelian subgroup $I \subsetneq R$, such that for all $x \in R, a \in I$, the product xa belongs to I.

Remark 4.4. A quotient space R/I is a ring (prove this). Also, for any ring homomorphism, its kernel is an ideal.

Definition 4.8. **A maximal ideal** is an ideal $I \subset R$, such that for any other ideal $I' \supsetneq I$, $I' \ni 1$.

Exercise 4.11. Show that any ideal is contained in a maximal ideal (use Zorn's lemma).[1]

Exercise 4.12. Show that an ideal $I \subset R$ is maximal if and only if the quotient R/I is a field.

Exercise 4.13 (*). Find all maximal ideals in the ring of smooth functions on a compact manifold.

Definition 4.9. A ring is called **local** if it contains only one maximal ideal.

Exercise 4.14. Prove that a ring of rational numbers $\frac{m}{n}$, where m, n are integer, and n odd, is local. Find its quotient by the maximal ideal.

Exercise 4.15. Let F be a ring of rational functions (functions $\frac{P}{Q}$, where P and $Q \in \mathbb{C}[t_1, ..., t_n]$ are polynomials) without a pole in 0. Show that this ring is local. Find its quotient by a maximal ideal.

Exercise 4.16 (!). Are the following rings local?

a. The ring of germs of smooth functions.

b. The ring of germs of polynomial functions on $\mathbb{R}^n$.

c. The ring of germs of functions of differentiability class C^i, $i \geqslant 0$.

d. The ring of germs of continuous functions.

e. The ring of germs of real analytic functions on $\mathbb{R}^n$.

Exercise 4.17. Show that a ring with a maximal ideal I is local iff each element $r \notin I$ is invertible.

Definition 4.10. **Zero divisors** in a ring are non-zero elements r_1, r_2, saisfying $r_1 r_2 = 0$. **Nilpotent** is $r \in R$ such that $r^n = 0$ for some n.

[1] You are not required to prove Zorn's lemma in this exercise.

Exercise 4.18. Find whether the following rings have zero divisors.

a. The ring of germs of smooth functions.

b. The ring of germs of polynomial functions.

c. The ring of germs of continuous functions.

Definition 4.11. A continuous function f on $\mathbb{R}^n$ is called **piecewise polynomial** if $\mathbb{R}^n$ is represented as a union of polyhedra, and on each of these polyhedra, f is polynomial.

Exercise 4.19. Let $\mathcal{F}$ – a sheaf of piecewise polynomial functions on $\mathbb{R}$, S – a ring of its germs at 0.

a. Find out whether S is a local ring.

b. Show that S is isomorphic to $\mathbb{R}[t_1, t_2]/(t_1 t_2 = 0)$.

Exercise 4.20 (!). Let R be a local ring, $\mathfrak{m}$ its maximal ideal, and $K(R) := \bigcap_i \mathfrak{m}^i$. Prove that it is an ideal. Find whether this ideal is zero for

a. The ring of germs of smooth functions.

b. The ring of germs of real analytic functions.

c. The ring of germs of continuous functions.

Exercise 4.21 (*). Let $R = k[t_1, ..., t_n]$ be a ring of polynomials over a field, and $I \subset R$ an ideal.[2] Prove that $\bigcap_i I^i = 0$.

Exercise 4.22. Let R be a ring of germs of smooth functions in x, $\mathfrak{m}$ its maximal ideal, and $K(R) := \bigcap_i \mathfrak{m}^i$. Prove that for all $f \in K(R)$, all derivatives of f in zero (of any order) vanish.

Exercise 4.23. Let $x_1, ..., x_n$ be coordinates on $\mathbb{R}^n$, and f a function with all derivatives of any order vanishing. Show that $\frac{f}{(\sum_i x_i^2)^p}$ is continuous for any $p > 0$.

Exercise 4.24 (!). Under assumptions of the previous exercise, prove that the function $\frac{f}{\sum_i x_i^2}$ is smooth.

Exercise 4.25 (!). Let R be a ring of germs of smooth functions in $x \in \mathbb{R}^n$, $K(R) := \bigcap_i \mathfrak{m}^i$ the ideal defined above. Prove that $K(R)$ is an ideal of functions with vanishing derivatives of any order at x.

[2] The ideals in R are tacitly assumed to be $\neq R$.

Hint. Use the previous exercise.

Exercise 4.26 (*). Let $R/K(R)$ be the ring defined above.

a. Are there non-zero nilpotents in $R/K(R)$?

b. Are there zero divisors in $R/K(R)$?

4.3 Soft sheaves

Definition 4.12. Let $(M, \mathcal{F})$ be a topological space ringed by a sheaf of functions, and $X \subset M$ its subset. Consider a diagram indexed by open subsets $U_i \subset M$ containing X, with arrows corresponding to inclusions $U_j \subset U_i$, and associate with each U_i the corresponding section space $\mathcal{F}(U_i)$. A direct limit of this diagram is called **the ring of germs of $\mathcal{F}$ in X,** and denoted as $\mathcal{F}(X)$.

Exercise 4.27 (*). Let $(M, C^\infty M)$ be a manifold ringed by a sheaf of smooth functions, and $X \subset M$. Suppose that the space of germs of $C^\infty M$ in X is a local ring. Prove that X is a point.

Definition 4.13. A ring of functions $\mathcal{F}$ on M is called **soft** if for any closed subset $X \subset M$, the natural map from the space of global sections $\mathcal{F}(M)$ to the space of germs $\mathcal{F}(X)$ is surjective.

Exercise 4.28. Show that the sheaf of real analytic functions on $\mathbb{R}^n$ is not soft.

Exercise 4.29. Show that a constant sheaf on a manifold is not soft.[3]

Exercise 4.30. Find a topological space M and a sheaf of functions $\mathcal{F}$ on it such that the restriction map from $\mathcal{F}(M)$ to the space of germs of $\mathcal{F}$ in a point is always surjective, but the sheaf $\mathcal{F}$ is not soft.

Exercise 4.31. Let $N, N' \subset M$ be two closed subsets of a metric space, $N \cap N' = \emptyset$. Prove that there exist non-intersecting neighbourhoods $U \supset N$, $U' \supset N'$.

Exercise 4.32 (!). Let M be a manifold admitting a partition of unity, $N \subset M$ a closed subset, and $U \supset N$ its neighbourhood. Prove that M has a locally finite cover $\{U_i\}$ such that all U_i which intersect N are contained in U.

Hint. Prove that M admits a metric, and use the previous exercise.

Definition 4.14. Support of a function f is the set of all points where $f \neq 0$. A function is called **supported in** U if its support is contained in U.

[3] All manifolds are tacitly assumed to be of positive dimension.

Exercise 4.33. Let $U \subset M$ be an open subset of a manifold, $U' \Subset M$ an open subset satisfying $\bar{U}' \subset U$, and f a smooth function on U with support in U'. Prove that f can be extended to a smooth function on M.

Exercise 4.34 (*). Let M be a manifold admitting a partition of unity. Prove that the sheaf of smooth functions on M is soft.

Hint. Given a smooth function f on $U \supset N$, find a cover $\{U_i\}$, $i \in I$ as in previous exercise, and let $\{\psi_i\}$ be a subordinate partition of unity. Let $A \subset I$ be the set of indices $\alpha \in I$ such that $U_\alpha \cap N \neq 0$. Prove that the function $f' := \sum_{\alpha \in A} \psi_\alpha f$ is supported in $U' \Subset U$, can be extended smoothly to the whole M, and equal f on N.

Definition 4.15. Let $f \in \mathcal{F}(M)$ be a section of a sheaf $\mathcal{F}$ on M. **Support** of f is the set of all points $x \in M$ such that there is no neighbourhood $U \ni x$ such that $f\big|_U = 0$.

Exercise 4.35. Prove that support of any section is closed.

Definition 4.16. A sheaf $\mathcal{F}$ on M is called **fine** if for any locally finite cover $\{U_\alpha\}$ of an open set $U \subset M$ indexed by $\alpha \in I$ and any section $f \in \mathcal{F}(U)$ there exists a collection of sections $f_\alpha \in \mathcal{F}(U)$ indexed by the same set I such that a support of any f_α is contained in U_α, and $\sum_I f_\alpha = f$.

Remark 4.5. Essentially the fine sheaves are sheaves which admit partition of unity.

Exercise 4.36 (*). Let M be a smooth manifold. Prove that the sheaf of smooth functions is fine.

Exercise 4.37 (*). Let M be a smooth manifold. Prove that the sheaf of smooth functions is soft.

Hint. Use the previous exercise.

Exercise 4.38 (**). Let M be a metrizable topological space. Prove that the sheaf of continuous functions is fine.

Exercise 4.39 (**). Let M be a metrizable topological space. Find a soft sheaf on M which is not fine.

Exercise 4.40 (**). Let $\mathcal{F}$ be a soft sheaf of functions, with the rings of germs local at all points. Prove that $\mathcal{F}$ is fine, or find a counterexample.

Exercise 4.41 (**). Let M be a metrizable topological space. Prove that any fine sheaf on M is soft.