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Lista 1

Problem 1 Let $\Phi = \{\varphi_i, U_i\}_{i\in I}$ be a locally finite atlas of X, $\mathcal{K} = \{K_i\}_{i\in I}$ a family of compact sets $K_i \subset U_i$, $\Psi = \{\psi_i, V_i\}$ an atlas of Y, $\varepsilon = \{\varepsilon_i\}_{i\in I} \subset \mathbb{R}^+$ and $f \in C^\infty(X, Y)$ such that $f(K_i) \subset V_i$. Let

$$W^k(f;\Phi,\Psi,\mathcal{K},\varepsilon) = \left\{ g \in C^\infty(X,Y) : \begin{array}{c} g(K_i) \subset V_i, \\ \|D^r(\psi_i \circ f \circ \varphi_i^{-1})(x) - D^r(\psi_i \circ g \circ \varphi_i^{-1})(x)\| < \varepsilon_i, \\ \forall x \in K_i, \qquad \forall i \in I, \qquad \forall r \text{ s.t. } 0 \leqslant r \leqslant k \end{array} \right\}$$

Prove that the collection of all sets of this form is a base for the C^k topology on $C^\infty(X, Y)$.

Solution. Fix U a C^∞-open set and $f \in U$. We need to show that there is a set of the form $W^k(f;\Phi,\Psi,\mathcal{K},\varepsilon)$ contained in U.

Since $f \in U$, by definition of the C^∞ topology there must be a set

$$M(V) = \{g \in C^\infty(X,Y) : j^k g(x) \in V \ \forall x \in X\} \subset U$$

for some V open in $J^k(X,Y)$. We need to construct a set W contained in $M(V)$. Fix for now any atlases Φ, Ψ and set of compact sets $\mathcal{K}$ and set of numbers ε. To get the contention $W := W^k(f;\Phi,\Psi,\mathcal{K},\varepsilon) \subset M(V)$ we must show that any $g \in W$ satisfies $j^k g(X) \subset V$, which "is the definition of being in V".

Suppose that $V = J^k(\tilde{U}, \tilde{V})$ for some open sets $\tilde{U} \subset X$ and $\tilde{V} \subset Y$. Equivalently, $V \overset{\text{dif}}{\simeq} \tilde{U} \times \tilde{V} \times \tilde{W}$ for some open set $\tilde{W}$ in an euclidean space (constructed as a polynomial vector space). The condition $j^k g(x) \in V \forall x \in X$ may be restated as: the source of g is in $\tilde{U}$ (though this is obvious if we take the jet $j^k g(x)$ for $x \in \tilde{U}$), its target is in $\tilde{V}$, and its derivatives in $\tilde{W}$.

□

Problem 2 A function is said to be ***closed*** if the image of every closed set is closed. Prove that the set $\{f \in C^\infty(\mathbb{R},\mathbb{R}) : f \text{ is closed}\}$ is closed in the C^∞ topology.

Solution. Choose a non-closed function f and around it find a C^k-neighbourhood of non-closed functions, for some k. This means that when we put the C^k glasses, we see less clearly, we can barely distinguish functions up to degree k (further than that they could be different but we won't notice).

If f is non-closed then it cannot be a submersion: a submersion is open. Then there is a point where $d_p f$ is not surjective. I want a neighbourhood of functions that have the same derivative as f (but they are still k-close to f). So they can be different in second derivative. So all these functions are the same function in $k = 1$ topology. Do I not need

C^0 topology for that? To distinguish them. But C^0 topology gives me the functions that map $x \mapsto y = f(x)$. $\square$

Problem 3 Let X and Y be manifolds and $\ell \geqslant k$. Prove that there exists a natural fiber bundle $J^\ell(X,Y) \to J^k(X,Y)$ and compute the dimension of its fiber.

Solution. Consider the map

$$\begin{aligned} \pi : J^\ell(X,Y) &\longrightarrow J^k(X,Y) \\ j^\ell f(p) &\longmapsto j^k f(p) \end{aligned}$$

for any smooth function $f \in C^\infty(X,Y)$ and $p \in X$.

First notice that π is a submersion. Fix a point $p \in X$ and a jet $\sigma := j^\ell f(p)$. After fixing local coordinates on X and Y, σ has a coordinate representation

$$\Big((x_1,\dots,x_n),(y_1,\dots,y_m),T_\ell f_1(p),\dots,T_\ell f_m(p)\Big)$$

where T_ℓ gives the truncated Taylor polynomials at p of the coordinate functions of f with respect to the chosen coordinates. Then the coordinates of $\pi\sigma$ are given simply by composing T_ℓ with T_k, which basically means "forgetting" the coefficients of the Taylor polynomials for degrees above k. This just says that the differential will be the identity in the coordinates of the points in X and Y, and also for the first k coordinates of the Taylor polynomials. The rest of the matrix will have zeroes, but it will be full rank since the dimension of $J^k(X,Y)$ is smaller than that of $J^\ell(X,Y)$.

Now pick a jet τ downstairs and let's look at $\pi^{-1}(\tau)$. It's the manifold given by all the ℓ-jets that coincide with τ up to order k. Computing its dimension is analogous of computing the dimension of jet spaces in general: it will be the product of the dimensions of X, Y, and the dimension of a certain polynomial space. For the polynomial coordinates we need to consider how the Taylor expansion of the coordinate functions can vary in degrees between $k+1$ and ℓ. It is $m\sum_{r=k+1}^{\ell}\binom{r+n-1}{n-1}$ according to the following combinatorial argument.

> For each coordinate function of f, we consider its Taylor polynomial at p, which is a polynomial in n variables. We need to put a number (a coordinate) at every monomial. Every monomial is determined by the exponents we put in each indeterminate. The exponents should add up to the degree of the monomial, say r where $k < r \leqslant \ell$. Thus different monomials are different choices of n nonnegative integers that add up to r. That's like putting r balls in n boxes, which is like putting $n-1$ "divisions" among the r objets. So it's a choice of $n-1$ things among $r+n-1$ things. Taking into account the m polynomials, this gives the above number.

To conclude we must show that π admits local trivializations. Let $\tau \in J^k(X,Y)_{p,q}$. An open neighbourhood W of τ is a product of neighbourhoods of p and q in their respective spaces, and $\mathbb{R}^N$ for some crazy N. The inverse image $\pi^{-1}(W)$ only differs from W in the polynomial part, where now we consider polynomials up to degree ℓ instead of only k.

The key observation here is that the fiber $\pi^{-1}(\tau)$ is given precisely by varying the last $\ell - k$ coordinates of the polynomial part. This yields a local trivialization $\pi^{-1}(W) \cong W \times F$. $\square$

Problem 4 Let M be a non-compact manifold.

(a) Prove that multiplication by scalar $\mathbb{R} \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ is not continuous in the C^∞ topology.

(b) Prove that addition and multiplication of functions are continuous in the C^∞ topology.

Solution.

(a) Fix $f \in C^\infty(M, \mathbb{R})$. Then we have a multiplication map $\mu_f : \mathbb{R} \to C^\infty(M, \mathbb{R}$. Following [GG74], pp. 46-47, if μ_f was continuous, we'd have that $\lim_n f/n = 0$ in the C^∞ topology. If M is not compact and we choose f to have noncompact support, by definition we must have that given any compact set K there is a point x_0 outside the support of f, i.e. such that $f(x_0) \neq 0$.

A contradiction follows from

Exercise 8.5 ([Muk15], Chpt. 8, Sec. 2, p. 240) Let X be a paracompact space, and Y a metrizable space. Show that a sequence $\{f_n\}$ in $C^\infty(X, Y)$ converges in the $C^\infty(X, Y)$ topology iff there exists a compact set $K \subset X$ such that $f_n = f$ [outside K] for all but finitely many n, and the sequence $\{f_n|_K\}$ converges uniformly to $f|_K$.

Indeed: if $f/n \xrightarrow{C^\infty} 0$ there would be a compact outside of which $f/n = 0$ for all but finitely many n, but this is impossible since f has to be nonzero somewhere outside any compact set K.

Now let's use Problem 1 from this homework to solve Exercise 8.5 above.

Solution of Exercise 8.5 in [Muk15]. Suppose that $f_n \xrightarrow{C^\infty} f$. This means that for every basic open neighbourhood $W := W^k(f; \Phi, \Psi, \mathcal{K}, \varepsilon)$ of f there is a natural numer N such that for every $n > N$, $f_n \in W$. This says that

$$
\begin{gathered}
f_n(K_i) \subset V_i, \\
\|D^r(\psi_i \circ f \circ \varphi_i^{-1})(x) - D^r(\psi_i \circ f_n \circ \varphi_i^{-1})(x)\| < \varepsilon_i, \\
\forall x \in K_i, \qquad \forall i \in I, \qquad \forall r \text{ s.t. } 0 \leqslant r \leqslant k.
\end{gathered}
$$

So we get uniform convergence on *any* compact set K_i immediately by taking $r = 0$. But **why should f_n and f coincide outside *any* of the compact sets?** $\square$

(b) First notice (thanks to ChatGPT) that addition map

$$
A : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), \qquad (f, g) \mapsto f + g
$$

induces a bundle map

$$
\tilde{A} : J^k(M, \mathbb{R}) \times J^k(M, \mathbb{R}) \longrightarrow J^k(M, \mathbb{R})
$$

which is smooth. Indeed, addition of two sections $j^k f$ and $j^k g$ is smooth since addition of real numbers is smooth. (To define this map formally we fix a point of M and map two k-jets at x to their sum at x, which in coordinates gives a polynomial whose coefficients are sums of the coefficients of the original polynomials.)

Then we show that addition of smooth functions is C^∞-continuous like this: Let $U \subset C^\infty(M, \mathbb{R})$ open, then $A^{-1}(U)$ is some set in $C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R})$. Take a point (f, g) in there. Since $f + g$ is in the open set U, there is a k such that $f + g \in M(V)$ for some V open in $J^k(M, \mathbb{R})$. The preimage $\tilde{A}^{-1}(V) := V_1 \times V_2$ is open in $J^k(M, \mathbb{R}) \times J^k(M, \mathbb{R})$. Consider the C^∞-open set $M(V_1) \times M(V_2) \ni (f, g)$.

To conclude we must check that $M(V_1) \times M(V_2) \subset A^{-1}(M(V))$. This means that the sum of any pair of smooth functions in $M(V_1) \times M(V_2)$ remains in $M(V)$. But any two smooth induce a sum of k-jets that remains in V by construction.

Multiplication of functions is analogous; this time we should use a bundle map $\tilde{M}$ which is also continuous since multiplication of k-jets is locally multiplication of polyonimals.

□

Problem 5 Let X be a submanifold of $\mathbb{R}^n$ and $k \geqslant \operatorname{codim} X$. Prove that almost every subspace of dimension k intersects X transversally, i.e. the set of all subsets of dimension k that don't intersect X transversally has measure zero.

Solution. For this exercise I use [GP10] since this problem is exercise 7 in Chpt. 2, Sec. 3. As a preliminary remark notice that the condition $k \geqslant \operatorname{codim} X$ doesn't say much—by definition of transversality we can't have transversal intersections without this condition. Now let's start by stating

The Transversality Theorem ([GP10], p. 68) Suppose that $F : X \times S \to Y$ is a smooth map of manifolds, where only X has boundary, and let Z be any boundaryless submanifold of Y. If both F and ∂F are transversal to Z, then for almost every $s \in S$, both f_s and ∂f_s are transversal to Z.

In order to use this result recall that in Lecture 7 of our course it was proved that

Lemma 4.6 ([GG74], p.53) Let X, B and Z be smooth manifolds with W a submanifold of Z. Let $j : B \to C^\infty(X, Y)$ be a mapping (not necessarily continuous) and define $\Phi : X \times B \to Y$ by $\Phi(x, b) = j(b)(x)$. Assume that Φ is smooth and that $\Phi \pitchfork W$. Then the set $\{b \in B : j(b) \pitchfork W\}$ is dense in B. **And in my lecture notes I find that:** the set actually has total measure.

I shall follow the notation from [GP10].

A hint from [GP10] tells us to consider the map

$$F : \mathbb{R}^k \times S \longrightarrow \mathbb{R}^n$$
$$\Big((t_1, \dots, t_k), \{v_i\}_{i=1}^k\Big) \longmapsto \sum_{i=1}^k t_i v_i$$

where

$$S := \{\text{set of } k\text{-frames of } \mathbb{R}^n\},$$

and show it is a submersion. (A ***frame*** is a set of linearly independent vectors.) The hint suggests that S is an open set of $(\mathbb{R}^n)^k$—we need to give S the structure of a smooth manifold! Though I'm not sure how to show it is an open set, we could consider only the *orthonormal* frames, which coincide with orthonormal matrices, which may be given the structure of a smooth manifold via inverse function theorem using the map $\mathsf{GL}(n) \to \mathrm{Mat}(n), A \mapsto A^T A$.

Moving on, if we can show that F is transversal to $X \subset \mathbb{R}^n$, we obtain that F_s is transversal to X for almost every k-frame $s := \{v_i\}$. The latter is exactly what we want: the map F_s is an embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ putting $\mathbb{R}^k$ in $\mathbb{R}^n$ as the space generated by the frame s.

As suggested in the hint, to show that F is transversal to X it's enough to show it is a submersion. To this end we write the coordinate functions of F by

$$F(\vec{t}, \{v_i\}) = t_1 \begin{pmatrix} v_1^1 \\ \vdots \\ v_1^n \end{pmatrix} + \dots + t_k \begin{pmatrix} v_k^1 \\ \vdots \\ v_k^n \end{pmatrix} = \begin{pmatrix} F_1(\vec{t}, \{v_i\}) \\ \vdots \\ F_n(\vec{t}, \{v_i\} \end{pmatrix}$$

where $\vec{t} = (t_1, \dots, t_k)$. For further clarity let's write

$$\begin{aligned}
F_1(\vec{t}, \{v_i\}) &= t_1 v_1^1 + t_2 v_2^1 + \dots + t_k v_k^1 \\
F_2(\vec{t}, \{v_i\}) &= t_1 v_1^2 + t_2 v_2^2 + \dots + t_k v_k^2 \\
&\vdots \\
F_n(\vec{t}, \{v_i\}) &= t_1 v_1^n + t_2 v_2^n + \dots + t_k v_k^n
\end{aligned}$$

To compute the differential of F first notice that the derivatives of the coordinate functions F_i with respect to the t_i variables gives a matrix whose columns are the coordinates of the vectors v_i:

$$\left(\frac{\partial F_i}{\partial t^j}\right)_{i,j} = \begin{pmatrix} v_1^1 & v_2^1 & \dots & v_k^1 \\ v_1^2 & v_2^2 & \dots & v_k^2 \\ \vdots & & & \\ v_1^n & v_2^n & \dots & v_k^n \end{pmatrix}$$

which is of rank k because $\{v_i\}$ is a frame.

The rest of the matrix $DF_{\vec{t}, \{v_i\}}$ is given by differentiating with respect to each of the coordinates of each of the basis vectors $\{v_i\}$. So for each coordinate function we differentiate with respect to all the coordinates of the first vector, then w.r.t. the coordinates of the second vector and so on.

This gives the following matrix:

$$\left(\frac{\partial F_i}{\partial v^j}\right)_{i,j} = \begin{pmatrix} \overbrace{\begin{matrix} t_1 & 0 & \cdots & 0 \\ 0 & t_1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & t_1 \end{matrix}}^{v_1} & \overbrace{\begin{matrix} t_2 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & t_2 \end{matrix}}^{v_2} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \overbrace{\begin{matrix} t_k & 0 & \cdots & 0 \\ 0 & t_k & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & t_k \end{matrix}}^{v_k} \end{pmatrix}$$

Each of the blocks of this matrix is a multiple of the identity of $n \times n$. In conclusion, the differential of F at $(\vec{t}, \{v_i\})$ is the $n \times (k + nk)$ matrix

$$DF_{(\vec{t},\{v_i\})} = \left(\left(\frac{\partial F_i}{\partial t^j}\right)_{i,j} \quad \left(\frac{\partial F_i}{\partial v^j}\right)_{i,j} \right)$$

And it is surjective in virtue of the second block, which is of rank n provided not all t_i are zero. □

References

[GG74] M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities*. Graduate texts in mathematics. Springer, 1974.

[GP10] V. Guillemin and A. Pollack. *Differential Topology*. AMS Chelsea Publishing. AMS Chelsea Pub., 2010.

[Muk15] A. Mukherjee. *Differential Topology*. Springer International Publishing, 2015.