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Practice exercises on smooth manifolds

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Plan for today: correct an exercise from last time (Partition of unity), two more exercises from Partition of unity. Homework 1 from Differential Topology course (3 exercises). Questions about Homework 2 from Complex Surfaces.

1 Partition of unity

Exercise 2.30 (!) Let $U \subset M$ be an open subset with compact closure, and $V \supset M \backslash U$ another open subset. Prove that there exists $U' \subset U$ such that the closure of U' is contained in U, and $V \cup U' = M$.

Hint. Use the previous exercise.

Solution. (Using ChatGPT.) Define the ***boundary*** ∂A of a set A in a topological space X to be the set of points $x \in X$ such that every neighbourhood of x contains a point of A and a point of $X \backslash A$.

The boundary ∂U of our open set with compact closure U is compact: it is contained in the closure of U (since all its points are limit points of U), and it is closed: every point in its complement has a neighbourhood that stays inside its complement; whether it is in U, or in $M \backslash \bar{U}$.

Now let's use Exercise 2.29. We can separate $K_1 := \partial U$ and $K_2 := \overline{U \backslash V}$. Both are compact since they are closed sets in the compact set $\overline{U}$. And they are disjoint: if $x \in \partial U \cap \overline{U \backslash V}$, then x cannot be in U since it is a point of the boundary, meaning that $x \in V$, and since V is open, there is a neighbourhood W of x contained in V. But $x \in \overline{U \backslash V} = U \backslash V \cup \partial U \backslash V$, so that $x \in \partial U \backslash V$ since $x \notin U$. So every neighbourhood of x intersects $U \backslash V$. So there is a point of W not in V, a contradiction.

Then we use Exercise 2.29 to obtain disjoint neighbourhoods U_1 and U_2 of K_1 and K_2.

Now let's show that $U_2 \cap U := U'$ is the open set we are looking for, that is, that its closure is contained in U and $V \cup U' = M$. If a point in the closure of U' was outside U, then such a limit point would be in the boundary of U: any open neighbourhood must contain a point of U since it is a limit point of U, and also a point outside it, the limit point itself! But the boundary of U is disjoint from U_2.This shows that the closure of U' is inside U.

To show that $V \cup U' = M$ pick a point in $M \backslash V$. Then $U' := U_2 \cap U \supset K_2 := U \backslash V$ contains it. □

Exercise 2.31 (!) Let $\{U_\alpha\}$ be a countable locally finite cover of a Hausdorff topological space, such that a closure of each U_α is compact. Prove that there exists another cover $\{V_\alpha\}$ indexed by the same set, such that $V_\alpha \Subset U_\alpha$.

Hint. Use induction and the previous exercise.

Solution. In order to use Exercise 2.30 consider for every α the set $W_\alpha = \bigcup_{\beta \neq \alpha} U_\beta$. Then $W_\alpha \supset M \backslash U_\alpha$, so that there exists $U'_\alpha \Subset U_\alpha$ and $W_\alpha \cup U'_\alpha = M$. It remains to show that $\{U'_\alpha\}$ is a cover. Let $x \in M$ be any point. but how?

That's why the hint says use induction. We go one by one: consider U_1, an open set. The rest of the cover yields an open set like V from the last exercise, which contains the complement of U. Then that exercise yields a set $U'_1 \Subset U$ st $V \cup U'_1 = M$.

Now take $n = 2$. But don't use the original open cover: *substitute* U_1 *by* U'_1. Obviously. (It works basically because of the second condition, explaining why we went through so much hustle to construct the set U', anyway moving on.) The point is that now we get a set $U'_2 \Subset U_2$ which covers M along with U'_1 and the rest of the U_α.

This works for all α: there is $U'_\alpha \Subset U_\alpha$ such that $U'_\alpha \cup U'_{\alpha-1} \cup \ldots \cup U'_1 \cup \bigcup_{i>\alpha} U_i$ covers M.

Let's show that $\{U'_\alpha\}$ is a cover. Suppose there's a point x outside U'_α for all α. Then it is in $\bigcup_{i>\alpha} U_i$ for all α, meaning x is in a infinite ammount of open sets of the locally finite cover U_i. □

Exercise 2.34 Show that all derivatives of $e^{-\frac{1}{x^2}}$ at 0 vanish.

Solution. First notice that the function $e^{-x^{-2}}$ is not defined at 0. However, the limit as $x \to 0$ is zero, so that defining the function to be 0 at $x = 0$ preserves continuity. The same will happen with its derivatives.

The first derivative is

$$\frac{d}{dx} e^{-x^{-2}} = 2x^{-3} e^{-x^{-2}}.$$

Since exponential decay is faster than polynomial decay, the limit as $x \to 0$ is zero.

The second derivative is

$$\begin{aligned} \frac{d^2}{dx^2} e^{-x^{-2}} &= 2\left(x^{-3} \frac{d}{dx} e^{-x^{-2}} - 3x^{-4} e^{-x^{-2}}\right) \\ &= 2\left(x^{-3} 2x^{-3} e^{-x^{-2}} - 3x^{-4} e^{-x^{-2}}\right) \\ &= P_2(x) e^{-x^{-2}} \end{aligned}$$

where $P_2(x)$ is some polynomial, so that again the limit as $x \to 0$ is zero. Proceeding by induction suppose that the n-th derivative is the product of some polynomial $P_n(x)$

times $e^{-x^{-2}}$. Then the $(n+1)$-th derivative is

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}e^{-x^{-2}} &= \frac{d}{dx}\left(\frac{d^n}{dx^n}e^{-x^{-2}}\right)\\ &= \frac{d}{dx}\left(P_n(x)e^{-x^{-2}}\right)\\ &= P_n'(x)e^{-x^{-2}} + P_n(x)\frac{d}{dx}e^{-x^{-2}}\\ &= P_n'(x)e^{-x^{-2}} + P_n(x)P_1(x)e^{-x^{-2}}.\end{aligned}$$

□

2 Homework 1 from Differential Topology

Problem 1 Let $G(k,n)$ be the set of dimension k vector subspaces of $\mathbb{R}^n$. Construct a smooth structure on $G(k,n)$ and compute its dimension.

Solution. (Ideas from StackExchange, [?].) Every k-vector subspace of $\mathbb{R}^n$ has an orthonormal base. $\mathsf{O}(k)$ acts on the set $V(n,k)$ of orthonormal frames of $\mathbb{R}^n$ since its transformations preserve the inner product. The equivalence relation $\alpha \sim \beta$ iff $\exists A \in \mathsf{O}(k)$ st $A\alpha = \beta$ for $\alpha, \beta \in V(n,k$ is the same as asking that two bases generate the same vector space, so that the quotient space is $G(n,k)$.

$V(n,k)$ has a smooth structure making it a manifold of dimesion $nk-(k+1)k/2$ [Wolfram].

In order to use the Quotient Manifold Theorem (thm 21.10 in [?]) we need to check that the action $\mathsf{O}(n) \curvearrowright V(n,k)$ is smooth, free and proper. The latter follows from $\mathsf{O}(n)$ being compact (which in turn is because it is closed by being a level set of $\mathsf{GL}(n) \to \mathsf{Mat}(n)$, $A \mapsto A^{\mathsf{T}}A$, and bounded the normality condition). Freeness and smoothness are immediate.

The dimension of the quotient manifold $G(n,k)$ is $\dim V(n,k) - \dim \mathsf{O}(k)$. Since the dimension of $\mathsf{O}(k)$ is $k(k-1)/2$, substracting we get $\dim G(n,k) = k(n-k)$. □

Next I show my progress following the proof of [?], which unfortunately is incomplete. Feel free to skip it.

Proof. (Proof in [?].) First we topologize $G(n,k)$ as follows. We identify it with the quotient $W/\sim$, the set of k-frames (sets of k linearly independent vectors of $\mathbb{R}^n$) modulo the equivalence relation of spanning the same vector space. Since $W \subset (\mathbb{R}^n)^k$ it has a subspace topology, and $G(n,k)$ has a quotient topology.

Now fix $V \in G(n,k)$. To construct a chart consider the set

$$W_V = \{U \in G(n,k) : \text{orthogonal projection } U \to V \text{ is bijective}\}$$

and the function

$$\begin{aligned}\rho_V : W_V &\longrightarrow \mathrm{Hom}(V, V^\perp)\\ U &\longmapsto \pi_{U,V^\perp} \circ \pi_{U,V}^{-1}\end{aligned}$$

where $\pi_{X,Y}$ is the orthogonal projection from X to Y. Since the set $\mathrm{Hom}(V, V^\perp)$ is the space of matrices of $\dim V \times \dim V^\perp = k(n-k)$, we may write $\mathrm{Hom}(V, V^\perp) = \mathbb{R}^{k(n-k)}$.

To complete the proof we must confirm that: (1) $G(n, k)$ is Hausdorff and second countable, (2) W_V is open for all V, (3) ρ_V is a homeomorphism for all V, (4) transition maps $\rho_V \circ \rho_{V'}^{-1}$ are smooth.

1. By Theorem 7.7 in [**?**], it's enough to show that the quotient projection $q : W \to W/\sim$ is an open map and that the graph $\Gamma = \{(x, y) \in W : x \sim y\}$ of $\sim$ is closed in $W \times W$.

   (q is open.) Let $Z \subset W$ be an open set. Showing that q is open means that $q^{-1}(q(Z))$ is open. Let $\{z_i\}_{i=1}^k$ be an element (a frame) in $q^{-1}(q(Z))$. We will show that there is an (open) neighbourhood of $\{z_i\}$ contained in $q^{-1}(q(W))$.

   Since $\{z_i\} \in q^{-1}(q(Z))$, $\{z_i\}$ generates the same vector space as some frame $\{z_i'\}$ in Z. Because Z is open there is an open neighbourhood Z' of $\{z_i'\}$ contained in Z. The fact that $\mathrm{span}(z_i) = \mathrm{span}(z_i')$ means that there is a linear transformation $A \in \mathrm{GL}(k)$ mapping $z_i \mapsto z_i'$.

   Notice that A does not act on elements of W pointwise: they are nk-dimensional vectors! However, A does act on elements of W when these are seen as frames within the k-dimensional vector space they span. Explicitly, a frame given as an $n \times k$ matrix of rank k may be multiplied by A (a $k \times k$ matrix) giving a $n \times k$ matrix. The range of the product of this matrices is k since A is invertible, i.e. another frame.

   Since A is a homeomorphism, the set $A(Z') \subset Z$ is the required open neighbourhood of $\{z_i\}$.

   (Graph of $\sim$ is closed.) [Incomplete]

2. To see that W_V is open define $\widetilde{W}_V$ to be the set of k-frames $\{u_i\}_{i=1}^k$ of $\mathbb{R}^n$ such that the orthogonal projection from $\mathrm{span}(u_i)$ onto V is bijective. Then $\widetilde{W}_V/\sim = W_V$. Since we have shown that the quotient map q is open, it's enough to show that $\widetilde{W}_V$ is open.

   Fix $\{u_i\}_{i=1}^n \in \widetilde{W}_V$. It is clear from elementary properties of euclidean space that there is a neighbourhood of every u_i such that the vector space obtained by choosing one vector in each of these neighbourhoods orthogonally-projects bijectively onto V. Since we are using the product topology on W, it follows that $\widetilde{W}_V$ is open.

3. [Incomplete]

4. [Incomplete]

□

Problem 2 Let M and N be manifolds of dimension m and n, respectively, and let $f : M \to N$ be a smooth function whose rank is k for every point in an open set $\tilde{U} \subset M$. Prove that for each point $p \in \tilde{U}$, there exist charts (U, ϕ) and (V, ψ) centered at p and $f(p)$ such that $f(U) \subset V$ and

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_k, x_{x+1}, \ldots, x_m) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

Solution. (Adapted from the proof of Theorem B.4 in [**?**].) Since $D_p f$ has rank k at p, we may assume the first k columns are linearly independent. Define a locally invertible map of $\mathbb{R}^m$ to itself by

$$G(x_1, \ldots, x_k, y_1, \ldots, y_{m-k}) = \Big(f_1(x, y), \ldots, f_k(x, y), y\Big).$$

where f_i are the coordinate functions of f for some charts of p and $f(p)$. G is locally invertible since it has nonsingular derivative at p. Notice that $f \circ G^{-1}$ maps

$$(x, y) \mapsto \Big(x, f_{k+1} \circ G^{-1}(x, y), \ldots, f_n \circ G^{-1}(x, y)\Big).$$

Notice that $f \circ G^{-1}$ does not depend on y in a neighbourhood of p: since G is locally a diffeomorphism, the rank of $f \circ G^{-1}$ must be the same as that of f, and its derivative is

$$D_q(f \circ G^{-1}) = \begin{pmatrix} \mathrm{Id} & 0 \\ \frac{\partial (f\circ G^{-1})_i}{\partial x^j} & \frac{\partial (f\circ G^{-1})_i}{\partial y^j} \end{pmatrix}, \quad \text{for } k \leqslant i \leqslant n$$

so that the matrix $\frac{\partial (f\circ G^{-1})_i}{\partial y^j}$ must be singular **for all** q **in a neighbourhood of** p (here we use that f has **constant** rank k).

This allows us to define the function of (some subsets of) R^n to itself

$$F(x, y) = \Big(x, y_1 - f_{k+1} \circ G^{-1}(x), \ldots, y_n - f_n \circ G^{-1}(x)\Big)$$

which is locally invertible: its derivative is

$$D_{f(p)} F(x, y) = \begin{pmatrix} \mathrm{Id} & 0 \\ * & \mathrm{Id} \end{pmatrix}$$

using that $f_i \circ G^{-1}$ does not depend on y near p. Thus we may restrict our domains as necessary to obtain open sets $U \ni p$ and $V \ni f(p)$ such that

$$F \circ \hat{f} \circ G^{-1}(x, y) = F\Big(x, f_{k+1} \circ G^{-1}(x, y), \ldots, f_n \circ G^{-1}(x, y)\Big) = (x, 0)$$

where $\hat{f} = (f_1, \ldots, f_n)$ is the coordinate representation of f with which we started.

□

Problem 3 Let M be a compact manifold. Prove that does not exist a submersion $F : M \to \mathbb{R}^k$, $k > 0$.

Solution. Since M is compact any real-valued function is bounded, so the composition of F with the modulus function $\| \cdot \| \circ F$ is bounded and so is F. Now let $x_0 \in M$ a point such that $\|F(x)\|$ is maximum over M.

Let γ be the curve in $\mathbb{R}^k$ given by $\gamma(t) = F(x_0) + tF(x_0)$. It corresponds to a vector at $F(x_0)$ pointing in the direction of $F(x_0)$, so that the norm of points on γ for positive t is larger than that of $F(x_0)$

Since $D_{x_0}F$ is surjective, there is a vector such that its image under $D_{x_0}F$ is $[\gamma]$. Namely, $[F^{-1} \circ \gamma]$ for some local inverse of F. (Indeed: $F_*[F^{-1} \circ \gamma] = \frac{d}{dt}\Big|_{t=0} F \circ F^{-1} \circ \gamma$.)

Now let $x_1 := F^{-1} \circ \gamma(t_1)$ for some $t_1 > 0$. Then $F(x_1)$ has a bigger norm than $F(x_0)$:

$$\|F(x_1)\| = \|\gamma(t_1)\| = \|F(x_0) + t_1 F(x_0)\| > \|F(x_0)\|$$

but $\|F(x_0)\|$ is maximum.

Remark After consulting the literature I figured an easier proof: submersion is an open map since it is locally invertible in every point of its image (yielding at every such point an open neighbourhood that stays within $f(X)$). Then $f(X)$ is both open and closed, making f surjective since $\mathbb{R}$ is connected. A contradiction, since $f(X)$ must be bounded.

□

3 Homework 2 from Complex Surfaces