Differential Topology 2025, Problem set 1

Problem 1: Let $G(k, n)$ be the set of dimension k vector subspaces of $\mathbb{R}^n$. Construct a smooth structure on $G(k, n)$ and compute its dimension.

Problem 2: Let M and N be manifolds of dimension m and n, respecticely, and let $f : M \to N$ be a smooth function whose rank is k for every point in an open set $U \subset M$. Prove that for each point $p \in U$, there exist charts (U, ϕ) and (V, ψ) centered at p and $f(p)$ such that $f(U) \subset V$ and

$$\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

Problem 3: Let M be a compact manifold. Prove that there does not exist a submersion $F : M \to \mathbb{R}^k$, for $k > 0$.