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Practice exercises on smooth manifolds

A pdf file with the questions may also be found here.

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1 Remedial topology

1.1 Topological spaces

Definition 1.1 A set of all subsets of M is denoted 2^M. ***Topology*** on M is a collection of subsets $S \subset 2^M$ called ***open subsets***, and satifsying the following conditions:

1. Empty set and M are open
2. A union of any number of open sets is open
3. An intersection of a finite number of open subsets is open.

A complement of an open set is called ***closed***. A set with topology on it is called a ***toplogical space***. An ***open neighbourhood*** of a point is an open set containig this point.

Definition 1.2 A map $\phi : M \to M'$ of topological spaces is called ***continuous*** if a preimage of each open set $U \subset M'$ is open in M. A bijective cintunuous map is called a ***homeomorphism*** if its inverse is also continuous.

Exercise 1.1 Let M be a set, and S a set of all subsets of M. Prove that S defines a topology on M. This topology is called ***discrete***. Describe the set of all continuous maps form M to a given topological space.

Solution. Since all sets are open, topology axioms are satisfied by S. All maps from M to a given topological space are continuous. $\square$

Exercise 1.2 Let M be a set, and $S \subset 2^M$ a set of two subsets: empty set and M. Prove that S defines a topology on M. This topology is called ***codiscrete***. Describe the set of all continuous maps from M to a space with discrete topology.

Solution. It's trivial that S satisfies the axioms of topology. Continuous maps from M to a space with discrete topology are constant maps. Such maps are continuous since the preimage of any (open) set is open: either it contains the value of the map at all points, in which case the preimage is all of M, or it doesn't, in which case the preimage is empty. Conversely, if a map has more than one value, the preimage of such a value cannot be neither M nor the empty set. □

Definition 1.3 Let M be a topological space, and $Z \subset M$ its subset. ***Open subsets*** of Z are subsets obtained as $Z \cap U$, where U is open in M. This topology is called ***induced topology***.

Definition 1.4 A ***metric space*** is a set M equipped with a ***distance function*** $d : M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying the following acioms.

1. $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. (triangle inequality) $d(x, y) + d(y, z) \geqslant d(x, z)$.

An ***open ball*** of raduis r with center in xx is $\{y \in M : d(x, y) < r\}$.

Definition 1.5 Let M be a metric space. A subset $U \subset M$ is called ***open*** if it is obtained as a union of open balls. This topology is called ***induced by the metric***.

Definition 1.6 A topological space is called ***metrizable*** if its topology can be induced by a metric.

Exercise 1.3 Show that discrete topology can be induced by a metric, and codiscrete cannot.

Solution. To induce the discrete metric define the distance between any two distinct points to be 1. This clearly satisfies the three axioms of metric, and the ball of radius $1/2$ is an open set that contains only its center, making any point and thus any subset an open set.

If a metric space contains at least two points at distance d, the ball with radius $d/2$ at any of these points is an open set distinct from the empty set and the total, so the topology induced by the metric cannot be discrete. □

Exercise 1.4 Prove that an intersection of any collection of closed subsets of a topological space is closed.

Solution. As I recall this is due to de Morgan laws stating that for any collection F_α of subsets

$$\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha F^c \tag{1}$$

where superscript c means set complement. If this is true then we are done because if F_α are closed, we see that the intersection is also closed as its complement is open.

$\square$

Definition 1.7 An intersection of all closed supersets $Z \subset M$ is called ***closure*** of Z

Definition 1.8 A ***limit point*** of a set $Z \subset M$ is a point $x \in M$ such that any neighbourhood of M contains a point of Z other than x. A ***limit*** of a sequence $\{x_i\}$ of points in M is a point $x \in M$ such that any neighbourhood of $x \in M$ contains all x_i for ll i except a finite number. A sequence which has a limit is called ***convergent***.

Exercise 1.5 Show that a closure of a set $Z \subset M$ is a union of Z and all its limit points.

Solution. It's enough to show that the union of Z and all its limits points W is a closed set and that it is contained in any closed set containing Z.

To see W is closed chose a point in its complement $p \in W^c$. Since p is not a limit point of Z nor a point of Z, there is a neighbourhood of p not intersecting Z. This means that such neighbourhood is contatined in W^c. We can do this for all points in W^c, thus obtaining a W^c as a union of open sets, which is open, and then W is closed.

To see W is contained in any closed set containing Z, suppose F contains Z but not W. Then there must be a limit point of Z that is not in F. But then F cannot be closed because there is no neighbourhood of such a limit point contained in F^c, which should be open. Indeed, if F^c is open then every point contains a neighbourhood contained in F^c: at least F^c itself! $\square$

Exercise 1.7 Let $f : M \to M'$ be a map of metrizable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous.

Solution. It is equivalent that the preimage of every open set is open (definition of f being continuous) with the preimage of every closed subset is closed: for any closed set $M' \backslash U$ with U open, $f^{-1}(M \backslash U) = f^{-1}(M') \backslash f^{-1}(U)$ is closed.

Consider the closed set $F \subset M'$ and let's check that its preimage is also closed. By the same reasoning as in Exercise 1.5, to show closedeness it's enough to show the set contains all its limit points. Take a limit point $p \in f^{-1}(F)$. We construct a convergent sequence $\{x_n\}$ taking balls of radius $\frac{1}{n}$ around p, each of which must contain a point in $f^{-1}(F)$. This gives a sequence in F, which by hypothesis must converge to a limit point $\lim_i f(x_i) = f(\lim_i x_i) \in F$. This means $p = \lim_i x_i$ is in the inverse image of F. $\square$

Exercise 1.8* Find a counterexample to the previous problem for non-metrizable, Hausdorff topological spaces (see the next subsection of a definition of Hausdorff).

Sketch of solution. Probably Sørgenfrey line is a counter-example? I should look for its definition to make sure it is Hausdorff (and how is it defined exactly—I think open sets are positive rays). $\square$

Exercise 1.9** Let $f : M \longrightarrow M'$ be a map of countable topological spaces, such that $f(\lim_i x_i) = \lim_i f(x_i)$ for any convergent sequence $\{x_i \in M\}$. Prove that f is continuous, or find a counterexample.

Sketch of solution. Is a ***countable space*** a space whose cardinality is $\mathbb{N}$? What are the possible topologies on $\mathbb{N}$? Discrete topology gives that every map is continuous. Other topologies are maybe, again, rays. $\square$

Exercise 1.10* Let $f : M \longrightarrow N$ be a bijective map inducing homeomorphisms on all countable subsets of M. Show that it is a homeomorphism, or find a counterexample.

Sketch of solution. If we suppose that M and M' are metrizable, we can use Exercise 1.7 as follows. Choose any convergent sequence $\{x_i \in M\}$. Then the countable set $\{x_i\} \cup \{\lim_i f(x_i)\}$ is mapped homeomorphically to $\{f(x_i)\} \cup \{f(\lim_i x_i)\}$. This implies that $f(\lim_i f(x_i)) = f(\lim_i x_i)$, so f is continuous. The same holds for f^{-1}, so f is a homeomorphism.

Probably the statement isn't true in general, so let's look for a counter-example. $\square$

1.2 Hausdorff spaces

Definition 1.9 Let M be a topological space. It is called ***Hausdorff*** or ***separable***, if any two distinct points $x \neq y \in M$ can be ***separated*** by open subsets, that is, there exist open neighbourhoods $U \ni x$ and $V \ni y$ wuch that $U \cap V = \varnothing$.

Remark 1.1 In topology, the Hausdorff axiom is usually assumed by default. In subsequent handouts, it will be always assumed (unless stated otherwise).

Exercise 1.11 Let M be a Hausdorff topological space. Prove that all points in M are closed subsets.

Solution. Fix a point $x \in M$. For every $y \in M$ distinct from x we have the neighbourhoods $U_y \ni x$ and $V_y \ni y$ with $U_y \cap V_y = \varnothing$. Then $X \backslash \{x\} = \bigcup_{y \neq x} V_y$, which is open. $\square$

Exercise 1.12 (Points are closed in Hausdorff) Let M be a Hausdorff topological space. Prove that all points in M are closed subsets.

Solution. Let $x \in M$ and let's see that $M \backslash \{x\}$ is open. Choose a point $y \in M \backslash \{x\}$. Then there are open sets $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$. Then $V \subset M \backslash \{x\}$. $\square$

Exercise 1.13 Let M be a topological space, with all points of M closed. Prove that M is Hausdorff, or find a counterexample.

Solution. No solution yet... □

Exercise 1.14 Count the number of non-isomorphic topologies on a finite set of 4 elements. How many of these topologies are Hausdorff.

Solution. For any set S of subsets of $\{1,2,3,4\}$ we can consider the ***topology generated by*** S, which consists of all unions and intersections of elements in S, along with the total space and the empty set.

For the following choices of S we get non-isomorphic topologies:

1. $S = \varnothing$ (codiscrete topology).
2. $S = \big\{\{1\}\big\}$
3. $S = \big\{\{1\},\{2\}\big\}$.
4. $S = \big\{\{1\},\{2\},\{3\}\big\}$.
5. $S = \big\{\{1\},\{2\},\{3\},\{4\}\big\}$ (discrete topology).
6. $S = \big\{\{1,2\}\big\}$.
7. $S = \big\{\{1,2\},\{3\}\big\}$
8. $S = \big\{\{1,2\},\{3,4\}\big\}$.
9. $S = \big\{\{1,2,3\}\big\}$.
10. $S = \big\{\{1,2,3\},\{4\}\big\}$.

There are some topologies missing... □

Exercise 1.5 (!) Let Z_1, Z_2 be nonintersecting closed subsets of a metrizable space M. Find open subsets $U \supset Z_1, V \supset Z_2$ which do not intersect.

Solution. Consider the distance between Z_1 and Z_2:

$$d(Z_1, Z_2) := \inf\{d(z_1, z_2) : z_1 \in Z_1, z_2 \in Z_2\}.$$

We must argue that $d(Z_1, Z_2) \neq 0$. Suppose by contradiction that $d(Z_1, Z_2) = 0$ Then for every $n \in \mathbb{N}$ there is a pair of points z_1^n and z_2^n such that $d(z_1^n, z_2^n) < 1/n$. That distance need not be zero! There must be another way to do this... □

Definition 1.10 Let M, N be topological spaces. ***Product topology*** is a topology on $M \times N$, with open sets obtained as unions $\bigcup_\alpha U_\alpha \times V_\alpha$, where U_α is open in M and V_α is open in N.

Exercise 1.16 Prove that a topology on X is Hausdorff if and only if the diagonal $\Delta := \{(x,y) \in X \times X | x = y\}$ is closed in the product topology.

Solution. ($\Longrightarrow$) Suppose that X is Hausdorff. To check that Δ is closed suppose that $(x,y) \in X \times X$ is a limit point of Δ. We need to show that $(x,y) \in \Delta$, i.e. that $x = y$. If $x \neq y$ we can separate x and y by disjoint open subsets $U \ni x$ and $V \ni y$. Then the open set $U \times V$ contains (x,y), and since (x,y) is a limit point of Δ there must be a point $(z,z) \in U \times V$. Then $z \in U$ and $z \in V$, which is a contradiction.

($\Longleftarrow$) Suppose Δ is closed in the product topology and choose two different points $x \neq y$ in X. Then $(x,y) \in (X \times X) \backslash \Delta$, which is an open set by hypothesis. Then by definition of product topology there must be two open sets in X, $U \ni x$ and $V \ni y$. Suppose there is a point in the intersection $z \in U \cap V$. Then $(z,z) \in (U \times V) \cap \Delta$, a contradiction. $\square$

Definition 1.11 Let $\sim$ be an equivalence relation on a topological space M. ***Factor-topology*** (or ***quotient topology***) is a topology on the set $M/\sim$ of equivalence classes such that a subset $U \subset M/\sim$ is open whenever its preimage in M is open.

Exercise 1.17 Let G be a finite group acting (continuously) on a Hausdorff topological space M.Prove that the quotient map is closed (i.e. puts closed subsets to closed subsets).

Solution. The quotient map is $\pi : M \to M/\sim$ where $x \sim y$ if $y = gx$ for some $g \in G$. To show π is closed pick $F \subset M$ closed. We need to show that $\pi(F)$ is closed, so we may show its complement is open. According to the definition of factor topology we want to show that

$$\pi^{-1}\Big((M/\sim)\backslash\pi(F)\Big) = M\backslash\pi^{-1}(\pi(F))$$

is open. Now $\pi^{-1}(\pi(F))$ is the set of points that are G-related to points in F, namely $\bigcup_{g\in G} gF$. Since G is finite and acts by homeomorphisms, this set is a finite union of closed sets, which is closed. Looks like the Hausdorff hypothesis is not necessary. $\square$

Exercise 1.18* Let $\sim$ be an equivalence relation on a topological space M, and $\Gamma \subset M \times M$ its ***graph***, that is, the set $\{(x,y) \in M \times M | x \sim y\}$. Suppose that the map $M \longrightarrow M/\sim$ is open, and that Γ is closed in $M \times M$. Show that $M/\sim$ is Hausdorff.

Hint . Prove that diagonal is closed in $M \times M$.

Solution. Notice that any open surjective map is closed: let $f : X \twoheadrightarrow Y$ be an open surjective map and $F \subset X$ closed, then $f(X\backslash F) = f(X)\backslash f(F) = Y\backslash f(F)$.

Our objective is to show that the diagonal $\tilde{\Delta}$ in $(M/\sim) \times (M/\sim)$ is closed. The projection of the graph Γ is $\tilde{\Delta}$. Since Γ is closed, by the remark above it follows that $\tilde{\Delta}$ is closed in $(M/\sim) \times (M/\sim)$ as we needed. $\square$

Exercise 1.19 Let G be a finit group acting on a Hausdorff topological space M. Prove that M/G with the quotient topology is Hausdorff,

(a) (!) when M is compact.

(b) (*) for abitrary M.

Hint. Use the previous exercise.

Sketch of solution. To use the previous exercise first notice that the action of G induces an equivalence relation on M; this follows from group axioms. Then it's enough to show that the projection is closed and that the graph Γ of the equivalence relation is closed in $M \times M$. But by Exercise 1.17 we already know that the projection is closed, so it's enough to show that Γ is closed.

Notice that $\Gamma = \bigcup_{x \in X} (Gx) \times (Gx)$, that is, the union of cartesian products of every orbit with itself. Each of these cartesian products is a finite set because G is finite. If M is compact, then ... □

Exercise 1.20** Let $M = \mathbb{R}$, and $\sim$ an equivalence relation with at most two elements in each equivalence class. Prove that $\mathbb{R}/\sim$ is Hausdorff, or find a counterexample.

Solution. By Exercise 1.19, if this equivalence relation is induced by a finite-group action, we know the quotient space is Hausdorff. Let's try to show that there always exists a group inducing this equivalence relation. Since every orbit has at most two elements, we can produce a function

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} y & \exists y \sim x, y \neq x \\ x & \text{else} \end{cases} \end{aligned}$$

This function satisfies $g^2 = \mathsf{id}$. So the group $G = \{\mathsf{id}, g\}$ acts on $\mathbb{R}$ producing the equivalence relation we began with. But is g continuous? □

Exercise 1.21* (Gluing of closed subsets) Let M be a metrizable topological space, and $Z_i \subset M$ a finite number of closed subsets which do not intersect, grouped into pairs of homeomorphic $Z_i \sim Z_i'$. Let $\sim$ be an equivalence relation generated by these homeomorphisms. Show that $M/\sim$ is Hausdorff.

Solution. ? □

1.3 Compact spaces

Definition 1.12 A ***cover*** of a topological space M is a collection of open subsets $\{U_\alpha \in 2^M\}$ such that $\bigcup U_\alpha = M$. A ***subcover*** of a cover $\{U_\alpha\}$ is a subset $\{U_\beta\} \subset \{U_\alpha\}$. A topological space is called ***compact*** if any cover of this space has a finite subcover.

Exercise 1.22 (Closed subset of compact is compact) Let M be a compact topological space, and $Z \subset M$ a closed subset. Show that Z is also compact.

Solution. Choose a cover $\{U_\alpha\}$ of Z. Complete to a cover $\{U_\alpha\} \cup (M \backslash Z)$ of M since $M \backslash Z$ is open by hypothesis. Since M is compact then there is a finite subcover $\{U_\beta\}$ of M. This is also a finite subcover of Z. □

Exercise 1.23 (Countable metrizable $\implies$ contains convergent subseq. or is discrete) Let M be a countable, metrizable topological space. Show that either M contains a converging sequence of pairwise different elements, or M is discrete.

Solution. Suppose M is not discrete. Then there is a point z_0 such that $\{z_0\}$ is not an open set. Then every open set containing z_0 contains another point. Choose for every $n \in \mathbb{N}$ a point z_n different from z_0 inside the a ball $B_{1/n}(z_0)$. Taking a subsequence if necessary, we obtain a sequence of pairwise different elements $\{z_i\}$ converging to z_0.

If M is discrete, it's clear that it cannot have a convergent sequence of pairwise disjoint elements: if the limit point $\{z_0\}$ was open, $M \backslash \{z_0\}$ would be closed and thus it would contain all its limit points! ◻

Definition 1.13 A topological space is called ***sequentially compact*** if any sequence $\{z_i\}$ of points of M has a converging subsequence.

Exercise 1.24 (Metrizable compact $\implies$ sequentially compact) Let M be a metrizable compact topological space. Show that M is sequentially compact.

Solution. Let $\{z_i\}$ be a sequence. Since the restricition of a metric to a subset is also a metric, we may use Exercise 1.23 on the countable metric subspace $\{z_i\}$. Suppose by contradiction that $\{z_i\}$ has no limit point in M. In particular it has no limit point in $\{z_i\}$, so by Exercise 1.23 it is discrete. Then there are neighbourhoods $U_i \ni z_i$ such that $U_i \cap \{z_j\}_{j \neq i} = \emptyset$. Then $\{U_i\} \cup (M \backslash \{U_i\})$ is an open cover of M, which has a finite subcover. By the pigeon principle, at least one of the U_i contains an infinite number of points in $\{z_i\}$, which is not possible. ◻

Definition (Folland, *Real Analysis*, p. 14-15) A sequence $\{x_n\}$ in a metric space (X, ρ) is called ***Cauchy*** if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$. A subset E of X is called ***complete*** if every Cauchy sequence in E converges and its limit is in E. E is called ***totally bounded*** if for every $\varepsilon > 0$, E can be covered by finitely many balls of radius ε.

Theorem 1.6.5 (Burago-Burago-Ivanov, *A course in metric geometry*) Let X be a metric space. Then the following statements are equivalent:

1. X is compact.
2. Any sequence in X has a converging subsequence.
3. Any infinite subset of X has an accumulation point.
4. X is complete and totally bounded.

[No proof]

Theorem 0.25 (Folland, *Real Analysis*) If E is a subset of the metric space (X, ρ), the following are equivalent:

(a) E is complete and totally bounded.

(b) (**Bolzano-Weierstrass property**) Every sequence in E has a subsequence that converges to a point of E.

(c) (**The Heine-Borel Property**) If $\{V_\alpha\}_{\alpha \in A}$ is a cover of E by open sets, there is a finite set $F \subset A$ such that $\{V_\alpha\}_{\alpha \in F}$ covers E.

Plan of proof. (a) and (b) are equivalent, and (a) and (b) together imply (c). □

Exercise 1.25* Construct an example of a Hausdorff topological space which is sequentially compact, but not compact.

Exercise 1.26* Construct an example of a Hausdorff topological space which is compact, but not sequentially compact.

Definition 1.14 A ***topological group*** is a topological space with group operations $G \times G \longrightarrow G, x, t \mapsto xy$ and $G \longrightarrow G, x \mapsto x^{-1}$ which are continuous. In a similar way, one defines ***topolocial vector spaces***, ***topological rings*** and so on.

Exercise 1.27* Let G be a compact topological group, acting on a topological space M in such a way that the map $M \times G \longrightarrow M$ is continuous. Prove that the quotient space is Hausdorff.

Solution. □

Exercise 1.28 (Continuous function maps compact to compact) Let $f : X \to Y$ be a continuous map of topological spaces with X compact. Prove that $f(X)$ is also compact.

Solution. Choose an open cover $\{U_\alpha\}$ of $f(X)$. Then $\left\{f^{-1}(U_\alpha)\right\}$ is an open cover of X since f is continuous, and thus it has an open subcover $\{f^{-1}(U_\beta)\}$. I claim that $\{U_\beta\}$ is a cover of $f(X)$: if there was a point $f(x) \notin \bigcup U_\beta$, then x couldn't be in any of the $f^{-1}(U_\beta)$, which cover X. □

Exercise 1.29 (Compact subset of Hausdorff is closed) Let $Z \subset Y$ be a compact subset of a Hausdorff topological space. Prove that it is closed.

Solution. Recall that a set is closed if it contains all its limit points (any point that is not a limit point has a neighbourhood not intersecting the set, making the complement open).

Let z_0 be a limit point of Z. Choose for every point $z \in Z$ neighbourhoods $U_z \ni z$ and $V_z \ni z_0$ such that $U_z \cap V_z = \varnothing$. If $z_0 \notin Z$, then $\{U_z\}$ is an open cover of Z, so there exists a finite subcover $U_{z_1}, \ldots, U_{z_n}$. The set $\bigcup_{i=1}^n V_{z_i}$ is an open neighbourhood of z_0 that does not intersect Z, a contradiction. □

Exercise 1.30 Let $f : X \to Y$ be a continuous, bijective map of topological spaces, with X compact and Y Hausdorff. Prove that it is a homeomorphism.

Solution. We need to see that f^{-1} is continuous, i.e. that $(f^{-1})^{-1}(U)$ is open for any $U \subset Y$ open. Since f is bijective, $(f^{-1})^{-1}(U) = f(U)$; so we must check f is open. Equivalently, we can check f is closed: if $f(F)$ is closed for any closed $F \subset X$, then for any open set $U \subset X$, we see $f(X \backslash U) = Y \backslash f(U)$ is closed.

To see f is closed note that since X is compact and f is bijective, $f(X) = Y$ is also compact by Exercise 1.28. By Exercise 1.22 a closed subset F of X is compact. Again by continuity, $f(F)$ is compact in Y. Finally by Exercise 1.29, since Y is Hausdorff and $f(F)$ is compact, it must be closed. □

Definition 1.15 A topological space M is called ***pseudocompact*** if any continuous function $f : M \to \mathbb{R}$ is bounded.

Exercise 1.31 Prove that any compact topological space is pseudocompact.

Solution. We must show that any continuous function $f : M \to \mathbb{R}$ is bounded, in the sense that its image contained in a ball of finite radius (c.f. Exercise 1.33). The image of any such function is compact by Exercise 1.28. But compact sets of $\mathbb{R}$ are bounded: if for every $r > 0$, the image $f(X)$ is not contained in the ball of radius r centered at zero, $B_r(0)$, then $\{B_r(0)\}$ is an open cover of $f(X)$ (since its union is all of $\mathbb{R}$) without a finite subcover. □

Exercise 1.32 Show that for any continuous function $f : M \longrightarrow \mathbb{R}$ on a compact space there exists $x \in M$ such that $f(z) = \sup_{z \in M} f(z)$.

Solution. As in Exercise 1.31, the image of f is a bounded set of $\mathbb{R}$, which means the supremum is well-defined. To see it is attained at a point in M notice that $f(M)$ is compact and thus closed. This means that all its limit points belong to $f(M)$: if a limit point z_0 of the closed set $f(M)$ is not in $f(M)$, then z_0 has no neighbourhood contained in $\mathbb{R} \backslash f(M)$, but $\mathbb{R} \backslash f(M)$ is open. In particular $\sup_{z \in M} f(z)$ is in $f(M)$.

(Extra: proof for sequentially compact.) We construct a sequence of points in $f(X)$ whose limit is $\sup_{z \in M} f(z)$. (Take balls of radius $1/n$.) This gives a sequence in X which must have a convergent subsequence within M. The limit of such sequence is the desired point. □

Exercise 1.33 Consider $\mathbb{R}^n$ as a metric space, with the standard (Euclidean) metric. Let $Z \subset \mathbb{R}^n$ be a closed, bounded set (***bounded*** means contained in a ball of finite radius). Prove that Z is sequentially compact.

Solution. First consider the case $n = 1$. If $\{z_n\}$ is a sequence contained in a closed and bounded set Z, then $\sup_{n \in N} z_n$ is a well-defined element in Z. Taking balls of radius $1/m$ we construct a subsequence of $\{z_n\}$ converging to the $\sup$.

Now let's show that $[0, 1]^n \subset \mathbb{R}^n$ is also sequentially compact. Chose a sequence $\{z_n\} \subset [0, 1]^n$. This gives a sequence in every coordinate, each of which must have a convergent subsequence, but this doesn't give a convergent subsequence in the product... □

Exercise 1.34** Find a pseudocompact Hausdorff topological space which is not compact.

Definition 1.16 A map of topological spaces is called ***proper*** if a preimage of any compact subset is compact.

Exercise 1.35* Let $f : X \to Y$ be a continuous proper, bijective map of metrizable topological spaces. Prove that f is a homeomorphism, or find a counterexample.

Exercise 1.36* Let $f : X \to Y$ be a continuous, proper map of metrizable topological spaces. Show that f is closed, or find a counterexample.

2 Manifolds and sheaves

2.1 Topological manifolds

Remark 2.1 Manifolds can be smooth (of a given "class of smoothness"), real analytic, or topological (continuous). ***Topological manifold*** is easiest to define, it is a topological space which is locally homeomorphic to an open ball in $\mathbb{R}^n$.

Definition 2.1 ***An action*** of a group on a manifold is silently assumed to be continuous. Let G be a group acting on a set M. The ***stabilizer*** of $x \in M$ is the subgroup of all elements in G that fix x. An action is ***free*** if the stabilizer of every point os trivial. The ***quotient space*** M/G is the space of orbits, equipped with the following topology: an open set $U \subset M/G$ is open if its preimage in M is open.

Exercise 2.1 (!) Let G be a finite group acting freely on a Hausdorff manifold M. Show that the quotient space M/G is a topological manifold.

Solution.

□

Exercise 2.2 Construct an example of a finite group G acting non-freely on a topological manifold M m such that M/G is not a topological manifold.

Solution. Consider the manifold $\mathbb{R}^2$ and a group G generated by some rotation about the origin. This makes the origin the only fixed point of G. Any other point has a neighbourhood in which it is the only element in its orbit; such a neighbourhood is mapped diffeomorphically onto the quotient, giving a coordinate chart.

However, any neighbourhood of the origin contains at least two G-related elements. Suppose there is a coordinate chart of $\bar{0}$ in the quotient. The preimage of this open set is an open set in $\mathbb{R}^2$ containing 0. The composition of the quotient map and the coordinate chart produce a non-injective map from $\mathbb{R}^2$ to $\mathbb{R}^2$. If such a map was smooth at 0, by the inverse function theorem there would be a neighbourhood of 0 in which the map would be invertible, a contradiction. □

Exercise 2.3 Consider the quotient of $\mathbb{R}^2$ by the action of $\{\pm 1\}$ that maps x to $-x$. Is the quotient space a topological manifold?

Solution. Yes (it's not *smooth* but it is topological). A homeomorphism between $\mathbb{R}^2/\{\pm 1\}$ and $\mathbb{R}^2$ is $re^{i\theta} \mapsto re^{i2\theta}$ where the angle θ in the domain is in the interval $[0, \pi)$ and $r \in [0, \infty)$. This map is clearly bijective (any equivalence class has a unique representative of the given form) and its continuous inverse is $re^{i\varphi} \mapsto re^{i\varphi/2}$ for $\varphi \in [0, 2\pi)$. □

Exercise 2.4* Let M be path connected, Hausdorff topological manifold and G a group of all its homeomorphisms. Prove that G acts transitively.

Exercise 2.5** Prove that any closed subgroup $G \subset \mathsf{GL}(n)$ of a matrix group is homeomorphic to a manifold, or find a counterexample.

Remark 2.2 In the above definition of a manifold, it is not required to be Hausdorff. Nevertheless, in most cases, manifolds are tacitly assumed to be Hausdorff.

Exercise 2.6 Construct an example of a non-Hausdorff manifold.

Exercise 2.7 Show that $\mathbb{R}^2/\mathbb{Z}^2$ is a manifold.

Solution. Let $\bar{z}$ be a point in $\mathbb{R}^2/\mathbb{Z}^2$. Its preimage is the lattice $\{z + (a, b) : a, b \in \mathbb{Z}\}$. A ball of radius $\frac{1}{2}$ centered at any representative of $\bar{z}$ contains only one representative of any other class, so that the restriction of the projection is bijective (and continuous by definition of quotient topology). The inverse map is also continuous by definition of quotient topology: an open set in the ball on $\mathbb{R}^2$ is mapped to an open set in the quotient because its preimage is open. □

Exercise 2.8 Let α be an irrational number. The group $\mathbb{Z}^2$ acts on $\mathbb{R}$ by the formula $t \mapsto t + m + n\alpha$. Show that this action is free, but the quotient $\mathbb{R}/\mathbb{Z}^2$ is not a manifold.

Solution. This action is obviously free since any nonzero pair of integers "moves" any number t. The idea is to show that the orbit of every point is dense. This would mean that any open set in the quotient is homeomorphic to $\mathbb{R}$, preventing it from having local neighbourhoods homeomorphic to balls, i.e. being a topological manifold. □

2.2 Smooth manifolds

Definition 2.2 A ***cover*** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a ***refinement*** of a cover $\{U_i\}$ if every V_i is contained in some U_i.

Exercise 2.11 Show that any two covers of a topological space admit a common refinement.

Solution. Let $\{U_i\}$ and $\{U'_i\}$ be covers of a topological space X. Then $\{V_{ij} := U_i \cap U'_j\}$ is a common refinement. It is obvious that V_{ij} is contained in U_i and U_j, so it is a subcover of both covers. And it is also a cover: it $x \in X$ then x must be in some U_i and some U'_j, so that it is in V_{ij}. $\square$

Definition 2.3 A cover $\{U_i\}$ is an ***atlas*** if for every U_i we have a map $\varphi_i : U_i \to \mathbb{R}^n$ giving a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. The ***transition maps***

$$\phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are induced by the above homeomorphisms. An atlas is ***smooth*** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), ***smooth of class*** C^i if all transition functions are of differentiability class C^i and ***real analytic*** if all transition maps admita a Taylor expansion at each point.

Definition 2.4 A ***refinement of an atlas*** is a refinement of the corresponding cover $V_i \subset U_i$ equipped with the maps $\varphi_i : V_i \to \mathbb{R}^n$ that are the restricitions of $\varphi_i : U_i \to \mathbb{R}^n$. Two atlases (U_i, φ_i) and (U_i, ψ_i) of class C^∞ of C^i (with the same cover) are ***equivalent*** in this class if, for all i, the map $\psi_i \circ \varphi_i^{-1}$ defined on the corresponding open subset in $\mathbb{R}^n$ belongs to the mantioned class. Two arbitratry atlases are ***equivalent*** if the corresponding covers possess a common refinement giving equivalent atlases.

Definition 2.5 A ***smooth structure*** on a manifold (of class C^∞ of C^i) is an atlas of class C^∞ or C^i considered up to the above equivalence. A ***smooth manifold*** is a topological manifold equipped with a smooth structure.

Remark 2.3 Terrible, isn't is?

Exercise 2.12* Construct an example of two nonequivalent smooth structures on $\mathbb{R}^n$.

Definition 2.6 A ***smooth function*** on a manifold M is a function f whose restriction to the chart (U_i, φ_i) gives a smooth function $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{R}$ for each open subset $\varphi_i(U_i) \subset \mathbb{R}^n$.

Remark 2.4 There are several ways to define a smooth manifold. The above way is most standard. It is not the most convenient one but you should know it. Two other ways (via sheaves of functions and via Whitney's theorem) are presented further in these handouts.

Definition 2.7 A ***presheaf of functions*** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1$.

Definition 2.8 A presheaf of functions $\mathcal{F}$ is called a ***sheaf of functions*** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of memebers of the cover. Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.

Remark 2.5 A *presheaf of functions* is a collection of subrings of functions on open subsets, compatible with restrictions. A *sheaf of functions* is a presheaf allowing "gluing" of a function on a bigger open set if its restriction to smaller open sets lies in the presheaf.

Definition 2.9 A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \ldots$ of homomorphisms of abelian grous or vector spaces is called ***exact*** if the image of each map is the kernel of the next one.

Exercise 2.13 Let $\mathcal{F}$ be a presheaf of functions. Show that $\mathcal{F}$ is a sheaf if and only if for every open cover $\{U_i\}$ of an oen subset $U \subset M$ the sequence of restriction maps

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi_1} \prod_i \mathcal{F}(U_i) \xrightarrow{\varphi_2} \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta|_{U_i \cap U_j}$ and $-\eta|_{U_j \cap U_i}$.

Solution. The key observation is that elements of $\ker \varphi_2$ are collections of functions $f_i \in \mathcal{F}(U_i)$ satisfying compatibility in pairwise intesections, i.e.,

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

To achieve this I think we must define φ_2 by

$$(\ldots, \quad f_i, \quad \ldots, \qquad f_j, \quad \ldots) \longmapsto (\ldots, \quad f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}, \quad \ldots).$$

Then elements in $\ker \varphi_2$ satisfy the desired compatibility condition.

($\Longrightarrow$) Suppose the the sequence above is exact. To show $\mathcal{F}$ is a sheaf fix $f_i \in \mathcal{F}(U_i)$ for every i satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every i, j. This is equivalent to choosing an element in $\ker \varphi_2$. By exactness this element is in $\operatorname{img} \varphi_1$. This means that f_i is the restriction of some $f \in \mathcal{F}(U)$ for every i as desired.

($\Longleftarrow$) Suppose $\mathcal{F}$ is a sheaf. Injectivity of φ_1 is immediate: if $f \in \mathcal{F}(U)$ is mapped to zero under φ_1, meaning $f_i|_{U_i} = 0$ for all i, it must be zero since $\{U_i\}$ is a cover. Exactness in the second ring is equivalent to the definition of sheaf by the remarks above. $\square$

Exercise 2.14 Show that the following spaces of functions on $\mathbb{R}^n$ define sheaves of functions.

(a) Space of continuous functions.

(b) Space of smooth functions.

(c) Space of functions of differentiability class C^i.

(d) (*) Space of functions which are pointwise limits of sequences of continuous functions.

(e) Space of functions vanishing outside a set of measure 0.

Solution. Injectivity of φ_1 is immediate in all cases: a function that vanishes on every subset of an open cover vanishes identically.

(a) Define a global function f on U by $x \mapsto f_i(x)$ for any $f_i \in \mathcal{F}(U_i)$ such that $x \in U_i$. Continuity follows from continuity of f_i, and the fact that f is well-defined follows from the gluing condition of $\mathcal{F}$.

(b) Like above: smoothness follows from smoothness of f_i.

(c) Like above.

(d)

(e) Not sure (uncountable union of measure-zero sets may have positive measure).

□

Exercise 2.15 Show that the following spaces of functions on $\mathbb{R}^n$ are presheaves, but not sheaves

(a) Space of constant functions.

(b) Space of bounded functions.

(c) Space of functions vanishing outside of a bounded set.

(d) Space of continuous functions with finite $\int |f|$.

Solution. The presheaf condition, that the restriction of a function to

(a) Open sets with two connected components may not glue to a global constant function.

(b) Unbounded functions may be bounded in open subsets! Take the open set $(0, \infty) \subset \mathbb{R}$ and the cover $U_i = (1/i, \infty)$. Define the bounded function $f_i(x) = 1/x$ in every U_i.

In $\mathbb{R}^n$ we may do the same trick using the half-spaces with bounded last coordinate $U_i = \{(x_1, \ldots, x_n) : x_n \geqslant 1/i\}$ and taking $f_i(x) = 1/\|x\|$.

(c) Let the open set U be all of $\mathbb{R}^n$. An open cover is given by balls B_i of radius $2/3$ with center in $i \in \mathbb{Z}^n$. For every $i \in \mathbb{Z}^n$ define functions f_i that vanish only outside a ball of very small radius, say $1/6$, with center in i. These functions coincide (they vanish) in the intersections of the cover, but the function obtained by gluing cannot vanish outside any bounded set: it is non-zero in the union of balls of radius 1/6 with centers in $\mathbb{Z}^n$.

(d) Item (c) works if we manage to make the functions continuous. This can be done partition of unity. Also we must require that the values of the integrals in the smaller balls of radius 1/6 do not tend to zero (this way the global integral is an infinite sum of numbers that do not tend to zero, so it cannot be finite).

$\square$