Problem set 2

Differential topology 2025

Due: 4 February 2025

Problem 1: Let $\Phi = \{\varphi_i, U_i\}_{i\in I}$ be a locally finite atlas of X, $\mathcal{K} = \{K_i\}_{i\in I}$ a family of compact sets $K_i \subset U_i$, $\Psi = \{\psi_i, V_i\}$ an atlas of Y, $\varepsilon = \{\varepsilon_i\}_{i\in I} \subset \mathbb{R}^+$ and $f \in C^\infty(X,Y)$ such that $f(K_i) \subset V_i$. Let

$$W^k(f;\Phi,\Psi,\mathcal{K},\varepsilon) = \left\{g \in C^\infty(X,Y) \mid g(K_i) \subset V_i, \|D^r(\psi_i \circ f \circ \varphi_i^{-1}) - D^r(\psi_i \circ g \circ \varphi_i^{-1})\|_{C^0(\varphi_i(K_i))} < \varepsilon_i, \forall i \in I, 0 \le r \le k\right\}.$$

Prove that the collection of all sets of this form is a base for the C^k topology on $C^\infty(X,Y)$.

Problem 2: A function is said to be closed if the image of every closed set is closed. Prove that the set

$$\{f \in C^\infty(\mathbb{R},\mathbb{R}) | f \text{ is closed}\}$$

is closed in the C^∞ topology.

Problem 3: Let X and Y be manifolds and $l \ge k$. Prove that there exists a natural fiber bundle $J^l(X,Y) \to J^k(X,Y)$ and compute the dimension of its fiber.

Problem 4: Let M be a non-compact manifold.

(a) Prove that multiplication by scalar $\mathbb{R} \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ is not continuous in the C^∞ topology.

(b) Prove that addition and multiplication of functions are continuous in the C^∞ topology.

Problem 5: Let X be a submanifold of $\mathbb{R}^n$ and $k \ge \operatorname{codim}(X)$. Prove that almost every subspace of dimension k intersects X transversely, i.e. the set of all subsets of dimension k that don't intersect X transversely has measure zero.