

# algebraic topology exercises

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## Homework 1

### 0 Preliminaries

In the category of sets there is a bijection  $\text{Hom}(X \times Y, Z) \cong \text{Hom}(X, \text{Hom}(Y, Z))$  that depends naturally on  $X$ ,  $Y$  and  $Z$ . The notions related to this bijection are “Cartesian closed category”, “currying” and “internal Hom”.

**Definition.** A category  $\mathcal{C}$  is *Cartesian closed* if:

1.  $\mathcal{C}$  has all finite products (Caveat: some require that  $\mathcal{C}$  has all finite limits)
2. For any object  $Y$  the functor  $- \times Y$  has a right adjoint, which we will denote by  $\text{Map}(Y, -)$  or by  $-^Y$ .

**Remark.** By section 3 [here](#), the second property above implies that we get a functor  $\text{Map}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ , and moreover we get natural isomorphisms  $\text{Hom}(X, \text{Map}(Y, Z)) \cong \text{Hom}(X \times Y, Z)$  and  $\text{Map}(X, \text{Map}(Y, Z)) \cong \text{Map}(X \times Y, Z)$ .

**Lemma (Yoneda, [wiki](#)).** Let  $F$  be a functor from a locally small category  $\mathcal{C}$  to  $\mathbf{Set}$ . Then for each object  $X$  of  $\mathcal{C}$ , the natural transformations  $\text{Nat}(\text{Hom}(X, -), F)$  are in one-to-one correspondence with the elements of  $F(X)$ , that is

$$\text{Nat}(\text{Hom}(X, -), F) \cong F(X)$$

Moreover, this isomorphism is natural in  $A$  and  $F$  when both sides are regarded as functors from  $\mathcal{C} \times \mathbf{Set}^{\mathcal{C}}$  to  $\mathbf{Set}$ . ( $\mathbf{Set}^{\mathcal{C}}$  denotes the category of functors from  $\mathcal{C}$  to  $\mathbf{Set}$ .)

There is a contravariant version of Yoneda lemma asserting that if  $F$  is a contravariant functor from  $\mathcal{C}$  to  $\mathbf{Set}$ ,

$$\text{Nat}(\text{Hom}(-, X), F) \cong F(X).$$

**Corollary.**  $\text{Nat}(\text{Hom}(-, X), \text{Hom}(-, Y)) \cong \text{Hom}(X, Y)$ .

**Remark.** The correspondence  $X \mapsto \text{Hom}(-, X)$  is fully faithful, that is, the correspondence  $\text{Hom}(X, X') \rightarrow \text{Nat}(\text{Hom}(-, X), \text{Hom}(-, X'))$  is injective and bijective.

**Exercise (a).** Let  $\mathcal{C}$  be any category. Show that if for some objects  $X$  and  $X'$  we have  $\text{Hom}(X, Y) \cong \text{Hom}(X', Y)$  for all objects  $Y$ , with isomorphisms being natural in  $Y$ , then  $X \cong X'$ . Dually, if  $\text{Hom}(Y, X) \cong \text{Hom}(Y, X')$  naturally in  $Y$ , then also  $X \cong X'$ .

*Solution.* The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude  $X \cong X'$ . The dual statement follows from the analogue formulation of Yoneda lemma.  $\square$

**Exercise (b).** Let  $\mathcal{C}$  be a Cartesian closed category and  $\text{pt}$  be the terminal object. Show that for any object  $X$  we have  $X \cong \text{Map}(\text{pt}, X)$ .

*Solution.* Using item (a) with  $X$  and  $X' = \text{Map}(\text{pt}, X)$ , it suffices to show that

$$\text{Hom}(Y, X) \cong \text{Hom}(Y, \text{Map}(\text{pt}, X))$$

for all objects  $Y$  and isomorphisms natural in  $Y$ .

Since  $\mathcal{C}$  is Cartesian closed, we have isomorphisms [natural](#) in  $Y$

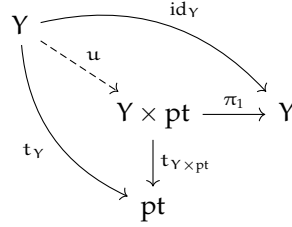
$$\text{Hom}(Y, \text{Map}(\text{pt}, X)) \cong \text{Hom}(Y \times \text{pt}, X) \cong \text{Hom}(Y, X)$$

since  $\text{pt}$  is a terminal object. Indeed:

**Claim.** In a Cartesian closed category  $\mathcal{C}$  with terminal object  $\text{pt}$ , we have that  $Y \times \text{pt} \cong Y$  for any object  $Y$ .

*Proof of claim.* (**From [StackExchange](#)**) The universal property of the product  $Y \times \text{pt}$  shows that the maps  $\text{id}_Y$  and  $t_Y : Y \rightarrow \text{pt}$  must factor through some  $u : Y \rightarrow Y \times \text{pt}$ , making

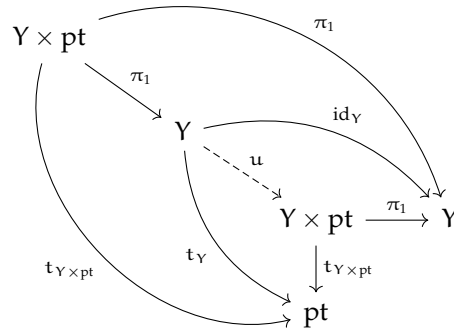
$$\pi_1 \circ u = \text{id}_Y.$$



It is also true that  $u \circ \pi_1 = \text{id}_{Y \times \text{pt}}$ , since

- $\pi_1 \circ u \circ \pi_1 = \text{id}_Y \circ \pi_1 = \pi_1$  and
- $t_{Y \times \text{pt}} \circ u \circ \pi_1 = t_{Y \times \text{pt}}$

so by uniqueness of the universal property we get that  $u \circ \pi_1 = \text{id}_{Y \times \text{pt}}$ .



□

□

## 1 Based spaces and smash product

**Definition.** The appropriate analogue of the Cartesian product in the category of based spaces is the *smash product*  $X \wedge Y$  defined by

$$X \wedge Y = X \times Y / X \vee Y.$$

Here  $X \vee Y$  is viewed as the subspace of  $X \times Y$  consisting of those pairs  $(x, y)$  such that either  $x$  is the basepoint of  $X$  or  $y$  is the basepoint of  $Y$ .

**Exercise.** For a based space  $(X, x_0)$  let  $\Sigma X$  be  $[0, 1] \times X / \{1\} \times X \cup \{0\} \times X \cup [0, 1] \times \{x_0\}$ . Check that  $\Sigma X \cong S^1 \wedge X$ . In particular  $S^n \cong S^1 \wedge S^{n-1} \cong (S^1)^{\wedge n}$ .

**Remark.** Another way of defining the reduced suspension  $\Sigma X$  (I think) is

$$\Sigma X = (I \times X) / (t, x) \sim (0, y) \sim (1, y) \quad \forall y \in X.$$

*Proof.* To see that  $\Sigma X \cong S^1 \wedge X$  simply notice that “both spaces are the quotient  $X \times I$  with  $X \times \partial I \cup \{x_0\} \times I$  collapsed to a point” (Hatcher, ex. 0.10). This is clear for  $\Sigma X$ . For  $X \wedge S^1$ , notice that collapsing  $X \times \partial I$  to a point in  $X \times I$  amounts to taking  $X \times S^1$  and collapsing one copy of  $X$  to a point. Further, collapsing  $x_0 \times I$  to a point amounts to collapsing the copy of  $S^1$  in  $X \vee S^1$  to a point.

Let’s try induction on  $n$ . If  $n = 2$ , the smash product  $S^1 \wedge S^1$  is easily seen to be  $S^2$  since it consists on collapsing the boundary  $S^1 \vee S^1$  of the square whose quotient yields  $S^1 \times S^1$ . For the inductive step *Still incomplete...*  $\square$

## 2 Mapping cylinders and Hurewicz cofibrations

**Definition** (wikipedia). Let  $X$  be a topological space and let  $A \subset X$ . We say that the pair  $(X, A)$  has the *homotopy extension property* if for any space  $Y$ , any homotopy  $g_\bullet : A \rightarrow Y^I$  and any map  $\tilde{g}_0 : X \rightarrow Y$  such that  $\tilde{g}_0 \circ \iota = g_0$ , there exists an *extension* of  $f_\bullet$  to a homotopy  $\tilde{g}_\bullet : X \rightarrow Y^I$  such that  $\tilde{g}_\bullet \circ \iota = g_\bullet$ .

$$\begin{array}{ccc} A & \xrightarrow{g_\bullet} & Y^I \\ \downarrow \iota & \nearrow \tilde{g}_\bullet & \downarrow \pi_0 \\ X & \xrightarrow{\tilde{g}_0} & Y \end{array}$$

A *Hurewicz cofibration* is a map  $\iota : A \rightarrow X$  satisfying the homotopy extension property.

**Exercise (a).** Prove that an inclusion  $f : A \rightarrow X$  is a Hurewicz cofibration if and only if  $A \times I \cup X \times \{0\}$  is a retract of  $X \times I$ .

**Remark.** A little late I noticed the comment on Telegram that we may assume  $A$  to be a closed subspace. Maybe I wouldn’t have tried the solution following Miller if I had knew this earlier, hehe— still it was nice to see two different solutions.

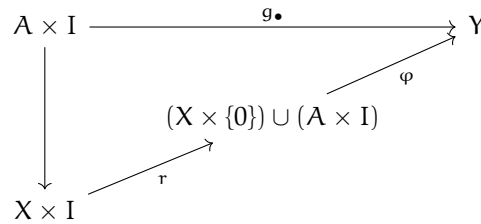
*Solution following Hatcher.* ( $\implies$ ) According to the former definition, choose  $Y = (X \times \{0\}) \cup (A \times I)$ . The inclusion  $A \times I \hookrightarrow Y$  is an homotopy  $g_\bullet$  from  $A$  to  $Y$ . Also, the inclusion  $X \times \{0\} \hookrightarrow Y$  is an extension  $\tilde{g}_0$ . Then there exists an extension  $\tilde{g}_\bullet$  of the whole homotopy, which is just a map from  $X \times I$  to  $Y$ . We have thus produced a retraction:

$$\begin{array}{ccc} (X \times \{0\}) \cup (A \times I) & \xrightarrow{\text{id}} & (X \times \{0\}) \cup (A \times I) = Y \\ \downarrow & \nearrow & \\ X \times I & & \end{array}$$

( $\impliedby$ ) Now suppose that  $(X \times \{0\}) \cup (A \times I)$  is a retract of  $X \times I$ . Let  $Y$  be any space,  $g_\bullet : A \rightarrow Y^I$  an homotopy and  $\tilde{g}_0$  a map such that  $\tilde{g}_0 = g_0 \circ f$ .

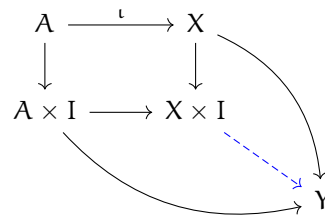
The homotopy  $g_\bullet$  along with  $\tilde{g}_0$  yield a map  $\varphi : (A \times I) \cup (X \times \{0\}) \rightarrow Y \cup (X \times \{0\})$ . The key observation is that if  $A$  is closed in  $X$ , then this map is continuous by the *gluing lemma*.

Then we simply compose the given retraction  $r$  with this map to obtain the homotopy extension:

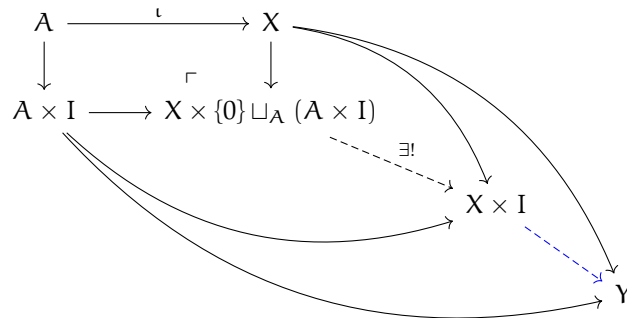


A complicated argument in Hatcher's appendix shows that such a function is continuous even without the assumption that  $A$  is closed.  $\square$

*Solution following Miller.* The homotopy extension property may be defined as a map  $\iota : A \rightarrow X$  such that for any solid-arrow diagram as below, a dotted blue arrow exists making the whole diagram commute:



Now consider the pushout corresponding to  $\iota$  and the inclusion  $A \rightarrow A \times I$ . By the universal property of the pushout, the former diagram must factor by the pushout, and we get the following diagram:



The implication  $(\implies)$  of our exercise again follows by choosing  $Y = (X \times \{0\}) \cup (A \times I)$ . For the implication  $(\impliedby)$  it appears that we have the same problem as before: we need to construct the blue dashed arrow from the rest of the diagram (using that the black dashed arrow has a left inverse), but it seems that the natural thing to do is defining this function from the two pieces just like before, and we must make sure it is continuous.  $\square$

**Definition.** Let  $f : X \rightarrow Y$  be a map. Let  $M_f = X \times [0, 1] \cup_f Y$  be the *mapping cylinder of  $f$* , i.e. the pushout of  $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0, 1]$  and of  $f : X \rightarrow Y$ . Let  $g : X \rightarrow M_f$  be the map  $X \xrightarrow{\cong} X \times \{1\} \rightarrow M_f$ . Let  $h : M_f \rightarrow Y$  be the map that is induced by  $X \times [0, 1] \rightarrow Y : (x, t) \mapsto f(x)$  and  $\text{id}_Y : Y \rightarrow Y$ . Observe that  $f$  is the composition of  $g$  and  $h$ .

**Remark.** In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e.  $(\text{colim } A_i) \times B$  is canonically homeomorphic with  $\text{colim}(A_i \times B)$ .

**Exercise.**

- Show that  $h$  is a deformation retract, and in particular is a homotopy equivalence.
- Show that  $g : X \rightarrow M_f$  is a cofibration. You may use exercise (a), but the direct proof might be simpler.

*Solution.*

- We have that

$$\begin{array}{ccc}
 X \times \{0\} & \xrightarrow{f} & Y \\
 \text{id} \times 1 \downarrow & \searrow g & \downarrow \text{id}_Y \\
 X \times I & \xrightarrow{\quad} & M_f \\
 & \searrow (x,t) \mapsto f(x) & \downarrow h \\
 & & Y
 \end{array}$$

We must show that there is a homotopy between the identity map on  $M_f$  and a retraction from  $M_f$  to  $Y$ . So we want  $h : M_f \times I \rightarrow M_f$  such that

$$h(-, 0) = \text{id}_{M_f}, \quad \text{img } h(-, 1) \subset Y \quad \text{and} \quad h(-, 1)|_Y = \text{id}_Y$$

Since  $M_f$  is a pullback, we can see it as a colimit, that is

$$M_f = \text{colim}(X \times I \leftarrow X \rightarrow Y)$$

and, since colimits commute with products in CGWH, we get

$$M_f \times I = \text{colim}(X \times I \times I \leftarrow X \times I \rightarrow Y \times I)$$

that is,

$$\begin{array}{ccc}
 X \times \{0\} \times I & \longrightarrow & Y \times I \\
 \downarrow & \lrcorner & \downarrow ? \\
 X \times I \times I & \xrightarrow{\quad ? \quad} & M_f \times I \\
 & \searrow (x,t,s) \mapsto f(x) & \downarrow \text{dashed} \\
 & & M_f
 \end{array}$$

[I certainly got stuck in concluding...]

- c. [Also in progress...] Consider the following lifting problem:

$$\begin{array}{ccc} X & \xrightarrow{H} & Z^I \\ g \downarrow & \nearrow \text{dashed} & \downarrow \pi_0 \\ M_f & \xrightarrow{h} & Z \end{array}$$

□

### 3 Path spaces and fibrations

#### Exercise.

- a. Show that  $\text{Map}(I, Y)$  deformation retracts on  $\text{Map}(\text{pt}, Y)$ . Most likely you'll have to find a correct map  $I \times I \rightarrow I$ . Also show that  $\text{Map}(I, Y) \rightarrow \text{Map}(\text{pt}, Y)$  is a Hurewicz fibration. The key map will be of the form  $I \times I \rightarrow I \times I$ .

*Solution.*

- a. ( $\text{Map}(I, Y) \rightarrow \text{Map}(\text{pt}, Y)$  is a **Hurewicz fibration.**) Let  $A$  be any space. We must show that for any homotopy  $H$  and lift  $h_0$  there exists an homotopy  $\tilde{H}$  as in the following diagram:

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{h_0} & \text{Map}(I, Y) \\ \downarrow & \nearrow \tilde{H} & \downarrow p \\ A \times I & \xrightarrow{H} & \text{Map}(0, Y) \end{array}$$

From the isomorphism  $\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$  we may rewrite the problem as

$$\begin{array}{ccc} (A \times \{0\}) \times I & & \\ & \searrow H_0 & \\ (A \times I) \times I & \xrightarrow{\tilde{H}} & Y \\ & \nearrow H & \\ (A \times I) \times \{0\} & & \end{array}$$

So we define the dashed arrow by

$$(a, s, t) \mapsto \begin{cases} H_0(a, 0, s - t) & \text{when } s - t \geq 0 \\ H(a, t - s, 0) & \text{when } s - t \leq 0 \end{cases}$$

so that when  $s = t$  the functions coincide, when  $s = 0$  we get  $H$  and when  $t = 0$  we get  $H_0$ .

( $\text{Map}(I, Y)$  **deformation retracts on**  $\text{Map}(\text{pt}, Y)$ .) We must show there is a homotopy

$$h : \text{Map}(I, Y) \times I \longrightarrow \text{Map}(I, Y)$$

such that

$$h(-, 0) = \text{id}_{\text{Map}(I, Y)}, \quad h(-, 1) \subset \text{Map}(\text{pt}, Y)$$

$$\text{and} \quad h(-, 1)|_{\text{Map}(\text{pt}, Y)} = \text{id}_{\text{Map}(\text{pt}, Y)}.$$

Consider the map

$$\begin{aligned} I \times I &\rightarrow I \\ (s, t) &\mapsto s - st \end{aligned}$$

Our deformation retract may be written like

$$\begin{aligned} h : \text{Map}(I, Y) \times I &\longrightarrow \text{Map}(I, Y) \\ (f(s), t) &\longmapsto f(s - st) \end{aligned}$$

Then for  $t = 0$  we have the identity on  $\text{Map}(I, Y)$ , and when  $t = 1$  we have  $\text{ev}_0$ .

□

Let  $f : X \rightarrow Y$  be a map. Let  $E_f$  be the pullback of  $f : X \rightarrow Y$  and of  $\text{ev}_0 : \text{Map}(I, Y) \rightarrow Y$ . Let  $h : X \rightarrow E_f$  be the map that sends  $x$  to  $(x, \text{const}(f(x)))$ , where  $\text{const}(f(x)) : I \rightarrow Y$  is the constant path at  $f(x)$ . Let  $g : E_f \rightarrow Y$  be the composition of projection map  $E_f \rightarrow \text{Map}(I, Y)$  with  $\text{ev}_1 : \text{Map}(I, Y) \rightarrow Y$ .

**Exercise.** b. Show that  $h : X \rightarrow E_f$  is an inclusion of a deformation retract.

c. Show that  $g : E_f \rightarrow Y$  is a fibration.

## Homework 1.5

### 1 Exercise on model categories

**Exercise (3.1.8 from Riehl).** Verify that the class of morphisms  $\mathcal{L}$  characterized by the left lifting property against a fixed class of morphisms  $\mathcal{R}$  is closed under coproducts, closed under retracts, and contains the isomorphisms.

*Solution. (Coproducts.)* Suppose the maps  $\ell_i : A_i \rightarrow B_i$  are in  $\mathcal{L}$ . Then their coproduct in the arrow category is the obvious map  $\coprod A_i \rightarrow \coprod B_i$ .



Explicitly, their coproduct is an arrow  $\coprod \ell_i$  and a collection of maps  $f_i : \ell_i \rightarrow \coprod \ell_i$  such that for any other object  $m : A \rightarrow B$  in the arrow category and a map  $g : \ell \rightarrow m$ , the following diagram is completed uniquely:

$$\begin{array}{ccc} \ell_i & \xrightarrow{f_i} & \coprod \ell_i \dashrightarrow^{\exists!} m \\ & \searrow g & \uparrow \\ & & \end{array} \quad \forall i$$

So we conclude that the source of  $\coprod \ell_i$  is  $\coprod A_i$  and its target  $\coprod B_i$ . Indeed, we really looking at

$$\begin{array}{ccc} A_i & \xrightarrow{\ell_i} & B_i \\ f_i^1 \downarrow & & \downarrow f_i^2 \\ \coprod A_i & \xrightarrow{\coprod \ell_i} & \coprod B_i \\ \exists! \downarrow & & \downarrow \exists! \\ A & \xrightarrow{m} & B \end{array}$$

Now consider the following lifting problem with respect to a morphism  $r \in \mathcal{R}$ :

$$\begin{array}{ccc} \coprod A_i & \longrightarrow & \bullet \\ \coprod \ell_i \downarrow & & \downarrow r \in \mathcal{R} \\ \coprod B_i & \longrightarrow & \bullet \end{array}$$

Since  $\ell_i \in \mathcal{L}$ , we have maps

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$

which in turn means we have a unique map

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$

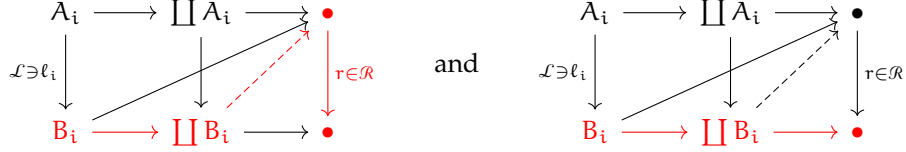
by the universal property of the coproduct  $\coprod B_i$ .

To conclude we need to check that the triangles below and above the dashed arrow in the former diagram commute. This follows from the universal property of the coproducts

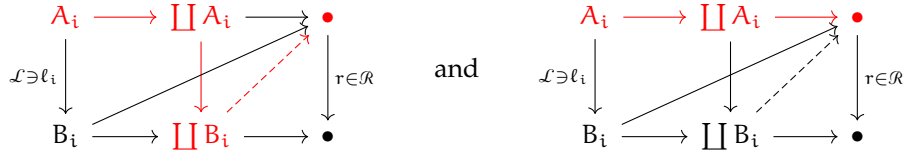
$\coprod A_i$  and  $\coprod B_i$  since, *in general*,

$$\mathrm{Hom}\left(\coprod X_i, Y\right) \cong \prod \mathrm{Hom}(X_i, Y).$$

More explicitly, we now that the red paths in the following diagrams are the same:



and also



so the conclusion follows from the former comment.

**(Closed under retracts.)** Let us at least state what a retract of a morphism  $g$  should be in the arrow category. Recall that a retract is just

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & X \\ & & \searrow & \text{id}_X & \nearrow \end{array}$$

So in the arrow category we get

$$\begin{array}{ccccc} \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet \\ \downarrow f & & \downarrow g & & \downarrow f \\ \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet \\ & & \searrow & \text{id} & \nearrow \end{array}$$

□

## 2 Hatcher's exercise on Whitehead's theorem

**Theorem 1 (Whitehead, May).** If  $X$  is a CW complex and  $e : Y \rightarrow Z$  is an  $n$ -equivalence, then  $e_* : [X, Y] \rightarrow [X, Z]$  is a bijection if  $\dim X < n$  and surjection if  $\dim X = n$ .

**Theorem 2 (Whitehead, May).** An  $n$ -equivalence between CW complexes of dimension less than  $n$  is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

**Theorem 3** (Whitehead (4.5), Hatcher). If a map  $f : X \rightarrow Y$  between connected CW complexes induces isomorphisms  $f_* : \pi_n(X) \rightarrow \pi_n(Y)$  for all  $n$ , then  $f$  is a homotopy equivalence. In case  $f$  is the inclusion of a subcomplex  $X \hookrightarrow Y$ , the conclusion is stronger:  $X$  is a deformation retract of  $Y$ .

**Exercise** (Hatcher 4.1.12). Show that an  $n$ -connected,  $n$ -dimensional CW complex is contractible.

*Solution.* Just recall that  $n$ -connectedness means that  $\pi_i(X) = 0$  for all  $i \leq n$ , which means that  $X$  is contractible by theorem 2.  $\square$

## Homework 2

**Definition** (H-space, Hatcher p. 281).  $X$  is an **H-space**, 'H' standing for Hopf, if there is a continuous multiplication map  $\mu : X \times X \rightarrow X$  and an identity element  $e \in X$  such that the two maps  $X \rightarrow X$  given by  $x \mapsto \mu(x, e)$  and  $x \mapsto \mu(e, x)$  are homotopic to the identity through maps  $(X, e) \rightarrow (X, e)$ .

**Exercise** (4.1.3). For an H-space  $(X, x_0)$  with multiplication  $\mu : X \times X \rightarrow X$ , show that the group operation in  $\pi_n(X, x_0)$  can also be defined by the rule  $(f + g)(x) = \mu(f(x), g(x))$ .

*Solution.* According to the **Eckmann-Hilton argument**, we may show that  $\pi_n(X, x_0)$  with the usual operation  $+$  and the operation  $\oplus$  given by  $(f \oplus g)(x) = \mu(f(x), g(x))$  coincide if we manage to show that for all  $a, b, c, d \in \pi_n(X, x_0)$

$$(a + b) \oplus (c + d) = (a \oplus c) + (b \oplus d).$$

This follows from definitions. Recall that for  $f, g \in \pi_n(X, x_0)$ ,

$$(f + g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \dots, s_n) & s_1 \in [1/2, 1] \end{cases}$$

so

$$\begin{aligned} (a \oplus c) + (b \oplus d) &= \begin{cases} (a \oplus c)(2s_1, s_2, \dots, s_n) & s_1 \in [0, 1/2] \\ (b \oplus d)(2s_1 - 1, s_2, \dots, s_n) & s_1 \in [1/2, 1] \end{cases} \\ &= \begin{cases} \mu(a(2s_1, s_2, \dots, s_n), c(2s_1, s_2, \dots, s_n)) & s_1 \in [0, 1/2] \\ \mu(b(2s_1 - 1, s_2, \dots, s_n), d(2s_1 - 1, s_2, \dots, s_n)) & s_1 \in [1/2, 1] \end{cases} \\ &= \mu(a + b, c + d) \\ &= (a + b) \oplus (c + d) \end{aligned}$$

$\square$

**Exercise** (4.1.19). Consider the equivalence relation  $\simeq_w$  generated by weak homotopy equivalence:  $X \simeq_w Y$  if there are spaces  $X = X_1, X_2, \dots, X_n = Y$  with weak homotopy

equivalences  $X_i \rightarrow X_{i+1}$  or  $X_i \leftarrow X_{i+1}$  for each  $i$ . Show that  $X \simeq_w Y$  iff  $X$  and  $Y$  have a common CW approximation.

*Solution.* ( $\Leftarrow$ ) Suppose  $Z$  is a common CW approximation of  $X$  and  $Y$ , that is,  $Z$  is a CW complex and there are weak homotopy equivalences  $Z \rightarrow X$  and  $Z \rightarrow Y$ . Then the sequence of spaces  $X = X_1$ ,  $Z = X_2$  and  $Y = X_3$  shows that  $X \simeq_w Y$ .

( $\Rightarrow$ ) Suppose  $Z$  is a CW approximation of  $X$  and let's show it can be made (somehow) into a CW approximation of  $Y$ . There is a weak homotopy equivalence  $Z \rightarrow X$ , and also a weak homotopy equivalence either  $X = X_1 \rightarrow X_2$  or  $X = X_1 \leftarrow X_2$ . I wonder if this implies that the composition  $Z \rightarrow X = X_1 \rightarrow X_2$  is also a weak homotopy equivalence  $\square$

**Exercise (4.2.1).** Use homotopy groups to show that there is no retraction  $\mathbb{R}P^n \rightarrow \mathbb{R}P^k$  for  $n > k > 0$ .

*Solution (in progress...)* Suppose there is a retraction

$$\begin{array}{ccccc} \mathbb{R}P^k & \hookrightarrow & \mathbb{R}P^n & \longrightarrow & \mathbb{R}P^k \\ & & \searrow \text{id} & \nearrow & \\ & & & & \end{array}$$

it induces isomorphisms

$$\begin{array}{ccccc} \pi_i(\mathbb{R}P^k) & \longrightarrow & \pi_i(\mathbb{R}P^n) & \longrightarrow & \pi_i(\mathbb{R}P^k) \\ & & \searrow \cong & \nearrow & \\ & & & & \end{array}$$

and then just notice that that map is zero. Indeed, recall that the homotopy groups of a covering space are the same as the base (because the homotopy groups of the fibers are trivial), so we have that  $\pi_k(\mathbb{R}P^k) \rightarrow \pi_k(\mathbb{R}P^n) = 0 \rightarrow \pi_k(\mathbb{R}P^k)$  will be zero.  $\square$

**Exercise (4.2.2).** Show that the action of  $\pi_1(\mathbb{R}P^n)$  on  $\pi_n(\mathbb{R}P^n) \cong \mathbb{Z}$  is trivial for  $n$  odd and nontrivial for  $n$  even.

*Solution.* (I have used [StackExchange](#) as suggested in Telegram chat, and also [this internet pdf](#).)

By exercise 4.1.4 the action of  $\pi_1(\mathbb{R}P^n)$  on  $\pi_n(\mathbb{R}P^n)$  can be identified with the action of deck transformations  $G(S^n) \cong \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ . The nontrivial element in such a group is the antipodal map, and it acts on  $\pi_n(\mathbb{R}P^n)$  by concatenation with the map induced on homotopy by the deck transformation (and a change of base point isomorphism).

Then we use Corollary 4.25, which says that the degree map  $\pi_n(S^n) \rightarrow \mathbb{Z}$  is an isomorphism. Then our action, which is a map  $\pi_1(\mathbb{R}P^n) \rightarrow \text{Aut}_{\deg} \pi_n(S^n) \cong \mathbb{Z}$  is trivial for  $n$  odd and nontrivial for  $n$  even since antipodal map multiplies degree by  $(-1)^{n+1}$ .  $\square$

**Exercise (4.2.8).** Show that the suspension of an acyclic CW complex is contractible.

*Solution.* (Warning: there are acyclic spaces with non-trivial homotopy groups.) Let's try to use Hurewicz theorem. Recall that by Freudenthal suspension theorem (coro 4.24) that if  $X$  is  $n$ -connected, then  $\pi_k(X) \rightarrow \pi_{k+1}(\Sigma X)$  is an isomorphism for  $k \leq 2n$ . This makes  $\pi_1$  of the suspension trivial.  $\square$

**Exercise (4.2.12).** Show that a map  $f : X \rightarrow Y$  of connected CW complexes is a homotopy equivalence if it induces an isomorphism on  $\pi_1$  and if a lift  $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$  to the universal covers induces an isomorphism on homology. [The latter condition can be restated in terms of homology with local coefficients as saying that  $f_* : H_*(X; \mathbb{Z}[\pi_1 X]) \rightarrow H_*(Y; \mathbb{Z}[\pi_1 Y])$  is an isomorphism].

**Exercise (4.2.13).** Show that a map between connected  $n$ -dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on  $\pi_i$  for  $i \leq n$ . [Pass to universal covers and use homology.]

*Solution.* Let  $X$  and  $Y$  be  $n$ -dimensional CW complexes and  $f : X \rightarrow Y$  such that  $f_* : \pi_i(X) \rightarrow \pi_i(Y)$  is an isomorphism for  $i \leq n$ . Let's try to use Hurewicz theorem, which states that a map between simply connected CW complexes is a homotopy equivalence if it induces isomorphisms on all homology groups.

Consider the universal covers  $\tilde{X}$  and  $\tilde{Y}$ , which are simply connected and also [have CW structures](#). By prop. 4.1, the cover projections induce isomorphisms in the homotopy groups for all  $i \geq 2$ . By [StackExchange](#) there is a unique lift  $\tilde{f}$  to the universal covers making the diagram on the left commute, and by functoriality the diagram on the right also commutes.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} \pi_i(\tilde{X}) & \xrightarrow{\tilde{f}_*} & \pi_i(\tilde{Y}) \\ p_* \downarrow \cong & & \cong \downarrow q_* \\ \pi_i(X) & \xrightarrow{f_*} & \pi_i(Y) \end{array} \quad i \geq 2$$

We conclude that  $\tilde{f}$  is a **weak homotopy equivalence**, and by prop. 4.21 it induces isomorphisms on homology groups. Finally, by Hurewicz theorem (coro. 4.33) it is an homotopy equivalence and so is  $f$ .  $\square$

**Exercise (4.2.15).** Show that a closed simply connected 3-manifold is homotopy equivalent to  $S^3$ .

*Solution.* Since both  $S^3$  and  $M$  are simply connected, by Whitehead's theorem it suffices to construct a map  $M \rightarrow S^3$  that induces isomorphisms on  $\pi_n(X, x_0)$ . To construct the map first notice that  $M$  is 2-connected. To see that  $\pi_2(M) = 0$  we notice that  $H^2(M) \cong H_1(M) \cong \pi_1^{\text{ab}}(X) \cong 0$  by Poincaré duality. By Universal Coefficient Theorem (?), we see that **(the free-torsion part is the same in homology and cohomology, yielding)**  $H_2(M) = 0$  too. Now we use Hurewicz theorem, which tells us that the first non-zero homotopy group is isomorphic to the first non-zero homology group via the Hurewicz map  $h$  :

$\pi_3(M) \cong H_3(M)$ . Further, since  $M$  is simply-connected, it is orientable by prop. 3.25, and by thm 3.26  $H_3(M) \cong \mathbb{Z}$ .

The generator of  $\pi_3(M)$  is the map we need to apply Whitehead's theorem. Indeed, it is a map  $f : S^3 \rightarrow M$  such that  $h[f] = f_*(\alpha)$  with  $\alpha$  a generator of  $H_n(D^n, \partial D^n)$ , is a generator of  $H_3(M)$  by definition of the Hurewicz map. In other words,  $f_*$  maps generator to generator and thus is an isomorphism. Since the other homotopy groups are zero, we are done.  $\square$

**Exercise (4.2.31).** (This was solved using comments from exercise lecture and again [this internet pdf.](#))

For a fiber bundle  $F \rightarrow E \rightarrow B$  such that the inclusion  $F \hookrightarrow E$  is homotopic to a constant map, show that the long exact sequence of homotopy groups breaks into split short exact sequences giving isomorphisms  $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$ . In particular, for the Hopf bundles  $S^3 \rightarrow S^7 \rightarrow S^4$  and  $S^7 \rightarrow S^{15} \rightarrow S^8$  this yields isomorphisms

$$\begin{aligned}\pi_n(S^4) &\cong \pi_n(S^7) \oplus \pi_{n-1}(S^3) \\ \pi_n(S^8) &\cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)\end{aligned}$$

Thus  $\pi_7(S^4)$  and  $\pi_5(S^{15}) \oplus \pi_{n-1}(S^7)$  contain  $\mathbb{Z}$  summands.

*Solution.* Consider the long exact sequence in homotopy,

$$\cdots \rightarrow \pi_i(F) \rightarrow \pi_i(E) \rightarrow \pi_i(B) \rightarrow \pi_{i-1}(F) \rightarrow \pi_{i-1}(E) \rightarrow \cdots$$

This yields

$$0 \rightarrow \pi_i(E) \rightarrow \pi_i(B) \rightarrow \pi_{i-1}(F) \rightarrow 0.$$

To show that this sequence splits we show there is an arrow that goes backward in the last part of the previous sequence. Let  $f$  represent a homotopy class in  $\pi_{i-1}(F)$  and construct the following diagram:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & F \\ \downarrow & & \downarrow \\ D^n & \dashrightarrow & E \\ \downarrow & & \downarrow \\ D^n/S^{n-1} \cong S^n & \rightarrow & B \end{array}$$

where the dashed arrow is defined by

$$D^n \ni v \mapsto \begin{cases} H(f(v/|v|), |v|) & v \neq 0 \\ \text{basepoint} & v = 0 \end{cases}$$

with respect to a nullhomotopy  $H : F \times I \rightarrow E$  of the inclusion  $F \hookrightarrow E$ .  $\square$

**Exercise (4.2.32).** Show that if  $S^k \rightarrow S^m \rightarrow S^n$  is a fiber bundle, then  $k = n - 1$  and  $m = 2n - 1$ . [Look at the long exact sequence of homotopy groups.]

*Solution.* From the previous exercise we have

$$\pi_i(S^n) = \pi_i(S^m) \oplus \pi_{i-1}(S^k).$$

Notice that the inclusion of the fiber in the total space is homotopic to a constant map because this is a fibration, ie. there are local neighbourhoods in the base where the preimage looks like  $\mathbb{R}^n \times S^n$ , implying that  $n + k = m$ , that is,  $k < m$ . So  $\pi_k(S^m) = 0$ .

Now if we take  $i = n$ , we get that

$$\mathbb{Z} \cong \pi_n(S^m) \oplus \pi_{n-1}(S^k).$$

Now observe that

- If  $k = 0$  and  $n = m > 1$  then  $S^n = S^m$  is simply-connected and there is no non-trivial covering  $S^m \rightarrow S^n$ .
- $k = 0$  and  $m = n = 1$ , then there is  $S^0 \rightarrow S^1 \rightarrow S^1$ .
- $k > 0$  then  $n < m$ , so  $\pi_n(S^m) = 0$  and then  $\mathbb{Z} \cong \pi_{n-1}(S^k)$ . This means that  $n - 1 \geq k$ .

Now choose  $i = k + 1$ . We get that

$$\pi_{k+1}(S^n) \cong \pi_{k+1}(S^m) \oplus \mathbb{Z}.$$

This means that, since  $m = n + k$  because fiber bundle, we have  $n + k \geq 2k + 1 > k + 1$ . This implies that  $\pi_{k+1}(S^m) = 0$ . Finally  $\pi_{k+1}(S^n) \cong \mathbb{Z} \implies k + 1 \geq n$ .  $\square$

**Exercise (4.2.34).** Let  $p : S^3 \rightarrow S^2$  be the Hopf bundle and let  $q : T^3 \rightarrow S^3$  be the quotient map collapsing the complement of a ball in the 3-dimensional torus  $T^3 = S^1 \times S^1 \times S^1$  to a point. Show that  $pq : T^3 \rightarrow S^2$  induces the trivial map on  $\pi_*$  and  $\tilde{H}_*$ , but is not homotopic to a constant map.

*Solution.* First let's show that  $pq$  induces a trivial map on  $\pi_*$  and  $\tilde{H}_*$ . Recall that the product behaves good in homotopy groups, so that  $\pi_1(T^3) \cong \mathbb{Z}^3$  and  $\pi_i(T^3) \cong 0$  for  $i > 1$ .

Now, notice that fiber bundles are Hurewicz fibrations (over second countable manifolds). This gives us a lift

$$\begin{array}{ccc} T^3 & \xrightarrow{\quad} & S^3 \\ \downarrow & \nearrow \gamma & \downarrow \\ T^3 \times I & \xrightarrow{\quad} & S^2 \end{array}$$

We get a map  $g : T^3 \rightarrow S^3$  that factors through the fiber

$$\begin{array}{ccc} g : H_3(T^3) & \xrightarrow{\quad} & H_3(S^3) \\ & \searrow \quad \swarrow & \\ & H_3(S^1) & \end{array}$$

which makes  $f_* : H_3(T^3) \rightarrow H_3(S^3)$  an isomorphism. □

**Exercise.** There is a fiber sequence  $U(n) \hookrightarrow U(n+1) \rightarrow U(n+1)/U(n) \cong S^{2n+1}$ . Use this to show that  $\pi_k(U(n)) \rightarrow \pi_k(U(n+1))$  is isomorphism for  $n > k/2$ . Compute  $\pi_k(U(n))$  for  $n \geq 2$  and  $k = 1, 2, 3$ . In fact, if  $k$  is even then  $\pi_k(U(N)) = 0$  and if  $k$  is odd then  $\pi_k(U(N)) = \mathbb{Z}$ , where again  $N > k/2$ . These equalities are known as Bott periodicity.

*Solution.* The required isomorphisms  $\pi_k(U(n)) \rightarrow \pi_k(U(n+1))$  follow simply from the fact that  $S^{2n+1}$  is  $2n+1$ -connected: in the long homotopy sequence of the fiber bundle we have

$$\pi_{k+1}(S^{2n+1}) \longrightarrow \pi_k(U(n)) \longrightarrow \pi_k(U(n+1)) \longrightarrow \pi_k(S^{2n+1})$$

so when  $2n+1 > k+1 \iff n > k/2$  the homotopy groups of the spheres vanish and we have an isomorphism.

The group  $\pi_1(U(n))$  is isomorphic to  $\mathbb{Z}$ . This follows from the fact that  $U(1)$  is homeomorphic to a circle and by induction using the former isomorphism  $\pi_1(U(n)) \cong \pi_1(U(n+1))$ . We also have  $\pi_2(U(1)) = 0$ , so that again by induction we get  $\pi_2(U(n)) = 0$ . Finally, a similar argument shows  $\pi_3(U(n)) = 0$ . □

## Cohomology ring of $\mathbb{CP}^n$

**Exercise.** Show that

$$H^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$$

where  $\alpha$  has degree 2.

*Proof.* The CW structure of  $\mathbb{CP}^n$  consists of one cell for every even dimension. This gives us the following chain complex:

$$\begin{array}{ll} \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \dots \longrightarrow \mathbb{Z}, & \text{if } n \text{ is even} \\ \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \dots \longrightarrow 0, & \text{if } n \text{ is odd} \end{array}$$

which yields the cohomology

$$H^i(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}, & i = 0, 2, 4, \dots, 2n \\ 0, & i \text{ otherwise} \end{cases}$$



so that

$$H^\bullet(\mathbb{CP}^n) = H^0(\mathbb{CP}^n) \oplus H^2(\mathbb{CP}^n) \oplus \dots \oplus H^{2n}(\mathbb{CP}^n)$$

This means that the underlying group of the cohomology ring is the same as that of

$$\mathbb{Z}[\alpha]/(\alpha^{n+1})$$

where  $\alpha$  has degree 2. To show that these groups are also isomorphic as algebras we can use Poincaré duality as follows.

Consider the case  $n = 2$ , where we may immediately multiply the generator of second cohomology group with itself:

$$\begin{aligned} H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2) &\rightarrow H^4(\mathbb{CP}^2) \\ (\alpha, \alpha) &\mapsto \alpha \smile \alpha = \alpha^2 \end{aligned}$$

By Poincaré duality this map is a nondegenerate symmetric bilinear form, so it must map generator to a generator. **The fact that the product of the generator in degree 2 is the generator of degree 4** yields an homomorphism

$$\begin{aligned} \varphi : \mathbb{Z}[\alpha] &\rightarrow H^\bullet(\mathbb{CP}^n) \\ \alpha &\mapsto \alpha \in H^2(\mathbb{CP}^n) \end{aligned}$$

with kernel  $(\alpha^{n+1})$  as desired.

Now the case of  $\mathbb{CP}^3$  is:

$$\begin{aligned} H^2(\mathbb{CP}^3) \times H^4(\mathbb{CP}^3) &\rightarrow H^6(\mathbb{CP}^3) \\ (\alpha, \alpha^2) &\mapsto \alpha \smile \alpha^2 = \alpha^3 \end{aligned}$$

which also maps generator to generator, producing the desired algebra isomorphism. Notice we have used the group isomorphism  $H^4(\mathbb{CP}^3) \approx H^4(\mathbb{CP}^2)$  when denoting the generator of  $H^4(\mathbb{CP}^3)$  as  $\alpha^2$ . Such an isomorphism is induced by inclusion  $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$  via relative cohomology exact sequence.

The case for dimension  $n$  follows by induction. □

## Spectral sequences

### 1 Wang spectral sequence

Let's use course notes from S. Burkin and [wikipedia](#) to understand Wang spectral sequence.

We will use the following result:

**Theorem 4 (Serre spectral sequence for homology).** Let  $F \rightarrow X \rightarrow B$  be a fibration with  $B$  path-connected. If  $\pi_1(B)$  acts trivially on  $H_*(F; G)$ , then there is a spectral sequence  $\{E_{p,q}^r, d_r\}$  with

•

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-q,q+r-1}^r \quad \text{and} \quad E_{p,q}^{r+1} = \ker d_{p,q}^r / \text{img } d_{p,q}^r$$

• The groups

$$F_p H_n := \text{img}(H_n(X_p) \rightarrow H_n(X))$$

where the map  $H_n(X_p) \rightarrow H_n(X)$  is just the induced map by inclusion  $X_p \hookrightarrow X$ , form a filtration

$$0 \subset F_0 H_n \subset \dots \subset F_n H_n = H_n(X; G)$$

of  $H_n(X; G)$  such that

$$E_{p,n-p}^\infty \cong F_p H_n / F_{p-1} H_n.$$

Another way to write this is

$$H_{p,q}^\infty = F_p H_{p+q} / F_{p-1} H_{p+q}.$$

•

$$E_{p,q}^2 \cong H_p(B; H_q(F; G))$$

To start consider a fibration over a sphere  $F \hookrightarrow E \rightarrow B$ . According to our theorem for Serre spectral sequences, we know that the  $n$ -th page looks as follows:

$n$	$H_n(F)$	$H_n(F)$
	$\vdots$	$\vdots$
$2$	$H_2(F)$	$H_2(F)$
$1$	$H_1(F)$	$H_1(F)$
$0$	$H_0(F)$	$H_0(F)$
	$0 \quad \dots \quad n$	

We can see there can be nontrivial differentials only on the  $n$ th page, so that  $E^{n+1} = E^n$ . Moreover, the differentials that don't vanish are of the form

$$E_{n,q}^n \xrightarrow{d_{n,q}^n} E_{0,q+n-1}^n$$

For  $d_{n,q-n}^n$  we obtain

$$0 \longrightarrow \ker d_{n,q-n}^n \xrightarrow{\text{inclusion}} E_{n,q-n}^n \xrightarrow{d_{n,q-n}^n} E_{0,q-1}^n \xrightarrow{\text{quotient}} \text{coker } d_{n,q-n}^n \longrightarrow 0$$

But all that is just

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker d_{n,q-n}^n & \xrightarrow{\text{inclusion}} & E_{n,q-n}^n & \xrightarrow{d_{n,q-n}^n} & E_{0,q-1}^n \xrightarrow{\text{quotient}} \text{coker } d_{n,q-n}^n \longrightarrow 0 \\
& & \parallel & & \parallel & & \parallel \\
0 & \longrightarrow & \frac{\ker d_{n,q-n}^n}{\text{img } d_{n,q-n}^{n-1}} & \longrightarrow & E_{n,q-n}^2 & \longrightarrow & E_{0,q-1}^2 \longrightarrow \frac{E_{n,q-n}^n}{\text{img } d_{n,q-n}^n} \longrightarrow 0 \\
& & \parallel & & \parallel & & \parallel \\
0 & \longrightarrow & E_{n,q-n}^\infty & \longrightarrow & H_{q-n}(F) & \longrightarrow & H_{q-1}(F) \longrightarrow E_{0,q-1}^\infty \longrightarrow 0
\end{array}$$

This is the first "half" of the Wang sequence. For the other half recall that by the theorem of Serre spectral sequence for homology we know that

$$E_{p,n-p}^{\infty} = F_p H_n / F_{p-1} H_n$$

for a filtration on the  $n$ -th homology group of the total space

$$\emptyset = F_{-1} \subset F_0 \subset F_1 \subset \dots \subset F_n = H_n(E).$$

And recall that we may also write

$$E_{p,q}^{\infty} = F_p H_{p+q} / F_{p-1} H_{p+q}.$$

Then we can see that

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{0,q}^{\infty} & \xrightarrow{i^*} & H_q(E) & \longrightarrow & E_{n,q-n}^{\infty} \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \frac{F_0 H_q(F)}{0 F_{-1} H_q(F)} & & & & \frac{F_n H_q(E)}{F_{n-1} H_q(E)} \end{array}$$

But

$$E_{0,q}^{\infty} = \frac{F_0 H_q(F)}{0 F_{-1} H_q(F)} \stackrel{\text{why?}}{=} H_q(F)$$

This is the other "half" of the Wang sequence. It only remains to put both halves together and conclude that

$$\dots \rightarrow H_q(F) \xrightarrow{i^*} H_q(E) \rightarrow E_{n,q-n}^{\infty} \xrightarrow{d_{n,q-n}^n} H_{q-1}(F) \rightarrow H_{q-1}(E) \rightarrow E_{0,q-1}^{\infty} \rightarrow \dots$$

But we may remove the  $E^{\infty}$  terms to get

$$\dots \rightarrow H_q(F) \xrightarrow{i^*} H_q(E) \rightarrow H_{q-n}(F) \xrightarrow{d_{n,q-n}^n} H_{q-1}(F) \rightarrow H_{q-1}(E) \rightarrow H_{q-n-1}(F) \rightarrow \dots$$

## 2 Extra exercise on spectral sequences

**Exercise (June 20).** Let  $F \hookrightarrow E \rightarrow B$  be a Serre fibration,  $\mathbb{F}$  a field and suppose the  $\pi_1(B) \curvearrowright H_*(F, \mathbb{F})$  is trivial. Suppose  $\chi(B)$  and  $\chi(F)$  exist. Show that  $\chi(E) = \chi(F) \cdot \chi(B)$ .

*Solution.* Recall that for a topological space  $X$ ,

$$\chi(X) = \sum_i (-1)^i \dim H_i(X, \mathbb{F}).$$

Also recall that when the coefficients in a spectral sequence are a field, we have the following nice expression for the associated graded:

$$H_n(X) \cong \bigoplus_p E_{p,n-p}^{\infty} = \bigoplus_p F_p H_n / F_{p-1} H_n$$

Then we have that

$$\begin{aligned}
\chi(E) &= \sum_i (-1)^i \dim H_i(E) \\
&= (-1)^i \sum_i \dim \left( \bigoplus_p F_p / F_{p-1} \right) \\
&= (-1)^i \sum_i \dim E_{p,n-p}^\infty
\end{aligned}$$

On the other hand, we have that

$$\chi(F) \cdot \chi(B) = \sum_i (-1)^i \dim H_i(F) \cdot \sum_j (-1)^j \dim H_j(B) = \sum_{i,j} (-1)^{i+j} E_{j,i}^r.$$

If we understand the last equation, our exercise is solved once we show that

**Claim.** The expression  $\sum_{i,j} (-1)^{i+j} E_{j,i}^r$  does not depend on  $r$ .

Here's a few facts that will turn out helpful

- $H_i(X, \mathbb{F}^k) \cong \bigoplus_k H_i(X, \mathbb{F})$
- $H_i(X, V) = V \otimes H_i(X, \mathbb{F})$  for a vector space  $V$  over  $\mathbb{F}$ .
- $H_p(B, H_q(F, \mathbb{F})) \cong H_q(F, \mathbb{F}) \otimes H_p(B, \mathbb{F})$ .
- For a chain complex

$$\cdots \rightarrow C_k \rightarrow C_{k-1} \rightarrow \cdots$$

we have

$$\sum_i (-1)^i \dim C_i = \sum_i (-1)^i \dim H_i(C_\bullet)$$

□

## Exam questions

### 1 Computation of $\pi_4(S^3)$

#### 1.1 Solution from [StackExchange](#)

#### 1.2 Solution from [754notes.pdf](#)

When we apply the Postnikov tower to  $S^3$ , we obtain from the third and fourth levels a fibration

$$K(\pi_4(S^3), 4) \hookrightarrow Y_4 \rightarrow Y_3 = K(\mathbb{Z}, 3).$$

Now we do the homology spectral sequence of this fibration. We get the following second page

$$\begin{array}{c|cccc} 4 & & & & \\ 3 & & & & \\ 2 & & & & \\ 1 & & & & \\ 0 & \pi_4 & & & \pi_4 \\ \hline E^2 & 0 & 1 & 2 & 3 \end{array}$$

where  $\pi_4 = \pi_4(S^3)$ . But we don't know what lies further up nor right.

We should look at the 5th page, since the differential

$$d_{5,0}^5 : E_{5,0}^5 \rightarrow E_{0,4}^5$$

has chances of being non-trivial.

$$\begin{array}{c|cccccc} 4 & \pi_4 & & & & \pi_4 \\ 3 & & & & & \\ 2 & & & & & \\ 1 & & & & & \\ 0 & \pi_4 & & & \pi_4 & ? \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

Recall that a CW decomposition of  $S^3$  is given by a 3-cell and a 0-cell. Since  $Y_4$  is obtained from  $S^3$  by attaching cells of dimension  $n+2$ , we get that

$$H_4(Y_4) = H_5(Y_5) = 0.$$

We know there is a filtration on homology

$$0 = F_{-1}H_n \subset F_0H_n \subset \dots \subset F_nH_n = H_n(Y_4)$$

such that

$$E_{p,q}^\infty = F^p H^{p+q} / F^{p-1} H^{p+q}.$$

But the 4th and 5th homology groups are zero, so the filtration is zero and so are the groups  $E_{p,q}^\infty$  with  $p+q = 4, 5$ . And of course the groups  $E_{5,0}^6$  and  $E_{0,4}^6$  are (always) already  $E_{5,0}^\infty$  and  $E_{0,4}^\infty$ , meaning

$$0 = E_{5,0}^6 = \frac{\ker d^5}{\text{img } d^5} \implies \pi_4(S^3) \cong H_5(K(\mathbb{Z}, 3)).$$

...so finally we are convinced that we are looking for  $H^5(K(\mathbb{Z}, 3), \mathbb{Z})$ .

To find such a group let's compute the cohomology spectral sequence of the fibration

$$\Omega K(\mathbb{Z}, 3) \hookrightarrow PK(\mathbb{Z}, 3) \rightarrow K(\mathbb{Z}, 3).$$

We immediately notice  $\pi_i(\Omega K(\mathbb{Z}, 3)) \cong \pi_{i+1}(K(\mathbb{Z}, 3))$  by the long exact sequence in homotopy. So, another name for  $\Omega K(\mathbb{Z}, 3)$  is  $K(\mathbb{Z}, 2)$  which in turn is just  $\mathbb{CP}^\infty$ . To find out how the second page looks like we use the fact that homology groups of  $\mathbb{CP}^\infty$  are finitely generated, **so that by the universal coefficients theorem we have**

$$E_2^{p,q} = H^p(K(\mathbb{Z}, 3); H^q(\mathbb{CP}^\infty)) \cong H^p(K(\mathbb{Z}, 3)) \otimes H^q(\mathbb{CP}^\infty).$$

This gives us

4	$a^2\mathbb{Z}$	$s\mathbb{Z} \otimes a^2\mathbb{Z}$					
3							
2	$a\mathbb{Z}$	$s\mathbb{Z} \otimes a\mathbb{Z}$					
1							
0	$\mathbb{Z}$	$s\mathbb{Z} \otimes \mathbb{Z}$					?
	0	1	2	3	4	5	6

where we have chosen generators for each cohomology group in  $H^\bullet(K(\mathbb{Z}, 3)) \otimes H^\bullet(\mathbb{CP}^\infty)$  according to its product structure.

ow, again by the **universal coefficient theorem for cohomology** we have a short exact sequence

$$0 \longrightarrow \text{Ext}(H_{i-1}(X; \mathbb{R}), G) \longrightarrow H^i(X; G) \xrightarrow{h} \text{Hom}(H_i(X; \mathbb{R}), G) \longrightarrow 0$$

which splits, giving for  $i = 5$

$$H^5 = \text{Hom}(H_5, \mathbb{Z}) \oplus \text{Ext}(H_4, \mathbb{Z}),$$

where the homology and cohomology groups correspond to  $K(\mathbb{Z}, 3)$ .

Since cohomology groups are finitely generated, **we may write**

$$H_5 = \text{Free part of } H^5 \oplus \text{Torsion part of } H^6.$$

"and because the generator squared is minus the generator, the group  $H^5$  must be  $\mathbb{Z}/2$ . And  $H^6$  is zero."

## References

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